Adding Threshold Concepts to the Description Logic $\mathcal{EL}$

Von der Fakultät für Mathematik und Informatik
der Universität Leipzig
genanmmene

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

im Fachgebiet
Informatik

Vorgelegt
von M.Sc. Oliver Fernández Gil

geboren am 11. Juni 1982 in Pinar del Río, Kuba

Die Annahme der Dissertation wurde empfohlen von:

1. Prof. Dr.-Ing. Franz Baader, TU Dresden
2. Prof. Dr. Gerhard Brewka, University of Leipzig
3. Prof. Dr. habil. Frank Wolter, University of Liverpool

Die Verleihung des akademischen Grades erfolgt mit Bestehen
der Verteidigung am 18.05.2016 mit dem Gesamtprädikat

magna cum laude
To Margot
Acknowledgements

First of all, I would like to thank my supervisors Gerhard Brewka and Franz Baader for their continuous support during the last three years. They taught many new things and were always open to talk to me. I am also grateful to Prof. Frank Wolter for reviewing this thesis.

I am indebted to the Deutsche Forschungsgemeinschaft (DFG) for supporting me financially within the scope of the Graduiertenkolleg 1763 “Quantitative Logics and Automata”.

To my friends, wherever they are, thanks for everything. I always know that we will meet again some sunny day, :).

Thank you Julia and Santiago for being so beautiful and special to me, and for giving me so much innocent love in such a short time.

Finally, and most importantly, I want to thank my parents Hugo and Bárbara, my brothers Huguito and Ale and my sister Lauri. You have always been there for me no matter how far you are. I love you all so much!
## Contents

1 Introduction
   1.1 Related work
      1.1.1 Fuzzy DLs
      1.1.2 The logic \textit{sim-ALCQO}
      1.1.3 Concept similarity measures
   1.2 Structure of the Thesis

2 The Description Logic \(\mathcal{EL}\)
   2.1 Characterization of membership in \(\mathcal{EL}\)

3 The Logic \(\tau\mathcal{EL}(m)\)
   3.1 Description graphs and homomorphisms in \(\tau\mathcal{EL}(m)\)
   3.2 Deciding the existence of a \(\tau\)-homomorphism

4 The membership function \(\text{deg}\)
   4.1 The membership function \(\text{deg}\)
   4.2 Two useful properties of \(\text{deg}\)
   4.3 Relation to the Description Logic \(\mathcal{ALC}\)
      4.3.1 Full negation is not expressible in \(\tau\mathcal{EL}(\text{deg})\)
      4.3.2 Expressing \(\tau\mathcal{EL}(\text{deg})\) concept descriptions in \(\mathcal{ALC}\)

5 Reasoning in \(\tau\mathcal{EL}(\text{deg})\)
   5.1 Terminological reasoning
   5.2 Assertional reasoning

6 Adding Terminologies to \(\tau\mathcal{EL}(m)\)
   6.1 \(\mathcal{EL}\) TBoxes
   6.2 TBoxes for \(\tau\mathcal{EL}(m)\) and \(\tau\mathcal{EL}(\text{deg})\)
   6.3 Models of non-polynomial size
   6.4 Reasoning with respect to acyclic \(\tau\mathcal{EL}(\text{deg})\) TBoxes
      6.4.1 Lower bounds
      6.4.2 Normalization
      6.4.3 Upper bounds
      6.4.4 Reasoning with acyclic knowledge bases

7 Concept similarity measures, relaxed instance queries and \(\tau\mathcal{EL}(m)\)
   7.1 Defining membership degree functions
   7.2 Reasoning in \(\tau\mathcal{EL}(\text{med})\)
      7.2.1 Undecidability
      7.2.2 Decidability

vii
Chapter 1
Introduction

Description Logics (DLs) \[\text{BCM}^+\text{03}\] are a family of logic-based knowledge representation formalisms, which can be used to represent the conceptual knowledge of an application domain in a structured and formally well-understood way. They allow their users to define the important notions of the domain as concepts by stating necessary and sufficient conditions for an individual to belong to the concept. These conditions can be atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). The expressivity of a particular DL is determined by what sort of properties can be required and how they can be combined.

The DL $\mathcal{EL}$, in which concepts can be built using concept names as well as the concept constructors conjunction ($\sqcap$), existential restriction ($\exists r.C$), and the top concept ($\top$), has drawn considerable attention in the last decade since, on the one hand, important inference problems such as the subsumption problem are polynomial in $\mathcal{EL}$, even with respect to expressive terminological axioms [Bra04]. On the other hand, though quite inexpressive, $\mathcal{EL}$ can be used to define biomedical ontologies, such as the large medical ontology SNOMED CT.\footnote{see http://www.ihtsdo.org/snomed-ct/} In $\mathcal{EL}$ we can, for example, define the concept of a happy man as a male human that is healthy and handsome, has a rich and intelligent wife, a son and a daughter, and a friend:

$$\text{Human} \sqcap \text{Male} \sqcap \text{Healthy} \sqcap \text{Handsome} \sqcap \exists \text{spouse.}(\text{Rich} \sqcap \text{Intelligent} \sqcap \text{Female}) \sqcap \exists \text{child}.\text{Male} \sqcap \exists \text{child}.\text{Female} \sqcap \exists \text{friend}.\top$$

(1.1)

For an individual to belong to this concept, all the stated properties need to be satisfied. However, maybe we would still want to call a man happy if most, though not all, of the properties hold. It might be sufficient to have just a daughter without a son, or a wife that is only intelligent but not rich, or maybe an intelligent and rich spouse of a different gender. But still, not too many of the properties should be violated.

In this thesis, we introduce a DL extending $\mathcal{EL}$ that allows us to define concepts in such an approximate way. The main idea is to use a graded membership function $m$, which instead of a Boolean membership value 0 or 1 yields a membership degree from the interval $[0, 1]$. We can then require a happy man to belong to the $\mathcal{EL}$ concept (1.1) with degree at least .8. More generally, if $C$ is an $\mathcal{EL}$ concept, then the threshold concept $C_t$, for $t \in [0, 1]$ collects all the individuals that belong to $C$ with degree at least $t$. In
addition to such upper threshold concepts, we will also consider lower threshold concepts $C_{\leq t}$ and allow the use of strict inequalities in both. For example, an unhappy man could be required to belong to the $\mathcal{EL}$ concept (1.1) with a degree less than $0.2$. Using these constructors and defining their underlying semantics based on a graded membership function, we define the family of DLs $\tau \mathcal{EL}(m)$ where $m$ is a parameter of the logic representing the chosen function.

We then go further and define a particular membership degree function $\deg$. Its definition is a natural extension of the homomorphism characterization of crisp membership in $\mathcal{EL}$. Basically, an individual is punished (in the sense that its membership degree is lowered) for each missing property in a uniform way. For instance, suppose that some individual $d$ belongs to the sets corresponding to $\text{Human}$ and $\text{Healthy}$ under some interpretation, but does not belong to the ones corresponding to $\text{Handsome}$ and $\text{Male}$. Then, regarding the concept description $\text{Human} \sqcap \text{Male} \sqcap \text{Healthy} \sqcap \text{Handsome}$, the computation of $\deg$ will punish $d$ for the two missing properties, and give the value $\deg(d, \text{Human} \sqcap \text{Male} \sqcap \text{Healthy} \sqcap \text{Handsome}) = 1/2$ as the degree of membership of $d$ in that concept (see Chapter 4 for the precise details).

From a technical point of view, this function is akin to the similarity measures for $\mathcal{EL}$ concepts introduced in [LT12, Sun13], though only [Sun13] directly draws its inspiration from the homomorphism characterization of subsumption in $\mathcal{EL}$. The threshold logic $\tau \mathcal{EL}(\deg)$ induced by $\deg$ constitutes the main subject of study in Chapters 5 and 6, where we investigate the complexity of reasoning in $\tau \mathcal{EL}(\deg)$ with respect to the empty terminology and to a particular form of acyclic TBoxes.

The last part of the thesis is devoted to better understand the relationship between concept similarity measures and our threshold logic formalism. We will describe a particular form of constructing membership degree functions from concept similarity measures, which then originates a wide family of threshold Description Logics. In this way, we obtain a variety of logics that could be useful in diverse scenarios according to the specific properties of their underlying similarity measures.

The remainder of this introduction is concerned with an overview on related work, and a more detailed summary of the subsequent chapters in this document.

1.1 Related work

We now provide an overview of some of the existing approaches to represent imprecise knowledge in Description Logics. We consider the ones that we believe look closest to our work. Nevertheless, there exists a vast number of other proposals. See for example [PZ13] for an extension of $\mathcal{EL}$ with the notion of rough sets, [LS10] for a family of probabilistic DLs, and [LS08] for a survey on managing uncertainty and vagueness in Description Logics.

1.1.1 Fuzzy DLs

The use of membership degree functions with values in the interval $[0, 1]$ may remind the reader of fuzzy logics. However, there is no strong relationship between this work and the work on fuzzy DLs [BDP15] for two reasons. First, in fuzzy DLs the semantics is extended to fuzzy interpretations where concept and role names are interpreted as fuzzy
sets and relations, respectively. Basically, given an interpretation \( \mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}) \) and a set of truth values \( \mathfrak{B} \):

- concept names \( A \) are interpreted as fuzzy sets \( A^\mathcal{I} : \Delta^\mathcal{I} \to \mathfrak{B} \), and
- role names \( r \) as binary fuzzy relations \( r^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to \mathfrak{B} \).

The membership degree of an individual to belong to a complex concept is then computed using fuzzy interpretations of the concept constructors (e.g., conjunction is interpreted using an appropriate triangular norm \( \otimes \)).

In our setting, we consider crisp interpretations of concept and role names, and directly define membership degrees for complex concepts based on them. Second, we use membership degrees to obtain new concept constructors, but the threshold concepts obtained by applying these constructors are again crisp rather than fuzzy. Additionally, for our threshold logics the membership degree value in a complex concept need not be systematically determined by the membership degree values of its parts. Let us illustrate this situation with a simple example. Consider the following two fragments of the concept (1.1):

\[
\text{Human} \sqcap \text{Male} \quad \text{and} \quad \text{Human} \sqcap \text{Handsome}
\]

For an individual \( x \) in an interpretation \( \mathcal{I} \) that only belongs to the concept \( \text{Human} \), the intuition explained above for \( \deg \) yields the membership degrees:

\[
\deg(x, \text{Human} \sqcap \text{Male}) = 1/2 \quad \text{and} \quad \deg(x, \text{Human} \sqcap \text{Handsome}) = 1/2
\]

Now, the relevant aspect is that, when computing \( \deg(x, \text{Human} \sqcap \text{Male} \sqcap \text{Human} \sqcap \text{Handsome}) \), we do not want to count the fact that \( x \) is a \( \text{Human} \) twice, but rather give \( 1/3 \) as the membership degree of \( x \) in the composed concept. This intuition will be captured in one of two conditions that membership functions are required to satisfy. Hence, if a specific t-norm \( \otimes \) were to be used to interpret conjunction in this particular scenario, it would satisfy \( 1/2 \otimes 1/2 = 1/3 \). However, let us further consider two more concepts:

\[
\text{Human} \sqcap \text{Male} \quad \text{and} \quad \text{Handsome} \sqcap \text{Healthy}
\]

If in addition, \( x \) also belongs to \( \text{Healthy} \) we obtain similar values as before:

\[
\deg(x, \text{Human} \sqcap \text{Male}) = 1/2 \quad \text{and} \quad \deg(x, \text{Handsome} \sqcap \text{Healthy}) = 1/2
\]

The difference is that the membership degree of \( x \) in the concept representing the composition of these two concepts, as already explained above, is \( \deg(x, \text{Human} \sqcap \text{Male} \sqcap \text{Handsome} \sqcap \text{Healthy}) = 1/2 \). Obviously, this is not consistent with the initial definition of \( \otimes \) for the pair \((1/2, 1/2)\), as required before.

### 1.1.2 The logic \( \text{sim-ALCQO} \)

In [LWZ03], the authors introduced the Description Logic \( \text{sim-ALCQO} \) for expressing “vague” concepts and reasoning about them. This logic is obtained as the result of combining the DL \( \text{ALCQO} \) [HS01], and the logic \( \mathcal{M} \mathcal{S} \) introduced in [WZ03] for reasoning about metric spaces. In particular, the integration of \( \mathcal{M} \mathcal{S} \) with \( \text{ALCQO} \) allows to
express concepts of the form $E^{\leq a}C$ (among others). The interpretation of such a concept description collects the set of all individuals $x$ that are similar to at least one instance $y$ of $C$ with degree at most $a$. Here, $x$ and $y$ are elements of a domain $W$ and the similarity is measured by a distance function $d$, where $\langle W, d \rangle$ is a metric space.

In principle we could try to express a threshold concept $C \geq t$ as the $\text{sim-}\mathcal{ALCQO}$ concept $E^{\leq (1-t)}C$. This is based on the idea that by using a distance function, the closer two individuals are the more similar they are. Therefore, if $y$ is an instance of $C$ and $d(x, y) \leq 1-t$, we could interpret this as that $x$ is an element of $C \geq t$. However, the most important difference to our approach is that [LWZ03] do not fix a specific distance function $d$, but reason with respect to all possible such functions. This, for example, includes distance functions which do not take into account the conceptual structure of the domain elements in an interpretation to measure the distance between them.

1.1.3 Concept similarity measures

In the last few years, the idea of measuring similarity between concepts described in DLs has received considerable attention. Many concept similarity measures have been proposed to approach problems from a different/new perspective in very dissimilar domains. See for example [BWH05] for an early survey on the topic, and [SPF08, PFE06, EPT15, Sun13] for recently proposed measures and their applications.

One particular application of concept similarity measures to DLs is suggested in [EPT14, EPT15]. Instead of requiring that an individual is an instance of a query concept, the authors only require that it is an instance of a concept that is “similar enough” to the query concept. A somehow related approach has been presented in [TS14], but following the ideas exposed in [Sun13]. As we will show in Chapter 7, such kind of relaxed instance queries can be expressed as instance queries with respect to threshold concepts of the form $C \geq t$. However, the new family of DLs introduced in this thesis is considerably more expressive than just such threshold concepts since we also allow the use of comparison operators other than $>$ in threshold concepts, and the threshold concepts can be embedded in complex $\mathcal{EL}$ concepts.

1.2 Structure of the Thesis

In the following, we briefly describe the contents of each chapter of the thesis.

- Chapter 2 formally introduces the lightweight Description Logic $\mathcal{EL}$. We start by presenting the syntax and semantics of $\mathcal{EL}$, as well as defining some technical notions that will be important for the rest of the thesis. To conclude, we then recall the well-known characterization of element-hood in $\mathcal{EL}$ concepts via existence of homomorphisms between $\mathcal{EL}$ description graphs (which can express both $\mathcal{EL}$ concepts and interpretations in a graphical way).

- In Chapter 3 we introduce our new family of DLs $\tau\mathcal{EL}(m)$. We extend $\mathcal{EL}$ by new threshold concept constructors which are based on an arbitrary, but fixed graded membership function $m$ (hence the name $\tau\mathcal{EL}(m)$). We will impose some minimal requirements on such membership functions, and show the consequences
that these conditions have for our threshold logic. Afterwards, we define description graphs and the notion of $\tau$-homomorphisms for $\tau\mathcal{E}\mathcal{L}(m)$. Based on them we show that membership in $\tau\mathcal{E}\mathcal{L}(m)$ concept descriptions can be characterized by the existence of $\tau$-homomorphisms. Such a characterization is independent of the used graded membership function, and will be crucial for the study of the computational complexity of inference problems carried out in subsequent chapters. Finally, we provide algorithms that for finite interpretations, can be used to decide membership in $\tau\mathcal{E}\mathcal{L}(m)$ concepts according to the given characterization.

- Chapter 4 introduces the graded membership function $\text{deg}$. We show that $\text{deg}$ is well-defined and satisfies the properties required for membership functions in Chapter 3. In the last part of the chapter, we look at the relationship between its induced threshold logic $\tau\mathcal{E}\mathcal{L}(\text{deg})$ and the DL $\mathcal{ALC}$ [SS91]. On the one hand, we show that full negation is not expressible in $\tau\mathcal{E}\mathcal{L}(\text{deg})$, and thus there are $\mathcal{ALC}$ concept descriptions that cannot be expressed in $\tau\mathcal{E}\mathcal{L}(\text{deg})$. On the other hand, we prove that $\tau\mathcal{E}\mathcal{L}(\text{deg})$ is a fragment of $\mathcal{ALC}$.

- Chapter 5 investigates the computational properties of $\tau\mathcal{E}\mathcal{L}(\text{deg})$. We start by considering satisfiability and subsumption as the standard reasoning tasks concerning terminological reasoning. In contrast to $\mathcal{E}\mathcal{L}$, the satisfiability problem is not trivial and it turns out to be NP-hard. A matching upper bound is obtained due to the existence of polynomial size models for all satisfiable concepts. Then, we demonstrate, that the ideas used to construct such small models can be extended to concepts of the form $\hat{C} \sqcap \neg \hat{D}$ where $\hat{C}$ and $\hat{D}$ are $\tau\mathcal{E}\mathcal{L}(\text{deg})$ concepts. Since $\tau\mathcal{E}\mathcal{L}(\text{deg})$ cannot express negation of $\tau\mathcal{E}\mathcal{L}(\text{deg})$ concepts, this comes in handy to prove that subsumption is a complete problem for the class coNP. Finally, we are able to extend these ideas further to deal with assertional knowledge, and show that ABox consistency is NP-complete whereas the instance problem is coNP-complete (w.r.t. data complexity).

- Chapter 6 is concerned with extending our logic $\tau\mathcal{E}\mathcal{L}(\text{deg})$ to consider concept descriptions defined in a background TBox. We first extend well-defined graded membership functions to compute membership degrees with respect to acyclic $\mathcal{E}\mathcal{L}$ TBoxes. Subsequently, $\tau\mathcal{E}\mathcal{L}(m)$ and $\tau\mathcal{E}\mathcal{L}(\text{deg})$ TBoxes are defined taking into account some necessary restrictions. We will see that the presence of TBoxes apparently increases the computational complexity of the satisfiability and subsumption problems, namely, they become $\Pi^p_2$- and $\Sigma^p_2$-hard, respectively. These hardness results hold already with respect to acyclic $\tau\mathcal{E}\mathcal{L}(\text{deg})$ TBoxes. Regarding upper bounds, we design a non-deterministic polynomial space algorithm that solves both problems, thus providing membership in PSPACE for both of them. Moreover, these PSPACE upper bounds carry over to reasoning with respect to acyclic $\tau\mathcal{E}\mathcal{L}(\text{deg})$ knowledge bases.

- In Chapter 7, we study the relationship between our threshold DLs $\tau\mathcal{E}\mathcal{L}(m)$ and concept similarity measures. The chapter is organized into three main parts. To start, we show that a variant of the relaxed instance query approach of [EPT14] can be used to turn a similarity measure $\bowtie$ into a well-defined graded membership
function $m_{\infty}$, and consequently $\triangleright$ induces a threshold logic $\tau\mathcal{E}\mathcal{L}(m_{\infty})$. In addition, we show that the relaxed instance queries of [EPT14] can be expressed as instance queries w.r.t. threshold concepts of the form $C_{>t}$. The second part of the chapter explores the computational complexity landscape of reasoning in such a big family of threshold logics. We obtain undecidability and decidability results, as well as more precise complexity results for logics induced by a particular class of measures satisfying certain properties. Last, we present the framework $\text{simi}$ introduced in [LT12] for defining similarity measures, and identify a concrete subclass of its instances exhibiting those properties. Moreover, it turns out that, applied to a simple instance $\triangleright^{1}$ of $\text{simi}$, our construction actually yields our membership function $\deg$.

- In Chapter 8, we summarize our results and point out several directions for future work.
- Appendix A contains missing proofs of some results needed along this document.

The results of this thesis consisting of the definition of the family of DLs $\tau\mathcal{E}\mathcal{L}(m)$, the graded membership function $\deg$ and the computational properties of $\tau\mathcal{E}\mathcal{L}(\deg)$ studied in Chapter 5 have previously been published in [BBG15a] and [BBG15b].
Chapter 2

The Description Logic $\mathcal{EL}$

We start by introducing the Description Logic $\mathcal{EL}$. Starting with finite sets of concept names $\mathbb{NC}$ and role names $\mathbb{NR}$, the set $\mathcal{CE}_\mathcal{EL}$ of $\mathcal{EL}$ concept descriptions is obtained by using the concept constructors \textit{conjunction} ($C \cap D$), \textit{existential restriction} ($\exists r.C$) and \textit{top} ($\top$), in the following way:

$$C ::= \top \mid A \mid C \cap C \mid \exists r.C$$

where $A \in \mathbb{NC}$, $r \in \mathbb{NR}$ and $C \in \mathcal{CE}_\mathcal{EL}$.

An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ consists of a non-empty domain $\Delta^\mathcal{I}$ and an interpretation function $\mathcal{I}$ that assigns subsets of $\Delta^\mathcal{I}$ to each concept name and binary relations over $\Delta^\mathcal{I}$ to each role name. The interpretation function $\mathcal{I}$ is inductively extended to concept descriptions in the usual way:

$$\top^\mathcal{I} := \Delta^\mathcal{I}$$

$$(C \cap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}$$

$$(\exists r.C)^\mathcal{I} := \{x \in \Delta^\mathcal{I} \mid \exists y.[(x, y) \in r^\mathcal{I} \land y \in C^\mathcal{I}]\}$$

Given $C, D \in \mathcal{CE}_\mathcal{EL}$, we say that $C$ is \textit{subsumed by} $D$ (denoted as $C \subseteq D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ for every interpretation $\mathcal{I}$. These two concept descriptions are \textit{equivalent} (denoted as $C \equiv D$) iff $C \subseteq D$ and $D \subseteq C$. Finally, $C$ is \textit{satisfiable} iff $C^\mathcal{I} \neq \emptyset$ for some interpretation $\mathcal{I}$.

Information about specific individuals can be expressed in an ABox. An ABox $\mathcal{A}$ is a finite set of assertions of the form $C(a)$ or $r(a,b)$, where $C$ is an $\mathcal{EL}$ concept description, $r \in \mathbb{NR}$, and $a,b$ are individual names. For example, if HUGUITO, JULIA and SANTIAGO are individual names, one can state that HUGUITO is a human male, JULIA his daughter and SANTIAGO his son, through the following ABox $\mathcal{A}$:

$$\mathcal{A} := \{\text{Human(HUGUITO)}, \text{Male(HUGUITO)}, \text{Male(SANTIAGO)}, \text{Female(JULIA)}, \text{child(HUGUITO, JULIA)}, \text{child(HUGUITO, SANTIAGO)}\} \quad (2.1)$$

Concerning the semantics, in addition to concept and role names, an interpretation $\mathcal{I}$ now assigns domain elements $a^\mathcal{I}$ to individual names $a$. An assertion $C(a)$ is satisfied by $\mathcal{I}$ iff $a^\mathcal{I} \in C^\mathcal{I}$, and $r(a,b)$ is satisfied by $\mathcal{I}$ iff $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$. The interpretation $\mathcal{I}$ is a \textit{model} of $\mathcal{A}$ iff $\mathcal{I}$ satisfies all assertion in $\mathcal{A}$. The ABox $\mathcal{A}$ is \textit{consistent} iff it has a model, and the individual $a$ is an \textit{instance} of the concept $C$ in $\mathcal{A}$ iff $a^\mathcal{I} \in C^\mathcal{I}$ holds in all models of $\mathcal{A}$. We denote the set of individual names occurring in $\mathcal{A}$ as $\text{Ind}(\mathcal{A})$. 

7
We now define some notions related to $\mathcal{EL}$ concept descriptions that will be useful for subsequent chapters.

**Definition 2.1 (sub-description).** Let $C$ be an $\mathcal{EL}$ concept description. The set $\text{sub}(C)$ of sub-descriptions of $C$ is defined in the following way:

$$\text{sub}(C) := \begin{cases} \{C\} & \text{if } C = \top \text{ or } C \in \mathbb{N}_C, \\ \{C\} \cup \text{sub}(C_1) \cup \text{sub}(C_2) & \text{if } C \text{ is of the form } C_1 \sqcap C_2, \\ \{C\} \cup \text{sub}(D) & \text{if } C \text{ is of the form } \exists r.D. \end{cases}$$

Note that the number of sub-descriptions $|\text{sub}(C)|$ of a concept $C$ is linear in the size of $C$. Next, we define the role depth of a concept description $C$.

**Definition 2.2 (role depth).** The role depth $\text{rd}(C)$ of an $\mathcal{EL}$ concept description $C$ is inductively defined as follows:

$$\begin{align*} \text{rd}(\top) &= \text{rd}(A) := 0, \\ \text{rd}(C_1 \sqcap C_2) &= \max(\text{rd}(C_1), \text{rd}(C_2)), \\ \text{rd}(\exists r.C) &= \text{rd}(C) + 1. \end{align*} \quad \diamondsuit$$

A concept description is called an atom iff it is a concept name or an existential restriction. The set of all $\mathcal{EL}$ atoms is denoted by $\mathbb{N}_A$. Additionally, every $\mathcal{EL}$ concept description is a conjunction $C_1 \sqcap \ldots \sqcap C_n$ of atoms. These conjuncts are called the top-level atoms of $C$ and the set $\{C_1, \ldots, C_n\}$ is denoted as $\text{tl}(C)$.

Finally, given two interpretations $I$ and $J$, we say that $I$ is contained in $J$ (denoted $I \subseteq J$) iff $\Delta_I \subseteq \Delta_J$ and $X^I \subseteq X^J$ for all $X \in (\mathbb{N}_C \cup \mathbb{N}_R)$.

### 2.1 Characterization of membership in $\mathcal{EL}$

Our definition of graded membership will be based on graphical representations of concepts and interpretations, and on homomorphisms between such representations. For this reason, we recall these notions together with the pertinent results. They are all taken from [BKM99, Kus01, Baa03].

**Definition 2.3 ($\mathcal{EL}$ description graph).** An $\mathcal{EL}$ description graph is a graph of the form $G = (V_G, E_G, \ell_G)$ where:

- $V_G$ is a set of nodes.
- $E_G \subseteq V_G \times \mathbb{N}_R \times V_G$ is a set of edges labeled by role names,
- $\ell_G : V_G \to 2^{\mathbb{N}_C}$ is a function that labels nodes with sets of concept names. \quad \diamondsuit

The empty label corresponds to the top concept $\top$. In particular, an $\mathcal{EL}$ description tree $T$ is a description graph that is a tree with a distinguished element $v_0$ representing its root. In [BKM99], it was shown the correspondence that exists between $\mathcal{EL}$ concept descriptions and $\mathcal{EL}$ description trees, i.e., every $\mathcal{EL}$ concept description $C$ can be translated into a corresponding description tree $T_C$ and vice versa. Furthermore, every interpretation $I = (\Delta^I, \mathbb{X}^I)$ can be translated into an $\mathcal{EL}$ description graph $G_I = (V_I, E_I, \ell_I)$ in the following way [Baa03]:
2.1 Characterization of membership in $\mathcal{EL}$

\[ T_C: \quad v_0: \{A\} \quad \quad G_T: \quad a_1: \{A\} \]

\[ v_1: \{A, B\} \quad v_3: \{A\} \quad \quad a_2: \{A, B\} \]

\[ r \quad \quad r \quad \quad r \]

\[ v_2: \{} \quad \quad \quad \quad \quad a_3: \{B\} \]

Figure 2.1: $\mathcal{EL}$ description graphs.

- $V_I = \Delta^I$,
- $E_I = \{(vrw) \mid (v, w) \in r^I\}$,
- $\ell_I(v) = \{A \mid v \in A^I\}$ for all $v \in V_I$.

The following example illustrates the relation between concept descriptions and description trees, and interpretations and description graphs.

**Example 2.4.** The $\mathcal{EL}$ concept description

\[ C := A \sqcap \exists r. (A \sqcap B \sqcap \exists r. T) \sqcap \exists r. A \]

yields the $\mathcal{EL}$ description tree $T_C$ depicted on the left-hand side of Figure 2.1. The description graph on the right-hand side corresponds to the following interpretation:

- $\Delta^I := \{a_1, a_2, a_3\}$,
- $A^I := \{a_1, a_2\}$ and $B^I := \{a_2, a_3\}$,
- $r^I := \{(a_1, a_2), (a_2, a_3), (a_3, a_1)\}$.  

Now, we generalize homomorphisms between $\mathcal{EL}$ description trees \cite{BKM99} to arbitrary graphs.

**Definition 2.5 (Homomorphisms on $\mathcal{EL}$ description graphs).** Let $G = (V_G, E_G, \ell_G)$ and $H = (V_H, E_H, \ell_H)$ be two $\mathcal{EL}$ description graphs. A mapping $\varphi : V_G \rightarrow V_H$ is a homomorphism from $G$ to $H$ if and only if the following conditions are satisfied:

1. $\ell_G(v) \subseteq \ell_H(\varphi(v))$ for all $v \in V_G$, and
2. $vrw \in E_G$ implies $\varphi(v)r\varphi(w) \in E_H$.

This homomorphism is an isomorphism if and only if $\varphi$ is bijective, equality instead of just inclusion holds in [1], and biimplication instead of just implication holds in [2].

In Example 2.4, the mapping $\varphi$ with $\varphi(v_i) = a_{i+1}$ for $i = 0, 1, 2$ and $\varphi(v_3) = a_2$ is a homomorphism. Homomorphisms between $\mathcal{EL}$ description trees can be used to characterize subsumption in $\mathcal{EL}$. 

\[ a_1: \{A\} \quad \quad a_2: \{A, B\} \]

\[ r \quad \quad r \quad \quad r \]

\[ a_3: \{B\} \]
Chapter 2. The Description Logic $\mathcal{EL}$

Theorem 2.6 ([BKM99]). Let $C, D$ be $\mathcal{EL}$ concept descriptions and $T_C, T_D$ the corresponding $\mathcal{EL}$ description trees. Then $C \subseteq D$ iff there exists a homomorphism from $T_D$ to $T_C$ that maps the root of $T_D$ to the root of $T_C$.

The proof of this result can be easily adapted to obtain a similar characterization of element-hood in $\mathcal{EL}$, i.e., whether $d \in C^I$ for some $d \in \Delta^I$.

Theorem 2.7. Let $I$ be an interpretation, $d \in \Delta^I$, and $C$ an $\mathcal{EL}$ concept description. Then, $d \in C^I$ iff there exists a homomorphism $\varphi$ from $T_C$ to $G_I$ such that $\varphi(v_0) = d$.

In Example 2.4 the existence of the homomorphism $\varphi$ defined above thus shows that $a_1 \in C^I$. Equivalence of $\mathcal{EL}$ concept descriptions can be characterized via the existence of isomorphisms, but for this the concept descriptions first need to be normalized by removing redundant existential restrictions. To be more precise, the reduced form of an $\mathcal{EL}$ concept description is obtained by applying the rewrite rule $\exists r.C \sqcap \exists r.D \rightarrow \exists r.C$ if $C \subseteq D$ as long as possible. This rule is applied modulo associativity and commutativity of $\sqcap$, and not only on the top-level conjunction of the description, but also under the scope of existential restrictions. Since every application of the rule decreases the size of the description, it is easy to see that the reduced form can be computed in polynomial time. We say that an $\mathcal{EL}$ concept description is reduced iff this rule does not apply to it. In our Example 2.4 the reduced form of $C$ is the reduced description $A \sqcap \exists r.(A \sqcap B \sqcap \exists r.\top)$.

Theorem 2.8 ([Küs01]). Let $C, D$ be $\mathcal{EL}$ concept descriptions, $C^r, D^r$ their reduced forms, and $T_{C^r}, T_{D^r}$ the corresponding $\mathcal{EL}$ description trees. Then $C \equiv D$ iff there exists an isomorphism between $T_{C^r}$ and $T_{D^r}$.
Chapter 3

The Logic $\tau\mathcal{EL}(m)$

Our new logic will allow us to take an arbitrary $\mathcal{EL}$ concept $C$ and turn it into a threshold concept. To this end we introduce a family of constructors that are based on the membership degree of individuals in $C$. For instance, the threshold concept $C_{>0.8}$ represents the individuals that belong to $C$ with degree $>0.8$. The semantics of the new threshold concepts depends on a (graded) membership function $m$. Given an interpretation $\mathcal{I}$, this function takes a domain element $d \in \Delta^\mathcal{I}$ and an $\mathcal{EL}$ concept $C$ as input, and returns a value between 0 and 1, representing the extent to which $d$ belongs to $C$ in $\mathcal{I}$.

The choice of the membership function obviously has a great influence on the semantics of the threshold concepts. In Chapter 4 we will propose one specific such function $\deg$, but we do not claim this is the only reasonable way to define such a function. Rather, the membership function is a parameter in defining the logic. To highlight this dependency, we call the logic $\tau\mathcal{EL}(m)$.

Nevertheless, membership functions are not arbitrary. There are two properties we require such functions to satisfy:

**Definition 3.1.** A graded membership function $m$ is a family of functions that contains for every interpretation $\mathcal{I}$ a function $m^\mathcal{I}: \Delta^\mathcal{I} \times \mathcal{C}_{\mathcal{EL}} \to [0,1]$ satisfying the following conditions (for $C,D \in \mathcal{C}_{\mathcal{EL}}$):

- $M1 : d \in C^\mathcal{I} \iff m^\mathcal{I}(d,C) = 1$ for all $d \in \Delta^\mathcal{I}$,
- $M2 : C \equiv D \iff \forall \mathcal{I} \forall d \in \Delta^\mathcal{I} : m^\mathcal{I}(d,C) = m^\mathcal{I}(d,D)$.  

Property $M1$ requires that the value 1 is a distinguished value reserved for proper containment in a concept. Property $M2$ requires equivalence invariance. It expresses the intuition that the membership value should not depend on the syntactic form of a concept, but only on its semantics. Note that the right to left implication in $M2$ is already a consequence of $M1$: suppose for a contradiction that $C \not\equiv D$. This would imply that for some interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$, $d \in C^\mathcal{I}$ and $d \notin D^\mathcal{I}$ (or the opposite). Then, by $M1$ and the right-hand side of $M2$ we would obtain $m^\mathcal{I}(d,C) = 1 = m^\mathcal{I}(d,D)$, which clearly yields a contradiction against $d \notin D^\mathcal{I}$ and property $M1$.

We now turn to the syntax of $\tau\mathcal{EL}(m)$. Given finite sets of concept names $\mathcal{N}_C$ and role names $\mathcal{N}_R$, $\tau\mathcal{EL}(m)$ concept descriptions are defined as follows:

$$\widehat{C} ::= \top \mid A \mid \widehat{C} \cap \widehat{C} \mid \exists r.\widehat{C} \mid E_{\sim t}$$

where $A \in \mathcal{N}_C$, $r \in \mathcal{N}_R$, $\sim \in \{<,\leq,>,\geq\}$, $t \in [0,1] \cap \mathbb{Q}$, $E$ is an $\mathcal{EL}$ concept description.
Chapter 3. The Logic \(\tau\mathcal{EL}(m)\)

and \(\hat{C}\) is a \(\tau\mathcal{EL}(m)\) concept description. Concepts of the form \(E_{\sim t}\) are called threshold concepts. We denote by \(\hat{N}_E\) the set of all threshold concepts.

Using this newly introduced constructors, we can define ABoxes in \(\tau\mathcal{EL}(m)\) as a natural extension of \(\mathcal{EL}\) ABoxes. A \(\tau\mathcal{EL}(m)\) ABox is an \(\mathcal{EL}\) ABox that, in addition, is allowed to contain assertions of the form \(\hat{C}(a)\), where \(\hat{C}\) is a \(\tau\mathcal{EL}(\text{deg})\) concept description. Hence, if we know that another individual JACINTA is healthy and handsome with degree at least .8, we can now enrich the information provided in the ABox (2.1) by adding the assertion \(((\text{Healthy} \sqcap \text{Handsome})_{\geq 0.8})(\text{JACINTA})\).

The semantics of the new threshold concepts is defined in the following way:

\[
(E_{\sim t})^I := \{d \in \Delta^I \mid m^I(d, E) \sim t\}
\]

The extension of \(\mathcal{T}\) to more complex concepts is defined as in \(\mathcal{EL}\) by additionally considering the underlying semantics of the newly introduced threshold concepts.

Requiring property \(M1\) has the following consequences for the semantics of threshold concepts.

**Proposition 3.2.** For every \(\mathcal{EL}\) concept description \(E\) we have

\[
E_{\geq 1} \equiv E \quad \text{and} \quad E_{< 1} \equiv \neg E,
\]

where the semantics of negation is defined as usual, i.e., \([\neg E]^I := \Delta^I \setminus E^I\).

The second equivalence basically says that \(\tau\mathcal{EL}(m)\) can express negation of \(\mathcal{EL}\) concept descriptions. This does not imply that \(\tau\mathcal{EL}(m)\) is closed under negation since the threshold constructors can only be applied to \(\mathcal{EL}\) concept descriptions. Thus, negation cannot be nested using these constructors. A formal proof that \(\tau\mathcal{EL}(\text{deg})\) for the membership function \(\text{deg}\) introduced in the next section cannot express full negation can be found in Section 4.3.1. However, atomic negation (i.e., negation applied to concept names) can obviously be expressed. Consequently, unlike \(\mathcal{EL}\) concept descriptions, not all \(\tau\mathcal{EL}(m)\) concept descriptions are satisfiable (i.e., can be interpreted by a non-empty set). A simple example is the concept description \(A_{\geq 1} \sqcap A_{< 1}\), which is equivalent to \(A \sqcap \neg A\).

Last, some other notions defined for \(\mathcal{EL}\) in Chapter 2 extend naturally to \(\tau\mathcal{EL}(m)\):

- **role depth**: extends to \(\tau\mathcal{EL}(m)\) concept descriptions by defining \(\text{rd}(E_{\sim t}) := 0\) for all threshold concept \(E_{\sim t} \in \hat{N}_E\).
- **sub-description**: for all \(E_{\sim t} \in \hat{N}_E\), \(\text{sub}(E_{\sim t}) := \{E_{\sim t}\}\).

### 3.1 Description graphs and homomorphisms in \(\tau\mathcal{EL}(m)\)

Our next goal is to extend the characterization of membership in \(\mathcal{EL}\) (see Theorem 3.8) to \(\tau\mathcal{EL}(m)\). In addition, we will show that given a \(\tau\mathcal{EL}(m)\) ABox \(A\) and an interpretation \(\mathcal{I}\), the satisfaction relation \(\mathcal{I} \models A\) can also be characterized by the existence of homomorphisms. Such characterizations will be useful later on to provide decision procedures for specific instances of \(\tau\mathcal{EL}(m)\).
3.1 Description graphs and homomorphisms in $\tau\mathcal{EL}(m)$

We start by extending the notion of description graphs from $\mathcal{EL}$ to $\tau\mathcal{EL}(m)$. This is done by allowing the use of threshold concepts as labels.

**Definition 3.3 (\(\tau\mathcal{EL}(m)\) description graph).** A $\tau\mathcal{EL}(m)$ description graph is a graph of the form $\hat{G} = (V_G, E_G, \hat{\ell}_G)$ where:

- $V_G$ is a set of nodes,
- $E_G \subseteq V_G \times \mathbb{N}_R \times V_G$ is a set of edges labeled by role names, and
- $\hat{\ell}_G : V_G \rightarrow 2^{\mathbb{N}_C \cup \hat{\mathbb{N}}_E}$ is a function that labels nodes with subsets of $\mathbb{N}_C \cup \hat{\mathbb{N}}_E$.

Likewise for $\mathcal{EL}$ (see Definition 2.3), a $\tau\mathcal{EL}(m)$ description tree $\hat{T}$ is a $\tau\mathcal{EL}(m)$ description graph that is a tree with a distinguished element $v_0$ representing its root. Therefore, we can establish a similar relationship between concept descriptions and description trees in $\tau\mathcal{EL}(m)$, i.e., every $\tau\mathcal{EL}(m)$ concept description $\hat{C}$ can be translated into a $\tau\mathcal{EL}(m)$ description tree $\hat{T}_C$ and vice versa. The following example illustrates such a relationship.

**Example 3.4.** Let $E$ be an $\mathcal{EL}$ concept description. The $\tau\mathcal{EL}(m)$ concept description $\hat{C} := A \sqcap E_{>0.8} \sqcap \exists r.(A \sqcap B \sqcap E_{\leq 0.5} \sqcap \exists r.E_{<1}) \sqcap \exists r.A$ yields the $\tau\mathcal{EL}(m)$ description tree $\hat{T}_C$ depicted on the left-hand side of Figure 3.1.

Now, for ABoxes, the use of individual names and role assertions excludes the possibility of representing them as a description trees. Individuals in the ABox may have no relation at all or it could also happen that role assertions enforce the existence of a cycle involving some of them. In fact, the translation of concept descriptions into description trees in $\mathcal{EL}$ is adapted in [KM02] for an ABox $\mathcal{A}$ into a description graph $\hat{G}(\mathcal{A})$.

We lift the very same translation (see Section 3 in [KM02]) to ABoxes and description graphs in $\tau\mathcal{EL}(m)$. Some of the notation used in [KM02] is slightly changed for the sake of readability within this document.

**Definition 3.5 (ABoxes and \(\tau\mathcal{EL}(m)\) description graphs).** Let $\mathcal{A}$ be a $\tau\mathcal{EL}(m)$ ABox. $\mathcal{A}$ is translated into a $\tau\mathcal{EL}(m)$ description graph $\hat{G}(\mathcal{A})$ in the following way:
Chapter 3. The Logic $\tau\mathcal{EL}(m)$

\[ \hat{C}_{a_1} := A \sqcap B \quad \hat{C}_{a_3} := \top \quad \hat{G}(A) : \quad a_1 : \{A, B\} \]
\[ \hat{C}_{a_2} := E_{<1} \quad \hat{C}_{a_4} := \exists r.A \quad a_2 : \{E_{<1}\} \]
\[ \hat{C}_{a_3} := \top \quad a_3 : \{\} \]
\[ \hat{C}_{a_4} := \exists r.A \quad a_4 : \{\} \]
\[ r \quad s \quad r \quad a_2 : \{E_{<1}\} \quad a_3 : \{\} \quad v : \{A\} \]

Figure 3.2: $\tau\mathcal{EL}(m)$ description graph associated to an ABox.

- For all $a \in \text{Ind}(A)$, the $\tau\mathcal{EL}(m)$ concept description $\hat{C}_a$ is defined as:
  \[ \hat{C}_a := \bigcap_{\hat{D}(a) \in A} \hat{D} \]
  If there exists no assertion of the form $\hat{D}(a)$ in $A$, then $\hat{C}_a := \top$.

- For all $a \in \text{Ind}(A)$, let $\hat{T}(a) = (V_a, E_a, a, \ell_a)$ be the $\tau\mathcal{EL}(m)$ description tree corresponding to the concept $\hat{C}_a$ where $a$ itself represents its root. Without loss of generality let the sets $V_a$ with $a \in \text{Ind}(A)$ be pairwise disjoint. Then, $\hat{G}(A) = (V_A, E_A, \ell_A)$ is defined as:
  \[ V_A := \bigcup_{a \in \text{Ind}(A)} V_a, \]
  \[ E_A := \bigcup_{a \in \text{Ind}(A)} E_a \cup \{arb \mid r(a, b) \in A\}, \]
  \[ \ell_A(v) := \ell_a(v) \text{ for } v \in V_a. \]

The following example shows the idea of the previous construction.

\textbf{Example 3.6.} Let $E$ be an $\mathcal{EL}$ concept description and $A$ the following ABox:
\[ A := \{A(a_1), B(a_1), E_{<1}(a_2), (\exists r.A)(a_4), r(a_1, a_2), r(a_2, a_3), s(a_3, a_1)\} \]

The corresponding $\tau\mathcal{EL}(m)$ description graph $\hat{G}(A)$ is depicted in Figure 3.2.

Based on the notion of $\tau\mathcal{EL}(m)$ description graphs, we define homomorphisms from $\tau\mathcal{EL}(m)$ description graphs to the associated $\mathcal{EL}$ description graph of an interpretation $I$. To differentiate these kinds of homomorphisms from the classical ones, we name them $\tau$-homomorphisms and use the Greek letter $\phi$ (possibly with subscripts) to denote them.

\textbf{Definition 3.7.} Let $\hat{H} = (V_H, E_H, \ell_H)$ be a $\tau\mathcal{EL}(m)$ description graph and $I$ an interpretation. The mapping $\phi : V_H \to V_I$ is a $\tau$-homomorphism from $\hat{H}$ to $G_I$ iff:

1. $\phi$ is a homomorphism from $\hat{H}$ to $G_I$ in the sense of Definition 2.5 (ignoring threshold concepts in the labeling of $V_H$), and
2. for all $v \in V_H$: if $E_{\sim t} \in \ell_H(v)$, then $\phi(v) \in (E_{\sim t})^I$. 

\[ \boxed{\phi} \]
We denote by $\text{dom}(\phi)$ and $\text{img}(\phi)$ the domain and image of $\phi$, respectively. This is, $\text{dom}(\phi) := V_H$ and $\text{img}(\phi) := \{\phi(v) \mid v \in V_H\}$.

We now use $\tau$-homomorphisms to characterize membership in $\tau\mathcal{EL}(m)$. Such a characterization is based on the existence of a $\tau$-homomorphism and generalizes Lemma 2.7 from $\mathcal{EL}$ to $\tau\mathcal{EL}(m)$.

**Theorem 3.8.** Let $\widehat{C}$ be a $\tau\mathcal{EL}(m)$ concept description and $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{T})$ an interpretation. The following statements are equivalent for all $d \in \Delta^\mathcal{I}$:

1. $d \in \widehat{C}^\mathcal{I}$.

2. there exists a $\tau$-homomorphism $\phi$ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.

**Proof.** The $1) \to 2)$ direction is shown by induction on the role depth of $\widehat{C}$, while the other direction is proved by induction on the number of nodes in $T_{\widehat{C}}$. The details of the proof are deferred to the Appendix A. □

Using the previous lemma we give a similar characterization for the satisfaction relation between interpretations and ABoxes in $\tau\mathcal{EL}(m)$.

**Theorem 3.9.** Let $\mathcal{A}$ be a $\tau\mathcal{EL}(m)$ ABox and $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{T})$ be an interpretation. The following statements are equivalent:

1. $\mathcal{I}$ is a model of $\mathcal{A}$.

2. there exists a $\tau$-homomorphism $\phi$ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ such that $\phi(a) = a^\mathcal{I}$ for all $a \in \text{Ind}(\mathcal{A})$.

**Proof.** $1) \to 2)$. Assume that $\mathcal{I}$ is a model of $\mathcal{A}$. Then, $a^\mathcal{I} \in \widehat{D}^\mathcal{I}$ and $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$ hold for all assertions $\widehat{D}(a) \in \mathcal{A}$ and $r(a, b) \in \mathcal{A}$, respectively. Consequently, by definition of $\widehat{C}_{a}$ we have that $a^\mathcal{I} \in (\widehat{C}_{a})^\mathcal{I}$ for all $a \in \text{Ind}(\mathcal{A})$. Hence, we can apply Theorem 3.8 to obtain a $\tau$-homomorphism $\phi_a$ from $\widehat{T}(a)$ to $G_{\mathcal{I}}$ with $\phi_a(a) = a^\mathcal{I}$ (recall that $a$ is the root of $\widehat{T}(a)$).

Finally, since $a^\mathcal{I} \in \widehat{D}^\mathcal{I}$ for all $a \in \mathcal{A}$, and the sets $V_a$ used in the construction of $\widehat{G}(\mathcal{A})$ are pairwise disjoint, it is easy to verify that $\phi := \bigcup_{a \in \text{Ind}(\mathcal{A})} \phi_a$ is a $\tau$-homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ such that $\phi(a) = a^\mathcal{I}$ for all $a \in \text{Ind}(\mathcal{A})$.

$2) \to 1)$. Assume that the statement 2) holds. We show that $\mathcal{I}$ satisfies all assertions in $\mathcal{A}$:

- $r(a, b) \in \mathcal{A}$. By construction of $\widehat{G}(\mathcal{A})$ we know that $arb \in E_{\mathcal{A}}$. Since $\phi$ is a homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$, this means that $\phi(a)r\phi(b) \in E_{\mathcal{I}}$ as well. Consequently, it follows from $\phi(a) = a^\mathcal{I}$ and $\phi(b) = b^\mathcal{I}$ that $a^\mathcal{I}rb^\mathcal{I} \in E_{\mathcal{I}}$. Thus, $(a^\mathcal{I}, b^\mathcal{I}) \in r_{\mathcal{I}}$.

- $\widehat{D}(a) \in \mathcal{A}$. By construction of $\widehat{G}(\mathcal{A})$ one can see that the description tree $\widehat{T}(a)$ is a sub-graph of $\widehat{G}(\mathcal{A})$. Therefore, $\phi$ is also a $\tau$-homomorphism from $\widehat{T}(a)$ to $G_{\mathcal{I}}$ with $\phi(a) = a^\mathcal{I}$. An application of Theorem 3.8 then yields: $a^\mathcal{I} \in (\widehat{C}_{a})^\mathcal{I}$. Thus, since $\widehat{D}$ is one of the conjuncts in the definition of $\widehat{C}_{a}$, it follows that $a^\mathcal{I} \in \widehat{D}^\mathcal{I}$. □
3.2 Deciding the existence of a $\tau$-homomorphism

If the interpretation $\mathcal{I}$ is finite and $m$ is computable, then the existence of a $\tau$-homomorphism can be decided. We present two algorithms that (under the previous conditions) can be used to decide the relations characterized by Theorems 3.8 and 3.9. Our starting point is the polynomial time algorithm (Algorithm 1 below) introduced in [BKM99] to decide the existence of a homomorphism between two $\mathcal{E}\mathcal{L}$ description trees.

**Algorithm 1** Homomorphisms between $\mathcal{E}\mathcal{L}$ description trees.

**Input:** Two $\mathcal{E}\mathcal{L}$ description trees $T_1$ and $T_2$.

**Output:** “yes”, if there exists a homomorphism from $T_1$ to $T_2$; “no”, otherwise.

1: Let $T_1 = (V_1, E_1, v_0, \ell_1)$ and $T_2 = (V_2, E_2, w_0, \ell_2)$. Further, let $\{v_1, \ldots, v_n\}$ be a post-order sequence of $V_1$, i.e., $v_1$ is a leaf and $v_n = v_0$.
2: Define a labeling $\delta : V_2 \rightarrow 2^{V_1}$ as follows.
3: Initialize $\delta$ by $\delta(w) := \emptyset$ for all $w \in V_2$.
4: for all $1 \leq i \leq n$ do
5:  for all $w \in V_2$ do
6:      if $(\ell_1(v_i)) \subseteq \ell_2(w)$ and for all $v_r v \in E_1$ there is $w' \in V_2$ such that $v \in \delta(w')$ and $w w' w' \in E_2$ then
7:          $\delta(w) := \delta(w) \cup \{v_i\}$
8:      end if
9:  end for
10: end for
11: If $v_0 \in \delta(w_0)$ then return “yes”, else return “no”.

Theorem 3.8 characterizes elementhood in $\tau\mathcal{E}\mathcal{L}(m)$ concept descriptions via the existence of a $\tau$-homomorphism from a $\tau\mathcal{E}\mathcal{L}(m)$ description tree $T_\hat{C}$ to an $\mathcal{E}\mathcal{L}$ description graph $G_\hat{T}$ associated to an interpretation $\mathcal{I}$. If $\mathcal{I}$ is finite, then Algorithm 1 can be used to decide whether there exists a mapping satisfying Condition 1 in Definition 3.7. One needs only to replace the last line by $v_0 \in \delta(d)$ for some $d \in \Delta^\mathcal{I}$, since now $T_2$ becomes $G_\mathcal{T}$. In order to verify the second condition in Definition 3.7, we modify the test in line 6 to also consider whether $m^\mathcal{T}(d, E) \sim t$ for all $E \dashv \Delta \in \ell_{T_\hat{C}}(v_i)$. Algorithm 2 implements this modification.

Then, if one wants to know whether a precise element $e \in \Delta^\mathcal{I}$ belongs to $(\hat{C})^\mathcal{I}$, Algorithm 2 shall be invoked on $T_\hat{C}$ and $\mathcal{I}$. Note that a simple modification in line 12, namely testing whether $v_0 \in \delta(e)$, adapts the algorithm to answer the question for $e$.

Now, the main difference between Algorithms 1 and 2 is that the latter might need to compute $m^\hat{T}$ to verify whether $m^\hat{T}(d, E) \sim q$. Therefore, its computational complexity may depend on how difficult is to compute $m^\hat{T}$ for a chosen $m$. In particular, if $m^\hat{T}$ can be computed in polynomial time as for the graded membership function $deg$ introduced in the next section, Algorithm 2 will run in polynomial time.

Regarding the characterization given in Theorem 3.9 for the satisfaction relation between interpretations and ABoxes, note that the description graph $\hat{G}(A)$ associated to an ABox $A$ is not necessarily a tree. Therefore, finding a $\tau$-homomorphism $\phi$ from $\hat{G}(A)$ to $G_\mathcal{T}$ includes finding a homomorphism between two graphs, which in general
3.2 Deciding the existence of a \( \tau \)-homomorphism

Algorithm 2 \( \tau \)-homomorphism from a \( \tau \mathcal{E}(m) \) description tree to \( G_I \).

**Input:** A \( \tau \mathcal{E}(m) \) description tree \( \hat{T} \) and a finite interpretation \( I \).

**Output:** “yes”, if there exists a \( \tau \)-homomorphism from \( \hat{T} \) to \( G_I \); “no”, otherwise.

1. Let \( \hat{T} = (V_T, E_T, v_0, \hat{\ell}_T) \) and \( G_I = (V_I, E_I, \ell_I) \). Further, let \( \{v_1, \ldots, v_n\} \) be a post-order sequence of \( V_T \), i.e., \( v_1 \) is a leaf and \( v_n = v_0 \).
2. Define a labeling \( \delta : V_I \rightarrow 2 \) \( V_T \) as follows.
3. Initialize \( \delta \) by \( \delta(w) = \emptyset \) for all \( w \in V_I \).
4. for all \( 1 \leq i \leq n \) do
5.   for all \( d \in \Delta^I \) do
6.     if \( (\ell_T(v_i) \subseteq \ell_I(d) \) and \( \exists t \in \hat{\ell}_T(v_i) \Rightarrow m^I(d, E) \sim t) \) then
7.       \( \delta(d) := \delta(d) \cup \{v_i\} \)
8.     end if
9.   end for
10. end for
11. If there exists \( d \in \Delta^I \) such that \( v_0 \in \delta(d) \) then return “yes”, else return “no”.

is an NP-complete problem [GJ79]. However, by Definition 3.5 it can be seen that \( \hat{G}(A) \) has a particular form where cycles only involve nodes and edges corresponding to the individual elements and role assertions, respectively, occurring in \( A \). Moreover, since Theorem 3.9 requires \( \phi(a) = a^I \) for all \( a \in \text{Ind}(A) \), this means that the wanted \( \tau \)-homomorphism is partially fixed with respect to those elements. Hence, it suffices to check whether the interpretation of the individual names satisfies the role assertions in \( A \) and \( a^I \in (\hat{C}_a)^I \) (see Definition 3.5), for all \( a \in \text{Ind}(A) \). The following algorithm uses Algorithm 2 to decide whether a finite interpretation \( I \) satisfies an ABox \( A \).

Algorithm 3 \( \tau \)-homomorphisms for ABoxes and interpretations.

**Input:** An ABox \( A \) and a finite interpretation \( I \).

**Output:** “yes”, if there exists a \( \tau \)-homomorphism \( \phi \) from \( \hat{G}(A) \) to \( G_I \) with \( \phi(a) = a^I \) for all \( a \in \text{Ind}(A) \); “no”, otherwise.

1. Let \( \hat{G}(A) \) be as in Definition 3.5
2. for all \( r(a,b) \in A \) do
3.   if \( (a^T, b^T) \notin r^T \) then
4.     return “no”
5.   end if
6. end for
7. for all \( a \in \text{Ind}(A) \) do
8.   if \( a^T \notin (\hat{C}_a)^I \) then // this can be checked using Algorithm 2
9.     return “no”
10. end if
11. end for
12. return “yes”
Chapter 4

The membership function $\deg$

To make things more concrete, we now introduce a specific membership function, denoted $\deg$. Given an interpretation $\mathcal{I}$, an element $d \in \Delta^\mathcal{I}$, and an $\mathcal{EL}$ concept description $C$, this function is supposed to measure to which degree $d$ satisfies the conditions for membership expressed by $C$. To come up with such a measure, we use the homomorphism characterization of membership in $\mathcal{EL}$ concepts as starting point (see Theorem 2.7). Basically, we consider all partial mappings from $T_C$ to $G_\mathcal{I}$ that map the root of $T_C$ to $d$ and respect the edge structure of $T_C$. For each of these mappings we then calculate to which degree it satisfies the homomorphism conditions, and take the degree of the best such mapping as the membership degree $\deg^\mathcal{I}(d,C)$.

Example 4.1. Figure 4.1 shows the $\mathcal{EL}$ description tree corresponding to the $\mathcal{EL}$ concept description $C := A \sqcap B \sqcap \exists s. (B_1 \sqcap \exists r. B_3 \sqcap \exists r. B_2)$ and a fragment of an interpretation graph $G_\mathcal{I}$. In addition, it depicts two mappings from $V_{T_C}$ to $V_\mathcal{I}$. The one represented by the dashed lines and a variation represented with the dotted lines. One can see that none of them is a homomorphism from $T_C$ to $G_\mathcal{I}$ in the sense of Definition 2.5. In fact, since obviously $d \notin C^\mathcal{I}$, by Theorem 2.7 there exists no such homomorphism.

To compute the membership value induced by an specific mapping, we count the number of properties of $v_0$ (say $m$), see how many of those does $d$ in $\mathcal{I}$ actually have (say $n$) and give $\frac{n}{m}$ as the membership degree value. In our example $v_0$ has three properties, e.g., $A$, $B$ and the existence of an $s$-successor (represented by $v_1$) with certain properties. Interesting to see is that for both mappings, the selected $s$-successor of $d$ does not satisfy all the properties of $v_1$. Should we just assume that $d$ does not have this last property and give $\frac{1}{3}$ as the membership degree value? Instead of that, we would like to compute a value that expresses to which degree the $s$-successor of $d$ (to which $v_1$ is mapped to), satisfies the conditions for membership expressed by the subtree of $T_C$ rooted at $v_1$. This will be done using the very same idea recursively.

As mentioned before, we consider partial mappings rather than total ones since one of the violations of properties demanded by $C$ could be that a required role successor does not exist at all.

Example 4.2. Consider the description tree $T_C$ and the interpretation $\mathcal{I}$ depicted in Figure 4.2. Obviously, there exists no total mapping from $T_C$ to $G_\mathcal{I}$ since neither $d_1$ nor $d_2$ have a successor. Thus, restricting to consider only total mappings would give zero as the membership degree value of $d$ in $C$. This is not desired, since just like concept names may be missing and the membership value does not become zero, also role successors (required by $C$) may be missing and the membership degree need not be zero.
4.1 The membership function $\text{deg}$

To formalize the previously exposed ideas, we first define the notion of partial tree-to-graph homomorphisms from description trees to description graphs. In this definition, the node labels are ignored (they will be considered in the next step).

**Definition 4.3 (Partial tree-to-graph homomorphisms).** Let $T = (V_t, E_t, \ell_t, v_0)$ and $G = (V_g, E_g, \ell_g)$ be a description tree (with root $v_0$) and a description graph, respectively. A partial mapping $h : V_t \to V_g$ is a partial tree-to-graph homomorphism ($\text{ptgh}$) from $T$ to $G$ iff the following conditions are satisfied:

1. $\text{dom}(h)$ is a subtree of $T$ with root $v_0$, i.e., $v_0 \in \text{dom}(h)$ and if $(v, r, w) \in E_t$ and $w \in \text{dom}(h)$, then $v \in \text{dom}(h)$;

2. for all edges $(v, r, w) \in E_t$, $w \in \text{dom}(h)$ implies $(h(v), r, h(w)) \in E_g$.

To abbreviate, from now on we will write $\text{ptgh}(\text{ptghs})$ for the plural instead of partial tree-to-graph homomorphism.

In order to measure how far away from a homomorphism (in the sense of Definition 2.5) such a $\text{ptgh}$ is, we define the notion of a weighted homomorphism between a finite $\mathcal{EL}$ description tree and an $\mathcal{EL}$ description graph.
Definition 4.4. Let $T$ be a finite $\mathcal{EL}$ description tree, $G$ an $\mathcal{EL}$ description graph and $h: V_T \rightarrow V_G$ a ptgh from $T$ to $G$. We define the weighted homomorphism induced by $h$ from $T$ to $G$ as a recursive function $h_w: \text{dom}(h) \rightarrow [0..1]$ in the following way:

$$h_w(v) := \begin{cases} 1 & \text{if } |\ell_T(v)| + k^*(v) = 0 \\ \frac{|\ell_T(v) \cap \ell_G(h(v))| + \sum_{i \leq k} h_w(v_i)\ |\ell_T(v)| + k^*(v)} & \text{otherwise.} \end{cases}$$

The elements used to define $h_w$ have the following meaning. For a given $v \in \text{dom}(h)$, $k^*(v)$ denotes the number of successors of $v$ in $T$, and $v_1, \ldots, v_k$ with $0 \leq k \leq k^*(v)$ are the children of $v$ in $T$ such that $v_i \in \text{dom}(h)$.

It is easy to see that $h_w$ is well-defined. In fact, $T$ is a finite tree, which ensures that the recursive definition of $h_w$ is well-founded. In addition, the base case of the definition guarantees that division by zero is avoided. Using value 1 in this case is justified since then no property is required. In the second case, missing concept names and missing successors decrease the weight of a node since then the required name or successor contributes to the denominator, but not to the numerator. Required successors that are there are only counted if they are successors for the correct role, and then they do not contribute with value 1 to the numerator, but only with their weight (i.e., the degree to which they match the requirements for this successor).

When defining the value of the membership function $\text{deg}^I(d, C)$, we do not use the concept $C$ directly, but rather its reduced from $C^\tau$. This will ensure that $\text{deg}$ satisfies property $M2$.

Definition 4.5. Let $I = (\Delta^I, \tau)$ be an interpretation, $d$ an element of $\Delta^I$ and $C$ an $\mathcal{EL}$ concept description with reduced form $C^\tau$. In addition, let $\mathcal{H}(T_{C^\tau}, G_I, d)$ be the set of all ptghs from $T_{C^\tau}$ to $G_I$ with $h(v_0) = d$. The set $\mathcal{V}^I(d, C^\tau)$ of all relevant values is defined as:

$$\mathcal{V}^I(d, C^\tau) := \{ q \mid h_w(v_0) = q \text{ and } h \in \mathcal{H}(T_{C^\tau}, G_I, d) \}$$

Then we define $\text{deg}^I(d, C) := \max \mathcal{V}^I(d, C^\tau)$.

In case the interpretation $I$ is infinite, there may exist infinitely many ptghs from $T_{C^\tau}$ to $G_I$ with $h(v_0) = d$. Therefore, it is not immediately clear whether the maximum in the above definition actually exists, and thus whether $\text{deg}^I(d, C)$ is well-defined. To prove that the maximum exists also for infinite interpretations, we show that the set $\mathcal{V}^I(d, C^\tau)$ is actually a finite set. To this end, we introduce canonical interpretations induced by ptghs.

Definition 4.6 (Canonical interpretation). Let $I = (\Delta^I, \tau)$ be an interpretation, $C$ an $\mathcal{EL}$ concept description and $h$ be a ptgh from $T_{C^\tau}$ to $G_I$. The canonical interpretation $I_h$ induced by $h$ is the one having the description tree $T_{I_h} = (V_{I_h}, E_{I_h}, v_0, \ell_{I_h})$ with

$$V_{I_h} := \text{dom}(h),$$
$$E_{I_h} := \{ vw \in E_{T_{C^\tau}} \mid v, w \in \text{dom}(h) \}$$
$$\ell_{I_h}(v) := \ell_{T_{C^\tau}}(v) \cap \ell_I(h(v)) \text{ for all } v \in \text{dom}(h).$$
Remark 4.7. One can see that $T_{I_h}$ satisfies $V_{T_h} \subseteq V_{T_{C^r}}$, $E_{T_h} \subseteq E_{T_{C^r}}$, $\ell_{T_h}(v) \subseteq \ell_{T_{C^r}}(v)$ and $\ell_{T_h}(v) \subseteq \ell_{T}(h(v))$ for all $v \in \text{dom}(h)$. Moreover, the construction of $T_{I_h}$ verifies that the mapping $h$ is a homomorphism from $T_{I_h}$ to $G_T$. \hfill

Lemma 4.8. Let $I = (\Delta^I, I)$ be an interpretation, $d \in \Delta^I$ and $C$ an $\mathcal{EL}$ concept description. The set $V^I_d(d, C)$ contains finitely many elements.

Proof. Let $I_H$ be the set of all canonical interpretations induced by all $h \in \mathcal{H}(T_{C^r}, G_I, d)$, i.e.,

$$I_H := \{ I_h \mid h \in \mathcal{H}(T_{C^r}, G_I, d) \}$$

From Remark 4.7, we have that $V_{T_h} \subseteq V_{T_{C^r}}$, $E_{T_h} \subseteq E_{T_{C^r}}$ and $\ell_{T_h}(v) \subseteq \ell_{T_{C^r}}(v)$ for all $v \in \text{dom}(h)$. This implies that the description graph $T_{I_h}$ induced by $I_h$ is a subtree of $T_{C^r}$. Hence, the set $I_H$ must be finite, i.e., there are only finitely many different canonical interpretations induced by $ptgh$ $h \in \mathcal{H}(T_{C^r}, G_I, d)$.

Now, consider any $h \in \mathcal{H}(T_{C^r}, G_I, d)$ and let $i^{T_h} : \text{dom}(h) \to V_{T_h}$ be a mapping such that $i^{T_h}(v) = v$ for all $v \in \text{dom}(h)$. Note that $i^{T_h}$ is well-defined by definition of $I_h$, and it is easy to see that it is a $ptgh$ from $T_{C^r}$ to $T_{I_h}$. Furthermore, let $V^{I_H}_d$ be the set:

$$V^{I_H}_d := \{ q \mid i^{T_h}_w(v_0) = q \text{ for all } h \in \mathcal{H}(T_{C^r}, G_I, d) \}$$

Since $\text{dom}(h) \subseteq V_{T_{C^r}}$, there are finitely many sets that could act as the source for a mapping $i^{T_h}$. Moreover, $I_H$ is a finite set of finite interpretations. Hence, there can only be finitely many different mappings $i^{T_h}$. Consequently, the set $V^{I_H}_d$ must be finite. In addition, one can see that the following three properties hold:

1. $\text{dom}(i^{T_h}) = \text{dom}(h)$,
2. $\ell_{T_h}(i^{T_h}(v)) = \ell_{T_{C^r}}(v) \cap \ell_{T}(h(v))$ for all $v \in \text{dom}(h)$, and
3. for all $v, w \in \text{dom}(h)$: if $vw \in E_{T_{C^r}}$, then $h(v)rh(w) \in E_I$ and $i^{T_h}(v)ri^{T_h}(w) \in E_{T_h}$.

