Synthetic notions of curvature and applications in graph theory

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Introduction

1.1 Synthetic approach of curvature

Curvature is a parameter which indicates how different the geometry of a manifold is from the Euclidean space. In Riemannian geometry, it is defined from derivatives of the Riemnannian metric tensor, that is, in a purely analytic way. (See Definition 1 below.) The analytic approach means introducing coordinates, derivatives and computing geometric objects such as geodesics in equations etc. The restriction of this approach is that it only works on smooth spaces. In order to generalize the concept of curvature to nonsmooth spaces, people usually take a synthetic approach. That means, one first find a characteristic property of curvature or curvature bounds which can also be expressed in a nonsmooth setting, and then transform this property as a definition of curvature or curvature bounds. For a nice survey of the opposition of analytic and synthetic approach, we refer to Chapter 26 in Villani [114].

Actually, the approach of geometry by ancient Greeks is close to the synthetic approach. We describe some geometric intuitions of curvature in that spirit now. Let's first think about the manifolds with constant curvature, more precisely, the simply connected space forms. If the curvature is equal to zero, then the space is Euclidean. In fact the geometry of spaces with constant curvature, which we also refer to as non-Euclidean geometry, was created along with the efforts to prove Euclid's Fifth Postulate, i.e. the parallel postulate, as a consequence of the other four in the book Elements. As we know, those efforts all failed and it turns out that different parallel postulates produce different geometry of spaces. See e.g. the historical introduction in Ryan [105] for more details. If we look back, the different parallel postulates also give the very first intuition for non-zero curvatures.

When we think of non-Euclidean geometry, we should often keep in mind that (the following statements may not be precise)

1. two parallel lines could intersect at some point;

2. the sum of the inner angles of a triangle could be not equal to π.

Let's make it more precise now. In Euclidean space, i.e., the case that curvature equals to zero, two lines or geodesics with a common perpendicular can never intersect, see case E in Figure 1.1. Moreover, the distance between two walkers moving along those two lines in the same direction with the same velocity satisfies

$$d(x(t), y(t)) = d(x, y),$$

where $x(t)$, $y(t)$ is the position of the two walkers at time t, and $x(0) = x$, $y(0) = y$.

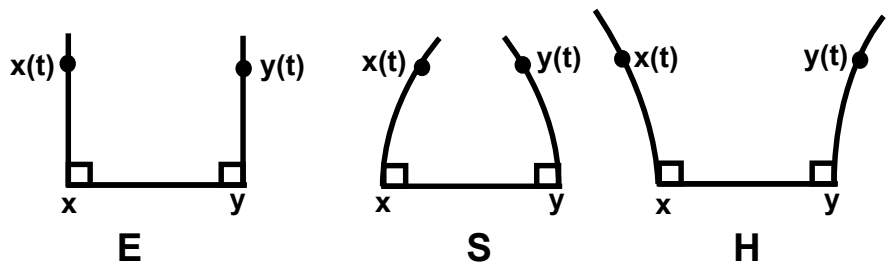

Figure 1.1: Different parallel postulates

But if the curvature is nonzero, see the case S, H in Figure 1.1, we have

$$I_1 := d(x(t), y(t)) - d(x, y) \neq 0. \tag{1.1.1}$$

If $I_1 < 0$, then the sum of the inner angles of a triangle $\Sigma_\triangle$ is larger than π. Thinking of the case S in Figure 1.1, we will immediately get a triangle whose sum of the inner angles is larger than π when the two walkers meet each other at some time. Similarly, we see that if $I_1 > 0$, the sum of the inner angles of a triangle $\Sigma_\triangle$ should be smaller than π. That is we in fact have another intuition of non-zero curvature as follows

$$I_2 := \pi - \Sigma_\triangle \neq 0. \tag{1.1.2}$$

The above two indices (1.1.1) and (1.1.2) are exactly the starting point of the synthetic curvatures we will study in this thesis, Ollivier-Ricci curvature and combinatorial curvature, respectively.

In the sequel, we will present an analytic calculation about (1.1.1) on a Riemannian manifold leading to Ollivier's proposition [92] on Ricci curvature. For an earlier lower bounded Ricci curvature version, see Theorem 1.5 in von Renesse-Sturm [103]. The following materials are basic calculations in Riemannian geometry and we explain in this process the relation between curvature and Jacobi fields, Hessian of distance functions, variational formulas of arc length. In fact,

we will calculate the following Taylor expansion,

$$d(x(t),y(t)) = d(x,y) + t\left.\frac{d}{dt}\right|_{t=0} d(x(t),y(t)) + \frac{t^2}{2}\left.\frac{d^2}{dt^2}\right|_{t=0} d(x(t),y(t)) + O(t^3), \tag{1.1.3}$$

for which we need to do the first order and second order variations of the arc length. We will use the following notations on a Riemannian manifold M with a metric tensor g:

$\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$,	scalar product;
$\lvert\cdot\rvert$,	the corresponding norm;
d,	the distance function;
T_pM,	the tangent space at p;
∇_X,	the covariant differentiation w.r.t. a vector field X;
∇,	the gradient;
$Hess$,	the Hessian;
exp,	the exponential map.

For details of those basic concepts and variational formulas of arc length, one can refer to textbooks on Riemannian geometry, see e.g. Jost [70], Gallot-Hulin-Lafontaine [56], Bai-Shen-Shui-Guo [1].

Let's now describe Figure 1.2. Suppose $x, y \in M$ are two points that can be

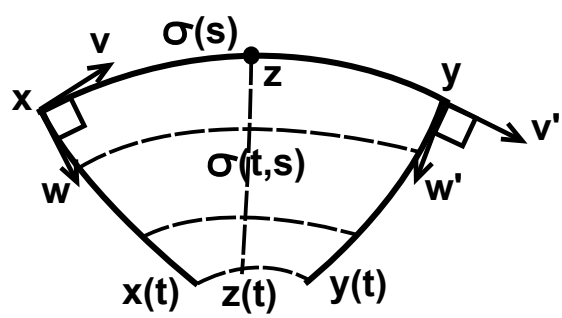

Figure 1.2: Variational vector fields

connected by a shortest geodesics $\sigma(s)$ whose length is sufficiently small. Denote $v := \sigma'(0)$. We suppose further that σ is normal, that is , $|v| = 1$. For small t, let

$$x(t) = \exp_x tw, \;\; y(t) = \exp_y tw',$$

where $w \in T_xM$, $|w| = 1$, $w' \in T_yM$ such that w' is the parallel tranport of w along the geodesic $\sigma(s)$. By the definition of exponential map, we know $\nabla_w w = \nabla_{w'} w' = 0$. We only need to consider the case $\langle w, v\rangle = 0$ here. Denote $v' = \sigma'(d(x,y))$. Since parallel transport is an isometry of tangent spaces as inner product spaces, we know $\langle w', v'\rangle = 0$ and $|w'| = 1$.

Now we specify the variational vector fields. We denote a family of geodesics connecting $x(t)$ and $y(t)$ by

$$\sigma(t,s) = \sigma_t(s) : \mathbb{R} \times \mathbb{R} \longrightarrow M.$$

Then we have

$$T := \sigma_* \left(\frac{\partial}{\partial t} \right), \ S := \sigma_* \left(\frac{\partial}{\partial s} \right).$$

For specified t, we denote the length of the geodesic $\sigma_t(s)$, $0 \leq s \leq d(x,y)$ by $L(t)$. Then we calculate the first order variation at $t = 0$

$$\begin{aligned} \frac{d}{dt} d(x(t), y(t)) &= \frac{d}{dt} L(t) = \frac{d}{dt} \int_0^{d(x,y)} \langle S, S \rangle^{\frac{1}{2}} ds \\ &= \int_0^{d(x,y)} \frac{\langle S, \nabla_T S \rangle}{|S|} ds = \int_0^{d(x,y)} \frac{\langle S, \nabla_S T \rangle}{|S|} ds \\ &= \langle S, T \rangle |_o^{d(x,y)} = \langle w, v \rangle - \langle w', v' \rangle = 0, \end{aligned} \tag{1.1.4}$$

where we use the facts that $\nabla_T S = \nabla_S T$ and $\nabla_S S = 0$.

For later purpose, we present another point of view for this fact. Let's consider the product manifold $M \times M$ with the metric

$$\langle U_1 \oplus U_2, V_1 \oplus V_2 \rangle := \langle U_1, V_1 \rangle + \langle U_2, V_2 \rangle,$$

where $U_1, V_1 \in T_x M$, $U_2, V_2 \in T_y M$. Let $z(t) \in \sigma(t, s_0)$ for some $0 < s_0 < d(x,y)$ and $z := z(0)$. Then we have

$$d((x,y),(z,z)) = d(x,z) + d(z,y) = d(x,y).$$

Here we don't distinguish the notion for distance function on $M \times M$ and that on M. Those kind of calculation in this view can be found in Jäger-Kaul [69].

Then we calculate

$$\begin{aligned} &\left. \frac{d}{dt} \right|_{t=0} d(x(t), y(t)) \\ =& \langle \nabla d((x,y),(z,z)), w \oplus w' \rangle = \langle -v \oplus v', w \oplus w' \rangle = 0. \end{aligned}$$

In the above we used the Gauss' lemma (see e.g. Corollary 4.2.3 on page 184 of Jost [70]). Note that $-v \oplus v'$ is the outward tangent vector at (x,y) in $M \times M$.

Now we turn to the second order variation term, the following calculation is also at $t = 0$.

$$\begin{aligned} &\frac{d^2}{dt^2} d(x(t), y(t)) = \int_0^{d(x,y)} \frac{\langle S, \nabla_S T \rangle}{|S|} ds \\ =& \int_0^{d(x,y)} \frac{1}{|S|^2} \left\{ \nabla_T \langle S, \nabla_S T \rangle |S| - \frac{\langle S, \nabla_S T \rangle}{|S|} \right\} ds \\ =& \int_0^{d(x,y)} \left\{ |\nabla_S T|^2 + \langle S, \nabla_T \nabla_T S \rangle - (\nabla_S \langle S, T \rangle)^2 \right\} ds. \end{aligned} \tag{1.1.5}$$

For the second term in the integral above, we have

$$\begin{aligned}\nabla_T\nabla_T S = \nabla_T\nabla_S T &= \nabla_T\nabla_S T - \nabla_S\nabla_T T - \nabla_{[S,T]}T + \nabla_S\nabla_T T \\ &:= R(T,S)T + \nabla_S\nabla_T T, \end{aligned} \tag{1.1.6}$$

where R is the Riemannian curvature tensor. Note that here the Lie bracket $[S,T]=0$ and

$$\int_0^{d(x,y)} \langle S, \nabla_S\nabla_T T\rangle ds = \langle S, \nabla_T T\rangle|_0^{d(x,y)} = 0. \tag{1.1.7}$$

We then consider the third term in the integral of (1.1.5). In fact T is a Jacobi field (see e.g. Definition 4.2.1 on page 178 of Jost [70]), because it satisfies the equation

$$\nabla_S\nabla_S T = R(S,T)S.$$

(This in fact follows easily from $\nabla_S\nabla_S T = \nabla_S\nabla_T S - \nabla_T\nabla_S S = R(S,T)S$.) Therefore,

$$\frac{d^2}{ds^2}\langle T, S\rangle = \langle \nabla_S\nabla_S T, S\rangle = \langle R(S,T)S, S\rangle = 0.$$

Hence we can find $a, b \in \mathbb{R}$, such that $T(s)$ can be written as

$$T(s) = T^{\perp} + (as+b)\frac{\partial}{\partial s}, \text{ where } \langle T^{\perp}, \frac{\partial}{\partial s}\rangle = 0. \tag{1.1.8}$$

Noting the fact that,

$$\langle T, \frac{\partial}{\partial s}\rangle|_{s=0} = \langle w, v\rangle = 0, \ \langle T, \frac{\partial}{\partial s}\rangle|_{s=d(x,y)} = \langle w', v'\rangle = 0,$$

we have in (1.1.8) $a = b = 0$. That is

$$\langle T, S\rangle|_{t=0} = \langle T, \frac{\partial}{\partial s}\rangle = 0. \tag{1.1.9}$$

Now inserting (1.1.6), (1.1.7) and (1.1.9) into (1.1.5), we get,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} d(x(t), y(t)) = \int_0^{d(x,y)} \left\{|\nabla_S T|^2 - \langle R(T,S)S, T\rangle\right\} ds. \tag{1.1.10}$$

Let's recall the definition of sectional curvature and Ricci curvature on a Riemannian manifold.

Definition 1. *Let T_pM be the tangent space with an orthonormal basis $X_1, X_2, \ldots, X_n$. Then the sectional curvature of the subspace spanned by X_i, X_j, $i \neq j$ is*

$$K(X_i, X_j) = \langle R(X_i, X_j)X_j, X_i\rangle.$$

The Ricci curvature along a direction X_i is

$$Ric(X_i, X_i) = \sum_{j=1, j\neq i}^{n} K(X_i, X_j).$$

We first think of (1.1.10) in manifolds with constant sectional curvature K. In fact from the view of product manifolds, one can get (see Jäger-Kaul [69])

$$\frac{d^2}{dt^2}\bigg|_{t=0} d(x(t),y(t)) = Hess\, d(w\oplus w', w\oplus w') = 2\frac{s_K'(d(x,y))-1}{s_K(d(x,y))}, \tag{1.1.11}$$

where

$$s_k(s) := \begin{cases} \frac{\sin\sqrt{K}s}{\sqrt{K}}, & K>0; \\ s, & K=0; \\ \frac{\sinh\sqrt{-K}s}{\sqrt{-K}}, & K<0. \end{cases}$$

The Taylor expansion of the right hand side of (1.1.11) gives

$$\frac{d^2}{dt^2}\bigg|_{t=0} d(x(t),y(t)) = -Kd(x,y) + O(d^2(x,y)). \tag{1.1.12}$$

In order to deal with the case when sectional curvature is not constant, we look into the details of (1.1.11) here. First observe

$$\langle R(T,S)S,T\rangle = \langle \nabla_T\nabla_S S - \nabla_S\nabla_T S, T\rangle = -\langle \nabla_S\nabla_S T, T\rangle$$

and therefore by (1.1.10) we get

$$\frac{d^2}{dt^2}\bigg|_{t=0} d(x(t),y(t)) = \int_0^{d(x,y)} \nabla_S(\langle T,\nabla_S T\rangle)ds = \langle T,\nabla_S T\rangle|_0^{d(x,y)}. \tag{1.1.13}$$

Meanwhile we know

$$\nabla_S|T| = \frac{\langle T,\nabla_S T\rangle}{|T|}.$$

This implies that the calculation boils down to writing precisely out the norm of the Jacobi field T.

Let's explore the Jacobi fields on constant curved manifolds now. Recall $\langle T,S\rangle = 0$ and note the fact that all 2-planes in constant curved manifolds are totally geodesic, we can suppose T has the form $T(s) = h(s)A(s)$ where $A(s)$ is parallel along $\frac{\partial}{\partial s}$ with norm 1. Then one can calculate

$$\begin{aligned} h''(s) = |T|''(s) = \nabla_S\left(\frac{\langle T,\nabla_S T\rangle}{|T|}\right) &= \frac{\langle T,\nabla_S\nabla_S T\rangle + \langle \nabla_S T,\nabla_S T\rangle}{|T|} - \frac{\langle T,\nabla_S T\rangle^2}{|T|^3} \\ &= -\frac{\langle R(T,S)S,T\rangle}{|T|} + \frac{|\nabla_S T|^2|T|^2 - \langle T,\nabla_S T\rangle^2}{|T|^3} = -Kh(s). \end{aligned}$$

That is $h(s)$ is the solution of

$$\begin{cases} h''(s) + Kh(s) = 0, \\ h(0) = 1, h(d(x,y)) = 1. \end{cases}$$

Therefore we have

$$h(s) = \frac{1}{s_K(d(x,y))}(s_K(d(x,y)-s) + s_K(s)) \tag{1.1.14}$$

Combining (1.1.14) and (1.1.13), we get (1.1.11).

We can also look at (1.1.10) directly. Noting $|h'(0)|^2 = O(d(x,y))$, we have

$$\int_0^{d(x,y)} |\nabla_S T|^2 ds = \int_0^{d(x,y)} |h'(s)|^2 ds = \int_0^{d(x,y)} (|h'(0)|^2 + O(d(x,y)))ds = O(d^2(x,y)).$$

and then get (1.1.12). So for constant curved manifolds, we arrive at

$$d(x(t), y(t)) = d(x,y) - \frac{t^2}{2} K d(x,y) + O(t^2 d^2(x,y)) + O(t^3). \tag{1.1.15}$$

Now for the nonconstant curvature case, we have $\langle R(T,S)S, T\rangle|_x = K(v,w) + O(d(x,y))$ and then

$$d(x(t), y(t)) = d(x,y) - \frac{t^2}{2} K(v,w) d(x,y) + \frac{t^2}{2} \int_0^{d(x,y)} |\nabla_S T|^2 ds + O(t^2 d^2(x,y)) + O(t^3). \tag{1.1.16}$$

When $d(x,y)$ is sufficiently small, $K(T,S)$ is constant up to higher order terms. Then by the theory of continuous dependence on data for solutions of ODEs, we can get

$$\int_0^{d(x,y)} |\nabla_S T|^2 ds) = O(d^2(x,y)). \tag{1.1.17}$$

Or we see the above fact by the Hessian comparison theorem. When $d(x,y)$ is sufficiently small, we can suppose $K_1 \leq K(T,S) \leq K_2$, and $K_1, K_2 = K(v,w) + O(d(x,y))$. Then we have

$$-K_2 d(x,y) + O(d(x,y)) \leq \left.\frac{d^2}{dt^2}\right|_{t=0} d(x(t), y(t)) \leq -K_1 d(x,y) + O(d(x,y)).$$

Then recalling (1.1.10), we get (1.1.17).

Therefore we finally get

Proposition 1 (see e.g. Ollivier [92])**.** *In Figure 1.2, we have*

$$d(x(t), y(t)) = d(x,y) - \frac{t^2}{2} K(v,w) d(x,y) + O(t^2 d^2(x,y)) + O(t^3). \tag{1.1.18}$$

For fixed $x, y \in M$, we define a function $f : B_\epsilon^T(x) \subset T_x M \longrightarrow \mathbb{R}$ on the tangent space at x as

$$f(W) := d(\exp_x W, exp_y W'),$$

where W' is the parallel transport of W along the geodesic $\sigma(s)$. Then $f(tw) = d(x(t), y(t))$, recall in Figure 1.2, $w = T|_x$ is a unit vector.

Suppose T_xM has an orthonormal basis $v, T_1, \ldots, T_{n-1}$, then any $W \in T_xM$ could be written as

$$W = w_0 v + \sum_{i=1}^{n-1} w_i T_i.$$

Hence we can also see W as the vector $(w_0, \ldots, w_{n-1})$. By Taylor expansion we have

$$f(W) = f(0) + W^T \nabla f(0) + \frac{1}{2} W^T Hess f(0) W + \cdots .$$

Let's try to calculate $\frac{1}{vol(B_\epsilon^T(x))} \int_{B_\epsilon(x)} f(W) dW$. By Proposition 1 we already know $\nabla f(0) = 0$. (In fact, here even $\int_{B_\epsilon^T(x)} w_i dW = 0$.) Noting moreover $\int_{B_\epsilon^T(x)} w_i w_j dW = 0$ and

$$\int_{B_\epsilon^T(x)} w_i^2 dW = \frac{1}{n} \int_{B_\epsilon^T(x)} \sum_{i=0}^{n-1} w_i^2 dW = \frac{\epsilon^2}{n+2} vol(B_\epsilon(x)),$$

we get

$$\begin{aligned} &\frac{1}{vol(B_\epsilon^T(x))} \int_{B_\epsilon^T(x)} f(W) dW \\ =& f(0) + \frac{1}{2} \frac{\epsilon^2}{n+2} \left(\sum_{i=1}^{n-1} \nabla_{T_i} \nabla_{T_i} f(0) + \nabla_v \nabla_v f(0) \right) + O(\epsilon^3), \end{aligned}$$

where

$$\sum_{i=1}^{n-1} \nabla_{T_i} \nabla_{T_i} f(0) = -Ric(v, v) d(x, y) + O(\epsilon^2 d^2(x, y)),$$

and

$$\nabla_v \nabla_v f(0) = Hess\, d(v \oplus v', v \oplus v') = 0.$$

That is we arrive at

$$\begin{aligned} &\frac{1}{vol(B_\epsilon^T(x))} \int_{B_\epsilon^T(x)} d(\exp_x W, \exp_y W') dW \\ =& d(x, y) - \frac{\epsilon^2}{2(N+2)} Ric(v, v) d(x, y) + O(\epsilon^4 d^2(x, y)) + O(\epsilon^3). \end{aligned} \tag{1.1.19}$$

It turns out that the left hand side of (1.1.19) is closely related to the optimal transportation distance between the two probability measures

$$dm_x^\epsilon = \frac{dvol|_{B_\epsilon}(x)}{vol(B_\epsilon(x))}, \; dm_y^\epsilon = \frac{dvol|_{B_\epsilon}(y)}{vol(B_\epsilon(y))},$$

which is defined as

$$W_1(m_x^\epsilon, m_y^\epsilon) := \inf_{\xi \in \prod(m_x^\epsilon, m_y^\epsilon)} \int_{M\times M} d(x', y') d\xi(x', y').$$

The $\prod(m_x^\epsilon, m_y^\epsilon)$ is the set of probability measures on $M \times M$ projecting to m_x^ϵ and m_y^ϵ. The ξ which attains the infimum is called an optimal coupling of the two measures. For topics in optimal transportation, see e.g. Villani [113, 114], Evans [54].

First note that in (1.1.19), the measure

$$\frac{dW|_{B_\epsilon^T(x)}}{vol(B_\epsilon^T(x))}$$

is equal to m_x^ϵ up to higher order terms. And von Renesse-Sturm [103] proved that the parallel coupling

$$(m_x^\epsilon, (\exp_y \circ Par_\sigma \circ \exp_x^{-1})_\sharp m_x^\epsilon),$$

where Par_σ is the parallel transport along σ, is a coupling of m_x^ϵ and m_y^ϵ up to higher order terms. Further in [92], Ollivier proved up to higher order terms, the parallel coupling is optimal and then get

Proposition 2 (Ollivier [92])**.**

$$W_1(m_x^\epsilon, m_y^\epsilon) = d(x, y)\left(1 - \frac{\epsilon^2}{2(n+2)} Ric(v, v)\right) + \textit{higher order terms}. \quad (1.1.20)$$

Obviously, this proposition can also be stated on a metric space (X, d), on which each point is associated with a probability measure m_x, ignoring the higher order terms. Ollivier [92] transformed this proposition as a definition of Ricci curvature on this more general setting. In fact, he defined the curvature along two points $x, y \in X$ as

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}.$$

We see from the above calculations that the analytic definition of Ricci curvature involves the second order variation of arc length (or second order derivatives of distance function), while Ollivier's synthetic notion only refer to the metric and measure on the space. Observe that (1.1.20) does not involve the geodesics of the space. So Ollivier's synthetic notion works in particular on discrete spaces.

Something similar happens to (1.1.2). To be precise, let's first figure out the definition of angles on a Riemannian manifold (M, g) (see e.g. Section 3.6.5 in Burago-Burago-Ivanov [13]).

Definition 2. *Let x, y, z be three distinct points in (M, g). The comparison angle $\widetilde{\measuredangle} xyz$ is defined as*

$$\widetilde{\measuredangle} xyz = \arccos \frac{d(x,y)^2 + d(y,z)^2 - d(x,z)^2}{2d(x,y)d(y,z)} \tag{1.1.21}$$

In fact, this angle is the corresponding angle in the comparison triangle $\triangle_{\bar{x}\bar{y}\bar{z}}$, by which we refer to the triangle in $\mathbb{R}^2$ satisfying $d(\bar{x}, \bar{y}) = d(x, y)$, $d(\bar{x}, \bar{z}) = d(x, z)$, $d(\bar{y}, \bar{z}) = d(y, z)$ (this is unique up to isometry).

If we connect each two of x, y, z by geodesics, then we get a geodesic triangle $\triangle_{xyz}$. Suppose $\alpha : [0, d(x, y)] \to M$ and $\beta : [0, d(y, z)] \to M$ are two geodesics such that $\alpha(0) = \beta(0) = y$, $\alpha(d(x, y)) = x$ and $\beta(d(y, z)) = z$. Then we define the angle $\measuredangle xyz$ as

$$\measuredangle xyz = \lim_{s,t \to 0} \widetilde{\measuredangle} \alpha(s) y \beta(t) \tag{1.1.22}$$

if the limit exists.

In fact we have the following characteristic proposition of nonnegative sectional curvature, which is a precise version of (1.1.2).

Proposition 3. *A Riemannian manifold (M, g) has nonnegative sectional curvature if and only if every point in M has a neighborhood such that, for every triangle $\triangle_{xyz}$ contained in this neighborhood, the angles $\measuredangle xyz$, $\measuredangle xzy$, $\measuredangle yxz$ are well defined and satisfy*

$$\measuredangle xyz \geq \widetilde{\measuredangle} xyz, \ \measuredangle xzy \geq \widetilde{\measuredangle} xzy, \ \measuredangle yxz \geq \widetilde{\measuredangle} yxz,$$

and, in addition, the following holds: for any two shortest geodesics γ_1, γ_2 connecting a, b and c, d respectively where c is an inner point of γ_1, one has $\measuredangle acd + \measuredangle bcd = \pi$.

This proposition can be stated in metric spaces. It was taken as one of the definitions of nonnegative curvature in the synthetic approach of Alexandrov. (For this and the following contents about Alexandrov geometry, we refer to Burago-Burago-Ivanov [13].)

To explain more about this synthetic approach of curvature, we first recall some basic concepts in metric geometry. A curve γ in a metric space (X, d) is a continuous map $\gamma : [a, b] \to X$. The length of a curve γ is defined as

$$L(\gamma) = \sup \left\{ \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)) : \text{any partition } a = y_0 < y_1 < \ldots < y_N = b \right\}.$$

A curve γ is called rectifiable if $L(\gamma) < \infty$. Given $x, y \in X$, denote by $RC(x, y)$ the set of rectifiable curves joining x and y. A metric space (X, d) is called a length space if $d(x, y) = \inf_{\gamma \in RC(x,y)} \{L(\gamma)\}$, for any $x, y \in X$. A curve $\gamma : [a, b] \to X$ is called a geodesic if $d(\gamma(a), \gamma(b)) = L(\gamma)$. It is always true by the definition of

the length of a curve that $d(\gamma(a), \gamma(b)) \leq L(\gamma)$. A geodesic is a shortest curve (or shortest path) joining the two end points. A geodesic space is a length space (X, d) satisfying that for any $x, y \in X$, there is a geodesic joining x and y. In a geodesic space (X, d), the geodesic triangle, comparison triangle, comparison angle and angle can be well defined as their definitions only involve the distance function and geodesics. We denote by γ_{xy} one of the geodesics joining x and y, for any x, $y \in X$.

There are several equivalent definitions of nonnegative sectional curvature on complete geodesics spaces. The one frequently used is the following, which employs the distance comparison instead of angle comparison.

Definition 3. *A complete geodesic space (X, d) is called an Alexandrov space with nonnegative sectional curvature (SecX ≥ 0 for short) if for any $p \in X$, there exists a neighborhood U_p of p such that for any $x, y, z \in U_p$, any geodesic triangle $\triangle_{xyz}$, and any $w \in \gamma_{yz}$, letting $\bar{w} \in \gamma_{\bar{y}\bar{z}}$ be in the comparison triangle $\triangle_{\bar{x}\bar{y}\bar{z}}$ satisfying $d(\bar{y}, \bar{w}) = d(y, w)$ and $d(\bar{w}, \bar{z}) = d(w, z)$, we have*

$$d(x, w) \geq d(\bar{x}, \bar{w}).$$

In other words, an Alexandrov space (X, d) is a geodesic space which locally satisfies the Toponogov triangle comparison theorem for the sectional curvature. It is proved in Burogo-Gromov-Perelman [14] that the Hausdorff dimension of an Alexandrov space (X, d), $\dim_H(X)$, is an integer or infinity. One dimensional Alexandrov spaces are: straight line, S^1, ray and closed interval.

There is another equivalent definition of nonnegative curvature which employs angles comparison.

Definition 4. *A complete geodesic space (X, d) is called an Alexandrov space with SecX ≥ 0 if for any $p \in X$, there exists a neighborhood U_p such that for any collection of four different points $a, b, c, d \in U_p$ the following condition is satisfied:*

$$\widetilde{\measuredangle} bac + \widetilde{\measuredangle} cad + \widetilde{\measuredangle} dab \leq 2\pi. \tag{1.1.23}$$

Note that (1.1.23) does not involve the geodesics of the space. Therefore it is suitable to be modified as curvature notion on discrete spaces.

Let's constrain ourselves on 2-manifolds, on which the Ricci curvature along any tangent direction is equal to the sectional curvature. Consider the semiplanar graphs, which are graphs embedded into 2-manifolds without self-intersections (see exact definition in Section 3.1.1). The combinatorial curvature at a vertex x is defined as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where d_x is the degree of the vertex x, $\deg(\sigma)$ is the number of sides of the face σ, and the sum is taken over all faces incident to x. This is closely related to

(1.1.23), since if we treat every face incident to x as a regular polygon with the same number of sides in a plane, $\Phi(x) \geq 0$ is equivalent to $2\pi - \Sigma_x \geq 0$, where Σ_x is the sum of angles incident to x (called total angle at x). We will systematically explore this point in Chapter 3.

1.2 A brief history of synthetic Ricci curvatures on graphs

In Riemannian geometry, Ricci curvature is a fundamental local concept on a manifold which implies many significant global geometric and analytic results. There are many efforts to explore a synthetic approach of Ricci curvature on nonsmooth spaces, in particular on discrete space.

The synthetic Ricci curvature on graphs derives from several different sources.