Therefore, from Definition 4.4 it follows that $h_w(v_0) = i^{T_h}_w(v_0)$. This means that for all $h \in \mathcal{H}(T_{C^r}, G_I, d)$ it is the case that $h_w(v_0) \in V^{I_H}_d$. Hence, $V^{I}_d(d, C^r) \subseteq V^{I_H}_d$ and $V^{I}_d(d, C^r)$ is a finite set.

Thus, max $V^{I}_d(d, C^r)$ exists and $deg^I_d(d, C)$ is well-defined. \hfill

If the interpretation $I$ is finite, $deg^I_d(d, C)$ can be computed in polynomial time for all $d \in \Delta^I$ and all $\mathcal{EL}$ concept descriptions $C$. The polynomial time algorithm described below (Algorithm 4) is inspired by the polynomial time algorithm for checking the existence of a homomorphism between $\mathcal{EL}$ description trees [BKM98, BKM99], and similar to the algorithm for computing the similarity degree between $\mathcal{EL}$ concept descriptions introduced in [Sun13].

Algorithm 4 considers each pair $(v, e)$ with $v \in V_{T_{C^r}}$ and $e \in \Delta^I$ only once. Therefore, it is easy to see that it runs in polynomial time in the size of $C$ and $I$. The following lemma shows that Algorithm 4 computes the value of $deg^I_d$, i.e., $S(v_0, d) = deg^I_d(d, C^r)$ (see Appendix A).
Algorithm 4 Computation of $deg^I$.

**Input:** An $\mathcal{EL}$ concept description $C$, a finite interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$.

**Output:** $deg^I(d, C)$.

1: Let $C^r$ be the reduced form of $C$, $G^I = (V^I, E^I, \ell^I)$ and $\{v_1, \ldots, v_n\}$ be a post-order sequence of $V^I_{\mathcal{C}^r}$ where $v_n = v_0$.

2: The assignment $S : V^I_{\mathcal{C}^r} \times V^I \to [0, 1]$ is computed as follows:

3: for all $1 \leq i \leq n$ do

4: \quad if $|\ell^I_{\mathcal{C}^r}(v_i)| + k^*(v_i) = 0$ then

5: \quad \quad $S(v_i, e) := 1$ for all $e \in \Delta^I$

6: \quad else

7: \quad \quad for all $e \in \Delta^I$ do

8: \quad \quad \quad $c := |\ell^I_{\mathcal{C}^r}(v_i) \cap \ell^I(e)|$

9: \quad \quad \quad for all $v_i r v \in E_{\mathcal{C}^r}$ do

10: \quad \quad \quad \quad $c := c + \max\{S(v, e') \mid (e, e') \in r^I\}$

11: \quad \quad \quad end for

12: \quad \quad $S(v_i, e) := \frac{c}{|\ell^I_{\mathcal{C}^r}(v_i)| + k^*(v_i)}$

13: \quad \quad end for

14: end if

15: end for

16: return $S(v_0, d)$

---

**Lemma 4.9.** Let $C$ be an $\mathcal{EL}$ concept description, $\mathcal{I}$ a finite interpretation and $d \in \Delta^\mathcal{I}$. Then, Algorithm 4 terminates on input $(C, \mathcal{I}, d)$ and outputs $deg^I(d, C)$, i.e., $S(v_0, d) = deg^I(d, C^r)$.

Finally, it remains to show that $deg$ satisfies the properties required for graded membership functions.

**Proposition 4.10.** The function $deg$ satisfies the properties M1 and M2.

**Proof.** We first show that M1 is satisfied by $deg$. Assume that $d \in C^\mathcal{I}$. Since $C$ is equivalent to its reduced form, we also have $d \in (C^r)^\mathcal{I}$. The application of Theorem 2.7 yields that there exists a homomorphism $\varphi$ from $\mathcal{C}^r$ to $G^I$ with $\varphi(v_0) = d$. Then it is easy to verify from Definition 4.4 that $\varphi_w(v_0) = 1$ and consequently, $\max V^I(d, C^r) = 1$. Thus, $deg^I(d, C) = 1$. Conversely, assume that $deg^I(C, d) = 1$. This means that there exists a ptgh $h$ from $\mathcal{C}^r$ to $G^I$ with $h(v_0) = d$ and $h_w(v_0) = 1$. Similar as before, it is easy to see that $h$ is a homomorphism according to Definition 2.5. The application of Theorem 2.7 yields $d \in (C^r)^\mathcal{I}$ and consequently, $d \in C^\mathcal{I}$.

Concerning M2, as mentioned in Chapter 3, the right to left implication is already a consequence of M1, which we just proved to be satisfied by $deg$. Assume that $C \equiv D$, then by Theorem 2.8 there exists an isomorphism $\psi$ between $\mathcal{C}^r$ and $\mathcal{D}^r$. Consider an arbitrary interpretation $\mathcal{I}$ and any element $d \in \Delta^\mathcal{I}$. We show that $deg^I(d, C^r) = deg^I(d, D^r)$, which obviously implies $deg^I(d, C) = deg^I(d, D)$ (see Definition 4.5).

Let $h$ be a ptgh from $\mathcal{C}^r$ to $G^I$ with $h(v_0) = d$ and $h_w(v_0) = \max V^I(d, C^r)$. Since $\psi$ is an isomorphism, the composition $h \circ \psi$ is a ptgh from $\mathcal{D}^r$ to $G^I$, with $(h \circ \psi)(v_0) = d$.
and \((h \circ \psi)_w(v_0) = h_w(v_0)\). This means that \(\text{deg}^T(d, C^r) \leq \text{deg}^T(d, D^r)\). The same reasoning can be applied starting with \(T_{D^r}\) to obtain \(\text{deg}^T(d, D^r) \leq \text{deg}^T(d, C^r)\). Thus, we have shown that \(\text{deg}^T(d, C^r) = \text{deg}^T(d, D^r)\).

Note that \(M2\) follows from the fact that we use the reduced form of a concept description rather than the description itself. Otherwise, \(M2\) would not hold. For example, consider the concept description \(C := \exists r.A \land \exists r.(A \land B)\), which is equivalent to its reduced form \(C^r = \exists r.(A \land B)\). Let \(d\) be an individual that has a single \(r\)-successor belonging to \(A\), but not to \(B\). Then using \(C\) instead of \(C^r\) would yield membership degree \(\frac{3}{4}\), whereas the use of \(C^r\) yields the degree \(\frac{1}{2}\).

### 4.2 Two useful properties of \(\text{deg}\)

The following lemma shows that \(\text{deg}\) satisfies a monotonicity property with respect to two interpretations \(I\) and \(J\) which are related by a homomorphism.

**Lemma 4.11.** Let \(I\) and \(J\) be two interpretations such that there exists a homomorphism \(\varphi\) from \(G_I\) to \(G_J\). Then, for any individual \(d \in \Delta^I\) and any \(\mathcal{EL}\) concept description \(C\) it holds: \(\text{deg}^I(d, C) \leq \text{deg}^J(\varphi(d), C)\).

**Proof.** Let \(C^r\) be the reduced form of \(C\) and \(h\) be any \(ptgh\) from \(T_{C^r}\) to \(G_I\) with \(h(v_0) = d\). Since \(\varphi\) is a homomorphism from \(G_I\) to \(G_J\), the mapping \(\varphi \circ h\) is a \(ptgh\) from \(T_{C^r}\) to \(G_J\) with \((\varphi \circ h)(v_0) = \varphi(d)\).

Then, we have that for each \(v \in \text{dom}(h)\) the homomorphism \(\varphi\) makes \(\ell_I(h(v)) \subseteq \ell_J((\varphi \circ h)(v))\). In addition, for each \(r\)-successor \(w \in \text{dom}(h)\) of \(v\) in \(T_{C^r}\), we have that \(h(w)\) is an \(r\)-successor of \(h(v)\) in \(G_I\). Therefore, \((\varphi \circ h)(w)\) is also an \(r\)-successor of \((\varphi \circ h)(v)\) in \(G_J\). Hence, it follows from Definition 4.4 that \(h_w(v_0) \leq (\varphi \circ h)_w(v_0)\) for all \(ptghs \ h\) from \(T_{C^r}\) to \(G_I\) with \(h(v_0) = d\).

Thus, we can conclude that \(\text{deg}^I(d, C) \leq \text{deg}^J(\varphi(d), C)\).

Now, using this monotonicity property and elements from the proof of Lemma 4.8 we can show that the value \(\text{deg}^I(d, C)\) is preserved by the canonical interpretation corresponding to a \(ptgh\) \(h\) such that \(h_w(v_0) = \text{deg}^I(d, C)\).

**Lemma 4.12.** Let \(I = (\Delta^I, I)\) be an interpretation, \(d\) be an individual of \(\Delta^I\) and \(C\) an \(\mathcal{EL}\) concept description. Let \(h\) be a \(ptgh\) from \(T_{C^r}\) to \(G_I\) such that \(h(v_0) = d\) and \(h_w(v_0) = \text{deg}^I(d, C)\). In addition, let \(I_h\) be the canonical interpretation induced by \(h\). Then, \(\text{deg}^I_h(v_0, C) = \text{deg}^I(d, C)\).

**Proof.** Assume that \(\text{deg}^I(d, C) = q\). From Definition 4.5 we have:

\[
\text{deg}^I(d, C) = \max V^I(d, C^r) = h_w(v_0) = q
\]

In the proof of Lemma 4.8, we saw that \(i^I_h\) is a \(ptgh\) from \(T_{C^r}\) to \(T_{I_h}\) with \(i^I_h(v_0) = v_0\) and \(h_w(v_0) = i^I_w(v_0)\). Hence, \(\text{deg}^I_h(v_0, C) \geq q\). Remark 4.7 tells us that \(h\) is a homomorphism from \(T_{I_h}\) to \(G_I\) with \(h(v_0) = d\). Then, the application of Lemma 4.11 yields:

\[
\text{deg}^I_h(v_0, C) \leq \text{deg}^I(d, C)
\]

Thus, \(\text{deg}^I_h(v_0, C) = q = \text{deg}^I(d, C)\).
4.3 Relation to the Description Logic $ALC$

We now investigate the relation between our threshold logic $\tau\mathcal{EL}(deg)$ and the DL $ALC$ \cite{ss91}. On the one side, we show that full negation of $\mathcal{EL}$ concept descriptions cannot be expressed in $\tau\mathcal{EL}(deg)$, and consequently there are $ALC$ concept descriptions that cannot be expressed in $\tau\mathcal{EL}(deg)$. On the other side, we will see that every $\tau\mathcal{EL}(deg)$ concept description has its corresponding equivalent concept in $ALC$, but the provided translation involves an exponential blow up.

Let us start by briefly introducing the DL $ALC$. The set of $ALC$ concept descriptions is the smallest set such that:

- $\top$ is an $ALC$ concept description,
- if $A \in N_C$, then $A$ is an $ALC$ concept description,
- if $C, D$ are $ALC$ concept descriptions and $r \in N_R$, then $\neg C$, $C \sqcap D$ and $\exists r.C$ are $ALC$ concept descriptions.

The semantics of the negation constructor under an interpretation $I$ is given as:

$$(\neg C)^I := \{d \in \Delta^I | d \notin C^I\}$$

As usual, $\forall r.C$ is an abbreviation for $\neg \exists r.\neg C$ and $C \sqcup D$ for $\neg (\neg C \sqcap \neg D)$.

Before going on to the main details of this section, we need to define the notion of a concept part of an $\mathcal{EL}$ concept description.

**Definition 4.13 (concept part).** Let $C$ be an $\mathcal{EL}$ concept description. The set of concept parts $\text{Part}(C)$ of $C$ is the smallest set such that:

- $\{\top, C\} \subseteq \text{Part}(C)$.
- if $\exists r.D \in \text{Part}(C)$, then $\exists r.D' \in \text{Part}(C)$ where $D' \in \text{Part}(D)$.
- if $C_1 \sqcap C_2 \in \text{Part}(C)$, then $C_1' \sqcap C_2' \in \text{Part}(C)$ where $C_1' \in \text{Part}(C_1)$ and $C_2' \in \text{Part}(C_2)$.

4.3.1 Full negation is not expressible in $\tau\mathcal{EL}(deg)$

In Chapter 3 we mentioned that although $\tau\mathcal{EL}(m)$ can express negation of $\mathcal{EL}$ concept descriptions, negation cannot be nested using the constructors of $\tau\mathcal{EL}(m)$. We prove that full negation cannot be expressed in $\tau\mathcal{EL}(deg)$, by showing that $\tau\mathcal{EL}(deg)$ cannot express the simple $ALC$ concept description $\forall r.A$.

The semantics of $\forall r.C$ can be expressed as follows:

$$(\forall r.C)^I := \{d \in \Delta^I | \forall e \in \Delta^I, ((d, e) \in r^I \Rightarrow e \in C^I)\}$$

**Lemma 4.14.** In $\tau\mathcal{EL}(deg)$, there is no concept description $\hat{C}$ such that $\forall r.A \equiv \hat{C}$, where $A \in N_C$. 


Proof. Suppose that there exists a $\tau\mathcal{EL}(\text{deg})$ concept description $\hat{C}$ such that $\forall r. A \equiv \hat{C}$. Then, for all interpretations $I$ we have $(\forall r. A)^\mathcal{I} = \hat{C}^\mathcal{I}$. Consider the interpretation $I_0 = (\{d\}, X_0)$ such that $X^\mathcal{I}_0 = \emptyset$ for all $X \in \mathcal{N}_\mathcal{E} \cup \mathcal{N}_R$. Obviously, $d \in (\forall r. A)^{I_0}$ and by our initial assumption it also holds $d \in \hat{C}^\mathcal{I}_0$.

By Theorem 3.8 there exists a $\tau$-homomorphism $\phi$ from $T_\mathcal{E}$ to $G_{I_0}$ with $\phi(v_0) = d$. Since $d$ has no $\tau$-successors in $\Delta^\mathcal{I}_0$ nor it is an instance of any concept name, this means that $\hat{C}$ must be of the following form:

$$(E^1)_{\sim t_1} \cap \ldots \cap (E^q)_{\sim t_q}$$

where each $E^i$ is an $\mathcal{EL}$ concept description. Let us now consider the interpretations $I_1$ and $I_2$ which have the description graphs shown below.

\[
\begin{align*}
I_0 & : d : \{\} & I_1 & : d_1 : \{\} & I_2 & : d_2 : \{\} \\
& \quad r & & \quad r & & \quad r \\
& \quad d_3 : \{\} & & \quad d_4 : \{A\}
\end{align*}
\]

In addition to $d \in (\forall r. A)^{I_0}$, it is also the case that $d_2 \in (\forall r. A)^{I_2}$. Hence, since $\hat{C} \equiv \forall r. A$, we also have $d_2 \in \hat{C}^{I_2}$. This means that $d \in [(E^i)_{\sim t_i}]_{I_0}$ and $d_2 \in [(E^i)_{\sim t_i}]^{I_2}$ for all $1 \leq i \leq q$. Further, it is easy to see that Lemma 4.11 can be applied to obtain for all $1 \leq i \leq q$:

$$\text{deg}^{I_0}(d, E^i) \leq \text{deg}^{I_1}(d_1, E^i) \leq \text{deg}^{I_2}(d_2, E^i)$$

Therefore, it is immediate to see that $d_1 \in [(E^i)_{\sim t_i}]^{I_1}$ for all conjuncts $(E^i)_{\sim t_i}$ of $\hat{C}$, and consequently $d_1 \in \hat{C}^{I_1}$. Our initial assumption $\forall r. A \equiv \hat{C}$ implies that $d_1 \in (\forall r. A)^{I_1}$, but this is a contradiction since $d_1$ has an $r$-successor $d_3$ and $d_3 \not\in A^{I_1}$. Thus, there is no $\tau\mathcal{EL}(\text{deg})$ concept description $\hat{C}$ such that $\hat{C} \equiv \forall r. A$.\qed

Lemma 4.14 implies that full negation of $\mathcal{EL}$ concept descriptions cannot be expressed in $\tau\mathcal{EL}(\text{deg})$. Otherwise, since $\forall r. A \equiv \neg \exists r. \neg A$, there would be a $\tau\mathcal{EL}(\text{deg})$ concept description $\hat{D}$ such that $\hat{D} \equiv \neg \exists r. \neg A$ contradicting the lemma. Moreover, since $\exists r. \neg A \equiv \exists r. A_{<1}$, this implies that neither negation of $\tau\mathcal{EL}(\text{deg})$ concept descriptions can be expressed.

4.3.2 Expressing $\tau\mathcal{EL}(\text{deg})$ concept descriptions in $\mathcal{ALC}$

Since $\mathcal{EL}$ is a fragment of $\mathcal{ALC}$, the concept constructors we need to look at are the ones corresponding to threshold concepts $E_{\sim t}$. In particular, a direct consequence of the semantics corresponding to such constructors are the equivalences:

$$E_{<t} \equiv \neg E_{\geq t} \quad \text{and} \quad E_{\leq t} \equiv \neg E_{>t}$$

The four possibilities are gathered in the following proposition.
Proposition 4.15. Let \( E_\sim \) be a threshold concept. The negated concept \( \neg E_\sim \) is equivalent to the threshold concept \( E_{\chi(\sim)} \), where \( \chi \) is the following mapping:

\[
\chi(<) := >, \quad \chi(\leq) := >, \quad \chi(>) := >.
\]

Thus, since \( ALC \) allows full negation of concept descriptions, we can restrict our attention to threshold concepts of the form \( E_\sim \) with \( \sim \in \{>, \geq\} \). For all \( rE\mathcal{L}(deg) \) concept descriptions \( \hat{C} \), its corresponding \( ALC \) concept description \( [\hat{C}]^* \) is recursively defined as follows:

\[
[T]^* := T
\]
\[
[A]^* := A, \text{ if } A \in \mathbb{N}_C
\]
\[
[E_\sim]^* := \neg[E_{\chi(\sim)}]^*, \text{ if } \sim \in \{>, \leq\}
\]
\[
[\hat{C} \cap \tilde{D}]^* := [\hat{C}]^* \cap [\tilde{D}]^*
\]
\[
[\exists r.\hat{C}]^* := \exists r.[\hat{C}]^*
\]

It remains to define the transformation \([E_\sim]^*\) for \( E_\sim \), when \( \sim \in \{>, \geq\} \). We show that such threshold concepts \( E_\sim \) can be equivalently expressed in \( ALC \) as a disjunction of \( E\mathcal{L} \) concept descriptions \( E_i \sqcup \ldots \sqcup E_q \), such that \( E_i \) (\( 1 \leq i \leq q \)) is a concept part of \( E^r \).

Let \( I \) be an interpretation and \( d \in \Delta^I \) such that \( d \in (E_\sim)^I \). We make two observations about path \( h \) and their induced interpretations \( I_h \).

1. Let \( h \) be path in \( H(T_{E^r}, G_T, d) \) and \( I_h \) its induced canonical interpretation. Since \( T_{I_h} \) is a tree, we can speak of its associated \( E\mathcal{L} \) concept description \( C_{I_h} \). Furthermore, from Remark 4.7 we know that \( h \) is a homomorphism from \( T_{I_h} \) to \( G_T \). Thus, since \( h(v_0) = d \), we can apply Theorem 2.7 to obtain \( d \in (C_{I_h})^I \).

2. It is clear from the construction of \( I_h \) in Definition 4.6 that \( C_{I_h} \) is a concept part of \( E^r \).

In view of Lemma 4.12 and the fact that \( \sim \in \{>, \geq\} \), the first observation tells us that: \( d \in (E_\sim)^I \) iff \( d \in C^I \) for some \( E\mathcal{L} \) concept description \( C \), whose associated \( E\mathcal{L} \) description tree \( T_C \) corresponds to an interpretation \( I_C \) such that \( \text{deg}^{T_C}(v_0, E) \sim t \). In addition, the second observation implies that it is sufficient to consider concept parts of \( E^r \). We now formally define the set of such relevant concepts.

Definition 4.16. Let \( E_\sim \) be a threshold concept with \( \sim \in \{>, \geq\} \). For all \( X \in \text{Part}(E^r) \), we assign to \( X \) the value \( v(X) \in [0..1] \) computed as:

\[
v(X) := \text{deg}^{T_X}(v_0, E)
\]

Then, the subset \( \mathcal{R}(E_\sim) \) of relevant concepts in \( \text{Part}(E^r) \) is defined as follows:

\[
\mathcal{R}(E_\sim) := \{ X \mid X \in \text{Part}(E^r) \text{ and } v(X) \sim t \} \quad \diamond
\]

The following lemma shows that membership in \( E_\sim \) is equivalent to membership in at least one concept description from \( \mathcal{R}(E_\sim) \).
Lemma 4.17. Let $\mathcal{I}$ be an interpretation, $d \in \Delta^\mathcal{I}$ and $E_{\sim t}$ a threshold concept with $\sim \in \{>, \geq\}$. The following statements are equivalent.

1. $d \in (E_{\sim t})^\mathcal{I}$.

2. There exists $X \in \mathcal{R}(E_{\sim t})$ such that $d \in X^\mathcal{I}$.

Proof. 1) $\rightarrow$ 2). Assume that $d \in (E_{\sim t})^\mathcal{I}$. Then, there exists a ptgh $h$ in $\mathcal{H}(T_{E^t}, G^\mathcal{I}, d)$ such that:

$h(v_0) = d$ and $h_w(v_0) = \deg^\mathcal{I}(d, E) \sim t$

By Lemma 4.12, the canonical interpretation $I_h$ satisfies $

\deg^{I_h}(v_0, E) = \deg^\mathcal{I}(d, E) \sim t$

As observed above, we have that $d \in (C_{I_h})^\mathcal{I}$ and $C_{I_h}$ is a concept part of $E^t$. Hence, since $v(C_{I_h}) = \deg^{I_h}(v_0, E) \sim t$, this means that $C_{I_h} \in \mathcal{R}(E_{\sim t})$.

2) $\rightarrow$ 1). Assume that there exists $X \in \mathcal{R}(E_{\sim t})$ such that $d \in X^\mathcal{I}$. By definition of $\mathcal{R}(E_{\sim t})$ we know that $\deg^{\mathcal{I}_X}(v_0, E) \sim t$. Moreover, since $d \in X^\mathcal{I}$, there exists a homomorphism $\varphi$ from $T_{\mathcal{I}_X}$ to $G^\mathcal{I}$ with $\varphi(v_0) = d$ (Theorem 2.7). Hence, the application of Lemma 4.11 to $I_X$ and $I$ yields $\deg^{\mathcal{I}_X}(v_0, E) \leq \deg^\mathcal{T}(d, E)$.

Thus, $\deg^\mathcal{I}(d, E) \sim t$ and $d \in (E_{\sim t})^\mathcal{I}$. $\square$

The previous lemma tells us how to build an equivalent $\mathcal{ALC}$ concept description $[E_{\sim t}]^*$ for $E_{\sim t}$. The existential quantification in the second statement is expressed using disjunction, and since $\mathcal{R}(E_{\sim t})$ is a finite set, we translate $E_{\sim t}$ into the following $\mathcal{ALC}$ concept description:

$$[E_{\sim t}]^* := \bigsqcup_{X \in \mathcal{R}(E_{\sim t})} X$$

One can still reduce the size of $[E_{\sim t}]^*$. Let $(\mathcal{R}(E_{\sim t}), \sqsubseteq)$ be the partially ordered set defined by $\sqsubseteq$ on $\mathcal{R}(E_{\sim t})$. Using Lemma 4.11 and the characterization of subsumption in $\mathcal{EL}$ (Theorem 2.6), it is easy to prove that for all pairs of concepts $X, Y \in \mathcal{R}(E_{\sim t})$

$$X \sqsubseteq Y \Rightarrow v(X) \geq v(Y)$$

This means that it is enough to consider the concept descriptions in $\mathcal{R}(E_{\sim t})$ that are maximal (or the most general ones) with respect to $\sqsubseteq$. Let $\mathcal{R}_{\text{max}}(E_{\sim t})$ be the set of maximal concepts in $\mathcal{R}(E_{\sim t})$ with respect to $\sqsubseteq$. We redefine $[E_{\sim t}]^*$ as:

$$[E_{\sim t}]^* := \bigsqcup_{X \in \mathcal{R}_{\text{max}}(E_{\sim t})} X$$

Lemma 4.18. Let $E_{\sim t}$ be a $\tau\mathcal{EL}(\deg)$ threshold concept with $\sim \in \{>, \geq\}$. Then,

$$E_{\sim t} \equiv [E_{\sim t}]^*$$
4.3 Relation to the Description Logic ALC

Proof. Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$. Assume that $d \in (E_{\sim \ell})^\mathcal{I}$. The application of Lemma 4.17 yields that there exists $X \in \mathcal{R}(E_{\sim \ell})$ such that $d \in X^\mathcal{I}$. Obviously, there exists $Y \in \mathcal{R}_{\max}(E_{\sim \ell})$ such that $X \subseteq Y$. Consequently, $d \in Y^\mathcal{I}$ and $d \in [(E_{\sim \ell})^\mathcal{I}]^\mathcal{I}$. Therefore, $(E_{\sim \ell})^\mathcal{I} \subseteq [(E_{\sim \ell})^\mathcal{I}]^\mathcal{I}$.

Conversely, suppose that $d \in [(E_{\sim \ell})^\mathcal{I}]^\mathcal{I}$. This means that there is $X \in \mathcal{R}_{\max}(E_{\sim \ell})$ such that $d \in X^\mathcal{I}$. Hence, since $\mathcal{R}_{\max}(E_{\sim \ell}) \subseteq \mathcal{R}(E_{\sim \ell})$, the application of Lemma 4.17 yields $d \in (E_{\sim \ell})^\mathcal{I}$.

Thus, we have shown that $E_{\sim \ell} = [(E_{\sim \ell})^\mathcal{I}]^\mathcal{I}$. □

Lemma 4.18 completes the translation $[.]^\mathcal{I}$ presented above. Then, one can easily show by induction on the structure of $\hat{C}$ that $\hat{C} \equiv [\hat{C}]^\mathcal{I}$ for all $\tau\mathcal{E}\mathcal{L}(\deg)$ concept descriptions $\hat{C}$. Thus, we have shown that $\tau\mathcal{E}\mathcal{L}(\deg)$ is a fragment of the DL ALC. However, as we will see in the following, this translation may produce a concept $[\hat{C}]^\mathcal{I}$ of size exponential in the size of $\hat{C}$.

Let $C_n \ (n \geq 1)$ be the $\mathcal{E}\mathcal{L}$ concept description $C_n^r \cap C_n^s$, where $C_n^x \ (x \in \{r, s\})$ is inductively defined as follows:

$$C_n^x := \begin{cases} \exists x. A & \text{if } n = 1 \\ \exists x. (A \cap C_{n-1}^x) & \text{if } n > 1 \end{cases}$$

The size of $C_n$ is linear in $n$, i.e., $s(C_n) = \mathcal{O}(n)$. Our translation into ALC of the threshold concept $(C_n)_{\geq \frac{1}{2}}$ yields an ALC concept description $[(C_n)_{\geq \frac{1}{2}}]^\mathcal{I}$ of size exponential in $n$. Let us explain this in the following example for $n = 3$.

Example 4.19. The $\mathcal{E}\mathcal{L}$ description tree depicted on the right-hand side of Figure 4.3 corresponds to the concept description $C_3$. Now, the left-hand side of the same figure contains the representation of four $\mathcal{E}\mathcal{L}$ description trees. In particular, we can say the following about $T_3^4$:

- its associated concept description $D_4^3$ is a concept part of $C_3$, and
- $\deg T_3^4 (v_0, C_3) = \frac{1}{2}$, where $T_3^4$ denotes the interpretation with description graph $T_3^4$ and $v_0$ its root. Consequently, there exists $h \in \mathcal{H}(T_{C_3}, T_3^4, v_0)$ such that $h_w(w_0) = 1/2$.

Therefore, $D_4^3 \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$. Furthermore, the $r$-branch in $T_{C_3}$ is fully present for $v_0$ in $T_3^4$, whereas the $s$-branch is completely missing. This means that they contribute to the top-level of the computation of $h_w(w_0)$ with the values $v_r = 1$ and $v_s = 0$, respectively. Hence, $D_4^3$ must be maximal in $\mathcal{R}((C_3)_{\geq \frac{1}{2}})$ with respect to $\subseteq$, for otherwise any concept $X \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$ satisfying $D_4^3 \subseteq X$ and $X \not\subseteq D_4^3$ is such that $v_r < 1$ and $v_s = 0$. This would imply that $\deg X (v_0, (C_3)_{\geq \frac{1}{2}}) < \frac{1}{2}$ which contradicts $X \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$. Thus, $D_4^3 \in \mathcal{R}_{\max}((C_3)_{\geq \frac{1}{2}})$ and it is one of the disjuncts in $[(C_3)_{\geq \frac{1}{2}}]^\mathcal{I}$.

The same conclusion can be drawn for the other three description trees. Basically, the values of the pair $(v_r, v_s)$ for $T_3^4$ and $T_3^1$ will be $(3/4, 1/4), (2/4, 2/4)$ and $(1/4, 3/4)$, respectively. Then, finding a more general concept $X$ for $D_4^i \ (1 \leq i \leq 3)$ would mean
that one of \(v_r, v_s\) decreases while the other one remains the same. Overall, this means that \(D_1 \sqcup D_2 \sqcup D_3 \sqcup D_4\) is a fragment of \([[(C_3) \geq 1]]^*\).

Thus, generalizing this idea for all \(n\) we obtain that \(s(((C_n) \geq 1/2)^*) \geq 2^{n-1}\). \(\diamondsuit\)

In conclusion, the DL \(\tau\mathcal{EL}(deg)\) is a fragment of \(\mathcal{ALC}\), but so far we do not know whether it is more succinct than \(\mathcal{ALC}\).
Chapter 5

Reasoning in $\tau\mathcal{EL}(deg)$

We now study the complexity of reasoning in the DL $\tau\mathcal{EL}(deg)$. We start with investigating the complexity of terminological reasoning (satisfiability, subsumption), and then turn to assertional reasoning (consistency, instance checking).

Using a very simple reduction from a variant of the propositional satisfiability problem, we show that satisfiability and non-subsumption in $\tau\mathcal{EL}(deg)$ are NP-hard. To provide an NP upper bound for satisfiability, we establish a polynomial bounded model property for satisfiable $\tau\mathcal{EL}(deg)$ concept descriptions. A key ingredient to obtain this property is the characterization of membership in $\tau\mathcal{EL}(deg)$ concept descriptions settled in Theorem 3.8. Afterwards, starting with a polynomial size model of a concept $\hat{C}$, we show how to extend it into a model of a concept $\hat{C} \sqcap \neg \hat{D}$ that is still polynomial in the size of $\hat{C}$ and $\hat{D}$. This will give us membership in NP for the non-subsumption problem, and thus a matching coNP upper bound for subsumption in $\tau\mathcal{EL}(deg)$.

Regarding assertional reasoning, the consistency problem can be tackled in a similar way as the satisfiability problem, by using Theorem 3.9 as a characterization of the satisfaction relation for $\tau\mathcal{EL}(deg)$ ABoxes. Then, similar to our treatment of subsumption, the bounded model of an ABox can be used to obtain a bounded model property for the non-instance problem. Therefore, we obtain that ABox consistency is NP-complete and the instance problem is coNP-complete (w.r.t. data complexity).

5.1 Terminological reasoning

We start by recalling the first two decision problems we will look at:

- **concept satisfiability**: Let $\hat{C}$ be a $\tau\mathcal{EL}(deg)$ concept description $\hat{C}$. The concept $\hat{C}$ is satisfiable iff there exists an interpretation $I$ such that $\hat{C}^I \neq \emptyset$.

- **subsumption**: Let $\hat{C}$ and $\hat{D}$ be two $\tau\mathcal{EL}(deg)$ concept descriptions. $\hat{C}$ is subsumed by $\hat{D}$ iff $\hat{C}^I \subseteq \hat{D}^I$ for every interpretation $I$.

The size $s(\hat{C})$ of a $\tau\mathcal{EL}(deg)$ concept description $\hat{C}$ is the number of occurrences of symbols needed to write $\hat{C}$.

In contrast to $\mathcal{EL}$, where every concept description is satisfiable, we have seen in Chapter 3 that there are unsatisfiable $\tau\mathcal{EL}(deg)$ concept descriptions such as $A_{\geq 1} \sqcap A_{< 1}$. Thus, the satisfiability problem is non-trivial in $\tau\mathcal{EL}(deg)$. In fact, by a simple reduction from the well-known NP-complete problem ALL-POS ONE-IN-THREE 3SAT (see [GJ79, page 259]) we can show that testing $\tau\mathcal{EL}(deg)$ concept descriptions for satisfiability is actually NP-hard.
Definition 5.1 (ALL-POS ONE-IN-THREE 3SAT). Let $U$ be a set of propositional variables and $C$ be a finite set of propositional clauses over $U$ such that:

- each clause in $C$ is a set of three literals over $U$, and
- no $c \in C$ contains a negated literal.

ALL-POS ONE-IN-THREE 3SAT is the problem of deciding whether there exists a truth assignment to the variables in $U$, such that each clause in $C$ has exactly one true literal.

Let $C = \{c_1, \ldots, c_n\}$ be a set of clauses over $U$. We now show how to build a $\tau\mathcal{EL}(\deg)$ concept description $\hat{C}_C$ such that $U$ has a truth assignment where exactly one literal per clause in $C$ is true iff $\hat{C}_C$ is satisfiable. Each propositional variable $u \in U$ is identified with the concept name $A_u$. In addition, to each clause $c_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ in $C$ we associate an $\mathcal{EL}$ concept description $D_i$ of the form $A_{u_{i_1}} \land A_{u_{i_2}} \land A_{u_{i_3}}$. Then the concept $\hat{C}_C$ is defined as follows:

$$\hat{C}_C := \prod_{i=1}^n [(D_i)_{\leq \frac{1}{3}} \land (D_i)_{\geq \frac{1}{3}}]$$

The main idea underlying this reduction is that for any three distinct concept names $A_i, A_j, A_k$, an individual belongs to $(A_i \land A_j \land A_k)_{\leq \frac{1}{3}} \land (A_i \land A_j \land A_k)_{\geq \frac{1}{3}}$ iff it belongs to exactly one of these three concepts.

Lemma 5.2. $\hat{C}_C$ is satisfiable iff there exists a truth assignment to the variables in $U$ such that each clause in $C$ has exactly one true literal.

Proof. ($\Rightarrow$) Assume that $\hat{C}_C$ is satisfiable. Then, there exists an interpretation $I$ such that $(\hat{C}_C)^I \neq \emptyset$, i.e., $d \in (\hat{C}_C)^I$ for some $d \in \Delta^I$. We construct an assignment $t$ for $U$ in the following way:

$$t(u) = \text{true} \text{ iff } d \in (A_u)^I, \text{ for all } u \in U \quad (5.1)$$

Now, let $c_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ be any clause in $C$. Since $d \in (\hat{C}_C)^I$, this means that $d \in [(D_i)_{\leq \frac{1}{3}}]_I$ and $d \in [(D_i)_{\geq \frac{1}{3}}]_I$. Therefore, $\deg^I(d, D_i) = \frac{1}{3}$ and by definition of $\deg^I$, $d$ is an instance of exactly one of the concept names $A_{u_{i_1}}, A_{u_{i_2}}, A_{u_{i_3}}$. Thus, by construction of $t$ in (5.1), exactly one literal in $c_i$ is assigned to true.

($\Leftarrow$) We assume that there exists a truth assignment $t$ to the variables in $U$ such that exactly one literal is true for each clause in $C$. Then, we build a single-pointed interpretation $I = (\{d\}, \hat{I})$ in the following way:

$$d \in (A_u)^I \text{ iff } t(u) = \text{true}, \text{ for all } u \in U$$

The properties satisfied by $t$ (with respect to $C$) imply that $d$ is an instance of exactly one concept name in the definition of $D_i$. Hence, for all $1 \leq i \leq n$ we have $\deg^I(d, D_i) = \frac{1}{3}$. Thus, $d \in (\hat{C}_C)^I$ and $I$ satisfies $\hat{C}_C$. \hfill $\square$

This also yields coNP-hardness for subsumption in $\tau\mathcal{EL}(\deg)$ since unsatisfiability can be reduced to subsumption: $\hat{C}$ is unsatisfiable iff $\hat{C} \subseteq A_{\geq 1} \land A_{< 1}$.

Lemma 5.3. In $\tau\mathcal{EL}(\deg)$, satisfiability is NP-hard and subsumption is coNP-hard.
To show an NP upper bound for satisfiability, we use the \( \tau \)-homomorphism characterization of membership for \( \tau \mathcal{EL}(m) \) concept descriptions introduced in Chapter 3. Using Theorem 5.3, we prove a bounded model property for \( \tau \mathcal{EL}(\text{deg}) \) concept descriptions.

**Lemma 5.4.** Let \( \hat{C} \) be a \( \tau \mathcal{EL}(\text{deg}) \) concept description of size \( s(\hat{C}) \). If \( \hat{C} \) is satisfiable, then there exists an interpretation \( J \) such that \( \hat{C}^J \neq \emptyset \) and \( |\Delta^J| \leq s(\hat{C}) \).

**Proof.** Since \( \hat{C} \) is satisfiable, there exists an interpretation \( I \) such that \( d \in \hat{C}^I \) for some \( d \in \Delta^I \). Therefore, there exists a \( \tau \)-homomorphism \( \phi \) from \( T_{\hat{C}} \) to \( G_I \) with \( \phi(v_0) = d \) (Theorem 3.8). The idea is to use \( \phi \) and small fragments of \( I \) to build \( J \) and a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_J \), and then apply Theorem 3.8 to \( \hat{C} \) and \( J \).

The interpretation \( J \) is built in two steps. We first use as base interpretation \( I_0 \) the one associated to the description tree \( T_{\hat{C}} \), where we ignore the labels of the form \( E_{\sim-t} \) (i.e. the description tree \( T_C \), see Figure 3.1). It is easy to see that the identity mapping \( \phi_{id} \) is a homomorphism from \( T_{\hat{C}} \) to \( G_{I_0} \). However, this interpretation and homomorphism need not satisfy Condition 2 of Definition 3.7. There may exist \( v \in \Delta^{I_0} \) such that \( E_{\sim-t} \in \tilde{\ell}_{T_{\hat{C}}}(v) \), but \( v \notin (E_{\sim-t})^{I_0} \). To repair this we extend \( I_0 \) to \( J \) by adding appropriate fragments of \( I \).

More precisely, for such a node \( v \in I_0 \) we know that \( \phi(v) \in (E_{\sim-t})^I \), and consequently \( \text{deg}^I(\phi(v), E) \sim t \). By Lemma 4.12 we do not need all of \( I \) to obtain \( \text{deg}^J(\phi(v), E) \) for \( v \) in \( J \). It is sufficient to use the canonical interpretation \( I_h \) induced by a \( \text{ptgh} \) \( h \) from \( T_{E^I} \) to \( G_I \) such that:

- \( h(w_0) = \phi(v) \), and
- \( \text{deg}^J(\phi(v), E) = h_w(w_0) \).

Here, \( w_0 \) is the root of \( T_{E^I} \). We rename it as \( v \) (the corresponding problematic node in \( I_0 \)) for the rest of the proof. We denote \( I_h \) as \( I_v^E \) and the \( \text{ptgh} \) \( h \) which induces \( I_h \) as \( h_w^E \).

Now, let \( J \) be the family of all interpretations \( I_v^E \) needed to repair the inconsistencies in \( I_0 \), i.e.,

\[
J := \{ I_v^E \mid v \in \Delta^{I_0}, E_{\sim-t} \in \tilde{\ell}_{T_{\hat{C}}}(v) \text{ and } v \notin (E_{\sim-t})^{I_0} \}
\]

For all pairs \( I_v^E, I_w^E \in J \) we assume \( \Delta^{I_v^E} \) and \( \Delta^{I_w^E} \) to be pairwise disjoint in the following sense: if \( v \neq w \) they have no element in common, otherwise only \( v \) is shared. In addition, for all \( I_v^E \in J \) the sets \( \Delta^{I_v^E} \) share only the distinguished element \( v \) with \( \Delta^{I_0} \). Once these disjointness assumptions have been established, \( J \) is constructed as follows:

- \( \Delta^J := \Delta^{I_0} \cup \bigcup_{K \in \mathcal{J}} \Delta^K \)
- \( X^J := X^{I_0} \cup \bigcup_{K \in \mathcal{J}} X^K \) for all \( X \in (\mathcal{N}_C \cup \mathcal{N}_R) \).

We now prove that Condition 2 of Definition 3.7 is satisfied by \( \phi_{id} \) and \( J \). For all \( v \in V_{T_{\hat{C}}} \) and \( E_{\sim-t} \in \tilde{\ell}_{T_{\hat{C}}}(v) \), we distinguish two cases:

- \( \sim \in \{>, \geq \} \). Suppose that \( v \in (E_{\sim-t})^{I_0} \). Since \( I_0 \subseteq J \), this makes Lemma 4.11 to be applicable to \( I_0 \), \( J \) and \( t^I \). Hence, we have \( \text{deg}^{I_0}(v, E) \leq \text{deg}^J(v, E) \) and

\footnote{The identity mapping from \( \Delta^{I_0} \) to \( \Delta^J \) is a homomorphism from \( G_{I_0} \) to \( G_J \) (recall the definition of \( I \subseteq J \) in Chapter 3).}
obviously \( v \in (E_\prec)\mathcal{J} \). Conversely, assume that \( v \not\in (E_\prec)^{I_0} \). The selection of \( \mathcal{I}_v^E \) to build \( J \) and the application of Lemma 4.12 yields:

\[
\deg^{\mathcal{I}_v^E}(v, E) = \deg^J(\phi(v), E)
\]

This means that \( v \in (E_\prec)^{\mathcal{I}_v^E} \), since \( \phi(v) \in (E_\prec)^J \). Moreover, from the construction of \( J \) it follows that \( \mathcal{I}_v^E \subseteq \mathcal{J} \). Then, a second application of Lemma 4.11 to \( \mathcal{I}_v^E \), \( J \) and \( v \) yields \( v \in (E_\prec)^{\mathcal{J}} \).

- \( \sim \in \{<, \leq \} \). Since \( \phi(v) \in (E_\prec)^J \), we intend to use again Lemma 4.11 with respect to \( J \) and \( \mathcal{I} \). For this, we build a mapping \( \varphi \) from \( V_J \to V_I \) such that \( \varphi(w) = \phi(w) \) for all \( w \in \Delta^{I_0} \), and show that it is a homomorphism from \( G_J \) to \( G_I \).

\[
\varphi := \phi \cup \bigcup_{\mathcal{I}_w^E \in \mathcal{J}} h_w^E \]

Recall from Remark 4.7 that \( h_w^E \) is a homomorphism from \( T_{\mathcal{I}_w^E} \) to \( G_I \). Consequently, \( \varphi \) is defined for all \( d \in \Delta^J \) and since \( \phi \) and each \( h_w^E \) have \( V_I \) as their images, \( \varphi \) is certainly a mapping from \( V_J \) to \( V_I \). In addition, by the disjointness assumptions made to build \( \Delta^J \) and the fact that \( h_w^E \) is chosen such that \( h_w^E(w) = \phi(w) \), we further have that \( \varphi \) is unambiguous and \( \varphi(w) = \phi(w) \) for all \( w \in \Delta^{I_0} \).

Let us now see why \( \varphi \) is really a homomorphism in the sense of Definition 2.5:

- For all \( d \in \Delta^J \), we either have \( d \in \Delta^{I_0} \) and

\[
\ell_J(d) = \ell_{I_0}(d) \cup \bigcup_{\mathcal{I}_w^E \in \mathcal{J}} \ell_{\mathcal{I}_w^E}(d) \quad (w = d)
\]

or \( d \in \Delta^{I_w^E} \) for some \( \mathcal{I}_w^E \in \mathcal{J} \) and \( \ell_J(d) = \ell_{\mathcal{I}_w^E}(d) \), where \( w \neq d \). Since \( \phi \) is a homomorphism from \( T_{\mathcal{I}_w} \) to \( G_I \), this means that \( \ell_{I_0}(d) \subseteq \ell_I(\phi(d)) \). Since each \( h_w^E \) is also a homomorphism from \( T_{\mathcal{I}_w^E} \) to \( G_I \), this means that \( \ell_{\mathcal{I}_w^E}(d) \subseteq \ell_I(h_w^E(d)) \). Hence, by the way \( \varphi \) has been defined we can conclude that \( \ell_J(d) \subseteq \ell_I(\varphi(d)) \) for all \( d \in \Delta^J \).

- \( d_1 r d_2 \in E_J \). If \( d_1, d_2 \in \Delta^{I_0} \), then \( \phi \) implies that \( \varphi(d_1) r \varphi(d_2) \in E_I \). Otherwise, \( d_1, d_2 \in \Delta^{I_w^E} \) for some \( \mathcal{I}_w^E \in \mathcal{J} \). Then, the corresponding homomorphism \( h_w^E \) guarantees that \( \varphi(d_1) r \varphi(d_2) \in E_I \).

Consequently, \( \varphi \) is a homomorphism from \( G_J \) to \( G_I \). Since \( \varphi(w) = \phi(w) \) for all \( w \in \Delta^{I_0} \) and \( v \in \Delta^{I_0} \), the rest relies in applying Lemma 4.11 with respect to \( J \), \( I \) and \( v \) to obtain \( v \in (E_\prec)^{\mathcal{J}} \).

Thus, we have shown that \( \phi_\text{id} \) is \( \tau \)-homomorphism from \( T_{\mathcal{I}} \) to \( G \). Since \( \phi_\text{id}(v_0) = v_0 \), the application of Theorem 3.8 yields \( v_0 \in \hat{C}^{\mathcal{J}} \).

To conclude, we look at the size of \( J \). By construction of \( J \) we have:

\[
|\Delta^J| = |\Delta^{I_0}| + \sum_{\mathcal{K} \in \mathcal{J}} |\Delta^K|
\]
It is not hard to see that the size of $I_0$ is bounded by the size of $\hat{C}$ (without counting the threshold concepts). In addition, any occurrence of a threshold concept $E_{\sim t}$ in $\hat{C}$ is considered at most once to build $J$. Moreover, each canonical interpretation $I^E_r \in I$ is selected with respect to $E^r$ and its size is bounded by the size of $E^r$ (see Definition 4.6). Since $E^r$ is obviously not bigger than $E$, this implies $|\Delta^I_r| \leq s(E_{\sim t})$. Thus, it is clear that $|\Delta^J| \leq s(\hat{C})$.

This lemma yields a standard guess-and-check NP-algorithm to decide satisfiability of a concept $\hat{C}$. The algorithm first guesses an interpretation $J$ of size at most $s(\hat{C})$, and then checks whether there exists a $\tau$-homomorphism from $T_{\hat{C}}$ to $G_J$. To verify the existence of a $\tau$-homomorphism it uses Algorithm 2 in Section 3.2. Since $\text{deg}$ can be computed in polynomial time (Chapter 4), Algorithm 2 runs in polynomial time with respect to $\text{deg}$.

**Remark 5.5.** We would like to point out that the construction presented in the previous lemma yields a tree-shaped interpretation $J$, i.e., $G_J$ is a tree. The base interpretation $I_0$ is tree-shaped since its description graph has the structure of $T_{\hat{C}}$, and so are the canonical interpretations used to extend $I_0$ into $J$. This combined with the applied disjointness assumptions guarantee that the resulting graph $G_J$ is a tree. Additionally, the element $v_0 \in \Delta^J$ corresponding to the root of $G_J$ satisfies $v_0 \in \hat{C}^J$.

A coNP upper bound for subsumption cannot directly be obtained from the fact that satisfiability is in NP. In fact, though we have $\hat{C} \subseteq \hat{D}$ if $\hat{C} \cap \sim \hat{D}$ is unsatisfiable, this equivalence cannot be used directly since $\sim \hat{D}$ need not be a $\tau \mathcal{EL}(\text{deg})$ concept description as shown in Section 4.3.1. Nevertheless, we can extend the ideas used in the proof of Lemma 5.4 to obtain a bounded model property for satisfiability of concepts of the form $\hat{C} \cap \sim \hat{D}$.

**Lemma 5.6.** Let $\hat{C}$ and $\hat{D}$ be $\tau \mathcal{EL}(\text{deg})$ concept descriptions of respective sizes $s(\hat{C})$ and $s(\hat{D})$. If $\hat{C} \cap \sim \hat{D}$ is satisfiable, then there exists an interpretation $J$ such that $\hat{C}^J \setminus \hat{D}^J \neq \emptyset$ and $|\Delta^J| \leq s(\hat{C}) \times s(\hat{D})$.

**Proof.** Assume that $\hat{C} \cap \sim \hat{D}$ is satisfiable. Then, there exists an interpretation $I$ such that $d \in \hat{C}^I$ and $d \notin \hat{D}^I$ for some $d \in \Delta^I$. We first apply the construction used in Lemma 5.4 to build (with respect to $I$) an interpretation $J_0$ such that $\hat{C}^{J_0} \neq \emptyset$ and $|\Delta^{J_0}| \leq s(\hat{C})$. From Lemma 5.4 we know:

- $G_{J_0}$ is a tree and $v_0 \in \hat{C}^{J_0}$.
- $\phi$ is a $\tau$-homomorphism from $T_{\hat{C}}$ to $G_I$ with $\phi(v_0) = d$.
- $\phi_I$ is a $\tau$-homomorphism from $T_{\hat{C}}$ to $G_{J_0}$.
- $\varphi$ is a homomorphism from $G_{\sim J_0}$ to $G_I$ with $\varphi(w) = \phi(w)$ for all $w \in \Delta^{J_0}$.

Since $\varphi(v_0) \notin \hat{D}^I$, the idea is to use $\varphi$ to extract from $I$ the necessary information to extend $J_0$ into an interpretation $J$ that falsifies $\hat{D}$ in $v_0$, while keeping $v_0 \in \hat{C}^J$. In order to do this, we consider all the nodes in $\Delta^{J_0}$ in a top-down manner starting with the root $v_0$. 

Chapter 5. Reasoning in \( \tau \mathcal{EL}(\deg) \)

We construct a series of pairs \((\mathcal{J}_0, S_0), (\mathcal{J}_1, S_1), \ldots\), where each \(\mathcal{J}_i\) is an interpretation and \(S_i\) is a set of pairs of the form \((v, \hat{F})\) being \(v \in \Delta^{\mathcal{J}_0}\) and \(\hat{F}\) a \(\tau \mathcal{EL}(\deg)\) concept description. The initial pair \((\mathcal{J}_0, S_0)\) is set as \((\mathcal{J}_0, \{(v_0, \hat{D})\})\). The sequence is built such that \(\varphi(v) \notin \hat{F}^{\tau}\) represents an invariant for all pairs \((v, \hat{F}) \in S_i\). This will then be used to show that \(v \notin F^\mathcal{J}\), and hence \(v_0 \notin \hat{D}^\mathcal{J}\).

Each pair \((\mathcal{J}_i, S_i)\) \((i > 0)\) is computed from the pair \((\mathcal{J}_{i-1}, S_{i-1})\) as follows:

- First, an auxiliary set \(S_i^\ast\) is computed with the purpose to decompose concepts \(\hat{F}\) of the form \(\hat{F}_1 \cap \ldots \cap \hat{F}_n\). More precisely, for all \((v, \hat{F}) \in S_{i-1}\) exactly one conjunct \(\hat{F}_j^\prime = \hat{F}_j \ (1 \leq j \leq n)\) is selected such that \(\varphi(v) \notin (\hat{F}_j)^\tau\). The set \(S_i^\ast\) is defined as follows:

\[
S_i^\ast := S_{i-1} \cup \bigcup_{(v, \hat{F}) \in S_{i-1}} \{ (v, \hat{F}^\prime) \}
\]

- Then, \(S_i\) is obtained from \(S_i^\ast\) as:

\[
S_i := \{(v, \hat{F}) \mid (v, \exists r, \hat{F}) \in S_i^\ast, (v, u) \in r^{\mathcal{J}_0} \text{ and } u \in \hat{F}^{\mathcal{J}_0} \}
\]

- Regarding \(\mathcal{J}_i\), for all \((v, E_{\sim \cdot}) \in S_i^\ast\) such that \(v \in (E_{\sim \cdot})^{\mathcal{J}_0}\) we select a canonical interpretation \(I_v^E\) (as for the proof of Lemma 5.4) with \(h_v^E(w_0) = \varphi(v)\). Now, let \(\mathcal{J}_i\) be the following set:

\[
\mathcal{J}_i := \{ I_v^E \mid (v, E_{\sim \cdot}) \in S_i^\ast \text{ and } v \in (E_{\sim \cdot})^{\mathcal{J}_0} \}
\]

Using the same disjointness assumptions as in Lemma 5.4, \(\mathcal{J}_i\) is built as follows:

\[
- \Delta^{\mathcal{J}_i} := \Delta^{\mathcal{J}_{i-1}} \cup \bigcup_{K \in \mathcal{I}_i} \Delta^K,
- X^{\mathcal{J}_i} := X^{\mathcal{J}_{i-1}} \cup \bigcup_{K \in \mathcal{I}_i} X^K \text{ for all } X \in (N_C \cup N_R).
\]

As we will see later, whenever \((v, \hat{F}_1 \cap \ldots \cap \hat{F}_n) \in S_{i-1}\) for some \(i > 0\), there always exists \(1 \leq j \leq n\) such that \(\varphi(v) \notin (\hat{F}_j)^\tau\). Moreover, the tree shape of \(\mathcal{J}_0\) makes this construction to consider every node in \(\Delta^{\mathcal{J}_0}\) at most once in the following sense. On the one hand, a node \(v\) does not occur in more than one set \(S_i\) \((i \geq 0)\). Therefore, at some point the iteration terminates for some \(p\) where \(S_p = \emptyset\). On the other hand, if \((v, \hat{F}) \in S_i\), there is no other pair \((v, \_\,\_)\) occurring in \(S_i\). This further implies that at most one canonical interpretation is added for each \(v \in \Delta^{\mathcal{J}_0}\). Moreover, observe that for all \((v, \hat{F}) \in S_i^\ast\) the concept description \(\hat{F}\) is a sub-description of \(\hat{D}\). In particular, for \((v, E_{\sim \cdot}) \in S_i^\ast\) it follows that \(|\Delta^{\mathcal{J}_p}| \leq s(\hat{D})\). Then, since \(|\Delta^{\mathcal{J}_0}| \leq s(\hat{C})\), once the iteration finishes we will have \(|\Delta^{\mathcal{J}_p}| \leq s(\hat{C}) \times s(\hat{D})\).

The next step is to show that \(v_0 \notin \hat{C}^{\mathcal{J}_p}\) and \(v_0 \notin \hat{D}^{\mathcal{J}_p}\). Consider the mapping \(\varphi^*\) from \(V_{\mathcal{J}_p}\) to \(V_{\hat{I}}\):

\[
\varphi^* := \varphi \cup \bigcup_{i=1}^{p} \bigcup_{I_v^E \in \mathcal{I}_i} h_v^E
\]

One can show that \(\varphi^*\) is a homomorphism from \(G_{\mathcal{J}_p}\) to \(G_{\hat{I}}\) with \(\varphi^*(w) = \phi(w)\) for all \(w \in \Delta^{\mathcal{J}_0}\). The proof uses the same arguments showing that \(\varphi\) is a homomorphism.
from \( G_J \) to \( G_I \) in Lemma 5.4. Then, \( \phi_{id} \) remains a \( \tau \)-homomorphism from \( T_{\hat{F}_0} \) to \( G_{\hat{J}_p} \). Similar as in Lemma 5.4, one can use Lemma 4.11 to prove that \( v \in (E_{\sim})^{\hat{J}_p} \) for all \( E_{\sim} \in \hat{L}_{\hat{T}_G}(v) \). If \( \sim \in \{ >, \geq \} \), it follows from the fact that \( J_0 \subseteq J_p \) and \( v \in (E_{\sim})^{\hat{J}_0} \). Otherwise, the argument relies on the homomorphism \( \varphi^* \) from \( G_{\hat{J}_p} \) to \( I \), \( \varphi^*(v) = \phi(v) \) and \( \phi(v) \in (E_{\sim})^I \). We thus have \( v_0 \in \hat{C}^{\hat{J}_p} \).

Before going into the main details of why \( v_0 \notin \hat{D}^{\hat{J}_p} \), we clarify why the invariant mentioned before is satisfied along the construction of \( J_p \):

\[
(v, \hat{F}) \in S_i \Rightarrow \varphi(v) \notin \hat{F}^T \tag{5.2}
\]

The initial pair \((v_0, \hat{D})\) satisfies it, since \( \varphi(v_0) = d \) and \( d \notin \hat{D}^I \). By definition, all the pairs in \( S^*_i \) clearly satisfy the property. Now, let \((v, \exists r. \hat{F}) \in S^*_i \). Starting with \( \varphi(v) \notin (\exists r. \hat{F})^I \), for any \( r \)-successor \( u \) of \( v \) the homomorphism makes \( (\varphi(v), \varphi(u)) \in r^I \).

Therefore, \( \varphi(u) \notin \hat{F}^I \) and \((u, \hat{F})\) satisfies the property as well. Consequently, \( S_i \) satisfies (5.2). Applying the same reasoning inductively shows that (5.2) remains invariant for all \( S_i \).

Note that this additionally implies that \( \hat{F}^T \) can always be selected when constructing \( S^*_i \).

To finally prove that \( v_0 \notin \hat{D}^{\hat{J}_p} \), we show the following more general claim.

Claim: for all \( 0 < i \leq p \), if \((v, \hat{F}) \in S^*_i \) then \( v \notin \hat{F}^{\hat{J}_p} \).

The proof goes by induction on the structure of \( \hat{F} \). Let \((v, \hat{F}) \in S^*_i \) for some \( 0 < i \leq p \):

- \( \hat{F} \) is of the form \( \top \) or \( A \in N_\mathcal{C} \). The case \( \hat{F} = \top \) never occurs, since \( \varphi(v) \notin \hat{F}^I \).

  Otherwise, if \( \hat{F} = A \) this means that \( \varphi(v) \notin A^I \). Since \( \varphi^* \) is a homomorphism from \( G_J \) to \( G_I \) with \( \varphi^*(v) = \varphi(v) \) for all \( v \in \Delta^{\hat{J}_0} \), it must be that \( v \notin A^{\hat{J}_p} \).

- \( \hat{F} \) is of the form \( E_{\sim} \). By (5.2) we have \( \varphi(v) \notin (E_{\sim})^I \).

  Moreover, we know that \( J_0 \subseteq J_p \) and \( \varphi^* \) is a homomorphism from \( G_J \) to \( G_I \) with \( \varphi^*(v) = \varphi(v) \) for all \( v \in \Delta^{\hat{J}_0} \). Hence, when \( v \notin (E_{\sim})^{\hat{J}_0} \), Lemma 4.11 ensures that \( v \notin (E_{\sim})^{\hat{J}_p} \). Otherwise, \( v \in (E_{\sim})^{\hat{J}_0} \) and the construction of \( J_p \) adds an interpretation \( I^E \) such that \( \text{deg}^E(v, E) = \text{deg}^I(\varphi(v), E) \).

  Since \( I^E \subseteq J_p \), again we obtain \( v \notin (E_{\sim})^{\hat{J}_p} \).

- \( \hat{F} = \hat{F}_1 \sqcap \ldots \sqcap \hat{F}_n \). By construction of \( S^*_i \) there is \( \hat{F}_j \) (\( 1 \leq j \leq n \)), such that \( \varphi(v) \notin (\hat{F}_j)^I \) and \((v, \hat{F}_j) \in S^*_i \). The application of induction to \( \hat{F}_j \) yields \( v \notin (\hat{F}_j)^{\hat{J}_p} \).

  Hence, \( v \notin \hat{F}^{\hat{J}_p} \).

- \( \hat{F} \) is of the form \( \exists r. \hat{F}^r \). Since each node is considered only once while building \( J_p \), one can see that each direct \( r \)-successor of \( v \) in \( G_J \) is a node in \( \Delta^{\hat{J}_0} \). Let \( u \in \Delta^{\hat{J}_0} \) such that \((v, u) \in r^{\hat{J}_0} \). We distinguish two cases:

  - \( u \in (\hat{F}^r)^{\hat{J}_0} \). This means that \((u, \hat{F}^r) \in S_i \) and consequently \((u, \hat{F}^r) \in S^*_i+1 \).

    Then, the application of induction hypothesis yields \( u \notin (\hat{F}^r)^{\hat{J}_p} \).

  - \( u \notin (\hat{F}^r)^{\hat{J}_0} \). This means that \( u \) is not relevant to obtain \( S_i \) from \( S^*_i \), and since \( G_{\hat{J}_0} \) is a tree neither of its successors is considered in the construction of \( J_p \). Therefore, the elements reachable from \( u \) in \( J_p \) are exactly the same as in \( J_0 \). Suppose now that \( u \in (\hat{F}^r)^{\hat{J}_p} \), then by Theorem 3.8 there exists a
τ-homomorphism \( \phi' \) from \( \hat{T}_p \) to \( G_{\mathcal{J}_p} \), with \( \phi'(w_0) = u \) (where \( w_0 \) is the root of \( \hat{T}_p \)).

But then, it would also be a \( \tau \)-homomorphism from \( T_p \) to \( G_{\mathcal{J}_0} \) contradicting \( u \not\in (\hat{F})^{\mathcal{J}_0} \). Consequently \( u \not\in (\hat{F})^{\mathcal{J}_p} \).

In conclusion, we have that for any \( u \in \Delta_{\mathcal{J}_0} \) such that \( (v, u) \in r_{\mathcal{J}_p} \) it is the case that \( u \not\in (\hat{F})^{\mathcal{J}_p} \). Hence, \( v \not\in (\exists \mathcal{J}_p)^{\mathcal{J}_p} \).

The lemma yields an obvious guess-and-check NP-algorithm for non-subsumption, which shows that subsumption is in coNP. Like for the satisfiability problem, the algorithm guesses an interpretation \( \mathcal{J} \) of size \( s(\hat{C}) \times s(\hat{D}) \), and then checks if \( d \in \hat{C}^{\mathcal{J}} \) and \( d \not\in \hat{D}^{\mathcal{J}} \) for some element \( d \in \Delta^{\mathcal{J}} \). This can obviously be done, in polynomial time, by using Algorithm 2.

Overall, we thus have shown:

**Theorem 5.7.** In \( \tau\mathcal{E}\mathcal{L}(deg) \), satisfiability is NP-complete and subsumption is coNP-complete.

### 5.2 Assertional reasoning

Let us now look at reasoning in the presence of \( \tau\mathcal{E}\mathcal{L}(deg) \) ABoxes. We study the following two decision problems.

- **ABox consistency:** Let \( \mathcal{A} \) be a \( \tau\mathcal{E}\mathcal{L}(deg) \) ABox. The ABox \( \mathcal{A} \) is consistent iff there exists an interpretation \( \mathcal{I} \) which is a model of \( \mathcal{A} \) (denoted \( \mathcal{I} \models \mathcal{A} \)).

- **instance checking:** Let \( \mathcal{A} \) be \( \tau\mathcal{E}\mathcal{L}(deg) \) ABox, \( \hat{C} \) a \( \tau\mathcal{E}\mathcal{L}(deg) \) concept description and \( a \) an individual. The individual \( a \) is an instance of \( \hat{C} \) in \( \mathcal{A} \) (denoted \( \mathcal{A} \models \hat{C}(a) \)) iff \( a^{\mathcal{I}} \in \hat{C}^{\mathcal{I}} \) holds in all models of \( \mathcal{A} \).

We define the size \( s(\mathcal{A}) \) of an ABox \( \mathcal{A} \) as:

\[
s(\mathcal{A}) := \sum_{\hat{C}(a) \in \mathcal{A}} s(\hat{C}) + \sum_{r(a,b) \in \mathcal{A}} 1
\]

Since satisfiability can obviously be reduced to consistency (\( \hat{C} \) is satisfiable iff \( \{\hat{C}(a)\} \) is consistent), and subsumption to the instance problem (\( \hat{C} \subseteq \hat{D} \) iff \( \{\hat{C}(a)\} \models \hat{D}(a) \)), the lower bounds from Lemma 5.3 also hold for assertional reasoning.

**Lemma 5.8.** In \( \tau\mathcal{E}\mathcal{L}(deg) \), ABox consistency is NP-hard and instance checking is coNP-hard.

Regarding upper bounds, we proceed in the same way as for concept satisfiability and subsumption. We first show a bounded model property for consistent ABoxes, which yields an NP upper bound for ABox consistency. Then, similar to our treatment of subsumption, this bounded model can be used to obtain a bounded model property for
the complement of the instance problem (a is not an instance of \(\hat{C}\) in \(A\)). However, as we will show, the bound of the model has the size of \(\hat{C}\) in the exponent. For this reason, we obtain a coNP upper bound for the instance problem only if we consider data complexity [DLNS94], where the size of the query concept \(\hat{C}\) is assumed to be constant.

The consistency problem can be tackled in a similar way as the satisfiability problem. As we have shown in Section 3.1, based on the translation given in [KM02], \(\tau EL(m)\) ABoxes can be translated into \(\tau EL(\text{deg})\) description graphs and consistency can be characterized using \(\tau\)-homomorphisms (see Theorem 3.9). We use this characterization to prove the following bounded model property.

**Lemma 5.9.** Let \(A\) be an ABox in \(\tau EL(\text{deg})\) of size \(s(A)\). If \(A\) is consistent, then there exists an interpretation \(J\) such that \(J \models A\) and \(|\Delta^J| \leq s(A)\).

**Proof.** Assume that \(A\) is consistent, then there exists an interpretation \(I\) such that \(I \models A\). Therefore, there exists a \(\tau\)-homomorphism \(\phi\) from \(\hat{G}(A)\) to \(G_I\) such that \(\phi(a) = a^I\) for all \(a \in \text{Ind}(A)\) (Theorem 3.9).

We proceed in the same way as in Lemma 5.4. The base interpretation \(I_0\) is the one having the description graph \(\hat{G}(A)\), where we ignore the labels of the form \(E_{\sim t}\). Again, the identity mapping \(\phi_{id}\) is a homomorphism from \(\hat{G}(A)\) to \(G_{I_0}\), but need not satisfy Condition 2 of Definition 3.7. The interpretation \(I_0\) has the following shape:

![Diagram](image)

Here, \(\{a_1, a_2, \ldots, a_n\} = \text{Ind}(A)\) and \(T(a_1), T(a_2), \ldots, T(a_n)\) are the \(\tau EL(m)\) description trees corresponding to \(\hat{C}_{a_1}, \hat{C}_{a_2}, \ldots, \hat{C}_{a_n}\), respectively (see Definition 3.5). The inner area of the diagram consists of the role assertions in \(A\), i.e.,

\[(a, b) \in E_A\text{ if } r(a, b) \in A\]

We extend \(I_0\) into \(J\) using the same construction of Lemma 5.4, i.e., a canonical interpretation \(T^E_v\) is attached to \(I_0\) for all \(v \in V_A\) such that \(E_{\sim t} \in \hat{E}_{A^+}(v)\) and \(v \notin (E_{\sim t})^{I_0}\). Note that besides the structure required by the role assertions in \(A\), the rest of \(G_{I_0}\) consists of disjoint description trees whose roots are individual elements of \(A\). Therefore, for two different individuals \(a, b \in \text{Ind}(A)\), reparations needed in \(T(a)\) and \(T(b)\) can be done independently of each other. Then, one can show that there is also a homomorphism \(\varphi\) from \(G_J\) to \(G_I\) with \(\varphi(w) = \phi(w)\) for all \(w \in \Delta^{I_0}\). Once we have this homomorphism, the same arguments used in Lemma 5.4 will show that \(\phi_{id}\) is a \(\tau\)-homomorphism from \(\hat{G}(A)\) to \(G_J\). Finally, by setting \(a^J = a\) we obtain \(\phi_{id}(a) = a^J\) for all \(a \in \text{Ind}(A)\). Thus, the application of Theorem 3.9 yields \(J \models A\).