Based on the study of optimal transportation distance on the space of probability measures on a Riemannian manifold, several beautiful works define synthetic Ricci curvature on metric measure spaces, and get further analogue properties as in Riemannian case. In 2006, Sturm [110, 111] and Lott-Villani [89] independently defined a notion of Ricci lower boundedness for a large class of metric measure spaces. Later, Ollivier [92] defined a notion of Ricci curvature taking values on every pair of points in a metric space associated with Markov chains. See also Ohta [91], Ollivier [93], Joulin-Olliver [72] etc. It turns out that Lott-Sturm-Villani's notion could not be applied on graphs directly since it involves the geodesics of the space. (However, In Bonciocat-Sturm [11], and recently Erbar-Maas [52], it was modified to the discrete setting.) And Ollivier's Ricci curvature works particular well on graphs as we pointed out in the last section. See also Ollivier-Villani [94] for a discussion of both notions on the discrete hypercube.

Another criterion for Ricci lower boundedness is studied by Bakry-Émery [3, 4, 2] in 1980s. Their notion works on measure spaces associated with a Markov semigroup P_t of probability measures. Starting from the infinitesimal generator of P_t, they define a curvature dimension inequality, which could be considered as a generalization of the Bochner formula in Riemannian geometry. It turns out that their inequality can imply many functional inequalities and geometric theorems (one can see Bakry [2] and the references therein). Recently, in their spirit, Lin-Yau [88] studied such kind of inequality on graphs and proved a lower bound for this Ricci curvature notion. Lin-Yau's work is also some kind of generalization of an earlier Ricci-flat notion due to Chung-Yau [31]. Chung-Yau's Ricci flat notion is also in the spirit of generalizing Bochner's formula, or more precisely, flat Ricci curvature implies the higher order derivatives commute. Note that Forman [55] defined a Ricci curvature notion on a cell complex also by generalizing the Bochner formula.

There is also an approach to define Ricci curvature on discrete spaces using deficiency of angles. We call this notion combinatorial curvature in the following. This idea usually works on a cell complex. This kind of idea has already appeared in Regge [102] in 1961, where he defined some curvature notion on higher dimension analogs of polyhedra and discussed the Bianchi identities and Einstein equations in that setting. In 1976, Stone [109] defined notions of Ricci curvature on a cell complex which is a 2-dimensional or a higher n-dimensional manifold aiming at a discrete analogue of Myers' theorem. Note that on a 2-dimensional Riemannian manifold, Ricci curvature coincides with the sectional curvature. In fact in this 2-dimensional case, the curvature at a vertex x given by Stone is just $(2\pi - \Sigma_x)/\pi$, where Σ_x is the total angle over x. This notion seems also to have other lines of development. Gromov [60] introduced a notion of combinatorial curvature to study hyperbolic groups in 1987. It was later modified by Ishida [68] and defined on embedded planar graphs in Higuchi [62] in 2001 which is in fact the same as Stone's notion for the 2-dimensional case. Higuchi studied the isoperimetric inequalities under the negative curvature constraints. In fact Woess [118] had studied those things in 1998 using this curvature notion under the name characteristic number of vertices. Woess also study the characteristic number of edges and tiles (for the characteristic number of tiles, see also Żuk [124]). After that the negative or nonpositive curvature on different classes of planar graphs and sometimes finer curvature notions were studied intensively by Baues-Peyerimhoff [9, 10], Klassert-Lenz-Peyerimhoff-Stollmann [77], Keller [74, 75], Keller-Peyerimhoff [76] etc. And the curvature bounded from below for some classes of graphs embedded into 2-dimensional manifolds was studied by Sun-Yu [112], DeVos-Mohar [46], Chen-Chen [20], Chen [19] etc.

We also point out that Dodziuk-Karp [51] also suggest an analogue of Ricci curvature lower boundedness on graphs by generalizing the Laplacian comparison theorem in Riemannian geometry.

1.3 Outline of this thesis

The interaction between the study of geometric and analytic aspects of Riemannian manifolds and that of graphs is a very amazing subject. On the one hand, the points of view and methods in Riemannian geometry and geometric analysis give new insights to the study of graphs. See for instance the works of Chung-Yau [28, 29, 30, 32, 33, 34, 35, 36], Coulhon-Grigor'yan [41], Delmotte [43, 44, 45]. On the other hand, the study of graphs is also beneficial to that of Riemannian manifolds. One example in this aspects is the graph constructed from the ϵ-net of a Riemannian manifold shares some properties with the Riemannian manifold (see Kanai [73], Coulhon-Saloff-Coste [42]). If the property on the graph is easier to check, then this will simplify the proofs in Riemannian case. The study of graphs can also be a way of constructing examples for

the study of Riemannian manifolds (see e.g. Barlow-Coulhon-Grigor'yan [6]). In fact there are also unified studies of Riemannian manifolds and graphs, see Chung-Yau [31], Chung-Grigor'yan-Yau [25, 26, 27] etc. The study of synthetic curvature on graphs adds new contributions to this aspect. We will mainly study Ollivier-Ricci curvature on locally finite graphs and the combinatorial curvature on infinite semiplanar graphs. In both cases, we will focus on the case that the curvature is bounded from below. Our results on the one hand can be seen as analogs of results in the Riemannian case, and on the other hand have their own combinatorial character and shed new light on the existing concepts in graph theory.

In Chapter 2, we study the Ollivier-Ricci curvature on locally finite graphs. The setting in this chapter are undirected, weighted, connected graphs. We allow the existence of self-loops. The number of edges connecting to every vertex is finite. The local clustering coefficient is introduced by Watts-Strogatz [117] to study small-world networks. It is defined as the relative proportion of connected neighbors among all the neighbors of a vertex. The vertex itself and a pair of connected neighbors form a triangle intuitively. Therefore, roughly speaking, the local clustering coefficient represents the local abundance of triangles. In a general setting, this coefficient would also relate weights and self-loops (a self-loop also is a kind of "triangle"). For simplicity we will state the outline of this chapter in the simple graph case, i.e. an unweighted graph without of self-loops. In Section 2.1 we give the preliminaries about Ollivier-Ricci curvature, graph Laplace operator and Bakry-Émery's curvature dimension inequalities. In Section 2.2, we derive lower and upper Ollivier-Ricci curvature bounds on graphs in terms of number of triangles, which is sharp for instance for complete graphs. Then we study the relation between Ollivier-Ricci curvature and Watts-Strogatz's local clustering coefficient. Furthermore, positive lower boundedness of Ollivier-Ricci curvature for neighboring vertices imply the existence of at least one triangle. It turns out that the existence of triangles can also improve Lin-Yau's [88] curvature dimension inequality on graphs and then produce an implication from Ollivier-Ricci curvature lower boundedness to the curvature dimension inequality. This is the content we present in Section 2.3. Note that a graph with triangles can't be a bipartite one. And the largest eigenvalue of a finite graph equals 2 if and only if the graph is bipartite. Therefore it is natural that Ollivier-Ricci curvature is closely related to the largest eigenvalue estimates. In Section 2.4, we combine the neighborhood graph method developed by Bauer-Jost [7] to study the spectrum estimates of a finite graph based on the idea related to number of triangles.

In Chapter 3, we study the nonnegative combinatorial curvature on infinite semiplanar graphs. A semiplanar graph is a graph that can be embedded into a connected 2-dimensional manifold without self-intersections of edges and each face is homeomorphic to a closed disk with finite edges as the boundary. We also suppose the degree of every vertex and every face is finite and larger than or equal to 3. Unlike the previous Gauss-Bonnet formula approach (see e.g. DeVos-

Mohar [46], Chen-Chen [20], Chen [19]), we systematically explore an Alexandrov approach based on the observation that the nonnegative combinatorial curvature on a semiplanar graph G is equivalent to nonnegative Alexandrov curvature on the surface $S(G)$ obtained by replacing each face by a regular polygon of side length one with the same facial degree and gluing the polygons along common edges. This observation enable us to carry over the results in Alexandrov geometry to the graph setting. In Section 3.1, we present preliminaries about semiplanar graphs, combinatorial curvature, previous Gauss-Bonnet formula approach and some basic facts about semigraphs with nonnegative curvature. The fact that the graph will have a rather special structure of linear volume growth like a one-side cylinder once the maximal degree of a face is at least 43 will be quite useful in the following. In Section 3.2, we introduce the Alexandrov approach by proving the bi-Lipschitz equivalence of several metrics and the equivalence of two kinds of curvature. In Section 3.3, by carrying over Cheeger-Gromoll splitting theorem in Alexandrov geometry to our graph setting, we give a metric classification of infinite semiplanar graphs with nonnegative curvature. We also construct the graphs embedded in the projective plane minus one point. After that we further prove the volume doubling property and Poincaré inequality which make the running of Nash-Moser iteration possible in Section 3.4 and 3.6. In Section 3.5, we explore the volume growth behavior on Archimedean tilings on a plane and prove that they satisfy a weak version of relative volume comparison with constant 1, the condition (R) (see precise statement in that section). At last, in Section 3.7, we discuss several applications of the volume growth property and Poincaré inequality. We obtain, on a semiplanar graph with nonnegative curvature, the Liouville theorem for positive harmonic functions, the parabolicity, and the dimension estimates for polynomial growth harmonic functions.

1.4 Lower bounded Ricci curvature in Riemannian geometry

It turns out that a Riemannian manifold (M, g) with Ricci curvature bounded from below has a fair amount of significant properties. One one hand, the geometric quantities, such as diameter, volume, can be controlled in some way. On the other hand, many analytic properties like eigenvalue estimates of the Laplace-Beltrami operator and from this the Poincaré inequalities on balls could be obtained. Poincaré inequalities combined with the volume estimates make the running of the powerful analytic tool of Nash-Moser iteration possible and many analytic consequences follow then.

We hope that from the synthetic Ricci curvature on general spaces, we can derive some analogous results. As can be seen in the outline in Section 1.3, we will explore some of them for Ollivier-Ricci curvature and combinatorial curvature on

graphs in this thesis. In this section, we list the results related to our study in the following chapters in Riemannian geometry. On a Riemnnain manifold (M, g), we will write $Ric \geq k$ for the condition that $Ric(v, v) \geq k|v|^2$ for any $x \in M$ and any $v \in T_xM$.

1.4.1 Geometry aspects

First note that whether the lower bound k of the Ricci curvature is positive or not impacts the geometry of the manifold a lot. When $k > 0$, Myers [90] proved the following famous theorem.

Theorem 1. *Let (M, g) be an n-dimensional complete Riemannian manifold such that $Ric \geq k > 0$, then the diameter of the M*

$$diam(M) \leq \pi\sqrt{\frac{n-1}{k}}. \tag{1.4.1}$$

Note that a complete Riemannian manifold with $Ric > 0$ could be non-compact (see e.g. Gallot-Hulin-Lafontaine [56]).

Let's consider the volume of geodesic balls in M. Denote the volume of geodesic balls of radius r in a simply connected space form of constant curvature a by $V^a(r)$. The Bishop-Gromov volume comparison theorem states as follows (see e.g. Zhu [123]).

Theorem 2. *On an n-dimensional complete Riemannian manifold (M, g) with $Ric \geq k$, for any $p \in M$, $0 < r < R$, we have*

1. *Gromov's relative volume comparison theorem.*

$$\frac{vol(B_R(p))}{vol(B_r(p))} \leq \frac{V^{\frac{k}{n-1}}(R)}{V^{\frac{k}{n-1}}(r)}. \tag{1.4.2}$$

2. *Bishop volume comparison theorem.*

$$vol(B_r(p)) \leq V^{\frac{k}{n-1}}(r). \tag{1.4.3}$$

When $k = 0$, that is the Ricci curvature is nonnegative, (1.4.2) implies the volume doubling property,

$$vol(B_{2r}(p)) \leq 2^n vol(B_r(p)). \tag{1.4.4}$$

For noncompact manifolds, Yau [120] proved the following theorem.

Theorem 3. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature, then*

$$vol(B_r(p)) \geq cr, \tag{1.4.5}$$

for some $c > 0$.

Cheeger-Gromoll [18] have the following splitting theorem.

Theorem 4. *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature. The M is the isometric product $\overline{M} \times \mathbb{R}^l$ where $\overline{M}$ contains no lines and $\mathbb{R}^l$ has its standard flat metric.*

In the above, a line means a complete geodesic $\gamma : (-\infty, +\infty) \longrightarrow M$ which realizes the distance between any two of its points. In particular, Cheeger-Gromoll's splitting theorem can also be stated as follows.

Theorem 5. *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature. If M contains a line, then M splits isometrically as $M = M' \times \mathbb{R}$, where M' is a Riemannian manifold with nonnegative curvature.*

1.4.2 Bochner's formula

Bochner's formula is a very powerful formula which encodes deep analytic properties of Ricci curvature, see e.g. Jost [70], Gallot-Hulin-Lafontaine [56]. The formula involves the Laplace-Beltrami operator Δ on (M, g). For a smooth function f on M, it is defined in local coordinates $x_1, x_2, \ldots, x_n$ by

$$\Delta f(x) = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x^j} f \right)(x),$$

where $\det(g)$ stands for the determinant of the matrix $(g_{ij})_{i,j=1,\ldots,n}$ and g^{ij} stands for the (i, j) term in the inverse matrix.

Proposition 4 (Bochner's formula)**.** *For any smooth function f on a Riemannian manifold (M, g), we have*

$$\frac{1}{2}\Delta(|\nabla f|^2) = |Hess\ f|^2 + \langle \nabla(\Delta f), \nabla f \rangle + Ric(\nabla f, \nabla f). \tag{1.4.6}$$

This formula reflects the role that Ricci curvature plays in the commutativity of order of higher derivatives for functions on M. By using Schwarz's inequality, we have

$$|Hess\ f|^2 \geq \frac{(\Delta f)^2}{n},$$

where n is the dimension constant. So $Ric \geq k$ implies the following curvature-dimension inequality.

$$\frac{1}{2}\Delta(|\nabla f|^2) \geq \frac{(\Delta f)^2}{n} + \langle \nabla(\Delta f), \nabla f \rangle + k|\nabla f|^2. \tag{1.4.7}$$

Bakry-Émery [2, 3, 4] take this inequality as the starting point and directly use the operators to define curvature bounds. Starting from an operator Δ, which

is a generator of a Markov semigroup, they define iteratively,

$$\begin{aligned}\Gamma_0(f,g) &= fg,\\ \Gamma(f,g) &= \frac{1}{2}\{\Delta\Gamma_0(f,g) - \Gamma_0(f,\Delta g) - \Gamma_0(\Delta f, g)\},\\ \Gamma_2(f,g) &= \frac{1}{2}\{\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\}.\end{aligned}$$

In fact, $\Gamma(f,f)$ is an analogue of $|\nabla f|^2$, and $\Gamma_2(f,f)$ is an analogue of $\frac{1}{2}\Delta|\nabla f|^2 - \langle\nabla(\Delta f), \nabla f\rangle$ in (1.4.7).

Definition 5. *We say an operator Δ satisfies a curvature-dimension inequality $CD(n,K)$ if for all functions f in the domain of the operator*

$$\Gamma_2(f,f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K(x)\Gamma(f,f)(x),\ \forall x, \tag{1.4.8}$$

where $n \in [1,+\infty]$ is the dimension parameter, $K(x)$ is the curvature function.

Bakry-Émery's curvature-dimension inequality contains plentiful information and implies a lot of functional inequalities including spectral gap inequalities, Sobolev inequalities, and logarithmic Sobolev inequalities and many celebrated geometric theorems (see Bakry [2] and the references therein).

Let's go back to the Riemannian case now and describe a little about the eigenvalue problem here. For more details, we refer to Gallot-Hulin-Lafontaine [56] and the references therein. A real number λ is called an eigenvalue of the Dirichlet problem for $\Omega \subset M$ if there exists a smooth function $f \not\equiv 0$ that solves the equation

$$\Delta f = -\lambda f, \tag{1.4.9}$$

with the boundary condition that $f|_\Omega = 0$. And we call it an eigenvalue of the Neumann problem if we replace the boundary condition by $\partial f/\partial\nu = 0$, where ν is the outward normal direction of the boundary. The solution f is called eigenfunctions associated to λ. If we replace Ω by a compact Riemannian manifold with boundary, the definition still works. For the simpler case of a closed Riemannian manifold (compact manifold without boundary), we don't need to think about the boundary problems.

On a compact Riemannian manifold, for both Dirichlet and Neumann cases, we can list the eigenvalues in the form

$$0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,\ \lim_{n\to+\infty}\lambda_n = +\infty, \tag{1.4.10}$$

where each eigenvalues is counted according to its multiplicity. The corresponding eigenfunctions are then denoted by $f_0, f_1, f_2, \ldots, f_n, \ldots$. For the Neumann case or a closed manifold case, $\lambda_0 = 0$ and $f_0 = constant$. So λ_1 is the first nonzero eigenvalue. But for the Dirichlet case, $\lambda_0 > 0$ is the first nonzero eigenvalues. A

variational characterization of the eigenvalues is given by the following formulas (we only give those for λ_0, λ_1).

$$\lambda_0 = \inf_{f \not\equiv 0} \frac{\int_M |\nabla f|^2 dvol}{\int_M f^2 dvol}, \tag{1.4.11}$$

$$\lambda_1 = \inf_{\langle f, f_0 \rangle = 0} \frac{\int_M |\nabla f|^2 dvol}{\int_M f^2 dvol} = \inf_f \frac{\int_M |\nabla f|^2 dvol}{\int_M (f - f_M)^2 dvol}, \tag{1.4.12}$$

where $f_M := \frac{1}{vol(M)} \int_M f dvol$ is the mean value of f on M. Note that the admissible space for f is a Hilbert space. For the Dirichlet case it is the completion of $C_0^\infty(M)$ under the norm $\int_M |\nabla f|^2 dvol$, and for the Neumann case the completion of $C^\infty(M)$.

Now let's go back to Bochner's formula and focus on the eigenvalue problem on closed manifolds. Then λ_1 is the first nonzero eigenvalues and f_1 is not a constant function. Applying (1.4.6) to eigenfunction f_1 and integrating it over M gives immediately

$$0 = \int_M |Hess f_1|^2 - \lambda_1 \int_M \langle \nabla f_1, \nabla f_1 \rangle + \int_M Ric(\nabla f_1, \nabla f_1).$$

If $Ric \geq k > 0$, this implies easily $\lambda_1 \geq k$. In fact a more careful discussion about the Hessian term gives the following estimate (see e.g. Gallot-Hulin-Lafontaine [56]).

Theorem 6 (Lichnerowicz)**.** *Let (M, g) be an n-dimensional closed Riemannian manifold. Suppose the lower bound of the Ricci curvature $k > 0$, then*

$$\lambda_1 \geq \frac{n}{n-1} k. \tag{1.4.13}$$

Another powerful method to estimate λ_1, the so-called coupling method by Chen-Wang [23], utilizes the heat equation, see also the surveys Chen [21, 22]. We sketch their ideas here for closed manifold case. Consider the equation

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \\ f(x,0) = f_1(x). \end{cases}$$

Then the solution is

$$f(x,t) = f_1(x) e^{-\lambda_1 t},$$

which evolves information from both eigenvalue and eigenfunction. Then we have

$$e^{-\lambda_1 t} |f_1(x) - f_1(y)| = |f(x,t) - f(y,t)| = |P_t f_1(x) - P_t f_1(y)|, \tag{1.4.14}$$

where $(P_t)_{t \geq 0}$ is the heat semigroups on M. There are probability measures $m_x^t(\cdot)$ associated to every $x \in M$, $t \geq 0$, such that

$$P_t f_1(x) = \int_M f_1(z_1) dm_x^t(z_1).$$

In fact when $Ric \geq k$, the Bochner's formula could lead to gradient estimates for solutions of heat equation. For example the one by Bakry-Émery [4] (see also von Renesse-Sturm [103]) tell us that $\sup_M |\nabla P_t f_1| \leq e^{-kt} \sup_M |\nabla f_1|$. By this results one can go on to get

$$e^{-\lambda_1 t}|f_1(x) - f_1(y)| = |P_t f_1(x) - P_t f_1(y)| \leq e^{-kt} \sup_M |\nabla f_1| d(x,y). \tag{1.4.15}$$

Let $t \longrightarrow +\infty$, we know the above forces that $\lambda_1 \geq k$.

The point for Chen-Wang's method is that they can get better estimate for $|P_t f_1(x) - P_t f_1(y)|$ and then improve the eigenvalue estimates. Note

$$\begin{aligned} |P_t f_1(x) - P_t f_1(y)| &= \left| \int_M f_1(z_1) dm_x^t(z_1) - \int_M f_1(z_2) dm_x^t(z_2) \right| \\ &\leq \sup_M |\nabla f_1| \int_{M \times M} d(z_1, z_2) d\xi_{x,y}^t(z_1, z_2), \end{aligned} \tag{1.4.16}$$

where $\xi_{x,y}^t$ is the coupling of m_x^t and m_y^t. Obviously the last term is closely related to the optimal transportation distance. Then on one hand one can try to find optimal couplings $\xi_{x,y}^t$ and on the other hand to find better choice of distance function d used in (1.4.16) to improve the estimates. (Note that one do not change the metric of the Riemannian manifold. The new distance function here is chosen such that f_1 is Lipschitz w.r.t. it.)

1.4.3 Poincaré inequality

Poincaré inequality basically tells that one can bound a function by the bounds of its first derivatives and the geometry of the domain on which it is defined. We are interested in the Poincaré inequality on balls which are stated in the form

$$\int_{B_r} |f - f_{B_r}|^2 dvol \leq P_N(B_r) \int_{B_r} |\nabla u|^2 dvol, \quad f \in C^\infty(B_r). \tag{1.4.17}$$

Recall (1.4.12), we can see this Poincaré inequality is closely related to the estimates of the first nonzero eigenvalue of Neumann problems. More precisely, the inverse of a lower bound of the first nonzero eigenvalues gives an upper bound for $P_N(B_r)$ in (1.4.17). By (1.4.11), the Poincaré inequality related to Dirichlet eigenvalue estimates is

$$\int_{B_r} |f|^2 dvol \leq P_D(B_r) \int_{B_r} |\nabla u|^2 dvol, \quad f \in C_0^\infty(B_r). \tag{1.4.18}$$

If we replace the L^2 norm of f and ∇f by the L^p norm in (1.4.17) or (1.4.18), we call it the corresponding L^p version.

By estimating the lower bound of the first nonzero eigenvalue, under the hypothesis $Ric \geq -k$, $k \geq 0$, Li-Yau [84] proved on a compact manifold, $P_N(M) \leq$

$e^{C(n)(1+\sqrt{k}diam(M))}diam^2(M)$, where $C(n)$ is a constant depends only on the dimension n. By proving a lower bound for the Cheeger constant which in turn bounds the first nonzero eigenvalue, Buser [15] proved (1.4.17) with $P_N(B_r) \leq e^{C(n)(1+\sqrt{k}r)}r^2$. Li-Schoen [83] proved an L^p $(1 \leq p < +\infty)$ version of (1.4.18) with similar constants. Saloff-Coste [106] further proved a generalized L^p version of Buser's result.

So, the lower bounded Ricci curvature implies a Poincaré inequality. The important point is that the Poincaré inequality (1.4.17) together with the doubling property (1.4.4) is enough for the running of Nash-Moser iteration (for this aspects, see Grigor'yan [57], Saloff-Coste [106]), which is powerful enough to get many analytic results such as global properties of harmonic functions.

1.5 Basic facts in Alexandrov geometry

Alexandrov geometry can be seen as a natural generalization of Riemannian geometry, and many fundamental results of Riemannian geometry extend to the more general Alexandrov setting. In particular, several results in the last section still hold in Alexandrov geometry. For later purpose, we will review them in this section. Readers are referred to Burago-Gromov-Perelman [14], Burago-Burago-Ivanov [13].

Let (X, d) be an Alexandrov space, $B_R^X(p)$ denote the closed geodesic ball centered at $p \in X$ of radius $R > 0$, i.e. $B_R^X(p) = \{x \in X : d(p, x) \leq R\}$. The well known Bishop-Gromov volume comparison theorem holds on Alexandrov spaces (see Burago-Burago-Ivanov [13]).

Theorem 7. *Let (X, d) be an n-dimensional Alexandrov space with nonnegative curvature, i.e. $SecX \geq 0$. Then for any $p \in X, 0 < r < R$, it holds that*

$$\frac{H^n(B_R^X(p))}{H^n(B_r^X(p))} \leq \left(\frac{R}{r}\right)^n, \tag{1.5.1}$$

$$H^n(B_{2R}^X(p)) \leq 2^n H^n(B_R^X(p)), \tag{1.5.2}$$

$$H^n(B_R^X(p)) \leq C(n)R^n, \tag{1.5.3}$$

where H^n is the n-dimensional Hausdorff measure.

We call (1.5.1) the relative volume comparison and (1.5.2) the volume doubling property.

A curve $\gamma : (-\infty, \infty) \to X$ is called an infinite geodesic if for any $s, t \in (-\infty, \infty)$, $d(\gamma(s), \gamma(t)) = L(\gamma|_{[s,t]})$, i.e. every restriction of γ to a subinterval is a geodesic (shortest path). For two metric spaces $(X, d_X), (Y, d_Y)$, the metric

product of X and Y is a product space $X \times Y$ equipped with the metric $d_{X\times Y}$ which is defined as

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)},$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. The Cheeger-Gromoll splitting theorem holds on Alexandrov spaces with nonnegative curvature [13].

Theorem 8. *Let (X, d) be an n-dimensional Alexandrov space with* $\mathrm{Sec}X \geq 0$. *If it contains an infinite geodesic, then X is isometric to a metric product $Y \times \mathbb{R}$, where Y is an $(n-1)$-dimensional Alexandrov space with* $\mathrm{Sec}Y \geq 0$.

Let (X, d) be an n-dimensional Alexandrov space with $\mathrm{Sec}X \geq 0$. The tangent space at each point $p \in X$ is well defined, denoted by T_pX, which is the pointed Gromov-Hausdorff limit of the rescaling sequence $(X, \lambda d, p)$ as $\lambda \to \infty$ (see Burago-Burago-Ivanov [13]). A point $p \in X$ is called regular (resp. singular) if T_pX is (resp. not) isometric to $\mathbb{R}^n$. Let $S(X)$ denote the set of singular points in X. It is known that $H^n(S(X)) = 0$ (see Burago-Gromov-Perelman [14]). Otsu and Shioya [95] obtained the C^1-differential and C^0-Riemannian structure on the regular part of X, $X \setminus S(X)$. A function f defined on a domain $\Omega \subset X$ is called Lipschitz if there is a constant C such that for any $x, y \in \Omega$, $|f(x) - f(y)| \leq Cd(x, y)$. It can be shown that every Lipschitz function is differentiable H^n-almost everywhere and with bounded gradient $|\nabla f|$ (see Cheeger [17]). Let $Lip(\Omega)$ denote the set of Lipschitz functions on Ω. For any precompact domain $\Omega \subset X$ and $f \in Lip(\Omega)$, the $W^{1,2}$ norm of f is defined as

$$\|f\|^2_{W^{1,2}(\Omega)} = \int_\Omega f^2 + \int_\Omega |\nabla f|^2.$$

The $W^{1,2}$ space on Ω, denoted by $W^{1,2}(\Omega)$, is the completion of $Lip(\Omega)$ with respect to the $W^{1,2}$ norm. A function $f \in W^{1,2}_{loc}(X)$ if for any precompact domain $\Omega \subset\subset X$, $f|_\Omega \in W^{1,2}(\Omega)$. The Poincaré inequality was proved in Kuwae-Machigashira-Shioya [79], Hua [65].

Theorem 9. *Let (X, d) be an n-dimensional Alexandrov space with* $\mathrm{Sec}X \geq 0$ *and $u \in W^{1,2}_{loc}(X)$, then*

$$\int_{B^X_R(p)} |u - u_{B_R}|^2 \leq C(n)R^2 \int_{B^X_R(p)} |\nabla u|^2, \tag{1.5.4}$$

where $u_{B_R} = \frac{1}{H^n(B^X_R(p))} \int_{B^X_R(p)} u$.

At last, we explain the concept ends on geodesic spaces. Let (X, d) be a geodesic space and $\{B^X_{R_i}(p)\}_{i=1}^\infty$ be an exhaustion of X, i.e. $B^X_{R_i}(p) \subset B^X_{R_{i+1}}(p)$ for any $i \geq 1$ and $X = \bigcup_{i=1}^\infty B^X_{R_i}(p)$, equivalently $R_i \leq R_{i+1}$ and $R_i \to \infty$ as

$i \to \infty$. A connected component E of $X \setminus B^X_{R_i}(p)$ is called connecting to infinity if there is a sequence of points $\{q_j\}_{j=1}^{\infty}$ in E such that $d(p, q_j) \to \infty$ as $j \to \infty$. The number of connected components of $X \setminus B^X_{R_i}(p)$ connecting to infinity, denoted by N_i, is nondecreasing in i. Then the limit $N(X) = \lim_{i\to\infty} N_i$ is well defined and called the number of ends of X. It is easy to show that $N(X)$ does not depend on the choice of the exhaustion of X, $\{B^X_{R_i}(p)\}_{i=1}^{\infty}$.

Chapter 2

Ollivier-Ricci curvature on locally finite graphs

The contents of this chapter are essentially included in the submitted paper Jost-Liu [71] and Bauer-Jost-Liu [8].

As we see in Section 1.1, Ricci curvature controls how fast geodesics diverge on average. Equivalently, it controls the amount of overlap of two distance balls in terms of their radii and the distance between their centers. In fact, such upper bounds follow from a lower bound on the Ricci curvature. In this chapter, we want to explore the implications of such ideas in graph theory. The geometric idea is that a lower Ricci curvature bound prevents geodesics from diverging too fast and balls from growing too fast in volume. On a graph, the analogue of geodesics starting in different directions, but eventually approaching each other again, would be a triangle. Therefore, it is natural that the Ricci curvature on a graph should be related to the relative abundance of triangles. The latter is captured by the local clustering coefficient introduced by Watts-Strogatz [117]. Thus, the intuition of Ricci curvature on a graph should play with the relative frequency of triangles a vertex shares with its neighbors. In fact, more precisely, since the local clustering coefficient averages over the neighbors of a vertex, this should really related to some notion of scalar curvature, as an average of Ricci curvatures in different directions, that is, for different neighbors of a given vertex.

Recall in Section 1.1, Ollivier-Ricci curvature is formulated in terms of the transportation distance between local measures:

$$\kappa(x, y) := 1 - W_1(m_x, m_y), \tag{2.0.1}$$

where x, y are vertices in our graph that are neighbors (written as $x \sim y$) and the measure $m_x = \frac{1}{d_x}$, where d_x is the degree of x, puts equal weight on all neighbors. When two balls strongly overlap, as is the case in Riemannian geometry when the Ricci curvature has a large lower bound, then it is easier to transport the mass of one to the other. Analogously, in the graph case, when the two vertices share many triangles, then the transportation distance should be smaller, and the

curvature therefore correspondingly larger. This is the idea of Ollivier's definition as we see it and explore in this chapter. We shall obtain both upper and lower bounds for Ollivier's Ricci curvature on graphs in Section 2.2, which are optimal on many graphs.

Let us now formulate our main result on simple graphs, i.e. unweighted graphs without self-loops (recalled and proved below in a general setting as Theorem 13).

Theorem 10. *On a locally finite graph, we put for any pair of neighboring vertices x, y,*

$$\sharp(x,y) := \textit{number of triangles which include } x,y \textit{ as vertices} = \sum_{x_1, x_1\sim x, x_1\sim y} 1.$$

We then have

$$\kappa(x,y) \geq -\left(1-\frac{1}{d_x}-\frac{1}{d_y}-\frac{\sharp(x,y)}{d_x\wedge d_y}\right)_+ - \left(1-\frac{1}{d_x}-\frac{1}{d_y}-\frac{\sharp(x,y)}{d_x\vee d_y}\right)_+ + \frac{\sharp(x,y)}{d_x\vee d_y}. \tag{2.0.2}$$

where $s_+ := \max(s,0), s\vee t := \max(s,t), s\wedge t := \min(s,t)$.

This equality is sharp for instance for a complete graph of n vertices where the left and the right hand side both equal to $\frac{n-2}{n-1}$.

The local clustering coefficient introduced by Watts-Strogatz [117] is

$$c(x) := \frac{\text{number of edges between neighbors of } x}{\text{number of possible existing edges between neighbors of } x}, \tag{2.0.3}$$

which measures the extent to which neighbors of x are directly connected, i.e.,

$$c(x) = \frac{1}{d_x(d_x-1)} \sum_{y, y\sim x} \sharp(x,y). \tag{2.0.4}$$

Thus, this local clustering coefficient is an average over the $\sharp(x,y)$ for the neighbors of x. Thus, we might also introduce some kind of scalar curvature (suggested in Problem Q in Ollivier [93]) as

$$\kappa(x) := \frac{1}{d_x} \sum_{y, y\sim x} \kappa(x,y). \tag{2.0.5}$$

For illustration, let us consider the case where our graph is d-regular, that is, $d_z = d$ for all vertices z. When $1 \geq \frac{2}{d} + \frac{\sharp(x,y)}{d}$ for all $y \sim x$, we would then get

$$\kappa(x) \geq -2 + \frac{4}{d} + \frac{3(d-1)}{d} c(x). \tag{2.0.6}$$

This example nicely illustrates the relation between Ollivier's curvature and the Watts-Strogatz's local clustering coefficient.

Without the triangle terms $\sharp(x, y)$, Theorem 10 is due to Lin-Yau [88, 85], and we take their proof as our starting point. Lin-Yau also obtain analogues of Bochner type inequalities in the spirit of Bakry-Émery and eigenvalue estimates as known from Riemannian geometry.

In this chapter, we also want to find relations on locally finite graphs between Ollivier's Ricci curvature and Bakry-Émery's curvature dimension inequalities, which represent the geometric and analytic aspects of graphs respectively. Again, this is inspired by Riemannian geometry where one may attach a Brownian motion with a drift to a Riemannian metric [92]. We also mention that the definitions given by Sturm and Lott-Villani are also consistent with that of Bakry-Émery [110, 111, 89]. So exploring the relations on nonsmooth spaces may provide a good point of view to connect Ollivier's definition to Sturm and Lott-Villani's (in this aspect, see also Ollivier-Villani [94]). In Section 2.3, we use the local clustering coefficient again to establish more precise curvature dimension inequalities than those of Lin-Yau [88]. And with this in hand, we prove curvature dimension inequalities under the condition that Ollivier's Ricci curvature of the graph is positive.

Further analytical results following from curvature dimension inequalities on finite graphs have been described in [85], and Lin-Lu-Yau [86] study a modified definition of Ollivier's Ricci curvature on graphs. Recently, Paeng [96] study upper bounds of diameter and volume for finite simple graphs in terms of Ollivier's Ricci curvature.

We point out that, as in Riemannian geometry, both Ollivier's Ricci curvature and Bakry-Émery's curvature dimension inequality can give lower bound estimates for the first eigenvalue λ_1 of the Laplace operator (see Ollivier [92], Bakry [2]). Therefore our results in fact relate λ_1 to the Watts-Strogatz's local clustering coefficient, or the number of cycles with length 3. In [47], Diaconis and Stroock obtain several geometric bounds for eigenvalues of graphs, one of which is related to the number of odd length cycles. For more geometric quantities and methods concerning eigenvalue estimates in the study of Markov chains, see [48, 43, 44] and the references therein.

In the last section, Section 2.4, we utilize techniques inspired by Riemannian geometry and the theory of stochastic processes in order to control eigenvalues of graphs. In particular, we shall quantify the deviation of a (connected, undirected, weighted, finite) graph G from being bipartite (a bipartite graph is one without cycles of odd lengths; equivalently, its vertex set can be split into two classes such that edges can be present only between vertices from different classes) in terms of a spectral gap. The operator whose spectrum we shall consider here is the normalized graph Laplacian Δ. This is the operator underlying random walks on graphs, and so, this leads to a natural connection with the theory of stochastic processes. We observe that on a bipartite graph, a random walker, starting at a vertex x at time 0 and at each step hopping to one of the neighbors of the vertex where it currently sits, can revisit x only at even times. This

connection then will be explored via the eigenvalues of Δ. More precisely, the largest eigenvalue λ_{N-1} of Δ is 2 iff G is bipartite and is < 2 else. Therefore, $2 - \lambda_{N-1}$ quantifies the deviation of G from being bipartite, and we want to understand this aspect in more detail. In more general terms, we are asking for a quantitative connection between the geometry (of the graph G) and the analysis (of the operator Δ, or the random walk encoded by it). Now, such connections have been explored systematically in Riemannian geometry, and many eigenvalue estimates are known there that connect the corresponding Laplace operator with the geometry of the underlying space M, see e.g. Li-Yau [84], Chavel [16]. The crucial role here is played by the Ricci curvature of M. Ollivier's notion again turns out to be most useful for our purposes. In his paper [92], Olliver actually showed that the eigenvalues of the normalized Laplace operator satisfy

$$k \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} \leq 2 - k.$$

In fact, one of the main points of Section 2.4 is to relate lower bounds for λ_1 and upper bounds for λ_{N-1} via random walks. As in Bauer-Jost [7], this relationship is translated into the geometric concept of a neighborhood graph. The idea here is that in the t-th neighborhood graph $G[t]$ of G, vertices x and y are connected by an edge with a weight given by the probability that a random walker starting at x reaches y after t steps times the degree of x. We note that even though the original graph may have been unweighted, the neighborhood graphs $G[t]$ are necessarily weighted. In addition, they will in general possess self-loops, because the random walker starting at x may return to x after t steps. Therefore, we need to develop our theory on weighted graphs with self-loops even though the original G might have been unweighted and without such loops. Since Ollivier's curvature is defined in terms of transportation distances (Wasserstein metrics), we can then use our neighborhood graphs in order to geometrically control the transportation costs and thereby to estimate the curvature of the neighborhood graphs in terms of the curvature of the original graph. As it turns out that lower bounds for the smallest eigenvalue of $G[2]$ are related to upper bounds for the largest eigenvalue of G, we obtain the following more general estimate

$$1 - (1 - k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1 - k[t])^{\frac{1}{t}}, \forall \text{ integers } t \geq 1.$$

For non-bipartite graphs, this will always produce nontrivial estimates when t is large enough even if $k = k[1]$ is nonpositive.

For controlling the smallest eigenvalue, besides Ollivier [92], we also refer to Lin-Yau [88]. Recalling in Section 2.2, we could relate λ_1 to the local clustering coefficient introduced in Watts-Strogatz [117]. The local clustering coefficients measures the relative local frequency of triangles, that is, cycles of length 3. Since bipartite graphs cannot possess any triangles (in fact any cycles with odd lengths), this then is obviously related to our question about quantifying the deviation of the given graph G from being bipartite. In fact, in Section 2.2, this

local clustering will be controlled in terms of Ollivier-Ricci curvature. And $k[t]$ further encodes the frequency of longer cycles besides triangles. Thus, in the last section we are closing the loop between the geometric properties of a graph G, the spectrum of its graph Laplacian, random walks on G, and the generalized curvature of G, drawing upon deep ideas and concepts originally developed in Riemannian geometry and the theory of stochastic processes.

The estimates of the smallest nonzero eigenvalue of the corresponding Laplace operator has also been studied on Alexandrov spaces, see Qian-Zhang-Zhu [101] and on Finsler manifolds, see Wang-Xia [115, 116].

In this chapter, $G = (V, E)$ will denote an undirected, weighted, connected graph. We do not exclude self-loops, i.e. we permit the existence of an edge between a vertex and itself. V denotes the set of vertices and E denotes the set of edges. If two vertices $x, y \in V$ are connected by an edge, we say x and y are neighbors, in symbols $x \sim y$. The associated weight function w: $V \times V \to \mathbb{R}$ satisfies $w_{xy} = w_{yx}$ (because the graph is undirected) and we assume $w_{xy} > 0$ whenever $x \sim y$ and $w_{xy} = 0$ iff $x \not\sim y$. For a vertex $x \in V$, its degree d_x is defined as $d_x := \sum_{y \in V} w_{xy}$.

Sometimes for simplicity and in order to see more geometry, we will consider unweighted graphs, i.e., the case that $w_{xy} = 1$ whenever $x \sim y$. Or even simple graphs, i.e., the unweighted graphs which do not have self-loops.

For the results related to curvature only, we require that the graphs are locally finite, i.e., for every $x \in V$, the number of edges connected to x is finite. But when consider the problems about the eigenvalues of the Laplacian operator in Section 2.4, we will concentrate on finite graphs, i.e. the number of vertices is finite. In that case we denote the number of vertices by N.

2.1 Ollivier-Ricci curvature and curvature dimension inequality

Ollivier's Ricci curvature works on a general metric space (X, d), on which we attach to each point $x \in X$ a probability measure $m_x(\cdot)$. We denote this structure by (X, d, m).

For a locally finite graph $G = (V, E)$, we define a metric d on the set of vertices V as follows. For neighbors x, y, $d(x, y) = 1$. For general distinct vertices x, y, $d(x, y)$ is the length of the shortest path connecting x and y, i.e. the number of edges of the path. We attach to each vertices $x \in V$ a probability measure

$$m_x(y) = \begin{cases} \frac{w_{xy}}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

An intuitive illustration of this is a random walker that sits at x and then chooses amongst the neighbors of x with equal probability $\frac{w_{xy}}{d_x}$. Then we get a structure (V, d, m) in this case.

Remark 1. *Note that $x \sim x$ is possible when x has a loop. On a graph G without loops, we can also consider a lazy random walk. A lazy random walk is a random walk that does not move with a certain probability, i.e. for some x we might have $m_x(x) \neq 0$. In this case, the lazy random walk on G is equivalent to the usual random walk on the graph G^{lazy} that is obtained from G by adding for every vertex x a loop with a weight $d_x m_x(x)$.*

In the following of this section, we will present some basic facts about Ollivier-Ricci curvature and the curvature dimension inequality. Essentially all those contents work on the general structure (X, d, m). However, we will only use the case (V, d, m).

2.1.1 Ollivier-Ricci curvature

Definition 6 (Ollivier)**.** *On (X, d, m), for any two distinct points $x, y \in X$, the (Ollivier-) Ricci curvature of (X, d, m) along (xy) is defined as*

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}. \tag{2.1.2}$$

Here, $W_1(m_x, m_y)$ is the optimal transportation distance between the two probability measures m_x and m_y, defined as follows (cf. Villani [113, 114], Evans [54]).

Definition 7. *For two probability measures μ_1, μ_2 on a metric space (X, d), the transportation distance between them is defined as*

$$W_1(\mu_1, \mu_2) := \inf_{\xi \in \prod(\mu_1, \mu_2)} \int_{X \times X} d(x, y) d\xi(x, y), \tag{2.1.3}$$

where $\prod(\mu_1, \mu_2)$ is the set of probability measures on $X \times X$ projecting to μ_1 and μ_2.

In other words, ξ satisfies

$$\xi(A \times X) = \mu_1(A), \;\; \xi(X \times B) = \mu_2(B), \;\; \forall A, B \subset X.$$

Remark 2. *Intuitively, this distance measures the minimal cost to transport one pile of sand to another one with the same mass. For case of a graph structure (V, d, m), the supports of m_x and m_y are finite discrete sets, and thus, ξ is just a matrix with terms $\xi(x', y')$ representing the mass moving from $x' \in$ support of m_x to $y' \in$ support of m_y. We will use the notation $\xi^{x,y}$ to stress the dependence on x, y. Then, in this case,*

$$W_1(m_x, m_y) = \inf_{\xi^{x,y}} \sum_{x', x' \sim x} \sum_{y', y' \sim y} d(x', y') \xi^{x,y}(x', y'),$$

where the infimum is taken over all matrices $\xi^{x,y}$ which satisfy

$$\sum_{x',x'\sim x} \xi^{x,y}(x',y') = \frac{w_{yy'}}{d_y}, \quad \sum_{y',y'\sim y} \xi^{x,y}(x',y') = \frac{w_{xx'}}{d_x}.$$

We also call such a $\xi^{x,y}$ a transfer plan between m_x and m_y, or a coupling of two random walks governed by m_x and m_y respectively. Those $\xi^{x,y}$ ($\xi^{x,y}$ might not be unique) which attain the infimum value in (2.1.3), are called optimal couplings. The optimal coupling exists in a very general setting. For locally finite graphs the existence follows from a simple and interesting argument in Remark 14.2 in Levin-Peres-Wilmer [80].

If we can find a particular transfer plan, we then get an upper bound for W_1 and therefore a lower bound for κ.

A very important property of transportation distance is the Kantorovich duality (see, e.g. Theorem 1.14 in Villani [113]). We state it here in our particular graph setting

Proposition 5 (Kantorovich duality)**.**

$$W_1(m_x, m_y) = \sup_{f,1-Lip} \left[\sum_{z,z\sim x} f(z)m_x(z) - \sum_{z,z\sim y} f(z)m_y(z) \right],$$

where the supremum is taken over all functions on G that satisfy

$$|f(x) - f(y)| \leq d(x,y),$$

for any $x, y \in V$, $x \neq y$.

From this property, a good choice of a 1-Lipschitz function f will yield a lower bound for W_1 and therefore an upper bound for κ.

Remark 3. *We list some basic first observations about this curvature concept (see Ollivier [92]):*

- $\kappa(x,y) \leq 1$.
- *Rewriting (2.1.2) gives $W_1(m_x, m_y) = d(x,y)(1 - \kappa(x,y))$, which is analogous to the expansion in the Riemannian case. (Recall Proposition 2 in Chapter 1.)*
- *A lower bound $\kappa(x,y) \geq k$ for any $x, y \in X$ implies*

$$W_1(m_x, m_y) \leq (1-k)d(x,y), \tag{2.1.4}$$

which can be seen as some kind of Lipschitz continuity of measures.

In Riemannian geometry, Ricci curvature controls how fast the geodesics emanating from the same point diverge on average, or equivalently how fast the volume of distance balls grows as a function of the radius. In the following we will translate those ideas into a combinatorial setting and show that Ollivier-Ricci curvature on a locally finite graph reflects the relative abundance of triangles, which is captured by the local clustering coefficient introduced by Watts-Strogatz [117].

2.1.2 Ollivier-Ricci curvature from a probabilistic view

In this subsection we will further consider the lower curvature bound condition on (V, d, m). Let k be a lower bound for the Ollivier-Ricci curvature, i.e.

$$\kappa(x, y) \geq k, \ \forall x \sim y. \tag{2.1.5}$$

Remark 4. *By Proposition 19 in Ollivier [92], it follows that if k is a lower curvature bound for all neighbors x, y then it is a lower curvature bound for all pairs of vertices. This also follows from Theorem 11 below.*

Remark 5. *By definition, the lower bound k for the curvature κ is no larger than one. In fact, such a lower bound k always exists. Since the largest possible distance between points from the supports of two measures m_x and m_y at a pair of neighbors x, y is 3, we can easily estimate $\kappa(x, y) \geq -2$. We will derive a more precise lower bound for κ on a locally finite graph with loops in the next section, see also Lin-Yau [88] for related results.*

We could also write (2.1.5) as

$$W_1(m_x, m_y) \leq (1 - k)d(x, y) = 1 - k, \ \forall x \sim y, \tag{2.1.6}$$

which is essentially equivalent to the well known path coupling criterion on the state space of Markov chains used to study the mixing time of them (see Bubley-Dyer [12] or Levin-Peres-Wilmer [80], Peres [97]). We will utilize this idea to interpret the lower bound of the Ollivier-Ricci curvature as a control on the expectation value of the distance between two coupled random walks.

We first introduce the following notation. For a probability measure μ, we denote

$$\mu P(\cdot) := \sum_x \mu(x) m_x(\cdot).$$

Let δ_x be the Dirac measure at x, then we can write $\delta_x P^1(\cdot) := \delta_x P(\cdot) = m_x(\cdot)$. Therefore the distribution of a t-step random walk starting from x with a transition probability m_x is

$$\delta_x P^t(\cdot) = \sum_{x_1, \ldots, x_{t-1}} m_x(x_1) m_{x_1}(x_2) \cdots m_{x_{t-1}}(\cdot) \tag{2.1.7}$$

for $t > 1$.

We reformulate Bubley-Dyer's theorem (see [12] or [80], [97]) in our language.

Theorem 11 (Bubley-Dyer)**.** *On (V, d, m), if for each pair of neighbors $x, y \in V$, we have the contraction*

$$W_1(m_x, m_y) \leq (1 - k)d(x, y) = 1 - k,$$

then for any two probability measures μ and ν on V, we have

$$W_1(\mu P, \nu P) \leq (1 - k) W_1(\mu, \nu).$$

With this at hand, it is easy to see that if for any pair of neighbors x, y, $\kappa(x, y) \geq k$, then for any time t and any two $\bar{x}, \bar{y}$, which are not necessarily neighbors, the following is true,

$$W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t) \leq (1-k)^t d(\bar{x}, \bar{y}). \tag{2.1.8}$$

We consider two coupled discrete time random walks $(\bar{X}_t, \bar{Y}_t)$, whose distributions are $\delta_{\bar{x}} P^t$, $\delta_{\bar{y}} P^t$ respectively. They are coupled in a way that the probability

$$p(\bar{X}_t = \bar{x}', \bar{Y}_t = \bar{y}') = \xi_t^{\bar{x},\bar{y}}(\bar{x}', \bar{y}'),$$

where $\xi_t^{\bar{x},\bar{y}}(\cdot, \cdot)$ is the optimal coupling of $\delta_{\bar{x}} P^t$ and $\delta_{\bar{y}} P^t$. In this language, we can interpret the term $W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t)$ as the expectation value of the distance $\mathbf{E}^{\bar{x},\bar{y}} d(\bar{X}_t, \bar{Y}_t)$ between the coupled random walks $\bar{X}_t$ and $\bar{Y}_t$.

Corollary 1. *On (V, d, m), if $\kappa(x, y) \geq k$, $\forall x \sim y$, then we have for any two $\bar{x}, \bar{y} \in V$,*

$$\mathbf{E}^{\bar{x},\bar{y}} d(\bar{X}_t, \bar{Y}_t) = W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t) \leq (1-k)^t d(\bar{x}, \bar{y}). \tag{2.1.9}$$

2.1.3 The normalized graph Laplace operator

In this subsection, we recall the definition of the normalized graph Laplace operator. Let $C(V)$ denote the space of all real-valued functions on the set V.

Definition 8. *The normalized graph Laplace operator Δ is defined as*

$$\Delta f(x) = \sum_{y \in V} f(y) m_x(y) - f(x), \quad \forall f \in C(V). \tag{2.1.10}$$

This operator is an analogue of the Laplace-Beltrami operator in Riemannian geometry.

With the family of probability measures (2.1.1), Δ is just the normalized graph Laplace operator studied by many authors, see e.g. Grigoryan [59], Dodziuk-Karp [51], Banerjee-Jost [5], Bauer-Jost [7], Lin-Yau [88] and is unitarily equivalent to the Laplace operator studied in Chung [24].

2.1.4 Curvature-dimension inequality

In the Riemannian case, many analytical consequences of a lower bound of the Ricci curvature are obtained through Bochner's formula. In fact a lower bound of the Ricci curvature implies a curvature dimension inequality. Bakry-Émery generalize this inequality to generators of Markov semigroups. Recall their construction we described in Section 1.4.2. As studied in Lin-Yau [88], applying this construction to the operator (2.1.10) gives (in order to show the generality

of the following calculations, we decide to use the integral notation although for the graph setting it is nothing else but a summation)

$$\Gamma(f,f)(x) = \frac{1}{2}\int_V (f(y)-f(x))^2 dm_x(y), \tag{2.1.11}$$

In fact generally

$$\Gamma(f,g)(x) = \frac{1}{2}\int_V (f(y)-f(x))(g(y)-g(x))dm_x(y).$$

For the sake of convenience, we will denote

$$Hf(x) := \frac{1}{4}\int_V\int_V (f(x)-2f(y)+f(z))^2 dm_y(z)dm_x(y).$$

By the calculation in Lin-Yau [88] we get

$$\Delta\Gamma(f,f)(x) = 2Hf(x) - \int_V\int_V (f(x)-2f(y)+f(z))(f(x)-f(y))dm_y(z)dm_x(y),$$
$$2\Gamma(f,\Delta f)(x) = -(\Delta f(x))^2 - \int_V\int_V (f(z)-f(y))(f(x)-f(y))dm_y(z)dm_x(y).$$

and therefore we have the following Lemma.

Lemma 1. *For all $f \in C(V)$, we have*

$$\Gamma_2(f,f)(x) = Hf(x) - \Gamma(f,f)(x) + \frac{1}{2}(\Delta f(x))^2,\ \forall x \in V. \tag{2.1.12}$$

2.2 Ollivier-Ricci curvature and local clustering

In this section, we mainly prove lower bounds for Ollivier's Ricci curvauture on locally finite graphs. In particular we shall explore the implication between lower bounds of the curvature and the number of triangles including neighboring vertices; the latter is encoded in the local clustering coefficient. Recalling Remark 4, we only need to bound $\kappa(x,y)$ from below for neighboring x,y.

2.2.1 First oberservations

In this subsection, we first restrict attention to simple graphs. On those graphs, we will explore some first intuitions on the lower bound estimate of $\kappa(x,y)$.

In Lin-Yau [88], they prove a lower bound of Ollivier-Ricci curvature on locally finite graphs G. Here, for later purposes, we include the case where G may have vertices of degree 1 and get the following modified result.

Theorem 12. *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y,*

$$\kappa(x,y) \geq -2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+ = \begin{cases} -2 + \frac{2}{d_x} + \frac{2}{d_y}, & \text{if } d_x > 1 \text{ and } d_y > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 6. *Notice that if $d_x = 1$, then we can calculate $\kappa(x,y) = 0$ exactly. So, even though in this case $-2 + \frac{2}{d_x} = 0$, $\kappa(x,y) \geq \frac{2}{d_y}$ doesn't hold.*

For completeness, we state the proof of Theorem 12 here. It is essentially the one in Lin-Yau [88] with a small modification.

`Proof of Theorem` 12: Since $d(x,y) = 1$ for $x \sim y$, we have

$$\kappa(x,y) = 1 - W_1(m_x, m_y). \tag{2.2.1}$$

Using Kantorovich duality, we get

$$\begin{aligned} W_1(m_x, m_y) &= \sup_{f, 1-Lip}\left(\frac{1}{d_x}\sum_{z, z\sim x} f(z) - \frac{1}{d_y}\sum_{z', z'\sim y} f(z')\right) \\ &= \sup_{f, 1-Lip}\left(\frac{1}{d_x}\sum_{z, z\sim x, z\neq y}(f(z) - f(x)) - \frac{1}{d_y}\sum_{z', z'\sim y, z'\neq x}(f(z') - f(y))\right. \\ &\quad \left. + \frac{1}{d_x}(f(y) - f(x)) - \frac{1}{d_y}(f(x) - f(y)) + (f(x) - f(y))\right) \\ &\leq \frac{d_x - 1}{d_x} + \frac{d_y - 1}{d_y} + \left|1 - \frac{1}{d_x} - \frac{1}{d_y}\right| \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} + \left|1 - \frac{1}{d_x} - \frac{1}{d_y}\right|. \\ &= 1 + 2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+. \end{aligned} \tag{2.2.2}$$

Inserting the above estimate into (2.2.1) gives

$$\kappa(x,y) \geq -2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+.$$

□

Note that trees attain this lower bound. This coincides with the geometric intuition of curvature. Since trees have the fastest volume growth rate, it is plausible that they have the smallest curvature.

Proposition 6. *We consider a tree $T = (V, E)$. Then for any neighboring x, y, we have*

$$\kappa(x,y) = -2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+. \tag{2.2.3}$$

`Proof:` In fact with Theorem 12 in hand, we only need to prove that $1 + 2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+$ is also a lower bound of W_1. If one of x, y is a vertex of degree 1, say $d_x = 1$, it is obvious that $W_1(m_x, m_y) = 1$. So we only need to deal with the case $1 - \frac{1}{d_x} - \frac{1}{d_y} \geq 0$.

We can find a 1-Lipschitz function f on a tree as follows.

$$f(z) = \begin{cases} 0, & \text{if } z \sim y, z \neq x; \\ 1, & \text{if } z = y; \\ 2, & \text{if } z = x; \\ 3, & \text{if } z \sim x, z \neq x. \end{cases} \tag{2.2.4}$$

Since on a tree, the path joining two vertices are unique, there is no further path between neighbors of x and y. So this can be easily extended to a 1-Lipschitz function on the whole graph. Then by Kantorovich duality, we have

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x}(3(d_x - 1) + 1) - \frac{1}{d_y} \cdot 2 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y}. \end{aligned} \tag{2.2.5}$$

This completes the proof. □

In order to make clear the geometric meaning of the term $\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+$, and also to prepare the idea used in the next theorem, we give another method to get the upper bound of W_1. That works through a particular transfer plan. If

$$1 - \frac{1}{d_x} - \frac{1}{d_y} \geq 0, \text{ or } 1 - \frac{1}{d_y} \geq \frac{1}{d_x},$$

then for m_y, the mass at all z such that $z \sim y$, $z \neq x$ is larger than that of m_x at y. So we can move the mass $\frac{1}{d_x}$ at y to z, $z \sim y$, $z \neq x$ for distance 1. Symmetrically, we can move a mass of $\frac{1}{d_y}$ at the vertices z which satisfy $z \sim x$, $z \neq y$ to x for distance 1. The remaining mass of $\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)$ needs to be moved for distance 3. This gives

$$\begin{aligned} W_1(m_x, m_y) &\leq \left(\frac{1}{d_x} + \frac{1}{d_y}\right) \times 1 + \left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right) \times 3 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y}. \end{aligned} \tag{2.2.6}$$

If

$$1 - \frac{1}{d_x} - \frac{1}{d_y} \leq 0,$$

we only need to move the mass of m_x for distance 1 to the support of m_y. So we have in this case $W_1(m_x, m_y) = 1$. This gives the same upper bound as in (2.2.2).

From the view of transfer plans, the existence of triangles including neighboring vertices would save a lot of transport costs and therefore affect the curvature heavily. We denote for $x \sim y$,

$$\sharp(x,y) := \text{number of triangles which include } x, y \text{ as vertices} = \sum_{x_1, x_1 \sim x, x_1 \sim y} 1.$$

Remark 7. *This quantity $\sharp(x,y)$ is related to the local clustering coefficient introduced by Watts-Strogatz [117],*

$$c(x) := \frac{\textit{number of edges between neighbors of } x}{\textit{number of possible existing edges between neighbors of } x},$$

which measures the extent to which neighbors of x are directly connected. In fact, we have the relation

$$c(x) = \frac{1}{d_x(d_x-1)} \sum_{y, y \sim x} \sharp(x,y). \tag{2.2.7}$$

We will explore the relation between the curvature $\kappa(x,y)$ and the number of triangles $\sharp(x,y)$ in the following.

2.2.2 Estimates for Ollivier-Ricci curvature on locally finite graphs with loops

In this subsection, we will derive the estimates of Ollivier-Ricci curvature for locally finite graphs $G = (V, E)$ that may have loops.

We first fix some notations. For any two real numbers a, b,

$$a_+ := \max\{a, 0\},\ a \wedge b := \min\{a, b\}, \text{ and } a \vee b := \max\{a, b\}.$$

We denote $\tilde{N}_x := \{z \in V | z \sim x\}$ as the neighborhood of x and $N_x := \tilde{N}_x \cup \{x\}$. Then $N_x = \tilde{N}_x$ if x has a loop. For every pair of neighbors x, y, we divide N_x, N_y into disjoint parts as follows.

$$N_x = \{x\} \cup \{y\} \cup N_x^1 \cup N_{xy},\ N_y = \{y\} \cup \{x\} \cup N_y^1 \cup N_{xy}, \tag{2.2.8}$$

where

$$N_{xy} = N_{x \geq y} \cup N_{x<y}$$

and

$$\begin{aligned}
N_x^1 &:= \{z | z \sim x, z \not\sim y, z \neq y\}, \\
N_{x \geq y} &:= \{z | z \sim x, z \sim y, z \neq x, z \neq y, \frac{w_{xz}}{d_x} \geq \frac{w_{zy}}{d_y}\}, \\
N_{x<y} &:= \{z | z \sim x, z \sim y, z \neq x, z \neq y, \frac{w_{xz}}{d_x} < \frac{w_{zy}}{d_y}\}.
\end{aligned}$$

In Figure 2.1 we illustrate this partition of the vertex set.