Now, similarly as for \(\hat{C}\) in Lemma 5.4, the size of \(I_0\) is bounded by the size of \(A\) without counting the threshold concepts. Moreover, threshold concepts occurring in
concept assertions of $\mathcal{A}$ are also used at most once to build $\mathcal{J}$. Thus, it easily follows that $|\Delta^J| \leq s(\mathcal{A})$.

Using this lemma we can design an NP-algorithm to decide the consistency problem. The algorithm guesses an interpretation $\mathcal{J}$ of size at most $s(\mathcal{A})$. Afterwards, it checks using Algorithm 3 in polynomial time, whether there exists a $\tau$-homomorphism $\phi$ from $\hat{G}(\mathcal{A})$ to $G_{\mathcal{J}}$ with $\phi(a) = a^\mathcal{J}$ for all $a \in \text{ind}(\mathcal{A})$.

We now turn into the instance checking problem. The model $\mathcal{J}$ of $\mathcal{A}$ obtained in the previous lemma can be used as starting point to obtain a bounded model property for non-instance, i.e., $a$ is not an instance of $\hat{C}$ with respect to $\mathcal{A}$ iff $\mathcal{A} \cup \{\neg \hat{C}(a)\}$ is consistent. However, different from the interpretation $\mathcal{J}_0$ used in the construction of Lemma 5.6, the bounded model for an ABox obtained in Lemma 5.9 does not necessarily have a tree shape. As a consequence, using the procedure described in Lemma 5.6 to construct a model $\mathcal{J}$ of $\hat{C} \cap \neg \hat{D}$ would require to consider nodes from $\Delta^{\mathcal{J}_0}$ more than one time.

**Example 5.10.** Let $E$ be the $\mathcal{EL}$ concept description $\exists r. A \sqcap \exists r. B$. Consider the following ABox $\mathcal{A}$ and $\tau\mathcal{EL}(\text{deg})$ concept description $\hat{C}$:

$$\mathcal{A} := \{r(a,a)\} \quad \text{and} \quad \hat{C} := \exists r. \; \exists r. \; \underbrace{E < 1}_{p}$$

It is easy to see that $a$ is not an instance of $\hat{C}$ with respect to $\mathcal{A}$. The following single-pointed interpretation $\mathcal{K}$ (with $a^K = d$) is a model of $\mathcal{A}$ not satisfying $\hat{C}(a)$.

$$G_K : \begin{array}{c}
 r \\
 \{A, B\} \\
 d \end{array}$$

This means that $\mathcal{A} \cup \{\neg \hat{C}(a)\}$ is consistent. Let us now try to adapt the construction in Lemma 5.6 to $\mathcal{A}$ and $\neg \hat{C}(a)$. It starts by choosing $\mathcal{J}_0$ as the bounded model of $\mathcal{A}$ given by Lemma 5.9. Such a model has a similar shape as $\mathcal{K}$, but with $A^{\mathcal{J}_0} = B^{\mathcal{J}_0} = \emptyset$. The iteration is then guided by an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A}$ and $a^\mathcal{I} \notin \hat{C}$, and generates the following sequence of sets:

$$S_0 = \{(a, \hat{C})\}$$

$$\cdots$$

$$S_i = \{(a, \exists r \ldots \exists r. \; \underbrace{E_{<1}}_{p-i})\} \quad (1 \leq i < p)$$

$$\cdots$$

$$S_p = \{(a, E_{<1})\}$$

$$S_{p+1} = \emptyset$$

One can see that the iteration still terminates. The difference now is that not being $G_{\mathcal{J}_0}$ a tree, the element $a$ is considered several times. In particular, since $S_p = \{(a, E_{<1})\}$ and $a \in (E_{<1})^{\mathcal{J}_0}$, this means that $\mathcal{J}_0$ will be extended by adding a canonical interpretation which has the same description tree as $E$.
5.2 Assertional reasoning

\[
G_{J^p}: \quad \{A\} \xrightarrow{r} a \xrightarrow{r} \{B\}
\]

Since \(\varphi(a) \not\in (E_{<1})^J\), this will ensure that \(a \not\in (E_{<1})^J\). Unfortunately this is not sufficient to achieve \(a \not\in \hat{C}_{J^p}\). The set \(S_{p-1}\) contains the pair \((a, \exists r.E_{<1})\), which intuitively asks for \(a\) to satisfy \(a \not\in (\exists r.E_{<1})^J\). Clearly, the addition of the two \(r\)-successors of \(a\) implies that this is not the case. To repair this new problem, the natural extension of the procedure is to reconsider \(S_{p-1}\) with respect to the newly added elements. Such a repetition would then yield the following interpretation:

\[
G_{J^p}: \quad \{B\} \xrightarrow{r} \{A\} \xrightarrow{r} a \xrightarrow{r} \{B\} \xrightarrow{r} \{A\}
\]

Note that after fixing the problem for \(S_{p-1}\), the same issue will arise with respect to \((a, \exists r.E_{<1}) \in S_{p-2}\) and so on. Therefore, whenever a node \(v\) requires the addition of a canonical interpretation and has additional constraints (as just explained), the same idea needs to be recursively applied with respect to its new successors and those constraints.

Finally, one can see that this recursive application of the procedure leads to a model of size exponential in the size of \(\hat{C}\). This, however, does not necessarily imply that this is the best bound we can hope for. In fact, as illustrated above, \(K\) is a very small model satisfying \(A \cup \{\neg \hat{C}(a)\}\). It is just that the procedure does not realize that \(a\) can be an instance of \(A\) and \(B\) in \(J_0\) without contradicting \(J_0 \models A\). We do not yet know whether there is a better bound which applies to all possible combinations of \(A\) and \(\hat{C}\).

Based on the intuition given in Example 5.10, we extend the construction from Lemma 5.6 to ABoxes of the form \(A \cup \{\neg \hat{C}(a)\}\). We introduce a set of rules to transform \(A \cup \{\neg \hat{C}(a)\}\) into an ABox \(A'\), which contains additional assertions that (when consistent with \(A\)) are sufficient to falsify \(\hat{C}(a)\) in a model of \(A\). These rules are similar to some of the pre-processing rules defined in [BH91, Hol96], with the addition of specific rules to deal with the negation of threshold concepts. For the rest of this section we will use ABoxes that may also contain assertions of the form \(\neg \hat{C}(a)\). In case we want to refer to an ABox strictly in \(\tau EL(deg)\) we will mention it explicitly.

**Definition 5.11 (pre-processing rules).** Let \(A\) be an ABox. We define the following pre-processing rules:

- \(A \rightarrow_{\neg_1} A \cup \{\neg \hat{D}(a)\}\)
  - if \(\neg \hat{C}(a) \in A\) where \(\hat{C}\) is of the form \(\hat{C}_1 \cap \ldots \cap \hat{C}_n\), \(\neg \hat{C}_i(a) \not\in A\) for all \(i \in \{1 \ldots n\}\) and \(\hat{D} = \hat{C}_i\) for some \(i \in \{1 \ldots n\}\).

- \(A \rightarrow_{\neg_3} A \cup \{\neg \hat{D}(b)\}\)
  - if \((\neg \exists r.\hat{D})(a) \in A\), \(r(a,b) \in A\) and \(\neg \hat{D}(b) \not\in A\).

- \(A \rightarrow_{\neg_t} A \cup \{E_{\chi(\neg_t)}(a)\}\)
  - if \(\neg E_{\chi(\neg_t)}(a) \in A\) and \(E_{\chi(\neg_t)}(a) \not\in A\).
that no further rule application is possible over \( \mathcal{A} \). Note that if \( \mathcal{A} \) is a \( \tau \mathcal{EL}(\deg) \) ABox, the unique \textit{pre-processing} of \( \mathcal{A} \) is \( \mathcal{A} \) itself. The rules \( \rightarrow_{\sim} \) and \( \rightarrow_{\neg} \) are supported by the equivalences \( \neg E_{\sim} \equiv E_{\sim \sim} \) and \( \neg A \equiv A_{<1} \) (see Proposition 4.15 and Chapter 3). The rule \( \rightarrow_{\neg} \) has a non-deterministic flavor. It can be seen as the counterpart of the guided choice made in Lemma 5.6 to obtain a set \( S^* \). Regarding \( \rightarrow_{\sim} \), it has a similar aim as the construction of \( S \) from \( S^* \) in Lemma 5.6.

One can see that a rule application introduces neither a new individual nor a new role assertion. Therefore, \( \mathcal{A} \) and \( \mathcal{A}' \) have the same set of individuals and role assertions. Furthermore, only new assertions of the form \( \neg \mathcal{C}(a) \), \( E_{\sim \sim}(a) \) or \( A_{<1}(a) \) results from a rule application. In the first case \( \mathcal{C} \) is a sub-description of some concept \( \mathcal{D} \) such that \( \neg \mathcal{D}(b) \) is an assertion initially in \( \mathcal{A} \), whereas no rule is applicable to the other two cases. Hence, since \( \mathcal{A} \) is finite, there can never be an infinite sequence of rule applications.

Now, we can prove the following proposition (see Appendix A).

**Proposition 5.12.** Let \( \mathcal{A} \) be an ABox. Then, \( \mathcal{A} \) is consistent iff there exists a consistent \textit{pre-processing} \( \mathcal{A}' \) of \( \mathcal{A} \).

The following remark is a direct consequence from the proof of the previous proposition.

**Remark 5.13.** Let \( \mathcal{A} \) be an ABox and \( \mathcal{I} \) an interpretation. If \( \mathcal{I} \models \mathcal{A} \), then there exists a pre-processing \( \mathcal{A}' \) of \( \mathcal{A} \) such that \( \mathcal{I} \models \mathcal{A}' \).
Lemma 5.14. Let $\mathcal{A}$ be a consistent single-element ABox and $\mathcal{I}$ an interpretation such that $\mathcal{I} \models \mathcal{A}$. In addition, let $\mathcal{J}$ be the bounded model of $\mathcal{A}^+$ obtained in Lemma 5.9 with respect to $\mathcal{I}$. Then, there exists a tree-shaped interpretation $\mathcal{K}$ such that:

1. $\mathcal{K} \models \mathcal{A}$,

2. there exists a homomorphism $\varphi$ from $G^\mathcal{K}$ to $G^\mathcal{I}$ with $\varphi(a^K) = a^\mathcal{I}$, and

3. $|\Delta^K| \leq |\Delta^\mathcal{J}| \times p$, where:

$$p := \begin{cases} 1, & \text{if } A^- = \emptyset \\ \prod_{\neg \hat{D}(a) \in A^-} s(\hat{D}), & \text{otherwise}. \end{cases}$$

Once the previous lemma is applied to all the ABoxes $\mathcal{A}'(a)$, the second step is to combine all those models into a model of $\mathcal{A}$ of bounded size. More precisely, we show that the disjoint union of all those models together with the role assertions in $\mathcal{A}$ yield the wanted model. This is formalized in the following lemma (see Appendix A for its proof).

Lemma 5.15. Let $\mathcal{A}$ be an ABox, $\mathcal{I}$ an interpretation satisfying $\mathcal{A}$ and $\mathcal{A}'$ a pre-processing of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$. Moreover, for all $a \in \text{Ind}(\mathcal{A})$, let $\mathcal{I}_a$ be a tree-shaped interpretation satisfying the following:

- $\mathcal{I}_a \models \mathcal{A}'(a)$,

- there exists a homomorphism $\varphi_a$ from $G^\mathcal{I}_a$ to $G^\mathcal{I}$ with $\varphi_a(a^{\mathcal{I}_a}) = a^\mathcal{I}$.

Last, let $\mathcal{J}$ be the following interpretation:

- $\Delta^\mathcal{J} := \bigcup_{a \in \text{Ind}(\mathcal{A})} \Delta^\mathcal{I}_a$,

- $\mathcal{A}^\mathcal{J} := \bigcup_{a \in \text{Ind}(\mathcal{A})} \mathcal{A}^\mathcal{I}_a$ for all $A \in \mathcal{N}_C$,

- $r^\mathcal{J} := \{ (a^{\mathcal{I}_a}, b^{\mathcal{I}_a}) \mid r(a, b) \in \mathcal{A} \} \cup \bigcup_{a \in \text{Ind}(\mathcal{A})} r^\mathcal{I}_a$ for all $r \in \mathcal{N}_R$, and

- $a^\mathcal{J} := a^{\mathcal{I}_a}$, for all $a \in \text{Ind}(\mathcal{A})$.

where the sets $\Delta^\mathcal{I}_a$ are pairwise disjoint. Then, $\mathcal{J} \models \mathcal{A}$.

Using these two lemmas we can now established the final result. Recall that $\text{sub}(\hat{C})$ denotes the set of sub-descriptions of a concept description $\hat{C}$.

Lemma 5.16. Let $\mathcal{A}$ be an ABox in $\tau\mathcal{E}\mathcal{L}(\text{deg})$ of size $s(\mathcal{A})$, $\hat{C}$ a $\tau\mathcal{E}\mathcal{L}(\text{deg})$ concept description of size $s(\hat{C})$ and $a \in \mathcal{N}_I$. If $\mathcal{A} \cup \{ \neg \hat{C}(a) \}$ is consistent, then there exists an interpretation $\mathcal{J}$ such that:

1. $\mathcal{J} \models \mathcal{A} \cup \{ \neg \hat{C}(a) \}$,
2. \( \mathcal{J} \) is the result of the construction in Lemma 5.15.

3. for all \( a \in \text{Ind}(\mathcal{A}) \):

\[
|\Delta^{I_a}| \leq s(\mathcal{A}(a)) \times [s(\hat{C})]^u, \quad \text{where} \quad u = |\text{sub}(\hat{C})|
\]

Proof. Let \( I \) be an interpretation satisfying \( \mathcal{A} \cup \{ \neg \hat{C}(a) \} \). By Remark 5.13 there exists a pre-processing \( \mathcal{A}' \) of \( \mathcal{A} \cup \{ \neg \hat{C}(a) \} \) such that \( I \models \mathcal{A}' \). We apply Lemma 5.14 to \( \mathcal{A}'(a) \) for all \( a \in \text{Ind}(\mathcal{A}) \), and obtain a tree-shaped interpretation \( I_a \) such that:

- \( I_a \models \mathcal{A}'(a) \),
- there exists a homomorphism \( \varphi_a \) from \( G_{I_a} \) to \( G_I \) with \( \varphi_a(a^{T_a}) = a^I \).

Then, we can apply Lemma 5.15 to obtain an interpretation \( \mathcal{J} \) such that:

\[
\mathcal{J} \models \mathcal{A} \quad \text{and} \quad \Delta^{\mathcal{J}} = \bigcup_{a \in \text{Ind}(\mathcal{A})} \Delta^{I_a}
\]

We now look at the size of \( I_a \). For all \( a \in \text{Ind}(\mathcal{A}) \), let \( J_a \) denote the bounded model of \( \mathcal{A}^+(a) \) obtained in Lemma 5.9 with respect to \( I \). The construction of \( I_a \) in Lemma 5.14 yields:

\[
|\Delta^{I_a}| \leq |\Delta^{J_a}| \times \prod_{\neg D(a) \in \mathcal{A}^{-}(a)} s(D)
\]

One can see that each assertion in \( \mathcal{A}^+(a) \) is either of the form \( \hat{D}(a) \in \mathcal{A}(a) \) or \( E_{\chi(\sim t)} \). The latter case results from applications of the rules \( \to_{\sim} \) and \( \to_{\sim A} \). For the rule \( \to_{\sim A} \), \( A_{<1} \) corresponds to \( A_{(\geq 1)} \). In Lemma 5.9 the interpretation \( J_a \) is built starting with the interpretation \( I_0 \) which have the description graph \( \hat{G}(\mathcal{A}^+(a)) = (V_{\mathcal{A}^+(a)}, \ldots) \) (without threshold concepts), and it is then extended by considering the threshold concepts occurring in \( \hat{G}(\mathcal{A}^+(a)) \). We know the following about them:

- for all threshold concepts \( E_{\sim t} \) occurring in \( \mathcal{A}^+(a) \), either \( E_{\sim t} \) occurs in an assertion of \( \mathcal{A}(a) \) or it has been introduced by an application of \( \to_{\sim} \) or \( \to_{\sim A} \) (i.e., it is of the form \( E_{\chi(\sim t)}(a) \)),
- except for assertions of the form \( E_{\chi(\sim t)}(a) \),

\[
\hat{D}(a) \in \mathcal{A}^+(a) \quad \text{iff} \quad \hat{D}(a) \in \mathcal{A}(a)
\]

Thus, \( |V_{\mathcal{A}(a)}| = |V_{\mathcal{A}^+(a)}| \) and by construction of \( \mathcal{J} \) we obtain:

\[
|\Delta^{I_a}| \leq |V_{\mathcal{A}(a)}| + \sum_{E_{\sim t} \in \hat{G}(\mathcal{A}(a))} s(E_{\sim t}) + \sum_{E_{\chi(\sim t)}(a) \notin \hat{G}(\mathcal{A}(a))} s(E_{\chi(\sim t)}(a))
\]

Note that the partial sum of the first two elements in the right-hand side of the inequality is actually bounded by the size of \( \mathcal{A}(a) \). In addition, since \( s(E_{\chi(\sim t)}) > 1 \) we further have:
It is not hard to see that for all $\neg \hat{D}(a) \in A^{-}(a)$ and $E_{\sim t}(a) \notin \hat{G}(A(a))$, the concepts $\hat{D}$ and $E_{\sim t}$ (or $A$ for $\rightarrow_{a}$) are sub-descriptions of $\hat{C}$. Hence, the combinations of inequalities (5.3) and (5.4) yields:

$$|\Delta_{J}| \leq s(A(a)) \times \prod_{E_{\sim t}(a) \not\in \hat{G}(A(a))} s(E_{\sim t}(a))$$

Based on the previous results, we devise the following non-deterministic procedure to decide consistency of an ABox of the form $A \cup \{\neg \hat{C}(a)\}$.

1. For all $a \in \text{Ind}(A)$, guess an interpretation $I_{a}$ of size at most:

$$s(A(a)) \times [s(\hat{C})]^{u}$$

2. Construct $J$ using all the interpretations $I_{a}$ and $A$, as described in Lemma [5.15]

3. Check whether $J \models A$. This can be done in polynomial time (in the size of $J$ and $A$) by using Algorithm [3]. If it is not the case, then the algorithm answers “no”. Otherwise, it remains to verify whether $a^{J} \notin \hat{C}^{J}$.

4. To verify $a^{J} \notin \hat{C}^{J}$, by Theorem [3.8] it is enough to check that there is no $\tau$-homomorphism $\phi$ from $T_{\hat{C}}$ to $G_{J}$ with $\phi(v_{0}) = a^{J}$. This can also be checked in polynomial time by Algorithm [2]. If there is no such $\tau$-homomorphism the algorithm answers “yes”, and “no” otherwise.

If the size of $\hat{C}$ is considered as a constant, this algorithm becomes an NP-procedure for consistency of $A \cup \{\neg \hat{C}(a)\}$, and consequently a coNP-procedure to decide instance checking with respect to data complexity. Altogether, we thus have shown:

**Theorem 5.17.** In $\tau\mathcal{EL}(\deg)$, consistency is NP-complete, and instance checking is coNP-complete w.r.t. data complexity.

The instance problem becomes simpler if we consider only $\mathcal{EL}$ ABoxes and positive $\tau\mathcal{EL}(\deg)$ concept descriptions, i.e., concept descriptions $\hat{C}$ that only contain threshold concepts of the form $E_{\geq t}$ or $E_{> t}$. Basically, given an $\mathcal{EL}$ ABox $A$, a positive $\tau\mathcal{EL}(\deg)$ concept description $\hat{C}$, and an individual $a$, one considers the interpretation $I$ corresponding to the description graph $G(A)$ of $A$, and then checks whether there is a $\tau$-homomorphism $\phi$ from $T_{\hat{C}}$ to $G_{I}$ with $\phi(v_{0}) = a$. The following lemma supports the previous idea.

**Lemma 5.18.** Let $A$ be an $\mathcal{EL}$ ABox, $a \in \text{Ind}(A)$ and $\hat{C}$ a positive $\tau\mathcal{EL}(\deg)$ concept description. Additionally, let $I_{A}$ be the interpretation corresponding to the description graph $G(A)$ with $a^{I_{A}} = a$ for all $a \in \text{Ind}(A)$. Then, the following statements are equivalent:
1. \( \mathcal{A} \models \hat{C}(a) \), and
2. \( a \in \hat{C}^{\mathcal{I}_A} \).

Proof. 1) \( \rightarrow \) 2). Assume that \( \mathcal{A} \models \hat{C}(a) \). Then, for every model \( \mathcal{I} \) of \( \mathcal{A} \) we have \( a^\mathcal{I} \in \hat{C}^{\mathcal{I}} \). Since \( \mathcal{I}_A \) is obviously a model of \( \mathcal{A} \) and \( a^{\mathcal{I}_A} = a \), this means that \( a \in \hat{C}^{\mathcal{I}_A} \).

2) \( \rightarrow \) 1). Assume that \( a \in \hat{C}^{\mathcal{I}_A} \). The characterization for membership in \( \tau \mathcal{EL}(deg) \) given in Theorem 3.8 yields a \( \tau \)-homomorphism \( \phi \) from \( T_{\hat{C}} \) to \( G(A) \) with \( \phi(v_0) = a \). Now, consider any model \( \mathcal{I} \) of \( \mathcal{A} \). The application of Theorem 3.9 yields the existence of a \( \tau \)-homomorphism \( \varphi \) from \( G(A) \) to \( G_{\mathcal{I}} \) such that \( \varphi(a) = a^\mathcal{I} \) for all \( a \in \text{Ind}(\mathcal{A}) \). We then show that the mapping \( \varphi \circ \phi \) is a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_{\mathcal{I}} \):

- From \( \phi \) we know that \( \ell_{T_{\hat{C}}}(v) \subseteq \ell_{\mathcal{A}}(\phi(v)) \) for all \( v \in V_{T_{\hat{C}}} \). Similarly, \( \varphi \) implies that \( \ell_{\mathcal{A}}(a) \subseteq \ell_{\mathcal{I}}(\varphi(a)) \) for all \( a \in V_{\mathcal{A}} \). Hence, \( \ell_{T_{\hat{C}}}(v) \subseteq \ell_{\mathcal{I}}((\varphi \circ \phi)(v)) \) for all \( v \in V_{T_{\hat{C}}} \). The edge preserving relation can be verified in a similar way.

- Let \( v \in V_{T_{\hat{C}}} \) and \( E_{\sim t} \in \hat{\ell}_{T_{\hat{C}}}(v) \). Since \( \phi \) is a \( \tau \)-homomorphism, this means that \( \phi(v) \in (E_{\sim t})^{T_{\hat{C}}} \). Furthermore, the application of Lemma 4.11 to \( \mathcal{I}_A \), \( \mathcal{J} \) and \( \varphi \) yields:

\[
\text{deg}^{T_{\hat{C}}}(\phi(v), E) \leq \text{deg}^{\mathcal{I}}(\varphi(\phi(v)), E)
\]

Since \( \hat{C} \) is positive, this means that \( \sim \) is either > or \( \geq \). Consequently, \( (\varphi \circ \phi)(v) \in (E_{\sim t})^{\mathcal{I}} \).

Hence, \( \varphi \circ \phi \) is a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_{\mathcal{I}} \) with \( (\varphi \circ \phi)(v_0) = a^\mathcal{I} \). Altogether, this means that \( a^\mathcal{I} \in \hat{C}^{\mathcal{I}} \) for all models \( \mathcal{I} \) of \( \mathcal{A} \). Thus, \( \mathcal{A} \models \hat{C}(a) \).

Finally, since \( \mathcal{I}_A \) is linear on the size of \( \mathcal{A} \), checking whether \( a \in \hat{C}^{\mathcal{I}_A} \) can be done in polynomial time in the size of \( \mathcal{A} \) and \( \hat{C} \) by using Algorithm 2. Therefore, we obtain the following proposition.

**Proposition 5.19.** For positive \( \tau \mathcal{EL}(deg) \) concept descriptions and \( \mathcal{EL} \) \( \mathcal{A} \)Boxes, the instance checking problem can be decided in polynomial time.
Chapter 6

Adding Terminologies to $\tau\mathcal{EL}(m)$

Until now, we have only considered the basic concept language and background knowledge represented in the form of assertions about specific individuals. Nevertheless, most DLs also allow to store terminological knowledge about the application domain in a TBox. The aim of this chapter is to take initial steps in extending our logic $\tau\mathcal{EL}(deg)$ towards background knowledge represented in the form of axioms in a TBox.

We start by introducing $\mathcal{EL}$ TBoxes, and some related properties and technical notions concerning them. We will then turn to the definition of $\tau\mathcal{EL}(m)$ and $\tau\mathcal{EL}(deg)$ TBoxes. To accomplish this, there are two important aspects that we take into account. On the one hand, graded membership functions will now compute membership degrees to $\mathcal{EL}$ concepts defined with respect to an $\mathcal{EL}$ TBox. To handle this, we propose a general way to extend all functions $m$ through unfolding, and hence restrict threshold concepts $E \sim t$ to have $E$ defined with respect to an acyclic $\mathcal{EL}$ TBox. On the other hand, since we also intend to use TBoxes to define $\tau\mathcal{EL}(m)$ concept descriptions, further constraints are needed to exclude definitions of not well-formed $\tau\mathcal{EL}(m)$ concepts.

Once $\tau\mathcal{EL}(deg)$ TBoxes are defined, we direct our attention to study the computational complexity of reasoning in the presence of acyclic $\tau\mathcal{EL}(deg)$ TBoxes. It turns out that the possibility of succinctly representing exponentially large concept descriptions in a TBox, combined with the semantics of threshold concepts in $\tau\mathcal{EL}(deg)$, makes satisfiability and subsumption to be $\Pi^P_2$- and $\Sigma^P_2$-hard, respectively. Additionally, we provide a sound and complete non-deterministic PSPACE procedure to solve both problems, and later extend it to also consider assertional knowledge in an ABox. Such an extension keeps the use of space polynomial, an thus yields a PSPACE upper bound for all the standard reasoning tasks (including instance checking w.r.t. combined complexity).

6.1 $\mathcal{EL}$ TBoxes

A concept definition is of the form $A \doteq C_A$, where $A$ is a concept name and $C_A$ an $\mathcal{EL}$ concept description. An $\mathcal{EL}$ TBox $\mathcal{T}$ is a finite set of concept definitions such that no concept name occurs more than once on the left-hand side of a definition in $\mathcal{T}$. Concept names occurring on the left hand side of a definition are called defined concepts while all other concept names are called primitive concepts. The sets of defined and primitive concepts are denoted as $\mathcal{N}_{def}$ and $\mathcal{N}_{prim}$, respectively. A knowledge base (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox and an ABox.

We denote by $\Xi$ the set of all $\mathcal{EL}$ TBoxes. Given $\mathcal{T} \in \Xi$, $\text{def}(\mathcal{T})$ stands for the set of defined concepts in $\mathcal{T}$. Moreover, the size $s(\mathcal{T})$ of $\mathcal{T}$ corresponds to the following
expression:

\[ s(T) := |\text{def}(T)| + \sum_{A \in C_A \in T} s(C_A) \]

Finally, the size \( s(K) \) of a KB \( K = (T, A) \) is simply \( s(T) + s(A) \).

Concerning the semantics, an interpretation \( I \) is a model of a TBox \( T \) (in symbols \( I \models T \)) iff

\[ A^I = (C_A)^I \text{ for all } A \models C_A \in T \]

We denote as \( T(I) \subseteq T \) the set of all \( \mathcal{EL} \) TBoxes \( T \) such that \( I \models T \). The satisfaction relation for KBs is defined in the usual way: \( I \) is a model of a KB \( K = (T, A) \) iff \( I \) satisfies both \( T \) and \( A \). Now, given two \( \mathcal{EL} \) concept descriptions \( C \) and \( D \), \( C \) is satisfiable with respect to \( K \) iff \( C \models K D \) for all models \( I \) of \( K \). They are equivalent with respect to \( K \) (denoted \( C \equiv_K D \)) iff \( C \models K D \) and \( D \models K C \).

TBoxes can be classified regarding the dependencies between its concept definitions. More precisely,

**Definition 6.1 (\( \mathcal{EL} \) cyclic/acyclic TBoxes).** Let \( T \) be an \( \mathcal{EL} \) TBox. We define \( \rightarrow \) as a binary relation over the set \( \text{def}(T) \) to represent direct dependency between defined concepts in the following way.

A defined concept \( A \) directly depends on a defined concept \( B \) (denoted as \( A \rightarrow B \)) iff \( A \models C_A \in T \) and \( B \) occurs in \( C_A \). Let \( \rightarrow^+ \) be the transitive closure of \( \rightarrow \). The TBox \( T \) contains a terminological cycle iff there exists a defined concept \( A \) in \( T \) that depends on itself, i.e., \( A \rightarrow^+ A \). Then, \( T \) is called cyclic if it contains a terminological cycle. Otherwise, it is called acyclic.

For acyclic TBoxes, the relation \( \rightarrow^+ \) induces a well-founded partial order \( \preceq \) on the set \( \text{def}(T) \), i.e., \( A \preceq B \) iff \( B \rightarrow^+ A \). Furthermore, the unfolding \( u_T(C) \) of an \( \mathcal{EL} \) concept description \( C \) with respect to \( T \) can be defined as follows:

\[
\begin{align*}
    u_T(C \cap D) &:= u_T(C) \cap u_T(D) \\
    u_T(\exists r.C) &:= \exists r. u_T(C) \\
    u_T(A) &:= \begin{cases} A & \text{if } A \in \mathbb{N}_{\text{prim}}, \\ u_T(C_A) & \text{if } A \models C_A \in T. \end{cases}
\end{align*}
\]

It is well known that, regarding acyclic TBoxes, the meaning of concept descriptions follows directly from the meaning of their corresponding unfolded descriptions. The following is the equivalent, for \( \mathcal{EL} \), of Proposition 1 in \cite{Neb90}.

**Proposition 6.2.** For every acyclic \( \mathcal{EL} \) TBox \( T \), every \( \mathcal{EL} \) concept description \( C \) and every model \( I \) of \( T \):

\[ C^I = [u_T(C)]^I \]

As an immediate consequence of this equality, we obtain \( C \equiv_T u_T(C) \). This also has its counterpart from the model-theoretical point of view. Similar to Proposition 2 in \cite{Neb91}, we have the following for \( \mathcal{EL} \).
Proposition 6.3. Let $\mathcal{T}$ be an acyclic $\mathcal{EL}$ TBox. Any interpretation $\mathcal{J}$ of $N_{\text{prim}}$ and $N_{\mathcal{R}}$ can be uniquely extended to a model of $\mathcal{T}$.

These type of partial interpretations are called primitive. They do not assign any meaning to the defined concepts in $\mathcal{T}$. We say that an interpretation $\mathcal{I}$ is based on a primitive interpretation $\mathcal{J}$ iff it has the same domain as $\mathcal{J}$ and coincides with $\mathcal{J}$ on $N_{\mathcal{R}}$ and $N_{\text{prim}}$.

6.2 TBoxes for $\tau\mathcal{EL}(m)$ and $\tau\mathcal{EL}(\text{deg})$

We would now like to use sets of concept definitions to define $\tau\mathcal{EL}(m)$ concept descriptions. These concept definitions come in two forms. For example, the $\mathcal{EL}$ concept definition $E \doteq \exists r.A \sqcap \exists r.B$, can be used to build the threshold concept $E_{\leq 8}$. Furthermore, on top of that, one could also have $\tau\mathcal{EL}(m)$ concept definitions of the form:

$$\alpha \doteq \hat{\mathcal{C}}_{\alpha} \quad \text{(6.1)}$$

where $\alpha \in N_{\text{def}}$ and $\hat{\mathcal{C}}_{\alpha}$ is a $\tau\mathcal{EL}(m)$ concept description. For instance, the definition of $E$ together with $\alpha \doteq A \sqcap E_{\leq 8}$, can be used to define the $\tau\mathcal{EL}(m)$ concept description $\exists s.A \sqcap \exists r.\alpha$.

In what follows, we first revisit the notion of membership degree functions from Definition 3.1 and define $\tau\mathcal{EL}(m)$ TBoxes. Afterwards, we provide a general way to extend such functions to consider concept descriptions defined in acyclic $\mathcal{EL}$ TBoxes. In particular, these two aspects are combined to extend our logic $\tau\mathcal{EL}(\text{deg})$ towards $\tau\mathcal{EL}(\text{deg})$ TBoxes. Last, preliminary aspects related to reasoning in $\tau\mathcal{EL}(\text{deg})$ with respect to acyclic TBoxes are discussed as a starting point for subsequent sections.

The use of defined $\mathcal{EL}$ concepts to build threshold concepts compels us to revisit the definition of membership degree functions. In this new setting, the equivalence relation between concept descriptions is defined modulo the TBox definitions, i.e., $\equiv_{\mathcal{T}}$. Therefore, to maintain the equivalence invariance property, the condition $M2$ from Definition 3.1 should be redefined with respect to $\equiv_{\mathcal{T}}$. This means that the definition of $m^{\mathcal{T}}$ with respect to the set of definitions in a TBox $\mathcal{T}$ only makes sense for models $\mathcal{I}$ of $\mathcal{T}$.

Definition 6.4. A graded membership function $m$ is a family of functions that contains for every interpretation $\mathcal{I}$ a function $m^{\mathcal{T}} : \Delta^{\mathcal{T}} \times C_{\mathcal{EL}} \times \mathfrak{T}(\mathcal{I}) \to [0, 1]$ satisfying the following conditions (for $C, D \in C_{\mathcal{EL}}$ and $\mathcal{T} \in \mathfrak{T}(\mathcal{I})$):

$$M1^{\mathcal{T}} : d \in C^{\mathcal{T}} \iff m^{\mathcal{T}}(d, C, \mathcal{T}) = 1 \text{ for all } d \in \Delta^{\mathcal{T}},$$

$$M2^{\mathcal{T}} : C \equiv_{\mathcal{T}} D \iff \forall \mathcal{I} \models \mathcal{T} \forall d \in \Delta^{\mathcal{T}} : m^{\mathcal{T}}(d, C, \mathcal{T}) = m^{\mathcal{T}}(d, D, \mathcal{T}).$$

Note that this is a generalization of Definition 3.1 where $\mathcal{T}$ is the empty TBox. ♦

Now, our idea of a $\tau\mathcal{EL}(m)$ TBox is to combine definitions from $\mathcal{EL}$ TBoxes with concept definitions having the form of (6.1). The following example shows that such a combination should not be arbitrary, for otherwise not every defined concept will be a well-formed $\tau\mathcal{EL}(m)$ concept description.
Example 6.5. Let $T_{EL}$ be the following $EL$ TBox:

$$T_{EL} := \{ E \equiv \exists r.A \land \exists r.B \}$$

and $T_T$ the following set of concept definitions:

$$T_T := \{ \begin{array}{l} \alpha \equiv \exists s.A \land \exists r.\beta \\ \beta \equiv A \land E_{\leq 8} \end{array} \}$$

Here, the definition of $E$ corresponds to the $EL$ concept description $\exists r.A \land \exists r.B$. Moreover, $\alpha$ and $\beta$ define well-formed $\tau EL(m)$ concept descriptions. For example, the unfolding of $\alpha$ with respect to $T_T \cup T_{EL}$ yields the $\tau EL(m)$ concept description $\exists s.A \land \exists r.(A \land (\exists r.A \land \exists r.B)_{\leq 8})$. Suppose now, that $\alpha$ has the following definition in $T_T$:

$$\alpha \equiv \exists s.A \land \exists r.(\beta_{< 1})$$

Even though $\hat{C}_\alpha$ looks like a syntactically well-formed $\tau EL(m)$ concept description, the unfolding of $\alpha$ yields $\exists s.A \land \exists r.([\exists r.A \land \exists r.B)_{< 1}]$ and the problem arise immediately: $[A \land (\exists r.A \land \exists r.B)_{\leq 8}]_{< 1}$ is not a valid threshold concept. Consequently, $\alpha$ does not define a well-formed $\tau EL(m)$ concept description. Thus, we should require for all $E_{\sim t}$ occurring in definitions of $T_T$ that $E$ is not defined in terms of any other threshold concept.  

Definition 6.6. Let $\{N_{def}^r, N_{def}^0\}$ be a partition of $N_{def}$. A $\tau EL(m)$ TBox $\hat{T}$ is a pair $(T_T, T_{EL})$ satisfying the following conditions:

- $T_T$ is a set of concept definitions of the form $\alpha \equiv \hat{C}_\alpha$ such that:
  - $\alpha \in N_{def}^r$ and $\hat{C}_\alpha$ is a $\tau EL(m)$ concept description.
  - for all threshold concepts $E_{\sim t}$ occurring in a definition of $T_T$, $E$ is defined over $N_{def}^0 \cup N_{prim}$.

- $T_{EL}$ is an $EL$ TBox such that:
  - $E \in N_{def}^0$, for all defined concepts in $T_{EL}$.
  - for all $\alpha \in N_{def}^r$, $\alpha$ does not occur in any definition of $T_{EL}$.  

Restricting threshold concepts to be defined over $N_{def}^0 \cup N_{prim}$ and $T_{EL}$ not to contain occurrences of defined concepts in $T_T$ guarantees the $\alpha$ always defines a well-formed $\tau EL(m)$ concept description for all $\alpha \equiv \hat{C}_\alpha \in T_T$.

Remark 6.7. We have not been very precise about well-formed $\tau EL(m)$ concept descriptions externally defined over the signature of a $\tau EL(m)$ TBox $\hat{T}$. Hereafter, we understand by that any string of symbols $\hat{C}$ generated by the grammar in Chapter 3 such that $\hat{C}$ complies with the same restrictions imposed on the defined concepts in $\hat{T}$. More precisely, the set of definitions $\hat{T} \cup \{ \alpha = \hat{C} \}$ is still a $\tau EL(m)$ TBox, where $\alpha$ is a fresh concept name from $N_{def}^r$.

Consequently, we define a $\tau EL(m)$ knowledge base $K$ as a pair $K = (\hat{T}, A)$ where $\hat{T}$ is a $\tau EL(m)$ TBox, and for all $\hat{C}(a) \in A$ the concept description $\hat{C}$ is defined over
As we will introduce next, the satisfaction relation between interpretations and KBs allows to replace the assertions $\hat{C}(a)$ in $A$ with $\alpha_{\hat{C}}(a)$, by adding the concept definition $\alpha_{\hat{C}} \equiv \hat{C}$ to $T$, where $\alpha_{\hat{C}}$ is a fresh concept name from $\mathbb{N}_{\text{def}}$. Therefore, from now on we assume that all the concept assertions in $A$ are of the form $\alpha(a)$ where $\alpha \in \text{def}(\hat{T})$.

The satisfaction relation for $\tau\mathcal{EL}(m)$ TBoxes depends on the chosen function $m$. An interpretation $I$ satisfies a $\tau\mathcal{EL}(m)$ TBox $\hat{T} = (T_r, T_{\mathcal{EL}})$ iff $I \models T_{\mathcal{EL}}$ and $\alpha^{I} = (\hat{C}_{\alpha})^{I}$ in $\tau\mathcal{EL}(m)$ for all $\alpha \equiv \hat{C}_{\alpha} \in T_r$. Finally, $I$ satisfies a knowledge base $K = (\hat{T}, A)$ iff $I \models \hat{T}$ and $I \models A$.

We now turn to extending graded membership functions to deal with concept descriptions defined over acyclic $\mathcal{EL}$ TBoxes. Since $m$ is a parameter for the logic, besides the properties required in Definition 6.4, there is not much information about how defined concepts should be taken into account to compute membership degrees. Initially, one idea could be to treat defined concepts simply as concept names in the computation of $m$. This, however, would mean that the definition of a concept $E$ in $T_{\mathcal{EL}}$ is not really used to compute $m^{T}(d, E, T_{\mathcal{EL}})$ whenever $d \not\in E^{T}$, i.e., $m^{T}(d, E, T_{\mathcal{EL}}) = 0$, rather than giving a more approximate value based on the definition of $E$. The following example explains this situation for the membership function $deg$.

**Example 6.8.** Consider the following $\mathcal{EL}$ TBox $T_{\mathcal{EL}}$ and interpretation $I$:

\[
T_{\mathcal{EL}} := \left\{ \begin{array}{l}
E_1 \equiv \exists r.A \land \exists r.B \\
E_2 = A \land \exists s.E_3 \\
E_3 = B \land \exists r.E_2
\end{array} \right. \]

\[
G_{I}: \quad d_1 : \{A, E_2\} \quad d_2 : \{A\} \quad d_3 : \{B, E_3\}
\]

Here, $I \models T_{\mathcal{EL}}$ and $d_1 \not\in (E_1)^{I}$. Treating $E_1$ as a concept name in the computation of $deg$ yields $deg^{T}(d_1, E_1, T_{\mathcal{EL}}) = 0$. In other words, $deg$ would ignore the fact that $d_1$ has an $r$-successor which is an instance of $A$. In contrast, using directly the definition of $E_1$, we have $deg^{T}(d_1, \exists r.A \land \exists r.B, T_{\mathcal{EL}}) = 1/2$. This shows the limitations of treating defined concepts as concept names when computing $deg$, but more importantly it tells us that $deg$ would then violate property $M2^{T}$ since $E_1 \equiv T_{\mathcal{EL}} \exists r.A \land \exists r.B$.

One way to repair this is to consider the unfolding $u_{\mathcal{EL}}(E_1)$ of $E_1$ to compute $deg^{T}(d_1, E_1, T_{\mathcal{EL}})$, i.e., $deg^{T}(d_1, E_1, T_{\mathcal{EL}}) := deg^{T}(d_1, u_{\mathcal{EL}}(E_1))$. Obviously, this will not work for $E_2$ and $E_3$ since they are defined in a cyclic way, and this means that they cannot be unfolded.

Based on the previous arguments, we extend the computation of graded membership functions towards $\mathcal{EL}$ concept descriptions defined over acyclic $\mathcal{EL}$ TBoxes.

**Definition 6.9.** Let $T$ be an acyclic $\mathcal{EL}$ TBox, $C$ an $\mathcal{EL}$ concept description and $m$ a graded membership function. $m$ is extended to compute membership degree values with respect to $T$ as follows:

\[
m^{T}(d, C, T) := m^{T}(d, u_{T}(C))
\]

Being $m$ a graded membership function in the sense of Definition 6.1, it satisfies $M1$ and $M2$. Hence, since $C \equiv_{T} u_{T}(C)$, the definition of $m$ with respect to $T$ satisfies $M1^{T}$. 

\[\diamondsuit\]
Moreover, \( C \equiv_T \tau \) implies that \( u_\tau(C) \equiv u_\tau(D) \). From this it is easy to verify that \( m \) also satisfies \( M2^T \). Additionally, since the unfolding of an \( \mathcal{EL} \) concept description with respect to an acyclic TBox yields another \( \mathcal{EL} \) concept description, well-definedness of \( m \) implies its well-definedness with respect to \( T \).

So far, we have first restricted sets of definitions in order to avoid nesting of threshold concepts. Afterwards, we have extended the computation of membership degree functions to concept descriptions defined with respect to acyclic \( \mathcal{EL} \) TBoxes. Now, since \( \text{deg} \) is a well-defined graded membership function in the sense of Definition 6.9, its extension according to Definition 6.11 is also well-defined. Thus, we can now consider threshold concepts defined with respect to an background TBox in \( \tau \mathcal{EL}(\text{deg}) \). To emphasize that \( \text{deg} \) is defined for acyclic \( \mathcal{EL} \) TBoxes, we define a \( \tau \mathcal{EL}(\text{deg}) \) TBox as a \( \tau \mathcal{EL}(m) \) TBox \( \hat{T} = (\tau_T, \tau_{\mathcal{EL}}) \) where \( \tau_{\mathcal{EL}} \) is an acyclic \( \mathcal{EL} \) TBox.

Despite the acyclicity restriction on \( \tau_{\mathcal{EL}} \), \( \tau_T \) is allowed to have terminological cycles in the sense of Definition 6.1. This is considering the relation \( \rightarrow \) and its transitive closure \( \rightarrow^+ \) over defined concepts in \( \tau_T \). Consequently, we can talk about cyclic and acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBoxes. In particular, the notion of unfolding transfers naturally to acyclic ones. The constructors \( \sqcap \) and \( \exists r.C \) are treated in the same way and two new rules are added:

\[
\begin{align*}
\hat{u}_\tau(\alpha) &:= \hat{u}_\tau(\hat{C}_\alpha), \text{ for all } \alpha \vdash \hat{C}_\alpha \in \tau_T \\
\hat{u}_\tau(E \shortrightarrow t) &:= [u_{\tau_{\mathcal{EL}}}(E)] \shortrightarrow t
\end{align*}
\]

Since \( \tau_{\mathcal{EL}} \) is an acyclic TBox, this means that \( u_{\tau_{\mathcal{EL}}}(E) \) is an \( \mathcal{EL} \) concept description. Consequently, \( (u_{\tau_{\mathcal{EL}}}(E)) \shortrightarrow t \) is a well-formed threshold concept, and thus the unfolding \( \hat{u}_\tau(E \shortrightarrow t) \) is well-defined. Then, the counterparts of Propositions 6.2 and 6.3 also hold for acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBoxes.

**Proposition 6.10.** For every acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBox \( \hat{T} \), every \( \tau \mathcal{EL}(\text{deg}) \) concept description \( \hat{C} \) and every model \( I \) of \( \hat{T} \):

\[
\hat{C}^I = [\hat{u}_\tau(\hat{C})]^I
\]

**Proof.** The proof is the same as for \( \mathcal{EL} \), except that in addition, one has to consider the unfolding of threshold concepts \( E \shortrightarrow t \) where \( E \) is defined with respect to \( \tau_{\mathcal{EL}} \). This is not a problem, since \( E \equiv_{\tau_{\mathcal{EL}}} u_{\tau_{\mathcal{EL}}}(E) \) and therefore, property \( M2^T \) implies that \( E \shortrightarrow t \equiv_\tau [u_{\tau_{\mathcal{EL}}}(E)] \shortrightarrow t \).

**Proposition 6.11.** Let \( \hat{T} \) be an acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBox. Any primitive interpretation \( I \) can be uniquely extended to a model of \( \hat{T} \).

Proposition 6.10 tells us that the unfolding of concepts preserves equivalence, i.e., \( \hat{C} \equiv_\tau \hat{u}_\tau(\hat{C}) \). This together with Proposition 6.11 allow us to reduce reasoning with respect to acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBoxes to reasoning in the empty terminology, by using unfolding. However, there are two reasons why this could not yield worst-case optimal decision procedures for the different reasoning tasks. On the one hand, as shown in [Neb90], the unfolding of a concept description may result in a concept description of exponential size. This was actually shown for the description logic \( \mathcal{FL}_0 \). The following example shows the corresponding version for \( \mathcal{EL} \).
Example 6.12. For all $n \geq 0$, the $\mathcal{EL}$ TBox $T_n$ is inductively defined as follows:

$$
T_0 := \{ \alpha_0 \models \top \}
$$

$$
T_1 := T_0 \cup \{ \alpha_1 \models \exists r. \alpha_0 \land \exists s. \alpha_0 \}
$$

$$
\vdots
$$

$$
T_n := T_{n-1} \cup \{ \alpha_n \models \exists r. \alpha_{n-1} \land \exists s. \alpha_{n-1} \}
$$

Regarding the size of $T_n$ and $w_{T_n}(\alpha_n)$, we have $s(T_n) = \Theta(n)$ and $s(w_{T_n}(\alpha_n)) \geq 2^n$. ◦

On the other hand, concept satisfiability is NP-complete in $\tau\mathcal{EL}(\text{deg})$ with respect to the empty TBox. Therefore, given a $\tau\mathcal{EL}(\text{deg})$ acyclic TBox $\hat{T}$ and a $\tau\mathcal{EL}(\text{deg})$ concept description $\hat{C}$, unfolding $\hat{C}$ with respect to $\hat{T}$ and then using the NP decision procedure from Chapter 5, yields in general a non-deterministic exponential time algorithm for concept satisfiability with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes.

Two natural questions arise from the previous discussion: Can we do it better than in NEXP?, and more hopefully, could it still be decided in non-deterministic polynomial time? We give a positive and a negative answer, respectively, to these questions. In fact, we will see that the possibility of using acyclic TBoxes to express exponentially large concept descriptions in a succinct way combined with the use of threshold concepts, makes the concept satisfiability problem harder than all the problems in NP (unless NP=$\Pi^p_2$).

6.3 Models of non-polynomial size

We start by showing that, different from the empty TBox case, $\tau\mathcal{EL}(\text{deg})$ concept descriptions do not enjoy the polynomial model property when defined with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes. More precisely, for all $n \geq 0$ there is a TBox $\hat{T}_n$ and a defined concept $\alpha_n$ such that $\alpha_n$ is satisfiable with respect to $\hat{T}_n$, but not in models of size polynomial in $s(\hat{T}_n)$. There are two purposes in doing this. It automatically rules out the possibility of designing an algorithm that searches for a model of polynomial size, as for the case where $\hat{T} = \emptyset$. Further, the structure that an interpretation $I$ needs to have in order to satisfy $\alpha_n$ in $\hat{T}_n$ will be suitable to show that concept satisfiability is at least as hard as the problems contained in the class $\Pi^p_2$.

Consider the $\mathcal{EL}$ TBox $T_n$ from Example 6.12 and let $T_{\alpha_n}$ be the description tree corresponding to $w_{T_n}(\alpha_n)$. Additionally, let $I_{\alpha_n}$ be the primitive interpretation with description graph $T_{\alpha_n}$, and $d_0 \in \Delta^{\hat{T}_n}$ the element representing its root. $I_{\alpha_n}$ can be uniquely extended to a model of $T_n$ (Proposition 6.3), and this extension is such that $d_0 \in (\alpha_n)^{\hat{T}_n}$. Moreover, the following is an easy consequence from the definition of $T_n$.

**Proposition 6.13.** Let $I$ be a model of $T_n$ and $d \in \Delta^{\hat{T}}$. For all $0 \leq j \leq n$: if $d \in (\alpha_j)^{\hat{T}}$, then for each word $x \in \{r,s\}^j$ there exists a path $dx_1d_1 \ldots x_jd_j$ in $G_I$ such that $d_i \in (\alpha_{j-i})^{\hat{T}}$ for all $1 \leq i \leq j$.

The reason why $|\Delta^{\hat{T}_n}| \geq 2^n$ is that all pair of paths $\pi_1, \pi_2$ in $T_{\alpha_n}$ corresponding to two different words $x_{\pi_1}, x_{\pi_2} \in \{r,s\}^n$, are disjoint in their last nodes. This can obviously
be avoided, the interpretation $I_0$ (in the picture below) is a model of $T_n$ satisfying $\alpha_n$ and has size $O(1)$ with respect to the size of $T_n$:

$$G_{I_0} : \quad r, s$$

$$d_0 \leftarrow \alpha_0, \ldots, \alpha_n$$

In fact, in $\mathcal{EL}$ regardless which type of TBox is considered, every satisfiable concept description is satisfiable in an interpretation of polynomial size.

Our aim is to transform $T_n$ into a $\tau\mathcal{EL}(deg)$ TBox $\hat{T}_n$ such that each model $I$ of $\hat{T}_n$ satisfying $\alpha_n$ is of size at least $2^n$. To this end, we use $2n$ auxiliary primitive concept names $A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n$. The intention is to enforce for each word $x \in \{r, s\}^n$ the existence of at least one path $d_0x_1d_1 \ldots x_nd_n$ in $G_I$ such that:

$$d_n \in (A_i)^I \iff x_i = r \quad (1 \leq i \leq n) \quad (6.2)$$

Note that for two different words $x, y \in \{r, s\}^n$ and the corresponding paths $\pi_x = d_0x_1 \ldots x_nd_nx, \pi_y = d_0y_1 \ldots y_nd_ny$ in $G_I$ satisfying the equivalence in (6.2), there must exist $i$ such that $x_i \neq y_i$, and this would imply:

$$d_{nx} \in (A_i)^I \iff d_{ny} \notin (A_i)^I$$

Hence, $d_{nx}$ and $d_{ny}$ must be two different domain elements in $\Delta^I$. This argument extends to all pair of words in $\{r, s\}^n$. Thus, since there are $2^n$ words in $\{r, s\}^n$, in this way $\Delta^I$ would need to have at least $2^n$ elements.

So far, the structure of $T_n$ already guarantees the existence of a path from $d_0$ for all $x \in \{r, s\}^n$, but not the satisfaction of (6.2). One needs to be able to express within the logic the correct propagation of the concept names along each path. For example, for $n = 3$ and $x_1$, one possible way to do it is redefining $\alpha_3$ as:

$$\alpha_3 \equiv \exists r. (\alpha_2 \sqcap \bigwedge_{x_2, x_3 \in \{r, s\}} \forall x_2x_3.A_1) \sqcap \exists s. (\alpha_2 \sqcap \bigwedge_{x_2, x_3 \in \{r, s\}} \forall x_2x_3.\neg A_1) \quad (6.3)$$

Unfortunately, as shown in Section 4.3.1 the simple concept $\forall r. A$ cannot be expressed in $\tau\mathcal{EL}(deg)$. Moreover, in general this idea would require the use of exponentially many $\forall$-restrictions. Nevertheless, $\forall r. \neg A$ can actually be expressed, and this is where the concept names $\bar{A}_i$ come into play. Their role is to be complementary with $A_i$ at $d_n$.

Our first step is to assert a weaker version of the equivalence in (6.2) using $A_i$ and $\bar{A}_i$. For each $1 \leq j \leq n$, we define two TBoxes $T^j$ and $\bar{T}^j$ as follows. We select $T^j$ and $\bar{T}^j$ as two copies of $T_{j-1}$ (from Example 6.12), where each defined concept $\alpha_i$ ($0 \leq i \leq j - 1$) is renamed as $E_i^j$ and $E_i^j$, in $T^j$ and $\bar{T}^j$, respectively. Then, $E_0^j$ and $E_0^j$ are redefined as $E_0^j \equiv \bar{A}_{n-j+1}$ and $E_0^j \equiv A_{n-j+1}$. The union of all these TBoxes is denoted as $T_{n,paths}$:

$$T_{n,paths} := \bigcup_{j=1}^n (T^j \cup \bar{T}^j)$$
Let us illustrate this construction and explain why it will be useful.

**Example 6.14.** Let $n = 3$. Starting from $T_2, T_1$ and $T_0$, the $\mathcal{EL}$ TBoxes $T^3, T^2$ and $T^1$ consist of the following set of definitions, respectively:

- $E^3_2 \doteq \exists r.E^3_1 \land \exists s.E^3_1$
- $E^3_1 \doteq \exists r.E^2_1 \land \exists s.E^2_0$
- $E^3_0 \doteq \tilde{A}_3$
- $E^1_2 \doteq \exists r.E^2_0 \land \exists s.E^2_0$
- $E^3_1 \doteq \exists r.E^1_0 \land \exists s.E^0_0$
- $E^1_0 \doteq \tilde{A}_2$
- $E^1_1 \doteq \tilde{A}_1$

The TBox $T^3$ corresponds to the case where $j = 3$. Note that $n - j + 1 = 1$ matches the index of $\tilde{A}_j$ in the definition of $E^3_0$. The same applies for $E^3_2$ and $E^3_1$, since as $j$ decreases the index $i$ increases accordingly. TBoxes $T^3, T^2$ and $T^1$ have the same structure except that $A_1$, $A_2$ and $A_3$ are used instead.

Let now $T_{E^3_2}$ and $T_{E^3_1}$ be the $\mathcal{EL}$ description trees corresponding to the unfolding of $E^3_2$ and $E^3_1$ in $T^3$ and $T^1$, respectively:

\[
T_{E^3_2}:
\begin{array}{c}
\begin{array}{c}
\bar{A}_1 \\
\bar{A}_1
\end{array} \\
\begin{array}{c}
\bar{A}_1 \\
\bar{A}_1
\end{array}
\end{array}
\]

\[
T_{E^3_1}:
\begin{array}{c}
\begin{array}{c}
\bar{A}_1 \\
\bar{A}_1
\end{array} \\
\begin{array}{c}
\bar{A}_1 \\
\bar{A}_1
\end{array}
\end{array}
\]

We exploit the structure of these trees in two directions. First, both of them provide a succinct representation of all the possible paths corresponding to words in $\{r, s\}^2$. Second, for an interpretation $I$ the threshold concept $(E^3_2)_{\leq 0}$ tells the following about any $d \in \Delta^I$:

- if $dx_2d_2x_3d_3$ is a path in $G_I$ where $\{x_2, x_3\} \subseteq \{r, s\}$ and $d_3 \in (\bar{A}_1)^I$, then there is an equal path $vx_2v_2x_3v_3$ in $T_{E^3_2}$. Now, the partial mapping $h$ from $T_{E^3_2}$ to $G_I$ with $h(v) = d$ and $h(v_i) = d_i$ ($i = 2, 3$) satisfies $h_w(v) > 0$. Therefore, $d \not\in [(E^3_2)_{\leq 0}]^I$.

- Conversely, if no such path exists for $d$, then for all paths $vx_2v_2x_3v_3$ in $T_{E^3_1}$ and all partial mappings $h$ from $T_{E^3_1}$ to $G_I$ such that $v_3 \in \text{dom}(h)$, it is the case that $h(v_3) \not\in (\bar{A}_1)^I$. Therefore, by definition of $h_w$, it must be the case that $h_w(v) = 0$. Consequently, $deg^I(d, E^3_1, T^4) = 0$ and $d \in [(E^3_1)_{\leq 0}]^I$.

The same reasoning applies for $E^3_2$ and $A_1$. The good that comes from this is that we obtain the following equivalences:

\[(E^3_2)_{\leq 0} \equiv_{T^3} \forall x_2x_3.\neg \bar{A}_1 \quad \text{and} \quad (E^3_1)_{\leq 0} \equiv_{T^3} \forall x_2x_3.\neg A_1\]

Since $\neg \bar{A}_1$ is meant to represent $A_1$, as in (6.3) it would be possible to propagate correctly $A_1$ according to the value of $x_1$ in all paths. \(\diamond\)
Based on this, one can in general use the threshold concepts \((E^j_{j-1})^{\leq 0}\) and \((E^j_{j-1})^{< 0}\) to represent the generalization of the value restrictions used in [6.3] to arbitrary lengths.

**Proposition 6.15.** For all models \(I\) of \(T_{\text{npaths}}\), \(d \in \Delta^I\) and \(1 \leq j \leq n:\)

1. \(d \in [((E^j_{j-1})^{\leq 0})^I]\) iff \(d \in \left( \bigcap_{x \in \{r,s\}^{j-1}} \forall x. \neg \bar{A}_{n-j+1} \right)^I.\)

2. \(d \in [(E^j_{j-1})^{< 0})^I]\) iff \(d \in \left( \bigcap_{x \in \{r,s\}^{j-1}} \forall x. \neg \bar{A}_{n-j+1} \right)^I.\)

**Proof.** We only give the proof for the first statement (the second one can be shown using the same argument). We denote as \(T^j_{E^j_{j-1}}\) the description tree corresponding to the unfolding \(u_{T^j}(E^j_{j-1})\) of \(E^j_{j-1}\) in \(T^j\). For simplicity, we use just \(\ell\) (without subscript) to refer to the labeling of \(T^j_{E^j_{j-1}}\).

\((\Rightarrow)\) Assume that \(d \in [((E^j_{j-1})^{\leq 0})^I]\). Since \(E^j_{j-1}\) is a defined concept in \(T^j\), this implies:
\[
\deg^T(d, E^j_{j-1}, T^j) = \deg^T(d, u_{T^j}(E^j_{j-1})) = 0
\]

For a contradiction, suppose that:
\[
d \notin \left( \bigcap_{x \in \{r,s\}^{j-1}} \forall x. \neg \bar{A}_{n-j+1} \right)^I.
\]

Then, there is a word \(x_1 \ldots x_{j-1} \in \{r, s\}^{j-1}\) such that \(d \notin (\forall x_1 \ldots x_{j-1}. \neg \bar{A}_{n-j+1})^I\). The semantics of the value restriction constructor yields the existence of a path of the form \(dx_1d_1 \ldots x_{j-1}d_{j-1}\) in \(G^I\) such that \(d_{j-1} \in (\bar{A}_{n-j+1})^I.\)

By definition of \(E^j_{j-1}\) in \(T^j\), there is a path \(v_0x_1v_1 \ldots x_{j-1}v_{j-1}\) in \(T^j_{E^j_{j-1}}\) with \(\ell(v_{j-1}) = \{\bar{A}_{n-j+1}\}\), where \(v_0\) is the root of \(T^j_{E^j_{j-1}}\). Therefore, the \(ptgh\) \(h\) from \(T^j_{E^j_{j-1}}\) to \(G^I\) with \(h(v_0) = d\) and \(h(v_i) = d_i\) (\(1 \leq i \leq j-1\)) induces a weighted homomorphism \(h^w\) such that: \(h^w(v_0) > 0\). This contradicts our initial assumption since it implies:
\[
\deg^T(d, u_{T^j}(E^j_{j-1})) > 0
\]

Thus, the left to right implication holds.

\((\Leftarrow)\) Assume that:
\[
d \in \left( \bigcap_{x \in \{r,s\}^{j-1}} \forall x. \neg \bar{A}_{n-j+1} \right)^I.
\]

This implies that \(d \in (\forall x_1 \ldots x_{j-1}. \neg \bar{A}_{n-j+1})^I\) for all words \(x_1, \ldots, x_{j-1} \in \{r, s\}^{j-1}\). Hence, any path of the form \(dx_1d_1 \ldots x_{j-1}d_{j-1}\) in \(G^I\) is restricted to have:
\[
d_{j-1} \notin (\bar{A}_{n-j+1})^I
\]

Let now \(v_0x_1v_1 \ldots x_{j-1}v_{j-1}\) be any path in \(T^j_{E^j_{j-1}}\). By definition of \(E^j_{j-1}\) in \(T^j\) we know that \(x_1 \ldots x_{j-1} \in \{r, s\}^{j-1}\) and \(\ell(v_{j-1}) = \{\bar{A}_{n-j+1}\}\). Therefore, for all \(ptgh\) \(h\) from \(T^j_{E^j_{j-1}}\)
to $G_I$ having $h(v_0) = d$ and $v_{j-1} \in \text{dom}(h)$, it is the case that $A_{n-j+1} \not\in \ell_I(h(v_{j-1}))$. Hence, since $v_{j-1}$ is a leaf in $T_{E_j^{j-1}}$, this means that $h_w(v_{j-1}) = 0$.

Overall, we have shown that for all leaves $v$ in $T_{E_j^{j-1}}$ and all $ptgh$ with $v \in \text{dom}(h)$, it holds that $h_w(v) = 0$. Then, since $\ell(v) = \emptyset$ if $v$ is a non-leaf node, there is no possible way in which $h_w(v_0) > 0$. Consequently, it follows:

$$\text{deg}^I(d, u_I(E_j^{j-1})) = 0$$

Thus, $d \in [(E_j^{j-1})_{\leq 0}]^I$. \hfill $\square$

Having these equivalences, the next step is to generalize the intuition expressed by the combination of \([6.3]\) and Example \([6.14]\). More precisely, we integrate the threshold concepts of the form $(E_j^{j-1})_{\leq 0}$ and $(E_j^{j-1})_{\leq 0}$ into $T_n$ as follows. For all $1 \leq j \leq n$:

$$\alpha_j \doteq \exists r.(\alpha_{j-1} \cap (E_j^{j-1})_{\leq 0}) \cap \exists s.(\alpha_{j-1} \cap (E_j^{j-1})_{\leq 0})$$

We name the resulting TBox as $T_{n,\tau}$. Note that $T_{n,paths}$ is acyclic and $(T_{n,\tau}, T_{n,paths})$ satisfies the conditions required in Definition \([6.6]\). Therefore, $(T_{n,\tau}, T_{n,paths})$ is a $\tau\mathcal{EL}(\text{deg})$ TBox. We can now state for $(T_{n,\tau}, T_{n,paths})$ the equivalent of Proposition \([6.13]\).

**Proposition 6.16.** Let $I$ be a model of $(T_{n,\tau}, T_{n,paths})$ and $d \in \Delta^I$. For all $0 \leq j \leq n$: if $d \in (\alpha_j)^I$, then for each word $x \in \{r,s\}^I$ there exists a path $dx_1d_1...x_jd_j$ in $G_I$ such that for all $1 \leq i \leq j$,

- $d_i \in (\alpha_{j-i})^I$,
- $d_i \in [(E_{j-i}^{j-i+1})_{\leq 0}]^I$ if $x_i = r$, otherwise $d_i \in [(E_{j-i}^{j-i+1})_{\leq 0}]^I$.

We continue with the previous example to see how $T_{3,\tau}$ looks like, and explain what is still missing to achieve our goal.

**Continuation of Example 6.14.** After integrating the new threshold concepts into $T_3$, the $\tau\mathcal{EL}(\text{deg})$ TBox $T_{3,\tau}$ consists of the following set of definitions:

- $\alpha_3 \doteq \exists r.(\alpha_2 \cap (E_2^3)_{\leq 0}) \cap \exists s.(\alpha_2 \cap (E_2^3)_{\leq 0})$
- $\alpha_2 \doteq \exists r.(\alpha_1 \cap (E_1^2)_{\leq 0}) \cap \exists s.(\alpha_1 \cap (E_1^2)_{\leq 0})$
- $\alpha_1 \doteq \exists r.(\alpha_0 \cap (E_0^1)_{\leq 0}) \cap \exists s.(\alpha_0 \cap (E_0^1)_{\leq 0})$
- $\alpha_0 \doteq \top$

Let $I$ be an interpretation such that $I \models (T_{3,\tau}, T_{3,paths})$, $d_0 \in \Delta^I$ and $d_0 \in (\alpha_3)^I$. The addition of the threshold concepts gives us the following. For all words $x \in \{r,s\}^3$ there is at least one path $d_0x_1d_1x_2d_2x_3d_3$ such that:

$$x_1 = r \Rightarrow d_3 \not\in (A_1)^I$$
$$x_1 = s \Rightarrow d_3 \not\in (A_1)^I$$

(6.4)
For instance, the words rrr and srr yield two paths \(d_0rd_1rd_2rd_3\) and \(d_0se_1re_2re_3\) in \(G_I\), where \(d_3 \in (\neg A_1)^I\) and \(e_3 \in (\neg A_1)^I\). However, since no relationship has yet been established between \(\bar{A}_1\) and \(A_1\), there is no inconsistency between \(\neg A_1\) and \(\neg A_1\). Hence, \(d_3\) and \(e_3\) can be merged into one while keeping \(d_0 \in (\alpha_3)^I\). Overall only non-membership is required at the end of each path. Consequently, \(I_0\) is still good enough to be a model of \((T_{3,\tau}, T_{3,\text{paths}})\) satisfying \(\alpha_3\).

In general, if \(d_0 \in (\alpha_n)^I\), this construction tells us that for each word \(x \in \{r, s\}^n\) there exists at least one path \(d_0x_1d_1 \ldots x_nd_n\) in \(G_I\) such that:
\[
d_n \in (A_i)^I \Rightarrow x_i = r \tag{the contraposition of \[6.4\]}
\]
To have this implication also valid in the opposite direction, we use again threshold concepts to make \(A_i\) and \(\bar{A}_i\) complementary at \(d_n\), i.e., \(d_n \in (A_i)^I\) iff \(d_n \notin (\bar{A}_i)^I\). To each pair \((A_i, \bar{A}_i)\) we associate the concept definition \(F_i \equiv A_i \cap \bar{A}_i\). The TBox \(T_{n,\text{comp}}\) is defined as:
\[
T_{n,\text{comp}} := \bigcup_{i=1}^{n} \{F_i \equiv A_i \cap \bar{A}_i\}
\]
Using this, \(\alpha_0\) is redefined in \(T_{n,\tau}\) as:
\[
\alpha_0 \equiv n \prod_{i=1}^{n} \left[\left(F_i \leq \frac{1}{2}\right) \cap \left(F_i \geq \frac{1}{2}\right)\right]
\]

**Remark 6.17.** Defining \(\bar{A}_i \equiv (A_i)_{<1}\) \((1 \leq i \leq n)\) in \(T_{n,\tau}\), makes \(d \in (A_i)^I\) iff \(d \notin (\bar{A}_i)^I\) not only for \(d = d_n\), but for the whole interpretation domain. This is simpler than the definition of \(\alpha_0\). However, it would make \((E_j^{\downarrow})_{\leq 0}\) \((with \(j = n - i + 1\)) not a well-formed threshold concept since \(\bar{A}_i\) is used to define \(E_j^{\downarrow}\).

Finally, putting all these parts together, we end up with the \(\tau\mathcal{EL}(\text{deg})\) TBox \(\hat{T}_n := (T_{n,\tau}, T_{n,\mathcal{EL}})\) where:
\[
T_{n,\mathcal{EL}} := T_{n,\text{comp}} \cup T_{n,\text{paths}}
\]
We now proceed to show that satisfying \(\alpha_n\) in \(\hat{T}_n\) requires interpretations of size exponential in \(n\).

**Lemma 6.18.** For all \(n \geq 0\) and all interpretations \(I\) such that \(I \models \hat{T}_n\) and \((\alpha_n)^I \neq \emptyset\), we have \(\Delta^I \geq 2^n\).

**Proof.** Let \(I\) be an interpretation such that \(I \models \hat{T}_n\) and \((\alpha_n)^I \neq \emptyset\).

The case \(n = 0\) is trivial since \(\Delta^I\) is a non-empty domain. To see that the statement of the lemma is also true for an arbitrary \(n > 0\), we show that for all subsets \(X\) of \(\{A_1, \ldots, A_n\}\) there exists an element \(d_X \in \Delta^I\) such that:
\[
d_X \in (A_i)^I \text{ iff } A_i \in X \quad (1 \leq i \leq n) \tag{6.5}
\]
Let \(d \in \Delta^I\) be such that \(d \in (\alpha_n)^I\). In addition, let us fix a set \(Y \subseteq \{A_1, \ldots, A_n\}\) and
define its corresponding word \( y \in \{r, s\}^n \) as:

\[
y_i = r \text{ iff } A_i \in Y \quad (1 \leq i \leq n)
\]

By applying Proposition 6.16 to \( d \), we know that there is a path \( dy_1d_1 \ldots y_n d_n \) in \( G_{\overline{I}} \) such that for all \( 1 \leq i \leq n \):

- \( d_i \in (\alpha_{n-i})^T \),
- \( d_i \in [(E_{n-i-1}^{n-i+1})_{i \leq 0}]^T \) if \( y_i = r \), otherwise \( d_i \in [(E_{n-i-1}^{n-i+1})_{i > 0}]^T \)

In particular, the suffix \( d_i y_{i+1} \ldots y_{n} d_{n} \) is of length \( n - i \). Therefore, we further have:

\[
\begin{align*}
y_i = r &\implies d_i \in [(E_{n-i-1}^{n-i+1})_{i \leq 0}]^T \\
&\implies d_i \in (\forall y_{i+1} \ldots y_{n}, \neg \hat{A}_i)^T \quad \text{(Proposition 6.15 applied to } d_i \text{ and } E_{n-i-1}^{n-i+1}) \\
&\implies d_n \notin (\hat{A}_i)^T
\end{align*}
\]

Symmetrically, \( y_i = s \) implies \( d_n \notin (A_i)^T \). Now, we know that \( F_i = A_i \cap \hat{A}_i \in T_{n, \text{comp}} \) and \( \alpha_0 \) is of the form:

\[
\alpha_0 = \bigcap_{i=1}^{n} [(F_i)_{\leq \frac{1}{2}} \cap (F_i)_{\geq \frac{1}{2}}]
\]

Since \( d_n \in (\alpha_0)^T \) it follows:

\[
d_n \in (A_i)^T \text{ iff } y_i = r \quad (1 \leq i \leq n)
\]

From the way the word \( y \) is defined in (6.6), we can conclude that \( d_n \) is an element of \( \Delta^T \) satisfying (6.5) with respect to \( Y \).

Thus, one can easily see why \( \Delta^T \geq 2^n \).

To finally fulfill the initial aim of this section, it remains to show that \( \alpha_n \) is indeed satisfiable with respect to \( \hat{T}_n \).

**Lemma 6.19.** \( \alpha_n \) is satisfiable with respect to \( \hat{T}_n \).

**Proof.** We take the interpretation \( T_{\alpha_n} \) and extend it into a model \( \hat{T}_{\alpha_n} \) of \( \hat{T}_n \) satisfying \( \alpha_n \). By construction, \( T_{\alpha_n} \) is tree-shaped and has \( 2^n \) leaves. Moreover, there is a one-to-one correspondence between words in \( \{r, s\}^n \) and the leaves in \( T_{\alpha_n} \). The leaf \( d_x \) corresponding to the word \( x \) is the one reached from \( d_0 \) through the path \( d_0x_1d_1 \ldots x_n d_n \).

Let \( L_{\alpha_n} \) denote the set of leaves of \( T_{\alpha_n} \). The interpretation of \( A_i, \hat{A}_i \) under \( \hat{T}_{\alpha_n} \) is defined as follows. For all \( 1 \leq i \leq n \):

\[
\begin{align*}
(A_i)^{\hat{T}_{\alpha_n}} &:= \{ d_x \mid d_x \in L_{\alpha_n} \text{ and } x_i = r \} \\
(\hat{A}_i)^{\hat{T}_{\alpha_n}} &:= \{ d_x \mid d_x \in L_{\alpha_n} \text{ and } x_i = s \}
\end{align*}
\]

Hence, for all leaves \( d \) of \( T_{\alpha_n} \) and all \( i \in \{1 \ldots n\} \) we have:

\[
d \in (A_i)^{\hat{T}_{\alpha_n}} \text{ iff } d \notin (\hat{A}_i)^{\hat{T}_{\alpha_n}}
\]

(6.8)
Since \( T_{\eta,\tau} \) and \( T_{\eta,\mathcal{L}} \) are both acyclic, there is a unique way to extend \( \hat{T}_{\alpha_n} \) into a model of \( \hat{T}_n \). Having done so, let \( \eta(d) \) denote the height of a domain element \( d \) in \( T_{\alpha_n} \). We show by induction on \( \eta(d) \) the following claim:

\[
\text{for all } d \in \Delta \hat{T}_{\alpha_n} : d \in (\alpha_{\eta(d)}) \hat{T}_{\alpha_n}
\]

**Induction Base.** \( d \in \Delta \hat{T}_{\alpha_n} \) and \( \eta(d) = 0 \). Then, \( d \) is a leaf in \( T_{\alpha_n} \). Recall that \( \alpha_0 \) is defined in \( T_{\eta,\tau} \) as:

\[
\alpha_0 \overset{n}{=} \prod_{i=1}^{n} [(F_i)_{\leq \frac{1}{2}} \cap (F_i)_{\geq \frac{1}{2}}]
\]

Consequently, since \( F_i \) is defined in \( T_{\eta,\mathcal{L}} \) as \( F_i \overset{\eta}{=} A_i \cap \bar{A}_i \), using (6.8) we obtain \( d \in [(F_i)_{\leq \frac{1}{2}} \cap (F_i)_{\geq \frac{1}{2}}] \hat{T}_{\alpha_n} \). Thus, \( d \in (\alpha_0) \hat{T}_{\alpha_n} \) holds.

**Induction Step.** Let \( d \in \Delta \hat{T}_{\alpha_n} \) with \( 0 < \eta(d) \leq n \). We assume our claim holds for all \( e \in \Delta \hat{T}_{\alpha_n} \) with \( \eta(e) < \eta(d) \).

To start, \( \alpha_{\eta(d)} \) is defined in \( T_{\eta,\tau} \) as:

\[
\alpha_{\eta(d)} \overset{n}{=} \exists r.(\alpha_{\eta(d)-1} \cap (E^{\eta(d)}_{\eta(d)-1})_{\leq 0}) \cap \exists s.(\alpha_{\eta(d)-1} \cap (E^{\eta(d)}_{\eta(d)-1})_{\leq 0})
\]

By application of induction hypothesis to \( e \) yields \( e \in (\alpha_{\eta(d)-1}) \hat{T}_{\alpha_n} \).

Consider now any word \( y \in \{r,s\}^{\eta(d)-1} \). Since \( \eta(e) = \eta(d) - 1 \), by definition of \( T_{\alpha_n} \) there is a unique path of the form \( e y_1 e_1 \ldots y_{\eta(d)-1} e_{\eta(d)-1} \in T_{\alpha_n} \), where \( e_{\eta(d)-1} \) is a leaf. Moreover, such a path is suffix of a path \( d_0 x_1 d_1 \ldots j \ldots x_j d_n \), where \( d_j = d \), \( x_j = r \) and \( e_{\eta(d)-1} = d_n \). Then, we obtain the following equalities:

\[
n - (j + 1) = (\eta(d) - 1) - 1
\]

\[
n - \eta(d) + 1 = j
\]

Since \( x_j = r \), by (6.7) we obtain that \( d_n \in (A_{n-\eta(d)+1}) \hat{T}_{\alpha_n} \), and by (6.8) \( d_n \not\in (\bar{A}_{n-\eta(d)+1}) \hat{T}_{\alpha_n} \). Hence, as \( y \) was chosen arbitrarily from \( \{r,s\}^{\eta(d)-1} \), we have just shown that:

\[
e \in \left( \prod_{y \in \{r,s\}^{\eta(d)-1}} \forall y. \lnot \bar{A}_{n-\eta(d)+1} \right) \hat{T}_{\alpha_n}
\]

The application of Proposition 6.15 then yields \( e \in [(E^{\eta(d)}_{\eta(d)-1})_{\leq 0}] \hat{T}_{\alpha_n} \). For the \( \exists s \) restriction in the definition of \( \alpha_{\eta(d)} \), the corresponding result can be shown in the same way. Therefore, \( d \in (\alpha_{\eta(d)}) \hat{T}_{\alpha_n} \).

Using this result and the fact that \( d_0 \) is of height \( n \) in \( T_{\alpha_n} \), we can conclude that \( d_0 \in (\alpha_n) \hat{T}_{\alpha_n} \). Thus, \( \alpha_n \) is satisfiable with respect to \( \hat{T}_n \). □

Finally, let us look at the size of \( \hat{T}_n \). \( T_{\eta,\tau} \) and \( T_{\eta,\mathcal{L}} \) are both of size \( \mathcal{O}(n) \). Let us
6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes

recall the definition of $T_{n,\text{paths}}$:

$$T_{n,\text{paths}} = \bigcup_{j=1}^{n} (T^j \cup \bar{T}^j)$$

Then, it can be equivalently expressed as follows:

$$T_{n,\text{paths}} = T_{n-1,\text{paths}} \cup T^n \cup \bar{T}^n$$

Since $T^n$ and $\bar{T}^n$ have the same size as the $\mathcal{EL}$ TBox $T_{n-1}$, and $s(T_n) = O(n)$ for all $n \geq 1$, we obtain:

$$s(T_{n,\text{paths}}) = s(T_{n-1,\text{paths}}) + 2 * O(n - 1)$$

where $s(T_{1,\text{paths}}) = c \geq 1$ and $c \in \mathbb{N}$ is a constant.

Hence, $s(\hat{T}_n) = O(n^2)$ for all $n \geq 1$. Now let $I$ be a model of $\hat{T}_n$ satisfying $\alpha_n$. By Lemma 6.18 we know that $|\Delta^I| \geq 2^n$, and consequently $|\Delta^I| \geq 2^{O(\sqrt{s(\hat{T}_n)})}$. Therefore, the size of $I$ is not polynomial in the size of $\hat{T}_n$.

**Theorem 6.20.** For all $n \geq 0$ there exists a $\tau\mathcal{EL}(\text{deg})$ acyclic TBox $\hat{T}_n$ with a defined concept $\alpha_n$, such that $s(\hat{T}_n)$ is polynomial in $n$, but all models satisfying $\alpha_n$ are of size at least exponential in $n$.

### 6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes

The result obtained in Theorem 6.20 does not imply that there is no NP decision procedure for concept satisfiability. Even when, in general, models of polynomial size satisfying a concept description do not exist, it may well be the case that such very large models have abstract representations of polynomial size which can be used to design an NP procedure, or simply there is a different way to do it. Unfortunately, this seems to be very unlikely. We show that concept satisfiability and subsumption are $\Pi^P_2$-hard and $\Sigma^P_2$-hard, respectively, with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes. Additionally, we provide a PSPACE algorithm that is sound and complete for both problems. Finally, the algorithm will be extended to reasoning with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ KBs without giving up its polynomial space property.

### 6.4.1 Lower bounds

We reduce the problem $\forall \exists 3\text{SAT}$ to concept satisfiability with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes. This problem is well-known to be complete for the class $\Pi^P_2$ (see [Sto76], Section 4).

**Definition 6.21 ($\forall \exists 3\text{SAT}$).** Let $u = \{u_1, \ldots, u_n\}$ and $v = \{v_1, \ldots, v_m\}$ be two disjoint sets of propositional variables. Additionally, let $\varphi(u, v)$ be a formula in 3CNF defined over $u \cup v$, i.e., $\varphi(u, v)$ is a finite set of propositional clauses $C = \{c_1, \ldots, c_k\}$ such that:

- Each clause $c_i$ is a set of three literals $\{\ell_{i1}, \ell_{i2}, \ell_{i3}\}$ over $u \cup v$. 

A formula \((\forall u)(\exists v)\varphi(u, v)\) is satisfiable iff for all truth assignments \(t\) the variables in \(u\) there is an extension of \(t\) for the variables in \(v\) such that it satisfies \(\varphi(u, v)\). \(\forall 3\text{SAT}\) is then the problem of deciding whether given a 3CNF formula \(\varphi(u, v)\), the formula \((\forall u)(\exists v)\varphi(u, v)\) is satisfiable or not.

The idea for the reduction goes as follows. Each 3CNF formula \(\varphi(u, v)\) is translated into a \(\tau\mathcal{EL}(\deg)\) TBox \(\hat{T}_n^\varphi\) containing a defined concept \(\alpha_n\) such that: \((\forall u)(\exists v)\varphi(u, v)\) is satisfiable iff \(\alpha_n\) is satisfiable with respect to \(\hat{T}_n^\varphi\) (here, the value \(n\) corresponds to the number of universally quantified variables). We have seen in the proof of Lemma 6.18 that satisfiability of \(\alpha_n\) with respect to \(\hat{T}_n\) requires interpretations \(I\) containing for all subsets \(X\) of \(\{A_1, \ldots, A_n\}\) an element \(d_X \in \Delta^2\) such that: \(d_X \in (A_i)^2\) iff \(A_i \in X\).

We take advantage of this to encode the universal quantification \((\forall u)\). The existential quantification can be simulated by the very nature of the concept satisfiability problem. We will obtain \(\hat{T}_n^\varphi\) from \(\hat{T}_n\), by modifying the definition of \(\alpha_0\) in \(\hat{T}_{n,\tau}\), and adding new definitions to \(\hat{T}_{n,\text{comp}}\). In the following we provide the details of the translation and prove its correctness.

For each variable \(u_i \in u\), the literals \(u_i\) and \(\neg u_i\) are identified with the concept names \(A_i\) and \(\bar{A}_i\), respectively. Similarly, for literals over \(v\) we introduce new primitive concept names \(B_1, \bar{B}_1, \ldots, B_m, \bar{B}_m\). More formally, to each literal \(\ell\) over \(u \cup v\), the mapping \(\gamma\) assigns a primitive concept name as follows:

\[
\gamma(\ell) = \begin{cases} 
A_i & \text{if } \ell = u_i, \\
\bar{A}_i & \text{if } \ell = \neg u_i, \\
B_j & \text{if } \ell = v_j, \\
\bar{B}_j & \text{if } \ell = \neg v_j.
\end{cases}
\]

To encode \(\varphi(u, v)\), each clause \(c_i = \{\ell_{i1}, \ell_{i2}, \ell_{i3}\}\) in \(C\) is represented by the \(\mathcal{EL}\) concept description \(D_i := \gamma(\ell_{i1}) \cap \gamma(\ell_{i2}) \cap \gamma(\ell_{i3})\). Then, we define the \(\tau\mathcal{EL}(\deg)\) concept description \(\hat{C}_\varphi\) corresponding to \(\varphi(u, v)\) as:

\[
\hat{C}_\varphi := \bigcap_{i=1}^k (D_i)_{\geq \frac{1}{3}}
\]

The idea is that an individual \(d_X\) belongs to \((D_i)_{\geq \frac{1}{3}}\) iff it belongs to at least one concept name \(\gamma(\ell_{il})\) \((1 \leq l \leq 3)\). To constrain \(B_j\) and \(\bar{B}_j\) to be complementary at \(d_X\), for all \(1 \leq j \leq m\) we add to \(\hat{T}_{n,\text{comp}}\) the concept definition \(G_j = B_j \cap \bar{B}_j\). The last step is to adjust the definition of \(\alpha_0\) in \(\hat{T}_{n,\tau}\) to take into account the formula \(\varphi(u, v)\) and the newly introduced concept names corresponding to the variables in \(v\):

\[
\alpha_0 := \hat{C}_\varphi \cap \bigcap_{i=1}^m [(F_i)_{\leq \frac{1}{2}} \cap (F_i)_{\geq \frac{1}{2}}] \cap \bigcap_{j=1}^m [(G_j)_{\leq \frac{1}{2}} \cap (G_j)_{\geq \frac{1}{2}}]
\] (6.9)

To avoid confusions we denote by \(\hat{T}_{n,\tau}^\varphi\) and \(\hat{T}_{n,\text{comp}}^\varphi\) the modified TBoxes, and by \(\hat{\alpha}_0\) the altered \(\alpha_0\). Then, \(\hat{T}_n^\varphi := (\hat{T}_{n,\tau}^\varphi, \hat{T}_{n,\text{comp}}^\varphi \cup \hat{T}_{n,\text{paths}})\).
Lemma 6.22. Let \( u = \{u_1, \ldots, u_n\}, v = \{v_1, \ldots, v_m\} \) be sets of propositional variables, and \( \varphi(u, v) \) a formula in 3CNF defined over \( u \cup v \). Then, \( (\forall u)(\exists v) \varphi(u, v) \) is satisfiable iff \( \alpha_n \) is satisfiable with respect to \( \hat{T}_n^\varphi \).

Proof. (\( \Rightarrow \)) Assume that \( (\forall u)(\exists v) \varphi(u, v) \) is satisfiable. In the previous section (see Lemma 6.19) we have constructed an interpretation \( \hat{\mathcal{I}}_{\alpha_n} \) such that \( d_0 \in (\alpha_n)^{\hat{\mathcal{I}}_{\alpha_n}} \), and \( d \in (\alpha_0)^{\hat{\mathcal{I}}_{\alpha_n}} \) for all leaves \( d \) in \( T_{\alpha_n} \). We extend \( \hat{\mathcal{I}}_{\alpha_n} \) into a model \( \hat{\mathcal{I}}_{\alpha_n}^\varphi \) of \( \hat{T}_n^\varphi \) satisfying \( \alpha_n \).

The new interpretation \( \hat{\mathcal{I}}_{\alpha_n}^\varphi \) extends \( \hat{\mathcal{I}}_{\alpha_n} \) with the interpretation of the concept names \( B_1, \ldots, B_m, \hat{B}_1, \ldots, \hat{B}_m \). The positive side of using \( \hat{\mathcal{I}}_{\alpha_n} \) as a starting point is that since \( T_{n,\text{paths}} \) does not change and \( T_{n,\tau} \) only changes in the definition of \( \alpha_0 \), it is enough to extend \( \hat{\mathcal{I}}_{\alpha_n} \) in such a way that \( d \in (\alpha_0)^{\hat{\mathcal{I}}_{\alpha_n}} \) holds for all leaves \( d \) in \( T_{\alpha_n} \).

Let \( d \) be a leaf of \( T_{\alpha_n} \). We define the assignment \( t_d \) for \( u \cup v \) as follows. First,

\[
t_d(u_i) = \text{true} \text{ iff } d \in (A_i)^{\hat{\mathcal{I}}_{\alpha_n}} \quad (1 \leq i \leq n)
\]

Second, \( t_d \) assigns truth values to the variables in \( v \) such that it satisfies \( \varphi(u, v) \). This is always possible because \( (\forall u)(\exists v) \varphi(u, v) \) is satisfiable. If there is more than one possible way any of them can be used. Then, \( \hat{\mathcal{I}}_{\alpha_n}^\varphi \) extends \( \hat{\mathcal{I}}_{\alpha_n} \) as follows. For all \( 1 \leq j \leq m \):

\[
\begin{align*}
(B_j)^{\hat{\mathcal{I}}_{\alpha_n}^\varphi} &:= \{d \mid d \in L_{\alpha_n} \text{ and } t_d(v_j) = \text{true}\} \\
(\hat{B}_j)^{\hat{\mathcal{I}}_{\alpha_n}^\varphi} &:= \{d \mid d \in L_{\alpha_n} \text{ and } t_d(v_j) = \text{false}\}
\end{align*}
\]

(6.10)

Now, let us see why \( d \in (\alpha_0)^{\hat{\mathcal{I}}_{\alpha_n}} \) holds for all leaves \( d \) in \( T_{\alpha_n} \).

A similar relationship exists between \( d \) and \( A_i, \hat{A}_i \) (\( 1 \leq i \leq n \)), since \( d \in (\alpha_0)^{\hat{\mathcal{I}}_{\alpha_n}} \). Thus, we have:

\[
d \in \left( \prod_{i=1}^n (F_i)_{\leq \frac{1}{2}} \cap (F_i)_{\geq \frac{1}{2}} \right) \cap \left( \prod_{j=1}^m (G_j)_{\leq \frac{1}{2}} \cap (G_j)_{\geq \frac{1}{2}} \right)
\]

Regarding \( \hat{C}_\varphi \), let \( (D_l)_{\geq \frac{1}{2}} \) be any of its conjuncts and \( c_l = \{\ell_{i_1}, \ell_{i_2}, \ell_{i_3}\} \) its associated clause in \( \varphi(u, v) \). Since \( t_d \) satisfies \( \varphi(u, v) \), this means that there is \( \ell_{i_l} \) (\( 1 \leq l \leq 3 \)) such that \( t_d(\ell_{i_l}) = \text{true} \). Once we know that, the constructions of \( \gamma \) and \( t_d \) in combination with the properties of \( d \) mentioned above imply that \( d \in \left( \gamma(\ell_{i_l}) \right)^{\hat{\mathcal{I}}_{\alpha_n}} \). Consequently, \( d \in ([D_l]_{\geq \frac{1}{2}})^{\hat{\mathcal{I}}_{\alpha_n}} \) for all \( 1 \leq i \leq k \), and thus \( d \in \left( \hat{C}_\varphi \right)^{\hat{\mathcal{I}}_{\alpha_n}} \). Hence, \( d \in (\alpha_0)^{\hat{\mathcal{I}}_{\alpha_n}} \).

Since \( d \) is an arbitrary leaf, the same result is valid for all the leaves in \( T_{\alpha_n} \). As already mentioned, this guarantees that \( \alpha_n \) is satisfiable with respect to \( \hat{T}_n^\varphi \).

(\( \Leftarrow \)) Conversely, assume that \( \alpha_n \) is satisfiable with respect to \( \hat{T}_n^\varphi \). This means that there exists a model \( \mathcal{I} \) of \( \hat{T}_n^\varphi \) and \( d \in \Delta_{\mathcal{I}} \) such that \( d \in (\alpha_n)_{\mathcal{I}} \).

Let us fix a partial truth assignment \( t \) covering all the variables in \( u \). We show that \( t \) can be extended to \( v \) in such a way that it satisfies \( \varphi(u, v) \). The subset \( X_i \) of \( \{A_1, \ldots, A_n\} \) is induced by \( t \) as follows:

\[
X_i := \{A_i \mid t(u_i) = \text{true}\} \quad (1 \leq i \leq n)
\]
Now, since $\hat{\mathcal{T}}_n^\varphi$ only differs from $\hat{\mathcal{T}}_n$ in the definition of $\alpha_0$ and the inclusion of the $G_j$'s in $\mathcal{T}_{n,\text{comp}}$, Propositions 6.15 and 6.16 still apply to $I$ as a model of $\hat{\mathcal{T}}_n^\varphi$. Following the proof of Lemma 6.18 with respect to $I$ and $\hat{\mathcal{T}}_n^\varphi$, we obtain that there exists $d_i \in \Delta^I$ such that:

- $d_i \in (A_i)^I$ iff $A_i \in X_i$ (iff $t(u_i) = \top$),
- $d_i \in (\alpha_0^\varphi)^I$.

We use $d_i$ to extend $t$ to $v$ as follows. For all $1 \leq j \leq m$:

$$t(v_j) = \top \text{ iff } d_i \in (B_j)^I$$

Therefore, since $d_i$ satisfies the complementary restrictions required in the definition of $\alpha_0^\varphi$ for $A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n$ and $B_1, \ldots, B_m, \bar{B}_1, \ldots, \bar{B}_m$, we further obtain for all literals $\ell$ over $u \cup v$:

$$t(\ell) = \text{true } \text{ iff } d_i \in (\gamma(\ell))^I$$

(6.11)

Moreover, since $d_i \in (\hat{\mathcal{C}}_{\varphi})^I$ we have that $d_i \in [(D_i)_{\varphi}]^I$ for all $1 \leq i \leq k$. By definition of $\text{deg}$ and $D_i$ there must exist $\ell_d$ in $c_i$ such that $d_i \in (\gamma(\ell_d))^I$. It then follows from (6.11) that $t$ satisfies every clause $c_i \in \mathcal{C}$, and consequently it satisfies $\varphi(u, v)$.

Since the partial truth assignment $t$ for $u$ was chosen arbitrarily, we thus have shown that $(\forall u)(\exists v)\varphi(u, v)$ is satisfiable.

The construction of $\hat{\mathcal{T}}_n^\varphi$ modifies $\hat{\mathcal{T}}_n$ in two ways. First, $\mathcal{T}_{n,\text{comp}}$ is extended by adding the definitions of the concepts $G_j$ for all $1 \leq j \leq m$. This yields a TBox $\mathcal{T}_{n,\text{comp}}^\varphi$ such that:

$$s(\mathcal{T}_{n,\text{comp}}^\varphi) = s(\mathcal{T}_{n,\text{comp}}) + O(m)$$

Second, $\mathcal{T}_{n,\text{comp}}^\varphi$ results from $\mathcal{T}_{n,\varphi}$ by redefining $\alpha_0$ (renamed as $\alpha_0^\varphi$) as described in (6.9). It is not hard to see that the definition of $\alpha_0^\varphi$ is of size polynomial in $\varphi(u, v)$. Recall that $s(\hat{\mathcal{T}}_n)$ is a polynomial in $n$. Hence, since $n$ and $m$ are the number of variables in $u$ and $v$, respectively, this means that $s(\hat{\mathcal{T}}_n^\varphi)$ is a polynomial in the size of $(\forall u)(\exists v)\varphi(u, v)$. Thus, $\forall\exists 3\text{SAT}$ is polynomial-time reducible to satisfiability in $\tau\mathcal{EL}(\text{deg})$ with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes. The reduction of satisfiability to subsumption still holds, and therefore we obtain the following lower bounds.

**Lemma 6.23.** In $\tau\mathcal{EL}(\text{deg})$, satisfiability is $\Pi_2^P$-hard and subsumption is $\Sigma_2^P$-hard, with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes.

### 6.4.2 Normalization

To simplify the technical development of the decision procedures presented in the next section, it is convenient to use TBoxes in a special form. We now introduce **normalized** $\tau\mathcal{EL}(\text{deg})$ TBoxes in reduced form, and show that one can (without loss of generality) restrict the attention to this kind of TBoxes.

Let us start by recalling the normal form for $\mathcal{EL}$ TBoxes introduced in [Baa02]. An $\mathcal{EL}$ TBox $\mathcal{T}$ is said to be normalized iff $\alpha \models C_\alpha \in \mathcal{T}$ implies that $C_\alpha$ is of the form:

$$P_1 \cap \ldots \cap P_m \cap \exists r_1.\beta_1 \cap \ldots \cap \exists r_n.\beta_n$$
6.4 Reasoning with respect to acyclic $\tau \mathcal{EL}(deg)$ TBoxes

where $m, n \geq 0$, $P_1, \ldots, P_m \in N_{prim}$, and $\beta_1, \ldots, \beta_n \in N_{def}$. We extend this form to $\tau \mathcal{EL}(deg)$, and say that a $\tau \mathcal{EL}(deg)$ TBox $\hat{T} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ is normalized iff $\mathcal{T}_{\mathcal{EL}}$ is normalized and $\alpha \models \hat{C}_\alpha \in \mathcal{T}_\tau$ implies that $\hat{C}_\alpha$ is of the form:

$$\hat{P}_1 \sqcap \cdots \sqcap \hat{P}_m \sqcap \exists r_1. \beta_1 \sqcap \cdots \sqcap \exists r_n. \beta_n$$

where $m, n \geq 0$, for all $1 \leq i \leq m$ either $\hat{P}_i \in N_{prim}$ or it is of the form $E \sim t$ with $E \in N^0_{def}$, and $\beta_1, \ldots, \beta_n \in N^*_\tau \cup N^0_{def}$.

To illustrate this normalization process we start with a simpler version of Example 12 in [Baa02].

**Example 6.24.** Let $\mathcal{T}$ be the $\mathcal{EL}$ TBox consisting of the following definitions:

$$\alpha_1 \models P_1 \sqcap \alpha_2 \sqcap \exists r_1. \exists r_2. \alpha_3$$
$$\alpha_2 \models P_2 \sqcap \alpha_3 \sqcap \exists s. (\alpha_3 \sqcap P_3)$$
$$\alpha_3 \models P_4$$

Using auxiliary definitions we obtain a new TBox $\mathcal{T}'$:

$$\alpha_1 \models P_1 \sqcap \alpha_2 \sqcap \exists r_1. \beta_1$$
$$\beta_1 \models \exists r_2. \alpha_3$$
$$\alpha_2 \models P_2 \sqcap \alpha_3 \sqcap \exists s. \beta_2$$
$$\beta_2 \models \alpha_3 \sqcap P_3$$
$$\alpha_3 \models P_4$$

This step is formalized as the exhaustive application of the rule $R_3$.

**Condition:** applies to concept definitions of the form $\alpha \models C_1 \sqcap \cdots \sqcap C_n$ if there is an index $i \in \{1, \ldots, n\}$ with $C_i = \exists r. D$ and $D \notin N_{def}$.

**Action:** its application replaces the conjunct $C_i$ by $\exists r. \beta$, and introduces a new definition $\beta \equiv D$, where $\beta \in N_{def}$ is a fresh concept name.

Since $\alpha_1, \alpha_2$ and $\beta_2$ contain top-level atoms which are defined concepts, $\mathcal{T}'$ is not yet normalized. The original normalization process is devised to handle cyclic $\mathcal{EL}$ TBoxes that can be interpreted by different types of semantics. Consequently, the approach used to overcome this problem varies according to each semantics. In our case, however, this becomes simpler since the $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$ we are dealing with is acyclic. The solution for this follows from the discussion presented in [Baa02] for the general case, and consists of substituting these occurrences of defined concepts by their definitions. Following the example we obtain the following TBox:

$$\alpha_1 \models P_1 \sqcap P_2 \sqcap P_4 \sqcap \exists s. \beta_2 \sqcap \exists r_1. \beta_1$$
$$\beta_1 \models \exists r_2. \alpha_3$$
$$\alpha_2 \models P_2 \sqcap P_4 \sqcap \exists s. \beta_2$$
$$\beta_2 \models P_4 \sqcap P_3$$
\[ \alpha_3 \doteq P_4 \]

We name the corresponding rule \( R_\alpha \) and formally define it as follows.

**Condition:** applies to concept definitions of the form \( \alpha \doteq C_1 \sqcap \ldots \sqcap C_n \) if there is an index \( i \in \{1, \ldots, n\} \) with \( C_i = \beta \) and \( \beta \doteq C_\beta \in T \).

**Action:** its application replaces \( C_i \) by \( C_\beta \).

Then, once \( R_3 \) can no longer be applied, an exhaustive application of the rule \( R_\alpha \) will produce a normalized acyclic \( \mathcal{EL} \) TBox. However, to have a polynomial time procedure generating a new TBox of polynomial size, the sequence of applications of \( R_\alpha \) should not be arbitrary. This is achieved by following the order \( \preceq \) induced by \( \rightarrow^+ \), i.e., \( R_\alpha \) can be applied to a concept definition \( \alpha \doteq C_\alpha \) only if it has already been applied to all \( \beta \in \text{def}(T) \) such that \( \beta \preceq \alpha \).

Each application of \( R_3 \) replaces a top-level atom of the form \( \exists r.D \) with a new atom \( \exists r.\beta \), and introduces a simpler definition \( \beta \doteq D \). Concerning \( R_\alpha \), such an ordered sequence of rule applications will always terminate since we are dealing with acyclic TBoxes. Moreover, the *idempotency* of \( \sqcap \) can be exploited to avoid duplications. Hence, \( R_\alpha \) is only applied one time for each top-level atom of the form \( \beta \in N_{\text{def}} \) occurring in the TBox that results from the application of \( R_3 \), and it does not cause an exponential *blow-up* of the size of the TBox. Thus, the described normalization procedure runs in polynomial time and produces a TBox \( T' \) of size polynomial in the size of \( T \).

This procedure can be easily adapted to normalize acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBoxes. The rules \( R_3 \) and \( R_\alpha \) can be applied to \( T_\tau \) in the same way. The only difference is that to apply \( R_\alpha \) in \( T_\tau \), the definition \( \beta \doteq C_\beta \) may also occur in \( T_{\mathcal{EL}} \). Additionally, it is required that all occurrences of threshold concepts \( E_\sim t \) in \( \hat{T} \) are such that \( E \) is a defined concept in \( T_{\mathcal{EL}} \). For example, \( \alpha_1 \) could have been defined as:

\[ \alpha_1 \doteq P_1 \sqcap \exists r_1.([P_2 \sqcap \exists r_2.P_3]_{\leq 8}] \sqcap \exists r_1.\exists r_2.\alpha_3 \]

To handle this we use a new rule \( R_\sim \).

**Condition:** applies to concept definitions of the form \( \alpha \doteq \hat{C}_1 \sqcap \ldots \sqcap \hat{C}_n \in T_\tau \) if there is an index \( i \in \{1, \ldots, n\} \) with \( \hat{C}_i = D_\sim t \) and \( D \notin N_{\text{def}}^0 \).

**Action:** its application replaces the conjunct \( \hat{C}_i \) by \( (E_D)_{\sim t} \), and adds a new definition \( E_D \doteq D \) to \( T_{\mathcal{EL}} \), being \( E_D \) a fresh concept name in \( N_{\text{def}}^0 \).

Thus, the normalization will yield the \( \tau \mathcal{EL}(\text{deg}) \) TBox \( \hat{T} = (T_\tau, T_{\mathcal{EL}}) \) consisting of the following two sets of definitions:

\[
\begin{align*}
\alpha_1 & \doteq P_1 \sqcap \exists r_1.\beta_4 \sqcap \exists r_1.\beta_1 \\
\beta_4 & \doteq E_{\leq 8} \\
\beta_1 & \doteq \exists r_2.\alpha_3 \\
\alpha_2 & \doteq P_2 \sqcap P_4 \sqcap \exists s.\beta_2 \\
\beta_2 & \doteq P_4 \sqcap P_3 \\
\alpha_3 & \doteq P_4
\end{align*}
\]
and $\mathcal{T}_{\mathcal{EL}}$ the following set:

$$E \doteq P_2 \cap \exists r_2.P_3$$

Notice that in order to trigger the application of $R_\sim$, the concerned existential restriction in the definition of $\alpha_1$ had to be first decomposed by applying $R_\exists$. With this in mind, we define the normalization procedure for acyclic $\tau\mathcal{EL}(deg)$ TBoxes as the execution of the following steps.

1. Apply the rule $R_\exists$ exhaustively to $\mathcal{T}_\tau$.
2. Apply the rule $R_\sim$ to $\mathcal{T}_\tau$ as long as possible.
3. Normalize the augmented $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$.
4. Apply the rule $R_\alpha$ exhaustively to $\mathcal{T}_\tau$.

The applications of $R_\exists$ and $R_\alpha$ to $\mathcal{T}_\tau$ modify only $\mathcal{T}_\tau$, while no new threshold expressions are introduced. Regarding the second step, as $D_\sim$ is such that $D$ is defined over $N_{\text{def}}^0 \cup N_{\text{prim}}$, the threshold concept $(E_D)_\sim$ introduced by the application of the rule $R_\sim$ is still defined over $N_{\text{def}}^0 \cup N_{\text{prim}}$. Furthermore, adding $E_D \doteq D$ to $\mathcal{T}_{\mathcal{EL}}$ does not introduce any concept name $\alpha \in N_{\text{def}}^0$ in definitions of $\mathcal{T}_{\mathcal{EL}}$. Finally, the normalization of $\mathcal{T}_{\mathcal{EL}}$ only transforms the structure of $\mathcal{T}_{\mathcal{EL}}$. Therefore, $\hat{\mathcal{T}}'$ satisfies the restrictions required for $\tau\mathcal{EL}(m)$ TBoxes in Definition 4.11 and it is easy to see that no cycles are introduced in it.

Now, after the first step has been executed, all occurrences of threshold concepts in $\mathcal{T}_\tau$ appear as top-level atoms on its concept definitions. Consequently, the application of $R_\sim$ in the second step will cover all of them. Moreover, the normalization of $\mathcal{T}_{\mathcal{EL}}$ before the final step guarantees that $R_\exists$ need not be applied in case $R_\alpha$ applies to a defined concept in $\mathcal{T}_{\mathcal{EL}}$. Overall, this implies that the resulting TBox $\hat{\mathcal{T}}'$ is normalized.

Last, one can see that the rule $R_\sim$ is applied at most one time for each threshold concept $D_\sim$ occurring in a definition of the initial TBox $\mathcal{T}_\tau$. Consequently, at most polynomially many new definitions of the form $E_D \doteq D$ are added to $\hat{\mathcal{T}}$. Thus, using the same arguments given for the application of $R_\exists$ and $R_\alpha$ in the $\mathcal{EL}$ setting, the devised normalization procedure runs in polynomial time and yields a normalized acyclic $\tau\mathcal{EL}(deg)$ TBox $\hat{\mathcal{T}}'$ of size polynomial in the size of $\mathcal{T}$.

We now show that normalization preserves the unfolding of defined concepts.

**Lemma 6.25.** Let $\hat{T}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox and $\hat{T}'$ the $\tau\mathcal{EL}(deg)$ TBox that results from a single application of a normalization rule. Then, for all defined concepts $\alpha$ in $\hat{T}$, $u_{\hat{T}}(\alpha) = u_{\hat{T}'}(\alpha)$

**Proof.** Let $R$ be a normalization rule and $\beta \doteq \widehat{C}_\beta \in \hat{T}$ the concept definition that $R$ has been applied to. We use well-founded induction on the partial order induced by $\rightarrow^+$ on $\text{def}(\hat{T})$. For all defined concepts $\alpha$ in $\hat{T}$ we distinguish two cases:

- $\alpha \neq \beta$. This means that $R$ was not applied to $\alpha \doteq \hat{C}_\alpha$, and consequently $\alpha \doteq \hat{C}_\alpha \in \hat{T}'$. The top-down recursive application of unfolding through the structure of $\hat{C}_\alpha$
with respect to \( \mathcal{T} \) and \( \mathcal{T}' \) may only result in different concept descriptions if:

\[ u_{\mathcal{T}}(\alpha') \neq u_{\mathcal{T}'}(\alpha') \]

for some symbol \( \alpha' \) occurring in \( \mathcal{C}_\alpha \) that corresponds to a defined concept name in \( \mathcal{T} \). However, \( \alpha \rightarrow^+ \alpha' \) and the application of the induction hypothesis to \( \alpha' \) imply that this is never the case. Hence, \( u_{\mathcal{T}}(\alpha) = u_{\mathcal{T}'}(\alpha) \).

- \( \alpha = \beta \). Let \( \mathcal{C}_\beta \) be of the form \( \mathcal{C}_1 \sqcap \ldots \sqcap \mathcal{C}_n \). We analyze the outcome of applying each of the three possible rules to \( \beta \equiv \mathcal{C}_\beta \):

  - \( R_\omega \): the rule was applied to a conjunct \( \mathcal{C}_i \) such that \( \mathcal{C}_i = D_{\prec \omega} \) and \( D \notin N_{\text{def}} \).
    Its application replaces \( \mathcal{C}_i \) by \( (E_D)_{\sim \omega} \) in \( \mathcal{C}_\beta \), and adds \( E_D \equiv D \) to \( T_{\mathcal{E}L} \) where \( E_D \) is a fresh concept name. By definition of unfolding we have:
      \[
      u_{\mathcal{T}}(\beta) = \bigcap_{j=1}^{i-1} u_{\mathcal{T}}(\mathcal{C}_j) \sqcap (u_{\mathcal{T}}(D))_{\sim \omega} \sqcap \bigcap_{j=i+1}^{n} u_{\mathcal{T}}(\mathcal{C}_j)
      \]
    and,
    \[
    u_{\mathcal{T}'}(\beta) = \bigcap_{j=1}^{i-1} u_{\mathcal{T}'}(\mathcal{C}_j) \sqcap [u_{\mathcal{T}'}(E_D)]_{\sim \omega} \sqcap \bigcap_{j=i+1}^{n} u_{\mathcal{T}'}(\mathcal{C}_j)
    \]
    Applying the same inductive argument used above we obtain \( u_{\mathcal{T}}(\mathcal{C}_j) = u_{\mathcal{T}'}(\mathcal{C}_j) \) for all \( j \neq i \) (likewise for \( D \)). Thus, since \( u_{\mathcal{T}}(E_D) = u_{\mathcal{T}'}(D) \) it follows that \( u_{\mathcal{T}}(\beta) = u_{\mathcal{T}'}(\beta) \).

  - \( R_\exists \): the rule has been applied to an atom \( \mathcal{C}_i \) of the form \( \exists r \bar{D} \) such that \( D \notin N_{\text{def}} \). Hence, \( \mathcal{C}_i \) is substituted in \( \mathcal{C}_\beta \) by \( \exists r \mathcal{C}_1 \) with \( \mathcal{C}_1 \) being a fresh concept name and \( \mathcal{C}_1 \equiv \bar{D} \in \mathcal{T}' \). By definition of unfolding we have:
      \[
      u_{\mathcal{T}}(\beta) = \bigcap_{j=1}^{i-1} u_{\mathcal{T}}(\mathcal{C}_j) \sqcap \exists r.u_{\mathcal{T}}(\bar{D}) \sqcap \bigcap_{j=i+1}^{n} u_{\mathcal{T}}(\mathcal{C}_j)
      \]
    and,
    \[
    u_{\mathcal{T}'}(\beta) = \bigcap_{j=1}^{i-1} u_{\mathcal{T}'}(\mathcal{C}_j) \sqcap \exists r.u_{\mathcal{T}'}(\mathcal{C}_1) \sqcap \bigcap_{j=i+1}^{n} u_{\mathcal{T}'}(\mathcal{C}_j)
    \]
    We know that \( u_{\mathcal{T}'}(\mathcal{C}_1) = u_{\mathcal{T}'}(\bar{D}) \). Hence, the same reasoning used for \( R_\omega \) applies:
    \[
    u_{\mathcal{T}}(\mathcal{C}_1) = u_{\mathcal{T}'}(\bar{D}) \]
    yields \( u_{\mathcal{T}}(\alpha) = u_{\mathcal{T}'}(\alpha) \).

  - \( R_\alpha \): there is an index \( 1 \leq i \leq n \) such that \( \mathcal{C}_i \) is of the form \( \mathcal{C}_1 \), and \( \mathcal{C}_1 \equiv \mathcal{C}_{\beta_1} \in \mathcal{T} \). The application of \( R_\alpha \) to \( \beta_1 \) replaces it in \( \mathcal{C}_{\beta_1} \) with \( \mathcal{C}_{\beta_1} \). Since \( \beta \rightarrow^+ \beta_1 \), the application of induction hypothesis yields \( u_{\mathcal{T}}(\beta_1) = u_{\mathcal{T}'}(\beta_1) = u_{\mathcal{T}'}(\mathcal{C}_{\beta_1}) \).
    Again, \( u_{\mathcal{T}}(\mathcal{C}_j) = u_{\mathcal{T}'}(\mathcal{C}_j) \) for all \( j \neq i \), and the rest follows from the definition of unfolding on \( \beta \).
This property is then invariant under any number of rule applications. Therefore, the following proposition is a direct consequence of Lemma 6.25.

**Proposition 6.26.** Let \( \hat{T} \) be an acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBox and \( \hat{T}' \) the normal form of \( \hat{T} \). For all defined concepts \( \alpha \) in \( \hat{T} \), \( u_T(\alpha) = u_{\hat{T}}(\alpha) \).

Proposition 6.26 implies that reasoning with respect to an acyclic \( \tau \mathcal{EL}(\text{deg}) \) TBox \( \hat{T} \) can be reduced to reasoning with respect to its normal form \( \hat{T}' \). Therefore, from now on we only consider normalized TBoxes.

We still require one more transformation. Recall that for acyclic \( \mathcal{EL} \) TBoxes, the value of \( \text{deg}(d, C, \mathcal{T}) \) is defined in terms of applying the basic definition of \( \text{deg} \) (Chapter 4) to the unfolding of \( C \) in \( \mathcal{T} \). Moreover, \( \text{deg} \) needs to further translate \( u_T(C) \) into its reduced form \( [u_T(C)]^r \). Since \( u_T(C) \) may result in a concept of exponential size, it is certainly not a good idea to unfold and then compute the reduced form. To have this issue handled in a more transparent way by the decision procedures presented in the next section, we introduce the reduced form for acyclic \( \mathcal{EL} \) TBoxes. The ideas that follow are based on the results shown by Küsters in [Küst01].

**Definition 6.27.** Let \( \mathcal{T} \) be an acyclic \( \mathcal{EL} \) TBox and \( C \) an \( \mathcal{EL} \) concept description. Then, \( C \) is reduced with respect to \( \mathcal{T} \) iff:

- \( C \) is reduced according to Küsters’ definition modulo \( \sqsubseteq_T \) (i.e., \( \sqsubseteq_T \) is used to identify redundancies instead of \( \sqsubseteq \)).

We say that \( \mathcal{T} \) is in reduced form iff for all \( \alpha \models C_\alpha \in \mathcal{T} \) the concept \( C_\alpha \) is reduced with respect to \( \mathcal{T} \).

The benefit of using these type of TBoxes is that the unfolding of a defined concept will always result in a reduced concept description.

**Lemma 6.28.** Let \( \mathcal{T} \) be a normalized acyclic \( \mathcal{EL} \) TBox in reduced form. Then, for all \( \alpha \models C_\alpha \) the \( \mathcal{EL} \) concept description \( u_T(\alpha) \) is reduced.

**Proof.** We use well-founded induction on \( \rightarrow^+ \) over \( \text{def}(\mathcal{T}) \). Since \( \mathcal{T} \) is normalized, \( C_\alpha \) has the following structure:

\[
P_1 \cap \ldots \cap P_m \cap \exists r_1.\alpha_1 \cap \ldots \cap \exists r_n.\alpha_n
\]

Clearly, \( \alpha \rightarrow^+ \alpha_i \) for all \( 1 \leq i \leq n \). Therefore, the application of induction hypothesis yields that \( u_T(\alpha_i) \) is reduced. Now, since \( C_\alpha \) is reduced with respect to \( \mathcal{T} \), for all pairs \( (\exists r_i.\alpha_i, \exists r_j.\alpha_j) \) we have:

- \( r_i \neq r_j \), or
- \( \alpha_i \not\sqsubseteq_T \alpha_j \) and \( \alpha_j \not\sqsubseteq_T \alpha_i \).

In addition, we know that \( \alpha_i \equiv_T u_T(\alpha_i) \) and \( \alpha_j \equiv_T u_T(\alpha_j) \). This means that having \( r_i = r_j \), it will be the case that \( u_T(\alpha_i) \not\sqsubseteq u_T(\alpha_j) \) and \( u_T(\alpha_j) \not\sqsubseteq u_T(\alpha_i) \). Finally, since \( u_T(\alpha) \) is the following concept description

\[
P_1 \cap \ldots P_m \cap \exists r_1. u_T(\alpha_1) \cap \ldots \cap \exists r_n. u_T(\alpha_n)
\]

we can conclude that \( u_T(\alpha) \) is reduced. □
To translate acyclic EL TBoxes into its reduced form, the algorithm sketched in [Kis01] (derived from Proposition 6.3.1.) to compute the reduced form of EL concept descriptions comes in handy. By using \( \sqsubseteq_T \) instead of \( \subseteq \), it will be able to compute the reduced form \( C^{\tau(T)} \) of a concept \( C \) with respect to \( T \). Since \( \sqsubseteq_T \) is decidable in polynomial time in EL [Baa03], the modified procedure also runs in polynomial time. Moreover, the concept \( C^{\tau(T)} \) satisfies \( C \equiv_T C^{\tau(T)} \).

Based on this we can devise a very simple polynomial time transformation that given an acyclic EL TBox \( T \) outputs and equivalent TBox \( T' \) in reduced form. The translation and its correctness are given in the following lemma.

**Lemma 6.29.** Let \( T \) be a normalized acyclic EL TBox. The TBox \( T' \) obtained from \( T \) by the substitution of \( \alpha \overset{\tau}(\alpha) \) for \( \alpha \) (for all \( \alpha \overset{\tau}(\alpha) \in T \)) satisfies the following:

1. \( T \) and \( T' \) are equivalent.
2. \( T' \) is in reduced form.

**Proof.** 1) We show that every model of \( T \) is a model of \( T' \) and vice versa. Let \( \mathcal{I} \) be a model of \( T \), then \( \alpha^\mathcal{I} = (\alpha^\mathcal{I})^\tau \) for all \( \alpha \overset{\tau}(\alpha) \in T \). Since \( \alpha \equiv_T (\alpha^\tau) \), this means that \( \alpha^\mathcal{I} = [(\alpha^\tau)^\mathcal{I}]^\mathcal{I} \) for all \( \alpha \overset{\tau}(\alpha) \in T' \). Hence, \( \mathcal{I} \models T' \).

Conversely, let \( \mathcal{I}' \) be a model of \( T' \). We take a model \( \mathcal{I} \) of \( T \) such that \( \Delta^\mathcal{I} = \Delta^\mathcal{I}' \) and \( X^\mathcal{I} = X^\mathcal{I}' \), for all \( X \in N_{\text{prim}} \cup N_R \). Such a model exists because of Proposition 6.3. We prove that \( \alpha^\mathcal{I} = \alpha^\mathcal{I}' \) for all \( \alpha \overset{\tau}(\alpha) \in T \). The proof goes by induction on the partial order induced by \( \rightarrow^+ \). Since \( T \) is normalized, each top-level atom of \( \alpha \) is of the form \( A \in N_{\text{prim}} \) or \( \exists \gamma.\beta \), where \( \beta \equiv_T C \beta \). Moreover, the set of atoms occurring in \( (\alpha^\tau) \) is a subset of the corresponding set for \( \alpha \). Therefore, we distinguish two cases for all top-level atoms \( \alpha \) of \( \alpha \):

- \( \text{At} \) occurs in \( (\alpha^\tau) \). If \( \text{At} = A \), by selection of \( \mathcal{I} \) we have \( A^\mathcal{I} = A^\mathcal{I}' \). Otherwise, \( \text{At} = \exists \gamma.\beta \) and \( \alpha \rightarrow^+ \beta \). The application of induction yields \( \beta^\mathcal{I} = \beta^\mathcal{I}' \) and thus \( (\exists \gamma.\beta)^\mathcal{I} = (\exists \gamma.\beta)^\mathcal{I}' \). Hence, it is not hard to see that for all \( d \in \Delta^\mathcal{I} \), \( d \in \alpha^\mathcal{I} \) implies \( d \in \alpha^\mathcal{I}' \).

- \( \text{At} \) only occurs in \( \alpha \). There must be a top-level atom \( \text{At}' \) in \( \alpha \) such that \( \text{At}' \subseteq_T \text{At} \) and \( \text{At}' \) does occur in \( (\alpha^\tau) \). From the previous point we know that \( (\text{At}')^\mathcal{I} = (\text{At}')^\mathcal{I}' \). Therefore, if \( d \in (\text{At}')^\mathcal{I} \) we also have \( d \in (\text{At}')^\mathcal{I}' \) and \( d \in \text{At}^\mathcal{I} \). Hence, \( d \in \alpha^\mathcal{I} \) implies \( d \in \alpha^\mathcal{I}' \).

Thus, we have shown that \( \alpha^\mathcal{I} = \alpha^\mathcal{I}' \). This implies the following equalities:

\[
\alpha^\mathcal{T} = \alpha^\mathcal{T}' = (u_T(C))^{\tau(T)}
\]

Then, since \( \mathcal{I} \) and \( \mathcal{I}' \) have the same interpretation for \( N_{\text{prim}} \cup N_R \), this means that \( [u_T(C)]^\mathcal{I} = [u_T(C)]^\mathcal{I}' \). Hence, for all \( \alpha \overset{\tau}(\alpha) \in T \) we have:

\[
\alpha^\mathcal{T} = [u_T(C)]^\mathcal{T}'
\]

Consequently, \( \alpha^\mathcal{T}' = (C)^\mathcal{T}' \) and \( T' \) is a model of \( T \).
2) Assume that $\mathcal{T}'$ is not in reduced form. Then, there exists $\alpha \models (C_\alpha)^{\tau(\mathcal{T})} \in \mathcal{T}'$ such that $(C_\alpha)^{\tau(\mathcal{T})}$ is reducible with respect to $\mathcal{T}'$. This means that there are two top-level atoms $\text{At}_1$ and $\text{At}_2$ in $(C_\alpha)^{\tau(\mathcal{T})}$ such that $\text{At}_1 \sqsubseteq_{\mathcal{T}} \text{At}_2$. Since we just have shown that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent from a model-theoretic point of view, we also have $\text{At}_1 \sqsubseteq_{\mathcal{T}} \text{At}_2$. Hence, we obtain a contradiction against the fact that $(C_\alpha)^{\tau(\mathcal{T})}$ is reduced with respect to $\mathcal{T}$. Thus, $\mathcal{T}'$ is in reduced form.

To sum up, given an acyclic $\tau\mathcal{EL}(\text{deg})$ TBox $\hat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$, we have demonstrated the following along this section:

- $\hat{\mathcal{T}}$ can be normalized in polynomial time into an acyclic TBox $\hat{\mathcal{T}}' = (\mathcal{T}_{\tau}', \mathcal{T}_{\mathcal{EL}}')$, such that reasoning w.r.t. $\hat{\mathcal{T}}$ can be reduced to reasoning w.r.t. $\hat{\mathcal{T}}'$.
- The new TBox $\mathcal{T}_{\mathcal{EL}}'$ can be translated in polynomial time into an equivalent $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}''$ in reduced form.
- The computation of the reduced form only removes atoms from concept definitions. Therefore, $\mathcal{T}_{\mathcal{EL}}''$ remains normalized.

Hence, reasoning in $\tau\mathcal{EL}(\text{deg})$ with respect to acyclic TBoxes can be restricted to normalized acyclic TBoxes in reduced form.

**Proposition 6.30.** Satisfiability and subsumption on concepts defined in an acyclic $\tau\mathcal{EL}(\text{deg})$ TBox can be reduced in polynomial time to satisfiability and subsumption on concepts defined in a normalized acyclic $\tau\mathcal{EL}(\text{deg})$ TBox in reduced form.

### 6.4.3 Upper bounds

We now provide a PSPACE algorithm to decide satisfiability and subsumption with respect to an acyclic $\tau\mathcal{EL}(\text{deg})$ TBox $\hat{\mathcal{T}}$. Note that one can focus on satisfiability of concepts $\alpha$ and subsumption questions of the form $\alpha \sqsubseteq \beta_1 \sqsubseteq \beta_2$, where $\alpha, \beta_1, \beta_2 \in \text{def}(\hat{\mathcal{T}})$.

Any concept description $\hat{C}$ can be equivalently replaced with a fresh defined concept $\alpha_{\hat{C}}$, by adding the definition $\alpha_{\hat{C}} = \hat{C}$ to $\mathcal{T}_\tau$.

As explained in Section [6.2](#) by using unfolding, satisfiability and subsumption with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes can be reduced to reasoning with the empty TBox.

In addition, in Chapter 5 we showed that a concept description of the form $\hat{C} \sqcap -\hat{D}$ is satisfiable in $\tau\mathcal{EL}(\text{deg})$ iff there exists an interpretation $\mathcal{I}$ such that $\hat{C}^\mathcal{I} \setminus \hat{D}^\mathcal{I} \neq \emptyset$ and $|\Delta^\mathcal{I}| \leq s(\hat{C}) \times s(\hat{D})$ (see Lemma [5.6](#)). Hence, given an acyclic $\tau\mathcal{EL}(\text{deg})$ TBox $\hat{\mathcal{T}}$ and two of its defined concepts $\alpha_1$ and $\alpha_2$: $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable with respect to $\hat{\mathcal{T}}$ iff there exists an interpretation $\mathcal{I}$ over $\mathbb{N}_{\text{prim}} \cup \mathbb{N}_R$ such that:

- $[u_{\hat{\mathcal{T}}}(\alpha_1)]^\mathcal{I} \setminus [u_{\hat{\mathcal{T}}}(\alpha_2)]^\mathcal{I} \neq \emptyset$, and
- $|\Delta^\mathcal{I}| \leq s(u_{\hat{\mathcal{T}}}(\alpha_1)) \times s(u_{\hat{\mathcal{T}}}(\alpha_2))$.

Now, there is unique way to extend $\mathcal{I}$ into a model of $\hat{\mathcal{T}}$ (Proposition [6.11](#)). Therefore, we obtain the following bounded model property for satisfiability of concepts of the form $\alpha_1 \sqcap \neg \alpha_2$ in the presence of acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes.
Proposition 6.31. Let $\hat{T}$ be an acyclic $\tau\varepsilon\mathcal{L}(\text{deg})$ TBox and $\alpha_1, \alpha_2$ two defined concepts in $\hat{T}$. If $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable in $\hat{T}$, then there exists an interpretation $I \models \hat{T}$ such that $(\alpha_1)^T \setminus (\alpha_2)^T \neq \emptyset$ and $|\Delta^T| \leq s(u_{\hat{T}}(\alpha_1)) \times s(u_{\hat{T}}(\alpha_2))$.

For the empty terminology $|\Delta^T|$ is polynomial in the size of $\hat{C}$ and $\hat{D}$. This was used to provide an NP-algorithm for satisfiability of concepts of the form $\hat{C} \sqcap \neg \hat{D}$, which uses non-determinism to guess the whole interpretation $I$. Therefore, since $u_{\hat{T}}(\alpha_1)$ may result in a concept description of size exponential in $s(\hat{T})$, the same procedure applied to $u_{\hat{T}}(\alpha_1)$ or $u_{\hat{T}}(\alpha_1) \sqcap \neg u_{\hat{T}}(\alpha_2)$ would give a NEXP-algorithm for concept satisfiability and non-subsumption with respect to acyclic $\tau\varepsilon\mathcal{L}(\text{deg})$ TBoxes.

Our aim is to design a PSPACE algorithm that solves these problems. Obviously, such a procedure cannot store the whole interpretation $I$. However, the proof of Lemma 5.6 tells us the following:

- $I$ is tree-shaped,
- the depth of the associated description tree $T_I$ is bounded by:
  \[ \text{rd}(u_{\hat{T}}(\alpha_1)) + \text{rd}(u_{\hat{T}}(\alpha_2)) \]
- the domain element $d_0$ of $I$ corresponding to the root of $T_I$ satisfies:
  \[ d_0 \in [u_{\hat{T}}(\alpha_1) \sqcap \neg u_{\hat{T}}(\alpha_2)]^T \]

Fortunately, the depth $\text{rd}(u_{\hat{T}}(\alpha_1)) + \text{rd}(u_{\hat{T}}(\alpha_2))$ is always polynomial in $s(\hat{T})$. Thus, despite its size, it is possible to non-deterministically generate $I$ in a top-down fashion, while keeping the used space polynomial in $s(\hat{T})$. Let $d$ and $b \geq 0$ be natural numbers. The following procedure is meant to generate all the tree-shaped interpretations $I$ over $N_{\text{prim}} \cup N_R$, such that $|\Delta^T| \leq b$ and the depth of $T_I$ is not greater than $d$:

1: \textbf{procedure} $A(\delta : N, b : \text{binary})$
2: \hspace{1cm} $b := b - 1$ \hspace{1cm} // counts the individual represented by the current call
3: \hspace{1cm} non-deterministically choose a subset $P$ of $N_{\text{prim}}$
4: \hspace{1cm} \textbf{if} ($\delta \neq 0$) \textbf{and} ($b \neq 0$) \textbf{then}
5: \hspace{2cm} \textbf{for all} $r \in N_R$ \textbf{do}
6: \hspace{3cm} non-deterministically choose $0 \leq b_r \leq b$ \hspace{1cm} // $b_r : \text{binary}$
7: \hspace{3cm} $b := b - b_r$
8: \hspace{2cm} \textbf{for all} $1 \leq i \leq b_r$ \textbf{do}
9: \hspace{3cm} non-deterministically choose $0 \leq b_r^i \leq b$ \hspace{1cm} // $b_r^i : \text{binary}$
10: \hspace{3cm} $b := b - b_r^i$
11: \hspace{3cm} $A(\delta - 1, b_r^i + 1)$
12: \hspace{2cm} \textbf{end for}
13: \hspace{1cm} \textbf{end for}
14: \hspace{1cm} \textbf{end if}
15: \textbf{end procedure}

Note that each recursive call decreases the value of $\delta$, which implies that this is a terminating procedure executing at most $d$ nested recursive calls. Moreover, the parameter
6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes

declaration $b : \text{binary}$ states that $A$ works with the binary representation of the value $b$.

As mentioned above, we are dealing with interpretations that may have size exponential in $s(\tilde{T})$, and that is why the use of a binary counter to represent the value of $b$. Finally, the set of variables $b_r$ and $b_i^r$ can be reduced to two variables since they are only used within the scope of the for loops. Therefore, each run of $A$ uses space polynomial on $d$ and the number of bits needed to represent $b$.

The general idea of the procedure is as follows: each recursive call represents an individual of the domain and the recursion tree lays out the tree-shaped form of an interpretation. The set $\mathcal{P}$ contains the primitive concept names that a domain element is an instance of, the number $b_r$ stands for the number of $r$-successors, and $b_i^r$ means that the interpretation rooted at the $i$-th $r$-successor has at most $b_i^r + 1$ elements. To formalize this intuition we define the notion of a run of $A$.

**Definition 6.32.** A run $\rho$ of $A$ on $(d, b)$ is a tree of recursive calls $T_{(d,b)}$ such that:

- its root $v_0$ is labeled by the non-deterministic choices $\mathcal{P}$, $b_r$ for all $r \in \mathbb{N}_R$, and $b_i^r$ for all $1 \leq i \leq b_r$.

- for all $r \in \mathbb{N}_R$, there are exactly $b_r$ successors $v_{r1}, \ldots, v_{rb_r}$ of $v_0$ such that, the tree rooted at $v_{ri}$ is a run of $A$ on $(d - 1, b_i^r + 1)$.

Figure 6.1 depicts a run $\rho$ of $A$ (left-hand side). Such a run induces the $\mathcal{EL}$ description tree $T_\rho$ (right-hand side) with the same structure, where its nodes are labeled with the corresponding sets $\mathcal{P}$ chosen by $\rho$ and the edges with the role names generating the corresponding recursive call (line 5 in $A$). Therefore, we say that $\rho$ induces the interpretation $I_\rho$ that has the description tree $T_\rho$.

Conversely, for all tree-shaped interpretations $I$ of size at most $b$ and depth not greater than $d$, there is always a run of $A$ describing $I$.  

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (rho) {$\rho$};
\node [below of = rho] {$\mathcal{P} = \{A, B\}$};
\node [below of = $b_r = 2, b_s = 1$] {$b_i^1 = 1, b_i^2 = 3, b_i^3 = 0$};
\node [right of = rho] {$T_\rho$};
\node [below of = $v_0 : \{A, B\}$] {$v_0 : \{A, B\}$};
\node [below of = $r \ r \ r \ s$] {$v_1 : \{A\}$ \hspace{1cm} $v_2 : \{\}$ \hspace{1cm} $v_3 : \{A\}$};
\node [below of = $v_4 : \{B\}$] {$v_4 : \{B\}$};
\end{tikzpicture}
\caption{A run $\rho$ of $A$ and its induced $\mathcal{EL}$ description tree $T_\rho$.}
\end{figure}
Lemma 6.33. Let \( d \geq 0 \) and \( b > 0 \) be two natural numbers. For all tree-shaped interpretations \( \mathcal{I} \) over \( \mathbb{N}_{\text{prim}} \cup \mathbb{N}_R \) with at most \( b \) elements and depth not greater than \( d \), there exists a run \( \rho \) of \( A \) on \( (d, b) \) such that \( \mathcal{I} = \mathcal{I}_\rho \).