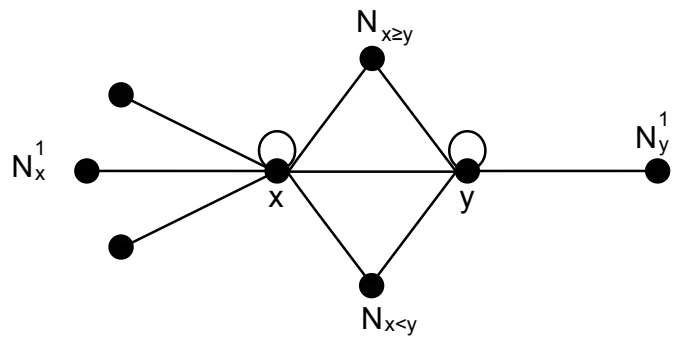

Figure 2.1: Partition of the vertex set

Theorem 13. *On $G = (V, E)$, we have for any pair of neighbors $x, y \in V$,*

$$\begin{aligned}\kappa(x,y) \geq k(x,y) := &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \vee \frac{w_{x_1y}}{d_y}\right)_+ \\ &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}\right)_+ \\ &+ \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y} + \frac{w_{xx}}{d_x} + \frac{w_{yy}}{d_y}.\end{aligned}$$

Moreover, this inequality is sharp for certain graphs.

Remark 8. *On an unweighted graph, the form $k(x, y)$ for $x \sim y$ becomes*

$$k(x,y) = -\left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \vee d_y}\right)_+ + \frac{\sharp(x,y)}{d_x \vee d_y} + \frac{s(x)}{d_x} + \frac{s(y)}{d_y},$$

where $\sharp(x, y) := \sum_{x_1 \in N_{xy}} 1$ is the number of triangles containing x, y, $s(x) = 0$ or 1 is the number of self-loops at x.

`Proof:` Since the total mass of m_x is equal to one, we obtain from (2.2.8) the following identity for neighboring vertices x and y:

$$1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} = \frac{w_{xx}}{d_x} + \sum_{x_1 \in N_x^1} \frac{w_{x_1x}}{d_x} \tag{2.2.9}$$

A similar identity holds for m_y.

We denote

$$\begin{aligned}A_{x,y} &:= 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \vee \frac{w_{x_1y}}{d_y}, \\ B_{x,y} &:= 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}.\end{aligned}$$

Obviously, $A_{x,y} \leq B_{x,y}$. We firstly try to understand these two quantities.

If $A_{x,y} \geq 0$, we have

$$1 - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1 y}}{d_y} \geq \frac{w_{xy}}{d_x} + \sum_{x_1 \in N_{x \geq y}} \left(\frac{w_{xx_1}}{d_x} - \frac{w_{x_1 y}}{d_y} \right), \tag{2.2.10}$$

i.e., using (2.2.9) we observe that the mass of m_y at y and N_y^1 is no smaller than that of m_x at y and the excess mass at $N_{x \geq y}$. Rewriting (2.2.10) in the form

$$\frac{w_{xy}}{d_y} + \sum_{x_1 \in N_{xy}} \frac{w_{x_1 y}}{d_y} \leq 1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{x \geq y}} \left(\frac{w_{xx_1}}{d_x} - \frac{w_{x_1 y}}{d_y} \right),$$

and subtracting the term $\sum_{x_1 \in N_{xy}} \frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}$ on both sides we obtain

$$\frac{w_{xy}}{d_y} + \sum_{x_1 \in N_{x<y}} \left(\frac{w_{x_1 y}}{d_y} - \frac{w_{xx_1}}{d_x} \right) \leq 1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{xx_1}}{d_x}, \tag{2.2.11}$$

i.e., the mass of m_x at x and N_x^1 is larger than that of m_y at x and the excess mass at $N_{x<y}$.

If $B_{x,y} \geq 0$, we have

$$1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{xx_1}}{d_x} + \sum_{x_1 \in N_{x \geq y}} \left(\frac{w_{xx_1}}{d_x} - \frac{w_{x_1 y}}{d_y} \right) \geq \frac{w_{xy}}{d_y}, \tag{2.2.12}$$

i.e., the mass of m_x at x and N_x^1 and the excess mass at $N_{x \geq y}$ is no smaller than that of m_y at x.

In the tree case of last subsection, it is explicitly described how much mass has to be moved from a vertex in N_x to which point in N_y, i.e. the exact value of $\xi^{x,y}(x', y')$, for any $x' \in N_x$, $y' \in N_y$. But in the case with loops it would be too complicated if we try to do the same thing. Instead, we adopt here a dynamic strategy. That is, we think of a discrete time flow of mass. After one unit time, the mass flows forward for distance 1 or stays there. We only need to determine the direction of the flow according to different cases.

We divide the discussion into 3 cases.

- $0 \leq A_{x,y} \leq B_{x,y}$. In this case we use the following transport plan: Suppose the initial time is $t = 0$.

  $t = 1$ Move all the mass at N_x^1 to x and the excess mass at $N_{x \geq y}$ to y. We denote the distribution of the mass after the first time step by m^1. We have

  $$W_1(m_x, m^1) \leq \left(1 - \frac{w_{xx}}{d_x} - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{xx_1}}{d_x} \right) \times 1 + \sum_{x_1 \in N_{x \geq y}} \left(\frac{w_{xx_1}}{d_x} - \frac{w_{x_1 y}}{d_y} \right) \times 1$$

$t = 2$ Move one part of the excess mass at x now to fill the gap at $N_{x<y}$ and the other part to y. By (2.2.11) the mass at x after $t = 1$ is enough to do so. The distribution of the mass is now denoted by m^2. We have

$$\begin{aligned} W_1(m^1, m^2) \leq & \sum_{x_1 \in N_{x<y}} \left(\frac{w_{x_1 y}}{d_y} - \frac{w_{x x_1}}{d_x} \right) \times 1 \\ & + \left[\left(1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{x x_1}}{d_x} \right) - \sum_{x_1 \in N_{x<y}} \left(\frac{w_{x_1 y}}{d_y} - \frac{w_{x x_1}}{d_x} \right) - \frac{w_{xy}}{d_y} \right] \times 1 \end{aligned}$$

$t = 3$ Move the excess mass at y now to N_y^1. We denote the mass after the third time step by $m^3 = m_y$. We have

$$\begin{aligned} W_1(m^2, m_y) \leq & \left[\left(1 - \frac{w_{xy}}{d_x} - \sum_{x_1 \in N_{xy}} \frac{w_{x x_1}}{d_x} \right) - \sum_{x_1 \in N_{x<y}} \left(\frac{w_{x_1 y}}{d_y} - \frac{w_{x x_1}}{d_x} \right) - \frac{w_{xy}}{d_y} + \frac{w_{xy}}{d_x} \right. \\ & \left. + \sum_{x_1 \in N_{x \geq y}} \left(\frac{w_{x x_1}}{d_x} - \frac{w_{x_1 y}}{d_y} \right) - \frac{w_{yy}}{d_y} \right] \times 1 \end{aligned}$$

By triangle inequality and (2.1.3), we get

$$\begin{aligned} W_1(m_x, m_y) \leq & W_1(m_x, m^1) + W_1(m^1, m^2) + W_1(m^2, m_y) \\ = & 3 - 2\frac{w_{xy}}{d_x} - 2\frac{w_{xy}}{d_y} - 2 \sum_{x_1 \in N_{xy}} \frac{w_{x x_1}}{d_x} \wedge \frac{w_{x_1 y}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x x_1}}{d_x} \vee \frac{w_{x_1 y}}{d_y} - \frac{w_{xx}}{d_x} - \frac{w_{yy}}{d_y}. \end{aligned}$$

Moreover, if the following function can be extended as a 1-Lipschitz function on the graph (i.e., if there are no paths of length 1 between N_x^1 and $N_{x<y}$, nor paths of length 1 between N_y^1 and $N_{x \geq y}$, nor paths of length 1 or 2 between N_x^1 and N_y^1),

$$f(z) = \begin{cases} 0, & \text{if } z \in N_y^1; \\ 1, & \text{if } z \in \{y\} \cup N_{x<y}; \\ 2, & \text{if } z \in \{x\} \cup N_{x \geq y}; \\ 3, & \text{if } z \in N_x^1, \end{cases}$$

then by Kantorovich duality, we can show that the inequality above is actually an equality. Recalling the definition of $\kappa(x, y)$, we have proved the theorem in this case.

- $A_{x,y} < 0 \leq B_{x,y}$. We use the following transfer plan:

  $t = 1$ We divide the excess mass of m_x at $N_{x \geq y}$ into two parts. One part together with the mass of m_x at y is enough to fill gaps at y and N_y^1. Since (2.2.10) doesn't hold in this case, this is possible. We move this part of mass to y and the other part to x. We also move all the mass of m_x at N_x^1 to x.

$t = 2$ We move the excess mass at x now to $N_{x<y}$ and the excess mass at y to N_y^1.

Applying this transfer plan, we can prove (we omit the calculation here)

$$W_1(m_x, m_y) \leq 2 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - 2 \sum_{x_1 \in N_{xy}} \left(\frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1y}}{d_y} \right) - \frac{w_{xx}}{d_x} - \frac{w_{yy}}{d_y}.$$

Moreover, if the following function can be extended as a function on the graph such that $\mathrm{Lip}(f) \leq 1$ (i.e., if there are no paths of length 1 between $N_x^1 \cup N_{x \geq y}$ and $N_y^1 \cup N_{x<y}$),

$$f(z) = \begin{cases} 0, & \text{if } z \in N_y^1 \cup N_{x<y}; \\ 1, & \text{if } z = x \text{ or } z = y; \\ 2, & \text{if } z \in N_x^1 \cup N_{x \geq y}, \end{cases}$$

then by Kantorovich duality, we can check that the inequality above is actually an equality.

- $A_{x,y} \leq B_{x,y} < 0$. We use the following transport plan:

  $t = 1$ Move the mass of m_x at N_x^1 and $N_{x \geq y}$ to x. Since now (2.2.12) doesn't hold, we need to move one part of the mass $m_y(y)$ to x and the other part to N_y^1 and $N_{x<y}$.

  Applying this transfer plan, we can calculate

$$W_1(m_x, m_y) \leq 1 - \sum_{x_1 \in N_{xy}} \left(\frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1y}}{d_y} \right) - \frac{w_{xx}}{d_x} - \frac{w_{yy}}{d_y}.$$

  Since the following function can be extended as a function on the graph such that $\mathrm{Lip}(f) \leq 1$,

$$f(z) = \begin{cases} 0, & \text{if } z \in \{x\} \cup N_{x<y} \cup N_y^1; \\ 1, & \text{if } z \in \{y\} \cup N_{x \geq y} \cup N_x^1, \end{cases}$$

  we can check the inequality above is in fact an equality by Kantorovich duality. That is, in this case for any $x \sim y$,

$$\kappa(x, y) = \sum_{x_1 \in N_{xy}} \left(\frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1y}}{d_y} \right) + \frac{w_{xx}}{d_x} + \frac{w_{yy}}{d_y}.$$

□

Remark 9. *From extending f to a 1-Lipschitz function, we see that the paths of length 1 or 2 between neighbors of x and y have an important effect on the curvature. That is, in addition to triangles, quadrangles and pentagons are also related to Ollivier's Ricci curvature. But polygons with more than 5 edges do not impact it.*

We also have an upper bound estimate.

Theorem 14. *On $G = (V, E)$, we have for every pair of neighbors x, y,*

$$\kappa(x, y) \leq \sum_{x_1 \in \{x\} \cup \{y\} \cup N_{xy}} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}.$$

`Proof:` $I := \sum_{x_1 \in \{x\} \cup \{y\} \cup N_{xy}} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}$ is exactly the mass of m_x which we need not move. The other mass needs to be moved for at least distance 1. So we have $W_1(m_x, m_y) \geq 1 - I$, which implies $\kappa(x, y) \leq I$, for $x \sim y$. □

Remark 10. *From the view of the metric measure space structure (V, d, m), the term $\sum_{x_1 \in \{x\} \cup \{y\} \cup N_{xy}} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}$ is exactly $m_x \wedge m_y(G) := m_x(G) - (m_x - m_y)_+(G)$, i.e. the intersection measure of m_x and m_y. From a metric view, the vertices N_{xy} constitute the intersection of the unit metric spheres $S_x(1)$ and $S_y(1)$.*

An immediate consequence of Theorem 14 is the following important observation.

Corollary 2. *If there exists two vertices $x \sim y$ in G such that $\sharp(x, y) = s(x) = s(y) = 0$ then $\kappa(x, y) \leq 0$.*

Example 1. *We consider a lazy random walk on an unweighted complete graph $\mathcal{K}_N$ with N vertices governed by $m_x(y) = 1/N, \forall x, y$. Or equivalently , we consider the graph $\mathcal{K}_N^{\text{lazy}}$. Using Theorem 13 and Theorem 14, we get for any x, y*

$$1 = \frac{N-2}{N} + \frac{1}{N} + \frac{1}{N} \leq \kappa(x, y) \leq \frac{1}{N} \cdot N = 1.$$

That is, in this case, both the lower and the upper bound are sharp.

2.2.3 Simple graph case

To see more geometric intuition, we restate the two theorems in the last subsection in the simple graph case.

Theorem 15. *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y,*

$$\kappa(x, y) \geq -\left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \vee d_y}\right)_+ + \frac{\sharp(x, y)}{d_x \vee d_y}.$$

Moreover, this inequality is sharp for certain graphs.

Remark 11. *If $\sharp(x, y) = 0$, then this lower bound reduces to the one in Theorem 12.*

Example 2. *On a complete graph $\mathcal{K}_N$ $(N \geq 2)$ with n vertices, $\sharp(x,y) = N-2$ for any x, y. So Theorem 15 implies*

$$\kappa(x,y) \geq \frac{N-2}{N-1}.$$

In fact, we can easily check that the above inequality is an equality. Also notice that on those graphs, the local clustering coefficient $c(x) = 1$ attains the largest value.

Let's explore more carefully here the different cases included in Theorem 15. First note now

$$A_{x,y} = 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \wedge d_y},$$
$$B_{x,y} = 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \vee d_y}.$$

Then

- $A_{x,y} \geq 0$ is equivalent to

$$d_x \wedge d_y > 1, \text{ and } \sharp(x,y) \leq d_x \wedge d_y - 1 - \frac{d_x \wedge d_y}{d_x \vee d_y}.$$

  Since $\sharp(x,y) \in \mathbf{Z}$, we know that $d_x \wedge d_y \geq 2$ and $\sharp(x,y) \leq d_x \wedge d_y - 2$. This means both x and y have at least one own neighbor.

- If $A_{x,y} < 0$, we get

$$d_x \wedge d_y - 1 - \frac{d_x \wedge d_y}{d_x \vee d_y} < \sharp(x,y) \leq d_x \wedge d_y - 1.$$

  I.e., $\sharp(x,y) = d_x \wedge d_y - 1$. This means the vertex with smaller degree has no own neighbors.

- If $A_{x,y} < 0 \leq B_{x,y}$ then on one hand $\sharp(x,y) = d_x \wedge d_y - 1$, on the other hand $B_{x,y} \geq 0$ is equivalent to

$$d_x \vee d_y \geq \frac{d_x \wedge d_y}{d_x \wedge d_y - 1} d_x \wedge d_y. \tag{2.2.13}$$

  In this case, one of x, y has no own neighbors, and if the other one has sufficiently many own neighbors, $B_{x,y} \geq 0$ will be satisfied.

From Theorem 15, we can force the curvature $\kappa(x,y)$ to be positive by increasing the number $\sharp(x,y)$.

Theorem 16. *On a locally finite graph $G = (V, E)$, for any neighboring x, y, we have*

$$\kappa(x, y) \leq \frac{\sharp(x, y)}{d_x \vee d_y}. \tag{2.2.14}$$

So if $\kappa(x, y) > 0$, then $\sharp(x, y)$ is at least 1. Moreover, if $\kappa(x, y) \geq k > 0$, we have

$$\sharp(x, y) \geq \lceil k d_x \vee d_y \rceil, \tag{2.2.15}$$

where $\lceil a \rceil := \min\{A \in \mathbf{Z} | A \geq a\}$, for $a \in R$.

We could define the average value of the curvature at x as a synthetic scalar curvature (see Problem Q in Ollivier [93]), i.e.

$$\kappa(x) := \frac{1}{d_x} \sum_{y, y \sim x} \kappa(x, y).$$

We will denote $D(x) := \max_{y, y \sim x} d_y$. By the relation (2.2.7), we can get immediately

Corollary 3. *The scalar curvature at x can be controlled by the local clustering coefficient at x,*

$$\frac{d_x - 1}{d_x} c(x) \geq \kappa(x) \geq -2 + \frac{d_x - 1}{d_x \vee D(x)} c(x).$$

Remark 12. *In fact in some special cases, we can get more precise lower bounds*

$$\kappa(x) \geq \begin{cases} -2 + \frac{2}{d_x} + \frac{2}{D(x)} + \left[\frac{(d_x - 1)}{d_x} + \frac{2(d_x - 1)}{d_x \vee D(x)}\right] c(x), & \text{if } A_{x,y} \geq 0 \text{ for all } y \sim x; \\ -1 + \frac{1}{d_x} + \frac{1}{D(x)} + \frac{2(d_x - 1)}{d_x \vee D(x)} c(x), & \text{if } A_{x,y} < 0 \leq B_{x,y} \text{ for all } y \sim x; \\ \frac{d_x - 1}{d_x \vee D(x)} c(x), & \text{if } B_{x,y} < 0 \text{ for all } y \sim x. \end{cases}$$

2.3 Curvature dimension inequalities

In this section, we establish curvature dimension inequalities on locally finite graphs. A very interesting one is the inequality under the condition $\kappa \geq k > 0$. Curvature dimension inequalities on locally finite graphs without self-loops are studied in Lin-Yau [88]. We first state a detailed version of their results in our setting, i.e. weighted graphs which may have loops. Let's denote

$$D_w(x) := \max_{y, y \sim x} \frac{4 d_y}{4 w_{yx} + w_{yy}}.$$

Notice that on an simple graph, this is the $D(x)$ we used in Section 2.2.3.

Theorem 17. *On a weighted locally finite graph $G = (V, E)$, the Laplace operator Δ satisfies*

$$\Gamma_2(f,f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{2}{D_w(x)} - 1\right)\Gamma(f,f)(x). \tag{2.3.1}$$

Remark 13. *Since in this case we attach measure (2.1.1), we get*

$$Hf(x) = \frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\frac{w_{xy}}{d_y}\sum_{z,z\sim y} w_{yz}(f(x) - 2f(y) + f(z))^2.$$

We only need to choose special $z = x$ and $z = y$ in the second sum and then (2.1.11) and Lemma 1 imply the theorem.

2.3.1 Simple graph case

We again restrict ourselves to simple graphs.

We observe that the existence of triangles causes cancellations in calculating the term $Hf(x)$. This gives

Theorem 18. *On a locally finite graph $G = (V, E)$, the Laplace operator satisfies*

$$\Gamma_2(f,f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}t(x) - 1\right)\Gamma(f,f)(x), \tag{2.3.2}$$

where

$$t(x) := \min_{y,y\sim x}\left(\frac{4}{d_y} + \frac{1}{D(x)}\sharp(x,y)\right).$$

Remark 14. *Notice that if there is a vertex y, $y \sim x$, such that $\sharp(x,y) = 0$, this will reduce to (2.3.1).*

`Proof:` Starting from (2.1.12), the main work is to compare $Hf(x)$ with

$$\Gamma(f,f)(x) = \frac{1}{2}\frac{1}{d_x}\sum_{y,y\sim x}(f(y) - f(x))^2.$$

First we try to write out $Hf(x)$ as

$$Hf(x) = \frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\left[\frac{4}{d_y}(f(x) - f(y))^2 + \frac{1}{d_y}\sum_{z,z\sim y,z\neq x}(f(x) - 2f(y) + f(z))^2\right].$$

If there is a vertex x_1 which satisfies $x_1 \sim x$, $x_1 \sim y$, we have

$$\begin{aligned}
&\frac{1}{d_y}(f(x)-2f(y)+f(x_1))^2+\frac{1}{d_{x_1}}(f(x)-2f(x_1)+f(y))^2\\
\geq&\frac{1}{D(x)}[(f(x)-f(y))^2+(f(y)-f(x_1))^2+2(f(x)-f(y))(f(x_1)-f(y))\\
&+(f(x)-f(x_1))^2+(f(y)-f(x_1))^2+2(f(y)-f(x_1))(f(x)-f(x_1))]\\
=&\frac{1}{D(x)}[(f(x)-f(y))^2+4(f(y)-f(x_1))^2+(f(x)-f(x_1))^2].\\
\geq&\frac{1}{D(x)}(f(x)-f(y))^2. \qquad (2.3.3)
\end{aligned}$$

So the existence of a triangle which includes x and y will give another term

$$\frac{1}{D(x)}(f(y)-f(x))^2$$

to the sum in $Hf(x)$. Since this effect is symmetric w.r.t. y and x_1, we can get

$$\begin{aligned}
Hf(x) &\geq \frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}\left(\frac{4}{d_y}+\frac{1}{D(x)}\sharp(x,y)\right)(f(y)-f(x))^2\\
&\geq t(x)\frac{1}{4}\frac{1}{d_x}\sum_{y,y\sim x}(f(y)-f(x))^2\\
&= t(x)\cdot\frac{1}{2}\Gamma(f,f)(x).
\end{aligned}$$

Inserting this into (2.1.12) completes the proof. □

Recalling Theorem 16 and the subsequent discussion, we get the following curvature dimension inequalities on graphs with positive Ollivier-Ricci curvature.

Corollary 4. *On a locally finite graph $G=(V,E)$, if $\kappa(x,y)>0$, then we have*

$$\Gamma_2(f,f)(x)\geq\frac{1}{2}(\Delta f(x))^2+\left(\frac{5}{2D(x)}-1\right)\Gamma(f,f)(x). \qquad (2.3.4)$$

Corollary 5. *On a locally finite graph $G=(V,E)$, if $\kappa(x,y)\geq k>0$, then we have*

$$\Gamma_2(f,f)(x)\geq\frac{1}{2}(\Delta f(x))^2+\left(\frac{1}{2}\min_{y,y\sim x}\left\{\frac{4}{d_y}+\frac{\lceil kd_x\vee d_y\rceil}{D(x)}\right\}-1\right)\Gamma(f,f)(x). \qquad (2.3.5)$$

Remark 15. *Observe that a rough inequality in this case is*

$$\Gamma_2(f,f)(x)\geq\frac{1}{2}(\Delta f(x))^2+\left(\frac{2}{D(x)}+\frac{kd_x}{2D(x)}-1\right)\Gamma(f,f)(x).$$

Comparing this one with (2.3.1), we see that positive κ increases the curvature function (recall Definition 5) here.

Remark 16. *We point out that the condition $\kappa(x,y) \geq k > 0$ implies that the diameter of the graph is bounded by $\frac{2}{k}$ (see Proposition 23 in Ollivier [92]). So in this case the graph is a finite one.*

Let us revisit the example of a complete graph $\mathcal{K}_N$ ($N \geq 2$) with N vertices. Recall in Example 2, we know

$$\kappa(x,y) = \frac{N-2}{N-1}, \ \forall\ x, y.$$

For the curvature dimension inequality on $\mathcal{K}_N$, Theorem 18 or Corollary 5 using the above κ implies

$$\begin{aligned}\Gamma_2(f,f) &\geq \frac{1}{2}(\Delta f)^2 + \left(\frac{2}{N-1} - 1 + \frac{1}{2}\frac{N-2}{N-1}\right)\Gamma(f,f) \\ &= \frac{1}{2}(\Delta f)^2 + \frac{4-N}{2(N-1)}\Gamma(f,f). \end{aligned} \tag{2.3.6}$$

Moreover, the curvature term in the above inequality cannot be larger. To see this, we calculate, using the same trick as in (2.3.3),

$$\begin{aligned} Hf(x) &= \frac{1}{4(N-1)^2}\sum_{y,y\sim x}\sum_{z,z\sim x}(f(x) - 2f(y) + f(z))^2 \\ &= \frac{N+2}{2(N-1)}\Gamma(f,f)(x) + \frac{1}{(N-1)^2}\sum_{(x_1,x_2)}(f(x_1) - f(x_2))^2, \end{aligned}$$

where $\sum_{(x_1,x_2)}$ means the sum over all unordered pairs of neighbors of x. Recalling Lemma 1, we get

$$\Gamma_2(f,f)(x) = \frac{1}{2}(\Delta f)^2(x) + \frac{4-N}{2(N-1)}\Gamma(f,f)(x) + \frac{1}{(N-1)^2}\sum_{(x_1,x_2)}(f(x_1) - f(x_2))^2. \tag{2.3.7}$$

For any vertex x, we can find a particular function $\overline{f}$,

$$\overline{f}(z) = \begin{cases} 2, & \text{when } z = x; \\ 1, & \text{when } z \sim x, \end{cases} \tag{2.3.8}$$

such that the last term in (2.3.7) vanishes, and $\Gamma(\overline{f},\overline{f}) \neq 0$. This means the curvature term in (2.3.6) is optimal for dimension parameter 2.

But the curvature term $\frac{4-N}{2(N-1)}$ behaves very differently from κ. In fact as $N \to +\infty$,

$$\frac{4-N}{2(N-1)} \searrow -\frac{1}{2} \ \text{ whereas } \ \kappa \nearrow 1.$$

To get a curvature dimension inequality with a curvature term which behaves like κ, it seems that we should adjust the dimension parameter. In fact, we have

Proposition 7. *On a complete graph $\mathcal{K}_N$ $(N \geq 2)$ with N vertices, the Laplace operator Δ satisfies for $n \in [1, +\infty]$,*

$$\Gamma_2(f,f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + \left(\frac{4-N}{2(N-1)} + \frac{n-2}{n}\right)\Gamma(f,f)(x). \tag{2.3.9}$$

Moreover, for every fixed dimension parameter n, the curvature term is optimal.

`Proof:` We have from (2.3.7)

$$\begin{aligned}\Gamma_2(f,f)(x) =& \frac{1}{n}(\Delta f)^2(x) + \frac{4-N}{2(N-1)}\Gamma(f,f)(x) \\ &+ \frac{1}{(N-1)^2}\sum_{(x_1,x_2)}(f(x_1)-f(x_2))^2 + \left(\frac{1}{2}-\frac{1}{n}\right)(\Delta f)^2.\end{aligned}$$

Let us denote the sum of the last two terms by I. Then we have

$$\begin{aligned}I =& \frac{1}{(N-1)^2}\Big\{\left(\frac{1}{2}-\frac{1}{n}\right)\sum_{y,y\sim x}(f(y)-f(x))^2 + \sum_{(x_1,x_2)}\Big[(f(x_1)-f(x))^2 + (f(x_2)-f(x))^2 \\ &+ \left(2\left(\frac{1}{2}-\frac{1}{n}\right)-2\right)(f(x_1)-f(x))(f(x_2)-f(x))\Big]\Big\} \\ =& \frac{1}{(N-1)^2}\Big[\left(\frac{1}{2}-\frac{1}{n}\right)\sum_{y,y\sim x}(f(y)-f(x))^2 + \left(1-\frac{n+2}{2n}\right)(N-2)\sum_{y,y\sim x}(f(y)-f(x))^2 \\ &+ \sum_{(x_1,x_2)}\frac{n+2}{2n}(f(x_1)-f(x_2))^2\Big] \\ =& \frac{n-2}{n}\Gamma(f,f)(x) + \frac{n+2}{2n(N-1)^2}\sum_{(x_1,x_2)}(f(x_1)-f(x_2))^2.\end{aligned}$$

This finishes the proof. □

An interesting point appears when we choose the dimension parameter n of $\mathcal{K}_N$ as $N-1$. Then we have

$$\Gamma_2(f,f) \geq \frac{1}{N-1}(\Delta f)^2 + \frac{1}{2}\frac{N-2}{N-1}\Gamma(f,f),$$

where the curvature term is exactly $\frac{1}{2}\kappa$. From the fact that $\mathcal{K}_N$ could be considered as the boundary of a $(N-1)$ dimensional simplex, the m we choose here seems also natural.

Remark 17. *We point out another similar fact here. On a locally finite graph with maximal degree D and minimal degree larger than 1, Theorem 12 and Theorem 17 imply that*

$$\kappa(x,y) \geq 2\left(\frac{2}{D}-1\right),\ \forall\ x,y, \tag{2.3.10}$$

and

$$\Gamma_2(f,f) \geq \frac{1}{2}(\Delta f)^2 + \left(\frac{2}{D} - 1\right)\Gamma(f,f), \tag{2.3.11}$$

respectively. It is not difficult to see that for regular trees with degree larger than 1*, the curvature term in (2.3.11) is optimal. (Just consider the extension of the function (2.3.8), taking values* 0 *on vertices which are not* x *and neighbors of* x *there.) So on regular trees, the curvature term is also exactly* $\frac{1}{2}\kappa$*.*

Remark 18. *In Erdös-Harary-Tutte [53], they define the dimension of a graph* G *as the minimum number* n *such that* G *can be embedded into a* n *dimensional Euclidean space with every edge of* G *having length* 1*. It is interesting that by their definition, the dimension of* $\mathcal{K}_N$ *is also* $n-1$ *and the dimension of any tree is at most* 2*.*

From the above observations, it seems natural to expect stronger relations between the lower bound of κ and the curvature term in the curvature dimension inequality if one chooses proper dimension parameters.

2.3.2 General case

We have similar results on the structure (V, d, m), with similar proofs.