Proof. Let \( \mathcal{I} \) be a tree-shaped interpretation of depth \( d(\mathcal{I}) \) such that \( |\Delta^{\mathcal{I}}| \leq b \) and \( d(\mathcal{I}) \leq d \). We show how to guide a run \( \rho \) of \( A \) such that \( \mathcal{I}_\rho = \mathcal{I} \). The proof goes by induction on the number \( d(\mathcal{I}) \).

Let \( d_0 \in \Delta^{\mathcal{I}} \) be the root of \( T^\mathcal{I} \). For all \( r \in \mathbb{N}_R \) we denote as \( r(d_0) = \{e_1, \ldots, e_n\} \) \((n \geq 0)\) the set of \( r \)-successors of \( d_0 \) in \( \mathcal{I} \). In addition, for an \( r \)-successor \( e_i \) of \( d_0 \), \( T^\mathcal{I}[e_i] \) denotes the subtree of \( T^\mathcal{I} \) rooted at \( e_i \), and \( \mathcal{I}_{e_i} \) the associated interpretation. Then, when \( A \) is invoked on \( (d, b) \) it makes the following non-deterministic choices:

- \( \mathcal{P} = \ell(\Delta(d_0)) \),
- for all \( r \in \mathbb{N}_R \): \( b_r = |r(d_0)| \),
- for all \( r \in \mathbb{N}_R \) and \( e_i \in r(d_0) \): \( b_r^i = |\Delta^{\mathcal{I}_{e_i}}| - 1 \),
- for all \( r \in \mathbb{N}_R \) and \( 1 \leq i \leq b_r \), the recursive call \( A(d-1, b_r^i+1) \) follows a run \( \rho_r^i \) such that \( \mathcal{I}_{\rho_r^i} = \mathcal{I}_{e_i} \).

Since \( |\Delta^{\mathcal{I}}| \leq b \) and \( d(\mathcal{I}) \leq d \), the first three choices are consistent with the execution of \( A \). Regarding the last choice, since \( T^\mathcal{I} \) is a tree we know that \( d(\mathcal{I}_{e_i}) < d(\mathcal{I}) \). Consequently, \( d(\mathcal{I}_{e_i}) \leq d-1 \) and the induction hypothesis can be applied to obtain the proper run \( \rho_r^i \). Therefore, \( \rho \) induces an \( \mathcal{E}\mathcal{L} \) description tree \( T^\rho \) such that:

- its root \( v_0 \) is labelled with \( \ell(\Delta(d_0)) \),
- for all \( r \in \mathbb{N}_R \): \( v_0 \) has exactly \(|r(d_0)| \) children \( v_1, \ldots, v_{|r(d_0)|} \), each edge \((v_0, v_i)\) \((1 \leq i \leq |r(d_0)|)\) is labelled with \( r \), and the subtree \( T^\rho[v_i] \) rooted at \( v_i \) in \( T^\rho \) is equal to \( T^\mathcal{I}[e_i] \).

Thus, we can conclude that \( \mathcal{I}_\rho = \mathcal{I} \). \( \square \)

The previous lemma ensures that, by choosing \( d \) as \( rd(u_{\mathcal{F}}(\alpha_1)) + rd(u_{\mathcal{F}}(\alpha_2)) \) and \( b \) as \( s(u_{\mathcal{F}}(\alpha_1)) \times s(u_{\mathcal{F}}(\alpha_2)) \), the set of runs of \( A \) on \( (d, b) \) covers all the relevant interpretations to find out if \( u_{\mathcal{F}}(\alpha_1) \cap u_{\mathcal{F}}(\alpha_2) \) is satisfiable. Therefore, it remains to see how to verify for each run \( \rho \) of \( A \), whether its induced interpretation \( \mathcal{I}_\rho \) fulfills \( d_0 \in [u_{\mathcal{T}}(\alpha_1) \cap u_{\mathcal{T}}(\alpha_2)]^{\mathcal{T}} \).

We have already provided an algorithm to do that (Algorithm 2 in Chapter 3), which is based on Theorem 3.8. Nevertheless, on the one hand it is not immediate to link Algorithm 2 to the way \( A \) generates \( \mathcal{I}_\rho \); and on the other hand due to the possible exponential size of \( u_{\mathcal{F}}(\alpha_1) \) and \( u_{\mathcal{F}}(\alpha_2) \), special care would be required in doing so.

To address these concerns, we go back to the initial formulation of our problem: search for a model \( \mathcal{J} \) of \( \mathcal{T} \) such that \((\alpha_1 \cap \neg \alpha_2)^{\mathcal{T}} \neq \emptyset \). Contrary to models of \( \mathcal{T} \), interpretations induced by runs of \( A \) do not interpret defined concepts in \( \mathcal{T} \). However, there is a unique way to extend each of them into a model of \( \mathcal{T} \) (see Proposition 6.11). Hence, since \( \mathcal{I}_\rho \) is tree-shaped, it is possible to compute such an extension in a bottom-up manner. From now on we will use indistinctively \( \mathcal{I}_\rho \) to identify both, a primitive interpretation and its unique extension into a model of \( \mathcal{T} \).
The idea is then to compute for all \( \alpha \models \hat{C}_\alpha \in \hat{\mathcal{T}} \), whether \( d_0 \in \alpha^{\mathcal{T}_\rho} \). To this end, we modify procedure A such that each run \( \rho \) additionally computes a set \( \text{Ex} \subseteq N_{\text{def}} \) with the following meaning:

\[
\text{Ex} := \{ \alpha \mid \alpha \models \hat{C}_\alpha \in \hat{\mathcal{T}} \text{ and } d_0 \in \alpha^{\mathcal{T}_\rho} \}
\]

The special forms introduced in Section 6.4.2 for acyclic TBoxes are of great help in computing \( \text{Ex} \). In particular, the normal form of \( \hat{\mathcal{T}} \) provides the following shape for \( \hat{C}_\alpha \):

\[
\hat{P}_1 \cap \ldots \cap \hat{P}_n \cap \exists r_1, \alpha_1 \cap \ldots \cap \exists r_m, \alpha_m
\]

Consequently for all \( d \in \Delta^{\mathcal{T}_\rho} \), \( d \in \alpha^{\mathcal{T}_\rho} \) iff:

1. \( d \in (\hat{P}_1)^{\mathcal{T}_\rho} \) for all \( 1 \leq i \leq n \), and
2. for all \( 1 \leq i \leq m \), there exists \( d_i \in \Delta^{\mathcal{T}_\rho} \) such that \((d, d_i) \in (r_i)^{\mathcal{T}_\rho} \) and \( d_i \in (\alpha_i)^{\mathcal{T}_\rho} \).

The computation of \( \text{Ex} \) will be based on checking these two conditions for \( d_0 \). If \( \hat{P}_i \) is of the form \( A \in N_{\text{prim}} \), verifying whether \( d_0 \in A^{\mathcal{T}_\rho} \) is simple since \( \mathcal{I}_\rho \) already contains that information (the non-deterministic choice in line 3). To check whether \( d_0 \in (E_{\sim t})^{\mathcal{T}_\rho} \), we further extend A to compute for all runs \( \rho \) an assignment \( D : \text{def}(\mathcal{T}_{\mathcal{E}_\mathcal{L}}) \to [0,1] \) such that:

\[
D(E) := \deg^{\mathcal{T}_\rho}(d_0, w_{\mathcal{E}_\mathcal{L}}(E))
\]

Once \( D \) is computed for \( d_0 \), it is immediate to verify whether \( d_0 \in (E_{\sim t})^{\mathcal{T}_\rho} \). Regarding Condition 2 as explained before the successors \( e \) of \( d_0 \) in \( \mathcal{I}_\rho \) are the roots of the interpretations induced by runs corresponding to the recursive calls triggered by \( \rho \). Hence, the sets \( \text{Ex}_e \) computed by such calls provide the necessary information to determine whether \( d_0 \in (\exists r_i, \alpha_i)^{\mathcal{T}_\rho} \) for all \( 1 \leq i \leq m \). However, since \( d_0 \) may have exponentially many direct successors in \( \mathcal{I}_\rho \), a PSPACE procedure cannot store all the corresponding sets \( \text{Ex}_e \). To deal with this, A will compute a relation of the form \( z \subseteq (N_{R} \times \text{def}(\hat{\mathcal{T}})) \cup (e \times N_{\text{prim}}) \) such that: \((r, \alpha) \in z \) iff there is \( e \in \Delta^{\mathcal{T}_\rho} \) satisfying \((d_0, e) \in r^{\mathcal{T}_\rho} \) and \( \alpha \in \text{Ex} \). In this way we can keep the relevant information needed to verify whether \( d_0 \in (\exists r_i, \alpha_i)^{\mathcal{T}_\rho} \), while using polynomial space.

Putting all these ideas together, we transform A into the following function:

\begin{verbatim}
1: function A(\partial : integer, b : binary)
2:   b := b - 1
3:   non-deterministically choose a subset \( \mathcal{P} \) of \( N_{\text{prim}} \)
4:   initialize \( v \) and \( z \)
5:   if (\partial \neq 0) and (b \neq 0) then
6:     for all \( r \in N_{R} \) do
7:       non-deterministically choose 0 \leq b_x := b
8:       \mathcal{b} := b - b_x
9:       for all \( 1 \leq i \leq \mathcal{b} \) do
10:      non-deterministically choose 0 \leq b^i_x \leq b
11:     \mathcal{b} := b - b^i_x
12:     (Ex^i_x, D^i_x) := A(\partial - 1, \mathcal{b} + 1)
13:   update v
\end{verbatim}
Chapter 6. Adding Terminologies to $\tau\mathcal{EL}(m)$

14: update $z$
15: end for
16: end for
17: end if
18: $D := \text{SUBdeg}(v)$
19: $\text{Ex} := \text{SUBex}(D, z)$
20: return ($\text{Ex}, D$)
21: end function

The subroutines $\text{SUBdeg}$ and $\text{SUBex}$ invoked in lines 18 and 19 correspond to the computation of the assignment $D$ and the set $\text{Ex}$, respectively. The execution of line 14 updates the relation $z$ using the content of $\text{Ex}_{i}$ after each recursive call has been executed. Regarding the symbol $v$ in line 13, as we explain below it represents a table used to help the computation of $D$.

Let us now move on to the details of the computation of $\text{Ex}$ and $D$. We start with the computation of $D$, and afterwards explain how to compute $\text{Ex}$.

Due to the normal form of $\hat{T}$, the $\mathcal{EL}$ concept $E$ in $E_{\sim t}$ is a defined concept in $T_{\mathcal{EL}}$. Therefore, by Definition 6.9 for all $d \in \Delta_{I_{\rho}}$:

$$d \in (E_{\sim t})^{I_{\rho}} \iff \text{deg}_{I_{\rho}}(d, u_{T_{\mathcal{EL}}}(E)) \sim t$$

Coming back to Chapter 4 we know that $\text{deg}_{I_{\rho}}(d, u_{T_{\mathcal{EL}}}(E))$ is the maximal value of $h_{w}(v_{0})$ among all $\text{ptgh}s h \in H(T_{u_{T_{\mathcal{EL}}}(E)}, G_{I_{\rho}}, d)$, where $v_{0}$ is the root of the description tree $T_{u_{T_{\mathcal{EL}}}(E)}$. Note that we use directly $T_{u_{T_{\mathcal{EL}}}(E)}$, since being $T_{\mathcal{EL}}$ in reduced form implies that $u_{T_{\mathcal{EL}}}(E)$ is reduced (see Lemma 6.28). Now, $E$ is defined in $T_{\mathcal{EL}}$ as follows:

$$E = P_{1} \cap \ldots \cap P_{m} \cap \exists r_{1}. E_{1} \cap \ldots \cap \exists r_{n}. E_{n}$$

This gives us the following information regarding $T_{u_{T_{\mathcal{EL}}}(E)}$:

- the label of $v_{0}$ in $T_{u_{T_{\mathcal{EL}}}(E)}$ is the set $\{P_{1}, \ldots, P_{m}\}$,
- $v_{0}$ has exactly $n$ ($n \geq 0$) successors $v_{1}, \ldots, v_{n}$ in $T_{u_{T_{\mathcal{EL}}}(E)}$,
- for all $1 \leq i \leq n$, the subtree $T_{u_{T_{\mathcal{EL}}}(E)}[v_{i}]$ of $T_{u_{T_{\mathcal{EL}}}(E)}$ rooted at $v_{i}$ is exactly the description tree associated to $u_{T_{\mathcal{EL}}}(E_{i})$.

Additionally, the computation of $h_{w}(v_{0})$ is based on the following expression:

$$h_{w}(v_{0}) = \begin{cases} 1 & \text{if } m + n = 0 \\ \frac{|\{P_{1}, \ldots, P_{m}\} \cap \text{deg}(d)| + \sum_{i \leq k} h_{w}(w_{i})}{m+n} & \text{otherwise.} \end{cases}$$

where $w_{1}, \ldots, w_{k}$ are the children of $v_{0}$ in $T_{u_{T_{\mathcal{EL}}}(E)}$ mapped by $h$. Now, regarding a $\text{ptgh}$ $h$ yielding a maximal value for $h_{w}(v_{0})$ we observe the following:

- if $(d, e) \in (r_{i})^{I_{\rho}}$ for some $e \in \Delta^{I_{\rho}}$, then we can assume that $v_{i} \in \text{dom}(h)$. 


6.4 Reasoning with respect to acyclic $\tau \mathcal{EL}(\text{deg})$ TBoxes

- Let $h(v_i) = e_i$ where $e_i \in \Delta^T$. Then, $h_w(v_i) = \text{deg}_r(e_i, u_{\tau_\mathcal{EL}}(E_i))$. This is a consequence of $v_i$ being the root of the description tree corresponding to $u_{\tau_\mathcal{EL}}(E_i)$, and the fact that $h_w(v_0)$ is maximal.

Therefore, $\text{deg}_r(d, u_{\tau_\mathcal{EL}}(E))$ can be expressed as:

$$\frac{|\{P_1, \ldots, P_m\} \cap \ell_{\tau}(d)| + \sum_{i=1}^{n} \max\{|\text{deg}_r(e, u_{\tau_{\mathcal{EL}}}(E_i))| \mid (d, e) \in (r_s)^T\}}{m + n}$$

(6.12)

Thus, knowing the values $\text{deg}_r(e, u_{\tau_\mathcal{EL}}(F))$ for all successors $e$ of $d$ in $\mathcal{I}_p$ and all $F \in \{E_1, \ldots, E_n\}$, it is straightforward to compute $\text{deg}_r(d, u_{\tau_\mathcal{EL}}(E))$. Therefore, similar to the computation of $\text{Ex}$ the assignment $D$ for $d_0$ can be computed by using all the assignments $D$ recursively computed for all successors of $d_0$ in $\mathcal{I}_p$. Once more, the problem related to the possible exponentially many successors of $d_0$ needs to be addressed. Here is where the aforementioned table $v$ comes into play. It is defined as $v : (N_R \times \text{def} (\tau_\mathcal{EL})) \cup \epsilon \times N_{\text{prim}} \rightarrow [0, 1]$ and each entry $v[r, E]$ stores the value $\max\{|D_c(E)| \mid (d_0, e) \in r^T\}$, where $D_c$ is the assignment $D$ for $c$, and $v[e, P] = 1$ iff $P \in \mathcal{P}$ (0 otherwise). The following fragment of pseudo-code updates $v$ within a run of $A$:

1. $v[r, E] = 0$ for all $(r, E) \in (N_R \times \text{def}(\tau_\mathcal{EL})) \cup \epsilon \times N_{\text{prim}}$  // Initialization
2. $v[e, P] = 1$ iff $P \in \mathcal{P}$
3. 
4. $D^i_r := A(d - 1, b^i_r + 1)$
5. for all $(E \in C_E \in \tau_{\mathcal{EL}})$ do
6.   if $D^i_r(E) > v[r, E]$ then
7.     $v[r, E] := D^i_r(E)$
8. end if
9. end for

Here, $D^i_r$ stands for the assignment $D$ corresponding to the root element of the interpretation induced by the recursive call. In other words, the $i$-th $r$-successor of $d_0$ in $\mathcal{I}_p$. After all the recursive calls have been executed, $v$ is used to compute $D$ as described in the following subroutine:

**procedure** SUBDEG($v : (N_R \times \text{def}(\tau_\mathcal{EL})) \cup \epsilon \times N_{\text{prim}} \rightarrow [0, 1]$)

for all $(E \in C_E \in \tau_{\mathcal{EL}})$ do
  $c := |\{P \mid P \in tl(C_E) \text{ and } v[e, P] = 1\}|$
  for all $\exists r.E' \in tl(C_E)$ do
    $c := c + v[r, E']$
  end for

$D(E) := \frac{c}{|\mathcal{H}(C_E)|}$
end for

return $D$
end procedure

It remains to see the details of the computation of $\text{Ex}$. The updating of the relation $z$ in $A$ is carried out as follows:
Chapter 6. Adding Terminologies to $\tau\mathcal{EL}(m)$

1: $z := \{(\epsilon, P) \mid P \in \mathcal{P}\}$ // Initialization
2: $
3: \text{Ex}_1 := A(\emptyset - 1, b^i + 1)
4: \text{for all } (\alpha \models \hat{\mathcal{C}}_\alpha \in \hat{T}) \text{ do}
5: \quad \text{if } \alpha \in \text{Ex}_1 \text{ then}
6: \quad \quad \text{z} := \text{z} \cup \{(r, \alpha)\}
7: \quad \text{end if}
8: \text{end for}

Then, using $D$ and $z$ Conditions 1 and 2 can be verified, and $\text{Ex}$ can be computed in the following way:

\textbf{procedure} SUBEx ($D : \text{def}(\mathcal{T}_{\mathcal{EL}}) \rightarrow [0, 1], z \subseteq (\mathbb{N}_R \times \text{def}(\hat{T})) \cup (\epsilon \times \mathbb{N}_{\text{prim}})$)

\begin{itemize}
  \item $s := \emptyset$
  \item $\text{for all } (\alpha \models \hat{\mathcal{C}}_\alpha \in \hat{T}) \text{ do}$
     \begin{itemize}
       \item $\text{if } ([P \in tl(\hat{\mathcal{C}}_\alpha)] \Rightarrow (\epsilon, P) \in z) \text{ and } ([E \sim t \in tl(\hat{\mathcal{C}}_\alpha)] \Rightarrow D(E) \sim t) \text{ and } ([\exists r, \beta \in tl(\hat{\mathcal{C}}_\alpha)] \Rightarrow (r, \beta) \in z)$ then
         \begin{itemize}
           \item $s := s \cup \{\alpha\}$
         \end{itemize}
     \end{itemize}
  \item $\text{end if}$
  \item $\text{end for}$
\end{itemize}

\textbf{return} $s$

\textbf{end procedure}

Thus, using the function $A$ we define our non-deterministic algorithm to decide satisfiability of concepts of the form $\alpha_1 \sqcap \neg \alpha_2$ with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes.

\textbf{Algorithm 5} Satisfiability of $\alpha_1 \sqcap \neg \alpha_2$ w.r.t. acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes.

\textbf{Input:} An acyclic $\tau\mathcal{EL}(\text{deg})$ TBox $\hat{T}$ and two defined concepts $\alpha_1, \alpha_2$ in $\hat{T}$.

\textbf{Output:} “yes”, if $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable in $\hat{T}$, “no” otherwise.

1: $b := s(u_{\hat{T}}(\alpha_1)) \times s(u_{\hat{T}}(\alpha_2))$ // $b$ is stored in binary
2: $d := \text{rd}(u_{\hat{T}}(\alpha_1)) + \text{rd}(u_{\hat{T}}(\alpha_2))$
3: $(\text{Ex}, D) := A(d, b)$
4: if $\alpha_1 \in \text{Ex}$ and $\alpha_2 \notin \text{Ex}$ then
5: \quad \text{return “yes”}
6: \text{end if}
7: \text{return “no”}

Since $A$ terminates, this implies that Algorithm 5 terminates as well. In the following, we show that Algorithm 5 is sound and complete. Let us start by showing that $A$ computes the right values for $D$ and $\text{Ex}$.

\textbf{Lemma 6.34.} Let $d \geq 0$ and $b > 0$ be two natural numbers, and $\rho$ be a run of $A$ on $(d, b)$. Then,

1. $D(E) = \text{deg}\mathcal{T}_\rho(d_0, u_{\mathcal{T}_{\mathcal{EL}}}(E)), \text{ for all } E \models C_E \in \mathcal{T}_{\mathcal{EL}}.$
2. $\text{Ex} = \{\alpha \mid \alpha \models \hat{\mathcal{C}}_\alpha \in \hat{T} \text{ and } d_0 \in \alpha^{\mathcal{T}_\rho}\}$
Proof. Let $d(I_\rho)$ denote the depth of $T_{\rho}$. We prove our claims by induction on $d(I_\rho)$. To start, we fix a role name $r \in N_R$ and define $r(d_0) = \{e_1,\ldots,e_n\}$ to be the set of $r$-successors of $d_0$ in $I_\rho$ (with $n \geq 0$). By construction of $T_{I_\rho}$, $\rho$ does exactly $n$ recursive calls $A(0 - 1, b_i)$ ($1 \leq i \leq n$). Let $\rho_i^r$ denote the run corresponding to the $i$-th call. Then, the interpretation $I_{\rho_i^r}$ induced by $\rho_i^r$ is the one having the description tree $T_{I_\rho}[e_i]$, i.e., the subtree of $T_{I_\rho}$ rooted at $e_i$.

The tree shape of $I_\rho$ implies that $d(I_{\rho_i^r}) < d(I_\rho)$. Therefore, induction hypothesis can be applied to obtain:

$$D_1^t(E) = deg^{I_{\rho_i^r}}(e_i, u_{T_{\rho}(E)})$$
$$Ex^i_\rho = \{\alpha \mid \alpha \models \hat{C}_\alpha \in \hat{T} \text{ and } e_i \in \alpha^{I_{\rho_i^r}}\}$$

The same reasoning applies for all the other role names $s \in N_R$. Note that since $I_{\rho_i^r}$ is a subtree of $I_\rho$, those two equalities are also valid for $I_\rho$, i.e.:

$$D_1^t(E) = deg^{I_{\rho_i^r}}(e_i, u_{T_{\rho_i^r}(E)})$$
$$Ex^i_\rho = \{\alpha \mid \alpha \models \hat{C}_\alpha \in \hat{T} \text{ and } e_i \in \alpha^{I_{\rho_i^r}}\}$$

Therefore, after all the recursive calls have been executed and the values in table $v$ and relation $z$ have been fully updated, we have for all $(r, E) \in N_R \times \text{def}(T_{\rho})$:

$$v[r, E] = \max\{deg^{I_{\rho_i^r}}(e, u_{T_{\rho_i^r}(E)}) \mid (d_0, e) \in r^{I_{\rho_i^r}}\} \quad (6.13)$$

and,

$$z = \{(r, \alpha) \mid e \in \Delta^{I_{\rho_i^r}}, (d_0, e) \in r^{I_{\rho_i^r}} \text{ and } e \in \alpha^{I_{\rho_i^r}}\} \quad (6.14)$$

Looking at the subroutine SUBdeg, for all $E = C_E \in T_{\rho}$ the value $D(E)$ is computed by the following expression:

$$D(E) = \frac{|\text{tl}(C_E) \cap \mathcal{P}| + \sum_{3r, E' \in \text{tl}(C_E)} v[r, E']}{\text{tl}(C_E)}$$

Now, by construction of $I_\rho$ we have that $\ell_{I_\rho}(d_0) = \mathcal{P}$. Hence, replacing $v[r, E']$ by the right-hand side of the equality in (6.13) we obtain the expression in (6.12). Consequently, we have shown that:

$$D(E) = deg^{I_{\rho_i^r}}(d_0, u_{T_{\rho_i^r}(E)})$$

Last, let $\alpha \models \hat{C}_\alpha \in \hat{T}$ with $\hat{C}_\alpha$ of the form:

$$\hat{P}_1 \sqcap \ldots \sqcap \hat{P}_n \sqcap \exists r_1.\alpha_1 \sqcap \ldots \sqcap \exists r_m.\alpha_m$$

According to SUBEX, $\alpha \in Ex$ iff:

- for all $1 \leq i \leq n$: if $\hat{P}_i$ is of the form $E \sqcap \sim t$ then $D(E) \sim t$, otherwise $\hat{P}_i \in \mathcal{P}$, and
- $(r_j, \alpha_j) \in z$, for all $1 \leq j \leq m$.

Since $\ell_{I_\rho}(d_0) = \mathcal{P}$ and $D(E) = deg^{I_{\rho_i^r}}(d_0, u_{T_{\rho_i^r}(E)})$, the first statement is equivalent
to have $d_0 \in (\bar{P}_i)^{T_{\rho}}$ ($1 \leq i \leq n$). Furthermore, (6.14) makes the second statement equivalent to having $d_0 \in (\exists j, \alpha_j)^{T_{\rho}}$ ($1 \leq j \leq m$). Thus, $\alpha \in \text{Ex}$ iff $d_0 \in \alpha^{T_{\rho}}$.

Note that the base case for the induction is already contained in the proof. □

Using Lemma 6.34 we now prove that Algorithm 5 is sound and complete.

**Lemma 6.35.** Let $\hat{T}$ be an acyclic $\tau\mathcal{EL}(\deg)$ $T$Box and $\alpha_1, \alpha_2$ two defined concepts in $\hat{T}$. Then,

Algorithm 5 answers “yes” iff $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable in $\hat{T}$.

**Proof.** ($\Rightarrow$) Suppose that the algorithm gives a positive answer and let $\rho$ be the run of function $A$ that leads to it. Then, we can talk about the interpretation $I_{\rho}$ induced by $\rho$. The “yes” answer means that for $\rho$, $\alpha_1 \in \text{Ex}$ and $\alpha_2 \not\in \text{Ex}$. Then, the application of Lemma 6.34 yields:

$$d_0 \in (\alpha_1 \sqcap \neg \alpha_2)^{T_{\rho}}$$

with $d_0 \in \Delta^{T_{\rho}}$. Hence, $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable with respect to $\hat{T}$.

($\Leftarrow$) Assume that $\alpha_1 \sqcap \neg \alpha_2$ is satisfiable with respect to $\hat{T}$. This means that there exists an interpretation $I$ such that $I \models \hat{T}$ and $(\alpha_1 \sqcap \neg \alpha_2)^{\hat{T}} \neq \emptyset$. By Proposition 6.31 and its subsequent remarks one can assume that $I$ is tree-shaped and satisfies the following properties:

1. $\Delta^I$ has at most $s(u_{\hat{T}}(\alpha_1)) \times s(u_{\hat{T}}(\alpha_2))$ elements,

2. the depth of $T_I$ is not greater than $rd(u_{\hat{T}}(\alpha_1)) + rd(u_{\hat{T}}(\alpha_2))$, and

3. its root element $d_0$ satisfies: $d_0 \in (\alpha_1 \sqcap \neg \alpha_2)^{I}$.

The selection of $\delta$ and $b$ in Algorithm 5 and the application of Lemma 6.33 guarantee the existence of a run $\rho$ of $A$ on $(\delta,b)$ generating the restriction of $I$ to $N_{\text{prim}} \cup N_R$. Hence, the application of Lemma 6.34 implies that the conditional in line 4 must evaluate to true for such a run $\rho$. Thus, Algorithm 5 answers “yes”. □

Algorithm 5 uses space polynomial in the size of $\hat{T}$ to store the binary representation of $b$. Furthermore, $z$ and $v$ are also stored within polynomial space, and the two subroutines run in polynomial time. Therefore, since each run $\rho$ of $A$ on $(\delta,b)$ does at most $\delta$ many nested recursive calls, $\rho$ uses space polynomial in $s(\hat{T})$. In addition, it is easy to see that both $b$ and $\delta$ can be computed in time polynomial in $s(\hat{T})$. Thus, Algorithm 5 is a non-deterministic polynomial space decision procedure for satisfiability of concepts of the form $\alpha_1 \sqcap \neg \alpha_2$ with respect to acyclic $\tau\mathcal{EL}(\deg)$ $T$Boxes. This means that satisfiability and non-subsumption are in NPSPACE. Then, by Savitch’s theorem [Sav70] and since PSPACE is closed under complement, we obtain the following results.

**Lemma 6.36.** In $\tau\mathcal{EL}(\deg)$, satisfiability and subsumption are in PSPACE, with respect to acyclic $\tau\mathcal{EL}(\deg)$ $T$Boxes.
6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}\text{(deg)}$ TBoxes

We show in this section that satisfiability and subsumption are still decidable in PSPACE with respect to acyclic knowledge bases. Furthermore, we also consider the consistency and the instance problem. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}\text{(deg)}$ knowledge base:

- $\mathcal{K}$ is consistent iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$.

Additionally, let $a \in \mathbb{N}_I$ be an individual name and $\alpha$ a defined concept in $\mathcal{A}$:

- $a$ is an instance of $\alpha$ with respect to $\mathcal{K}$ iff for all models $\mathcal{I}$ of $\mathcal{K}$ it holds that $a^\mathcal{I} \in \alpha^\mathcal{I}$.

Without loss of generality, we can restrict our attention to the consistency problem for KBs of the form $(\mathcal{T}, \mathcal{A} \cup \{\neg \alpha(a)\})$, since all the other problems can be reduced to it.

**Proposition 6.37.** Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}\text{(deg)}$ KB, $\alpha, \alpha_1$ and $\alpha_2$ defined concepts in $\mathcal{T}$ and $a \in \mathbb{N}_I$. Then,

1. $\alpha$ is satisfiable with respect to $\mathcal{K}$ iff $(\mathcal{T}, \mathcal{A} \cup \{\alpha(b)\})$ is consistent, where $b$ is an individual name not occurring in $\mathcal{A}$.

2. $\alpha_1$ is subsumed by $\alpha_2$ with respect to $\mathcal{K}$ (in symbols $\alpha_1 \sqsubseteq_K \alpha_2$) iff the knowledge base $(\mathcal{T}, \mathcal{A} \cup \{\alpha_1(b), \neg \alpha_2(b)\})$ is inconsistent, where $b$ is an individual name not occurring in $\mathcal{A}$.

3. $a$ is an instance of $\alpha$ in $\mathcal{K}$ (in symbols $\mathcal{K} \models \alpha(a)$) iff $(\mathcal{T}, \mathcal{A} \cup \{\neg \alpha(a)\})$ is not consistent.

Further, since $\mathcal{T}$ is acyclic, by using unfolding we can again get rid of the TBox and reduce reasoning to consistency with respect to the empty terminology. The unfolding of a $\tau\mathcal{EL}\text{(deg)}$ ABox $\mathcal{A}$ with respect to $\mathcal{T}$ is defined as follows:

$$u_{\mathcal{T}}(\mathcal{A}) := \bigcup_{\alpha(a) \in \mathcal{A}} \{[u_{\mathcal{T}}(\alpha)](a)\} \cup \bigcup_{r(a,b) \in \mathcal{A}} \{r(a,b)\}$$

**Proposition 6.38.** Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}\text{(deg)}$ KB, $\alpha$ a defined concept in $\mathcal{T}$ and $a \in \mathbb{N}_I$. $(\mathcal{T}, \mathcal{A} \cup \{\neg \alpha(a)\})$ is consistent iff $u_{\mathcal{T}}(\mathcal{A}) \cup \{[\neg u_{\mathcal{T}}(\alpha)](a)\}$ is consistent.

In what follows, we show how to reuse the idea of Algorithm 5 to decide consistency of $u_{\mathcal{T}}(\mathcal{A}) \cup \{[\neg u_{\mathcal{T}}(\alpha)](a)\}$. Lemma 6.16 tells us that if $u_{\mathcal{T}}(\mathcal{A}) \cup \{[\neg u_{\mathcal{T}}(\alpha)](a)\}$ is consistent, then it has a model $\mathcal{J}$ of the following form.
where \( \text{Ind}(A) = \{a_1, a_2, \ldots, a_p\} \) and \( \mathcal{I}_{a_1}, \mathcal{I}_{a_2}, \ldots, \mathcal{I}_{a_p} \) are tree-shaped interpretations. The inner area of the diagram consists of the satisfaction of the role assertions in \( \mathcal{A} \), i.e., \( (a^J, b^J) \in r^J \) \( \text{iff} \ r(a, b) \in \mathcal{A} \). Additionally, Lemma 5.14 provides an upper bound for the size of these tree-shaped interpretations. We will later talk about how big this bound could be, but for the moment let us focus in how to reuse Algorithm 5.

To start, it is clear that by choosing the appropriate values for \( d \) and \( b \), the interpretations \( \mathcal{I}_a \) can be independently generated using the function \( A \). It is important to keep in mind that \( a^I_a \) is the root of \( \mathcal{I}_a \). Consequently, a run \( \rho_a \) of \( A \) inducing \( \mathcal{I}_a \) will compute two sets \( Ex_a \) and \( D_a \) with the following meaning:

\[
D_a(E) = \deg^\mathcal{I}_a(a^\mathcal{I}_a, u_{\mathcal{E}\mathcal{L}}(E)), \quad \text{for all } E \models C_E \in T_{\mathcal{E}\mathcal{L}}
\]

\[
Ex_a = \{\beta \mid \beta \doteq \hat{C}_\beta \in \hat{T} \text{ and } a^\mathcal{I}_a \in \beta^\mathcal{I}_a\}
\]

Recall that technically \( \mathcal{I}_a \) (as generated by \( A \)) only interprets symbols from \( N_{\text{prim}} \cup N_R \), but when writing \( \beta^\mathcal{I}_a \) we meant its unique extension to a model of \( \hat{T} \). The veracity of the previous two equalities has been shown in Lemma 6.34. Now, the construction of the model \( \mathcal{J} \) depicted above combines all those interpretations in the following way (see Lemma 5.15):

- \( \Delta^J = \bigcup_{a \in \text{Ind}(A)} \Delta^\mathcal{I}_a \)
- \( A^J = \bigcup_{a \in \text{Ind}(A)} A^\mathcal{I}_a \) for all \( A \in N_{\text{prim}} \),
- \( r^J = \bigcup_{a \in \text{Ind}(A)} r^\mathcal{I}_a \cup \{(a^\mathcal{I}_a, b^\mathcal{I}_a) \mid r(a, b) \in \mathcal{A}\} \) for all \( r \in N_R \), and
- \( a^J = a^\mathcal{I}_a \) for all \( a \in \text{Ind}(A) \).

This means that given an individual \( a \in \text{Ind}(A) \), a defined concept \( \beta \) and an element \( d \in \Delta^\mathcal{I}_a \), it is not necessarily the case that \( d^J \in \beta^J \) \( \text{iff} \ \beta \in Ex_d \) (similarly for the membership degrees and the assignment \( D_d \)). The reason is that the role assertions between individual names are used to build \( \mathcal{J} \), but they are not taken into account by \( \rho_a \) to compute \( Ex_d \) and \( D_d \). Fortunately, this could only be the case for the domain elements \( a^J = a^\mathcal{I}_a \) corresponding to the individual names of \( \mathcal{A} \). This is a consequence of something that we have already observed in Chapter 5 for all \( a \in \text{Ind}(A) \) and \( d \in \Delta^\mathcal{I}_a \) such that \( d \neq a^\mathcal{I}_a \), no path in \( G_{\mathcal{J}} \) starting at \( d \) reaches a domain element \( b^\mathcal{I}_a \) (\( b \in \text{Ind}(A) \)). As a result we obtain the following:

\[
\deg^J(d, u^J(E)) = \deg^\mathcal{I}_a(d, u^\mathcal{I}_a(E)), \quad \text{for all } E \models C_E \in T_{\mathcal{E}\mathcal{L}}
\]

\[
d \in \beta^J \text{ iff } d \in \beta^\mathcal{I}_a, \quad \text{for all } \beta \doteq \hat{C}_\beta \in \hat{T}
\]

Therefore, if we can compute the correct content/values of \( Ex_a \) and \( D_a \) for the unique extension of \( \mathcal{J} \) satisfying \( \hat{T} \), it will be possible to verify whether \( \mathcal{J} \) satisfies \( u^\mathcal{F}(A) \cup \{[-u^\mathcal{F}(a)](a)\} \) (as it is done for subsumption in the previous section). There are two obstacles that we need to overcome. The first one is that \( Ex_a \) and \( D_a \), as computed by \( \rho_a \), do not contain enough information to obtain the ones corresponding to \( \mathcal{J} \).
6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}(\text{deg})$ TBoxes

Example 6.39. Let $a_1, a_2 \in \text{Ind}(\mathcal{A})$ and $r(a_1, a_2) \in \mathcal{A}$. Suppose that a run $\rho_{a_1}$ of $\mathcal{A}$ representing $\mathcal{I}_{a_1}$ yields $D_{a_1}(E) = t_1$ for some $E \models C_E \in \mathcal{T}_{\mathcal{EL}}$. Likewise, $D_{a_2}(E') = t_2$ for some run $\rho_{a_2}$ representing $\mathcal{I}_{a_2}$ and $E' \models C_{E'} \in \mathcal{T}_{\mathcal{EL}}$. In addition, there is a top-level atom in $C_E$ of the form $\exists v E'$.

As explained above, the value of $D_{a_2}(E')$ has not been considered in the computation of $D_{a_1}(E)$, and it may well be the case that it actually affects $D_{a_1}(E)$ in the big model $\mathcal{J}$, i.e., $\text{deg}^\mathcal{J}((a_1)^\mathcal{J}, u_{\mathcal{J}}(E)) > t_1$. This could happen if for all $r$-successors $d$ of $a_1$ in $\mathcal{I}_{a_1}$, we have that $\text{deg}^\mathcal{J}(d, u_{\mathcal{J}}(E')) < t_2$. Clearly, this is not something that can be inferred from $D_{a_1}$, but from the table $v$ computed for $a_1$ by $\rho_{a_1}$.

Similarly, assume that $\beta \notin \text{Ex}_{a_1}$ for some $\beta \models C_{\beta} \in \mathcal{T}_\mathcal{F}$. This means that $a^\mathcal{F}_\beta \notin \beta^\mathcal{F}_\mathcal{A}$. It could happen that $a_2$ satisfies properties in $\mathcal{J}$ that would make $(a_1)^\mathcal{J} \notin \beta^\mathcal{J}$. Then, we would need to look into the relation $z$ computed for $a_1$ by $\rho_{a_1}$, to discern such a change. ◊

To deal with that, we rearrange the structure of function $A$ such that it returns the pair $(z, v)$ instead of $(\text{Ex}, D)$. The following sketches how to modify $A$ accordingly.

1: function $A(\mathfrak{d} : \text{integer}, \mathfrak{b} : \text{binary})$

to (z, v)
12: end function

Note that in the previous version of $A$, the computation of $D^r_\mathfrak{d}$ and $\text{Ex}^r_\mathfrak{d}$ are the last operations executed inside the recursive call $A(\mathfrak{d} - 1, \mathfrak{b}^r_\mathfrak{d} + 1)$, and $v, z$ are updated right away after that. This order of actions is kept in the new definition given above. Since the computation of $D^r_\mathfrak{d}$ and $\text{Ex}^r_\mathfrak{d}$ only requires of $v^r_\mathfrak{d}$ and $z^r_\mathfrak{d}$, and these are returned by $A$, the new modifications preserve the properties of $A$.

The next step is to recompute $z_a$ and $v_a$ for all $a \in \text{Ind}(\mathcal{A})$ using the information provided by the role assertions in $\mathcal{A}$. Following Example 6.39, since $b^\mathcal{J}$ is related to $a^\mathcal{J}$ by the role name $r$, this means that $v_a$ and $z_a$ must be updated with respect to $r$, $\text{Ex}_a$, and $D_b$. Obviously, changes in $v_a$ and $z_a$ should be propagated to the individuals that $a$ is related to, and so on. The function $A$ can cope with such propagation in a bottom-up form, because it is computing a tree-shaped structure. However, this is no longer the case for the individuals in $\mathcal{A}$, since role assertions can define cycles involving them.

To solve this we appeal to the acyclic nature of $\mathcal{T}_\mathcal{F}$ and $\mathcal{T}_{\mathcal{EL}}$. It allows to traverse the structure of any defined concept (bottom-up) based on the partial order $\preceq$ induced by $\to^+$ on $\text{def}(\mathcal{T})$. Note that now we limit our attention to the fragment of $\mathcal{J}$ corresponding to the role assertions in $\mathcal{A}$, which is part of the input. Therefore, provided that $(z_a, v_a)$
has been computed for all $a \in \text{Ind}(\mathcal{A})$, the following subroutine updates all those pairs with respect to the combined interpretation $\mathcal{J}$.

1: **procedure** Update()  
2: compute $D_a := \text{SUBdeg}(v_a)$  
3: let $\{E_1, \ldots, E_n\}$ be a post-order of $\preceq$ (induced by $\rightarrow^+$ on $\text{def}(\mathcal{T}_{\mathcal{EL}})$)  
4: **for all** $1 \leq i \leq n$ **do**  
5: for all $(a, b) \in \mathcal{A}$ **do**  
6: if $D_b(E_i) > v_a[r, E_i]$ **then**  
7: $v_a[r, E_i] := D_b(E_i)$  
8: **end if**  
9: **end for**  
10: re-compute $D_a$  
11: **end for**  
12: compute $\text{Ex}_a := \text{SUBex}(D_a, z_a)$  
13: let $\{\beta_1, \ldots, \beta_n\}$ be a post-order of $\preceq$ on $\text{def}(\hat{T})$  
14: **for all** $1 \leq i \leq n$ **do**  
15: for all $(a, b) \in \mathcal{A}$ **do**  
16: if $\beta_i \in \text{Ex}_b$ **then**  
17: $z_a := z_a \cup \{(r, \beta_i)\}$  
18: **end if**  
19: **end for**  
20: re-compute $\text{Ex}_a$  
21: **end for**  
22: **end procedure**

Let us prove that Update does what we have claimed.

**Lemma 6.40.** For all $a \in \text{Ind}(\mathcal{A})$, let $\rho_a$ be a run of $\mathcal{A}$ and $I_a$ its induced interpretation. Moreover, based on these interpretations let $\mathcal{J}$ be the interpretation that results from the combination presented in Lemma 5.15. Then,

1. $D_a(E) = \text{deg}^\mathcal{J}(a^\mathcal{J}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$, for all $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$.

2. $\text{Ex}_a = \{\beta \mid \beta \doteq \hat{C}_\beta \in \hat{T} \text{ and } a^\mathcal{J} \in \beta^\mathcal{J}\}$

**Proof.** We give the proof for the assignments $D_a$. The case for $\text{Ex}_a$ can be done using the same idea and Lemma 6.34. To differentiate the final assignment $D_a$ from the initial one computed by $\rho_a$, we denote the latter as $D_a^0$ (likewise for $v_a$ and $v_a^0$). We show the claim by well-founded induction on the partial order $\preceq$.

Let $a \in \text{Ind}(\mathcal{A})$ and $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$. Since $\mathcal{T}_{\mathcal{EL}}$ is normalized, the concept description $C_E$ has the following structure:

$$P_1 \cap \ldots \cap P_n \cap \exists r_1. E_1 \cap \ldots \cap \exists r_m. E_m$$

Clearly, when $m = 0$ the value $\text{deg}^\mathcal{J}(a^\mathcal{J}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$ does not depend on any successor of $a^\mathcal{J}$. Moreover, by construction of $\mathcal{J}$ we know that $a^\mathcal{J} \in (P_i)^\mathcal{J}$ if $a^{I_a} \in (P_i)^{I_a}$ for all $1 \leq i \leq n$. This implies that:

$$\text{deg}^\mathcal{J}(a^\mathcal{J}, u_{\mathcal{T}_{\mathcal{EL}}}(E)) = \text{deg}^{I_a}(a^{I_a}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$
Then, by Lemma \[6.34\] we obtain that:

\[ D_0^E(a) = \text{deg}^J(a, u_{\tau\mathcal{L}}(E)) \]

Looking at SUBDEG one can see that the computation of \( D_0^E(E) \) depends only on the values \( v_0^b[\epsilon, P] \). Furthermore, it is easy to see that those values are never changed by a run of UPDATE. Hence, \( v_a[\epsilon, P] = v_0^b[\epsilon, P] \) and \( D_a(E) = D_0^E(E) \). Thus, \( D_a(E) \) is the right number.

Now, to show the claim for \( m > 0 \) we start by making some observations for all \( b \in \text{Ind}(A) \). Let \( F \) be a defined concept in \( T_{\tau\mathcal{L}} \):

- By Lemma \[6.34\] the initial table \( v_0^b \) satisfies the following:

\[ v_0^b[r, F] = \max\{\text{deg}^I_s(d, u_{\tau\mathcal{L}}(F)) \mid d \in \Delta^I_s \text{ and } (b^I_s, d) \in r^I_s\} \]

As explained above, since \( d \neq b^I_s \) it further satisfies:

\[ v_0^b[r, F] = \max\{\text{deg}^J(d, u_{\tau\mathcal{L}}(F)) \mid d \in \Delta^J_s \text{ and } (b^J_s, d) \in r^J_s\} \quad (6.15) \]

Additionally, let \( j \) be the index of \( F \) in the post-order created in line \[3\]. Then,

- the value of \( v_b[r, F] \) only changes at the \( j \)th iteration of the outer-loop in line \[4\].
- let \( k \) be the largest index of \( F' \) among all the top-level atoms of the form \( \exists r.F' \) in the definition of \( F \). Then, taking into account the previous statement, the value of \( D_b(F) \) never changes after the \( k \)th iteration of the outer-loop.
- since \( F' \preceq F \), this means that \( j > k \). Consequently, the final value of \( D_b(F) \) is computed before the iteration corresponding to \( F \).

Coming back to the defined concept \( E \), we know that \( E \preceq E_j \) for all \( 1 \leq j \leq m \).

Then, the application of induction hypothesis yields:

\[ D_a(E_j) = \text{deg}^J(a, u_{\tau\mathcal{L}}(E_j)) \quad (6.16) \]

Moreover, since at the moment of updating \( v_a[r, E_j] \) the value of \( D_b(E_j) \) is the one in \( (6.16) \) for all \( b \in \text{Ind}(A) \), using \( (6.15) \) we obtain:

\[ v_a[r, E_j] = \max\{\text{deg}^J(d, u_{\tau\mathcal{L}}(E_j)) \mid d \in \Delta^J_s \text{ and } (a^J, d) \in r^J_s\} \]

Thus, by the same arguments given in Lemma \[6.34\] it follows:

\[ D_a(E) = \text{deg}^J(a, u_{\tau\mathcal{L}}(E)) \]

By the previous lemma, once \( (E_a, D_a) \) has been computed by UPDATE for all \( a \in \text{Ind}(A) \), it is easy to verify whether \( J \) satisfies \( A \cup \{\neg \alpha(a)\} \). Therefore, it remains to make sure that enough candidates \( \mathcal{J} \) are considered to decide the satisfiability status of \( u_{\tau\mathcal{L}}(A) \cup \{\neg u_{\tau\mathcal{L}}(\alpha)|a\}\). This relies on estimating the appropriate values for \( d \) and \( b \).
• Let $m_{rd}(A)$ be the maximal role depth of a concept $\hat{D}$ occurring in an ABox $A$, i.e.,

$$m_{rd}(A) := \max\{\rd(\hat{D}) \mid \hat{D}(a) \in A\}$$

Coming back to Chapter 5, the construction described in Lemma 5.16 to obtain a bounded model $J$ of $A \cup \{\neg \hat{C}(a)\}$, uses Lemma 5.14 to obtain the interpretations $I_a$ for all $a \in \text{Ind}(A)$. Basically, $I_a$ is built by extending the description tree of $A(a)$ with canonical interpretations representing threshold concepts that either occur in $A$ or are sub-descriptions of $\hat{C}$. Hence, it is not hard to see that the depth $d(I_a)$ of $I_a$ can be bounded by:

$$d(I_a) \leq m_{rd}(A(a)) + \rd(\hat{C})$$

In the present context this means that $d_a$ can be chosen as:

$$m_{rd}(u_{\hat{T}}(A(a))) + \rd(u_{\hat{T}}(\alpha))$$

By Lemma 5.16 we have an upper-bound for $|\Delta I_a|$, namely,

$$|\Delta I_a| \leq s(A(a)) \times [s(\hat{C})]^u$$

where $u = |\text{sub}(\hat{C})|$. Translating this bound to our current setting, we obtain:

$$|\Delta I_a| \leq s(u_{\hat{T}}(A(a))) \times [s(u_{\hat{T}}(\alpha))]^{u^*}$$

with $u^*$ now being $|\text{sub}(u_{\hat{T}}(\alpha))|$.

Putting all the given arguments together, we devise Algorithm 6 as a non-deterministic procedure to decide consistency of $(\hat{T}, A \cup \{\neg \alpha(a)\})$. The following lemma shows that it is correct.

**Lemma 6.41.** Let $K = (\hat{T}, A)$ be an acyclic $\tau\mathcal{EL}(\deg)$ KB, $\alpha$ a defined concept in $\hat{T}$ and $a \in \text{Ind}(A)$. Then,

Algorithm 6 answers “yes” iff $(\hat{T}, A \cup \{\neg \alpha(a)\})$ is consistent.

**Proof.** ($\Rightarrow$) Suppose that the algorithm gives a positive answer, and for all $a \in \text{Ind}(A)$ let $\rho_a$ be the run of $A$ that leads to it. Then, we can talk about the interpretation $I_a$ induced by $\rho_a$. Now, let $J$ be the interpretation that results from the combination of all the fragments $I_a$ and the role assertions occurring in $A$ (as done in Lemma 5.15). A “yes” answer implies that the for loop described between lines 7 and 13 never falsifies $\beta \in \text{Ex}_b$ for all concept assertions $\beta(b) \in A$. By Lemma 6.40 this means that the extension of $J$ satisfying $\hat{T}$ is also a model of $A$.

In addition, the conditional in line 14 must evaluate to true. Consequently, for the same reasons explained above, we obtain that $a^J \notin \alpha^J$. Thus, $(\hat{T}, A \cup \{\neg \alpha(a)\})$ is consistent.

($\Leftarrow$) Conversely, assume that $(\hat{T}, A \cup \{\neg \alpha(a)\})$ is consistent. This means that there is an interpretation $J \models K$ such that $a^J \notin \alpha^J$. By Proposition 6.38 and Lemma 5.16
6.4 Reasoning with respect to acyclic $\tau EL(dg)$ TBoxes

By the selection of $d$ and $b$ in Algorithm 6 and an application of Lemma 6.33, there is always a run $\rho_a$ of $A$ generating $\mathcal{I}_a$ for all $a \in \text{Ind}(A)$. Then, by Lemma 6.40, after executing $\text{Update}$ none of the subsequent conditionals could evaluate to $false$. Thus, the algorithm answers “yes”.

---

Algorithm 6 Consistency of $(\widehat{T}, A \cup \{\neg \alpha(a)\})$.

**Input:** An acyclic KB $(\widehat{T}, A)$, a defined concept $\alpha$ in $\widehat{T}$ and $a \in \mathbb{N}_1$.

**Output:** “yes”, if $(\widehat{T}, A \cup \{\neg \alpha(a)\})$ is consistent, “no” otherwise.

1. $b := s(\alpha) \times [s(\alpha)]^{u^*}$ // $b$ is represented in binary
2. $d := m_{rd}(\alpha) + r_d(\alpha)$
3. for all $b \in \text{Ind}(A)$ do
4. $(z_b, v_b) := A(d, b)$
5. end for
6. Update( )
7. for all $b \in \text{Ind}(A)$ do
8. for all $\beta(b) \in A$ do
9. if $\beta \notin \text{Ex}_b$ then
10. return “no”
11. end if
12. end for
13. end for
14. if $\alpha \notin \text{Ex}_a$ then
15. return “yes”
16. end if
17. return “no”

Regarding the computational complexity of Algorithm 6 one can see that the value of $d$ is a polynomial in the size of $K$. Furthermore, since there are polynomially many individual names, this means that any run of the algorithm uses polynomial space (including the execution of $\text{Update}$), except maybe for the number of bits needed to represent $b$. Indeed, the expression that calculates $b$ is exponential in $u^*$. To give a preliminary approximation of how big $b$ could be, we observe that due to unfolding we may end up with the following worst-case lower bounds:

$$2^s(\widehat{T}) \leq s(\alpha) \text{ and } 2^s(\widehat{T}) \leq s(\alpha)$$

In particular, $u^*$ corresponds to the number of sub-descriptions of $\alpha(\alpha)$. Hence, in view of the lower bound for the size of $u(\alpha)$ one might think that the following lower bound
also holds:

\[ 2^{2^k} \leq [s(u_\bar{\varphi}(\alpha))]^u^* \]  

(6.17)

Therefore, in the worst-case we would end up with an EXPSPACE non-deterministic procedure. However, on the one side, a closer look at the reductions in Proposition 6.37 reveals that there are better choices for \( b \) depending on the reasoning problem. On the other side, the statement in (6.17) is actually false.

- **Knowledge base consistency and satisfiability:** in these cases the problem reduces to consistency of a \( \tau\mathcal{EL}(\text{deg}) \) ABox. Consequently, such double exponential explosion does not exist. Thus, \( b \) simply becomes \( s(u_\bar{\varphi}(\mathcal{A})) \) or \( s(u_\bar{\varphi}(\mathcal{A} \cup \{\alpha(b)\})) \).

- **Subsumption:** the reduction produces an ABox of the form:

\[ \mathcal{A} \cup \{\alpha_1(b), \neg\alpha_2(b)\} \]

The key aspect is that \( b \) does not occur in \( \mathcal{A} \). This means that the pre-processing propagation of the negative assertions does not go through the cycles that may occur in \( \mathcal{A} \). This obviously avoids the exponential explosion and \( b \) can be selected as:

\[ s(u_\bar{\varphi}(\mathcal{A})) + [s(u_\bar{\varphi}(\alpha_1)) \times s(u_\bar{\varphi}(\alpha_2))] \]

- **Instance checking:** According to (6.17), in this case the algorithm would need to store a value \( b \geq 2^{2^k} \). However, one can show that the number of sub-descriptions in \( u_\bar{\varphi}(\alpha) \) is actually bounded by \( s(\bar{T}) \) (see Corollary A.3 in Appendix A). Hence, the statement made in (6.17) is false and \( b \) can be chosen as:

\[ s(u_\bar{\varphi}(\mathcal{A})) \times [s(u_\bar{\varphi}(\alpha))]^s(\bar{T}) \]

Consequently, the binary representation of \( b \) needs only polynomially many bits in the size of \( \bar{T} \).

Thus, reasoning in \( \tau\mathcal{EL}(\text{deg}) \) with respect to acyclic KBs is in PSPACE.

**Theorem 6.42.** In \( \tau\mathcal{EL}(\text{deg}) \), satisfiability, subsumption, consistency and instance checking are in PSPACE with respect to acyclic \( \tau\mathcal{EL}(\text{deg}) \) knowledge bases.
Chapter 7

Concept similarity measures, relaxed instance queries and $\tau EL(m)$

This chapter consists of three sections. First, we show how to use the relaxed instance query approach from [EPT14] to turn a concept similarity measure (CSM) $\triangleleft$ into a membership degree function $m_\triangleleft$. Such a membership degree function, however, need not be well-defined. We present two properties that when satisfied by $\triangleleft$, are sufficient to obtain well-definedness for $m_\triangleleft$. Consequently, such CSMs induce a family of DLs $\tau EL(m_\triangleleft)$. Additionally, we show that the relaxed instance queries from [EPT14] can be expressed as instance queries with respect to threshold concepts of the form $C_{\geq t}$.

Afterwards, in Section 7.2 we investigate the computational properties of such induced family of threshold DLs. We will see that there are undecidable threshold logics, but also show that computability of a CSM $\triangleleft$ is sufficient to have a decidable DL $\tau EL(m_\triangleleft)$. Moreover, we will present more specific results for logics belonging to a particular subclass of the considered family.

Last, we present the framework simi introduced in [LT12], which can be used to define a variety of CSMs. It turns out that all instances of simi satisfy the properties required to obtain well-defined graded membership functions. Then, we consider their induced threshold DLs and see how the previously investigated computational properties apply to them. We further show that a particular instance $\triangleleft_1$ of this framework turns out to be equivalent to our membership degree function $deg$.

7.1 Defining membership degree functions

In its most general form, a concept similarity measure $\triangleleft$ is a function that maps each pair of concepts $C, D$ (of a given DL) to a value $C \triangleleft D \in [0, 1]$ such that $C \triangleleft C = 1$. Intuitively, the higher the value of $C \triangleleft D$ is, the more similar the two concepts are supposed to be. Such measures can in principle be defined for arbitrary DLs, but here we restrict the attention to CSMs between $EL$ concepts, i.e., a CSM is a mapping $\triangleleft : C_{EL} \times C_{EL} \rightarrow [0, 1]$.

Ecke et al. [EPT13, EPT15] use CSMs to relax instance queries, i.e., instead of requiring that an individual is an instance of the query concept, they only require that it is an instance of a concept that is “similar enough” to the query concept.

Definition 7.1 ([EPT14, EPT15]). Let $\bowtie$ be a CSM, $A$ an $EL$ ABox, and $t \in [0, 1)$. The individual $a \in N_I$ is a relaxed instance of the $EL$ query concept $Q$ w.r.t. $A$, $\bowtie$, and the threshold $t$ iff there exists an $EL$ concept description $X$ such that $Q \bowtie X > t$ and
\[ \mathcal{A} \models X(a). \] The set of all individuals occurring in \( \mathcal{A} \) that are relaxed instances of \( Q \) w.r.t. \( \mathcal{A} \), \( \bowtie \), and \( t \) is denoted by \( \text{Relax}_\bowtie^\circ(\tau, \mathcal{A}) \).

We apply the same idea on the semantic level of an interpretation rather than the ABox level to obtain graded membership functions from similarity measures.

**Definition 7.2.** Let \( \bowtie \) be a CSM. Then, for each interpretation \( I \), we define the function
\[
 m^\bowtie_I : \Delta^I \times C_{EL} \to [0,1]
\]
\[
 m^\bowtie_I(d,C) := \max \{ C \bowtie D \mid D \in C_{EL} \text{ and } d \in D^I \}. 
\]

For an arbitrary CSM \( \bowtie \), the maximum in this definition need not exist since \( D \) ranges over infinitely many concept descriptions. However, two properties that are satisfied by many similarity measures considered in the literature are sufficient to obtain well-definedness for \( m^\bowtie \). The first is equivalence invariance:

- The CSM \( \bowtie \) is **equivalence invariant** iff \( C \equiv C' \) and \( D \equiv D' \) implies \( C \bowtie D = C' \bowtie D' \) for all \( C, C', D, D' \in C_{EL} \).

To formulate the second property, we need to recall that the role depth of an \( EL \) concept description \( C \) is the maximum nesting of existential restrictions in \( C \) (see Chapter 2 for the formal definition); equivalently, it is the height of the description tree \( T_C \). The restriction \( C_k \) of \( C \) to role depth \( k \) is the concept description whose description tree is obtained from \( T_C \) by removing all the nodes (and edges leading to them) whose distance from the root is larger than \( k \). More formally,

\[
 C_k := C \\
 C_k := [C_1]_k \sqcap \ldots \sqcap [C_n]_k \\
 [\exists r.C]_k := \begin{cases} 
 \top & \text{if } k = 0, \\
 \exists r.[C]_{k-1} & \text{otherwise.}
\end{cases}
\]

- The CSM \( \bowtie \) is **role-depth bounded** iff \( C \bowtie D = C_k \bowtie D_k \) for all \( C, D \in C_{EL} \) and any \( k \) that is larger than the minimal role depth of \( C, D \).

Role-depth boundedness implies that, in Definition 7.2, we can restrict the maximum computation to concepts \( D \) whose role depth is at most \( \text{rd}(C) + 1 \). Since it is well-known that, up to equivalence, \( C_{EL} \) contains only finitely many concept descriptions of any fixed role depth (see Proposition 13 in [BST07]), these two properties yield well-definedness for \( m^\bowtie \). For \( m^\bowtie \) to be a graded membership function, it also needs to satisfy the properties \( M1 \) and \( M2 \). To obtain these two properties for \( m^\bowtie \), we must require that \( \bowtie \) satisfies the following additional property:

- The CSM \( \bowtie \) is **equivalence closed** iff the following equivalence holds:
  \( C \equiv D \) iff \( C \bowtie D = 1 \).

**Proposition 7.3.** Let \( \bowtie \) be an equivalence invariant, role-depth bounded, and equivalence closed CSM. Then \( m^\bowtie \) is a well-defined graded membership function.
7.1 Defining membership degree functions

Proof. Let $\mathcal{I}$ be an interpretation, $d \in \Delta^\mathcal{I}$ and $C$ an $\mathcal{EL}$ concept description of role-depth $k$. Since $\bowtie$ is role-depth bounded, this means that $m^\mathcal{I}_{\bowtie}(d, C)$ can be equivalently expressed as:

$$\max\{C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}}, d \in D^\mathcal{I} \text{ and } rd(D) \leq k + 1\}$$

Now, let $D_1$ be an $\mathcal{EL}$ concept description such that $d \in [D_1]^\mathcal{I}$. Since $\bowtie$ is equivalence invariant, this means that for any other $C$ in $\mathcal{C}_{\mathcal{EL}}$, $C \bowtie D_1$ and $C \bowtie D_2$ are the same. Therefore, since there are finitely many concepts in $\mathcal{C}_{\mathcal{EL}}$ of depth at most $k + 1$ (up to equivalence), it follows that the maximum always exists.

Since $\bowtie$ is equivalence closed, it easily follows that $m_{\bowtie}$ satisfies property $M1$. As mentioned in Chapter 3, the right to left implication in $M2$ already follows from $M1$. The left to right direction is a consequence of the definition of $m_{\bowtie}$ and the fact that $\bowtie$ is equivalence invariant. Hence, $m_{\bowtie}$ satisfies property $M2$.

Thus, $m_\bowtie$ is a well-defined graded membership function. □

Consequently, an equivalence invariant, role-depth bounded, and equivalence closed CSM $\bowtie$ induces a DL $\tau\mathcal{EL}(m_{\bowtie})$. Moreover, as we show in the following, computing instances of threshold concepts of the form $Q_{> t}$ in this logic corresponds to answering relaxed instance queries with respect to $\bowtie$.

**Proposition 7.4.** Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM, $\mathcal{A}$ an $\mathcal{EL}$ ABox, and $t \in [0, 1)$. Then

$$\text{Relax}_t^\bowtie(Q, \mathcal{A}) = \{a \mid \mathcal{A} \models Q_{> t}(a) \text{ and } a \text{ occurs in } \mathcal{A}\},$$

where the semantics of the threshold concept $Q_{> t}$ is defined as in $\tau\mathcal{EL}(m_{\bowtie})$.

Proof. ($\Rightarrow$) Let $a \in \text{Ind}(\mathcal{A})$ such that $a \in \text{Relax}_t^\bowtie(Q, \mathcal{A})$. Then, there exists an $\mathcal{EL}$ concept description $X$ such that $\mathcal{A} \models X(a)$ and $Q \bowtie X > t$. Since $\mathcal{A} \models X(a)$, this means that for each interpretation $\mathcal{J}$ such that $\mathcal{J} \models X$, it happens that $a^\mathcal{J} \in X^\mathcal{J}$. Hence, by definition of $m_{\bowtie}$, we have $m_{\bowtie}^\mathcal{J}(d, Q) > t$ for all models of $\mathcal{A}$. Thus, $\mathcal{A} \models Q_{> t}(a)$.

($\Leftarrow$) Conversely, assume that $\mathcal{A} \models Q_{> t}(a)$. By definition of $m_{\bowtie}$, we know that for each model $\mathcal{J}$ of $\mathcal{A}$ there exists $X^\mathcal{J}$ such that $a^\mathcal{J} \in (X^\mathcal{J})^\mathcal{J}$ and $Q \bowtie X^\mathcal{J} > t$. However, to guarantee that $a \in \text{Relax}_t^\bowtie(Q, \mathcal{A})$, we need to show that there exists one such concept which is common for all models of $\mathcal{A}$.

To this end, consider the description graph $G(\mathcal{A})$ induced by $\mathcal{A}$. Additionally, let $\mathcal{I}_\mathcal{A}$ denote the interpretation corresponding to $G(\mathcal{A})$ such that $a^\mathcal{I}_\mathcal{A} = a$ for all $a \in \text{Ind}(\mathcal{A})$. The following facts are easy consequences of Theorem 3.9.

- $\mathcal{I}_\mathcal{A} \models \mathcal{A}$, and
- for each $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{A}$, there exists a homomorphism $\varphi_\mathcal{J}$ from $G(\mathcal{A})$ to $G_\mathcal{J}$ with $\varphi(a) = a^\mathcal{J}$ for all $a \in \text{Ind}(\mathcal{A})$.

Since $\mathcal{I}_\mathcal{A} \models \mathcal{A}$, this means that there exists an $\mathcal{EL}$ concept description $X$ such that $Q \bowtie X > t$ and $a^\mathcal{I}_\mathcal{A} \in X^\mathcal{I}_\mathcal{A}$. The membership characterization via homomorphism in Theorem 2.7 yields the existence of a homomorphism $\varphi_1$ from $T_X$ to $G(\mathcal{A})$ with
\[\varphi_1(v_0) = a.\] Then, the composition \(\varphi_J \circ \varphi_1\) yields a similar homomorphism to each model \(J\) of \(A\), which implies \(a^J \in X^J\). Therefore, \(A \models X(a)\) and thus, \(a \in \text{Relax}^{\leq}(Q, A)\). \(\square\)

### 7.2 Reasoning in \(\tau \mathcal{E} \mathcal{L}(m_{\odot})\)

Definition 7.2 allows to create a wide range of well-defined graded membership functions \(m_{\odot}\) and their corresponding DLs \(\tau \mathcal{E} \mathcal{L}(m_{\odot})\). In this section, we carry out a preliminary study of the computational properties of such a big family of threshold DLs. We will present undecidability and decidability results, as well as more fine-grained complexity results for specific classes within this family.

#### 7.2.1 Undecidability

We present some uncomputability results concerning the type of CSMs being considered and their induced threshold DLs. To start, based on a specific kind of binary relations between \(\mathcal{E} \mathcal{L}\) concept descriptions, we introduce a very simple form of CSMs satisfying the three properties required in the previous section. We will see that it is not difficult to put a subset of such measures into a one-to-one correspondence with the power set of the natural numbers.

**Definition 7.5.** Let \(R\) be a binary relation over \(\mathcal{C}_{\mathcal{E} \mathcal{L}}\) and \(0 < a < 1\) a fixed rational number. Then, \(R\) induces the following CSM \(\triangleright_R\):

\[
C \triangleright_R D := \begin{cases} 
1 & \text{if } C \equiv D \\
\mu(C, D) & \text{otherwise.}
\end{cases}
\]

where \(\mu\) is defined as follows:

\[
\mu(C, D) := \begin{cases} 
a & \text{if } \text{rd}(C) = \text{rd}(D) \text{ and } (C, D) \in R \\
0 & \text{otherwise.}
\end{cases}
\]

In addition, we say that \(R\) is equivalence invariant (w.r.t. \(\equiv\)) if \(C \equiv C'\) and \(D \equiv D'\) implies:

\[(C, D) \in R \iff (C', D') \in R\]

For equivalence invariant relations \(R\), the induced CSM \(\triangleright_R\) satisfies the three properties required in Proposition 7.3.