Theorem 19. *On* (V, d, m)*, the Laplace operator satisfies*

$$\Gamma_2(f,f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}t_w(x) - 1\right)\Gamma(f,f)(x), \tag{2.3.12}$$

where

$$t_w(x) := \min_{y, y\sim x}\left\{\frac{4w_{xy} + w_{yy}}{d_y} + \sum_{x_1, x_1\sim x, x_1\sim y}\left(\frac{w_{xy}}{d_y} \wedge \frac{w_{xx_1}}{d_{x_1}}\right)\frac{w_{x_1y}}{w_{xy}}\right\}.$$

2.4 Spectrum of the normalized graph Laplace operator on finite graphs

In this section, we combine the Ollivier-Ricci curvature with the neighborhood graph method developed by Bauer-Jost [7] to explore the estimates of the spectrum of the normalized graph Laplace operator on finite graphs. We denote the number of vertices by N.

2.4.1 Basic properties of Eigenvalues

Let's first review some basic properties of the eigenvalues of a normalized graph Laplace operator.

Definition 9. *We call λ an eigenvalue of Δ if there exists some $f \not\equiv 0$ such that*

$$\Delta f = -\lambda f. \tag{2.4.1}$$

In fact, we have a natural measure μ on the whole set V,

$$\mu(x) := d_x,$$

which gives an inner product structure on $C(V)$.

Definition 10. *The inner product of two functions $f, g \in C(V)$ is defined as*

$$(f, g)_\mu = \sum_{x \in V} f(x)g(x)\mu(x). \tag{2.4.2}$$

Then $C(V)$ becomes a Hilbert space, and we can write $C(V) = l^2(V, \mu)$. By the definition of the degree and the symmetry of the weight function, we can check that

- μ is invariant w.r.t. $\{m_x(\cdot)\}$, i.e. $\sum_{x \in V} m_x(y)\mu(x) = \mu(y)$, $\forall y \in V$;
- μ is reversible w.r.t. $\{m_x(\cdot)\}$, i.e. $m_x(y)\mu(x) = m_y(x)\mu(y)$, $\forall x, y \in V$.

These two facts imply immediately that the operator Δ is nonpositive and self-adjoint on the space $l^2(V, \mu)$.

Remark 19. *The invariance of μ implies*

$$\sum_{x \in V} \Delta f(x)\mu(x) = 0, \ \ \forall f \in C(V). \tag{2.4.3}$$

As in Bakry-Émery [3], we can check

$$\Gamma(f, f) := \frac{1}{2}\Delta(f^2) - f\Delta f \geq 0.$$

Then we get the nonpositivity

$$(f, \Delta f)_\mu = -\sum_{x \in V} \Gamma(f, f)(x)\mu(x) \leq 0, \ \ \forall f \in C(V) = l^2(\mu).$$

Using the convention of eigenvalues we adopted here, it follows from the observation that Δ is self-adjoint and nonpositive that all its eigenvalues are real and nonnegative. In fact, it's well known that (see e.g. Chung [24])

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 2. \tag{2.4.4}$$

Since our graph is connected we actually have $0 < \lambda_1$. In Chung [24] it is shown, by proving a discrete version of the Cheeger inequality, that λ_1 is a measure for

how easy/difficult it is to cut the graph into two large pieces. Furthermore, it is well known that $\lambda_{N-1} = 2$ if and only if G is bipartite. In Bauer-Jost [7] a Cheeger type estimate for the largest eigenvalue λ_{N-1} was obtained. The results in Bauer-Jost [7] show that λ_{N-1} is a measure for how close a graph is to a bipartite one. In the following, we will call λ_1 the first eigenvalue and λ_{N-1} the largest eigenvalue of the operator Δ.

We also want to mention the Reyleigh formula for λ_1 here (see e.g. Chung [24]).

$$\lambda_1 = \inf_{f \in C(V)} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w_{xy}}{\sum_{x \in V} (f(x) - f_V)^2 d_x}, \tag{2.4.5}$$

where $f_V = \frac{1}{\sum_{x \in V} d_x} \sum_{x \in V} f(x) d_x$ is the mean value of f over the graph. This tells us that a lower bound of λ_1 implies a Poincaré type inequality. This point will be used in Chapter 3.

2.4.2 Neighborhood graphs

In this subsection, we recall the neighborhood graph method developed by Bauer-Jost [7].

As discussed above, the Laplace operator underlies random walks on graphs. In this section, we discuss the deep relationship between eigenvalues estimates for the Laplace operator Δ and random walks on the graph G by using neighborhood graphs.

Let's first recall the notation $\delta_x P^t$ in (2.1.7). In particular, using the measure (2.1.1), the probability that the random walk starting at x moves to y in t steps is given by

$$\delta_x P^t(y) = \begin{cases} \sum_{x_1, \ldots, x_{t-1}} \frac{w_{xx_1}}{d_x} \frac{w_{x_1 x_2}}{d_{x_1}} \cdots \frac{w_{x_{t-1} y}}{d_{x_{t-1}}}, & \text{if } t > 1; \\ \frac{w_{xy}}{d_x}, & \text{if } t = 1. \end{cases}$$

The idea is now to define a family of graphs $G[t]$, $t \geq 1$ that encodes the transition probabilities of the t-step random walks on the graph G.

Definition 11. *The neighborhood graph $G[t] = (V, E[t])$ of the graph $G = (V, E)$ of order $t \geq 1$ has the same vertex set as G and the weights of the edges of $G[t]$ are defined in terms of the transitions probabilities of the t-step random walk,*

$$w_{xy}[t] := \delta_x P^t(y) d_x. \tag{2.4.6}$$

In particular, $G = G[1]$ and $x \sim y$ in $G[t]$ if and only if there exists a path of length t between x and y in G.

Remark 20. *We note here that the neighborhood graph method is related to the discrete heat kernel $p_t(x, y)$ on graphs. For more details about the discrete heat*

kernel see for instance Grigor'yan [59]. We have

$$p_t(x,y) = \frac{w_{xy}[t]}{d_x d_y}.$$

Example 3. *We consider the following two examples. Note that the neighborhood*

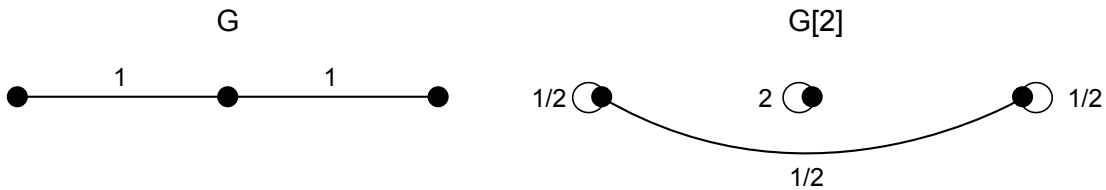

Figure 2.2: Neighborhood graph of a bipartite graph

graph $G[2]$ in Figure 2.2 is disconnected. In fact the next lemma shows that this is the case because G is bipartite. Note furthermore that $E(G) \nsubseteq E(G[2])$. For

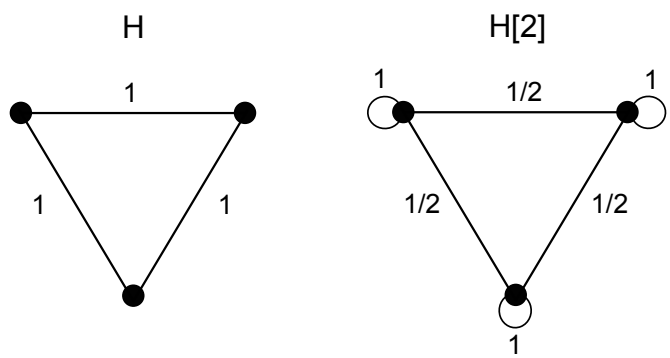

Figure 2.3: Neighborhood graph of a triangle

the example in Figure 2.3 we have $E(H) \subseteq E(H[2])$.

These examples shows that the neighborhood graph $G[t]$ is in general a weighted graph with loops, even if the original graph G is a simple graph.

Lemma 2. *The neighborhood graph $G[t]$ has the following properties (see Bauer-Jost [7]):*

(i) If t is even, then $G[t]$ is connected if and only if G is not bipartite. Furthermore, if t is even, $G[t]$ cannot be bipartite.

(ii) If t is odd, then $G[t]$ is always connected and $G[t]$ is bipartite iff G is bipartite.

(iii) $d_x[t] = d_x$ for all $x \in V$.

Note that (*iii*) implies that $l^2(V,\mu) = l^2(V,\mu[t])$ for all t. In particular, we have the same inner product for all neighborhood graphs $G[t]$. The crucial observation is the next theorem.

Theorem 20 (Bauer-Jost)**.** *The Laplace operator Δ on G and the Laplace operator $\Delta[t]$ on $G[t]$ are related to each other by the following identity:*

$$(\mathrm{id}+\Delta)^t - \mathrm{id} = \Delta[t].$$

The importance of this theorem comes from the following Corollary that establishes a connection between estimates for the smallest and the largest eigenvalue on G and $G[t]$ respectively.

Corollary 6 (Bauer-Jost)**.** *(i) Let $\mathcal{A}[t]$ be a lower bound for the eigenvalue $\lambda_1[t]$ of $\Delta[t]$, i.e., $\lambda_1[t] \geq \mathcal{A}[t]$. Then*

$$1-(1-\mathcal{A}[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1+(1-\mathcal{A}[t])^{\frac{1}{t}} \tag{2.4.7}$$

if t is even and

$$1-(1-\mathcal{A}[t])^{\frac{1}{t}} \leq \lambda_1 \tag{2.4.8}$$

if t is odd.

(ii) Let $\mathcal{B}[t]$ be an upper bound for the largest eigenvalue $\lambda_{N-1}[t]$ of $\Delta[t]$, i.e. $\lambda_{N-1}[t] \leq \mathcal{B}[t]$. (Since $\lambda_{N-1}[t] \leq 1$ for t even, we can assume in this case w.l.o.g. that $\mathcal{B}[t] \leq 1$ in this case.) Then all eigenvalues of Δ are contained in the union of the intervals

$$\left[0, 1-(1-\mathcal{B}[t])^{\frac{1}{t}}\right] \bigcup \left[1+(1-\mathcal{B}[t])^{\frac{1}{t}}, 2\right]$$

if t is even and

$$\lambda_{N-1} \leq 1-(1-\mathcal{B}[t])^{\frac{1}{t}}$$

if t is odd.

These results show how random walks on graphs (or equivalently neighborhood graphs) can be used to estimate eigenvalues of the Laplace operator.

Let's recall Corollary 2. It in fact shows that positive Ricci curvature is a quite strong requirement. For instance, in a loopless graph, already the existence of a single edge that is not contained in a triangle prevents the graph from having a positive Ricci curvature lower bound. We will show in the following that the neighborhood graph technique can be used to reduce the influence of such edges. This observation is particularly important in the next section when we study eigenvalue estimates in terms of the Ricci curvature.

Neighborhood graphs are nothing but coarse representations of the original graph. More precisely, the neighborhood graphs $G[t]$ encode the larger scale structure of the original graph G, where larger values of t stand for larger scales,

in the sense that an edge between two nodes in $G[t]$ is equivalent to the existence of a path of length t in the original graph G between these two nodes. In order to see how neighborhood graphs can reduce the influence of single edges, we state the following simple observations that follow immediately from the definition of the neighborhood graphs.

- Triangles and loops are preserved when we go to higher order neighborhood graphs, i.e. if (xyz) form a triangle in $G[s]$ (x has a loop in $G[s]$) then they from a triangle in $G[t]$ (x has a loop in $G[t]$) for all $t > s$.
- If t is even, every vertex has a loop in $G[t]$.
- If t is odd, the edge set of G is a subset of the edge set of $G[t]$, i.e. $E \subseteq E[t]$.
- If in G a vertex x is not contained in a triangle but contained in a cycle of length $3t$ then x is contained in a triangle in $G[t]$.
- If in G a vertex x is contained in a cycle of odd length $2l + 1$, $l \geq 1$, then x is contained in a triangle in $G[t]$ at least for $t \geq 2l - 1$. Moreover, every pair of vertices of the cycle will be connected then.
- If in G a vertex x is not contained in a triangle but $x \sim y$ where y is contained in a triangle, then x is also contained in a triangle in $G[t]$ for all $t \geq 2$.

These observations show that the number of triangles and loops will monotonically increase when we go from G to $G[t]$. Hence even though the Ricci curvature of the original graph is negative, Corollary 2 does not exclude that the Ricci curvature of the neighborhood graph $G[t]$ is positive. Indeed we will show in Theorem 22 that for all graphs that are not bipartite there exists a $s \in \mathbb{N}$ such that the Ricci curvature of the neighborhood graph $G[t]$ satisfies $k[t] > 0$ for all $t > s$.

2.4.3 Estimates of the spectrum in terms of Ollivier-Ricci curvature

In this section, we obtain nontrivial estimates for the extremal eigenvalues of the normalized Laplace operator in terms of the Olliver-Ricci curvature of the neighborhood graphs. In particular, our new estimates improve the eigenvalue estimates obtained by Olliver in [92].

In Proposition 30 of [92], Ollivier proved a spectral radius estimate which works on a general metric space with random walks. In particular, on finite graphs, it can be stated as follows.

Theorem 21 (Ollivier)**.** *On (V, d, m), if $\kappa(x, y) \geq k$, $\forall x \sim y$, then the eigenvalues of the normalized graph Laplace operator Δ satisfy*

$$k \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} \leq 2 - k.$$

The lower bound for λ_1 is a discrete analogue of the estimate for the smallest nonzero eigenvalue of the Laplace-Beltrami operator on a Riemannian manifold by Lichnerowicz. As pointed out in Ollivier [92], this result is also related to the coupling method for estimates of the first eigenvalue in the Riemannian setting developed by Chen-Wang [23] (which leads to a refinement of the eigenvalue estimate of Li-Yau [84]), see also the surveys Chen [21, 22]. The corresponding result of Corollary 1 in the smooth case, i.e., controlling the expectation distance of two coupled Markov chains in terms of the lower bound of Ricci curvature on a Riemannian manifold, is a key step in Chen-Wang's method.

A direct proof of Theorem 21 can be found in [92]. Here for readers' convenience, we present an analogue of Chen-Wang's method in the discrete setting, which motivated us to combine the Ollivier-Ricci curvature and the neighborhood graph method via random walks. It reflects the deep connection between eigenvalue estimates and random walks or heat equations.

`Proof:` We consider the transition probability operator $P : l^2(V, \mu) \to l^2(V, \mu)$ defined by $Pf(x) := \sum_y f(y) m_x(y) = \sum_y f(y) \delta_x P(y)$. Then we have $P^t f(x) = \sum_y f(y) \delta_x P^t(y)$. We construct a discrete time heat equation,

$$\begin{cases} f(x, 0) = f_1(x), \\ f(x, 1) - f(x, 0) = \Delta f(x, 0), \\ f(x, 2) - f(x, 1) = \Delta f(x, 1), \\ \cdots \\ f(x, t+1) - f(x, t) = \Delta f(x, t), \end{cases} \tag{2.4.9}$$

where $f_1(x)$ satisfies $\Delta f_1(x) = -\lambda f_1(x) = P f_1(x) - f_1(x)$ for $\lambda \neq 0$. Iteratively, one can find the solution of the above system of equations as

$$f(x, t) = P^t f_1(x) = (1 - \lambda)^t f_1(x). \tag{2.4.10}$$

We remark here that the solution of the heat equation on a Riemannian manifold with the eigenfunction as the initial value is $f(x, t) = f_1(x) e^{-\lambda t}$, which also involves information about both the eigenvalue λ and the eigenfunction $f_1(x)$.

Then we have for any $\bar{x}, \bar{y} \in V$

$$\begin{aligned} |1 - \lambda|^t |f_1(\bar{x}) - f_1(\bar{y})| &= |f(\bar{x}, t) - f(\bar{y}, t)| \\ &= |P^t f_1(\bar{x}) - P^t f_1(\bar{y})| \\ &\leq \sum_{\bar{x}', \bar{y}'} |f(\bar{x}') - f(\bar{y}')| \xi_t^{\bar{x}, \bar{y}}(\bar{x}', \bar{y}') \\ &\leq \mathrm{Lip}(f_1) \mathbf{E}^{\bar{x}, \bar{y}} d(\bar{X}_t, \bar{Y}_t) \\ &\leq \mathrm{Lip}(f_1)(1 - k)^t d(\bar{x}, \bar{y}). \end{aligned}$$

Here, $\mathrm{Lip}(f)$ is always finite since the underlying space V is a finite set. In the last inequality we used Corollary 1. From an analytic point of view, the above calculation can be seen as a gradient estimate for the solution of the heat equation.

Since the eigenfunction f_1 for the eigenvalue λ is orthogonal to the constant function, i.e. $(f_1, \mathbf{1})_\mu = 0$, we can always find $x_0, y_0 \in V$ such that $|f_1(x_0) - f_1(y_0)| > 0$. It follows that

$$0 < \left(\frac{1-k}{|1-\lambda|}\right)^t \mathrm{Lip}(f_1) d(x_0, y_0).$$

To prevent a contradiction when $t \to \infty$, we need

$$\frac{1-k}{|1-\lambda|} \geq 1, \tag{2.4.11}$$

which completes the proof. □

As an immediate consequence of Theorem 21 and Theorem 13 we obtain an estimate for the largest eigenvalue in terms of the number of triangles and loops in the graph.

Corollary 7. *On $G = (V, E)$, the largest eigenvalue satisfies*

$$\lambda_{N-1} \leq 2 - \min_{x \sim y} k(x, y),$$

where $k(x, y)$ is defined in Theorem 13.

Example 4. *On an unweighted complete graph $\mathcal{K}_N$ with N vertices, we have*

$$k = \kappa(x, y) = \frac{N-2}{N-1}, \forall x, y \text{ and } \lambda_1 = \cdots = \lambda_{N-1} = \frac{N}{N-1}.$$

Therefore,

$$k < \lambda_1 = \cdots = \lambda_{N-1} = 2 - k.$$

That is, the upper bound estimate for λ_{N-1} is sharp for unweighted complete graphs.

Example 5. *Let's revisit the graph $\mathcal{K}_N^{\mathrm{lazy}}$ in Example 1. We have*

$$k = \kappa(x, y) = 1, \forall x, y \text{ and } \lambda = \cdots = \lambda_{N-1} = 1.$$

Therefore,

$$k = \lambda_1 = \cdots = \lambda_{N-1} = 2 - k = 1.$$

That is, both estimates are sharp in this case.

The above two examples show the sharpness of Ollivier's estimates. However, from Corollary 2 we know that a positive lower curvature bound is a strong restriction on a graph. The open problem G of Olliver [93] asks for the possibility to relax this assumption. We will show in the following how to obtain nontrivial estimates for all graphs by using the neighborhood graph technique. This gives an answer to Ollivier's problem in the finite graph setting.

Before we show how one can improve Olliver's result by using the neighborhood graph technique, we use this technique to obtain upper bounds for λ_{N-1} from lower bounds for λ_1, which describes the connection between those two bounds. We do this by carefully comparing the Olliver-Ricci curvature on a graph G and its neighborhood graphs $G[t]$.

If we interpret the graph $G = (V, E)$ as a structure $(V, d, m = \{\delta_x P\})$, then by (3.7) its neighborhood graph $G[t] = (V, E[t])$ can be considered as a structure $(V, d[t], \{\delta_x P^t\})$. So the first step should be to estimate the variance of the metrics on neighborhood graphs.

Lemma 3. *For any $x, y \in V$, we have*

$$\frac{1}{t} d(x, y) \leq d[t](x, y). \tag{2.4.12}$$

`Proof:` For any $x, y \in V$, we set $d[t](x, y) = \infty$ if we cannot find a path connecting them in $G[t]$. Otherwise, we just choose a shortest path $x_0 = x, x_1, \ldots, x_l = y$, between x and y in $G[t]$, i.e. $l = d[t](x, y)$. For x_i, x_{i+1}, $i = 0, \ldots, l-1$, by definition of neighborhood graph, we have $d(x_i, x_{i+1}) \leq t$ in G. Equivalently,

$$\frac{1}{t} d(x_i, x_{i+1}) \leq 1 = d[t](x_i, x_{i+1}).$$

Summing over all i, we get

$$\frac{1}{t} \sum_{i=0}^{l-1} d(x_i, x_{i+1}) \leq d[t](x, y).$$

Then the triangle inequality of d on G gives (2.4.12). □

Remark 21. *In fact, when t is larger than the diameter D of the graph G, we have a better estimate*

$$\frac{1}{t} d(x, y) \leq \frac{1}{D} d(x, y) \leq 1 \leq d[t](x, y). \tag{2.4.13}$$

Lemma 4. *If $E \subseteq E[t]$, then $d[t](x, y) \leq d(x, y)$.*

`Proof:` The proof is obvious. □

The interesting point of Lemma 4 is that when the Ollivier-Ricci curvature of the graph G is positive, $E \subseteq E[t]$ is satisfied for all t and hence Lemma 4 is applicable. This can be seen as follows. Corollary 2 implies that if $k > 0$ then for all $(x, y) \in E$ we have $\sharp(x, y) \neq 0$ or $c(x) \neq 0$ or $c(y) \neq 0$ which in turn implies that $(x, y) \in E[t]$ for all t, see also Example 3.

Lemma 5. *Let k be a lower bound of κ on G. If $E \subseteq E[t]$, then the curvature $\kappa[t]$ of the neighborhood graph $G[t]$ satisfies*

$$\kappa[t](x,y) \geq 1 - t(1-k)^t, \ \ \forall x, y \in V. \tag{2.4.14}$$

`Proof:` By Lemma 4, Corollary 1 and Lemma 3, we get

$$\begin{aligned} W_1^{d[t]}(\delta_x P^t, \delta_y P^t) &\leq W_1^d(\delta_x P^t, \delta_y P^t) \\ &\leq (1-k)^t d(x,y) \\ &\leq t(1-k)^t d[t](x,y). \end{aligned}$$

We use $W_1^{d[t]}$, W_1^d here to indicate the different cost functions used in these two quantities. In the first inequality above we used that the transportation distance (2.2.6) is linear in the graph distance $d(\cdot,\cdot)$. Recalling the definition of the curvature, we have proved (2.4.14). □

Now we arrive at the point to give an geometric proof of the upper bound of the largest eigenvalue. Using Lemma 5 and $\lambda_1 \geq k$, we know on $G[t]$,

$$\lambda_1[t] \geq 1 - t(1-k)^t.$$

Then by using Corollary 6 (i), we get for any even number t,

$$\lambda_{N-1} \leq 1 + t^{\frac{1}{t}}(1-k).$$

Letting $t \to +\infty$, we get $\lambda_{N-1} \leq 2 - k$.

Using the neighborhood graph technique, we further obtain the following generalization of Theorem 21:

Theorem 22. *Let $k[t]$ be a lower bound of Ollivier-Ricci curvature of the neighborhood graph $G[t]$. Then for all $t \geq 1$ the eigenvalues of Δ on G satisfy*

$$1 - (1-k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1-k[t])^{\frac{1}{t}}. \tag{2.4.15}$$

Moreover, if G is not bipartite, then there exists a $t' \geq 1$ such that for all $t \geq t'$ the eigenvalues of Δ on G satisfy

$$0 < 1 - (1-k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1-k[t])^{\frac{1}{t}} < 2.$$

`Proof:` Combining Theorem 21, and Corollary 6 immediately yields (2.4.15).

The second part of this Theorem is proved in two steps. In the first step, we will show that if G is not bipartite then there exists a t' such that for all $t \geq t'$ the neighborhood graph $G[t]$ of G satisfies $w_{xy}[t] \neq 0$ for all $x, y \in V$, i.e. $G[t]$ is a complete graph and each vertex has a loop. In the second step, we show that any graph that satisfies $w_{xy} \neq 0$ for all $x, y \in V$ has a positive lower curvature bound, i.e. $k > 0$. This then completes the proof.

Step 1: By the definition of the neighborhood graph it is sufficient to show that for all $t \geq t'$ there exists a path of length t between any pair of vertices. Since G is not bipartite it follows from the definition of bipartiteness that there exists a path of even and a path of odd length between any pair of vertices in the graph. Given a path of length L between x and y then we can find a path of length $L+2$ between x and y as follows: We go in L steps from x to y and then from y to one of its neighbors and then back to y. This is a path of length $L+2$ between x and y. Since G is finite, it follows that there exists a t' such that for every pair of vertices there exists paths of length t for all $t \geq t'$.

Step 2: Given a graph that satisfies $w_{xy} \neq 0$ for all $x, y \in V$.

Since each vertex in the graph is a neighbor of all other vertices, it is clear that we can move the excess mass of m_x for distance 1 to anywhere. Therefore

$$W_1(m_x, m_y) \leq 1 - \sum_{x_1 \in V} \frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1 y}}{d_y},$$

which implies

$$\kappa(x, y) \geq \sum_{x_1 \in V} \frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}.$$

By Theorem 14, it follows that the above inequality is in fact an equality. Hence for all $x, y \in V$, we have

$$\kappa(x, y) = \sum_{x_1 \in V} \frac{w_{xx_1}}{d_x} \wedge \frac{w_{x_1 y}}{d_y} \geq N \frac{\min_{x,y} w_{xy}}{\max_x d_x} \geq \frac{\min_{x,y} w_{xy}}{\max_{x,y} w_{xy}} > 0,$$

since the weight is positive for every pair of vertices.

This completes the proof. □

In particular, unless G is bipartite, we can obtain nontrivial lower bounds for λ_1 and nontrivial upper bounds for λ_{N-1} even if $k = k[1]$ is nonpositive.

2.4.4 Estimates for the largest eigenvalue in terms of the number of joint neighbors

In Bauer-Jost [7] it is shown that the next lemma is a simple consequence of Theorem 20.

Lemma 6. *Let u be an eigenfunction of Δ for the eigenvalue λ. Then,*

$$2 - \lambda = \frac{(u, \Delta[2]u)_\mu}{(u, \Delta u)_\mu} = \frac{\sum_{x,y} w_{xy}[2](u(x) - u(y))^2}{\sum_{x,y} w_{xy}(u(x) - u(y))^2}. \tag{2.4.16}$$

Lemma 6 can be used to derive further estimates for the largest eigenvalue λ_{N-1} form above and below. We introduce the following notations:

Definition 12. *Let $\tilde{N}_x$ be the neighborhood of vertex x as in Section 2.2.2. The minimal and the maximal number of joint neighbors of any two neighboring vertices is defined as $\tilde{\sharp}_1 := \min_{x\sim y}(\sharp(x,y) + c(x) + c(y))$ and $\tilde{\sharp}_2 := \max_{x\sim y}(\sharp(x,y) + c(x) + c(y))$, respectively. Furthermore, we define $W := \max_{x,y} w_{xy}$ and $w := \min_{x,y;x\sim y} w_{xy}$.*

Theorem 23. *We have the following estimates for λ_{N-1}:*

(i) If $E(G) \subseteq E(G[2])$ then

$$\lambda_{N-1} \leq 2 - \frac{w^2}{W}\frac{\tilde{\sharp}_1}{\max_x d_x}.$$

(ii) If $E(G[2]) \subseteq E(G)$ then

$$2 - \frac{W^2}{w}\frac{\tilde{\sharp}_2}{\min_x d_x} \leq \lambda_{N-1}$$

Proof. On the one hand, we observe that if $E(G) \subseteq E(G[2])$ then for every pair of neighboring vertices $x \sim y$ in G

$$\frac{w_{xy}[2]}{w_{xy}} = \frac{\sum_z \frac{1}{d_z} w_{xz} w_{zy}}{w_{xy}} \quad \geq \quad \frac{w^2}{W}\frac{\tilde{\sharp}_1}{\max_x d_x}. \tag{2.4.17}$$

On the other hand if $E(G[2]) \subseteq E(G)$ then for every pair of neighboring vertices $x \sim y$ in $G(2)$ we have

$$\frac{w_{xy}[2]}{w_{xy}} = \frac{\sum_z \frac{1}{d_z} w_{xz} w_{zy}}{w_{xy}} \leq \frac{W^2}{w}\frac{\tilde{\sharp}_2}{\min_x d_x}. \tag{2.4.18}$$

Substituting the inequalities (2.4.17) and (2.4.18) in equation (2.4.16) completes the proof. □

For unweighted regular graphs, Theorem 23 (i) improves the estimate $\lambda_{N-1} \leq 2-k$. Since $\lambda_{N-1} \leq 2-k$ trivially holds if $k \leq 0$ we only consider the case when $k > 0$ is a lower curvature bound. The discussion after Lemma 4 shows that $k > 0$ implies that $E(G) \subseteq E(G[2])$ and hence we can apply Theorem 23 (i) in this case. From Theorem 14 it follows that for an unweighted graph

$$\kappa(x,y) \leq \frac{\sharp(x,y)}{d_x \vee d_y} + \frac{c(x)}{d_x} + \frac{c(y)}{d_y}$$

for all pairs of neighboring vertices x, y. In the case of a d-regular graph G this implies that a lower bound k for the Ollivier-Ricci curvature must satisfy,

$$k \leq \frac{\tilde{\sharp}_1}{d}.$$

Hence for an unweighted d-regular graph Theorem 23 implies

$$\lambda_{N-1} \leq 2 - \frac{\tilde{\sharp}_1}{d} \leq 2 - k.$$

We consider the following example.

Example 6. *For the complete unweighted graph $\mathcal{K}_N$, we have $E(\mathcal{K}_N) \subseteq E(\mathcal{K}_N[2])$. We have $\tilde{\sharp}_1 = N - 2$ and $\max_x d_x = N - 1$. Thus, Theorem 23 (i) yields*

$$\lambda_{N-1} \leq \frac{N}{N-1},$$

i. e. the estimate from above is sharp for complete graphs. Now we consider the unweighted complete graph on N vertices with N loops $\mathcal{K}_N^{\text{lazy}}$. We have $E(\mathcal{K}_N^{\text{lazy}}[2]) = E(\mathcal{K}_N^{\text{lazy}})$. Furthermore, we have $\tilde{\sharp}_2 = N$ and $\min_x d_x = N$. Thus, Theorem 23 (ii) yields

$$1 \leq \lambda_{N-1},$$

i. e. the estimate from below is sharp for complete graphs with loops.