**Lemma 7.6.** Let \(R \subseteq \mathcal{C}_{\mathcal{E} \mathcal{L}} \times \mathcal{C}_{\mathcal{E} \mathcal{L}}\) be equivalence invariant. Then, \(\triangleright_R\) is an equivalence invariant, role-depth bounded and equivalence closed CSM.

**Proof.** That \(\triangleright_R\) is equivalence closed follows directly from its definition. Let us look at the other two properties.

1. **equivalence invariance:** let \(C, C', D, D' \in \mathcal{C}_{\mathcal{E} \mathcal{L}}\) such that \(C \equiv C'\) and \(D \equiv D'\). According to the definition of \(\triangleright_R\) there are three possible cases for the value \(C \triangleright_R D\):
7.2 Reasoning in $\tau\mathcal{EL}(m_{\cong_R})$

- $C \bowtie_R D = 1$. This means that $C \equiv C' \equiv D \equiv D'$, and by definition $C \bowtie_R D = C' \bowtie_R D' = 1$.

- $C \bowtie_R D = 0$. There are two possibilities:
  - $\text{rd}(C) \neq \text{rd}(D)$. Since $C \equiv C'$ and $D \equiv D'$, this means that $\text{rd}(C') \neq \text{rd}(D')$. Hence, $C' \bowtie_R D' = 0$.
  - $(C, D) \not\in R$. Since $R$ is equivalence invariant, $C \equiv C'$ and $D \equiv D'$ imply that $(C', D') \not\in R$. Therefore, $C' \bowtie_R D' = 0$.

- $C \bowtie_R D = a$. Then, $C \not\equiv D$, $\text{rd}(C) = \text{rd}(D)$ and $(C, D) \in R$. Similarly as in the previous case, we obtain $C' \neq D'$, $\text{rd}(C') = \text{rd}(D')$ and $(C', D') \in R$. Thus, $C' \bowtie_R D' = a$.

2. role-depth boundedness: let $C, D \in \mathcal{C}_\mathcal{E}_\mathcal{L}$. Whenever $\text{rd}(C) = \text{rd}(D)$ the role-depth boundedness condition trivially holds for $C$ and $D$, since for any $k > \text{rd}(C)$ it is the case that $C = C_k$ and $D = D_k$. It remains to look at the case where $\text{rd}(C) \neq \text{rd}(D)$.

It follows from the definition of $\bowtie_R$ that $C \bowtie_R D = 0$. Now, without loss of generality, let $\text{rd}(C) < \text{rd}(D)$. For any value $k > \text{rd}(C)$ we have $\text{rd}(C_k) < \text{rd}(D_k)$.

Then, $\text{rd}(C_k) \neq \text{rd}(D_k)$, and consequently $C_k \bowtie_R D_k = 0 = C \bowtie_R D$.

Now, let us fix the sets $\mathcal{N}_C = \{A\}$ and $\mathcal{N}_R = \{r\}$. For all $N \subseteq \mathbb{N}$, its corresponding binary relation $R_N$ on $\mathcal{E}\mathcal{L}$ concept descriptions defined over $\mathcal{N}_C \cup \mathcal{N}_R$, is built as follows:

$$(C, D) \in R_N \iff \text{rd}(C) \in N \quad (7.1)$$

Obviously, since $C \equiv C'$ implies that $\text{rd}(C) = \text{rd}(C')$ and membership in $R_N$ only depends on the role depth of $C$, it follows that $R_N$ is equivalence invariant. Hence, each subset $N$ of the natural numbers induces an equivalence closed, equivalence invariant and role-depth bounded CSM $\bowtie_{R_N}$. More importantly, for all pairs of distinct subsets $N_1, N_2 \in \mathbb{N}$, the induced CSMs $\bowtie_{R_{N_1}}$ and $\bowtie_{R_{N_2}}$ are different. Just take a number $n$ such that $n \not\in N_1$ and $n \not\in N_2$ (or vice versa). Then, take two concepts $C$ and $D$ such that $\text{rd}(C) = \text{rd}(D) = n$ and $C \neq D$ (the fixed signature $\mathcal{N}_C \cup \mathcal{N}_R$ ensures that this is always possible). By definition we will obtain $C \bowtie_{R_{N_1}} D = a$ and $C \bowtie_{R_{N_2}} D = 0$.

Hence, there are as many CSMs of this type as subsets of the natural numbers, namely, uncountably many. Since there are only countable many Turing Machines, there must be non-computable CSMs which are equivalence invariant, role-depth bounded and equivalence closed.

**Proposition 7.7.** The set of equivalence invariant, role-depth bounded and equivalence closed CSMs on $\mathcal{E}\mathcal{L}$ concept descriptions, contains non-computable functions.

On the side of the induced threshold DLs, Proposition 7.3 implies that $m_{\bowtie_{R_N}}$ is a well-defined graded membership function for all $N \subseteq \mathbb{N}$, and it induces the DL $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$. Moreover, the very simple definition of $\bowtie_{R_N}$ makes possible to use satisfiability in $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$ as a component of an algorithm computing $\bowtie_{R_N}$. More precisely, given two $\mathcal{E}\mathcal{L}$ concept descriptions $C$ and $D$:

1. $C \equiv D \Rightarrow C \bowtie_{R_N} D = 1$, and $\text{rd}(C) \neq \text{rd}(D) \Rightarrow C \bowtie_{R_N} D = 0$. 
2. Otherwise, the computation of $C \bowtie_{R_N} D$ solely depends on whether $\text{rd}(C) \in N$. This can be alternatively solved by asking for satisfiability of the concept $C_{\leq a} \cap C_{\geq a}$ in $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{R_N}})$ whenever $C \neq \top$. A positive answer corresponds to $C \bowtie_{R_N} D = a$, while the opposite one yields $C \bowtie_{R_N} D = 0$. Why is this true?

- Satisfiability of $C_{\leq a} \cap C_{\geq a}$ implies that for some interpretation $I$ and $d \in \Delta^I$:
  \[ m_I^\bowtie (d, C) = a \]
  This means that for some concept $F$, $C \bowtie_{R_N} F = a$ which by definition of $\bowtie_{R_N}$ implies $d \in \Delta^I$.

- Conversely, let $C_{\leq a} \cap C_{\geq a}$ be unsatisfiable. Except for $C \equiv \top$, for all $\mathcal{E} \mathcal{L}$ concept descriptions $C$ there is always an interpretation $I$ and $d \in \Delta^I$ such that $d \notin C^I$. This means that $m_{\bowtie_{R_N}}^I (d, C) < 1$ for such a particular case. Since we are in the unsatisfiability case, it must be that $m_{\bowtie_{R_N}}^I (d, C) = 0$. Moreover, since $d \in \Delta^I$, the computation of $C \bowtie_{R_N} \top$ must have value 0. Thus, again by definition of $\bowtie_{R_N}$ it follows that $C \bowtie_{R_N} \top \notin N$.

3. If $C \equiv \top$, the dichotomy used in the previous step cannot be directly applied since $\top_{\leq a} \cap \top_{\geq a}$ is actually unsatisfiable. However, once the algorithm reaches the second step, the goal is to decide whether $\text{rd}(C) \in N$. Hence, since $\text{rd}(\top) = \text{rd}(A)$, $A$ can be used instead of $\top$ to solve the issue.

The first step of the previously describe procedure consists of solving “fairly” easy tasks. Consequently, it becomes clear that decidability of the satisfiability problem in a DL $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{R_N}})$ implies computability of the CSM $\bowtie_{R_N}$. Hence, the following undecidability result follows.

**Proposition 7.8.** Let $N \subseteq \mathbb{N}$ and $R_N$ its corresponding relation defined as in (7.1). If $\bowtie_{R_N}$ is a non-computable CSM, then it induces an undecidable threshold DL $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{R_N}})$.

Summing up, on the one hand, we have seen that there are non-computable CSMs that are equivalence invariant, role-depth bounded and equivalence closed. This has been established by setting a one-to-one correspondence with the power set of the natural numbers. On the other hand, a subset of all non-computable CSMs induces a set of undecidable DLs that are constructed as described in Definition 7.2. Nevertheless, it is not yet clear to us whether non-computability of a CSM $\bowtie_{R_N}$ always implies undecidability of the induced DL $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{\sigma}})$.

### 7.2.2 Decidability

We will now show that whenever $\bowtie_{\sigma}$ is computable, the standard reasoning problems in the corresponding logic $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{\sigma}})$ are decidable. To this end, we establish the following three properties for $m_{\bowtie_{\sigma}}$ and $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{\sigma}})$. First, we prove that computability of $\bowtie_{\sigma}$ implies that $m_{\bowtie_{\sigma}}$ is computable with respect to finite interpretations. Second, $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{\sigma}})$ enjoys the finite model property. Last, we show that there is a computable function that given a concept $\hat{C}$ finds a number representing a sufficiently large upper bound for the size of models satisfying $\hat{C}$. 

Lemma 7.9. Let ∞ be an equivalence invariant, role-depth bounded and equivalence closed CSM. Further, let I be a finite interpretation, C an EL concept description and \( d \in \Delta^I \). If ∞ is computable, then \( m^{\mathcal{I}}_{\mathcal{I}}(d, C) \) is computable.

Proof. By definition of \( m_{\mathcal{I}} \) we know that:

\[
m^{\mathcal{I}}_{\mathcal{I}}(d, C) = \max \{ C \bowtie D \mid D \in \mathcal{C}_{\mathcal{E}\mathcal{L}} \text{ and } d \in D^\mathcal{I} \}
\]

Since \( \bowtie \) is equivalence invariant and role-depth bounded, we can restrict our attention to concepts \( D \) in reduced form whose role depth is at most \( \text{rd}(C) + 1 \). As explained before, up to equivalence, \( \mathcal{C}_{\mathcal{E}\mathcal{L}} \) contains finitely many concept descriptions of role depth at most \( \text{rd}(C) + 1 \). Therefore, it is enough to consider the concepts \( D \) in reduced form identifying the corresponding equivalence classes.

Now, the set of such concept descriptions can be enumerated in finite time. Let \([C^k_{\mathcal{E}\mathcal{L}}]\) denote the set of all the representatives of role depth at most \( k \geq 0 \). For role depth 0, there are exactly \( 2^{\text{N}_C} \) equivalence classes. These are represented by all the concept descriptions of the form \( A_1 \cap \ldots \cap A_n \), where \( n \geq 0 \), \( \{A_1, \ldots, A_n\} \subseteq \text{N}_C \) and \( A_i \neq A_j \) (for all \( i \neq j \)). The particular case of \( n = 0 \) corresponds to the \( \top \) concept. Consequently, \([C^0_{\mathcal{E}\mathcal{L}}]\) is the following set:

\[
[C^0_{\mathcal{E}\mathcal{L}}] := \{ \top \} \cup \bigcup_{S \subseteq \text{N}_C} \left\{ \bigcap_{A \in S} A \right\}
\]

To continue the enumeration for larger values of \( k \), we inductively describe how to generate \([C^k_{\mathcal{E}\mathcal{L}}]\) from \([C^{k-1}_{\mathcal{E}\mathcal{L}}]\). First, every concept description \( C \) of role depth \( k > 0 \) is of the following form:

\[
A_1 \cap \ldots \cap A_n \cap \exists s_1.C_1 \cap \ldots \cap \exists s_q.C_q
\]

where \( n \geq 0 \), \( q \geq 1 \) and for all \( i \in \{1, \ldots, q\} \), \( \text{rd}(C_i) < k \). In addition, at least one \( C_i \) must have role depth equal to \( k - 1 \). Moreover, since we are interested only on concepts in reduced form, \( C \) satisfies the following conditions:

- for all \( 1 \leq i \leq q \), \( C_i \) is a concept in reduced form.
- for all \( s \in \text{N}_R \), let \( s(C) \) denote the following set:

\[
s(C) := \{ D \mid \exists s.D \in \text{tl}(C) \}
\]

Then, \( s(C) \) must be an antichain with respect to the subsumption relation, i.e., if \( C_1, C_2 \in s(C) \) neither \( C_1 \sqsubseteq C_2 \) nor \( C_2 \sqsubseteq C_1 \) holds. The same must be true for the set \( \{A_1, \ldots, A_n\} \).

Then, once \([C^{k-1}_{\mathcal{E}\mathcal{L}}]\) has been generated, it can be extended to \([C^k_{\mathcal{E}\mathcal{L}}]\) as follows:

1. \( \text{Aux} := \emptyset \)
2. Let \( \{r_1, \ldots, r_{|\text{N}_R|}\} \) be the enumeration of the role names in \( \text{N}_R \).
3. For all \( (S_0, S_1, \ldots, S_{|\text{N}_R|}) \in 2^{\text{N}_C} \times 2^{[C^{k-1}_{\mathcal{E}\mathcal{L}}]} \times \ldots \times 2^{[C^{k-1}_{\mathcal{E}\mathcal{L}}]} \) do
Chapter 7. Concept similarity measures, relaxed instance queries and $\tau\mathcal{E}\mathcal{L}(m)$

4: if $(S_i$ is an antichain for all $1 \leq i \leq |N_R|) \text{ and }$
5: $(\exists i \ni D \text{ s.t. } D \in S_i \text{ and } \text{rd}(D) = k - 1)$ then
6: construct the $\mathcal{E}\mathcal{L}$ concept description $X$ as follows:

\[
X := \bigsqcap_{A \in S_i} A \sqcap \bigcap_{i=1}^{|N_R|} \exists r_i Y
\]

7: $\text{Aux} := \text{Aux} \cup \{X\}$
8: end if
9: end for
10: $[\mathcal{C}_k^{\mathcal{E}\mathcal{L}}] := [\mathcal{C}_k^{\mathcal{E}\mathcal{L}}] \cup \text{Aux}$

Starting from $[\mathcal{C}_0^{\mathcal{E}\mathcal{L}}]$, the iteration of this procedure can be used to enumerate all the concepts $D$ identifying the equivalence classes in $\mathcal{C}_k^{\mathcal{E}\mathcal{L}}$. Hence, computing $m^I_{\text{deg}}(d, C)$ reduces to use this enumeration up to $\text{rd}(C) + 1$, and keep the maximum value $C \bowtie D$ among those satisfying $d \in D^I$. Checking for $d \in D^I$ in $\mathcal{E}\mathcal{L}$ can be done in polynomial time in the size of $D$ and $I$, whenever $I$ is finite. Thus, since $\bowtie$ is computable, $m^I_{\text{deg}}(d, C)$ can always be computed.

Let us now turn into the finite model property. We will see that the method used to provide a small model property for $\tau\mathcal{E}\mathcal{L}(\text{deg})$ can be used to establish the finite model property for $\tau\mathcal{E}\mathcal{L}(m_{\bowtie})$. The base argument for this comes again from the definition of $m_{\bowtie}$ and the basic properties required of $\bowtie$. There is always an $\mathcal{E}\mathcal{L}$ concept description $D$ of role depth at most $\text{rd}(C) + 1$ such that:

\[d \in D^I \text{ and } m^I_{\bowtie}(d, C) = C \bowtie D\]

Membership of $d$ into $D^I$ implies that the structure of $T_D$ can be extracted from $G_I$. The idea is that $T_D$ can play the same role as the logical interpretations $I_h$ do for $\text{deg}$, in the construction introduced in Lemma 5.4. In what follows, after formally defining the analogous of canonical interpretations for the current scenario, we show that such interpretations and $m_{\bowtie}$ exhibit the necessary properties to achieve the correctness of the construction in Lemma 5.4.

**Definition 7.10.** Let $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ be an interpretation, $d \in \Delta^\mathcal{I}$ and $D$ an $\mathcal{E}\mathcal{L}$ concept description such that $d \in D^\mathcal{I}$. The canonical interpretation $\mathcal{I}_D$ induced by $D$ is the one having the description tree $T_D$. \(\diamondsuit\)

Like Lemma 4.11 for $\text{deg}$, the monotonicity property generalizes easily to all graded membership functions $m_{\bowtie}$.

**Lemma 7.11.** Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM. Additionally, let $\mathcal{I}$ and $\mathcal{J}$ be two interpretations such that there exists a homomorphism $\varphi$ from $G_I$ to $G_J$. Then, for all $d \in \Delta^\mathcal{I}$ and all $\mathcal{E}\mathcal{L}$ concept descriptions $C$ it holds:

\[m^I_{\bowtie}(d, C) \leq m^J_{\bowtie}(\varphi(d), C)\]
7.2 Reasoning in $\tau\mathcal{EL}(m_{\bowtie})$

Proof. By definition of $m_{\bowtie}$ we know that:

$$m_{\bowtie}(d, C) = \max\{ C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^\mathcal{F} \}$$

Let $D$ be one such maximal concept description. Then, $d \in D^\mathcal{F}$ implies the existence of a homomorphism $\varphi_D$ from $T_D$ to $G_T$ such that $\varphi_D(v_0) = d$, where $v_0$ is the root of $T_D$. The composition $\varphi \circ \varphi_D$ yields a homomorphism from $T_D$ to $G_T$ with $(\varphi \circ \varphi_D)(v_0) = \varphi(d)$. Hence, $\varphi(d) \in D^\mathcal{F}$ and we have:

$$C \bowtie D \leq m_{\bowtie}(\varphi(d), C)$$

Thus, $m_{\bowtie}(d, C) \leq m_{\bowtie}(\varphi(d), C)$ follows.

The next step is to show that the value of $m_{\bowtie}(d, C)$ is preserved by canonical interpretations $I_D$ corresponding to a concept $D$, such that the value $C \bowtie D$ is the maximum with respect to the definition of $m_{\bowtie}$.

Lemma 7.12. Let $I = (\Delta^\mathcal{F}, \mathcal{F})$ be an interpretation, $d \in \Delta^\mathcal{F}$ and $C$ an $\mathcal{EL}$ concept description. For all equivalence invariant, role-depth bounded and equivalence closed CSM $\bowtie$:

$$m_{\bowtie}(v_0, C) = m_{\bowtie}(d, C)$$

for all $D \in \mathcal{C}_{\mathcal{EL}}$ such that $d \in D^\mathcal{F}$ and $m_{\bowtie}(d, C) = C \bowtie D$.

Proof. Since $I_D$ corresponds to $T_D$ and $d \in D^\mathcal{F}$, this means that there is a homomorphism $\varphi$ from $G_{T_D}$ to $G_T$ having $\varphi(v_0) = d$. Then, applying Lemma 7.11 we obtain:

$$m_{\bowtie}(v_0, C) \leq m_{\bowtie}(d, C)$$

On the other side, we know that $m_{\bowtie}(d, C) = C \bowtie D$. Since $v_0 \in D^\mathcal{F}$, the maximum in the definition of $m_{\bowtie}$ implies:

$$m_{\bowtie}(d, C) \leq m_{\bowtie}(v_0, C)$$

At this point we observe the following commonalities between $\text{deg}$ and $m_{\bowtie}$.

- The characterization of membership for $\tau\mathcal{EL}(m)$ given in Theorem 3.8 applies to all graded membership functions $m$. Therefore, it holds for $\tau\mathcal{EL}(m_{\bowtie})$ as well.
- Lemmas 7.11 and 7.12 are for $m_{\bowtie}$, the same as Lemmas 4.11 and 4.12 are for $\text{deg}$ in Section 4.2.
- If $d \in D^\mathcal{F}$, there is always a homomorphism $\varphi$ from $I_D$ to $I$ with $\varphi_D(v_0) = d$.
- Canonical interpretations $I_D$ are tree-shaped as $I_h$ in Definition 4.6.

Hence, the same construction used in Lemma 5.4 for $\tau\mathcal{EL}(\text{deg})$ applies to $\tau\mathcal{EL}(m_{\bowtie})$ by using interpretations $I_D$ instead of the canonical interpretations $I_h$. More precisely, suppose that $E_{\bowtie t} \in \hat{\mathcal{E}}_{I_0}(v)$ and $v \not\in (E_{\bowtie t})_{I_0}$ for some $v \in \Delta_{I_0}$. Then, the interpretation
\( \mathcal{I}_D \) used to repair this problem would be such that \( \phi(v) \in D^\mathcal{I} \) and \( m^\mathcal{I}_E(\phi(v), E) = E \bowtie D \). Since all those interpretations \( \mathcal{I}_D \) are finite and tree-shaped, we obtain a finite tree model property for \( \tau \mathcal{E}\mathcal{L}(m_\infty) \).

**Proposition 7.13.** Let \( \bowtie \) be an equivalence invariant, role-depth bounded, and equivalence closed CSM. For all \( \tau \mathcal{E}\mathcal{L}(m_\infty) \) concept descriptions \( \widehat{C} \), if \( \widehat{C} \) is satisfiable then there is a finite tree-shaped interpretation \( \mathcal{J} \) such that \( \widehat{C} \mathcal{J} \neq \emptyset \).

This form of finite model property is not sufficient to obtain decidability of the satisfiability problem in \( \tau \mathcal{E}\mathcal{L}(m_\infty) \). To achieve that, we show that a bound for the size of such models can always be computed. Let \( \mathcal{I} \) be a model of \( \widehat{C} \). Following the construction in Lemma 5.4, the size of the finite model \( \mathcal{J} \) resulting from Proposition 7.13 corresponds to the following expression:

\[
|\Delta^\mathcal{J}| = |\Delta^\mathcal{I}_0| + \sum_{I_D \in \mathcal{J}} |\Delta^{I_D}|
\]

Recall that \( \mathcal{I}_0 \) corresponds to the description tree \( T_C \) (this is \( T_\widetilde{C} \) without labels of the form \( E_{\sim t} \)), and \( \mathcal{J} \) the set of canonical interpretations used to extend \( \mathcal{I}_0 \) into \( \mathcal{J} \). The unclear part is to know how big \( \Delta^{I_D} \) can be. Those interpretations \( I_D \) are introduced for threshold concepts of the form \( E_{>t} \) or \( E_{\geq t} \) occurring in \( C \). Moreover, they correspond to the description tree of a concept \( D \) satisfying \( d \in D^\mathcal{I} \) and \( m^{\mathcal{I}_E}_D(d, E) = E \bowtie D \), for some \( d \in \Delta \). Hence, a trivial choice to provide such a bound is the size of the largest concept \( D \) in reduced form, whose role depth is at most \( rd(E) + 1 \).

We have already seen in Lemma 7.9 that the set of all those concepts can be enumerated in finite time. Then, the algorithm computing the bound \( b(\mathcal{C}) \) for the size of models satisfying \( \mathcal{C} \) does the following:

1. list the occurrences in \( \mathcal{C} \) of threshold concepts \( \mathcal{F}_1, \ldots, \mathcal{F}_q \) of the form \( E_{>t} \) or \( E_{\geq t} \).
2. let \( k_i = rd(\mathcal{F}_i) \) for all \( 1 \leq i \leq q \), and \( k \) the largest value among them.
3. enumerate the set of \( \mathcal{E}\mathcal{L} \) concept descriptions in \( [C^{k+1}] \). For all \( 1 \leq i \leq q \), let \( D_i \) be one of largest size among those with role depth at most \( k_i + 1 \).
4. the bound \( b(\mathcal{C}) \) for the size of \( \mathcal{J} \) is given as:

\[
|\Delta^{\mathcal{I}_0}| + \sum_{i=1}^{q} s(D_i)
\]

Thus, satisfiability of a concept \( \mathcal{C} \) in \( \tau \mathcal{E}\mathcal{L}(m_\infty) \) can be decided by first computing \( b(\mathcal{C}) \), and then looking for an interpretation \( \mathcal{J} \) of size at most \( b(\mathcal{C}) \) satisfying \( \mathcal{C} \). Checking whether \( \mathcal{J} \) satisfies \( \mathcal{C} \) can be done using Algorithm 2, since \( m_\infty \) has been proven to be computable in Lemma 7.9.

With respect to the other reasoning problems, observe that the interpretation \( \mathcal{J} \) obtained in Proposition 7.13 is tree-shaped. Therefore, it can be used as a base to extend decidability to the other reasoning problems by following the same constructions provided for \( \tau \mathcal{E}\mathcal{L}(deg) \) in Chapter 5.
7.2 Reasoning in $\tau EL(m_{\infty})$

- **non-subsumption**: Lemma 5.6 describes how to build a bounded model for satisfiable concepts of the form $\hat{C} \sqcap \neg \hat{D}$. The construction starts with a model $J$ of $\hat{C}$ that is extended into a model $J_p$ of $\hat{C} \sqcap \neg \hat{D}$, by attaching canonical interpretations $I_h$. The arguments used to show $(\hat{C} \sqcap \neg \hat{D})J_p \neq \emptyset$ that depends on the nature of $deg$ can be separated into two groups.

1. The results from Lemmas 4.11, 4.12 and 5.4. They all have a corresponding version for $\tau EL(m_{\infty})$.

2. The value $deg^I(d, C)$ only depends on the fragment of $G_I$ that is reachable from $d$. This property is exploited at the end of Lemma 5.6. Now, the computation of $m_{\infty}^I(d, C)$ depends on the $\mathcal{L}$ concept descriptions $D$ satisfying $d \in D_I^\tau$. Hence, we can also say that $m_{\infty}^I(d, C)$ only depends on the fragment of $G_I$ that is reachable from $d$.

Thus, the construction of Lemma 5.6 can be applied to $\tau EL(m_{\infty})$ to obtain finite models for satisfiable concepts of the form $\hat{C} \sqcap \neg \hat{D}$.

- **consistency**: uses Theorem 3.9 as a characterization of ABox satisfaction for all DLs $\tau EL(m)$. The construction of the corresponding bounded model in Lemma 5.9 uses basically the same arguments as the ones provided in Lemma 5.4 for satisfiability.

- **non-instance**: the bounded model obtained for $\tau EL(deg)$ is a combination of Lemmas 5.14 and 5.15 (see Lemma 5.16). The properties of $deg$ needed in the proofs are the same as the ones used for non-subsumption. Hence, the same construction is also valid for $\tau EL(m_{\infty})$.

A common aspect of all these constructions is that they extend $J$ by plugging canonical interpretations $I_h$. Moreover, the proofs in Chapter 5 are constructive and describe how those canonical interpretations are obtained from the threshold concepts occurring in an instance of a problem. Hence, the procedure computing $b(\hat{C})$ can be adapted to estimate a sufficient upper bound for the size of models satisfying concepts of the form $\hat{C} \sqcap \neg \hat{D}$ and/or ABoxes of the form $\mathcal{A} \cup \{\neg \hat{C}(a)\}$ in $\tau EL(m_{\infty})$.

**Theorem 7.14.** Let $\bowtie$ be an equivalence invariant, role-depth bounded and equivalence closed CSM. If $\bowtie$ is a computable function, then in $\tau EL(m_{\infty})$ satisfiability, subsumption, consistency and instance checking are decidable problems.

Overall, we have provided decidability for $\tau EL(m_{\infty})$ based on a strong form of the finite model property. It comes as a result of adapting the methods used to obtain “small” models for $\tau EL(deg)$. However, other than decidability, the previous construction does not give us much insight on how difficult is to reason in a particular logic $\tau EL(m_{\infty})$. In fact, the computation of the upper bound $b(\hat{C})$ is merely based on the structure of $\hat{C}$ and the described enumeration, not to mention how big it could be in its general form. In conclusion, CSMs are treated just as “black boxes” satisfying the properties required in Proposition 7.3 to induce the corresponding threshold DL, and none of its internal technicalities are taken into account.

For a particular CSM $\bowtie$ there are two main aspects to be considered in this regard:

- the complexity of computing $\bowtie$, 

Chapter 7. Concept similarity measures, relaxed instance queries and $\tau\mathcal{EL}(m)$

- the machinery that results from the interaction between the internal characteristics of $\gg$ and the maximization mechanism defining $\tau\mathcal{EL}(m_{\gg})$.

Obviously, the set of CSMs of this kind is very wide, and as shown in the previous section it even contains functions that are uncomputable. From now on we focus our attention to CSMs that can be computed in polynomial time. In the next two sections we intend to take some initial steps towards understanding the computational properties of a logic $\tau\mathcal{EL}(m_{\gg})$ where $\gg$ is polynomial time computable. First, we will show that this low-complexity family of CSMs has members whose induced threshold DL is at least PSPACE-hard. Notice that this will be the case, despite our initial requirement of CSMs to be defined over finite alphabets of concept and role names. Afterwards, we will provide a sufficient condition on CSMs that determines a better behaved (in terms of worst case complexity) family of threshold DLs.

7.2.3 A polynomial time CSM and its PSPACE-hard threshold DL

In the following, we define a polynomial time computable CSM satisfying the properties required in Proposition 7.3, such that the satisfiability problem in the induced threshold logic is at least PSPACE-hard. To define such a CSM we start by defining a particular relation $R_a$, and then follow the construction provided in Definition 7.5 to obtain $\gg_{R_a}$.

Note that the abstract definition of $\gg_{R_a}$ sets up a special connection between the value $a$ and membership in $R$. This permits to fix any subset of $\mathcal{EL}$ concepts (provided that the resulting relation $R$ is equivalence invariant) as the relevant ones to obtain the similarity value $a$ when compared to a concept description $C$. In particular, there are concepts in reduced form that grow exponentially (with respect to the size of $C$) in its width, having in this way description trees that represent exponentially large structures. Then, asking for satisfiability of the $\tau\mathcal{EL}(m)$ concept $C_{\leq a} \sqcap C_{\geq a}$ in $\tau\mathcal{EL}(m_{\gg})$ could be used to test whether $C$ satisfies a specific property on such type of structures. We will exploit this to obtain our PSPACE-hard threshold logic. The hardness result will be established by a reduction from the problem of deciding the validity of quantified Boolean formulas (QBF), which is introduced in the next definition.

Definition 7.15. A quantified Boolean formula consists of a pair $P, \varphi$ where:

- $\varphi$ is a propositional formula, and
- the prefix $P$ is a sequence of the form $Q_1x_1, \ldots, Q_nx_n$, where $x_1, \ldots, x_n$ are the propositional variables occurring in $\varphi$ and $Q_i \in \{\exists, \forall\}$ ($1 \leq i \leq n$). We say that $P$ is of length $n$.

A quantified Boolean formula $P, \varphi$ can be seen as a first-order logic closed formula, where its variables $x_1, \ldots, x_n$ are interpreted over a two-element domain $\{true, false\}$. For simplicity, we use the semantics defined in [DLNS94] (Section 5.2.2), where QBF is used to establish a PSPACE-hardness result for the DL $\mathcal{ALE}$.

A $P$-assignment is a mapping $t : \{x_1, \ldots, x_n\} \to \{true, false\}$. An assignment $t$ satisfies a literal $x_i$ if $t(x_i) = true$, and its negation $\neg x_i$ if $t(x_i) = false$. An assignment $t$ satisfies a clause $c$ if $t(\ell) = true$ for at least one literal $\ell$ of $c$. A set $S$ of $P$-assignments is canonical for $P$ if it satisfies the following conditions:
1. $S$ is non-empty,

2. $P = \exists x_1.P'$:
   - for all $t_1, t_2 \in S$, it holds $t_1(x_1) = t_2(x_1)$.
   - if $P'$ is non-empty, then the set $\{t|t(x_2,\ldots,x_n) | t \in S\}$ is canonical for $P'$.

3. $P = \forall x_1.P'$:
   - $S$ contains an assignment $t$ such that $t(x_1) = true$, and if $P'$ is not empty the set $\{t|t(x_2,\ldots,x_n) | t \in S \text{ and } t(x_1) = true\}$ is canonical for $P'$.
   - $S$ contains an assignment $t$ such that $t(x_1) = false$, and if $P'$ is not empty the set $\{t|t(x_2,\ldots,x_n) | t \in S \text{ and } t(x_1) = false\}$ is canonical for $P'$.

Then, $P.\varphi$ is valid if there exists a set $S$ of $P$-assignments that is canonical for $P$ such that every assignment in $S$ satisfies every clause in $\varphi$.  

QBF is a PSPACE-complete problem [GJ79], and this is still the case even if $\varphi$ is in conjunctive normal form (CNF) and the quantifiers in $P$ alternate. Moreover, by using dummy variables when needed, we can assume without loss of generality that $P$ begins with $\exists$. Consequently, we denote as $P_n$ “the” prefix of length $n$.

We now move into the details of the PSPACE-hardness result. First, we need to fix our particular threshold logic. To this end, a relation $R_s$ is defined to obtain the CSM $\triangledown_{R_s}$ (according to Definition 7.5), and the logic $\tau\mathcal{EL}(m_{\triangledown_{R_s}})$. Afterwards, we provide the translation reducing QBF to concept satisfiability in $\tau\mathcal{EL}(m_{\triangledown_{R_s}})$. The reduction is based on the following ideas.

- Each set of $P$-assignments $S$ that is canonical for $P$ can be represented as a concept description $D_S$. As one may expect, such a concept $D_S$ is of size exponential on the size of $P.\varphi$. Nevertheless, we want to stress that they are not involved in the translation, but are used to define the relation $R_s$ and subsequently the target DL $\tau\mathcal{EL}(m_{\triangledown_{R_s}})$.

- Likewise, a propositional formula $\varphi$ in CNF can be translated into an $\mathcal{EL}$ concept description $C_\varphi$, but this time $C_\varphi$ is polynomial on the size of $\varphi$.

- $R_s$ can be defined such that $(C_\varphi, D_S) \in R_s$ iff every assignment in $S$ satisfies $\varphi$. Then, the singularity of the value $a$ in the definition of $\triangledown_{R_s}$ can be used to link validity of $P.\varphi$ to satisfiability of $(C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a}$ in $\tau\mathcal{EL}(m_{\triangledown_{R_s}})$.

Let us start with the encoding of a set $S$ of $P$-assignments. Let $A$ be a distinguished
concept name, for all \( n > 0 \) we inductively define the string \( D_n := \exists r.D_0 \) as follows:

\[
D_0^0 := X_1^0 \sqcap \exists r.(A \sqcap D_0^0) \sqcap \exists s.D_2^1 \\
D_1^0 := \exists r.D_0^0 \\
D_2^0 := \exists r.D_1^1 \\
D_2^1 := \exists r.D_2^0 \\
D_3^0 := \exists r.D_1^1 \\
D_3^1 := \exists r.D_2^0 \\
\vdots
\]

\[
D_{2i}^j := X_{2i+1}^j \sqcap \exists r.(A \sqcap D_{2i}^j) \sqcap \exists s.D_{2i+1}^{j+1} \quad (0 \leq j < 2^i) \\
D_{2i+1}^j := \exists r.D_{2i}^{j+1} \quad (0 \leq j < 2^i) \\
\vdots
\]

\[
D_n^j := \begin{cases} 
\top & \text{if } n \text{ is even} \\
X_n^j & \text{otherwise.} 
\end{cases} \tag{7.2}
\]

The symbols \( X_{2i+1}^j \) represent variables that are to be instantiated to obtain \( \mathcal{E}\mathcal{L} \) concept descriptions. Let \( X_n \) be the set of all variables occurring in \( D_n \), i.e.:

\[
X_n = \{ X_{2i+1}^j \mid 1 \leq 2i + 1 \leq n \text{ and } 0 \leq j < 2^i \}
\]

We denote by \( X_n \) the set of all total mappings \( \theta : X_n \to \{\top, A\} \). The application of one such \( \theta \) to \( D_n \) is denoted as \( \theta[D_n] \) and consists of substituting each variable \( X_{2i+1}^j \) in \( D_n \) by \( \theta(X_{2i+1}^j) \). Then, \( X_n \) generates the following family of \( \mathcal{E}\mathcal{L} \) concept descriptions:

\[
\mathcal{D}_n := \{ \theta[D_n] \mid \theta \in X_n \}
\]

To be consistent later on, we define \( \mathcal{D}_0 \) as the empty set. Additionally, since the “branching” in the definition of the string \( D_n \) is defined using two different role names \( r \) and \( s \), one can see that all concept descriptions in \( \mathcal{D}_n \) are in reduced form. The purpose of these sets is that each concept in \( \mathcal{D}_n \) identifies a set of \( P \)-assignments that is canonical for the prefix \( P \) of length \( n \). The following example gives the intuition underlying such a correspondence.

**Example 7.16.** Let \( P_4 \) be the prefix of length 4, i.e., \( P_4 = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \). Let \( t_1, t_2, t_3 \) and \( t_4 \) be the following \( P_4 \)-assignments:

\[
t_1(x_1) = true \quad t_1(x_2) = true \quad t_1(x_3) = false \quad t_1(x_4) = true \\
t_2(x_1) = true \quad t_2(x_2) = true \quad t_2(x_3) = false \quad t_2(x_4) = false \\
t_3(x_1) = true \quad t_3(x_2) = false \quad t_3(x_3) = true \quad t_3(x_4) = true \\
t_4(x_1) = true \quad t_4(x_2) = false \quad t_4(x_3) = true \quad t_4(x_4) = false
\]

One can easily see that the set \( S = \{ t_1, t_2, t_3, t_4 \} \) is canonical for \( P_4 \). Now, the string \( D_4 \) contains the set of variables \( X_4 = \{ X_1^0, X_2^0, X_3^1 \} \). Let \( \theta \in X_4 \) be the mapping such that \( \theta(X_1^0) = A \), \( \theta(X_2^0) = \top \) and \( \theta(X_3^1) = A \). This yields the \( \mathcal{E}\mathcal{L} \) concept description \( D_S := \theta[D_4] \) having the description tree depicted on the left-hand side of Figure 7.1.
Consider the left-most path in $T_{D_S}$:

$$\pi_{lm} : \{\} \xrightarrow{r} \{A\} \xrightarrow{r} \{A\} \xrightarrow{r} \{\}$$

Denoting the nodes in $\pi_{lm}$ from left to right as $v_0, \ldots, v_4$, the assignment $t_1$ can be obtained as follows. For all $1 \leq i \leq 4$:

$$t_1(x_i) := \begin{cases} 
true & \text{if } \ell_{\pi_{lm}}(v_i) = \{A\} \\
false & \text{if } \ell_{\pi_{lm}}(v_i) = \{\}
\end{cases}$$

Conversely, the path $\pi_{lm}$ can be constructed from $t_1$ using the inverse correspondence between $\{true, false\}$ and $\{ A, \top \}$. The same relationship can also be established between the other three paths in $T_{D_S}$ and the assignments $t_2, t_3, t_4$, respectively. Hence, the idea is that for all sets $S$ that are canonical for $P_4$, there is an instance $\theta[D_4]$ such that each assignment $t \in S$ corresponds to a path $\pi_t$ in $T_{\theta[D_4]}$ and vice versa. For example, the variation of $S$ where $t_1(x_1) = t_2(x_1) = t_3(x_1) = t_4(x_1) = false$, $t_1(x_3) = t_2(x_3) = true$ and $t_3(x_3) = t_4(x_3) = false$, would correspond to the concept $\theta^*[D_4]$ where $\theta^*(X_3^0) = \top$, $\theta^*(X_3^1) = A$ and $\theta^*(X_3^2) = \top$. The description tree in the right-hand side of the same figure is the one associated to $\theta^*[D_4]$.

Let us formally define the correspondence illustrated in the previous example, and then show that it actually exists.

**Definition 7.17.** Let $n > 0$ be a natural number, $\pi = v_0r_1v_1 \ldots r_nv_n$ a path of length $n$ in some EL description tree $T$ and $t$ a truth assignment of the variables $x_1, \ldots, x_n$. We say that $\pi$ and $t$ are corresponding iff for all $1 \leq i \leq n$:

$$t(x_i) = true \Leftrightarrow \ell_T(v_i) \cap \{A\} = \{A\}$$
Chapter 7. Concept similarity measures, relaxed instance queries and $\tau\mathcal{EL}(m)$

Additionally, let $S$ be a set of $P_n$-assignments that is canonical for $P_n$ and $D \in \mathcal{D}_n$. We further say that $S$ and $D$ are corresponding iff:

- for all $t \in S$ exists a path $\pi_t$ in $T_D$ of length $n$, such that $t$ and $\pi_t$ are corresponding, and
- for all paths $\pi$ of length $n$ in $T_D$ there is $t_\pi \in S$, such that $t_\pi$ and $\pi$ are corresponding.

The proof of the following lemma is deferred to the Appendix A.

Lemma 7.18. Let $n > 0$ be a natural number. Then,

1. for all sets $S$ of $P_n$-assignments that are canonical for $P_n$, there exists $D_S \in \mathcal{D}_n$ such that $S$ and $D_S$ are corresponding, and
2. for all $D \in \mathcal{D}_n$, exists a set $S_D$ of $P_n$-assignments that is canonical for $P_n$ such that $S_D$ and $D$ are corresponding.

Next, we describe how to encode the structure of a propositional formula $\varphi$ in CNF into an $\mathcal{EL}$ concept description $C_\varphi$. We need to use an additional concept name $I$. Let $\varphi$ be the conjunction of clauses $c_1 \land \ldots \land c_q$, and $x_1, \ldots, x_n$ the propositional variables occurring in $\varphi$. For each $c_j (1 \leq j \leq q)$ we define its corresponding $\mathcal{EL}$ concept description $C_j$ as $\exists r_1^j . E_1^j$, where $E_1^j$ is of the following form:

$$
E_1^j := \gamma_1^j \land \exists r_2^j . E_2^j \\
\ldots \\
E_i^j := \gamma_i^j \land \exists r_{i+1}^j . E_{i+1}^j \quad (1 \leq i < n) \\
\ldots \\
E_n^j := \gamma_n^j
$$

Here $\gamma_i^j = I$ if $x_i$ does not occur in $c_j$. Otherwise $\gamma_i^j = A$ whenever $x_i$ is a literal in $c_j$ and $\gamma_i^j = \top$ for $\neg x_i$. One can assume that $x_i$ and $\neg x_i$ do not occur at the same time in any set $c_j$, since otherwise $c_j$ is always satisfied and it can be removed from $\varphi$. Regarding $r_1^j, \ldots, r_n^j$, they correspond to any of two fixed role names $r$ and $s$ as follows: if $\gamma_i^j = A$ or $\gamma_i^j = I$ then $r_i^j = r$. Otherwise, $r_i^j = s$. Then, the $\mathcal{EL}$ concept description $C_\varphi$ encoding the structure of $\varphi$ is defined as:

$$
C_\varphi := I \land \bigcap_{j=1}^q C_j
$$

Example 7.19. Let $\varphi$ be the following propositional formula in CNF:

$$
\{ x_1, \neg x_2, x_3 \} \land \{ \neg x_1, x_4, x_3 \} \land \{ \neg x_4, x_2, \neg x_3 \}
$$

A total of four propositional variables occur in $\varphi$. Then, the concept description $C_\varphi$ is the one having the following $\mathcal{EL}$ description tree:
Each branch of the tree corresponds to a clause in $\phi$. In particular, the nodes at the $i^{th}$ level (except for the root) tell us in which form the variable $x_i$ occurs in each clause of $\phi$. For $x_2$, the empty set (or $\top$) is used in the upper branch to represent a negative occurrence in $c_1$, $I$ in the middle branch to represent a negative occurrence in $c_2$, and $A$ is used in the last branch to state that $x_2$ occurs in $c_3$. The same idea applies for the rest of the variables occurring in $\phi$. 

So far, we have a way to encode propositional formulas and sets of $P$-assignments into concept descriptions. Then, the role of $R_s$ is to verify whether a formula $\phi$ is satisfied by all the assignments of a set $S$. The definition of $R_s$ is based on the representation of concepts as description trees. Let $T_1$ and $T_2$ be two $\mathcal{EL}$ description trees, and $\pi_1 = v_0 r_1 v_1 \ldots r_n v_n, \pi_2 = w_0 s_1 w_1 \ldots s_n w_n$ two paths of length $n$ in $T_1$ and $T_2$, respectively. We say that $\pi_1$ has a coincidence in $\pi_2$ (denoted $\pi_1 \triangleright \pi_2$) iff there is $0 \leq i \leq n$ such that:

$$\ell_{T_1}(v_i) \cap \{A\} = \ell_{T_2}(w_i) \cap \{A\} \text{ and } I \not\subseteq \ell_{T_1}(v_i) \quad (7.3)$$

For all $\mathcal{EL}$ description trees $T$ we denote by $\Pi(T)$ the set of all paths in $T$ starting at its root, and by $\Pi_p(T) \subseteq \Pi(T)$ the subset of those having length $p \geq 0$. Then, using the relation $\triangleright$ we define a family of binary relations $\triangleright_p$ (for all $p \in \mathbb{N}$) over the set of $\mathcal{EL}$ description trees as follows:

$$(T_1, T_2) \in \triangleright_p \iff \forall \pi_2 \in \Pi_p(T_2) \forall \pi_1 \in \Pi_p(T_1), \text{ it holds } \pi_1 \triangleright \pi_2$$

Using this family of relations together with the family of sets $\mathcal{D}_n$, the relation $R_s$ is defined as follows:

$$R_s := \{(C, D) \mid \exists D^* \in \mathcal{D}_{rd(C)} \text{ s.t. } D \equiv D^* \text{ and } (T_{C^\prime}, T_{D^*}) \in \triangleright_{rd(C)}\}$$

The following lemma shows that $R_s$ is equivalence invariant (see Definition 7.3).

**Lemma 7.20.** The relation $R_s$ is equivalence invariant.

**Proof.** Let $C, C', D$ and $D'$ be $\mathcal{EL}$ concepts such that $C \equiv C'$ and $D \equiv D'$. We need to show that $(C, D) \in R_s$ iff $(C', D') \in R_s$. Some simple facts follow:

- Since $C \equiv C'$, we have that $rd(C) = rd(C') = k$ for some $k \geq 0$. Moreover, by Theorem 2.8 there is an isomorphism between $T_{C^\prime}$ and $T_{(C^\prime)^r}$.
- Let $D^* \in \mathcal{D}_k$, $k > 0$. Then, $D \equiv D'$ implies that $D \equiv D^*$ iff $D' \equiv D^*$. 

For \( k = 0 \), the claim trivially holds since \( \mathcal{D}_0 \) is empty and there is no \( D^* \). Otherwise, suppose that \((T_{C^*}, T_{D^*}) \in \triangleright_k\). To see that also \((T_{(C')^*}, T_{D'}) \in \triangleright_k\), let \( \pi_{C'} \) and \( \pi_{D'} \) be arbitrary paths of length \( k \) in \( T_{(C')^*} \) and \( T_{D'} \), respectively. Using the isomorphism mentioned above, one can find a path \( \pi_C \) in \( T_{C^*} \) such that \( \pi_C \equiv \pi_{C'} \). Since \((T_{C^*}, T_{D^*}) \in \triangleright_k\), this means that \( \pi_C \triangleright \pi_{D'} \). Consequently \( \pi_{C'} \triangleright \pi_{D'} \), and we have thus shown that \((T_{C^*}, T_{D^*}) \in \triangleright_k \Rightarrow (T_{(C')^*}, T_{D'}) \in \triangleright_k\). The implication in the opposite direction can be obtained in a similar way.

All these elements combined imply that \( R_s \) is equivalence invariant.

Hence, \( R_s \) induces a CSM \( \bowtie_{R_s} \) that is equivalence invariant, role-depth bounded and equivalence closed (see Lemma 7.6). To see how difficult it is to compute \( \bowtie_{R_s} \), let us look at the abstract formulation of \( \bowtie_{R} \) in Definition 7.5. The computation of \( C \bowtie_{R} D \) first discriminates between whether \( C \equiv D \) or not. Checking equivalence of concept descriptions in \( \mathcal{EL} \) is a polynomial time problem. In case of a negative answer, \( C \bowtie_{R} D \) corresponds to the value \( \mu(C, D) \). Since verifying whether \( \text{rd}(C) = \text{rd}(D) \) is also a polynomial time issue, the difficulty of computing \( \bowtie_{R} \) will then depend on how hard it is to check for membership in \( R \).

Let \( \text{rd}(C) = k \). In particular, checking for membership in \( R_s \) consists of two steps.

1. Test whether there is \( D^* \in \mathcal{D}_k \) such that \( D \equiv D^* \). If such \( D^* \) exists, then there is an isomorphism between \((D^*)^r \) and \( D^* \) (by Theorem 2.8). The computation of reduced forms is in polynomial time, and we know that \( D^* \) is already in reduced form. Moreover, concepts in \( \mathcal{D}_k \) result from different instantiations of the variables occurring in \( D_k \). Fortunately, the definition of \( D_k \) follows a very simple pattern that simplifies the quest of whether such concept description \( D^* \) exists.

From a graphical point of view, this means that in \( T_{D^*} \) every node at an even level (the root node is at level 0) has exactly one \( r \)-successor, whereas the ones at odd levels have exactly one \( r \)-successor and one \( s \)-successor labeled with \{A\} and \{\}, respectively. Testing whether \( D^* \) has such a shape (up to isomorphism) can be done by traversing its structure. Since \( s(D^*) \leq s(D) \), this is a polynomial time procedure in the size of \( D \).

2. Check if \((T_{C^*}, T_{D^*}) \in \triangleright_k\). Note that \( D^* \) and \( D^* \) need not be syntactically equal, but they have nevertheless exactly the same paths. Hence, this step is equivalent to verifying whether \((T_{C^*}, T_{D^*}) \in \triangleright_k\).

- deciding \( \triangleright \) for paths of length \( k \) is obviously linear in \( k \).
- the number of paths of length \( k \) in \( T_{C^*} \) and \( T_{D^*} \) can be bounded by \( s(C^*) \) and \( s(D^*) \), respectively.

Therefore, checking whether \((T_{C^*}, T_{D^*}) \in \triangleright_k\) can be done in time \( \mathcal{O}(s(C^*) \times s(D^*) \times k) \).

Finally, counting that concepts in reduced form are the smallest elements in their equivalence classes, we have that \( \bowtie_{R_s} \) is computable in polynomial time.

Next, we continue Example 7.19 to see how all these pieces fit together.

**Continuation of Example 7.19.** Consider the set \( S = \{t_1, t_2, t_3, t_4\} \) from Example 7.16 and the corresponding concept description \( D_S \). One can easily see that \( t_1 \) satisfies the
7.2 Reasoning in $\tau\mathcal{E}\mathcal{L}(m_{\varphi})$

Clause $c_3$ in $\varphi$ whereas $t_3$ does not. From a graphical perspective, $c_3$ corresponds to the third path (from top to bottom) of the description tree $T_{C_\varphi}$ and $t_1$ ($t_3$) to the first (third) one (from left to right) in $T_{C_S}$, i.e.:

$$
\pi_{c_3} : \{I\} \xrightarrow{r} \{I\} \xrightarrow{r} \{A\} \xrightarrow{s} \{\} \xrightarrow{s} \{\}
$$

$$
\pi_{t_1} : \{\} \xrightarrow{r} \{A\} \xrightarrow{r} \{A\} \xrightarrow{r} \{A\} \xrightarrow{r} \{A\}
$$

$$
\pi_{t_3} : \{\} \xrightarrow{r} \{A\} \xrightarrow{s} \{\} \xrightarrow{r} \{A\} \xrightarrow{r} \{A\}
$$

Note that $\pi_{c_3}$ and $\pi_{t_1}$ agree on the third position according to $[7, 3]$, but this is not the case for any position regarding $\pi_{c_3}$ and $\pi_{t_3}$. This means that $\pi_{c_3} \not\succ \pi_{t_3}$ and $\pi_{c_3} \not\succ \pi_{t_1}$. The intuition here is that $\succ$ can be used to verify whether a clause $c$ is satisfied by an assignment $t$. Hence, membership/non-membership of $(T_{C_\varphi}, T_{D_S})$ in $\succ_4$ would determine whether every assignment in $S$ satisfies $\varphi$ or not. Contrary to $T_{D_S}$, for the description tree $T_{D_{\varphi}}$ corresponding to $\theta^*[D_4]$ (see Figure 7.1) it is the case that $(T_{C_\varphi}, T_{D_{\varphi}}) \in \succ_4$. This would mean that the canonical set corresponding to $\theta^*[D_4]$ certifies the validity of the formula $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \varphi$.

Overall, membership in $\succ_4$ leads to membership in $R_s$, and subsequently to the similarity value $a$ when computing $\succeq_{R_s}$. Since $R_s$ emphasizes that only those concepts in $\mathfrak{D}_4$ are relevant, this will make satisfiability of $(C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a}$ in $\tau\mathcal{E}\mathcal{L}(m_{\succ R_s})$ to be equivalent to validity of $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \varphi$.  

Obviously, the correctness of the previous idea relies on comparing all the paths in $T_{C_\varphi}$ to all the paths in $T_{D_S}$, when assessing the value of $C_\varphi \succeq_{R_s} D_S$. Since the definition of $R_s$ uses $\succ_p$ with respect to concept descriptions in reduced form, it would be a problem to have a reducible concept $C_\varphi$. The following lemma shows that the particular use of $r$ and $s$ when building $C_\varphi$ guarantees that this is never the case.

**Lemma 7.21.** Let $\varphi$ be a propositional formula in CNF. The concept description $C_\varphi$ is in reduced form.

**Proof.** Recall that we restrict our attention to clauses where $x$ and $\neg x$ do not occur at the same time. Additionally, one can also assume that no two clauses of $\varphi$ are equal. We denote by $V(c)$ the set of variables occurring in a clause $c$.

Now, let $c_i$ and $c_j$ be two clauses of $\varphi$. Following the construction of $C_\varphi$, they correspond to the top-level atoms $\exists x_i^1.E_i^1$ and $\exists x_j^1.E_j^1$ in $C_\varphi$. We want to prove that $r_i^1 = r_j^1$ implies $E_i^1 \nsubseteq E_j^1$ and $E_j^1 \nsubseteq E_i^1$. To this end, we distinguish two cases regarding $V(c_i)$ and $V(c_j)$:

- $V(c_i) \neq V(c_j)$. This means that there is at least one variable $x$ such that $x$ occurs in $c_i$ and not in $c_j$ (or vice versa). By construction of $C_\varphi$, $c_i$ and $c_j$ contribute to $T_{C_\varphi}$ with two paths of the form:
Chapter 7. Concept similarity measures, relaxed instance queries and $\tau\mathcal{EL}(m)$

$\pi_i : \{I\} \xrightarrow{r_1^i} \cdots \xrightarrow{r_x^i} \bullet \xrightarrow{x} \pi_j : \{I\}$

$\pi_j : \{I\} \xrightarrow{r_1^j} \cdots \xrightarrow{r} \{I\}$

where $\bullet$ stands for $\{A\}$ or $\{\}$ and $r_x^i$ for $r$ or $s$, depending on how $x$ occurs in $c_i$. By construction of $C_\varphi$, the possible combinations for $(\bullet, r_x^i)$ are $(\{\}, s)$ and $(\{A\}, r)$. Then, it is not hard to verify that no subsumption relation exists between $E_i^1$ and $E_j^1$.

- $V(c_i) = V(c_j)$. Since $c_i \neq c_j$, this means that there is $x \in V(c_i)$ such that $x$ occurs in $c_i$ and $\neg x$ in $c_j$ (or vice versa). Thus, the corresponding paths have the following structure:

$x$

$c_i : \{I\} \xrightarrow{r} \{A\}$

$c_j : \{I\} \xrightarrow{s} \{\}$

Again, it is immediate to see why $E_i^1 \not\subset E_j^1$ and $E_j^1 \not\subset E_i^1$.

Hence, these arguments prove that $C_\varphi$ is already in reduced form.

Altogether we have a polynomial time computable $\text{CSM} \bowtie_{R_\varphi}$ that is equivalence-invariant, role-depth bounded and equivalence closed. Moreover, $\bowtie_{R_\varphi}$ has been defined over the finite sets $\{A, I\}$ and $\{r, s\}$ of concept and role names, respectively. Consequently, it induces the DL $\tau\mathcal{EL}(m_{\bowtie_{R_\varphi}})$ as described in Section 7.1. Then, we reduce $\text{QBF}$ to satisfiability in $\tau\mathcal{EL}(m_{\bowtie_{R_\varphi}})$ as follows. Given a quantified Boolean formula $P.\varphi$, it is translated into the $\tau\mathcal{EL}(m)$ concept description $\widehat{C}_\varphi$ defined as follows:

$$\widehat{C}_\varphi := (C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a}$$

The following lemma shows the correctness of the reduction.

**Lemma 7.22.** Let $P.\varphi$ be a quantified Boolean formula. Then, $P.\varphi$ is valid iff $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_\varphi}})$.

**Proof.** Let $n > 0$ be the length of $P$, $c_1, \ldots, c_q$ the clauses of $\varphi$, and $x_1, \ldots, x_n$ the propositional variables occurring in $\varphi$.

$(\Rightarrow)$ Assume that $P.\varphi$ is valid. To prove that $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_\varphi}})$ we select the interpretation $I$ having the following $E\mathcal{L}$ description graph:

- $\{\} \xrightarrow{s} \{A\}
- \{\} \xrightarrow{r} \{\}
- \{\} \xrightarrow{s} \{A\}
- \{\} \xrightarrow{r} \{\}
- \{\} \xrightarrow{s} \{A\}$
Our goal is to show that $d_0 \in (\hat{C}_\varphi)^I$. By Definition 7.2

$$m^I_{\bowtie R_s}(d_0, C_\varphi) = \max\{C_\varphi \bowtie R_s D \mid D \in C_{\mathcal{EL}} \text{ and } d_0 \in D^I\}$$

Since $d_0 \notin I^I$, this means that for all the candidate concepts $D$ we have $C_\varphi \neq D$. Therefore, $m^I_{\bowtie R_s}(d_0, C_\varphi) < 1$. In particular, if $D \in \mathcal{D}_n$ it is easy to see that there is a homomorphism $\varphi$ from $T_D$ to $G_I$ mapping the root $v_0$ of $T_D$ to $d_0$. Hence, by Theorem 7.2, we obtain $d_0 \in D^I$.

Now, since $P_\varphi$ is valid, there is a set $S$ of $P$-assignments that is canonical for $P$ such that every truth assignment $t \in S$ satisfies $\varphi$. Let $D_S \in \mathcal{D}_n$ be an $\mathcal{EL}$ concept description such that $S$ and $D_S$ are corresponding (see Lemma 7.18). Then, we know the following about $C_\varphi$ and $D_S$:

- $rd(C_\varphi) = rd(D_S) = n$,
- $d_0 \in (D_S)^I$, and
- $C_\varphi \bowtie R_s D_S < 1$.

Let us now establish that $(T_{C_\varphi}, T_{D_S}) \in D_n$. Consider two arbitrary paths $\pi$ and $\pi_j$ of length $n$ in $T_{D_S}$ and $T_{C_\varphi}$, respectively. By construction of $C_\varphi$ and Definition 7.17, we have:

- $\pi_j$ corresponds to the clause $c_j$ in $\varphi$.
- since $S$ and $D_S$ are corresponding, there exists $t_\pi \in S$ such that $t_\pi$ and $\pi$ are corresponding.

As $t_\pi$ satisfies $c_j$, it must exist at least one literal $\ell$ in $c_j$ such that $t_\pi(\ell) = true$. Let $x_i$ be the variable corresponding to $\ell$. Then, for the $i^{th}$ node $v_i$ of $\pi_j$:

$$\ell_{T_{C_\varphi}}(v_i) = \{A\} \text{ if } \ell = x_i, \text{ and } \ell_{T_{C_\varphi}}(v_i) = \{\} \text{ if } \ell = \neg x_i$$

and (since $t_\pi$ and $\pi$ are corresponding) for the node $w_i$ of $\pi$:

$$t_\pi(x_i) = true \iff \ell_{T_{D_S}}(w_i) \cap \{A\} = \{A\}$$

Hence, one can see that $t_\pi$ satisfies $\ell$ iff $\ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{D_S}}(w_i) \cap \{A\}$. Consequently, $\pi_j \triangleright \pi$. Since these two paths have been chosen arbitrarily, we just have shown that $(T_{C_\varphi}, T_{D_S}) \in D_n$. Having $D_S \in \mathcal{D}_n$ further implies that $(C_\varphi, D_S) \in R_s$, and this means that $C_\varphi \bowtie R_s D_S = a$ (see the expression $\mu(C, D)$ in Definition 7.5). Thus, $m^I_{\bowtie R_s}(d_0, C_\varphi) = a$ and $d_0 \in (\hat{C}_\varphi)^I$.

$(\Leftarrow)$ Assume that $\hat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie R_s})$. Then, there exists an interpretation $I$ and $d \in \Delta^I$ such that $d \in (\hat{C}_\varphi)^I$. This means that $m^I_{\bowtie R_s}(d, C_\varphi) = a$. Thus, there exists a concept $D$ such that $d \in D^I$ and $C_\varphi \bowtie R_s D = a$. By definition of $\bowtie R_s$ this is the case if $(C_\varphi, D) \in R_s$, and membership in $R_s$ for $(C_\varphi, D)$ implies the existence of a concept $D^* \in \mathcal{D}_n$ such that:

$$D \equiv D^* \text{ and } (T_{C_\varphi}, T_{D^*}) \in D_n$$
Now, by Lemma 7.18 there is a set of $P$-assignments $S_{D^*}$ such that:

- $S_{D^*}$ is canonical for $P$, and
- $S_{D^*}$ and $D^*$ are corresponding.

The rest consists of proving that each assignment in $S_{D^*}$ satisfies $\varphi$. Let $t \in S_{D^*}$ and $c_j$ ($1 \leq j \leq q$) be a clause of $\varphi$. Again, by construction of $C_\varphi$ and Definition 7.17 we obtain:

- $c_j$ corresponds to a path $\pi_j$ in $T_{C_\varphi}$.
- since $S_{D^*}$ and $D^*$ are corresponding, there exists a path $\pi_t$ in $T_{D^*}$ of length $n$ such that $t$ and $\pi_t$ are also corresponding.

From $(T_{C_\varphi}, T_{D^*}) \in \triangleright_n$, it follows that there is $0 \leq i \leq n$ such that (7.3) holds with respect to $\pi_j$ and $\pi_t$, i.e.:

$$\ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{D^*}}(w_i) \cap \{A\} \text{ and } I \notin \ell_{T_{C_\varphi}}(v_i)$$

Note that $I \notin \ell_{T_{C_\varphi}}(v_0)$, and consequently $i > 0$. Let $\ell$ be the literal corresponding to the occurrence of the variable $x_i$ in $c_j$. The relationships from (7.4) and (7.5) are also valid in this case. Then, combining them with the previous equality ensures that $t$ satisfies $c_j$. This will always be the case for all $t \in S_{D^*}$ and $c_j$ in $\varphi$. Thus, $P.\varphi$ is valid.

Thus, we have shown PSPACE-hardness for satisfiability in $\tau\mathcal{EL}(m\triangleright_R_s)$.

**Corollary 7.23.** In $\tau\mathcal{EL}(m\triangleright_R_s)$, concept satisfiability is PSPACE-hard.

The standard reductions from satisfiability to the other reasoning problems (subsumption, consistency and instance) yield PSPACE-hardness for them as well.

Now, some additional information emerges from the previous result and its proof. The specific set of all satisfiable $\tau\mathcal{EL}(m\triangleright_R_s)$ concept descriptions of the form $\hat{C}_\varphi$ constitutes a PSPACE-hard language. Moreover, the proof of Lemma 7.22 shows that all these concepts are satisfiable in a model of polynomial size. Intuitively, the source of complexity resides on the fact that model checking $(C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a}$ requires to consider concepts of size exponential in $\hat{C}_\varphi$. These are the ones fixed by the definition of $R_s$, and its structure is succinctly encoded within the shape of the interpretation $I$ selected in Lemma 7.22.

Based on these observations, we now move into defining a bounding condition that is sufficient to safely disregard such a big concept descriptions.

### 7.2.4 Bounded CSMs

This section is organized as follows. We start right away by defining the bounded condition on CSMs. Afterwards, we shall restrict our attention to polynomially bounded CSMs, and study the computational aspects of the DLs induced by such a family of measures.

**Definition 7.24.** A filter $F$ is a subset of $\mathcal{C}_{\mathcal{EL}}$ such that for all $C, D \in F$:
7.2 Reasoning in \( \tau \mathcal{EL}(m_{\infty}) \)

- \( C \cap D \in F \), and
- \( C \sqsubseteq C' \) implies \( C' \in F \).

The set of all filters is denoted as \( \mathcal{F} \). Now, let \( g : \mathbb{N} \to \mathbb{N} \) be a function. We say that a CSM \( \bowtie \) is \( g \)-bounded iff for all \( \mathcal{EL} \) concept descriptions \( C \) and all filters \( f \in \mathcal{F} \), there is \( D \in f \) such that:

- \( C \bowtie D = \max \{ C \bowtie X \mid X \in f \} \), and
- \( s(D) \leq g(s(C)) \).

Establishing \( g \)-boundedness for a CSM \( \bowtie \) provides a form to estimate a more accurate bound for the size of models resulting from the general construction offered in Section 7.2.2. In particular, when \( g \) is a polynomial \( p(n) = n^k \), it yields a polynomial model property for the induced logic \( \tau \mathcal{EL}(m_{\infty}) \).

**Proposition 7.25.** Let \( p(n) = n^k \) be a polynomial and \( \hat{C} \) a \( \tau \mathcal{EL}(m) \) concept description. Moreover, let \( \bowtie \) be an equivalence invariant, role-depth bounded, equivalence closed and \( p(n) \)-bounded CSM. If \( \hat{C} \) is satisfiable in \( \tau \mathcal{EL}(m_{\infty}) \), then there exists a tree-shaped interpretation \( J \) such that \( \hat{C}_J \neq \emptyset \) and \( |\Delta^J| \leq p(s(\hat{C})) \).

**Proof.** Let \( (E_1)_{\sim_1}, \ldots, (E_n)_{\sim_n} \) be the threshold concepts occurring in \( \hat{C} \) with \( \sim \in \{ >, \geq \} \). Conversely, let \( (F_1)_{\sim_{s_1}}, \ldots, (F_q)_{\sim_{s_q}} \) be the ones where \( \sim \in \{ <, \leq \} \). Then, the size of \( \hat{C} \) can be expressed as:

\[
|\Delta^{I_0}| + \sum_{i=1}^{n} s((E_i)_{\sim_i}) + \sum_{j=1}^{q} s((F_j)_{\sim_{s_j}})
\]

Proposition 7.13 in the previous section shows that the construction used in Lemma 5.4 can be applied to \( \tau \mathcal{EL}(m_{\infty}) \) to obtain a finite interpretation \( J \) satisfying \( \hat{C} \) such that:

\[
|\Delta^J| = |\Delta^{I_0}| + \sum_{i=1}^{n} |\Delta^{I_{D_i}}|
\]

Here, \( I_{D_i} \) is a canonical interpretation in the sense of Definition 7.10 that by construction of \( J \) has been “extracted” from \( I \). This means that there is an element \( d \in \Delta^J \) such that \( d \in (D_i)^J \). Moreover, \( D_i \) satisfies \( m_{\bowtie}(d, E_i) = E_i \bowtie D_i \), and by definition of \( m_{\bowtie} \), we then have:

\[ E_i \bowtie D_i = \max \{ E_i \bowtie D \mid D \in C_{\mathcal{EL}} \text{ and } d \in D^J \} \quad (7.6) \]

Now, \( D_i \) generates the filter \( f_i \) containing all the concepts \( X \) such that \( D_i \sqsubseteq X \). Therefore, there are two things one can say about \( f_i \):

- \( d \in X^J \) for all \( X \in f_i \), and
- \( p(n) \)-boundedness of \( \bowtie \) yields a concept \( D_i^* \in f_i \) such that:
  - \( E_i \bowtie D_i^* = \max \{ E_i \bowtie X \mid X \in f_i \} \),
  - \( s(D_i^*) \leq p(s(E_i)) \).
Since $D_i \in f_i$ and $f_i$ is a subset of the set considered in (7.6), this means that $E_i \bowtie D_i = E_i \bowtie D_i^*$. Hence, Lemma 7.12 makes possible to choose $D_i^*$ in the place of $D_i$ to build $\mathcal{J}$. Therefore, without loss of generality, we can assume that $|\Delta J_i| \leq p(s(E_i))$ for all $1 \leq i \leq n$. Taking this into account, the size of $\mathcal{J}$ can be bounded by the following expression:

$$|\Delta \mathcal{J}| \leq |\Delta \mathcal{J}_0| + \sum_{i=1}^{n} p(s(E_i))$$

(7.7)

Thus, since $p(n) = n^k$, this means that $|\Delta \mathcal{J}| \leq p(s(\hat{C}))$.