2.4.5 An example

In this subsection, we explore a particular example, the circle $\mathcal{C}_5$ with 5 vertices. We show that our estimates using the neighborhood graph method can yield nontrivial estimates although the curvature of the original graph has a non-positive lower curvature bound. We also discuss the growth rate of the lower bound k for the curvature $\kappa[t]$ as $t \to \infty$ on $\mathcal{C}_5$.

We consider the unweighted graph $\mathcal{C}_5$ displayed in Figure 2.4. We know that

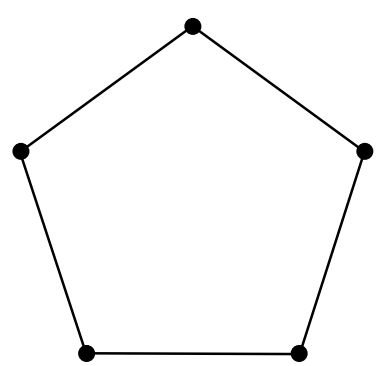

Figure 2.4: Unweighted graph $\mathcal{C}_5$

the first and largest eigenvalue of Δ on $\mathcal{C}_5$ are given by

$$\lambda_1 = 1 - \cos\frac{2\pi}{5} \doteq 0.6910, \ \lambda_4 = 1 - \cos\frac{4\pi}{5} \doteq 1.8090.$$

It is easy to check that the optimal lower bound k for the curvature is 0. So in this case Ollivier's estimates in Theorem 21 only yield trivial estimates.

Now we consider the neighborhood graph $\mathcal{C}_5[2]$ depicted in Figure 2.5 (we change the order of vertices). The weight of every dashed loop is 1 and the

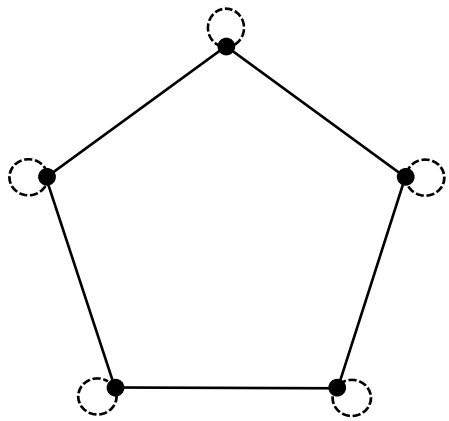

Figure 2.5: Neighborhood graph $\mathcal{C}_5[2]$

weight of every solid edge is 1/2. We can check that the optimal $k[2]$ is 1/4. Then Theorem 22 yields the nontrivial estimate,

$$\lambda_1 \geq 1 - \frac{\sqrt{3}}{2} \doteq 0.1340, \ \lambda_4 \leq 1 + \frac{\sqrt{3}}{2} \doteq 1.8660.$$

Moreover, the neighborhood graph $\mathcal{C}_5[4] = (\mathcal{C}[2])[2]$ is depicted in Figure 2.6. The weight of every dashed loop is 3/4 and the weight of every solid edge is

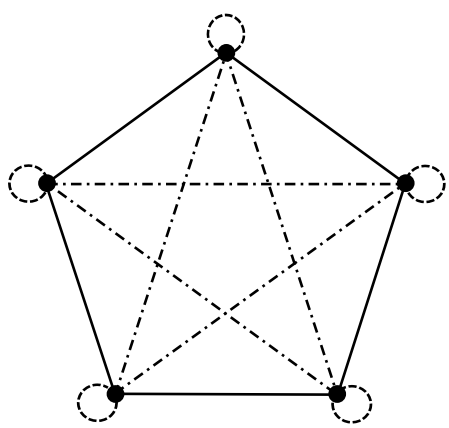

Figure 2.6: Neighborhood graph $\mathcal{C}_5[4]$

1/2 and every dash-dotted edge is 1/8. We can check that the optimal lower curvature bound is $k[4] = 1/2$. So Theorem 22 tells us that

$$\lambda_1 \geq 1 - \frac{1}{\sqrt[4]{2}} \doteq 0.1591, \ \lambda_4 \leq 1 + \frac{1}{\sqrt[4]{2}} \doteq 1.8409.$$

We can also consider the neighborhood graph of odd order $\mathcal{C}_5[3]$, which is depicted in Figure 2.7. The weight of every solid edge is 3/4 and the weight of

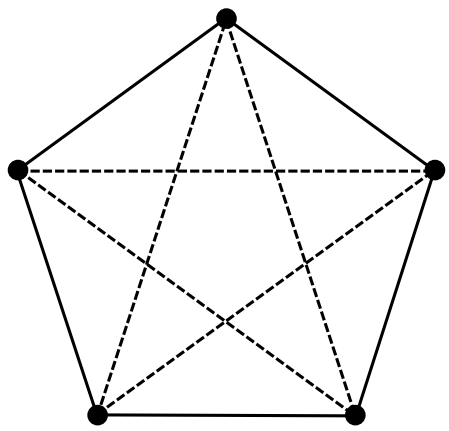

Figure 2.7: Neighborhood graph $\mathcal{C}_5[3]$

every dashed edge is 1/4. Then the optimal lower curvature bound is given by $k[3] = 3/8$. Theorem 22 implies

$$\lambda_1 \geq 1 - \left(\frac{5}{8}\right)^{\frac{1}{3}} \doteq 0.1450, \ \lambda_4 \leq 1 + \left(\frac{5}{8}\right)^{\frac{1}{3}} \doteq 1.8550.$$

From the above calculations, we see that the Ollivier-Ricci curvatures $\kappa[t]$ on the neighborhood graphs give better and better estimates in this example.

Another interesting problem is concerned with the limit of the neighborhood graphs. As shown in Bauer-Jost [7], since $\mathcal{C}_5$ is a regular non-bipartite graph, $\mathcal{C}_5[t]$ will converge to $\mathcal{C}_5[\infty] := \bar{\mathcal{C}}_5$ as $t \to \infty$. In this case $\bar{\mathcal{C}}_5$ is a complete graph and every vertex has a loop and the weight of every edge in $\bar{\mathcal{C}}_5$ is 2/5. We can then check that $\kappa[\infty] = 1$. By Theorem 22, for large enough t, we have

$$0 < 1 - (1 - k[t])^{\frac{1}{t}} \leq \lambda_1 = 1 - \cos\frac{2\pi}{5} < 1. \tag{2.4.19}$$

Therefore $0 < (1 - k[t])^{\frac{1}{t}} < 1$. Intuitively, $k[t]$ should become larger and larger as $t \to \infty$ since the graph $\mathcal{C}_5[t]$ converges to a complete graph and its weights become more and more the same. We suppose $\lim_{t\to\infty}(1 - k[t])^{\frac{1}{t}}$ exists (this is true at least for a subsequence). Then to avoid contradictions in (2.4.19), we know there exists a positive number a such that

$$\lim_{t\to\infty}(1 - k[t])^{\frac{1}{t}} = e^{-a} > 0.$$

That is,

$$\lim_{t\to\infty}\frac{\log(1 - k[t])}{t} = -a,$$

which means $k[t]$ behaves like $1 - P(t)e^{-at}$ as $t \to \infty$ where $P(t)$ is a polynomial in t.

Chapter 3

Nonnegative combinatorial curvature on infinite semiplanar graphs

The contents in this chapter except Section 3.5 are essentially included in the submitted paper Hua-Jost-Liu [67].

In this chapter, we study systematically (infinite) semiplanar graphs G of nonnegative curvature. This curvature condition can either be formulated purely combinatorically, as in the approach of Higuchi [62], or as an Alexandrov curvature condition on the polygonal surface $S(G)$ obtained by assigning length one to every edge and filling in faces. The fact that these two curvature conditions – nonnegative combinatorial curvature of G and nonnegative Alexandrov curvature of $S(G)$ – are equivalent will be systematically exploited in this chapter. First of all, we can then classify such graphs. Curiously, as soon as the maximal degree of a face is at least 43, the graph necessarily has a rather special structure. This will simplify our reasoning considerably. Secondly, as Alexandrov geometry is a natural generalization of Riemannian geometry, we can systematically carry over the geometric function theory of nonnegatively curved Riemannian manifolds to the setting of nonnegatively curved semiplanar graphs. Starting with two basic inequalities, the volume doubling property and the Poincaré inequality, which hold for such spaces, we obtain the Harnack inequality for harmonic functions by Moser's iteration scheme. Here, for defining (sub-, super-)harmonic functions, we use the normalized graph Laplacian operator of G. Our main results then say that a nonnegatively curved semiplanar graph is parabolic in the sense that it does not support any nontrivial positive superharmonic function (equivalently, Brownian motion is recurrent), and that the dimension of the space of harmonic functions of polynomial growth with exponent at most d is bounded for any d. This is an extension of the solution by Colding-Minicozzi [38] of a conjecture of Yau [122] in Riemannian geometry, see also Li [81].

Let us now describe the results in more precise technical terms. In this chap-

ter, we are interested in infinite graphs. Let G be an infinite graph embedded in a 2-manifold $S(G)$ such that each face is homeomorphic to a closed disk with finite edges as the boundary. This includes the case of a planar graph, and we call such a $G = (V, E, F)$ with its sets of vertices V, edges E, and faces F, a semiplanar graph. For each vertex $x \in V$, the combinatorial curvature at x is defined as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where d_x is the degree of the vertex x, $\deg(\sigma)$ is the degree of the face σ, and the sum is taken over all faces incident to x (i.e. $x \in \sigma$). The idea of this definition is to measure the difference of 2π and the total angle Σ_x at the vertex x on the polygonal surface $S(G)$ equipped with a metric structure obtained from replacing each face of G with a regular polygon of side lengths one and gluing them along the common edges. That is,

$$2\pi\Phi(x) = 2\pi - \Sigma_x.$$

Let $\chi(S(G))$ denote the Euler characteristic of the surface $S(G)$. The Gauss-Bonnet formula of G in DeVos-Mohar [46] reads as

$$\sum_{x \in G} \Phi(x) \leq \chi(S(G)),$$

whenever $\Sigma_{x \in G: \Phi(x) < 0} \Phi(x)$ converges. Furthermore, Chen and Chen [20] proved that if the absolute total curvature $\Sigma_{x \in G} |\Phi(x)|$ is finite, then G has only finitely many vertices with nonvanishing curvature. Then Chen [19] obtained the topological classification of infinite semiplanar graphs with nonnegative curvature: $\mathbb{R}^2$, the cylinder without boundary, and the projective plane minus one point. In addition, at the end of the paper [19], he proposed a question on the construction of semiplanar graphs with nonnegative curvature embedded in the projective plane minus one point.

We note that the definition of the combinatorial curvature is equivalent to the generalized sectional (Gaussian) curvature of the surface S(G). The semiplanar graph G has nonnegative combinatorial curvature if and only if the corresponding regular polygonal surface S(G) is an Alexandrov space with nonnegative sectional curvature, i.e. $\mathrm{Sec} S(G) \geq 0$ (or $\mathrm{Sec} G \geq 0$ for short).

This chapter will derive its insights from comparing these curvature notions, and we shall apply the Alexandrov geometry to study the geometric and analytic properties of semiplanar graphs with nonnegative curvature. Firstly, the Cheeger-Gromoll splitting theorem holds on Alexandrov spaces with nonnegative curvature (recall Theorem 8 in Section 1.5). Note that $S(G)$ is a 2-dimensional Alexandrov space with nonnegative curvature if G is a semiplanar graph with nonnegative combinatorial curvature. In Section 3.3, we shall apply this splitting theorem to the surface $S(G)$ and prove that if the semiplanar graph G with nonnegative curvature has at least two ends (geometric ends at infinity), then $S(G)$ is

isometric to the cylinder; this is interesting since we do not use the Gauss-Bonnet formula here. Moreover, we give the metric classification of $S(G)$ for semiplanar graphs G with nonnegative curvature. An orientable $S(G)$ is isometric to a plane, or a cylinder without boundary if it has vanishing curvature everywhere, and isometric to a cap which is homeomorphic but not isometric to the plane if it has at least one vertex with positive curvature. A nonorientable $S(G)$ is isometric to the metric space obtained by gluing in some way the boundary of $[0, a] \times \mathbb{R}$ with vanishing curvature everywhere (see Lemma 13).

Secondly, in Section 3.4, we prove that G inherits some geometric estimates from those of $S(G)$. Let d^G (resp. $d^{S(G)}$) denote the intrinsic metric on the graph G (resp. polygonal surface $S(G)$). It will be proved that these two metrics are bi-Lipschitz equivalent on G, i.e. for any $x, y \in V$,

$$Cd^G(x, y) \leq d^{S(G)}(x, y) \leq d^G(x, y).$$

We denote by $B_R(p) = \{x \in G : d^G(p, x) \leq R\}$ the closed geodesic ball in G and by $B_R^{S(G)}(p) = \{x \in S(G) : d^{S(G)}(p, x) \leq R\}$ the closed geodesic ball in $S(G)$ respectively. The volume of $B_R(p)$ is defined as $|B_R(p)| = \sum_{x \in B_R(p)} d_x$. Note that the Bishop-Gromov volume comparison holds on the n-dimensional Alexandrov space (X, d) with nonnegative curvature (recall Theorem 7 in Section 1.5). Let D_G denote the maximal degree of the faces in G, i.e. $D_G = \max_{\sigma \in F} \deg(\sigma)$ which is finite by Chen-Chen [20]. In this paper, for simplicity we also denote $D := D_G$ when it does not cause any confusion. The relative volume growth property for the graph G is obtained in the following theorem.

Theorem 24. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then for any $p \in G, 0 < r < R$, we have*

$$\frac{|B_R(p)|}{|B_r(p)|} \leq C(D) \left(\frac{R}{r}\right)^2, \tag{3.0.1}$$

$$|B_{2R}(p)| \leq C(D)|B_R(p)|, \tag{3.0.2}$$

$$|B_R(p)| \leq C(D)R^2, \qquad (R \geq 1) \tag{3.0.3}$$

where $C(D)$ is a constant only depending on D.

We also wonder whether the constant $C(D)$ in (3.0.1) could take the value 1 or not for $1 \leq r < R$, $r, R \in \mathbb{Z}$. We find in Section 3.5 that for Archimedean tilings on a plane the answer is not always affirmative but they do satisfy the following weak version which we will refer to as condition (R).

(R) There exists a sequence of integers $\{R_n\}_{n=1,2,\ldots}$ which satisfies

$$\lim_{n \to +\infty} R_n = +\infty \ \text{ and } |R_n - R_{n-1}| \leq c$$

such that for any $1 \leq r < R_n$, $r \in \mathbb{Z}$, we have

$$\frac{|B_{R_n}(p)|}{|B_r(p)|} \leq \left(\frac{R_n}{r}\right)^2.$$

In the above, c is an absolute constant.

Thirdly, in Section 3.6, we show that the Poincaré inequality holds on the semiplanar graph G with nonnegative curvature. The Poincaré inequality has been proved on Alexandrov spaces in Kuwae-Machigashira-Shioya [79], Hua [65], and also on some graphs constructed from the ϵ-nets of Riemannian manifolds with bounded geometry in Coulhon and Saloff-Coste [42] (see Proposition 6.10 there). For the case of Alexandrov spaces, recall Theorem 9 in Section 1.5. Here for any function $f : V \to \mathbb{R}$, we extend it to each edge of G by linear interpolation and then to each face nicely with controlled energy (see Lemma 17). So we get a local $W^{1,2}$ function on $S(G)$ which satisfies the Poincaré inequality (1.5.4), and then it implies the Poincaré inequality on the graph G.

Theorem 25. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then there exist two constants $C(D)$ and $C > 1$ such that for any $p \in G, R > 0, f : B_{CR}(p) \to \mathbb{R}$, we have*

$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 d_x \leq C(D) R^2 \sum_{x,y \in B_{CR}(p); x \sim y} (f(x) - f(y))^2, \tag{3.0.4}$$

where $f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x) d_x$.

Finally, in Section 3.7, we shall study some global properties of harmonic functions on the semiplanar graph G with nonnegative curvature. Let $f : V \to \mathbb{R}$ be a function on the graph G. We will use the normalized Laplace operator Δ defined in Subsection 2.1.3, i.e.

$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)).$$

A function f is called harmonic (subharmonic, superharmonic) if $\Delta f(x) = 0$ ($\geq 0, \leq 0$), for each $x \in G$.

A manifold or a graph is called parabolic if it does not admit any nontrivial positive superharmonic function. The question when a manifold is parabolic has been studied extensively in the literature; in fact, parabolicity is equivalent to recurrency for Brownnian motion (see Grigor'yan [58], Holopainen-Koskela [64], Rigoli-Salvatori-Vignati [104]). Noticing that the semiplanar graph G with nonnegative curvature has the quadratic volume growth (3.0.3), we obtain the following theorem in a standard manner (see Holopainen-Koskela [64]).

Theorem 26. *Any semiplanar graph G with* $\mathrm{Sec}G \geq 0$ *is parabolic.*

Since Yau [119] proved the Liouville theorem for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature, the study of harmonic functions on manifolds has been one of the central fields of geometric analysis. Yau conjectured in [121, 122] that the linear space of polynomial growth harmonic functions with a fixed growth rate on a Riemannian manifold with nonnegative Ricci curvature is of finite dimension. Colding-Minicozzi [38] gave an affirmative answer to the conjecture by the volume doubling property and the Poincaré inequality. After then, Li [81] provided a simplified argument by the mean value inequality (see also Colding-Minicozzi [37, 39]). In this chapter, we call this result the polynomial growth harmonic function theorem. Delmotte [43] proved it in the graph setting by assuming the volume doubling property and the Poincaré inequality. Kleiner [78] generalized it to Cayley graphs of groups of polynomial growth, by which he gave a new proof of Gromov's theorem in group theory. Hua [66] generalized it to Alexandrov spaces and gave the optimal dimension estimate analogous to the Riemannian manifold case.

Let G be a semiplanar graph with nonnegative curvature and $H^d(G) := \{u : Lu \equiv 0, |u(x)| \leq C(d^G(p,x)+1)^d\}$ be the space of polynomial growth harmonic functions of growth degree d on G. By the method of Colding-Minicozzi or Li (see [43]), the volume doubling property (3.0.2) and the Poincaré inequality (3.0.4) imply that $\dim H^d(G) \leq C(D)d^{v(D)}$ for any $d \geq 1$, where $C(D)$ and $v(D)$ depend on D. Instead of the volume doubling property (3.0.2), we use the relative volume comparison (3.0.1) to show that $\dim H^d(G) \leq C(D)d^2$. It seems natural that the dimension estimate of $H^d(G)$ should involve the maximal facial degree D because the relative volume comparison and the Poincaré inequality cannot avoid D. But the estimate is still not satisfactory since $C(D)$ here is only a dimensional constant in the Riemannian case.

Furthermore, we note that a semiplanar graph G with nonnegative curvature and $D_G \geq 43$ has a special structure of linear volume growth like a one-sided cylinder, see Theorem 32. Inspired by the work [108], in which Sormani proved that any polynomial growth harmonic function on a Riemannian manifold with one end and nonnegative Ricci curvature of linear volume growth is constant, we obtain the following theorem.

Theorem 27. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$ *and* $D_G \geq 43$. *Then for any $d > 0$,*

$$\dim H^d(G) = 1.$$

The final dimension estimate follows from combining the previous two estimates.

Theorem 28. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then for any $d \geq 1$,*

$$\dim H^d(G) \leq Cd^2,$$

where C is an absolute constant.

For convenience, we may change the values of the constants $C, C(D)$ from line to line in the sequel.

3.1 Preliminaries

3.1.1 Semiplanar graphs and combinatorial curvature

In this subsection, we make it precise what kind of graphs and curvature we will study in this chapter.

A graph is called planar if it can be embedded in the plane without self-intersection of edges. We define a semiplanar graph similarly. For a given graph $G = (V, E)$, let's consider each edge of E as a closed arc and denote $G_1 := V \cup E$. That is, G_1 is the corresponding 1-dimensional simplicial complex.

Definition 13. *A graph $G = (V, E)$ is called semiplanar if the following restrictions hold*

1. *G_1 can be embedded into a connected 2-manifold S without self-intersections;*

2. *$S \backslash G_1$ is a disjoint union of connected open sets which satisfy that the closure of each such open set has finite edges as the boundary and is homeomorphic to the closed disk.*

The embedding in the definition is called a strong embedding in Chen [19]. We call the closure of each connected open set in $S \setminus G_1$ a face, denoted by σ. Let F be the set of all faces of the graph. Then in the following we will use $G = (V, E, F)$ to denote the semiplanar graph with the set of vertices, V, edges, E and faces, F. Note that edges and faces are regarded as closed subsets of S here. Two objects from V, E, F are called incident if one is a proper subset of the other. In this chapter, essentially for simplicity, we shall always assume that the surface S has no boundary except in Remark 22 and G is a simple graph, i.e. without loops and multi-edges.

We denote by d_x the degree of the vertex $x \in V$ as in the last chapter and by $\deg(\sigma)$ the degree of the face $\sigma \in F$, i.e. the number of edges incident to σ. Further, we assume that $3 \leq d_x < \infty$ and $3 \leq \deg(\sigma) < \infty$ for each vertex x and face σ, which implies that G is a locally finite graph.

To show the second restriction in Definition 13 more clearly, let's look at some examples (or counterexamples). If a graph locally looks like Figure 3.1, then it can't be a semiplanar graph since the face σ surrounded by a, b, c, d, e, a, f, g, h, i, f is not homeomorphic to the closed disk. A finite planar graph like in Figure 3.2 can not be thought as a semiplanar graph embedded into the plane $\mathbb{R}^2$ since then the face σ' surrounded by j, k, l, j will not be homeomorphic to the closed disk. But we can see it as a semiplanar graph embedded into the sphere S^2.

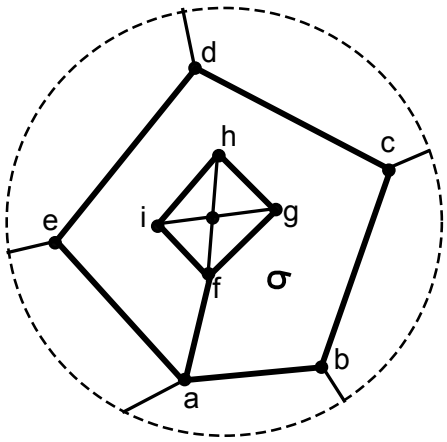

Figure 3.1: Not a semiplanar graph

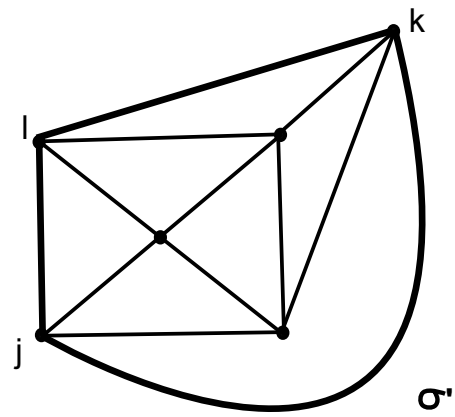

Figure 3.2: A finite planar graph

We only require the 2-manifold S to be connected. It could be noncompact and nonorientable. For example, the projective plane minus one point. For later purposes, we explain it here. It can be seen as the set of all lines going through the origin in $\mathbb{R}^3$ except one axis. Therefore it could also be obtained from a cylinder by identifying every two points which are on a straight line going through the origin, or equivalently, obtained from $[0, 2a] \times \mathbb{R}$ by gluing $(x, y) \in [0, 2a] \times \mathbb{R}$ and $(x - a, -y)$. This could be viewed equivalently further in two ways. One is to glue the boundary of $[0, a] \times \mathbb{R}$ by identifying the points $(0, y)$ and $(a, -y)$ where $x \in \mathbb{R}$. The other one is to glue the boundary S^1 of a half cylinder $S^1 \times [0, +\infty)$ by identifying the antipodal points. In fact, the last way will be the strategy we adopt to draw the figures in Section 3.3.

For each semiplanar graph $G = (V, E, F)$, there is a unique metric space, denoted by $S(G)$, which is obtained from replacing each face of G by a regular polygon of side length one with the same facial degree and gluing the faces along the common edges in S. $S(G)$ is called the regular polygonal surface of the semiplanar graph G. Please note that we have $V \subset G_1 \subset S(G)$. In fact, we have three metric spaces here: (V, d^G), (G_1, d^{G_1}) and $(S(G), d^{S(G)})$. d^G is the graph distance we use in the last chapter. d^{G_1} is the metric on G_1 obtained by assuming each edge has length one. And $d^{S(G)}$ is the intrinsic metric of the regular polygonal surface. For simplicity, we will write $d := d^{S(G)}$ in the following since it will appear frequently. This should not be confused with the simplified notation we use for d^G in Chapter 2. It is easy to see that $d^G(x, y) = d^{G_1}(x, y)$ for any $x, y \in V$. We will discuss the relations of those three metric further in Section 3.2.

Now we turn to the curvature notion we will study in this chapter.

Definition 14. *For a semiplanar graph G, the combinatorial curvature at each*

vertex $x \in G$ is defined as

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where the sum is taken over all the faces incident to x.

This curvature can be read from the corresponding regular polygonal surface $S(G)$ as,

$$2\pi\Phi(x) = 2\pi - \Sigma_x,$$

where Σ_x is the total angle of $S(G)$ at x. Positive curvature thus means convexity at the vertex. We shall prove that the semiplanar graph G has nonnegative curvature everywhere if and only if the regular polygonal surface $S(G)$ is an Alexandrov space with nonnegative curvature, which is a generalized sectional (Gaussian) curvature on metric spaces.

Another observation is that this curvature is closely related to the Gauss-Bonnet formula. Let's consider the simple case of a finite planar graph. Recall that a finite planar graph can be regarded as a semiplanar graph which is embedded into the sphere S^2. Then by a direct calculation and the Euler formula, we have

$$\sum_{x \in V} \Phi(x) = \sharp V - \sharp E + \sharp F = \chi(S^2) = 2.$$

That is, the Gauss-Bonnet formula holds in this simple case. Many previous studies originated from this observation, which we will review in the next subsection.

3.1.2 Gauss-Bonnet formula and basic facts

In the sequel, we will recall the Gauss-Bonnet formula and some basic facts about the combinatorial structure of semiplanar graphs. The Gauss-Bonnet formula for the semiplanar graph has been studied by DeVos-Mohar [46] and Chen-Chen [20]. The following one is due to DeVos-Mohar.

Theorem 29. *Let G be a semiplanar graph, $S(G)$ be the corresponding regular polygonal surface. If G has only finitely many vertices with negative curvature, then there exists a closed 2-manifold M, so that $S(G)$ is homeomorphic to M minus t points, and*

$$\sum_{x \in G} \Phi(x) \leq \chi(S(G)) := \chi(M) - t. \tag{3.1.1}$$

Note that the number t here in fact equal to $N(S(G))$ (recall the definition of end in Section 1.5).

With this formula in hand, DeVos-Mohar [46] proved the Higuchi's conjecture, that is, the semiplanar graph with positive curvature everywhere is finite. This is an analogue of Myers' theorem in Riemannian geometry. After that, Chen [19] further proved

Theorem 30. *If G is a semiplanar graph with nonnegative curvature everywhere, then the number of vertices with nonvanishing curvature is finite.*

In fact, by the Gauss-Bonnet formula, Chen [19] also gave the topological classification of semiplanar graphs with nonnegative curvature.

Theorem 31. *Let G be a semiplanar graph with nonnegative curvature everywhere and $S(G)$ be the regular polygonal surface. Then if G is finite, $S(G)$ is homeomorphic to: sphere, torus, projective plane or Klein bottle. If G is infinite, $S(G)$ is homeomorphic to: $\mathbb{R}^2$, the cylinder without boundary or the projective plane minus one point.*

Now, we turn to the basic facts about the combinatorial structure of semiplanar graphs with positive or nonnegative curvature. Let G be a semiplanar graph and $x \in V$. It is straightforward that $3 \leq d_x \leq 6$ if $\Phi(x) \geq 0$ and $3 \leq d_x \leq 5$ if $\Phi(x) > 0$.

A pattern of a vertex x is a vector $(\deg(\sigma_1), \deg(\sigma_2), \cdots, \deg(\sigma_{d_x}))$, where $\{\sigma_i\}_{i=1}^{d_x}$ are the faces incident to x ordered such that $\deg(\sigma_1) \leq \deg(\sigma_2) \leq \cdots \leq \deg(\sigma_{d_x})$. In fact all possible patterns of a vertex x with positive curvature can be listed as follows (see DeVos-Mohar[46], Chen-Chen [20]).

Patterns		$\Phi(x)$
$(3,3,k)$	$3 \leq k$	$= 1/6 + 1/k$
$(3,4,k)$	$4 \leq k$	$= 1/12 + 1/k$
$(3,5,k)$	$5 \leq k$	$= 1/30 + 1/k$
$(3,6,k)$	$6 \leq k$	$= 1/k$
$(3,7,k)$	$7 \leq k \leq 41$	$\geq 1/1722$
$(3,8,k)$	$8 \leq k \leq 23$	$\geq 1/552$
$(3,9,k)$	$9 \leq k \leq 17$	$\geq 1/306$
$(3,10,k)$	$10 \leq k \leq 14$	$\geq 1/210$
$(3,11,k)$	$11 \leq k \leq 13$	$\geq 1/858$
$(4,4,k)$	$4 \leq k$	$= 1/k$
$(4,5,k)$	$5 \leq k \leq 19$	$\geq 1/380$
$(4,6,k)$	$6 \leq k \leq 11$	$\geq 1/132$
$(4,7,k)$	$7 \leq k \leq 9$	$\geq 1/252$
$(5,5,k)$	$5 \leq k \leq 9$	$\geq 1/90$
$(5,6,k)$	$6 \leq k \leq 7$	$\geq 1/105$
$(3,3,3,k)$	$3 \leq k$	$= 1/k$
$(3,3,4,k)$	$4 \leq k \leq 11$	$\geq 1/132$
$(3,3,5,k)$	$5 \leq k \leq 7$	$\geq 1/105$
$(3,4,4,k)$	$4 \leq k \leq 5$	$\geq 1/30$
$(3,3,3,3,k)$	$3 \leq k \leq 5$	$\geq 1/30$

And all possible patterns of a vertex with vanishing curvature can be listed as (see Grünbaum-Shephard [61],Chen-Chen [20]):

$$(3,7,42),(3,8,24),(3,9,18),(3,10,15),(3,12,12),(4,5,20),(4,6,12),$$
$$(4,8,8),(5,5,10),(6,6,6),(3,3,4,12),(3,3,6,6),(3,4,4,6),(4,4,4,4),$$
$$(3,3,3,3,6),(3,3,3,4,4),(3,3,3,3,3,3).$$

With the help of those two list, we can immediately observe the following fact. Let $G=(V,E,F)$ be a semiplanar graph. We denote by $D_G := \sup\{\deg(\sigma) : \sigma \in F\}$ the maximal degree of faces in G. If G has nonnegative curvature everywhere, then $D_G < \infty$. This is because that by Theorem 30, G has at most finitely many vertices with nonvanishing curvature, and the patterns of vertices with vanishing curvature is finite.