Hence, $p(n)$-boundedness defines a family of CSMs for which the logic $\tau \mathcal{E} \mathcal{L}(m_{\bowtie})$ enjoys the polynomial model property. The proof of Proposition 7.25 describes how to compute such a bound, provided that $p(n)$ is known. Furthermore, contrary to $\bowtie_{R_s}$ and $\tau \mathcal{E} \mathcal{L}(m_{\bowtie_{R_s}})$, model checking a concept $\hat{C}$ on an interpretation $\mathcal{I}$ will not need to consider exponentially large concepts in the size of $\hat{C}$. We denote by $\mathcal{F}_{\bowtie_{\mathcal{E}}}[\text{poly}]$ the family of equivalence invariant, role-depth bounded and equivalence closed CSMs, that are polynomially bounded. Later in Section 7.3 we will identify a concrete set of CSMs that are part of this family.

Now, since we have focused our interest in CSMs that are computable in polynomial time, one might think of the algorithm presented in Chapter 5 (for $\text{deg}$) as a general purpose NP-algorithm to decide satisfiability in $\tau \mathcal{E} \mathcal{L}(m_{\bowtie})$. However, differently from $\tau \mathcal{E} \mathcal{L}(\text{deg})$ a polynomial bound does not ensure that such algorithm will always run in non-deterministic polynomial time. Intuitively, there are two reasons for this:

- $m_{\bowtie}$ is defined as maximization on top of $\bowtie$, and
- the NP-decision procedure from Chapter 5 uses Algorithm 2 to check the existence of a $\tau$-homomorphism. In particular, in line 6 it is required to check whether $m_{\bowtie}^2(d, E) \sim t$ for some $d \in \Delta \mathcal{J}$ and a threshold concept $E_{\bowtie t}$.

Where could the interaction of these two aspects be harmful? Being the threshold concept of the form $E_{\bowtie t}$ or $E_{\geq t}$, the maximization in the definition of $m_{\bowtie}$ allows to handle this by simply guessing an $E$ concept description $D$ such that $d \in D^I$ and $E \bowtie D > t$ (or $\geq$). Here, the second benefit of having $p(n)$-boundedness comes into play: the size of such $D$ is polynomial in the size of $E$. The problem arises, however, in the presence of threshold concepts of the form $E_{< t}$ or $E_{\leq t}$. In this case the same strategy will not suffice, since for instance having $E \bowtie D < t$ for a particular $D$ does not ensure $m_{\bowtie}^2(d, E) < t$.

A natural way to repair this problem is to use an NP-algorithm as an oracle, which verifies whether there exists $D$ such that $d \in D^I$ and $E \bowtie D \geq t$. Then, a “no” answer from the oracle will definitely certify that $m_{\bowtie}^2(d, E) < t$. The following lemma provides an NP$^{\text{NP}}$-algorithm that decides concept satisfiability in $\tau \mathcal{E} \mathcal{L}(m_{\bowtie})$, provided that $\bowtie$ is polynomially bounded and polynomial time computable.

Lemma 7.26. Let $p(n) = n^k$ be a polynomial and $\bowtie$ an equivalence invariant, role-depth bounded, equivalence closed and $p(n)$-bounded CSM. Additionally, $\bowtie$ can be computed in polynomial time. Then, in $\tau \mathcal{E} \mathcal{L}(m_{\bowtie})$ it is in NP$^{\text{NP}}$ to decide whether a concept is satisfiable.
Proof. Assume that we want to decide satisfiability of the concept \( \hat{C} \) in \( \tau \mathcal{EL}(m_{\text{deg}}) \). The NP-algorithm we are going to use as an oracle solves the following problem:

- **Instance:** A tuple \((I, d, C, t, \sim)\) where \( I \) is a finite interpretation, \( d \in \Delta^I \), \( C \) a concept description, \( t \in \mathbb{Q} \cap [0, 1] \) and \( \sim \in \{>, \geq\} \).

- **Question:** Is there an \( \mathcal{EL} \) concept description \( D \) such that \( d \in D^I \) and \( C \bowtie D \sim t \)?

Based on the definition of \( m_{\text{deg}} \) and the special properties satisfied by \( \bowtie \), we observe the following:

- \( m_{\text{deg}}(d, C) \sim t \) iff there exists such a concept \( D \). This is a consequence of the maximization used to define \( m_{\text{deg}} \) and \( \sim \in \{>, \geq\} \).

- \( p(n)\)-boundedness of \( \bowtie \) means that \( m_{\text{deg}}(d, C) = C \bowtie D' \), where \( d \in (D')^I \) and \( s(D') \leq p(s(C)) \).

These observations permit to reduce the search space to concepts of size at most \( p(s(C)) \). Moreover, testing for \( d \in D^I \) and computing \( C \bowtie D \) are both polynomial time tasks in the size of \( I, C \) and \( D \). Hence, the algorithm first guesses a concept description \( D \) of size at most \( p(s(C)) \), and then verifies whether \( d \in D^I \) and \( C \bowtie D \sim t \). It answers “yes” if both checks succeed, and “no” otherwise.

Now, the NP\(^{\text{NP}}\)-procedure behaves as follows:

1. Guess an interpretation \( J \) of size at most \( p(s(\hat{C})) \) (or use expression (7.7) for a tighter bound).

2. Use Algorithm 2 \(^{2}\) to check whether there exists a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_J \) with the following modifications. Whenever the test \( m_J^\tau(w, E) \sim t \) in line \(^{3}\) needs to be executed, it is handled by calling the oracle on \((J, w, E, t, \diamond)\) where \( \diamond \) is selected as \( \geq \) if \( \sim \in \{<, \geq\} \), or as \( > \) otherwise. The resulting pair \((\sim, \text{oracle's answer})\) determines the truth of the aforementioned test as follows:
   - the pairs \((<, \text{"no"}), (\leq, \text{"no"}), (\geq, \text{"yes"})\) and \((>, \text{"yes"})\) result in a positive answer to the question of whether \( m_J^\tau(w, E) \sim t \). In any other case the statement is false.

3. Answer \( \hat{C} \) is satisfiable iff there exists a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_J \).

By Proposition 7.25 it is sufficient to look only at interpretations of size at most \( p(s(\hat{C})) \). The characterization of membership for \( \tau \mathcal{EL}(m) \) given in Theorem 3.8 does not depend on which graded membership function \( m \) is considered. Hence, it is correct to use Algorithm 2 \(^{2}\) for \( \tau \mathcal{EL}(m_{\text{deg}}) \). Furthermore, it is not hard to see that the introduced modification is consistent with whether \( m_{\text{deg}}(w, E) \sim t \). For instance, to check \( m_{\text{deg}}^\tau(w, E) < t \) the oracle is invoked with \( \diamond = \geq \). A “no” answer means that there is no \( D \) such that \( w \in D^J \) and \( E \bowtie D \geq t \), which clearly implies \( m_{\text{deg}}^\tau(w, E) < t \). Finally, the number of calls to the oracle is at most \( s(\hat{C}) \times |\Delta^J| \). These arguments prove that concept satisfiability in \( \tau \mathcal{EL}(m_{\text{deg}}) \) is in NP\(^{\text{NP}}\). \( \square \)
The use of an oracle by this procedure is somehow forced by the fact that we stick to a specific approach, and in particular such method uses Algorithm 2 to decide a model checking problem. Moreover, as we will see in Section 7.3.2, deg is a function that can be obtained from a CSM $\triangleright_{\text{CM}}$ of the kind being considered, but its satisfiability problem is in NP. Therefore, an obvious question is whether this is a really “naive” way to decide satisfiability for the family of logics induced by such a class of CSMs.

We will now define a CSM that, at least in terms of worst-case complexity, suggests that the previous algorithm may not be so bad. More precisely, we slightly modify $R_s$ into $R^*_s$ such that the CSM $\triangleright_{R^*_s}$ obtained as in Definition 7.5, is polynomially bounded and induces a logic $\tau \mathcal{EL}(m \triangleright_{R^*_s})$ where concept satisfiability is both NP-hard and coNP-hard.

The problems we are going to use for the reductions are satisfiability and unsatisfiability of propositional formulas in conjunctive normal form, which are well-known to be complete for the classes NP and coNP, respectively [GJ79].

Since propositional satisfiability corresponds to validity of quantified Boolean formulas of the form $P.\varphi$ where $P$ only contains existential quantifiers, the elements defining $R_s$ can be modified to obtain $R^*_s$. Basically, starting with the relation $\triangleright$ a family of binary relations $\triangleright^*_p$ (for all $p \in \mathbb{N}$) is built as follows:

$$(T_1, T_2) \in \triangleright^*_p \iff \exists \pi_2 \in \Pi_p(T_2) \text{ s.t. } \forall \pi_1 \in \Pi_p(T_1), \text{ it holds } \pi_1 \triangleright \pi_2$$

Then, the relation $R^*_s$ is defined in the following form:

$$R^*_s := \{(C, D) \mid (T_{C^r}, T_{D^r}) \in \triangleright^*_p(\text{rd}(C))\}$$

There are two main differences between $R_s$ and $R^*_s$. First, the definition of $\triangleright^*_p$ poses an existential quantification over $\Pi_p(T_2)$. This has to do with the fact that for propositional satisfiability only one “good” assignment (path in a description tree) needs to be found. Second, the special concepts used to represent the structure of certificates for QBF are no longer needed. Therefore, the final definition of $R^*_s$ is limited to checking for membership into $\triangleright^*_p(\text{rd}(C))$.

Concerning its computational properties, checking for membership into $\triangleright^*_p$ requires at most the same number of comparisons as for $\triangleright_p$, and as explained in the previous section for $\triangleright_p$, it can be done in polynomial time. Therefore, $\triangleright_{R^*_s}$ is a polynomial time computable CSM. In addition, the following lemma shows that $\triangleright_{R^*_s}$ is polynomially bounded.

**Lemma 7.27.** The CSM $\triangleright_{R^*_s}$ is linear bounded.

**Proof.** Let us fix a filter $f \in \mathfrak{F}$ and a concept description $C$ with $\text{rd}(C) = k$. Now, let $D \in f$ such that $C \triangleright_{R^*_s} D = \max\{C \triangleright_{R^*_s} X \mid X \in f\}$. We make a case distinction on the possible values of $C \triangleright_{R^*_s} D$ (see Definition 7.5):

- $C \triangleright_{R^*_s} D = 1$. Then, $C \equiv D$ and obviously $C \in f$.
- $C \triangleright_{R^*_s} D = a$. This means that $C \neq D$, $\text{rd}(C) = \text{rd}(D)$ and $(C, D) \in R^*_s$. By definition of $R^*_s$ we know that $(T_{C^r}, T_{D^r}) \in \triangleright^*_p$. Then, there exists a path $\pi_D$ of...
length \( k \) in \( T_{D'} \) such that for all paths \( \pi_C \in \Pi_k(T_{C'}) \), it is true that \( \pi_C \triangleright \pi_D \).

Let \( \pi_{D'} \) be the path that results by replacing the labels in \( \pi_D \) by their intersection with \( \{A\} \) (as required in the definition of \( \triangleright \)). We denote as \( D' \) the \( \mathcal{E} \mathcal{L} \) concept description corresponding to \( \pi_{D'} \) (when seen as a description tree).

- Clearly, \( \pi_C \triangleright \pi_{D'} \) still holds for all paths \( \pi_C \in \Pi_k(T_{C'}) \). This means that

\[
(T_{C'}, T_{D'}) \in \triangleright_k^*, \quad \text{and consequently} \quad (C, D') \in R^*_s \quad \text{(notice that \( D' \) is already in reduced form)}.
\]

- It is not hard to see that \( D \sqsubseteq D' \). This implies that \( D' \in f \), and since \( D \) has been assumed to be maximal within \( f \) for \( C \triangleright_R^* D \), it must be the case that \( C \neq D' \). For otherwise, it would contradict \( C \triangleright_R^* D = a \).

Overall, this means that \( C \triangleright_R^* D' = a \).

\[ C \triangleright_R^* D = 0. \] Clearly, \( T \in f \) since \( D \sqsubseteq T \). Moreover, the maximality of \( D \) implies that \( C \triangleright_R^* T = 0 \).

In any case, there is always a concept \( D' \) such that \( D' \in f \), \( C \triangleright_R^* D' \) is maximal in \( f \) and \( s(D') \leq g(s(C)) \) where \( g(n) = n \). Thus, \( \triangleright_R^* \) is linear bounded.

Finally, likewise \( R_s \) the relation \( R^*_s \) is equivalence invariant.

**Lemma 7.28.** The relation \( R^*_s \) is equivalence invariant.

**Proof.** Let \( C, C', D \) and \( D' \) be \( \mathcal{E} \mathcal{L} \) concepts such that \( C \equiv C' \) and \( D \equiv D' \). We show that \( (C, D) \in R^*_s \) iff \( (C', D') \in R^*_s \). As pointed out for \( R_s \):

- Since \( C \equiv C' \) and \( D \equiv D' \), we have that \( r_d(C) = r_d(C') = k \) for some \( k \geq 0 \).

Moreover, by Theorem 2.8 there are isomorphisms between \( T_{C'} \) and \( T_{(C')'} \), and between \( T_{D'} \) and \( T_{(D')'} \).

Suppose that \( (T_{C'}, T_{D'}) \in \triangleright_k^* \). This means that there is a path \( \pi_D \) in \( T_{D'} \) of length \( k \), such that for all paths \( \pi_C \in \Pi_k(T_{C'}) \) it holds \( \pi_C \triangleright \pi_D \). Using the isomorphism mentioned above, one can find a path \( \pi_{D'} \) in \( T_{(D')'} \) such that \( \pi_D = \pi_{D'} \). To see that also \( (T_{(C')'}, T_{(D')'}) \in \triangleright_k^* \), let \( \pi_{C'} \) be an arbitrary path of length \( k \) in \( T_{(C')'} \). It will be enough to show that \( \pi_{C'} \triangleright \pi_{D'} \). Again, the isomorphism yields a path \( \pi_C \in \Pi_k(T_{C'}) \) such that \( \pi_C = \pi_{C'} \). Hence, since \( \pi_C \triangleright \pi_D \) and \( \pi_D = \pi_{D'} \), this implies that \( \pi_{C'} \triangleright \pi_{D'} \).

We have thus shown that \( (T_{C'}, T_{D'}) \in \triangleright_k^* \Rightarrow (T_{(C')'}, T_{(D')'}) \in \triangleright_k^* \). The implication in the opposite direction can be obtained in a similar way. Therefore, \( (C, D) \in R^*_s \Leftrightarrow (C', D') \in R^*_s \), and \( R^*_s \) is equivalence invariant.

Thus, \( m_{\triangleright_R^*} \) is a well-defined graded membership function and it induces the DL \( \tau\mathcal{E}\mathcal{L}(m_{\triangleright_R^*}) \). To show NP-hardness of satisfiability in \( \tau\mathcal{E}\mathcal{L}(m_{\triangleright_R^*}) \), we use exactly the same translation as in the previous section: given a propositional formula \( \varphi \) in conjunctive normal form its corresponding \( \tau\mathcal{E}\mathcal{L}(m) \) concept description \( \hat{C}_\varphi \) is of the form \( (C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a} \).

**Lemma 7.29.** Let \( \varphi \) be a propositional formula in CNF of the form \( c_1 \land \ldots \land c_q \), and \( x_1, \ldots, x_n \) the variables occurring in \( \varphi \). Then, \( \varphi \) is satisfiable iff \( \hat{C}_\varphi \) is satisfiable in \( \tau\mathcal{E}\mathcal{L}(m_{\triangleright_R^*}) \).
Proof. (⇒) Assume that $\varphi$ is satisfiable. To show that $\hat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\triangleright R^*_\mathcal{A}})$ we choose the interpretation $I$ that has the following $\mathcal{EL}$ description graph:

$$d_0 : \{\} \xrightarrow{r} d_1 : \{A\} \xrightarrow{r} d_2 : \{A\} \ldots \xrightarrow{r} d_n : \{A\}$$

We want to show that $d_0 \in (\hat{C}_\varphi)^I$. By Definition 7.2 we have:

$$m^I_{\triangleright R^*_\mathcal{A}}(d_0, C_\varphi) = \max\{C_\varphi \triangleright R^*_\mathcal{A} D \mid D \in \mathcal{EL} \text{ and } d_0 \in D^I\}$$

Now, since $\varphi$ is satisfiable, there is a truth assignment $t$ satisfying each clause in $\varphi$. Such an assignment induces the $\mathcal{EL}$ concept description $D_t = \exists r. F^t_1$:

$$F^t_1 := \lambda_1 \sqcap \exists r. F^t_2$$

$$\vdots$$

$$F^t_i := \lambda_i \sqcap \exists r. F^t_{i+1} \quad (1 \leq i < n)$$

$$\vdots$$

$$F^t_n := \lambda_n$$

where $t(x_i) = \text{true}$ implies $\lambda_i = A$, and $\lambda_i = \top$ otherwise. It is straightforward to see that there is a homomorphism from $T_{D_t}$ to $G_I$ mapping the root of $T_{D_t}$ to $d_0$. Therefore, $d_0 \in (D_t)^I$. Let us now look at the value $C_\varphi \triangleright R^*_\mathcal{A} D_t$.

- The description tree $T_{D_t}$ has a single path $\pi_t = w_0rw_1 \ldots rw_n$, and its labeling is determined by the values $\lambda_1, \ldots, \lambda_n$.

- There are $q$ paths $\pi_1, \ldots, \pi_q$ in $T_{C_\varphi}$ such that $\pi_j$ is induced by the top-level atom $C_j$ in $C_\varphi$. At the same time, $C_j$ corresponds to the clause $c_j$ of $\varphi$.

- Let $\ell$ be a literal in $c_j$ such that $t(\ell) = \text{true}$ and $x_i$ the corresponding variable (it exists because $t$ satisfies $\varphi$). By construction of $C_j$, we have two possibilities:

  - $\ell = x_i$ and $\ell_{T_{C_j}}(v_i) = \{A\}$. Since $t(\ell) = \text{true}$, this means that $\lambda_i = A$ and $\ell_{T_{C_j}}(v_i) = \ell_{T_{D_t}}(w_i) = \{\}$. Thus, according to (7.3) it follows $\pi_j \triangleright \pi_t$.

  - $\ell = \neg x_i$ and $\ell_{T_{C_j}}(v_i) = \{\}$. The same argument as before yields $\lambda_i = \top$ and $\ell_{T_{C_j}}(v_i) = \ell_{T_{D_t}}(w_i) = \{\}$. Consequently, $\pi_j \triangleright \pi_t$.

Overall, this means that $\pi_j \triangleright \pi_t$ for all $1 \leq j \leq q$. Hence, $(T_{C_\varphi}, T_{D_t}) \in \triangleright^*_n$ and $C_\varphi \triangleright R^*_\mathcal{A} D_t = a$. Moreover, $d_0 \notin I^D$ implies $C_\varphi \ntriangleleft D$ for all $D$ such that $d_0 \in D^I$. Thus, we can conclude that $m^I_{\triangleright R^*_\mathcal{A}}(d_0, C_\varphi) = a$ and $d_0 \in (\hat{C}_\varphi)^I$.

(⇐) Assume that $\hat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\triangleright R^*_\mathcal{A}})$. Then, there exists an interpretation $I$ and $d \in \Delta^I$ such that $d \in (\hat{C}_\varphi)^I$. This means that $m^I_{\triangleright R^*_\mathcal{A}}(d, C_\varphi) = a$. By definition of $m_{\triangleright R^*_\mathcal{A}}$ there must exist a concept $D$ such that $d \in D^I$ and $C \triangleright R^*_\mathcal{A} D = a$. Moreover, since $\triangleright R^*_\mathcal{A}$ is based on the relation $R^*_\mathcal{A}$ as constructed in Definition 7.3 we further have that $rd(C_\varphi) = rd(D) = n$ and $(T_{C_\varphi}, T_D) \in \triangleright^*_n$. Hence, there exists a path $\pi$ in $T_D$ such that for all $1 \leq j \leq q$ it holds $\pi_j \triangleright \pi$, where $\pi_j$ is the path in $T_{C_j}$ corresponding the clause $c_j$ of $\varphi$. 


Let \( \pi \) be of the form \( w_0r_1w_1 \ldots r_nw_n \), the assignment \( t_\pi \) is built as follows. For all \( 1 \leq i \leq n \):

\[
    t_\pi(x_i) := \begin{cases} 
        \text{true} & \text{if } A \in \ell_{T_{Dr}}(w_i) \\
        \text{false} & \text{otherwise.}
    \end{cases}
\]

Then, we show that \( t_\pi \) satisfies \( \varphi \). For any clause \( c_j \) of \( \varphi \), its corresponding top-level atom \( C_j \) in \( C_\varphi \) induces a path \( \pi_j = v_0r_1v_1 \ldots r_nv_n \) in \( T_{C_\varphi} \). We have already seen that \( \pi_j \triangleright \pi \), and this means that there is \( 0 \leq i \leq n \) such that:

\[
    \ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{Dr}}(w_i) \cap \{A\} \quad \text{and} \quad I \not\in \ell_{T_{C_\varphi}}(v_i)
\]

Since \( I \in \ell_{T_{C_\varphi}}(v_0) \), we know that \( i > 0 \). This means that the variable \( x_i \) occurs in \( c_j \). If \( x_i \) occurs in a positive form, by construction of \( C_\varphi \) we have that \( \ell_{T_{C_\varphi}}(v_i) = \{A\} \) and \( A \in \ell_{T_{Dr}}(w_i) \). Hence, it must be the case that \( t_\pi(x_i) = \text{true} \) and \( t_\pi \) satisfies \( c_j \). The case where \( \neg x_i \) occurs in \( c_j \) can be treated in a similar way.

Thus, we have shown that \( t_\pi \) satisfies \( \varphi \).

Next, we establish the coNP lower bound by a reduction from the non-satisfiability problem. Based on the previous reduction, notice that \((C_\varphi)_<\) represents that an assignment does not satisfy \( \varphi \). However, since unsatisfiability means that all possible assignments fail to satisfy \( \varphi \), we additionally need to ensure that all of them are taken into account. To this end, we introduce the concept \( C_{all} := \exists r.A^1 \) where \( A^1 \) is of the following form:

\[
    A^1 := A \sqcap \exists r.A^2 \\
    \ldots \\
    A^i := A \sqcap \exists r.A^{i+1} \quad (1 \leq i < n) \\
    \ldots \\
    A^n := A
\]

Then, given a propositional formula \( \varphi \) in CNF its corresponding \( \tau EL(m) \) concept description \( \hat{C}_\varphi^* \) has the following definition:

\[
    \hat{C}_\varphi^* := (C_\varphi <) \sqcap C_{all}^n
\]

**Lemma 7.30.** \( \varphi \) is unsatisfiable iff \( \hat{C}_\varphi^* \) is satisfiable in \( \tau EL(m_{\geq R}) \).

**Proof.** \((\Rightarrow)\) Assume that \( \varphi \) is unsatisfiable and let \( \mathcal{I} \) be the interpretation having the following description graph:

\[
    d_0 : \{\} \xrightarrow{r} d_1 : \{A\} \xrightarrow{r} d_2 : \{A\} \xrightarrow{r} \ldots \xrightarrow{r} d_n : \{A\}
\]

We want to show that \( d_0 \in (\hat{C}_\varphi^*)^T \). Notice that this is exactly the description tree associated to the concept \( C_{all}^n \), and consequently \( d_0 \in (C_{all}^n)^T \). Hence, it remains to show that \( d_0 \in [(C_\varphi <)^T \). By Lemma 7.29 we obtain that \( \hat{C}_\varphi \) is unsatisfiable in \( \tau EL(m_{\geq R}) \). Looking at the definition of \( \hat{C}_\varphi \), this means that for all interpretations \( \mathcal{J} \) and \( d \in \Delta \mathcal{J} \) it
Chapter 7. Concept similarity measures, relaxed instance queries and $\tau\mathcal{E}\mathcal{L}(m)$

holds:  
$$d \not\in [(C_\varphi)_{\leq a} \cap (C_\varphi)_{\geq a}]^J$$

Therefore, there are two possible scenarios for $d$:

$$m_\varphi^J (d, C_\varphi) < a \quad \text{or} \quad m_\varphi^J (d, C_\varphi) > a$$

Since the similarity values computed by $\vartriangleleft R_\ast$ range over the set $\{0, a, 1\}$, the second case is only valid when $d \in (C_\varphi)^J$ ($\vartriangleleft R_\ast$ is equivalence closed). Now, the concept name $I$ is a top-level atom of $C_\varphi$. This means, whenever $d \not\in I$ it must be that $m_\varphi^J (d, C_\varphi) < a$. This is actually the case for $d_0$ in $I$. Thus, $m_\varphi^J (d_0, C_\varphi) < a$ and $d_0 \in [(C_\varphi)_{< a}]^J$.

(⇐) Conversely, suppose that $\varphi$ is satisfiable. Based on a truth assignment $t$ satisfying $\varphi$, in the proof of Lemma 7.29 a concept $D_t$ is built such that:

$$C_\varphi \vartriangleleft R_\ast D_t = a$$

Moreover, it can also be seen that $D_t$ is such that $C_{all}^n \subseteq D_t$. Hence, for all interpretations $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$, having $d \in C_{all}^n$ implies:

$$m_\varphi^J (d, C_\varphi) \geq a$$

Thus, $\hat{C}_\varphi$ is unsatisfiable in $\tau\mathcal{E}\mathcal{L}(m_\varphi^J)$.

As a consequence of the previous two lemmas, we obtain the following computational lower bounds for satisfiability in $\tau\mathcal{E}\mathcal{L}(m_\varphi^J)$.

**Lemma 7.31.** In $\tau\mathcal{E}\mathcal{L}(m_\varphi^J)$, satisfiability is NP-hard and coNP-hard.

Overall, $p(n)$-boundedness of a CSM $\bowtie$ yields the following results for $\tau\mathcal{E}\mathcal{L}(m_\bowtie)$.

**Theorem 7.32.**

1. For all $\bowtie \in \mathcal{F}_\bowtie[\text{poly}]$, if $\bowtie$ is polynomial time computable and the polynomial $p(n)$ corresponding to its boundedness is known, then in $\tau\mathcal{E}\mathcal{L}(m_\bowtie)$ satisfiability is in $\Sigma^p_2$.

2. There is at least one CSM $\bowtie \in \mathcal{F}_\bowtie[\text{poly}]$ (for instance $\bowtie_{R^\ast}$), such that in $\tau\mathcal{E}\mathcal{L}(m_\bowtie)$ satisfiability is NP-hard and coNP-hard.

Similar to the decidability results from Section 7.2.2, the base model built in Proposition 7.25 and the $p(n)$-boundedness property can be used to obtain a polynomial model property for satisfiability of concepts of the form $\hat{C} \sqcap \neg \hat{D}$. Hence, the procedure described in Lemma 7.26 can easily be extended to obtain an NP$^{\text{NP}}$-decision procedure for the complement of the subsumption problem. Likewise, such a small model property exists also for consistency of ABoxes of the form $\mathcal{A} \cup \{\neg \hat{C}(a)\}$ with respect to the size of $\mathcal{A}$. Therefore, by using Algorithm 3 we obtain an NP$^{\text{NP}}$-algorithm to solve the consistency and the non-instance problem (data complexity). Thus, the first result in Theorem 7.32 can be extended to include the rest of the reasoning tasks.
Theorem 7.33. For all $\Delta \in F_{\infty}[\text{poly}]$, if $\Delta$ is polynomial time computable and the polynomial $p(n)$ corresponding to its boundedness is known, then in $\tau \mathcal{EL}(m_{\infty})$ consistency is in $\Sigma^p_2$, and subsumption and instance checking (data complexity) are in $\Pi^p_2$.

Summing up, based on the polynomial boundedness property we have obtained a family of DLs $\tau \mathcal{EL}(m_{\infty})$ with a satisfiability problem in $\Sigma^p_2$. This upper bound has been established by applying the methods introduced in Chapter 5 for $\tau \mathcal{EL}(\text{deg})$ to the class of polynomially bounded and polynomial time computable CSMs. Nevertheless, this only represents a sufficient condition to obtain our results, and it does not prevent CSMs outside $F_{\infty}[\text{poly}]$ to induce equally behaved threshold logics.

7.3 The simi framework

Lehmann and Turhan [LT12] introduced a framework (called simi framework) that can be used to define a variety of similarity measures between $\mathcal{EL}$ concepts satisfying the properties required by our Propositions 7.3 and 7.4. They first define a directional measure $\text{simi}_d$, and then use a fuzzy connector $\otimes$ to combine the values obtained by comparing the concepts in both directions with $\text{simi}_d$. Given two $\mathcal{EL}$ concepts $C$ and $D$, one could say that $\text{simi}$ uses $\text{simi}_d$ to measure how many properties of $C$ are present in $D$ and vice versa. Then, the bidirectional similarity measure $\text{simi}$ is defined as:

$$\text{simi}(C, D) := \text{simi}_d(C^r, D^r) \otimes \text{simi}_d(D^r, C^r)$$

The fuzzy connector is an operator $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ satisfying (among others) the following two properties (see [LT12]). For all $x, y \in [0, 1]$:

- $x \otimes y = y \otimes x$ (commutativity),
- $x \leq y \Rightarrow 1 \otimes x \leq 1 \otimes y$ (weak monotonicity).

In addition, $\otimes$ is monotonic if for all $x, y, z \in [0, 1]$:

- $x \leq y \Rightarrow x \otimes z \leq y \otimes z$.

Examples of monotonic fuzzy connectors are the average and minimum operators, and all bounded $t$-norms (see [LT12] for more information). In the following we recall the general definition of $\text{simi}_d$.

Definition 7.34 ([LT12]). Let $C, D$ be two $\mathcal{EL}$ concept descriptions. If one of these two concepts is equivalent to $\top$, then:

$$\text{simi}_d(C, D) := \begin{cases} 1 & \text{if } C \equiv \top \\ 0 & \text{if } C \not\equiv \top \text{ and } D \equiv \top \end{cases}$$

Otherwise, let $tl(C)$ and $tl(D)$ be the set of top-level atoms of $C$ and $D$, respectively.
Then, $\text{simi}_d$ is defined as follows:

$$
\text{simi}_d(C, D) := \begin{cases}
\frac{\sum_{C' \in \text{tl}(C)} \left[ g(C') \times \bigoplus_{D' \in \text{tl}(D)} \text{simi}_d(C', D') \right]}{\sum_{C' \in \text{tl}(C)} g(C')} & \text{if } |\text{tl}(C)| > 1 \text{ or } |\text{tl}(D)| > 1 \\
\text{pm}(C, D) & \text{if } C, D \in \text{N}_C \\
\text{pm}(r, s)[w + (1 - w)\text{simi}_d(E, F)] & \text{if } C = \exists r.E \text{ and } D = \exists s.F \\
0 & \text{otherwise.}
\end{cases}
$$

Let us now explain the meaning of the parameters used in the definition of $\text{simi}_d$.

- The symbol $g$ stands for a function mapping the set of $\mathcal{EL}$ atoms $N_A$ to a value in $\mathbb{R}_{>0}$. The idea is that $g : N_A \rightarrow \mathbb{R}_{>0}$ assigns a weight to each atom in $N_A$. This could be helpful, for instance, if one wants to express that some atom contributes more (or is more important) to the similarity than others.

- The purpose of the value $w \in (0, 1)$ is the following. Given two concept descriptions $\exists r.C$ and $\exists s.D$, if $\text{simi}_d(C, D) = 0$, having $w > 0$ allows to distinguish between the cases $r = s$ and $r \neq s$.

- $\text{pm} : (\text{N}_C \times \text{N}_C) \cup (\text{N}_R \times \text{N}_R) \rightarrow [0, 1]$ is a primitive measure for concept and role names satisfying the following basic properties (different from [LT12] we do not deal with role inclusion axioms):
  - $\text{pm}(A, B) = 1$ iff $A = B$ for all $A, B \in \text{N}_C$,
  - $\text{pm}(r, s) = 1$ iff $r = s$ for all $r, s \in \text{N}_R$.

In particular the default primitive measure $\text{pm}_d$ is defined as:

$$
\text{pm}_d(A, B) := \begin{cases}
1 & \text{if } A = B \\
0 & \text{otherwise.}
\end{cases}
$$

and

$$
\text{pm}_d(r, s) := \begin{cases}
1 & \text{if } r = s \\
0 & \text{otherwise.}
\end{cases}
$$

- Finally, the operator $\bigoplus$ represents a bounded triangular-conorm. One can find in [LT12] arguments in favor of using this type of operator. The max operator is a particular case of a bounded t-conorm.

The following two properties of $\text{simi}_d$ are presented in [LT12] (see Lemma 1). They will be useful later on to obtain our results. Let $C, D$ and $E$ be $\mathcal{EL}$ concept descriptions, then:

$$
\text{simi}_d(C, D) = 1 \text{ iff } D \sqsubseteq C \quad (7.8)
$$

$$
D \sqsubseteq E \Rightarrow \text{simi}_d(C, E) \leq \text{simi}_d(C, D) \quad (7.9)
$$
The proofs can be found in the extended version \cite{Leh12} of \cite{LT12} (Lemma 14 and Lemma 15). They indicate that these properties hold regardless of whether the concepts \(C, D\) and \(E\) are in reduced form or not.

Finally, one can easily see that \(\text{simi}\) only defines CSMs that are equivalence invariant, role-depth bounded and equivalence closed. The equivalence invariance property follows from the fact that \(\text{simi}_d\) is computed using the reduced forms of \(C\) and \(D\), and the fact that \(C \equiv C'\) implies that the structures of \(C^r\) and \((C')^r\) are isomorphic (see Theorem \ref{thm:equivalence}). In addition, its structural definition implies that \(\text{simi}\) is role-depth bounded. Regarding the third property, it has been shown already in \cite{LT12} that this is the case for any instance of \(\text{simi}\). Hence, for all instances \(\triangleright\triangleright\) of \(\text{simi}\) the induced \(m_{\triangleright\triangleright}\) is a well-defined graded membership function (Proposition \ref{prop:equivalence}). From now on, for any instance \(\triangleright\triangleright\) of \(\text{simi}\) we denote as \(\triangleright\triangleright_d\) the corresponding instance of \(\text{simi}_d\), and will use \(\triangleright\triangleright_d\) in infix notation.

7.3.1 A polynomially bounded family of instances of \(\text{simi}\)

We now identify a family of instances of \(\text{simi}\) that are polynomially bounded. Let \(\mathcal{F}_1\) be the family of CSMs that are instances of \(\text{simi}\), where \(\oplus\) is selected as \(\text{max}\), \(\otimes\) is a monotonic fuzzy connector and \(pm\) is the default primitive measure \(pm_d\). The following example gives an intuition of why CSMs in \(\mathcal{F}_1\) are polynomially-bounded.

Example 7.35. Let \(\triangleright\triangleright^x \in \mathcal{F}_1\) such that \(g\) assigns value 1 to every atom and \(w = 0.5\). In addition, let \(C\) and \(D\) be the following concept descriptions:

\[
C := A \sqcap B_1 \sqcap 3r. (A \sqcap 3r.B \sqcap 3s.A)
\]
\[
D := A \sqcap B_2 \sqcap 3r. (A \sqcap 3r.A \sqcap 3s.B) \sqcap 3rs.A
\]

Let us look at the atoms in \(D\) chosen by \(\oplus = \text{max}\) along the computation of \(C \triangleright\triangleright^x_d D\). To illustrate this, we use the following picture:

The left-hand side of the picture depicts the structure of \(C\) and the right-hand side does the same for \(D\). The superscripts are used to denote the pairings done by \(\oplus = \text{max}\) in the computation of \(C \triangleright\triangleright^x_d D\). For instance, at the top level of \(C\), \(A^1\) means that \(A\) is paired with the top-level atom of \(D\) exhibiting the 1 superscript (which is also \(A\)). The superscript 0 is used to denote that no such match exists, i.e., every possible match gives value 0. This is the case for \(B_1\) at the top-level of \(C\), since \(B_1 \triangleright\triangleright^x_d A = B_1 \triangleright\triangleright^x_d B_2 = B_1 \triangleright\triangleright^x_d 3r.(...) = 0\).
Our interest is to see what is the effect of removing the unmatched atoms from \(D\). In our example, doing that yields the following concept description:

\[
Y := A \sqcap \exists r. (A \sqcap \exists r. \top \sqcap \exists s. \top)
\]

From the definition of \(\text{simi}_d\) and the particular characteristics of \(\triangleright\triangleleft^\tau\), it is easy to see that \(C \triangleright\triangleleft^\tau_d D = C \triangleright\triangleleft^\tau_d Y = \frac{8}{9}\). This means that the unmatched atoms are actually irrelevant to obtain the value \(C \triangleright\triangleleft^\tau D\). However, this need not be the case for the computation of \(C \triangleright\triangleleft^\tau D\). In fact, one must not forget that \(\triangleright\triangleleft^\tau_d\) is used in both directions to compute \(C \triangleright\triangleleft^\tau D\). But still, there is something special in the structure of \(Y\): it is a concept part of both \(C\) and \(D\) (see Definition 4.13). Some consequences follow from it:

- \(s(Y) \leq s(C)\),
- \(D \sqsubseteq Y\). This means that for all filters \(f\), \(D \in f\) implies \(Y \in f\),
- \(C \sqsubseteq Y\). By property (7.8), it is the case that \(Y \triangleright\triangleright_d C = 1\).

Therefore, although the relationship between \(C \triangleright\triangleleft^\tau D\) and \(C \triangleright\triangleleft^\tau Y\) (if \(\otimes\) were not monotonic) is not clear in general, for a monotonic fuzzy connector it holds \(C \triangleright\triangleleft^\tau D \leq C \triangleright\triangleleft^\tau Y\). Consequently, even though \(C \triangleright\triangleleft^\tau Y\) may not preserve the value \(C \triangleright\triangleleft^\tau D\), the concept \(Y\) represents a better choice towards bounding \(\triangleright\triangleleft\) for \(C\) and a filter \(f\) containing \(D\) (as required in Definition 7.24).

Let us now generalize the intuition presented in the previous example. First, we show that such a concept \(Y\) always exists. Afterwards, we use its properties to establish that all CSMs in \(\mathcal{F}_1\) are linear bounded.

**Lemma 7.36.** Let \(\triangleright\triangleleft\) be a CSM in \(\mathcal{F}_1\). For all \(\mathcal{EL}\) concept descriptions \(C\) and \(D\), there exists a concept description \(Y\) such that:

- \(C \sqsubseteq Y\) and \(D \sqsubseteq Y\),
- \(C \triangleright\triangleleft_d D = C \triangleright\triangleleft_d Y\), and
- \(s(Y) \leq s(C)\).

**Proof.** We use induction on the structure of \(C\) to prove the claim.

- \(C\) is of the form \(A \in \mathbb{N}_C\) or \(\top\). For \(C = A\), the value \(C \triangleright\triangleleft_d D\) is the result of the following expression:

\[
g(A) \times \max \{ A \triangleright\triangleleft_d D' \mid D' \in \text{tl}(D) \} / g(A)
\]

The use of the primitive default measure in \(\triangleright\triangleleft\) implies that \(A \triangleright\triangleleft_d D = 1\) if \(A \in \text{tl}(D)\), otherwise \(A \triangleright\triangleleft_d D = 0\). Choosing \(Y := A\) or \(Y := \top\), accordingly, ensures that the claim is true. If \(C = \top\), then the definition of \(\text{simi}_d\) implies \(C \triangleright\triangleleft_d X = 1\) for all concept descriptions \(X\). Thus, setting \(Y := \top\) satisfies the claim.
7.3 The simi framework

- $C = C_1 \cap \ldots \cap C_n$ with $n > 1$. In this case we have:

$$C \triangleright_d D = \frac{\sum_{i=1}^{n} \left[ g(C_i) \times \max \{ C_i \triangleright_d D' \mid D' \in tl(D) \} \right]}{\sum_{i=1}^{n} g(C_i)}$$

Let $D_i (1 \leq i \leq n)$ be the top-level atom of $D$ that maximizes the value $C_i \triangleright_d D'$ among all $D' \in tl(D)$. The application of the induction hypothesis to $C_i$ and $D_i$ yields a concept description $Y_i$ such that:

- $C_i \subseteq Y_i$ and $D_i \subseteq Y_i$,
- $C_i \triangleright_d D_i = C_i \triangleright_d Y_i$,
- $s(Y_i) \leq s(C_i)$.

Obviously, $C_1 \cap \ldots \cap C_n \subseteq Y_1 \cap \ldots \cap Y_n$ and $D_1 \cap \ldots \cap D_n \subseteq Y_1 \cap \ldots \cap Y_n$. Therefore, the concept description $Y := Y_1 \cap \ldots \cap Y_n$ satisfies $C \subseteq Y$, $D \subseteq Y$ and $s(Y) \leq s(C)$. Now, the value of $C \triangleright_d Y$ is computed by the following expression:

$$C \triangleright_d Y = \frac{\sum_{i=1}^{n} \left[ g(C_i) \times \max \{ C_i \triangleright_d Y' \mid Y' \in tl(Y) \} \right]}{\sum_{i=1}^{n} g(C_i)}$$

Suppose that for some $C_i (1 \leq i \leq n)$, $C_i \triangleright_d Y_i$ is not the maximum among all the values $C_i \triangleright_d Y'$. Then, there is $Y_j \in tl(Y)$ such that $i \neq j$ and $C_i \triangleright_d Y_i < C_i \triangleright_d Y_j$. From this we obtain:

$$C_i \triangleright_d D_i = C_i \triangleright_d Y_i$$
$$< C_i \triangleright_d Y_j$$
$$\leq C_i \triangleright_d D_j$$

$$(D_j \subseteq Y_j \text{ and } [7.9])$$

Hence, it follows that $C_i \triangleright_d D_i < C_i \triangleright_d D_j$ which contradicts the maximality of $D_i$ with respect to $C_i$. Hence, $C_i \triangleright_d Y_i$ is actually the maximum and once this is true, it is easy to see that $C \triangleright_d D = C \triangleright_d Y$.

- $C$ is of the form $\exists r.C'$. Let $D^*$ be the top-level atom of $D$ maximizing the value $C \triangleright D^*$. If $D^*$ is not of the form $\exists r.D'$, then $C \triangleright_d D = 0$. This is a consequence of the general definition of simi$_d$ and the use of pm$_d$. Then, choosing $Y := T$ is enough. Otherwise, $C \triangleright_d D$ can be expressed as:

$$C \triangleright_d D = [w + (1 - w) \times (C' \triangleright_d D')]$$

The application of induction hypothesis to $C'$ and $D'$ yields a concept description $Y'$ such that:

- $C' \subseteq Y'$ and $D' \subseteq Y'$
- $C' \triangleright_d D' = C' \triangleright_d Y'$, and
124 Chapter 7. Concept similarity measures, relaxed instance queries and \( \tau \mathcal{EL}(m) \)

- \( s(Y') \leq s(C') \).

Then, for the concept \( Y := \exists r.Y' \) we have that \( C \subseteq Y, D \subseteq \exists r.D' \subseteq Y \) and \( s(Y) \leq s(C) \). Additionally,

\[
C \bowtie_d Y = [w + (1 - w) \times (C' \bowtie_d Y')] 
\]

Thus, \( C \bowtie_d D = C \bowtie_d Y \).

Next, using Lemma 7.36 we show linear boundedness for the family \( F_1 \).

**Corollary 7.37.** Let \( \bowtie \) be a CSM in \( F_1 \). Then, \( \bowtie \) is linear bounded.

**Proof.** Let \( f \) be a filter and \( C \) a concept description. Moreover, let \( D \in f \) be a concept description such that \( C \bowtie_d D = \max \{ C \bowtie X \mid X \in f \} \). From the abstract definition of \( simi \) we have:

\[
C \bowtie D = (C^r \bowtie_d D^r) \otimes (D^r \bowtie_d C^r) \tag{7.10}
\]

The application of Lemma 7.36 to \( C^r \) and \( D^r \) yields a concept description \( Y \) such that:

\[
C^r \subseteq Y, D^r \subseteq Y, C^r \bowtie_d D^r = C^r \bowtie_d Y \quad \text{and} \quad s(Y) \leq s(C^r)
\]

From \( C^r \subseteq Y \equiv Y^r \), it follows that \( Y^r \bowtie_d C^r = 1 \) (see property (7.8)). In addition, property (7.9) and \( Y \equiv Y^r \) imply that \( C^r \bowtie_d Y = C^r \bowtie_d Y^r \). Hence, \( C \bowtie Y \) can be expressed as follows:

\[
C \bowtie Y = (C^r \bowtie_d D^r) \otimes 1 \tag{7.11}
\]

Since fuzzy connectors are commutative, the monotonicity of \( \otimes \) implies that it is monotone in both arguments. Then, due to (7.10) and (7.11) we obtain \( (C \bowtie D) \leq (C \bowtie Y) \). Hence, \( (C \bowtie D) = (C \bowtie Y) \), for otherwise it would contradict the maximality of \( C \bowtie D \) (\( D^r \subseteq Y \) implies that \( Y \in f \)). Finally, since reduced forms are the smallest concepts in their equivalence classes, we have \( s(Y) \leq s(C^r) \leq s(C) \). Thus, the concept \( Y \) witnesses that \( \bowtie \) is linear bounded.

Corollary 7.37 implies that \( F_1 \subseteq F_{\bowtie}[ poly ] \). Then, since all its elements are linear bounded CSMs, the upper bounds shown in Section 7.2.4 with respect to \( F_{\bowtie}[ poly ] \) also apply for any DL \( \tau \mathcal{EL}(m_{\bowtie}) \) induced by a CSM \( \bowtie \in F_1 \).

Let us now continue Example 7.35 to illustrate that the same arguments failed for arbitrary primitive measures.

**Continuation of Example 7.35.** Let us slightly modify \( \bowtie^\tau \) such that \( pm(B_1, B_2) = pm(B_2, B_1) = 0.8 \). Now, \( B_2 \) becomes a relevant atom for the computation of \( C \bowtie D \), since \( B_1 \bowtie D \neq 0 \). Like in the first part of the example, the picture below shows the matches performed by \( \oplus = \max \).

```
\[
\begin{align*}
A^1 & \bowtie B_1^3 & \exists r(.)^2 \\
A^1 & \exists r(.)^2 & \exists s(.)^3 \\
B^0 & & A^0
\end{align*}
\]
```

\[
\begin{align*}
A^1 & \bowtie B_2^3 & \exists r(.)^2 \exists r(.) \\
A^1 & \exists r(.)^2 & \exists s(.)^3 \\
A & & B & \bowtie A
\end{align*}
\]
7.3 The simi framework

Following the same idea as before, \( Y \) becomes the following concept description:

\[
A \sqcap B_2 \sqcap \exists r.(A \sqcap \exists r. \top \sqcap \exists s. \top)
\]

Obviously, we still have \( C \bowtie^*_d D = C \bowtie^*_d Y \), but now \( C \not\subseteq Y \). Hence, \( Y^r \bowtie^*_d C^r < 1 \) and in principle one can no longer assume that \( (C \bowtie^*_d D) \leq (C \bowtie^*_d Y) \) as before. In fact, a closer inspection of the computation of \( \bowtie^*_d \) shows that \( Y^r \bowtie^*_d C^r < D^r \bowtie^*_d C^r \), and by monotonicity of \( \otimes \) it follows \( (C \bowtie^*_d Y) \leq (C \bowtie^*_d D) \). We will not enter into the details of the computation (they are very tedious), but let us briefly explain the idea behind this. The construction of \( Y \) excludes the right-most top-level atom of \( D \). However, one can see that the structure of \( \exists r. \exists s. A \) can be entirely "mapped" into the structure of \( C \). This means that as a top-level atom of \( D \), it contributes with value 1 to the computation of \( D' \bowtie^*_d C^r \). Therefore, we end up with the following expressions:

\[
Y^r \bowtie^*_d C^r = \frac{a}{3} \quad \text{and} \quad D^r \bowtie^*_d C^r = \frac{a + 1}{4}
\]

where \( a \) is a real value smaller than 3, which proves that \( Y^r \bowtie^*_d C^r < D^r \bowtie^*_d C^r \). Hence, throwing away the atom \( \exists r. \exists s. A \) decreases the value of the right to left comparison when computing \( \bowtie^*_d \). This means that the arguments used to prove linear boundedness in Corollary 7.37 are not valid in this case.

One could still wonder whether it is possible to remove less information from \( D \), while keeping the value \( C^r \bowtie^*_d D^r \) and the size of the resulting concept small enough. Notice that the concept description \( Y \sqcap \exists r. \exists s. A \) represents such a possibility. Nevertheless, this is a very particular case where the size of \( D \) is actually not much bigger than \( C \). Suppose for instance, that \( pm(r, s) = pm(s, r) = 0.9 \) and \( D \) is extended into \( D' \) as follows:

\[
D' := D \sqcap \exists r. \exists s. A \sqcap \exists s. A
\]

A consequence of having such a high similarity between \( r \) and \( s \) is that now \( \exists r. B \bowtie^*_d \exists s. B > \exists r. B \bowtie^*_d \exists r. A \). The picture below shows the change of scenario in the mapping corresponding to the top-level atoms of the second level.

Consequently, the same way of selecting \( Y \) would result in the following concept description:

\[
A \sqcap B_2 \sqcap \exists r.(A \sqcap \exists r. A \sqcap \exists s. B)
\]

Notice that all the newly added existential restrictions are irrelevant for the selection of \( Y \). In addition, there are at least exponentially many top-level atoms in \( D' \) (with respect to \( rd(C) \)), namely the ones corresponding to the atoms \( \exists r. \exists s. A, \exists r. \exists r. B, \exists s. \exists s. A \).
and $\exists s.\exists r.A$. Furthermore, due to the primitive similarity between $r$ and $s$, such atoms contribute with value $1$ (or very close to $1$) to the computation of $(D')^r \bowtie_0^C C^r$. Therefore, likewise for $\exists r.\exists s.A$ and $D$, throwing away any of them will decrease the value $(D')^r \bowtie_0^\triangledown C^r$.

To conclude, the previous example tells us that if $D'$ is the selected maximal concept with respect to $C$ and a filter $f$, one cannot use the idea from Corollary 7.37 to extract a “small” fragment $Y$ of $D'$ such that $C^r \bowtie_0^\triangledown D_r = C^r \bowtie_0^\triangledown Y_r$, and then exploit the monotonicity of $\otimes$ to obtain $C \bowtie^\triangledown D' \leq C \bowtie^\triangledown Y$.

At this moment, it is not clear to us whether for non-default primitive measures the resulting instances of $\text{simi}$ are polynomially bounded or not. An alternative could be to drop the requirement of having $C^r \bowtie_0^d (D')^r = C^r \bowtie_0^d Y^r$, but find a different method to build $Y$ such that at the end $C \bowtie D' \leq C \bowtie Y$ while keeping $Y$ small enough. We do not know if it is possible to do that by only knowing that the fuzzy connector $\otimes$ is monotonic.

### 7.3.2 Relation to the membership degree function $\text{deg}$

To conclude the section, we show that our graded membership function $\text{deg}$ can be obtained from a CSM $\bowtie^1$, using the construction in Definition 7.2. The function $\bowtie^1$ is defined as the following instance of $\text{simi}$:

- the fuzzy connector is defined as $\otimes = \min$ and the bounded t-conorm $\oplus$ as $\max$,
- the function $g$ maps every atom to $1$, $pm$ is the default primitive measure $pm_d$ and the value $w$ is selected as $0$.

There is a minor detail in the definition of $\bowtie^1$ regarding the $\text{simi}$ framework, namely, $w = 0$. The $\text{simi}$ framework defines $w \in (0, 1)$ for two reasons. First, using $w = 1$ would nullify the recursive computation of $\text{simi}$ on existential restrictions. Secondly, $w > 0$ is desired in order to be able to distinguish between different role names, as explained above. However, any instance of $\text{simi}$ with $w = 0$ still complies with the basic properties shown in [Leh12] that have been used so far. Therefore, $\bowtie^1$ is equivalence invariant, role-depth bounded, equivalence closed, and induces a well-defined graded membership function $m_{\bowtie^1}$. Moreover, since $\min$ is a monotonic fuzzy connector, this means that $\bowtie^1$ satisfies all the same properties as those CSMs in the family $\mathcal{F}_1$. Notice that the value of $w$ is irrelevant for the results shown for CSMs in $\mathcal{F}_1$, and one could say that $\bowtie^1 \in \mathcal{F}_1$.

Our main goal now is to show that $\text{deg} = m_{\bowtie^1}$. We start by proving that selecting $\otimes = \min$ makes the value $D^r \bowtie_0^1 C^r$ irrelevant for the computation of $C \bowtie^1 D$. The proof is supported by the application of Lemma 7.36 in the context of $\bowtie^1$.

**Lemma 7.38.** For all interpretations $I$, $d \in \Delta^I$, and $\mathcal{EL}$ concept descriptions $C$ we have:

$$m_{\bowtie^1}(d, C) = \max\{C^r \bowtie_0^1 D^r \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^I\}$$

**Proof.** By Definition 7.2

$$m_{\bowtie^1}(d, C) = \max\{C \bowtie^1 D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^I\}$$
For all concept descriptions $D$, by definition of $\bowtie^1_\bowtie$ we know that

$$C \bowtie^1_\bowtie D = \min\{C \bowtie^1_\bowtie D', D' \bowtie^1_\bowtie C'\}$$

Hence, it follows that $C \bowtie^1_\bowtie D \leq C' \bowtie^1_\bowtie D'$ and we obtain:

$$m^T_\bowtie(d, C) \leq \max\{C' \bowtie^1_\bowtie D' \mid D' \in C_{\bowtie} \text{ and } d \in D\} \quad (7.12)$$

Now, let $X$ be a concept description such that $d \in X_\bowtie$ and $C \bowtie^1_\bowtie X$ gives the maximum in (7.12). The application of Lemma 7.36 to $C'$ and $X'$, yields a concept $Y$ such that:

- $C' \subseteq Y$ and $X' \subseteq Y$,
- $C' \bowtie^1_\bowtie X' = C' \bowtie^1_\bowtie Y$.

Since $Y \equiv Y'$, having $C' \subseteq Y$ implies that $Y' \bowtie^1_\bowtie C' = 1$ (see (7.8)). Additionally, (7.9) further implies $C' \bowtie^1_\bowtie Y' = C' \bowtie^1_\bowtie Y'$. Hence, we obtain the following sequence of equalities:

$$C \bowtie^1_\bowtie Y = \min\{C' \bowtie^1_\bowtie Y', Y' \bowtie^1_\bowtie C'\} = C' \bowtie^1_\bowtie Y' = C' \bowtie^1_\bowtie Y \leq C' \bowtie^1_\bowtie X'$$

Moreover, $d \in X$ and $X' \subseteq Y'$ imply that $d \in Y'$. This means that $Y$ is one of the candidate concepts in the computation of $m^T_\bowtie(d, C)$. Therefore,

$$C' \bowtie^1_\bowtie X' \leq m^T_\bowtie(d, C) \quad (7.13)$$

Thus, by the way $X$ was chosen and the combination of (7.12) and (7.13), our claim follows.

Once we know that $D' \bowtie^1_\bowtie C'$ can be forgotten when computing $C \bowtie^1_\bowtie D$, a basic relationship between $\bowtie^1_\bowtie$ and $\deg$ is established in the following lemma.

**Lemma 7.39.** Let $X$ be an $\mathcal{EL}$ concept description and $T_X$ be the interpretation corresponding to the $\mathcal{EL}$ description tree $T_X$. Then, for each $\mathcal{EL}$ concept description $C$, it holds:

$$C' \bowtie^1_\bowtie X = \deg^{T_X}(d_0, C)$$

where $d_0$ is the domain element corresponding to the root of $T_X$.

**Proof.** We prove the claim by induction on the structure of $C$.

**Induction Base.** $C \in N_C$ or $C = \top$. Then, $C = C'$. If $C'$ is of the form $A$, then $A \bowtie^1_\bowtie X = 1$ when $A \in \operatorname{tl}(X)$ and 0 otherwise. A similar relationship holds for $\deg^{T_X}(d_0, A)$, but with respect to whether $d_0 \in A^{T_X}$. Since $A \in \operatorname{tl}(X)$ iff $d_0 \in A^{T_X}$, this means that $A \bowtie^1_\bowtie X = \deg^{T_X}(d_0, A)$. The case for $\top$ is trivial, since $\top \bowtie^1_\bowtie X = \deg^{T_X}(d_0, \top) = 1$.

**Induction Step.** We distinguish two cases:
• $C$ is of the form $\exists r.D$. Then, $C^r$ is of the form $\exists r.D^r$. By definition of $\triangleright_i^d$ and $\deg$, it is easy to see that whenever $X$ does not have a top-level atom of the form $\exists r.X'$, it is the case that:

$$\exists r.D^r \triangleright_i^d X = \deg^{I_X}(d_0, \exists r.D) = 0$$

Hence, without loss of generality, we focus on the cases where there exists at least one top-level atom in $X$ of the form $\exists r.X'$. Consequently, since $|tl(\exists r.D^r)| = 1$, we have:

$$\exists r.D^r \triangleright_i^d X = \max\{D^r \triangleright_i^d X' \mid \exists r.X' \in tl(X)\} \quad (7.14)$$

Since $I_X$ is induced by $T_X$, then for each atom $\exists r.X' \in tl(X)$ there exists a corresponding domain element $e \in \Delta I_X$ such that $(d_0, e) \in r I_X$. This correspondence also holds in the opposite direction. Moreover, it is easy to see that the tree rooted at $e$ in $T_X$ corresponds to the $\mathcal{E}L$ description tree $T_{X'}$. Hence, the application of induction hypothesis to $D$ yields:

$$D^r \triangleright_i^d X' = \deg^{T_X}(e, D), \text{ for all } \exists r.X' \in tl(X)$$

Therefore, it follows from the equality in (7.14):

$$\exists r.D^r \triangleright_i^d X = \max\{\deg^{T_X}(e, D) \mid (d_0, e) \in r I_X\} \quad (7.15)$$

Now, let $T_{\exists r.D^r}$ be the corresponding $\mathcal{E}L$ description tree of $\exists r.D^r$ and $v_0$ its root. Obviously, there exists exactly one $r$-successor $v_1$ of $v_0$ in $T_{\exists r.D^r}$ and moreover, the subtree of $T_{\exists r.D^r}$ rooted at $v_1$ is exactly the $\mathcal{E}L$ description tree $T_{D^r}$ associated to $D^r$. Consider, then, the set $\mathcal{H}(T_{\exists r.D^r}, G_{I_X}, d_0)$. By Definition 4.5 we have:

$$\deg^{T_X}(d_0, \exists r.D) = \max\{h_w(v_0) \mid h \in \mathcal{H}(T_{\exists r.D^r}, G_{I_X}, d_0)\} \quad (7.16)$$

Now, let $h$ be any $ptgh$ in $\mathcal{H}(T_{\exists r.D^r}, G_{I_X}, d_0)$ with $h(v_1) = e$, for some $e \in \Delta I_X$ such that $(d_0, e) \in r I_X$. We know that there exists at least one and any $ptgh$ $h'$ of a different form will not be interesting, since $h'_w(v_0) = 0$. By definition of $h_w$ (Definition 4.5), it follows that $h_w(v_0) = h_w(v_1)$. Additionally, for any $ptgh$ $h \in \mathcal{H}(T_{\exists r.D^r}, G_{I_X}, d_0)$ with $h(v_1) = e$, its restriction to $(V_{T_{\exists r.D^r}} \setminus \{v_0\})$ is a $ptgh$ in $\mathcal{H}(T_{D^r}, G_{I_X}, e)$. Conversely, any $ptgh$ $g$ in $\mathcal{H}(T_{D^r}, G_{I_X}, e)$ can be extended to a $ptgh$ in $\mathcal{H}(T_{\exists r.D^r}, G_{I_X}, d_0)$, by defining $g(v_0) = d_0$. Hence, (7.16) can be transformed into:

$$\deg^{T_X}(d_0, \exists r.D) = \max\{g_w(v_1) \mid g \in \mathcal{H}(T_{D^r}, G_{I_X}, e)\}$$

Finally, since for each $e \in \Delta I_X$ there exists a $ptgh$ $g \in \mathcal{H}(T_{D^r}, G_{I_X}, e)$ such that $\deg^{T_X}(e, D) = g_w(v_1)$ and $g_w(v_1)$ gives the maximum value, we further obtain the following equation:

$$\deg^{T_X}(d_0, \exists r.D) = \max\{\deg^{T_X}(e, D) \mid (d_0, e) \in r I_X\} \quad (7.17)$$
Thus, the combination of \((7.15)\) and \((7.17)\) yields
\[
\exists r. D^r \triangleright \!\! \!\! \!\! \!\! (d) X = \text{deg}^{I_X}(d, 0) \leq \text{deg}^T(d, C)
\]

- \(C\) is of the form \(C_1 \sqcap \ldots \sqcap C_k\). Then, its reduced form \(C^r\) is of the form \(D_1 \sqcap \ldots \sqcap D_n\), where \(1 \leq n \leq k\) and each \(D_j\) is the reduced form \([C_i]^r\) of some conjunct \(C_i\). Now, it is easy to see from the definition of \(\triangleright \!\! \!\! \!\!\!\! (d)\), that \(C^r \triangleright \!\! \!\! \!\!\!\! (d) X\) can be equivalently expressed as:
\[
C^r \triangleright \!\! \!\! \!\!\!\! (d) X = \frac{\sum_{j=1}^{n} (D_j \triangleright \!\! \!\! \!\!\!\! (d) X)}{n} \tag{7.18}
\]

Furthermore, though more involved, it is not hard to see from the definitions of \(\text{deg}\) and \(h_w\), that a similar situation occurs with respect to \(\text{deg}\):
\[
\text{deg}^{I_X}(d, 0) = \frac{\sum_{j=1}^{n} \text{deg}^{I_X}(d, D_j)}{n} \tag{7.19}
\]

Then, for each \(D_j\) one can apply the induction hypothesis to the atom \(C_i\) that has \([C_i]^r = D_j\) to obtain \(D_j \triangleright \!\! \!\! \!\!\!\! (d) X = \text{deg}^{I_X}(d, C_i)\). Since \(\text{deg}\) is equivalence invariant (in the sense of property \(M2\)), we have \(D_j \triangleright \!\! \!\! \!\!\!\! (d) X = \text{deg}^{I_X}(d, C) = \text{deg}^{I_X}(d, D_j)\). Hence, the combination of \((7.18)\) and \((7.19)\) yields \(C^r \triangleright \!\! \!\! \!\!\!\! (d) X = \text{deg}^{I_X}(d, C)\).

Finally, using the previous two results, one can show the equivalence between \(m_{\triangleright \!\! \!\! \!\!\!\! (d)}\) and \(\text{deg}\).

**Theorem 7.40.** For all interpretations \(I\), \(d \in \Delta^I\), and \(\mathcal{EL}\) concept descriptions \(C\) we have \(m_{\triangleright \!\! \!\! \!\!\!\! (d)}(d, C) = \text{deg}^T(d, C)\).

**Proof.** \((\Rightarrow)\) From Lemma 7.38 we know that there exists an \(\mathcal{EL}\) concept description \(X\) such that \(m_{\triangleright \!\! \!\! \!\!\!\! (d)}(d, C) = C^r \triangleright \!\! \!\! \!\!\!\! (d) X^r\) and \(d \in X^I\). The application of Lemma 7.39 to \(C\) and \(X\) yields:
\[
C^r \triangleright \!\! \!\! \!\!\!\! (d) X = \text{deg}^{I_X}(d, C)
\]

Recall that due to property \((7.9)\), \(C^r \triangleright \!\! \!\! \!\!\!\! (d) X = C^r \triangleright \!\! \!\! \!\!\!\! (X^r)\). Since \(d \in X^I\), the characterization of crisp membership in \(\mathcal{EL}\) yields the existence of a homomorphism \(\varphi\) from \(G_{I_X}\) (or \(T_X\)) to \(G_I\) with \(\varphi(d_0) = d\). Hence, the application of Lemma 4.11 to \(I_X\) and \(I\) implies \(\text{deg}^{I_X}(d_0, C) \leq \text{deg}^T(d, C)\). Therefore, we obtain:
\[
m_{\triangleright \!\! \!\! \!\!\!\! (d)}(d, C) \leq \text{deg}^T(d, C) \tag{7.20}
\]

\((\Leftarrow)\) Consider a \(ptgh\) \(h \in H(T_C^r, G^r, d)\) such that \(h_w(v_0) = \text{deg}^T(d, C)\). Let \(I_h\) be the canonical interpretation induced by \(h\). Since \(T_{I_h}\) is a tree, we can speak of its corresponding \(\mathcal{EL}\) concept description \(C_{I_h}\). Then, we obtain the following equalities:
\[
\text{deg}^T(d, C) = \text{deg}^{I_h}(v_0, C) \quad \text{(Lemma 4.12)}
\]
\[
= C^r \triangleright \!\! \!\! \!\!\!\! (I_h) C_{I_h} \quad \text{(Lemma 7.39)}
\]
\[
= C^r \triangleright \!\! \!\! \!\!\!\! (C_{I_h})^r \quad \text{(property 7.9)}
\]
Furthermore, it is easy to see that by definition of $I_h$, it holds that $d \in [C_{I_h}]^T$. Hence, Lemma 7.38 implies that $C^r \bowtie^1_d (C_{I_h})^r \leq m^{T\bowtie^1}(d, C)$ and consequently:

$$\text{deg}^T(d, C) \leq m^{T\bowtie^1}(d, C) \quad (7.21)$$

Thus, our claim follows from the combination of inequalities (7.20) and (7.21).

Once we have established this equivalence, Proposition 7.4 thus implies that answering of relaxed instance queries w.r.t. $\bowtie^1$ is the same as computing instances for threshold concepts of the form $Q_{> t}$ in $\tau\mathcal{EL}(\text{deg})$. Since such concepts are positive, Proposition 5.19 yields the following corollary.

**Corollary 7.41.** Let $A$ be an $\mathcal{EL}$ ABox, $Q$ an $\mathcal{EL}$ query concept, $a$ an individual name, and $t \in [0, 1)$. Then it can be decided in polynomial time whether $a \in \text{Relax}^{\bowtie^1}(Q, A)$ or not.

Note that Ecke et al. [EPT14, EPT15] show only an NP upper bound w.r.t. data complexity for this problem, albeit for a larger class of instances of the simi framework.
Chapter 8

Conclusions and Future Work

We have introduced a family of DLs $\tau\mathcal{EL}(m)$ parameterized with a graded membership function $m$, which extends the popular lightweight DL $\mathcal{EL}$ by threshold concepts that can be used to approximate classical concepts. Inspired by the homomorphism characterization of membership in $\mathcal{EL}$ concepts, we have defined a particular membership function $\deg$ and have investigated the complexity of reasoning in $\tau\mathcal{EL}(\deg)$. It turns out that the higher expressiveness takes its toll: whereas reasoning in $\mathcal{EL}$ can be done in polynomial time, it is NP- or coNP-complete in $\tau\mathcal{EL}(\deg)$, depending on which inference problem is considered.

The membership function $\deg$ has been further extended to consider $\mathcal{EL}$ concepts defined with respect to acyclic TBoxes. Based on this, we have defined $\tau\mathcal{EL}(\deg)$ TBoxes as pairs $(T_\tau, T_{\mathcal{EL}})$, where $T_\tau$ contains concept definitions that use threshold concepts defined over $T_{\mathcal{EL}}$. Obviously, reasoning with respect to acyclic $\tau\mathcal{EL}(\deg)$ TBoxes can already be handled by the basic approach through unfolding. We hoped that the possible exponential blow-up due to unfolding could be avoided, but unfortunately this is not the case. In fact, we have seen that the satisfiability and subsumption problems with respect to acyclic $\tau\mathcal{EL}(\deg)$ TBoxes are $\Pi_2^P$-hard and $\Sigma_2^P$-hard, respectively. In Section 6.4.3 a PSPACE decision procedure is provided to solve these problems, and it is later extended to tackle all the standard reasoning problems with respect to acyclic knowledge bases, while keeping the use of space polynomial in the size of the input.

We have also shown that concept similarity measures satisfying certain properties can be used to define graded membership functions. This extensive family of CSMs contains non-computable functions, and some of them induce undecidable threshold logics. On the positive side, however, a computable CSM $\bowtie$ always induces a decidable threshold DL $\tau\mathcal{EL}(m_{\bowtie})$. Decidability is achieved by adapting the decision procedures provided for $\tau\mathcal{EL}(\deg)$ to this more general class of DLs. To gain a preliminary insight into the computational complexity landscape exhibited by this family of decidable logics, we restricted our attention to polynomial time computable CSMs. It turns out that the maximization mechanism used to define a membership function $m_{\bowtie}$ may yield a PSPACE-hard logic $\tau\mathcal{EL}(m_{\bowtie})$. A sufficient bounding condition on CSMs is then defined to obtain a subfamily of logics whose satisfiability problem is in $\Sigma_2^P$.

Concrete examples of polynomially bounded CSMs have been presented in Section 7.3.1 as a particular subset of instances of the $\text{simi}$ framework of Lehmann and Turhan [LT12]. Their induced threshold logics inherit the computational complexity results derived for the whole class of polynomially bounded CSMs. In particular, our function $\deg$ can be constructed from a polynomially bounded CSM $\bowtie$. Nevertheless, our direct definition of $\deg$ based on homomorphisms is important since the partial tree-to-graph homomor-
phisms used there are the main technical tool for showing our decidability and complexity results. For instance, satisfiability in $\tau\mathcal{EL}(\text{deg})$ is shown to be NP-complete, in contrast to the general $\text{NP}^\text{NP}$ upper bound obtained from the polynomial boundedness property.

While introduced as a formalism for defining concepts by approximation, a possible use-case for $\tau\mathcal{EL}(\text{deg})$ is relaxation of instance queries, as motivated and investigated in [EPT14, EPT15]. Compared to the setting considered in [EPT14, EPT15], $\tau\mathcal{EL}(\text{deg})$ yields a considerably more expressive query language since we can combine threshold concepts using the constructors of $\mathcal{EL}$ and can also forbid that thresholds are reached. Restricted to the setting of relaxed instance queries, our approach actually allows relaxed instance checking in polynomial time. On the other hand, [EPT14, EPT15] can also deal with other instances of the simi framework.

8.1 Future Work

Last, we sketch some ideas and point out several directions for future work.

Membership functions for cyclic and general TBoxes. We would like to extend our function $\text{deg}$ to be able to compute membership degrees for concepts defined with respect to cyclic and general TBoxes. To do this, homomorphisms probably need to be replaced by simulations [Baa03]. On the side of concept similarity measures for DLs, a specific measure has been proposed in [EPT15] to deal with general TBoxes. In particular, such a CSM is akin to the simi framework in the sense that it also combines directional values to compute the similarity between two concepts. We believe that it is possible to exploit the ideas from [EPT15], and use the directional computation to extend $\text{deg}$ towards concepts defined with respect to general TBoxes. This is joint work in progress with Andreas Ecke.

Nesting of threshold concepts. Extending our introduced family of DLs with nesting of threshold concepts is an interesting topic for future work. To go further in this direction, the initial step is to understand how to come up with a well-defined and meaningful semantics to interpret the resulting concept descriptions. Since a graded membership function $m$ provides the interpretation for simple threshold concepts in a logic $\tau\mathcal{EL}(m)$, one idea that seems natural is to interpret nested threshold concepts by recursively applying the definition of $m$ bottom-up. More precisely, suppose we have a nested threshold concept $X_{>5}$ where $X$ is of the following form:

\[
\text{Healthy} \sqcap (\exists\text{spouse}.(\text{Rich} \sqcap \text{Intelligent} \sqcap \text{Female}))_{>7}
\]

To compute $m^Z(d, X)$ the function would first calculate $m^Z(d, \exists\text{spouse}.(\ldots))$ using the base definition of $m$ to obtain the corresponding value $t$. Afterwards, $m$ is applied one more time to compute the value $m^Z(d, X)$. Here, the inner threshold concept in $X_{>5}$ would be treated as an atom, where the previously computed value $t$ determines whether $d$ has the property $(\exists\text{spouse}.(\ldots))_{>7}$ or not. For example, let $d$ be the following element in some interpretation $Z$:
In our logic $\tau\mathcal{EL}(deg)$ we have $deg^T(d, \exists\text{spouse}(\ldots)) = 2/3$. This means that $d \not\in [(\exists\text{spouse}(\ldots))_{\geq 7}]^T$. Therefore, applying the idea presented above we would obtain $deg^T(d, X) = 1/2$, since $d \in (\text{Healthy})^T$ and $d \not\in [(\exists\text{spouse}(\ldots))_{\geq 7}]^T$. Thus, $d \not\in (X_{> 5})^T$. Obviously, one can object that $d$ is quite close to the crisp set defined by $(\exists\text{spouse}(\ldots))_{\geq 7}$, and consequently it should not be considered in such a way. Instead, maybe a more suitable idea could be to give a membership degree value for $d$ in $(\exists\text{spouse}(\ldots))_{\geq 7}$ and use it to compute $deg^T(d, X)$. At this moment it is still unclear to us which one would be a better choice or if both are useful in different scenarios.

Finally, from a computational point of view, one would expect the reasoning problems to become harder. In fact, looking at the equivalences in Proposition 3.2, it is not hard to see that one can express $\mathcal{ALC}$ concept descriptions by just using the threshold values $\{0, 1\}$. For instance, $\neg \exists r. \neg A$ would correspond to the nested threshold concept $(\exists r.A < 1 < 1)$. Cyclic $\tau\mathcal{EL}(deg)$ TBoxes. Since $deg$ is well-defined with respect to acyclic $\mathcal{EL}$ TBoxes $\mathcal{T}_{\mathcal{EL}}$, there is nothing to prevent us to have cyclic definitions in a TBox $\mathcal{T}_\tau$. We would like to consider this in the future. Note, that since $\forall r_1, \ldots, r_n. \neg A$ can be expressed in $\tau\mathcal{EL}(deg)$, it seems to be possible to encode cyclic TBoxes in the DL $\mathcal{FL}_0$ into cyclic $\tau\mathcal{EL}(deg)$ TBoxes. In particular, subsumption in $\mathcal{FL}_0$ for cyclic terminologies is PSPACE-complete w.r.t. descriptive semantics. This would give a preliminary PSPACE-hardness result for the subsumption problem in the presence of cyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Bounded CSMs. The polynomially bounded condition is still too strong to be satisfied by many useful CSMs. It would be important to find out how to relax it, without losing the good properties that it gives for a logic $\tau\mathcal{EL}(m_{\Diamond})$. This could, for example, provide more information about the logics $\tau\mathcal{EL}(m_{\Diamond})$ induced by instances of the $\text{simi}$ framework that use non-primitive measures $pm$.

Additionally, polynomial boundedness only gives a general NP upper bound for the satisfiability problem. It would be interesting to characterize which conditions a CSM in $\mathcal{F}_{\infty[\text{poly}]}$ must satisfy in order to have a satisfiability problem in NP, like it happens for $\forall^3$ and $\tau\mathcal{EL}(deg)$.

Open theoretical problems. The exact computational complexity of reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes (between $\Sigma_2^P / \Pi_2^P$ and PSPACE) remains open. Regarding the relationship between $\tau\mathcal{EL}(deg)$ and $\mathcal{ALC}$, we do not know whether $\tau\mathcal{EL}(deg)$ is exponentially more succinct than $\mathcal{ALC}$. 

\[ d : \{\text{Healthy}\} \xrightarrow{\text{spouse}} \{\text{Intelligent, Female}\} \]
Appendix A

Missing proofs

Missing proofs of Chapter 3

Theorem 3.8 Let \( \hat{C} \) be a \( \tau \mathcal{EL}(m) \) concept description and \( I = (\Delta^I, I) \) an interpretation. The following statements are equivalent for all \( d \in \Delta^I \):

1. \( d \in \hat{C}^I \).

2. there exists a \( \tau \)-homomorphism \( \phi \) from \( T_{\hat{C}} \) to \( G_I \) with \( \phi(v_0) = d \).

Proof. Let \( T_{\hat{C}} = (V_T, E_T, v_0, \hat{\ell}_T) \) be the description tree associated to \( \hat{C} \) and \( \hat{C} \) be of the form \( \hat{C}_1 \cap \ldots \cap \hat{C}_q \cap \exists r_1 \hat{D}_1 \cap \ldots \cap \exists r_n \hat{D}_n \), where each \( \hat{C}_i \) is either a concept name \( A \in N_C \) or a threshold concept \( E_{\sim t} \in N_E \).

\( (\Rightarrow) \) Assume that \( d \in \hat{C}^I \). Then, \( d \in (\hat{C}_i)^T \) and \( d \in (\exists r_j \hat{D}_j)^T \) for all \( 1 \leq i \leq q \) and \( 1 \leq j \leq n \). We show by induction on the role depth of \( \hat{C} \) that there exists a \( \tau \)-homomorphism \( \phi \) from \( T_{\hat{C}} \) to \( G_I \) with \( \phi(v_0) = d \).

Induction Base. \( \text{rd}(\hat{C}) = 0 \). Then, \( n = 0 \) and \( T_{\hat{C}} \) consists only of one node \( v_0 \) (the root), it has no edges and \( \hat{\ell}_T(v_0) = \{\hat{C}_1, \ldots, \hat{C}_q\} \). The mapping \( \phi(v_0) = d \) is a \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_I \). For each \( \hat{C}_i \) of the form \( A \in N_C \) we know that \( A \in \ell_I(d) \), and consequently \( \phi \) satisfies Condition 2 in Definition 3.7. In case \( \hat{C}_i \) is of the form \( E_{\sim t} \), the fact that \( d \in (\hat{C}_i)^T \) implies that \( \phi \) satisfies Condition 2 in Definition 3.7.

Induction Step. Assume that the claim holds for all the concepts with role depth smaller than \( k \). We show that it also holds for \( \text{rd}(\hat{C}) = k \). First, consider the concept \( \hat{D}_0 = \hat{C}_1 \cap \ldots \cap \hat{C}_q \). One can see that \( T_{\hat{D}_0} = (V_0, E_0, v_0, \hat{\ell}_0) \) is exactly the description tree with \( V_0 = \{v_0\} \), \( E_0 = \emptyset \) and \( \hat{\ell}_0(v_0) = \hat{\ell}_T(v_0) \). Since \( d \in (\hat{D}_0)^T \) and \( \text{rd}(\hat{D}_0) = 0 \), by induction hypothesis there exists a \( \tau \)-homomorphism \( \phi_0 \) from \( T_{\hat{D}_0} \) to \( G_I \) with \( \phi_0(v_0) = d \).

Now, consider any edge \( v_0 r_j v_j \) in \( E_T \). By the relationship between \( T_{\hat{C}} \) and \( \hat{C} \), there exists a top-level concept \( \exists r_j \hat{D}_j \) of \( \hat{C} \) such that \( T_{\hat{D}_j} = (V_j, E_j, v_j, \hat{\ell}_j) \) is precisely the subtree of \( T_{\hat{C}} \) with root \( v_j \). In addition, since \( d \in (\exists r_j \hat{D}_j)^T \) there exists \( d_j \in \Delta^T \) such that \( dr_j d_j \in E_T \) and \( d_j \in (\hat{D}_j)^T \). Since \( \text{rd}(\hat{D}_j) < k \), the application of the induction hypothesis on \( d_j \) and \( \hat{D}_j \) yields a \( \tau \)-homomorphism \( \phi_j \) from \( T_{\hat{D}_j} \) to \( G_I \) with \( \phi_j(v_j) = d_j \).

It is not hard to see that for all nodes \( v \in V_T \), there exists exactly one of such \( \tau \)-homomorphism \( \phi_j \) (\( 0 \leq j \leq n \)) such that \( v \in \text{dom}(\phi_j) \). Based on this, we build a mapping \( \phi \) from \( V_T \) to \( V_I \) as \( \phi = \bigcup_{j=0}^n \phi_j \). Note that \( \phi(v_0) = d \) by definition of \( \phi_0 \). Hence, it remains to show that \( \phi \) is \( \tau \)-homomorphism.
1. **φ is a homomorphism from \( T_C \) to \( G_I \):** Let \( v \) be any node in \( V_T \). We know that \( v \) is a node of one description tree \( T_{\hat{D}_j} \) and \( \phi(v) = \phi_j(v) \) for the corresponding mapping \( \phi_j \). Since \( \phi_j \) is a homomorphism, this means that \( \ell_j(v) \subseteq \ell_T(\phi_j(v)) \). Therefore, \( \ell(v) = \ell_j(v) \) implies \( \ell(v) \subseteq \ell_T(\phi(v)) \). Now, let \( vrw \) be any edge from \( E_T \). There are two possibilities:

- \( vrw \) is of the form \( v_0r_jv_j \). As explained before we have \( \phi(v_0) = d, \phi_j(v_j) = d_j \) and \( dr_jd_j \in E_T \). Hence, \( \phi(v_0)r_j\phi(v_j) \in E_T \).
- \( v, w \in \text{dom}(\phi_j) \) for some \( j \in \{1 \ldots n\} \). By construction of \( \phi \) and the fact that \( \phi_j \) is a homomorphism, it follows that \( \phi(v)r\phi(w) \in E_T \).

2. Condition 2 in Definition 3.7 follows from the fact that \( \phi \) is constructed using \( \tau \)-homomorphisms.

Thus, \( \phi \) is \( \tau \)-homomorphism from \( T_{\hat{C}} \) to \( G_I \) with \( \phi(v_0) = d \).

(\( \Leftarrow \)) Assume that there exists a \( \tau \)-homomorphism \( \phi \) from \( T_{\hat{C}} \) to \( G_I \) with \( \phi(v_0) = d \).

We show by induction on the size of \( V_T \) that \( d \in \hat{C}^\tau \).

**Induction Base.** \( |V_T| = 1 \). Then, \( \hat{C} \) is of the form \( \hat{C}_1 \cap \ldots \cap \hat{C}_q \) and \( \hat{\ell}_T(v_0) = \{\hat{C}_1, \ldots, \hat{C}_q\} \). We distinguish two cases for all \( \hat{C}_i \in \hat{\ell}_T(v_0) \):

- \( \hat{C}_i \) is of the form \( A \in N_C \). Since \( \phi \) is \( \tau \)-homomorphism, it is also a classical homomorphism in the sense of Definition 2.3 and hence, ignoring the labels of the form \( E_{\alpha,t} \) we have \( \ell_T(v_0) \subseteq \ell_T(d) \). Thus, \( d \in A^\tau \).
- \( \hat{C}_i \) is of the form \( E_{\alpha,t} \). By Definition 3.7 we also have \( d \in (E_{\alpha,t})^\tau \).

Thus, \( d \in (\hat{C}_i)^\tau \) for all conjuncts \( \hat{C}_i \) of \( \hat{C} \). Consequently, \( d \in \hat{C}^\tau \).

**Induction Step.** Assume that the claim holds for \( |V_T| < k \). We show that it also holds for \( |V_T| = k \). Since \( k > 0 \), there exist nodes \( v_1, \ldots, v_n \) in \( V_T \) such that \( v_0r_jv_j \in E_T \). This also means that \( \hat{C} \) is of the form \( \hat{C}_1 \cap \ldots \cap \hat{C}_q \cap \exists r_1.\hat{D}_1 \cap \ldots \cap \exists r_n.\hat{D}_n \) with \( n > 0 \), and the description tree \( T_{\hat{D}_j} = (V_j, E_j, v_j, \hat{\ell}_j) \) associated to \( \hat{D}_j \) is the subtree of \( T_{\hat{C}} \) rooted at \( v_j \). We consider the following two cases:

- \( q > 0 \). Then, \( d \in (\hat{C}_i)^\tau \) can be shown in the same way as for the base case.
- Consider any \( \exists r_j.\hat{D}_j \), with \( j \in \{1 \ldots n\} \). Since \( \phi \) is also a homomorphism from \( T_{\hat{C}} \) to \( G_I \) and \( v_0r_jv_j \in E_T \), then there exists \( e_j \in \Delta^\tau \) such that \( dr_je_j \in E_T \) and \( \phi(v_j) = e_j \). Moreover, it is clear that \( |V_j| < |V_T| \) and it is not difficult to see that the restriction of the domain of \( \phi \) to \( V_j \) is also a \( \tau \)-homomorphism from \( T_{\hat{D}_j} \) to \( G_I \) with \( \phi(v_j) = e_j \). Hence, the induction hypothesis can be applied to obtain that \( e_j \in (\hat{D}_j)^\tau \). Hence, \( d \in (\exists r_j.\hat{D}_j)^\tau \).

Thus, we have shown that \( d \in \hat{C}^\tau \). \( \square \)

**Missing proofs of Chapter 4**

**Definition A.1.** Let \( C \) be an \( EL \) concept description and \( T_C \) its corresponding \( EL \) description tree. For all nodes \( v \in V_{\tau_C} \), we denote by \( T_C[v] \) the subtree of \( T_C \) rooted at
v. Furthermore, the $\mathcal{EL}$ concept description $C[v]$ is the one having the description tree $T_C[v]$. Finally, the height $\eta(v)$ of a node $v$ in $T_C$ is the length of the longest path from $v$ to a leaf of $T_C$.

In the proof of Lemma 4.9 we will use concepts and description trees of the form $T_C[v]$ and $C[v]$. We would like to point out that for all concept descriptions $C'$ in reduced form, the concepts $C'[v]$ are also in reduced form (for all $v \in V_{T_{C'}}$). This is a consequence of the fact that to obtain the reduced form of a concept $C$ the rules are not only applied in the top-level conjunction of $C$, but also under the scope of existential restrictions (see Chapter 2).

**Lemma 4.9.** Let $C$ be an $\mathcal{EL}$ concept description, $I$ a finite interpretation and $d \in \Delta^I$. Then, Algorithm 4 terminates on input $(C, I, d)$ and outputs $deg^I(d, C)$, i.e., $S(v_0, d) = deg^I(d, C^r)$.

**Proof.** To see that the algorithm terminates, it is enough to observe that $T_{C^r}$ and $G_I$ are finite and the algorithm consists of nested iterations over the nodes and edges in $T_{C^r}$ and $G_I$. To show that $S(v_0, d) = deg^I(d, C^r)$, we prove a more general claim:

**Claim:** $S(v, e) = deg^I(e, C^r[v])$ for all $v \in V_{T_{C^r}}$ and $e \in \Delta^I$.

Note first, that for each pair $(v, e)$ the value of $S(v, e)$ is assigned only once during a run of the algorithm. We prove the claim by induction on the height $\eta(v)$ of each node $v$ in $T_{C^r}$.

**Induction Base.** $\eta(v) = 0$. Then $v$ is a leaf in $T_{C^r}$. This means that $v$ has no successors and for all $e \in \Delta^I$ there exists a unique path $h$ from $T_{C^r}[v]$ to $G_I$ with $h(v) = e$. One can see in Algorithm 4 that the special case where $|\ell_{T_{C^r}}(v)| + k^*(v) = 0$ is properly treated. Otherwise, we have $c = |\ell_{T_{C^r}}(v) \cap \ell_I(e)|$ and $S(v, e) = \frac{c}{|\ell_{T_{C^r}}(v)|}$. Note that this is exactly the value of $h_v(v)$ in Definition 4.4. Since $h$ is unique, this means that $deg^I(e, C^r[v]) = S(v, e)$.

**Induction Step.** $\eta(v) > 0$. Let $v_1, \ldots, v_k$ be the children of $v$ in $T_{C^r}$ such that if $v_1$ is an $r$-successor of $v$ in $T_{C^r}$, then $e$ has at least one $r$-successor in $G_I$. The application of the max operator in line 10 selects for each $r$-successor $v_i$ of $v$ an $r$-successor $e_i$ of $e$ in $\Delta_I$ that has the maximum value for $S(v_i, e_i)$. Such a value is then used in the computation of $c$. Let $(v_i, e_i)$ be the pairs representing such a selection for all $v_i$. Two observations are in order:

- Since $v_i$ is a child of $v$, it occurs first in the post-order selected in line 4. Therefore, the value of $S(v_i, e_i)$ is computed before the computation of $c$ for $(v, e)$.
- The value of $S(v, e)$ as computed by Algorithm 4 corresponds to the following expression:

  $$S(v, e) = \frac{|\ell_{T_{C^r}}(v) \cap \ell_I(e)| + \sum_{i=1}^k S(v_i, e_i)}{|\ell_{T_{C^r}}(v) + k^*(v)|} \quad (A.1)$$

- Since $\eta(v_i) < \eta(v)$, the application of the induction hypothesis yields

  $$S(v_i, e_i) = deg^I(e_i, C^r[v_i]) \quad (A.2)$$
Now, let $h_i$ be a $ptgh$ from $T_{C^r}[v_i]$ to $G_I$ such that $h_i(v_i) = e_i$ and $h_{i_w}(v_i) = \deg^I(\epsilon_i, C^r[v_i])$ for all $1 \leq i \leq k$. It is easy to see that the mapping $h = h_1 \cup \ldots \cup h_k \cup \{(v,e)\}$ is a $ptgh$ from $T_{C^r}[v]$ to $G_I$ with $h(v) = e$. Moreover, combining (A.1) and (A.2) it is also true that $h_{i_w}(v_i) = S(v,e)$. Hence, by Definition 4.5 we have $S(v,e) \in V^I(e, C^r[v])$. Suppose, however, that $S(v,e) < \max V^I(e, C^r[v])$. We show that this is not the case by reaching a contradiction.

Having $S(v,e) < \max V^I(e, C^r[v])$ implies the existence of $ptgh$ $h'$ from $T_{C^r}[v]$ to $G_I$ with $h'(v) = e$ such that $h'(v) > h(v)$. Looking at $h_w$ in Definition 4.4 the fact that $h(v) = h'(v)$ implies that the difference must be on the values of $h_w(v_i)$ and $h'_w(v_i)$. More precisely, there must exist at least one successor $v_i$ of $v$ such that $h'_w(v_i) > h_w(v_i)$. Based on this, we distinguish two cases:

- $h'(v_i) \neq h_i(v_i)$, i.e., the $ptgh$ $h'$ maps $v_i$ to a different element in $\Delta^I$. But, if that were the case, then the application of the max operator in line 10 would have chosen $h'(v_i)$ as the pairing for $v_i$, instead of $e_i$.
- $h'(v_i) = h_i(v_i) = e_i$. This case would contradict the induction hypothesis, since $h'_w(v_i) > h_w(v_i)$ would imply $S(v_i, e_i) < \deg^I(e_i, C^r[v_i])$.

Hence, we obtain by contradiction that $S(v,e) = \max V^I(e, C^r[v])$ and consequently, $S(v,e) = \deg^I(e, C^r[v])$. Since $S(v_0, d)$ is a particular case, we thus have shown that $S(v_0, d) = \deg^I(d, C^r)$.

**Missing proofs of Chapter 5**

**Proposition 5.12** Let $A$ be an ABox. Then, $A$ is consistent iff there exists a consistent pre-processing $\hat{A}'$ of $A$.

**Proof.** ($\Rightarrow$) Let $I$ be an interpretation such that $I \models A$. One can see that for any assertion $\neg \hat{C}(a)$ that a rule is applicable to, if $I \models \neg \hat{C}(a)$ there is a way to apply the rule such that $I$ also satisfies the newly introduced assertion. The case for $\rightarrow \neg$ is clear. For the rule $\rightarrow \neg$, if $I \models \neg \hat{C}(a)$ then there exists a conjunct $\hat{C}_i$ such that $I \models \neg \hat{C}_i(a)$. This can be the non-deterministic choice made by the application of $\rightarrow \neg$. Last, for assertions of the form $\neg E_{\neg \rightarrow}$ and $\neg A$ the applicable rules are $\rightarrow \neg$ and $\rightarrow \neg$, respectively. Since $\neg E_{\neg \rightarrow} \equiv E_{\neg \rightarrow} \land \neg A$, we have that $I$ satisfies $E_{\neg \rightarrow} \land \neg A \equiv A_{<1}$.

Thus, since $I$ satisfies every assertion in $A$ we can conclude that there exists a pre-processing $A'$ of $A$ such that $I \models A'$.

($\Leftarrow$) This direction is trivial since $A \subseteq A'$.

**Lemma 5.15** Let $A$ be an ABox, $I$ an interpretation satisfying $A$ and $A'$ a pre-processing of $A$ such that $I \models A'$. Moreover, for all $a \in \text{Ind}(A)$, let $I_a$ be a tree-shaped interpretation satisfying the following:

- $I_a \models A'(a)$,
- there exists a homomorphism $\varphi_a$ from $G_{I_a}$ to $G_I$ with $\varphi_a(a^{I_a}) = a^I$.
Last, let $\mathcal{J}$ be the following interpretation:

- $\Delta^\mathcal{J} := \bigcup_{a \in \text{Ind}(\mathcal{A})} \Delta^I_a$,
- $A^\mathcal{J} := \bigcup_{a \in \text{Ind}(\mathcal{A})} A^I_a$ for all $A \in \mathcal{N}_C$,
- $r^\mathcal{J} := \{(a^I_a, b^J_b) \mid r(a, b) \in \mathcal{A}\} \cup \bigcup_{a \in \text{Ind}(\mathcal{A})} r^I_a$ for all $r \in \mathcal{N}_R$, and
- $a^\mathcal{J} := a^I_a$, for all $a \in \text{Ind}(\mathcal{A})$.

where the sets $\Delta^I_a$ are pairwise disjoint. Then, $\mathcal{J} \models \mathcal{A}$.

Proof. We start by considering the following mapping from $V_\mathcal{J}$ to $V_\mathcal{I}$:

$$\varphi^* := \bigcup_{a \in \text{Ind}(\mathcal{A})} \varphi_a$$

Since the sets $\Delta^I_a$ are pairwise disjoint, the mapping $\varphi^*$ is unambiguous. Moreover, we know that $(a^I_a, b^I_b) \in r^I$ for all $r(a, b) \in \mathcal{A}$, $\varphi_a(a^I_a) = a^I$ and $\varphi_b(b^I_b) = b^I$. Therefore, $(\varphi^*(a^I_a), \varphi^*(b^I_b)) \in r^\mathcal{J}$ for all $(a^I_a, b^I_b) \in r^\mathcal{J}$. Consequently, it is clear that $\varphi^*$ is a homomorphism from $G_\mathcal{J}$ to $G_\mathcal{I}$ with $\varphi^*(a^J) = a^I$ for all $a \in \text{Ind}(\mathcal{A})$.

We now show that $\mathcal{J} \models \mathcal{A}'$. Since $\mathcal{A} \subseteq \mathcal{A}'$, this will imply $\mathcal{J} \models \mathcal{A}$. Recall that $r(a, b) \in \mathcal{A}'$ iff $r(a, b) \in \mathcal{A}$. By construction of $\mathcal{J}$ we have $(a^I_a, b^J_b) \in r^\mathcal{J}$ for all $r(a, b) \in \mathcal{A}$, and $a^J = a^I_a$ for all $a \in \text{Ind}(\mathcal{A})$. Hence, $r(a, b) \in \mathcal{A}'$ implies $(a^J, b^J) \in r^\mathcal{J}$. Thus, it remains to show that each concept assertion in $\mathcal{A}'$ is satisfied by $\mathcal{J}$.

We first prove that $\mathcal{J} \models \mathcal{A}^+$. Let $a \in \text{Ind}(\mathcal{A})$ and $\hat{C}(a) \in \mathcal{A}^+$. From $\mathcal{I}_a \models \mathcal{A}'(a)$ we know that $\mathcal{I}_a \models \hat{C}(a)$ and $a^I_a \in \mathcal{C}^\mathcal{I}_a$. Then, the application of Theorem 3.8 yields a $\tau$-homomorphism $\phi$ from $T_{\mathcal{I}_a}$ to $G_{\mathcal{I}_a}$ with $\phi(v_0) = a^I_a$. We want to show that $\phi$ is also a $\tau$-homomorphism from $T_{\mathcal{I}_a}$ to $G_{\mathcal{J}}$. The construction of $\mathcal{J}$ indicates that $\mathcal{I}_a \subseteq \mathcal{J}$. This means that $\phi$ is a classical homomorphism from $T_{\mathcal{I}_a}$ to $G_{\mathcal{J}}$, which means that Condition 1 in Definition 3.1 is satisfied. Hence, it remains to show that the second condition is also satisfied.

Since $\mathcal{I}_a$ is required to be tree-shaped, it is clear that $\phi(v) = a^\mathcal{I}_a$ only if $v = v_0$. Let $v \in V_{T_{\mathcal{I}_a}}$ and $E_{\sim} \in \ell_{T_{\mathcal{I}_a}}(v)$, we distinguish two cases:

- $v = v_0$. By the relationship that exists between $\tau R E \mathcal{L}(m)$ concept descriptions and $\tau R E \mathcal{L}(m)$ description trees (see Section 3.1), we have that $E_{\sim} \subseteq \mathcal{T}_{\mathcal{I}_a}$ is a top-level atom of $\hat{C}$. Therefore, $a^I_a \in (E_{\sim})^I_a$ and $a^\mathcal{I}_a \in (E_{\sim})^\mathcal{I}_a$. Additionally, we have that $\mathcal{I}_a \subseteq \mathcal{J}$ and $\varphi^*$ is homomorphism from $G_{\mathcal{I}_a}$ to $G_{\mathcal{J}}$ with $\varphi^*(a^\mathcal{I}_a) = a^\mathcal{J}$. Thus, Lemma 4.11 can be applied with respect to $\mathcal{I}_a$ and $\mathcal{J}$ (if $\sim \in \{<, \leq\}$) or to $\mathcal{J}$ and $\mathcal{I}$ (if $\sim \in \{>, \geq\}$), to obtain $a^\mathcal{J} \in (E_{\sim})^\mathcal{J}$.

- $v \neq v_0$. As said before, we have $\phi(v) = e$ with $e \neq a^I_a$ and $e \in \Delta^I_a$. Since $G_{\mathcal{I}_a}$ is a tree, the reachable elements from $e$ in $\Delta^\mathcal{J}$ through role relations are exactly the same as in $\Delta^\mathcal{I}_a$. Thus, it is easy to see that $\deg^\mathcal{I}_a(e, E) = \deg^\mathcal{J}(e, E)$, and $e \in (E_{\sim})^I_a$ implies $e \in (E_{\sim})^\mathcal{J}$.
Thus, $\phi$ is $\tau$-homomorphism from $T_{\tilde{C}}$ to $G_J$ with $\phi(v_0) = a^J$. The application of Theorem 3.8 yields $a^J \in \tilde{C}^J$. Since we have chosen $a$ and $\tilde{C}(a)$ arbitrarily, we can conclude that $J \models A^+$. We now turn into $A^-$, i.e., we prove $J \models \neg \tilde{C}(a)$ for all assertions $\neg \tilde{C}(a) \in A'$. The proof is very similar to the analogous case in Lemma 5.6 for the non-subsumption problem. We use induction on the structure of $\tilde{C}$.

- $\tilde{C}$ is of the form $E_{\tau,t}$ or $A$. Then, rules $\rightarrow_{\neg}$ and $\rightarrow_{\neg A}$ are applicable, and its application yields $E_{\chi(\neg) t}(a) \in A^+$ and $A_{<1}(a) \in A^+$. Since $-E_{\tau,t} = E_{\chi(\neg) t}$ and $\neg A \equiv A_{<1}$ (see Propositions 4.15 and 3.2, respectively) and $J \models A^+$, this means that $a^J \notin \tilde{C}^J$.

- $\tilde{C}$ is of the form $\tilde{C}_1 \sqcap \ldots \sqcap \tilde{C}_n$. By the definition of pre-processing, the rule $\rightarrow_{\neg}$ must have been applied adding an assertion of the form $\neg \tilde{C}_i(a)$ to $A'$ for some $i \in \{1, \ldots, n\}$. The application of the induction hypothesis to $\tilde{C}_i$ yields that $J \models \neg \tilde{C}_i(a)$ and $a^J \notin (\tilde{C}_i)^J$. Thus, $a^J \notin \tilde{C}^J$ and $J \models \neg \tilde{C}(a)$.

- $\tilde{C}$ is of the form $\exists r. \tilde{D}$ and $(\neg \exists r. \tilde{D})(a) \in A'$. Assume that $(a^J, d) \in r^J$ for some $d \in \Delta^J$. We have two cases:
  - $d = b^J$ for some $b \in \text{Ind}(A)$. By construction of $J$ we have $r(a, b) \in A$. Hence, the rule $\rightarrow_{\exists}$ is applicable and its application adds $\neg \tilde{D}(b)$ to $A'$. The application of induction to $\tilde{D}$ yields $J \models \neg \tilde{D}(b)$, and therefore $b^J \notin \tilde{D}^J$.
  - $d \neq b^J$ for all $b \in \text{Ind}(A)$. Then, by construction of $J$ we have $d \in \Delta^{I_a}$. Since, $(\neg \exists r. \tilde{D})(a) \in A'$ and $I_a \models A'(a)$, it holds that $d \notin \tilde{D}^{I_a}$. Now, suppose that $d \in \tilde{D}^J$. By Theorem 3.8 there exists a $\tau$-homomorphism $\phi$ from $T_{\tilde{D}}$ to $G_J$ with $\phi(v_0) = d$. But, if that is the case, by the disjointness assumptions made to build $J$ and the fact that $G_{I_a}$ is a tree, we would have that $\phi$ is also a $\tau$-homomorphism from $T_{\tilde{D}}$ to $G_{I_a}$, contradicting the fact that $d \notin \tilde{D}^{I_a}$. Thus, $d \notin \tilde{D}^J$.

Overall, we just have shown that for each $r$-successor $d$ of $a^J$ it is the case that $d \notin \tilde{D}^J$. Hence, $a^J \notin (\exists r. \tilde{D})^J$ and $J \models \neg \exists r. \tilde{D}(a)$.

Thus, $J \models A^-$ and consequently $J \models A'$.

Lemma 5.14 Let $A$ be a consistent single-element ABox and $I$ an interpretation such that $I \models A$. In addition, let $J$ be the bounded model of $A^+$ obtained in Lemma 5.9 with respect to $I$. Then, there exists a tree-shaped interpretation $K$ such that:

1. $K \models A$,

2. there exists a homomorphism $\varphi$ from $G_K$ to $G_J$ with $\varphi(a^K) = a^J$ and $\varphi(a^K) = a^J$,

3. $|\Delta^K| \leq |\Delta^J| \times p$, where:

$$
p := \begin{cases} 
1, & \text{if } A^- = \emptyset \\
\prod_{\neg D(a) \in A^-} s(D), & \text{otherwise}.
\end{cases}
$$
Proof. We start by recalling some elements from the proof of Lemma 5.9 that are useful to prove our claims.

- \( \phi \) is a \( \tau \)-homomorphism from \( \hat{G}(A^+) \) to \( G_I \) with \( \phi(a) = a^I \).
- \( \phi_{id} \) is a \( \tau \)-homomorphism from \( \hat{G}(A^+) \) to \( G_J \) with \( \phi_{id}(a) = a^J \).
- \( \varphi \) is a homomorphism from \( G_J \) to \( G_I \) with \( \varphi(v) = \phi(v) \) for all \( v \in V_{A^+} \).
- Since \( A \) contains only one individual name and no role assertions, this means that \( \hat{G}(A^+) \) is a tree and by construction of \( \mathcal{J} \) in Lemma 5.9, \( G_J \) is also a tree.

Let \( \#(A^+) \) denote the number of concept assertions occurring in \( A \). We prove our claim by induction on the number \( \#(A^-) \).

**Induction Base.** \( \#(A^-) = 0 \). Then, we have that \( A^- = \emptyset \). Therefore, \( A = A^+ \) is a \( \tau \mathcal{EL}(deg) \) ABox. We choose \( K \) to be the interpretation \( \mathcal{J} \). Hence, we have \( \mathcal{J} \models A \), \( |\Delta_J| \leq |\Delta_J^A| \), and as explained above \( G_J \) is a tree. Finally, \( \mathcal{J} \) interprets the individual name \( a \) as \( a^J = a \), which means that \( \varphi(a^J) = a^I \) (see the mappings \( \phi \) and \( \varphi \) above).

Thus, we have shown our claims for the chosen interpretation \( \mathcal{K} \).

**Induction Step.** Assume that the claim holds for all consistent single-element ABoxes \( B \) with \( 0 \leq \#(B^-) < k \). Then, we show that it also holds for consistent single-element ABoxes \( A \) with \( \#(A^-) = k \).

As in the base case, we know that \( \mathcal{J} \models A^+ \). However, \( \mathcal{J} \) need not satisfy \( A^- \) since the assertions from \( A^- \) were not taken into account to obtain it. The idea for the rest of the proof is to start with an ABox \( A_J \) reflecting the structure of \( \mathcal{J} \). Then, we will consider a pre-processing \( A' \) of \( A_J \cup A^- \) guided by \( \mathcal{I} \), and show how to use it to extend \( \mathcal{J} \) into an interpretation \( \mathcal{K} \) satisfying our claims.

Let \( G_J \) be the description graph associated to \( \mathcal{J} \) (recall that it is a tree). The ABox \( A_J \) is built as follows:

\[
A_J := \bigcup_{b \in V_J} \{A(b)\} \cup \bigcup_{b \in E_J} \{r(b, c)\}
\]

where \( \text{Ind}(A_J) = \Delta_J = V_J \).

We name the element \( a_J \) in \( J \) as \( a \) in the new ABox \( A_J \). In addition, for all \( b \in \text{Ind}(A_J) \) such that \( b \neq a \), we make \( b^J = b \). Then, since all the concept assertions in \( A_J \) are of the form \( A(a) \) with \( A \in N_C \), it is easy to see that \( \mathcal{J} \models A_J \). We now extend the interpretation \( \mathcal{I} \) to the individual names in \( A_J \) to make \( \mathcal{I} \) a model of \( A_J \), namely, \( b^I = \varphi(b^J) \) for all \( b \in \text{Ind}(A_J) \). Since \( \varphi(a^J) = a^I \), this means that the element \( a^I \) does not change. Hence, \( \varphi \) is a homomorphism from \( G_J \) to \( G_I \) with \( \varphi(b^J) = b^I \) for all \( b \in \text{Ind}(A_J) \). Using \( \varphi \), from \( b^J \in A_J \) we get \( b^I \in A_I \) for all \( (b) \in A_J \). Similarly, we obtain \( (a^J, b^J) \in r^J \) for all \( (a, b) \in A_J \). Thus, \( \mathcal{I} \models A_J \) and consequently \( \mathcal{I} \models A_J \cup A^- \).

By Remark 5.13, there exists a pre-processing \( A' \) of \( A_J \cup A^- \) such that \( \mathcal{I} \models A' \). Additionally, we have that \( \text{Ind}(A_J') = \text{Ind}(A') \) (recall that \( \text{Ind}(A) = \{a\} \)). Based on \( A' \), our first goal is to find interpretations \( \mathcal{I}_b \) for all \( b \in \text{Ind}(A_J) \), such that \( \mathcal{I}_b \models A'(b) \) and they can be combined using Lemma 5.15 into a model of \( A_J \cup A^- \).
For all individuals \( b \in \text{Ind}(A_J) \), let \( A'_b \) be the following ABox:
\[
A'_b := \bigcup_{E_{\chi(\neg_r)b} \in A'} \{E_{\chi(\neg_r)b}(b)\} \cup \bigcup_{\neg \exists r. \hat{D}(b) \in A'} \{\neg \exists r. \hat{D}(b)\}
\]

Here, \( E_{\chi(\neg_r)b}(b) \) is an assertion that results from the application of rule \( \to \neg_r \) or rule \( \to \neg A \). For the rule \( \to \neg A \), we also represent \( A_{<1} \) as \( E_{\chi(\neg_r)b} \), since it is obtained from \( \neg A \) and \( A \equiv A_{\geq 1} \). Then, the ABox \( A_b \) is defined as:
\[
A_b := A'_b \cup \bigcup_{A(b) \in A'} \{A(b)\}
\]

The difference between \( A_b \) and \( A'(b) \) is that \( A_b \) does not contain assertions of the form \( \neg \hat{C}(b) \) where \( \hat{C} \) is a conjunction or a threshold concept \( E_{\to_r} \). Let us now show that \( \#(A'_b) \leq \#(A^-) \).

- \( J \) is tree-shaped and \( \Delta_J = \text{Ind}(A_J) \).
- Let \( a \) be the individual in \( A_J \) corresponding to the root of \( T_J \). If \( E_{\chi(\neg_r)a} \in A' \), it must have been obtained by an application of \( \to \neg_r \) (\( \neg A(a) \)) to an assertion of the form \( \neg E_{\chi(\neg_r)}(a) \). Since \( a \) is the root element in the tree structure of \( A_J \cup A^- \), such a negative assertion is either initially in \( A^- \) or results from the application of \( \to \neg_r \) to \( \neg \hat{C}(a) \in A^- \). This last argument also applies to the assertions \( \neg \exists r. \hat{D}(a) \in A'_a \). Since \( \to \neg r \) can be applied only once to \( \hat{C}(a) \), this implies that \( \#(A'_a) \leq \#(A^-) \).
- Taking \( a \) as the base case, the same can be shown for the rest of the individuals using induction on the depth\(^1\) of each node in \( V_J \).

Once it is known that \( \#(A'_b) \leq \#(A^-) \), we can then find the interpretations \( I_b \). Let \( B \subseteq \text{Ind}(A_J) \) be a set such that \( b \in B \) if, and only if, \( A'_b \) contains at least one assertion of the form \( E_{\chi(\neg_r)b}(b) \). We distinguish two cases:

1. \( b \in B \). This means that \( \#(A'_b) < \#(A^-) \). Hence, we can apply induction to \( A_b \) to obtain a tree-shaped interpretation \( I_b \) and a homomorphism \( \varphi_b \) from \( G_{I_b} \) to \( G_J \) such that: \( I_b \models A_b \) and \( \varphi_b(b^{I_b}) = b^J \).

2. \( b \notin B \). Consider the single-pointed interpretation \( I_b = (\{b\}, \mathcal{T}_b) \) that is the restriction of \( J \) to \( \{b\} \). The ABox \( A_b \) contains only assertions of the form \( \neg \exists r. \hat{D}(b) \) or assertions from \( A_J \). Since \( J \models A_J \), it is clear that \( I_b \models A_b \) and \( \varphi_b \) with \( \varphi_b(b^{I_b}) = b^J \) is a homomorphism from \( G_{I_b} \) to \( G_J \).

To fulfill our intermediate goal it remains to show \( I_b \) also satisfies the rest of the assertions in \( A'(b) \). For assertions of the form \( \neg E_{\to_r}(b) \) and \( \neg A(b) \), the application of the rules \( \to \neg_r \) and \( \to \neg A \) ensures that \( E_{\chi(\neg_r)b}(b) \) and \( A_{<1}(b) \) are in \( A_b \). Since \( I_b \models A_b \), \( \neg E_{\to_r} \equiv E_{\chi(\neg_r)} \) and \( \neg A \equiv A_{\geq 1} \), it follows that \( I_b \models \neg E_{\to_r}(b) \) and \( I_b \models \neg A(b) \). The other

\(^1\)The depth of a node in a tree is the length of the path from the root of the tree to the node. The root of the tree has depth 0.
case corresponds to \(-\hat{C}(b) \in \mathcal{A}'\) where \(\hat{C}\) is of the form \(\hat{C}_1 \cap \ldots \cap \hat{C}_n\). By the application of the rule \(\rightarrow_{\tau_i}\) we know that there is some \(\hat{C}_i\) such that \(-\hat{C}_i(b) \in \mathcal{A}'\). Since \(-\hat{C}_i\) is of one of the previously considered forms, it then holds that \(I_b \models -\hat{C}(b)\).

Altogether, we have shown that \(I_b \models \mathcal{A}'(b)\) for all \(b \in \text{Ind}(\mathcal{A}_\mathcal{J})\). Therefore, considering the sets \(\Delta^\mathcal{J}_b\) pairwise disjoint, we can apply Lemma 5.15 to \(\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-\) to obtain an interpretation \(\mathcal{K}\) such that \(\mathcal{K} \models \mathcal{A}_\mathcal{J} \cup \mathcal{A}^-\). Thus, it remains to show that \(\mathcal{K}\) is a model of \(\mathcal{A}^+\) as well. Note that \(\mathcal{K}\) is the result of extending the base interpretation \(\mathcal{J}\) satisfying \(\mathcal{A}^+\), by attaching to it the interpretations \(I_b\). This means that \(\phi_{id}\) is also a classical homomorphism from \(\hat{G}(\mathcal{A}^+)\) to \(G\mathcal{K}\) with \(\phi_{id}(a) = a^\mathcal{J}\). To see that it is also a \(\tau\)-homomorphism we observe the following.

- \(\mathcal{J} \subseteq \mathcal{K}\).

- The homomorphism \(\varphi^*\) from \(G_\mathcal{K}\) to \(G_\mathcal{T}\) constructed in Lemma 5.15 is such that, \(\varphi^*(b) = \varphi_b(b) = b^T\) for all \(b \in \Delta^\mathcal{J}\). Moreover, \(b^T\) was defined as \(\varphi(b^\mathcal{J})\) and \(b^\mathcal{J} = b\). Hence, \(\varphi^*(b) = \varphi(b)\) for all \(b \in \Delta^\mathcal{J}\).

- By construction of \(\mathcal{J}\) in Lemma 5.9, we know that \(G(\mathcal{A}^+)\) is a subgraph of \(G_{\mathcal{J}}\) and \(\varphi(v) = \phi(v)\) for all \(v \in V_{\mathcal{A}^+}\). Hence, \(\varphi^*\) is a homomorphism from \(G_\mathcal{K}\) to \(G_\mathcal{T}\) such that \(\varphi^*(v) = \phi(v)\) for all \(v \in V_{\mathcal{A}^+}\).

Hence, similar to the way it is done for \(I_0\) and its extension \(\mathcal{J}\) in Lemma 5.9, we can use the monotonicity property of \(\text{deg}\) introduced in Lemma 4.11 to show that \(\phi_{id}\) is a \(\tau\)-homomorphism from \(\hat{G}(\mathcal{A}^+)\) to \(G_\mathcal{K}\) with \(\phi_{id}(a) = a^\mathcal{J}\). Thus, since \(a^\mathcal{J} = a^\mathcal{K}\) we can apply Theorem 3.9 to obtain \(\mathcal{K} \models \mathcal{A}^+\).

Next, to see that \(\mathcal{K}\) is tree-shaped, note that \(\mathcal{J}\) and all the interpretations \(I_b\) are tree-shaped. Consequently, since \(\mathcal{G}\) corresponds to the structure of \(\mathcal{A}_\mathcal{J}\), the construction in Lemma 5.15 yields a tree-shaped interpretation \(\mathcal{K}\).

Last, let us look at the size of \(\mathcal{K}\). If \(b \notin B\) we have \(|\Delta^\mathcal{J}_b| = 1\), otherwise \(I_b\) is obtained by the application of the induction hypothesis to \(\mathcal{A}_b\). Let \(\mathcal{J}_b\) be the bounded model for \(\mathcal{A}^+_b\) constructed in Lemma 5.9. Then,

\[
|\Delta^\mathcal{J}_b| \leq |\Delta^{\mathcal{J}_b}| \times \prod_{\sim \hat{D}(b) \in \mathcal{A}^-} s(\hat{D}) \tag{A.1}
\]

A closer look at \(\mathcal{A}^+_b\) shows that it only contains assertions of the form \(E_{\chi(\sim)b}(b)\), or \(A(b)\) with \(A(b) \in \mathcal{A}_\mathcal{J}\) and \(A \in \mathcal{N}_\mathcal{C}\). Furthermore, it contains exactly one individual name and no role assertions. Hence, the construction of \(\mathcal{J}_b\) in Lemma 5.9 yields:

\[
|\Delta^{\mathcal{J}_b}| \leq \sum_{E_{\chi(\sim)b}(b) \in \mathcal{A}^+_b} s(E_{\chi(\sim)b})
\]

Now, \(s(E_{\chi(\sim)b}) > 1\) allows to transform this inequality into the following one:

\[
|\Delta^{\mathcal{J}_b}| \leq \prod_{E_{\chi(\sim)b}(b) \in \mathcal{A}^+_b} s(E_{\chi(\sim)b}) \tag{A.2}
\]
Chapter A. Missing proofs

The ABox $\mathcal{A}_b$ is included in the pre-processing $\mathcal{A}' = \mathcal{A}_G \cup \mathcal{A}_{-b}$. Consequently, for all assertions $\neg \exists r. \hat{D}(b) \in \mathcal{A}_{-b}$, the concept $\hat{D}$ is a sub-description of a concept $\hat{C}$ such that $\neg \hat{C}(a) \in \mathcal{A}_-$. In addition, each threshold concept $E_{\chi(t)} \in \mathcal{A}_{+b}$ is the result of applying the rule $\rightarrow_{\sim}$ to a concept $\neg E_{\sim t}$. Again, $E_{\sim t}$ has to be a sub-description of a concept $\hat{C}$ such that $\neg \hat{C}(a) \in \mathcal{A}_-$. Since $G_J$ is a tree, each concept assertion $\neg \hat{C}(a) \in \mathcal{A}_{-b}$ contributes with at most one of these concepts to $\mathcal{A}_b'$. We have shown above that $\#(\mathcal{A}_b') \leq \#(\mathcal{A}_-)$.

Finally, since $|\Delta_J| = |\text{Ind}(\mathcal{A}_J)|$, the construction of $\Delta_K$ yields:

$$|\Delta_K| \leq \sum_{b \in \text{Ind}(\mathcal{A}_J)} |\Delta_I| \leq |\Delta_J| \times p \quad \square$$

Missing proofs of Chapter 6

Lemma A.2. Let $\mathcal{T}$ be an acyclic EL TBox in normal form. Then, for all $\alpha \in \text{def}(\mathcal{T})$ the number of sub-descriptions of $u_\mathcal{T}(\alpha)$ is at most $s(\mathcal{T})$.

Proof. Recall the definition of $\text{sub}(C)$ in Definition 2.1. Let $\text{sub}^*(C) \subseteq \text{sub}(C)$ be the following set:

$$\text{sub}^*(C) := \begin{cases} 
\{C\} & \text{if } C = \top \text{ or } C \in N_C, \\
\{C\} \cup \text{sub}^*(C_1) \cup \text{sub}^*(C_2) & \text{if } C \text{ is of the form } C_1 \sqcap C_2, \\
\{\exists r. D\} & \text{if } C \text{ is of the form } \exists r. D.
\end{cases}$$

Furthermore, for all $\alpha = C_\alpha \in \mathcal{T}$, let $\rightarrow^+(\alpha)$ denotes the set of defined concepts in $\mathcal{T}$ that $\alpha$ depends on, i.e.:

$$\rightarrow^+(\alpha) := \{\beta \mid \beta \in \text{def}(\mathcal{T}) \text{ and } \alpha \rightarrow^+ \beta\}$$

We prove the following claim about the set $\text{sub}(u_\mathcal{T}(\alpha))$:

$$\text{sub}(u_\mathcal{T}(\alpha)) = \text{sub}^*(u_\mathcal{T}(C_\alpha)) \cup \bigcup_{\beta \in \text{Ind}(\mathcal{T}), \beta \in \rightarrow^+(\alpha)} \text{sub}^*(u_\mathcal{T}(C_\beta)) \quad (A.3)$$

The proof is by well-founded induction on the partial order $\preceq$ induced by $\rightarrow^+$ on $\text{def}(\mathcal{T})$. Let $\alpha = C_\alpha \in \mathcal{T}$, due to the normal form of $\mathcal{T}$ the concept $C_\alpha$ has the following structure:

$$P_1 \sqcap \ldots P_q \sqcap \exists r_1. \beta_1 \sqcap \ldots \sqcap \exists r_n. \beta_n$$

The unfolding of $\alpha$ with respect to $\mathcal{T}$ is the following concept description:

$$u_\mathcal{T}(\alpha) = P_1 \sqcap \ldots P_q \sqcap \exists r_1. u_\mathcal{T}(\beta_1) \sqcap \ldots \sqcap \exists r_n. u_\mathcal{T}(\beta_n)$$
By the definitions of $\text{sub}$ and $\text{sub}^*$, we can express the set $\text{sub}(u_T(\alpha))$ as follows:

$$\text{sub}(u_T(\alpha)) = \text{sub}^*(u_T(C_\alpha)) \cup \bigcup_{i=1}^{n} \text{sub}(u_T(\beta_i)) \quad \text{(A.4)}$$

Now, the application of the induction hypothesis to each $\beta_i$ ($1 \leq i \leq n$) yields:

$$\text{sub}(u_T(\beta_i)) = \text{sub}^*(u_T(C_{\beta_i})) \cup \bigcup_{\beta \in C_{\beta_i} \in \mathcal{T}} \text{sub}^*(u_T(C_\beta))$$

Hence, substituting the previous equality in (A.4) we obtain the following one:

$$\text{sub}(u_T(\alpha)) = \text{sub}^*(u_T(C_\alpha)) \cup \bigcup_{i=1}^{n} \left[ \text{sub}^*(u_T(C_{\beta_i})) \cup \bigcup_{\beta \in C_{\beta_i} \in \mathcal{T}} \text{sub}^*(u_T(C_\beta)) \right]$$

Finally, since $\rightarrow^+(\alpha) = \bigcup_{i=1}^{n} (\{\beta_i\} \cup \rightarrow^+(\beta_i))$, it is clear that the set defined by the big union in the previous equality is equal to the one represented by the big union in (A.3). Thus, our claim in (A.3) is true.

According to the definition of $\text{sub}^*$, for a top-level atom $\exists r_i.\beta_i$ of $C_\alpha$ the set of concepts $\text{sub}^*(\exists r_i. u_T(\beta_i))$ corresponds to $\{\exists r_i. u_T(\beta_i)\}$. Hence, it is not hard to see that for all $\alpha \models C_\alpha \in \mathcal{T}$ it holds:

$$|\text{sub}^*(u_T(C_\alpha))| \leq s(C_\alpha)$$

Thus, using (A.3) we can conclude that $|\text{sub}(u_T(\alpha))| \leq s(\mathcal{T})$ for all $\alpha \in \text{def}(\mathcal{T})$. \qed

Now, since $\text{sub}(E_{\sim t})$ is equal to $\{E_{\sim t}\}$, the previous result also applies to acyclic $\tau \mathcal{E}(\text{deg})$ TBoxes.

**Corollary A.3.** Let $\hat{\mathcal{T}}$ be an acyclic $\tau \mathcal{E}(\text{deg})$ TBox in normal form. Then, for all $\alpha \in \text{def}(\mathcal{T})$ it holds:

$$|\text{sub}(u_{\hat{\mathcal{T}}}(\alpha))| \leq s(\hat{\mathcal{T}})$$

**Missing proofs of Chapter 7**

**Lemma 7.18.** Let $n > 0$ be a natural number. Then,

1. for all sets $S$ of $P_n$-assignments that are canonical for $P_n$, there exists $D_S \in \mathcal{D}_n$ such that $S$ and $D_S$ are corresponding, and

2. for all $D \in \mathcal{D}_n$, exists a set $S_D$ of $P_n$-assignments that is canonical for $P_n$ such that $S_D$ and $D$ are corresponding.

**Proof.** We prove the claim by induction on the number $n$. We start by considering two base cases:
\begin{itemize}
  \item \( n = 1 \). The prefix \( P_1 \) corresponds to \( \exists x_1 \). Therefore, there are only two sets of \( P_1 \)-assignments that are canonical for \( P_1 \). Namely, \( S_{true} = \{ \{ t(x_1) = true \} \} \) and \( S_{false} = \{ \{ t(x_1) = false \} \} \). Now, the string \( D_1 \) is of the form \( \exists r.X_1^0 \) (recall its definition in (7.2)). Hence, the instances \( \theta_{true}[D_1] \) and \( \theta_{false}[D_1] \) where \( \theta_{true}(X_1^0) = A \) and \( \theta_{false}(X_1^0) = \top \), are corresponding concepts in \( D_1 \) for \( S_{true} \) and \( S_{false} \), respectively.

  \item \( n = 2 \). The prefix \( P_2 \) is of the form \( \exists x_1.\forall x_2 \). In this case there are also two sets of \( P_2 \)-assignments that are canonical for \( P_2 \), but they are of the following form:

  \[
  S_{true} = \{ \{ t_1(x_1) = true, t_1(x_2) = false \}, \{ t_2(x_1) = true, t_2(x_2) = true \} \}
  \]

  \[
  S_{false} = \{ \{ t_1(x_1) = false, t_1(x_2) = false \}, \{ t_2(x_1) = false, t_2(x_2) = true \} \}
  \]

  The string \( D_2 \) is of the form \( \exists r.(X_1^0 \sqcap \exists r.A \sqcap \exists s.\top) \). Thus, \( \theta_{true}[D_2] \) and \( \theta_{false}[D_2] \) are also corresponding concepts in \( D_2 \) for \( S_{true} \) and \( S_{false} \), respectively.

  Notice, that in both cases the selected concepts from \( D_1 \) and \( D_2 \) are actually the only concepts contained in those sets. Therefore, the statement 2.) also holds for both base cases.

  \textit{Induction Step.} Let us assume that the claim holds for all natural numbers smaller than \( n \). We show that it also holds for all \( n > 2 \).

  1.) Let \( S \) be a set of \( P_n \)-assignments that is canonical for \( P_n \). Since \( P_n \) is of the form \( \exists x_1.P' \), by definition of canonical we have that the set \( S' = \{ t\{x_2,\ldots,x_n\} \mid t \in S \} \) is canonical for \( P' \). Moreover, \( P' \) is of the form \( \forall x_2.P'' \), and \( P'' \) is not empty because \( n > 2 \). Hence, there exist two sets \( S_{true} \) and \( S_{false} \) that are canoni- cal for \( P'' \) of the following form:

  \[
  S_{true} := \{ t\{x_3,\ldots,x_n\} \mid t \in S' \text{ and } t(x_2) = true \}
  \]

  \[
  S_{false} := \{ t\{x_3,\ldots,x_n\} \mid t \in S' \text{ and } t(x_2) = false \}
  \]

  Note that \( P'' \) is actually the prefix \( P_{n-2} \) when shifting the indexes of the variables \( \{x_3,\ldots,x_n\} \) to \( \{x_1,\ldots,x_{n-2}\} \). Therefore, we can apply the induction hypothesis to obtain two concept descriptions \( D_{S_1} \) and \( D_{S_2} \) in \( D_{n-2} \) such that they are corresponding concepts for \( S_{true} \) and \( S_{false} \), respectively. We now use these two concepts to construct a corresponding concept for \( S \). Let us start by observing the following facts about \( D_{S_1} \) and \( D_{S_2} \):

  \begin{itemize}
  \item There are mappings \( \theta_1, \theta_2 \in X_{n-2} \) such that \( \theta_1[D_{n-2}] = D_{S_1} \) and \( \theta_2[D_{n-2}] = D_{S_2}. \)

  \item For all \( i \) such that \( 1 \leq 2i + 1 \leq n - 2 \), \( D_{n-2} \) contains \( 2^i \) variables \( X_{2i+1}^0, \ldots, X_{2i+1}^{2^i-1}. \)

  \item \( D_{n-2} \) can be transformed into the strings \( \exists r.D_3^0 \) and \( \exists r.D_3^1 \) that are used to construct the string \( D_n \) by renaming its variables. We define two renamings \( r_1 \) and \( r_2 \) as follows. For all \( i \geq 0 \) and all \( j \) such that \( 1 \leq 2i + 1 \leq n - 2 \) and \( 0 \leq j < 2^i \):

    \[
    r_1(X_{2i+1}^j) := X_{2i+3}^j \quad \text{and} \quad r_2(X_{2i+1}^j) := X_{2i+3}^{2^i+j}
    \]

    It is not hard to see that applying \( r_1 \) (\( r_2 \)) to \( D_{n-2} \) yields the string \( \exists r.D_3^0 (\exists r.D_3^1). \)
  \end{itemize}
\end{itemize}
Based on $r_1$ and $r_2$, we define the mapping $\theta : X_n \rightarrow \{\top, A\}$ as:

$$\theta := r_1(\theta_1) \cup r_2(\theta_2) \cup \{(X_1^0, \lambda)\}$$

where $r_1(\theta_1)$ and $r_2(\theta_2)$ stands for the renaming of the variables in the domain sets of $\theta_1$ and $\theta_2$, and $\lambda = A$ if $x_1$ is mapped to true or $\top$ otherwise (recall that $x_1$ is mapped to the same truth value by all the assignments in $S$). Hence, $\theta \in \mathcal{X}_n$ and $\theta[D_n]$ has the following description tree:

$$T_{\theta[D_n]}: v_0 \xrightarrow{r} \{\lambda\} \xrightarrow{r} \{A\} \xrightarrow{s} \{\}$$

We now show that $\theta[D_n]$ and $S$ are corresponding.

- Let $t$ by an assignment in $S$. If $t(x_2) = true$, then the restriction $t_{true} := t_{\{x_3, \ldots, x_n\}}$ of $t$ is obviously an assignment in $S_{true}$. By induction hypothesis, there is a corresponding path of the form $\{\} r \pi$ in $T_{DS_1}$ for $t_{true}$. Since $D_{S_1} = \theta_1[D_{n-2}]$, by construction of $\theta$ the following is a path in $T_{\theta[D_n]}$:

$$v_0 : \{\} \xrightarrow{r} \{\lambda\} \xrightarrow{r} \{A\} \xrightarrow{s} \{\} \xrightarrow{r} \pi$$

Hence, taking into account the way $\lambda$ has been selected and the fact that $t(x_2) = true$, this is clearly a corresponding path for $t$. The case where $t(x_2) = false$ can be handled symmetrically.

- Conversely, let $\pi$ be a path in $T_{\theta[D_n]}$. Again, we can consider one of two symmetric cases. For example,

$$v_0 : \{\} \xrightarrow{r} \{\lambda\} \xrightarrow{s} \{\} \xrightarrow{r} \pi'$$

By construction of $\theta[D_n]$, $\{\} r \pi'$ is a path in $T_{DS_2}$. Again, by induction hypothesis there is an assignment $t' \in S_{false}$ such that $\{\} r \pi'$ and $t'$ are corresponding. Let $t$ be the truth value of $x_1$ in $S$, we build a $P_n$-assignment $t$ as $t' \cup \{(x_2, false), (x_1, t)\}$.

Obviously, $t \in S$, and moreover $t$ and $\pi$ are corresponding.

Thus, we have shown that $S$ and $\theta[D_n]$ are corresponding, and consequently our first claim is true. Regarding our second claim, a similar line of reasoning as the one just used can be applied. Basically, we start with a concept $\theta[D_n] \in \mathcal{D}_n$, the mapping $\theta$ yields two mappings $\theta_1, \theta_2 \in \mathcal{X}_{n-2}$, and then the induction hypothesis can be applied to obtain two $P_{n-2}$-assignments $S_{true}$ and $S_{false}$ with similar properties as the ones discussed above. From them, one can obtain a $P_n$-assignment $S$ such that it is canonical for $P_n$, and $S$ and $\theta[D_n]$ are corresponding.

\[\square\]
Bibliography


Bibliography


Scientific Career

October 2015 - Present  
Doctoral student in the DFG Research Training Group QuantLA

October 2012 - September 2015  
Doctoral Student at Universität Leipzig - Institut für Informatik  
Scholarship holder for the DFG Research Training Group 1763 QuantLA (Quantitative Logics and Automata)  
Supervisors: Prof. Dr. Gerhard Brewka and Prof. Dr.-Ing. Franz Baader

October 2012  
Master of Science  
European Master’s Program in Computational Logic  
Technical University of Dresden, Germany  
Free University of Bozen-Bolzano, Italy  
Final grade: 1.1 (excellent)

July 2006  
Bachelor’s degree in Computer Science  
Faculty of Mathematics and Computer Science  
University of Havana, Cuba  
Final grade: 4.62/5.0

School Year 1999-2000  
Senior High School Graduate  
Federico Engels Senior High School of Exact Sciences  
Pinar del Río, Cuba  
Final grade: 99.97/100
List of Publications


- Franz Baader, Oliver Fernández Gil, and Barbara Morawska. Hybrid $\mathcal{EL}$-unification is NP-complete. *In* Proceedings of the 26th International Workshop on Description Logics, Ulm, Germany, July 23-26, pages 29–40, 2013.

- Franz Baader, Oliver Fernández Gil, and Barbara Morawska. Hybrid unification in the Description Logic $\mathcal{EL}$. *In* UNIF@RTA/TLCA, pages 8–12, 2013.
List of Talks

   Talk: *Adding Threshold Concepts to the Description Logic EL*.

09.06.2015 DL 2015 - Athens, Greece.
   Talk: *Adding Threshold Concepts to the Description Logic EL*.

27.02.2015 Frontiers of Formal Methods 2015 - Aachen, Germany.
   Talk: *Threshold Concepts in a Lightweight Description Logic*.

18.07.2014 NMR 2014 - Vienna, Austria
   Talk: *On the Non-Monotonic Description Logic ALC+Tmin*

   Talk: *Hybrid unification in the Description Logic EL*.

23.07.2013 DL 2013 - Ulm, Germany.
   Talk: *Hybrid EL-unification is NP-complete.*
Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.


Oliver Fernández Gil