We recall a lemma in Chen-Chen[20] which is another consequence of the above two lists. For simplicity we denote $SecG \geq 0$ for nonnegative curvature everywhere in the following.

Lemma 7. *Let G be a semiplanar graph with $SecG \geq 0$ and σ be a face of G with* $\deg(\sigma) \geq 43$. *Then*

$$\sum_{x\in\sigma} \Phi(x) \geq 1.$$

Proof. For completeness, we give the proof of the lemma. Since the curvature is nonnegative and $\deg(\sigma) \geq 43$, the only possible patterns of the vertices incident to the face σ are: $(3,3,k),(3,4,k),(3,5,k),(3,6,k),(4,4,k)$ and $(3,3,3,k)$, where $k=\deg(\sigma)$. In each case, we have $\Phi(x) \geq \frac{1}{k}$, for $x \in \sigma$. Hence, we get

$$\sum_{x\in\sigma} \Phi(x) \geq 1.$$

□

Then we get the following lemma.

Lemma 8. *Let G be an infinite semiplanar graph with $SecG \geq 0$. Then either $D_G \leq 42$, or G has a unique face σ with* $\deg(\sigma) \geq 43$ *and has vanishing curvature elsewhere.*

Proof. If G has a face σ with $\deg(\sigma) \geq 43$, then by Lemma 7

$$\sum_{x\in\sigma} \Phi(x) \geq 1.$$

Since G is an infinite graph with nonnegative curvature, by the Gauss-Bonnet formula (3.1.1), we have

$$\sum_{x\in G} \Phi(x) \leq 1,$$

because $\chi(M) \leq 2$ and $t \geq 1$. Hence $\sum_{x\in\sigma} \Phi(x) = 1$ and $\Phi(y) = 0$ for any $y \notin \sigma$. Furthermore, the only possible patterns of the vertices incident to σ are: $(3,6,k),(4,4,k),(3,3,3,k)$, because the other three patterns $(3,3,k),(3,4,k)$, $(3,5,k)$ have curvature strictly larger than $\frac{1}{k}$, where $k = \deg(\sigma)$. $\square$

We will show that Lemma 8 implies the semiplanar graph G with nonnegative curvature and large face degree, i.e. $D_G \geq 43$, has a rather special structure, see Figure 3.3.

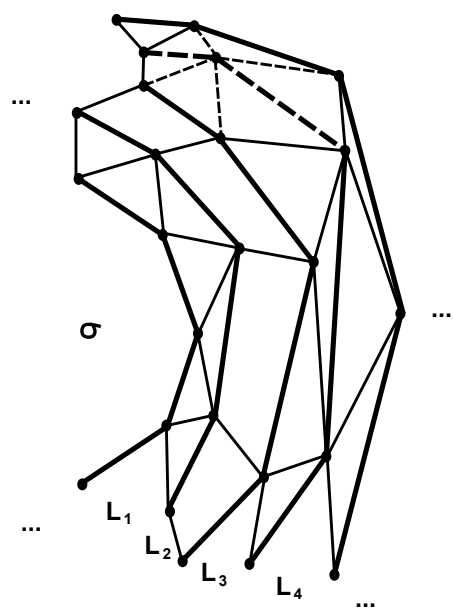

Figure 3.3: A special structure when $\deg(\sigma) \geq 43$

For this purpose, we first introduce a notation of graph operation. Let $\mathcal{G}$ denote the set of semiplanar graphs. We define a graph operation on $\mathcal{G}$, $T : \mathcal{G} \to \mathcal{G}$. For any $G \in \mathcal{G}$, we choose a (possibly infinite) subcollection of hexagonal faces of G, add new vertices at the barycenters of the hexagons, and join them to the vertices of the hexagons by new edges. In such a way, we obtain a new semiplanar graph, denoted by $T(G)$, which replaces each hexagon chosen in G by six triangles. We note that $T : \mathcal{G} \to \mathcal{G}$ is a multivalued map depending on which subcollection of hexagons we choose. The inverse map of T, denoted by $T^{-1} : \mathcal{G} \to \mathcal{G}$, is defined as a semiplanar graph $T^{-1}(G)$ obtained from replacing couples of six triangles incident to a common vertex of pattern $(3,3,3,3,3,3)$ in G by a hexagon (we require that the hexagons do not overlap). For example, in Figure 3.3, one possible value of T^{-1} is obtained by removing the dashed edges and their common vertex. It is easy to see that $S(T(G))$ and $S(T^{-1}(G))$ are isometric to $S(G)$ which implies that the graph operations T and T^{-1} preserve the curvature condition, i.e. $\mathrm{Sec}S(T(G)) \geq 0$ (or $\mathrm{Sec}S(T^{-1}(G)) \geq 0$) $\Longleftrightarrow$ $\mathrm{Sec}S(G) \geq 0$.

Now for a semiplanar graph G with $SecG \geq 0$ and $D_G \geq 43$, Lemma 8 shows that there is a unique large face σ such that $\deg(\sigma) = D_G = k \geq 43$ and the only patterns of vertices of σ are: $(3,6,k),(4,4,k)$ and $(3,3,3,k)$. Without loss of generality, by the graph operation T, it suffices to assume that the semiplanar graph G has no hexagonal faces. It is easy to show that if one of the vertices of σ is of pattern $(4,4,k)$ (or $(3,3,3,k)$), the other vertices incident to σ are of the same pattern. As shown in Figure 3.3, we denote by L_1 the set of faces attached to the large face σ, which are of the same type (triangle or square) and for which the boundary of $\sigma \cup L_1$ has the same number of edges as the boundary of σ. By Lemma 8, G has vanishing curvature except at the vertices incident to σ. Hence, $\sigma \cup L_1$ is in the same situation as σ. To continue the process, we denote by L_2 the set of faces attached to $\sigma \cup L_1$ which are of the same type (triangle or square). In this way, we obtain an infinite sequence of sets of faces, $\sigma, L_1, L_2, \ldots, L_m, \ldots,$ where L_m are the sets of faces of the same type (triangle or square) for $m \geq 1$. L_m and L_n $(m \neq n)$ may be different since they are independent.

Theorem 32. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$ *and $D_G \geq 43$, and let σ be the face of maximal degree. Then either G is constructed from a sequence of sets of faces, $\sigma, L_1, L_2, \cdots, L_m, \cdots$, where L_m are the sets of faces of the same type (triangle or square), denoted by $S(G) = \sigma \cup \bigcup_{m=1}^{\infty} L_m$, or $G = T^{-1}(G')$ where G' is constructed as above.*

3.2 An Alexandrov geometry approach

In the following, we will mainly focus on infinite semiplanar graphs with nonnegative curvature. Unlike the previous studies, we adopt another strategy, that is, an Alexandrov geometry approach. We will prove that any regular polygonal surface is a complete geodesic space and the combinatorial curvature definition is consistent with the sectional curvature in the sense of Alexandrov. This is the starting point of our approach.

Let G be a semiplanar graph, recall the three metric spaces associated to it: (V, d^G), (G_1, d^{G_1}) and $(S(G), d)$. We have already pointed out the restriction of d^{G_1} on V is the same as d^G. But the restriction of the intrinsic metric d of $S(G)$ on G_1 gives a different metric d on G_1. The following lemma says that this restriction metric is bi-Lipschitz equivalent with d^{G_1}. This point will be very useful in the subsequent arguments.

Lemma 9. *Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface of G. Then there exists a constant C such that for any $x, y \in G_1$,*

$$Cd^{G_1}(x,y) \leq d(x,y) \leq d^{G_1}(x,y). \tag{3.2.1}$$

To prove the lemma, we need the following lemma in Euclidean geometry.

Lemma 10. *Let $\triangle_n \subset \mathbb{R}^2$ be a regular n-polygon of side length one ($n \geq 3$). A straight line L intersects the boundary of $\triangle_n$ at two points, A and B. Denote by $|AB| = d$ the length of the segment AB, by l_1, l_2 the length of the two paths P_1, P_2 on the boundary of $\triangle_n$ joining A and B. Then we have*

$$C \min\{l_1, l_2\} \leq d \leq \min\{l_1, l_2\}, \tag{3.2.2}$$

where the constant C does not depend on n.

Before going into the details of the proof, keep in mind the fact that the distance d between two points on a circle and the length l of the corresponding shorter arc satisfies

$$\frac{d}{l} \geq \frac{2}{\pi}.$$

The proof of the lemma is in some sense a kind of technical modification of this fact.

Proof. It suffices to prove that $d \geq C \min\{l_1, l_2\}$. Without loss of generality, we may assume $l_1 \leq l_2$. If the shorter path P_1 contains no full edges of $\triangle_n$, i.e. A and B are on adjacent edges, then P_1 and AB form a triangle. Denote by a, b the lengths of the two sides in P_1 and by α the angle opposite to AB. Then we have $\alpha = \frac{(n-2)\pi}{n}$ and $l_1 = a + b$. By the cosine rule, we obtain that

$$d \geq a - b\cos\alpha,$$

$$d \geq b - a\cos\alpha.$$

Then it follows that

$$2d \geq (a+b)(1-\cos\alpha) \geq (a+b)(1-\cos\frac{\pi}{3}) = \frac{1}{2}l_1.$$

Hence

$$d \geq \frac{1}{4}l_1. \tag{3.2.3}$$

If $n = 3$, this lemma has been proved.

For $n \geq 4$, if P_1 contains at least one full edge, we consider the following cases.

Case 1. $n \leq 6$.

We choose one full edge in P_1 and extend it to a straight line, then project the path P_1 onto the line. It is easy to show that

$$d \geq |ProjP_1| \geq 1,$$

where $ProjP_1$ is the projection of the path P_1. Since $n \leq 6$, we have $l_1 \leq 3$ and

$$d \geq 1 \geq \frac{l_1}{3}. \tag{3.2.4}$$

Case 2. $n \geq 7$.

Denote by l the number of full edges contained in P_1. We draw the circumscribed circle of $\triangle_n$, denoted by C_n, with center O of radius R_n, where $2R_n \sin\frac{\pi}{n} = 1$. Let the straight line L (passing through A and B) intersect the circle C_n at C and D (C is close to A). Denote by d' the length of the segment CD, by θ the angle of $\measuredangle COD$ and by l' the length of the arc $\widehat{CD}$.

Case 2.1. $l \geq 3$.

On one hand, by $l \geq 3$, we have $\theta \geq l\frac{2\pi}{n} \geq 3\frac{2\pi}{n}$. Hence,

$$d' = 2R_n \sin\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\sin\frac{\pi}{n}} \geq \frac{\sin 3\frac{\pi}{n}}{\sin\frac{\pi}{n}} = 3 - 4\sin^2\frac{\pi}{n} \geq 3 - 4\sin^2\frac{\pi}{7} \geq 2.24.$$

On the other hand, by $|AC| \leq 1$ and $|BD| \leq 1$, we obtain that

$$d' - d = |AC| + |BD| \leq 2.$$

Then we have

$$\frac{d}{d'} \geq 1 - \frac{2}{d'} \geq 1 - \frac{2}{2.24} = C. \tag{3.2.5}$$

Since $d' = 2R_n \sin\frac{\theta}{2}$ and $l' = R_n\theta$, we have

$$\frac{d'}{l'} = \frac{2\sin\frac{\theta}{2}}{\theta} \geq \frac{2 \cdot \frac{2}{\pi} \cdot \frac{\theta}{2}}{\theta} = \frac{2}{\pi}. \tag{3.2.6}$$

In addition,

$$l' \geq l \geq l_1 - 2 \geq \frac{l_1}{3}, \tag{3.2.7}$$

where the last inequality follows from $l_1 \geq l \geq 3$.

Hence, by (3.2.5), (3.2.6) and (3.2.7), we have

$$d \geq Cl_1. \tag{3.2.8}$$

Case 2.2. $l = 1$.

We denote by EF the full edge contained in P_1 (E is close to A) and extend AE and BF to intersect at the point H. It is easy to calculate the angle $\measuredangle EHF = \pi - \frac{4\pi}{n} \geq \pi - \frac{4\pi}{7}$. By an argument similar to the beginning of the proof, we obtain that

$$d = |AB| \geq \frac{1}{2}(|AH| + |BH|)(1 - \cos(\pi - \frac{4\pi}{7})) \geq C(|AE| + |EF| + |FB|) = Cl_1, \tag{3.2.9}$$

where the last inequality follows from the triangle inequality.

Case 2.3. $l = 2$.

We denote by EF and FH the full edges contained in P_1 (E is close to A) and extend AE and BH to intersect at the point K. Easy calculation shows that $\measuredangle EKH = \pi - \frac{6\pi}{n} \geq \pi - \frac{6\pi}{7}$. By the same argument, we get

$$d = |AB| \geq \frac{1}{2}(|AK| + |BK|)(1 - \cos(\pi - \frac{6\pi}{7})) \geq Cl_1. \tag{3.2.10}$$

Hence, by (3.2.3), (3.2.4), (3.2.8), (3.2.9) and (3.2.10), we obtain that

$$d \geq Cl_1,$$

where C is an absolute constant. Then the lemma follows. □

Proof of Lemma 9. For any $x, y \in G_1$, it is obvious that $d(x,y) \leq d^{G_1}(x,y)$. Hence it suffices to show the inequality in the opposite direction. Let $\gamma : [a,b] \to S(G)$ be a geodesic joining x and y. By the local finiteness assumption of the graph G, there exist finitely many faces that cover the geodesic γ. There is a partition $\{y_i\}_{i=0}^N$ of $[a,b]$, where $a = y_0 < y_1 < \cdots < y_N = b$, such that $\gamma|_{[y_{i-1},y_i]}$ is a segment on the face σ_i and $\gamma(y_{i-1}), \gamma(y_i)$ are on the boundary of σ_i, for $1 \leq i \leq N$. For each $1 \leq i \leq N$, we choose the shorter path, denoted by l_i, on the boundary of the face σ_i which joins $\gamma(y_{i-1})$ and $\gamma(y_i)$. By Lemma 10, we get

$$CL(l_i) \leq d(\gamma(y_{i-1}), \gamma(y_i)) \leq L(l_i),$$

where $L(l_i)$ is the length of l_i. Connecting l_i, we obtain a path l in G_1 joining x and y. Then we have

$$L(l) = \sum_{i=1}^N L(l_i) \leq \frac{1}{C}\sum_{i=1}^N d(\gamma(y_{i-1}), \gamma(y_i)) = \frac{1}{C}d(x,y).$$

Hence,

$$d^{G_1}(x,y) \leq L(l) \leq \frac{1}{C}d(x,y).$$

□

Theorem 33. *Let $G = (V,E,F)$ be a semiplanar graph and $S(G)$ be the regular polygonal surface. Then $(S(G), d)$ is a complete metric space.*

Proof. We denote by $S(G) = \bigcup_{\sigma \in F} \sigma$ the regular polygonal surface of G, by $\overline{S(G)}$ the completion of $S(G)$ with respect to the metric d. Let $(\sigma)_{\epsilon_0}$ denote the ϵ_0-neighborhood of σ in $\overline{S(G)}$, for $\epsilon_0 > 0$. To prove the theorem, it suffices to show that there exists a constant ϵ_0 such that for any face $\sigma \in F$ we have $(\sigma)_{\epsilon_0} \subset S(G)$.

For any $\sigma \in F$, let $Q = \bigcup\{\tau \in F : \tau \cap \sigma \neq \emptyset\}$. By the local finiteness of G, Q is a union of finitely many faces and the boundary of Q, ∂Q, has finitely many edges. It is easy to see that $d^{G_1}(\partial Q, \partial\sigma) = \inf\{d^{G_1}(x,y) : x \in \partial Q, y \in \partial\sigma\} \geq 1$. By Lemma 9, we obtain that for any $x \in \partial Q, y \in \partial\sigma$,

$$d(x,y) \geq C = 2\epsilon_0,$$

where we choose $\epsilon_0 = \frac{C}{2}$. Then we have

$$d(\overline{S(G)} \setminus Q, \sigma) = \inf \left\{ d(x,y) : x \in \overline{S(G)} \setminus Q, y \in \sigma \right\} \geq 2\epsilon_0 > \epsilon_0.$$

Hence, it follows that

$$(\sigma)_{\epsilon_0} \subset Q \subset S(G).$$

□

Corollary 8. *Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface. Then G has nonnegative curvature everywhere if and only if $S(G)$ is an Alexandrov space with nonnegative curvature.*

Proof. By Theorem 33, $S(G)$ is a complete metric space. It is obvious that $S(G)$ is a geodesic space. Suppose G has nonnegative curvature everywhere. At each point except the vertices, there is a neighborhood which is isometric to the flat disk in $\mathbb{R}^2$. At the vertex $x \in G$, the curvature condition $\Phi(x) \geq 0$ is equivalent to $\Sigma_x \leq 2\pi$. Then there is a neighborhood of x (isometric to a conic surface in $\mathbb{R}^3$) satisfying the Toponogov triangle comparison with respect to the model space $\mathbb{R}^2$. Hence, $S(G)$ is an Alexandrov space with $\mathrm{Sec}S(G) \geq 0$. Conversely, if $S(G)$ is an Alexandrov space with $\mathrm{Sec}S(G) \geq 0$, then the total angle of each point of $S(G)$ is at most 2π, which implies the nonnegative curvature condition at the vertices. □

3.3 Cheeger-Gromoll splitting theorem and metric calssification of infinite semiplanar graphs with nonnegative curvature

In the following, we investigate the metric structure of regular polygonal surfaces by Alexandrov space methods, which is independent of the Gauss-Bonnet formula.

Lemma 11. *Let $G = (V, E, F)$ be a semiplanar graph, G_1 be the 1-dimensional simplicial complex and $S(G)$ be the regular polygonal surface. Then we have*

$$N(G_1) = N(S(G)).$$

Proof. It is easy to show that $N(S(G)) \leq N(G_1)$, since $G_1 \subset S(G)$. So it suffices to prove that $N(G_1) \leq N(S(G))$.

Let $\{B^{G_1}_{R_i}(p)\}_{i=1}^{\infty}$ be an exhaustion of G_1, such that $G_1 \setminus B^{G_1}_{R_i}(p)$ has N_i different connected components connecting to infinity, denoted by $E^i_1, \cdots, E^i_{N_i}$, and $N(G_1) = \lim_{i\to\infty} N_i$. By the local finiteness of G, $N_i < \infty$. For any $i \geq 1$, let $Q_i = \bigcup\{\sigma \in F : \sigma \cap B^{G_1}_{R_i}(p) \neq \emptyset\}$, i.e. the union of the faces attached to $B^{G_1}_{R_i}(p)$. By the local finiteness of G, Q_i is compact. We shall prove that $S(G) \setminus Q_i$ has

at least N_i different connected components connecting to infinity, then we have $N(S(G)) \geq N_i$ for any $i \geq 1$, which implies the lemma.

For fixed $i \geq 1$, let $H_j := E_j^i \cap (S(G) \setminus Q_i), j = 1, \cdots, N_i$. It is easy to see that $H_j \neq \emptyset$, since E_j^i is connecting to infinity for $1 \leq j \leq N_i$. We shall prove that for any $j \neq k, H_j$ and H_k are disconnected in $S(G) \setminus Q_i$. Suppose it is not true, then there exist $x \in H_j$, $y \in H_k$ and a curve $\gamma : [a, b] \to S(G)$ in $S(G) \setminus Q_i$ joining x and y, i.e.

$$\gamma \cap Q_i = \emptyset. \tag{3.3.1}$$

As in the proof of Lemma 9, we can find a curve $\gamma' : [a, b] \to G_1$ in G_1 such that γ' and γ pass through the same faces, i.e. for any $t \in [a, b]$, there is a face τ such that $\gamma(t) \in \tau$ and $\gamma'(t) \in \tau$. Since H_j and H_k are disconnected in $G_1 \setminus B_{R_i}^{G_1}(p)$, we have $\gamma'(t_0) \in B_{R_i}^{G_1}(p)$, for some $t_0 \in [a, b]$. Then there exists a face τ such that $\gamma(t_0) \in \tau$ and $\gamma'(t_0) \in \tau$. Hence $\tau \subset Q_i$ and $\gamma \cap Q_i \neq \emptyset$, which contradicts to (3.3.1).

□

By this lemma, we can apply the Cheeger-Gromoll splitting theorem to the polygonal surface of the semiplanar graph with nonnegative curvature.

Theorem 34. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$, *$S(G)$ be the regular polygonal surface. If $N(G_1) \geq 2$, then $S(G)$ is isometric to a cylinder without boundary.*

Proof. By Lemma 11, it follows from $N(G_1) \geq 2$ that $N(S(G)) \geq 2$. A standard Riemannian geometry argument proves the existence of an infinite geodesic $\gamma : (-\infty, \infty) \to S(G)$. Since $S(G)$ is an Alexandrov space with nonnegative curvature, the Cheeger-Gromoll splitting theorem, Theorem 8 in Section 1.5, shows that $S(G)$ is isometric to $Y \times \mathbb{R}$, where Y is a 1-dimensional Alexandrov space without boundary, i.e. straight line or circle. Because $N(S(G)) \geq 2$, Y must be a circle. Hence, $S(G)$ is isometric to a cylinder without boundary. □

Remark 22. *Since the Cheeger-Gromoll splitting theorem holds for Alexandrov spaces with boundary, we may formulate the above theorem in the case of regular polygonal surfaces with boundary (homeomorphic to a manifold with boundary). For the vertex x on the boundary, we define the combinatorial curvature as*

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{deg(\sigma)} = \frac{\pi - \Sigma_x}{2\pi},$$

where Σ_x is the total angle at x. Let G be a semiplanar graph with nonnegative curvature everywhere and $N(G_1) \geq 2$, then the polygonal surface $S(G)$ is isometric to either the cylinder without boundary or the cylinder with boundary, i.e. $[a, b] \times \mathbb{R}$.

Next we consider the tilings (or tessellations) of the plane (see Grünbaum-Shephard [61]) and the construction of semiplanar graphs with nonnegative curvature.

Let G be a semiplanar graph with nonnegative curvature and $S(G)$ be the regular polygonal surface of G. If $S(G)$ is isometric to the plane, $\mathbb{R}^2$, then G is just a tiling of the plane by regular polygons called a regular tiling. Then G has vanishing curvature everywhere. There are infinitely many tilings of the plane. A classification is possible only for regular ones. In this paper, we only consider regular tilings. A tiling is called monohedral if all tiles are congruent. The only three monohedral tilings are by triangles, squares or hexagons. There are 11 distinct tilings such that all vertices are of the same pattern:

$$(3^6), (3^4, 6), (3^3, 4^2), (3^2, 4, 3, 4), (3, 4, 6, 4), (3, 6, 3, 6), (3, 12^2), (4^4), (4, 6, 12),$$
$$(4, 8^2), (6^3).$$

They are called Archimedean tilings and they clearly include the three monohedral tilings.

If $S(G)$ has at least two ends, then by Theorem 34 it is isometric to a cylinder without boundary and G has vanishing curvature everywhere. If $S(G)$ is nonorientable, then by the Gauss-Bonnet formula (3.1.1) $S(G)$ is homeomorphic to the projective plane minus one point and G has vanishing curvature everywhere.

Conversely, if G has vanishing curvature everywhere, then so does $S(G)$. Hence, $S(G)$ is isometric to $\mathbb{R}^2$, or a cylinder if it is orientable. $S(G)$ is homeomorphic to the projective plane minus one point if it is nonorientable.

In addition, if G has positive curvature somewhere, then so does $S(G)$, which implies that $S(G)$ is not isometric to $\mathbb{R}^2$, but by the Gauss-Bonnet formula (3.1.1), it is homeomorphic to $\mathbb{R}^2$. We call it a cap.

An isometry of $\mathbb{R}^2$ is a mapping of $\mathbb{R}^2$ onto itself which preserves the Euclidean distance. All isometries of $\mathbb{R}^2$ form a group. It is well known that every isometry of $\mathbb{R}^2$ is of one of four types: 1. rotation, 2. translation, 3. reflection in a given line, 4. glide reflection, i.e. a reflection in a given line composed with a translation parallel to the same line (see Grünbaum-Shephard [61]).

For any tiling Σ, an isometry is called a symmetry of Σ if it maps every tile of Σ onto a tile of Σ. It is easy to see that all symmetries of Σ form a subgroup of isometries of $\mathbb{R}^2$. We denote by $S(\Sigma)$ the group of symmetries of Σ. For any $\iota \in S(\Sigma)$, we denote by $< \iota >$ the subgroup of $S(\Sigma)$ generated by the symmetry ι. The metric quotient of $\mathbb{R}^2$ by $< \iota >$, denoted by $\mathbb{R}^2/ < \iota >$, is a metric space with quotient metric obtained by the group action $< \iota >$ (see Burago-Burago-Ivanov [13]). The following lemma shows the construction of the tilings of a cylinder.

Lemma 12. *There is a correspondence between a planar tiling Σ with a translation symmetry T, (Σ, T) and a tiling of a cylinder.*

Proof. For any planar tiling Σ with a translation symmetry T, the metric quotient $\mathbb{R}^2/ < T >$ is isometric to a cylinder. The tiling Σ induces a tiling of $\mathbb{R}^2/ < T >$.

Conversely, given a tiling Σ' of a cylinder W, we lift W to its universal cover $\mathbb{R}^2$ by a map $\pi : \mathbb{R}^2 \to W$. It is easy to see that π is locally isometric, since W is flat. The tiling Σ' can be lifted by π to a tiling Σ of $\mathbb{R}^2$, which has a translation symmetry by construction. □

Next we consider the metric structure of the semiplanar graph with nonnegative curvature such that the corresponding regular polygonal surface is nonorientable, i.e. homeomorphic to the projective plane minus one point.

Lemma 13. *There is a correspondence between a planar tiling Σ with a glide reflection symmetry ι, (Σ, ι) and a tiling of the projective plane minus one point with nonnegative curvature.*

Proof. Let Σ be a planar tiling with symmetry of a glide reflection

$$\iota = T_{a,L} \circ F_L = F_L \circ T_{a,L},$$

where $a > 0$, L is a straight line, $T_{a,L}$ is a translation along L through distance a and F_L is a reflection in the line L. The metric quotient $\mathbb{R}^2/ < \iota >$ is isometric to the metric space obtained from gluing the boundary of $[0, a] \times \mathbb{R}$, which is perpendicular to the line L, by the glide reflection ι. It is easy to see that $\mathbb{R}^2/ < \iota >$ is homeomorphic to the projective plane minus one point and has vanishing curvature everywhere. Hence the planar tiling Σ and the symmetry ι of Σ induce a tiling of $\mathbb{R}^2/ < \iota >$.

Conversely, let Σ' be a tiling of $\mathbb{R}P^2 \backslash \{o\}$, with nonnegative curvature (actually with vanishing curvature everywhere). We construct a covering map of $\mathbb{R}P^2 \setminus \{o\}$ with a $\mathbb{Z}_2$ action,

$$\pi : S^2 \setminus \{S, N\} \to \mathbb{R}P^2 \setminus \{o\},$$

where S and N are the south and north pole of S^2. We lift the tiling Σ' to a tiling Σ'' of $S^2 \setminus \{S, N\}$. Since Σ' has vanishing curvature everywhere, so does the lifted tiling Σ''. Note that $S^2 \setminus \{S, N\}$ has two ends. By Theorem 34, the regular polygonal surface $S(\Sigma'')$ is isometric to a cylinder, denoted by $(\frac{a}{\pi} S^1) \times \mathbb{R}$. By Lemma 12, the tiling of a cylinder Σ'' induces a planar tiling Σ''' and a translation symmetry T_{2a} with T_{2a}-invariant domain $[0, 2a] \times \mathbb{R} \subset \mathbb{R}^2$. Since the $\mathbb{Z}_2$ action of π, the tiling Σ''' has a glide reflection symmetry

$$\iota = F_L \circ T_{a,L}$$

where L is parallel to the direction of the translation T_{2a}. □

By the discussion above, we obtain the metric classification of $S(G)$ for a semiplanar graph G with nonnegative curvature.

Theorem 35. *Let G be an infinite semiplanar graph with nonnegative curvature and $S(G)$ be the regular polygonal surface of G. If G has positive curvature somewhere, then $S(G)$ is isometric to a cap which is homeomorphic but not isometric to the plane. If G has vanishing curvature everywhere, then $S(G)$ is isometric to a plane, or a cylinder without boundary if it is orientable, and $S(G)$ is isometric to a metric space obtained from gluing the boundary of $[0, a] \times \mathbb{R}$ by a glide reflection, $\iota = T_{a,L} \circ F_L$, where L is perpendicular to the cylinder, if it is nonorientable.*

At the end of the paper [19], Chen raised a question on the classification of infinite graphs with nonnegative curvature everywhere which can be embedded into the projective plane minus one point. By Lemma 13, it suffices to find the planar tiling with a glide reflection symmetry.

Theorem 36. *The monohedral tilings of the projective plane minus one point with nonnegative curvature are of three types: triangle, square, hexagon.*

Proof. By Lemma 13, the monohedral tiling of the projective plane minus one point with nonnegative curvature is induced by the monohedral tiling of the plane of triangles, of squares or of hexagons and a glide reflection for the tiling. □

Chen [19] gave two classes of monohedral tilings of the projective plane with nonnegative curvature: PS_n (n is even) and PH_n (n is odd). PS_n is induced by the monohedral tiling of the plane of squares. In fact, PH_n (n is odd) is a proper subset of monohedral tilings of the projective plane minus one point which are induced by the monohedral tiling of the plane by hexagons. We give an example below (see Figure 3.4, 3.5) which is induced by the tiling of the plane by hexagons, but is not included in PH_n (n is odd). Let PT, PS, PH denote the tilings of the projective plane minus one point which are induced by the monohedral tiling of the plane of triangles, squares, hexagons and a glide reflection symmetry. They provide the complete classification of monohedral tilings of the projective plane minus one point with nonnegative curvature.

In addition, as the Archimedean tilings of the plane, we can classify the tilings of the projective plane minus one point with nonnegative curvature for which each vertex has the same pattern.

Theorem 37. *The tilings of the projective plane minus one point with nonnegative curvature such that the pattern of each vertex is the same are induced by the Archimedean tilings of the plane and a gilde reflection symmetry.*

We give two examples of tilings of the projective plane minus one point which are induced by the Archimedean tilings and glide reflection symmetries (see Figure 3.6, 3.7, 3.8, 3.9). It is easy to see that there are infinitely many tilings of the projective plane minus one point with nonnegative curvature because of the complexity of the tilings of the plane. Another way to see the complexity is that we can apply the graph operation T on the tiling of the projective plane minus one point with hexagonal faces to obtain a new one.

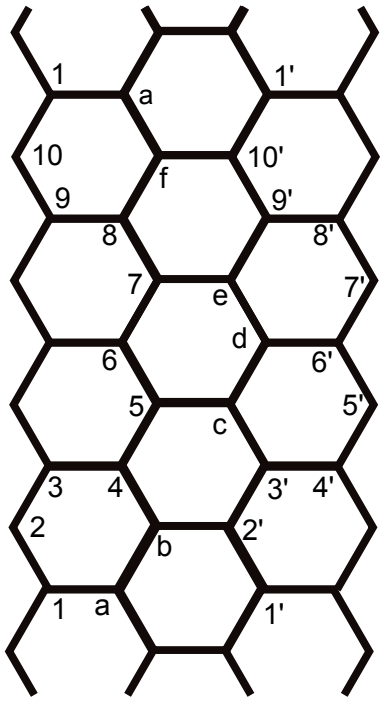

Figure 3.4: (6,6,6)

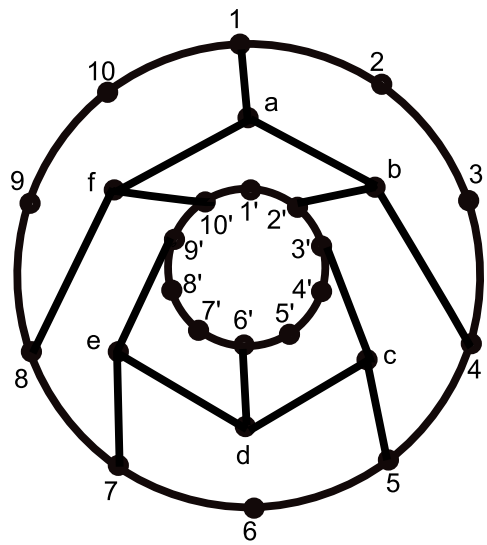

Figure 3.5: (6,6,6) in $\mathbb{R}P^2 \setminus \{o\}$

3.4 Volume doubling property

In this section, we shall prove the volume doubling property for semiplanar graphs with nonnegative curvature using the corresponding property Theorem 7 on the regular polygonal surface.

Let G be a semiplanar graph and $S(G)$ be the regular polygonal surface of G. For any $p \in G$ and $R > 0$, we denote by $B_R(p) = \{x \in V : d^G(p,x) \leq R\}$ the closed geodesic ball in the graph G, and by $B_R^{S(G)}(p) = \{x \in S(G) : d(p,x) \leq R\}$ the closed geodesic ball in the polygonal surface $S(G)$. The volume of $B_R(p)$ is defined as $|B_R(p)| = \sum_{x \in B_R(p)} d_x$, and the volume of $B_R^{S(G)}(p)$ is defined as $|B_R^{S(G)}(p)| = H^2(B_R^{S(G)}(p))$, where H^2 is the 2-dimensional Hausdorff measure. We denote by $\sharp B_R(p)$ the number of vertices in the closed geodesic ball $B_R(p)$. Recall that for any semiplanar graph G with nonnegative curvature, $3 \leq d_x \leq 6$, for any $x \in G$. Hence $|B_R(p)|$ and $\sharp B_R(p)$ are equivalent up to a constant, i.e. $3\sharp B_R(p) \leq |B_R(p)| \leq 6\sharp B_R(p)$, for any $p \in G$ and $R > 0$.

Theorem 38. *Let $G = (V, E, F)$ be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then there exists a constant $C_{rel}(D)$ depending on D, such that for any $p \in G$ and $0 < r < R$, we have*

$$\frac{|B_R(p)|}{|B_r(p)|} \leq C_{rel}(D) \left(\frac{R}{r}\right)^2. \tag{3.4.1}$$

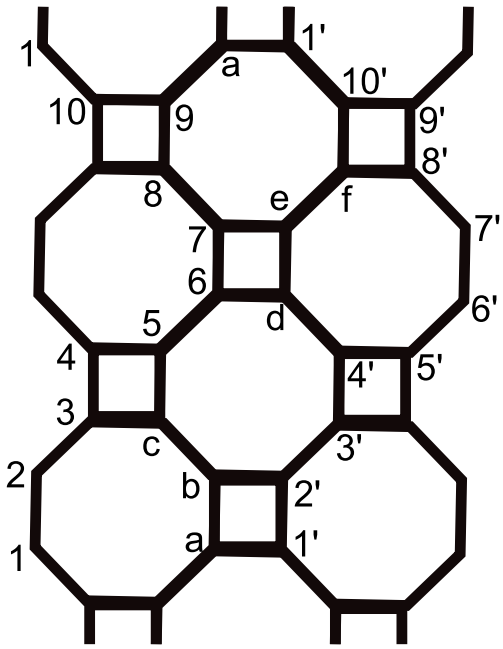

Figure 3.6: (4,8,8)

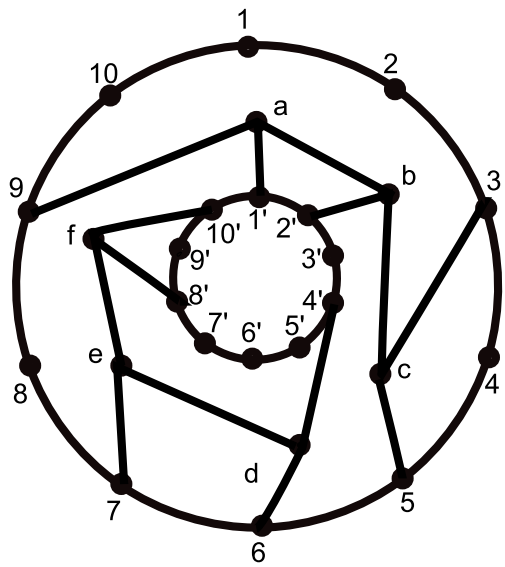

Figure 3.7: (4,8,8) in $\mathbb{R}P^2 \setminus \{o\}$

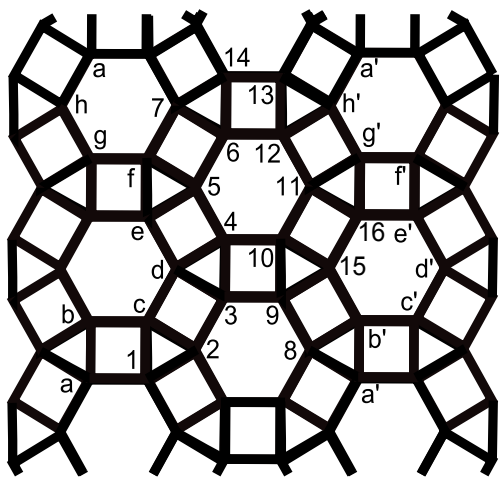

Figure 3.8: (3,4,6,4)

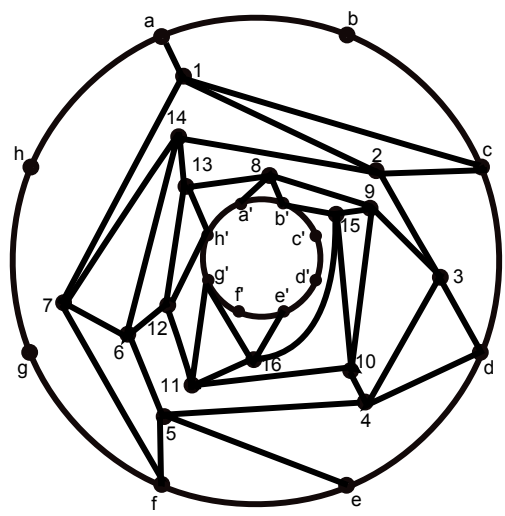

Figure 3.9: (3,4,6,4) in $\mathbb{R}P^2 \setminus \{o\}$

Proof. We denote $B_R := B_R(p)$ and $B_R^S := B_R^{S(G)}(p)$ for short. By Lemma 9, we have

$$B_{CR}^S \cap V \subset B_R \subset B_R^S \cap V.$$

For any $\sigma \in F$, $C_1 \leq |\sigma| := H^2(\sigma) \leq C_2(D)$. Let $H_R := \{\sigma \in F : \sigma \cap B_R \neq \emptyset\}$ denote the faces attached to B_R. Then

$$|B_R| = \sum_{x \in B_R} d_x \leq D \cdot \sharp H_R, \tag{3.4.2}$$

where $\sharp H_R$ is the number of faces in H_R. For any $\sigma \in F$, since the intrinsic diameter of σ is bounded, i.e. diam $\sigma := \sup\{d(x,y) : x, y \in \sigma\} \leq C_3(D)$, we have for any face $\sigma \in H_R$

$$\sigma \subset B^S_{R+\mathrm{diam}\sigma} \subset B^S_{R+C_3(D)}.$$

Hence it follows that

$$C_1 \sharp H_R \leq \sum_{\sigma \in H_R} |\sigma| \leq |B^S_{R+C_3(D)}|. \tag{3.4.3}$$

By the volume comparison (1.5.3) of $S(G)$ and (3.4.2), (3.4.3), we obtain

$$|B_R| \leq C(D)|B^S_{R+C_3(D)}| \leq C(D)(R + C_3(D))^2. \tag{3.4.4}$$

For $R \geq C_3(D)$, we have $|B_R| \leq 4C(D)R^2$. For $1 \leq R < C_3(D)$, we have $|B_R| \leq 6 \cdot 6^{C_3(D)} \leq C(D)R^2$. Hence, for any $R \geq 1$, the quadratic volume growth property follows

$$|B_R| \leq C(D)R^2. \tag{3.4.5}$$

For any $r > \frac{C_3(D)}{C}$, where C is the constant in Lemma 9, let $r' = Cr - C_3(D)$. We denote by $W_r = \{\sigma \in F : \sigma \cap B^S_{r'} \neq \emptyset\}$ the faces attached to $B^S_{r'}$, and by $\overline{W_r} = \bigcup_{\sigma \in W_r} \sigma$. For any vertex $x \in \overline{W_r} \cap G$, there exists a $\sigma \in W_r$ including x, such that $d(p,x) \leq r' + \mathrm{diam}\sigma \leq r' + C_3(D) = Cr$. By Lemma 9, we have $d^G(p,x) \leq C^{-1}d(p,x) \leq r$, which implies that $\overline{W_r} \cap G \subset B_r$. It is easy to see that

$$|B^S_{r'}| \leq |\overline{W_r}| = \sum_{\sigma \in W_r} |\sigma| \leq C_2(D)\sharp W_r, \tag{3.4.6}$$

where $\sharp W_r$ is the number of faces in W_r. Moreover, by $3 \leq \deg(\sigma) \leq D$ for any $\sigma \in F$,

$$3\sharp W_r \leq \sum_{\sigma \in W_r} \deg(\sigma) \leq \sum_{x \in \overline{W_r} \cap G} d_x \leq 6\sharp(\overline{W_r} \cap G), \tag{3.4.7}$$

where $\sharp(\overline{W_r} \cap G)$ is the number of vertices in $\overline{W_r} \cap G$.

Hence by (3.4.6) and (3.4.7), we have

$$|B^S_{r'}| \leq C(D)\sharp(\overline{W_r} \cap G) \leq C(D)\sharp B_r \leq C(D)|B_r|. \tag{3.4.8}$$

By the relative volume comparison (1.5.1) and (3.4.4), (3.4.8), we obtain that for any $r > \frac{C_3(D)}{C}$,

$$\frac{|B_R|}{|B_r|} \leq C(D)\frac{|B^S_{R+C_3(D)}|}{|B^S_{r'}|} \leq C(D)\left(\frac{R + C_3(D)}{Cr - C_3(D)}\right)^2.$$

Let $r_0(D) := \frac{2C_3(D)}{C}$. For $r_0(D) \leq r < R < \infty$, we have $r - \frac{C_3(D)}{C} \geq \frac{r}{2}$ and $R + C_3(D) \leq 2R$. Therefore

$$\frac{|B_R|}{|B_r|} \leq C(D)\left(\frac{R}{r}\right)^2. \tag{3.4.9}$$

For $0 < r < R \leq r_0(D)$, by (3.4.5), we have

$$\frac{|B_R|}{|B_r|} \leq \frac{|B_{r_0(D)}|}{|B_0|} \leq \frac{1}{3}C(D)r_0^2(D) \leq C(D)\left(\frac{R}{r}\right)^2. \tag{3.4.10}$$

For $0 < r < r_0(D) < R$, by (3.4.5), we have

$$\frac{|B_R|}{|B_r|} \leq \frac{C(D)R^2}{|B_0|} \leq C(D)r^2\left(\frac{R}{r}\right)^2 \leq C(D)r_0^2(D)\left(\frac{R}{r}\right)^2. \tag{3.4.11}$$

Hence it follows from (3.4.9), (3.4.10) and (3.4.11) that for any $0 < r < R$,

$$\frac{|B_R|}{|B_r|} \leq C(D)\left(\frac{R}{r}\right)^2.$$

□

From the relative volume comparison, it is easy to obtain the volume doubling property.

Corollary 9. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then there exists a constant $C(D)$ depending on D, such that for any $p \in G$ and $R > 0$, we have*

$$|B_{2R}(p)| \leq C(D)|B_R(p)|. \tag{3.4.12}$$

3.5 Relative volume growth on Archimedean tilings

An interesting question is the following. If we only consider $1 \leq r < R$, $r, R \in \mathbb{Z}$, could the constant $C_{rel}(D)$ in (3.4.1) take value 1 just like the case in Alexandrov geometry?

We will prove in this section that for the 11 kinds of Archimedean tilings, although the answer is not always yes, but a weak version of the relative volume comparison with $C_{rel} = 1$ holds. (For convenience, we will refer to it as (R).)

(R) There exists a sequence of integers $\{R_n\}_{n=1,2,\ldots}$ which satisfies

$$\lim_{n\to+\infty} R_n = +\infty \quad \text{and } |R_n - R_{n-1}| \leq c$$

such that for any $1 \leq r < R_n$, $r \in \mathbb{Z}$, we have

$$\frac{|B_{R_n}(p)|}{|B_r(p)|} \leq \left(\frac{R_n}{r}\right)^2.$$

In the above, c is an absolute constant.

We first describe the formula due to Préa [98, 99, 100] for distance sequence on Archimedean tilings on a plane. It turns out that one needs to recheck the proof in Préa [99] to reformulate three of them a little. Based on those formulas, we calculate the volume sequence and prove that all Archimedean tilings of a plane satisfy condition (R).

3.5.1 Distance sequence on Archimedean tilings

We will denote $A_i(p) := \sharp\partial B_i(p) = \sharp\{x \in V, d^G(x,p) = i\}$. The sequence $\{A_i(p)\}$ is called the distance sequence in Préa's work. We will write A_i for short since Archimedean tilings are all vertex-transitive. Obviously $A_0 = 1$. In the following we will adopt the notion $\lfloor a \rfloor$ for the floor of real a (the largest previous integer of a) and $\lceil a \rceil$ for the ceil of real a (the smallest following integer of a) and

$$\{a;b\}_c = \begin{cases} 1, & \text{if } b \equiv a(modc) \text{ and } b \geq a; \\ 0, & \text{otherwise,} \end{cases}$$

$$\{a_1, a_2, \ldots, a_n; b\}_c = \{a_1; b\}_c + \{a_2; b\}_c + \cdots + \{a_n; b\}_c.$$

We divide the 11 Archimedean tilings into 4 groups.

1. The distance sequence of the tiling (3^6) is

$$A_i = 6i, \quad i = 1, 2, \ldots.$$

The distance sequence of the tiling (4^4) is $A_i = 4i$, of the tiling (6^3) is $A_i = 3i$, of the tiling $(3,4,6,4)$ is $A_i = 4i$ and of the tiling $(3^3, 4^2)$ is $A_i = 5i$.

2. The distance sequence of the tiling $(4, 8^2)$ is

$$\begin{aligned} A_i =& 2i + \left\lfloor \frac{i}{3} \right\rfloor + \left\lfloor \frac{i+2}{3} \right\rfloor + i(mod3) \\ =& 3i - \left\lceil \frac{i-1}{3} \right\rceil, \quad \text{for } i = 1, 2, \ldots. \end{aligned}$$

The distance sequence of the tiling $(3^2, 4, 3, 4)$ is given by

$$\begin{aligned} A_i =& 4i + 1 - \{0; i\}_3 + 2\left(\left\lfloor \frac{i}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor\right) \\ =& 5i + \left\lceil \frac{i-1}{3} \right\rceil, \quad \text{for } i = 1, 2, \ldots. \end{aligned}$$

3. The distance sequence of the tiling $(3,6,3,6)$ is

$$A_1 = 4;$$
$$A_i = \begin{cases} 4i + 2, & \text{if } i \text{ is odd;} \\ 5i - 2, & \text{if } i \text{ is even,} \end{cases} \quad \text{for } i \geq 2.$$

The distance sequence of the tiling $(3, 12^2)$ is

$$A_1 = 3; \quad A_2 = 4;$$
$$A_i = \begin{cases} \frac{5i}{2} - 2, & \text{if } i \equiv 0(mod4); \\ \frac{9i+3}{4}, & \text{if } i \equiv 1(mod4); \\ 2i + 2, & \text{if } i \equiv 2(mod4); \\ \frac{9i-3}{4}, & \text{if } i \equiv 3(mod4). \end{cases} \quad \text{for } i \geq 3.$$

4. The distance sequence for the tiling $(4,6,12)$ is

$$
\begin{aligned}
A_1 =& 3;\\
A_i =& 4+\left\lfloor\frac{i+6}{10}\right\rfloor+\left\lfloor\frac{i+3}{10}\right\rfloor+\left\lfloor\frac{i+1}{10}\right\rfloor+\left\lfloor\frac{i-2}{10}\right\rfloor+2\left(\left\lfloor\frac{i+2}{6}+\left\lfloor\frac{i+7}{30}\right\rfloor\right\rfloor\right)\\
&+\left\lfloor\frac{i+3}{6}\right\rfloor+\left\lfloor\frac{i+1}{6}\right\rfloor+\left\lfloor\frac{i+8}{30}\right\rfloor+\left\lfloor\frac{i+6}{30}\right\rfloor\\
&+3\left(\left\lfloor\frac{i+5}{10}\right\rfloor+\left\lfloor\frac{i+4}{10}\right\rfloor+\left\lfloor\frac{i}{10}\right\rfloor+\left\lfloor\frac{i-1}{10}\right\rfloor\right)\\
&+2\{3;i\}_{30}+\{2,4,7,10,12,16,17,22;i\}_{30}+3\{8,9,13,15,18,21;i\}_{30}\\
&+4\{14,19,20;i\}_{30},\ \text{for } i\geq 2.
\end{aligned}
$$

The distance sequence for tiling $(6,3^4)$ is

$$
\begin{aligned}
A_1 =& 5; A_2=9;\\
A_i =& 2\left\lfloor\frac{i+4}{5}\right\rfloor+\{3,4,5;i\}_5+\left\lfloor\frac{i+2}{3}\right\rfloor+\left\lfloor\frac{i+4}{15}\right\rfloor+\{6,8,12;i\}_{15}\\
&+\left\lfloor\frac{i}{8}\right\rfloor+\left\lfloor\frac{i+5}{8}\right\rfloor+\left\lfloor\frac{i-5}{8}\right\rfloor+\left\lfloor\frac{i-6}{40}\right\rfloor\\
&+\{18,23,26,28,31,33,34,36,38,39,41,42,43,44;i\}_{40}\\
&+\{8,9,10;i\}_5+2\left\lfloor\frac{i+4}{5}\right\rfloor+\{0;i\}_5\\
&+2\left\lfloor\frac{i+2}{3}\right\rfloor+\{6;i\}_5+2\{9,12,14,15;i\}_{15}+2\left\lfloor\frac{i-2}{15}\right\rfloor\\
&+2\left\lfloor\frac{i}{5}\right\rfloor+2\left\lfloor\frac{i+1}{5}\right\rfloor+2\{2,3;i\}_5+\{4;i\}_5\\
&+\left\lfloor\frac{i+1}{3}\right\rfloor+\left\lfloor\frac{i}{15}\right\rfloor+\{7,10,12,13;i\}_{15}+\left\lfloor\frac{2i}{5}\right\rfloor+\left\lfloor\frac{2i-1}{5}\right\rfloor\\
&+\left\lfloor\frac{i}{3}\right\rfloor+\left\lfloor\frac{i+2}{15}\right\rfloor+\{5,8,10,11;i\}_{15}\\
=& 4\left\lfloor\frac{i+4}{5}\right\rfloor+2\left\lfloor\frac{2i+1}{5}\right\rfloor+\left\lfloor\frac{4i-1}{5}\right\rfloor+1+\{5;i\}_5+\{8,9,10;i\}_5\\
&+i+2\left\lfloor\frac{i+2}{3}\right\rfloor+\left\lfloor\frac{i}{8}\right\rfloor+\left\lfloor\frac{i+5}{8}\right\rfloor+\left\lfloor\frac{i+5}{8}\right\rfloor\\
&+\left\lfloor\frac{i-6}{40}\right\rfloor+\{18,23,26,28,31,33,34,36,38,39,41,42,43,44;i\}_{40}\\
&+\left\lfloor\frac{5i+1}{15}\right\rfloor-\{3,4,6,7,8,10;i\}_{15}-2\{11,13,16;i\}_{15},\ \text{for } i\geq 3.
\end{aligned}
$$

Remark 23. *Those kinds of formulas are listed in Préa [100] and the interesting proof is given in Préa [98, 99]. The formula for the tilings* $(4,8^2)$, $(3,12^2)$, $(6,3^4)$ *in Préa [100] need to be reformulated by checking the proof in [99] carefully to get the above ones. For the tiling* $(4,8^2)$, *the formula on page* 5 *in Préa [99] is*

correct, but the final form on page 6 *misses the term* $i(mod3)$. *For tiling* $(3, 12^2)$, *several formulas on page* $12-14$ *do not apply to* $i = 1, 2$. *Therefore one need to list them separately. The final formula for* $i \equiv 0(mod4)$ *misses* 1. *For the tiling* $(6, 3^4)$, *one need to recheck the term* $X(i)$ *on page* 19 *and several other formulas. On page* 20, *in* $Y(i)$, $\left\lfloor \frac{i-5}{40} \right\rfloor$ *should be* $\left\lfloor \frac{i-6}{40} \right\rfloor$; *in the formula for grey strips,* $\left\lfloor \frac{i}{5} \right\rfloor + 1$ *should be* $\{0; i\}_5$. *On page* 21, $Z(i)$ *should be* $2Z(i)$. *In the first paragraph of page* 22, *the cases for the left-hand side and the right-hand side should be different.*

For several cases, the idea of proof in Préa [98, 99] is to draw the graphs on a lined paper like in Figure 3.10 and 3.11. (One can compare the drawing of Figure 3.11 here with that of Figure 3.8)

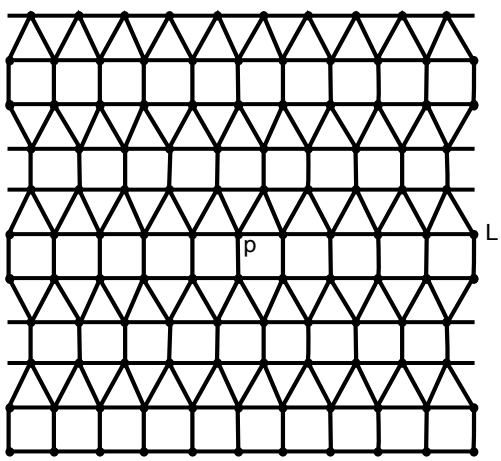

Figure 3.10: $(3^3, 4^2)$ on "lines"

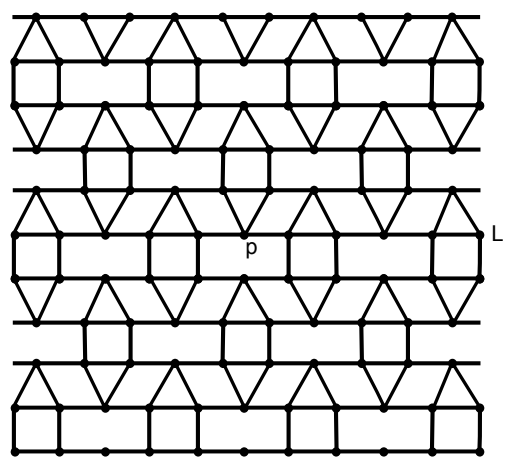

Figure 3.11: $(3, 4, 6, 4)$ on "lines"

3.5.2 (R) holds on Archimedean tilings

In this subsection we further calculate the volume sequence. We observe

$$|B_i| = d\left(1 + \sum_{k=1}^{i} A_i\right).$$

We also do this in 4 groups.

1. For the tilings (3^6), (4^4), (6^3), $(3, 4, 6, 4)$ and $(3^3, 4^2)$, we have $A_i = ai$, where a is a constant for each tiling. Therefore we can get

$$\frac{|B_i|}{d} = 1 + \frac{a}{2}i(i+1). \tag{3.5.1}$$

Then one can check

$$\frac{|B_{i+1}|}{|B_i|} < \left(\frac{i+1}{i}\right)^2 \tag{3.5.2}$$

which implies

$$\frac{|B_R|}{|B_r|} < \left(\frac{R}{r}\right)^2, \text{ for any } 1 \le r < R, r, R \in \mathbb{Z}. \tag{3.5.3}$$

2. For tiling $(4, 8^2)$, the volume satisfies

$$\frac{|B_i|}{3} = 1 + \sum_{k=1}^{i}\left(3k - \left\lceil\frac{k-1}{3}\right\rceil\right) = 1 + \frac{3}{2}i(i+1) - \sum_{k=1}^{i}\left\lceil\frac{k-1}{3}\right\rceil.$$

For $m \in \mathbb{Z}$, $m \ge 0$, we have

$$\sum_{k=1}^{i}\left\lceil\frac{k-1}{3}\right\rceil = \begin{cases} 3(1+2+\cdots+m) = \frac{3}{2}m(m+1), & \text{if } i-1 = 3m; \\ \frac{3}{2}m(m+1) + (m+1), & \text{if } i-1 = 3m+1; \\ \frac{3}{2}m(m+1) + 2(m+1), & \text{if } i-1 = 3m+2. \end{cases}$$

Hence we have

$$\frac{|B_i|}{3} = \begin{cases} 12m^2 + 12m + 4, & \text{if } i = 3m+1; \\ 12m^2 + 20m + 9, & \text{if } i = 3m+2; \\ 12m^2 + 28m + 17, & \text{if } i = 3m+3. \end{cases}$$

Now it is easy to check for all three cases that (3.5.2) holds and then (3.5.3).

For tiling $(3^2, 4, 3, 4)$, the volume

$$\begin{aligned} \frac{|B_i|}{5} &= 1 + \sum_{k=1}^{i}\left(5k + \left\lceil\frac{k-1}{3}\right\rceil\right) \\ &= \begin{cases} 24m^2 + 24m + 6, & \text{if } i = 3m+1; \\ 24m^2 + 40m + 17, & \text{if } i = 3m+2; \\ 24m^2 + 56m + 33, & \text{if } i = 3m+3. \end{cases} \end{aligned}$$

Then it is easy to check for all three cases that (3.5.2) holds and then (3.5.3).

3. For tiling $(3, 6, 3, 6)$, it is easy to check (3.5.2) for $i = 1$. Then for $i \ge 2$, if $i = 2m$, $m \ge 1$, we have

$$\begin{aligned} \sum_{k=2}^{i} A_k =& A_2 + A_4 + \cdots + A_{2m} \\ &+ A_3 + A_5 + \cdots + A_{2m-1} \\ =& 10\sum_{l=1}^{m} l + \sum_{l=2}^{m}(2l-1) - 2 \\ =& 9m^2 + 5m - 6. \end{aligned}$$

Then we arrive at

$$\frac{|B_i|}{4} = \begin{cases} 9m^2 + 5m - 1, & \text{if } i = 2m; \\ 9m^2 + 13m + 5, & \text{if } i = 2m + 1. \end{cases}$$

Now we can check when $i = 2m$,

$$\frac{|B_{i+1}|}{|B_i|} = \frac{9m^2 + 13m + 5}{9m^2 + 5m - 1} < \left(\frac{2m+1}{2m}\right)^2.$$

But when $i = 2m + 1$,

$$\frac{|B_{i+1}|}{|B_i|} = \frac{9m^2 + 23m + 13}{9m^2 + 13m + 5} > \left(\frac{2m+2}{2m+1}\right)^2, \text{ for } m \geq 4.$$

However, the good thing is that when $i = 2m + 1$, $m \geq 1$,

$$\frac{|B_i|}{|B_{i-2}|} = \frac{9m^2 + 13m + 5}{9m^2 - 5m + 1} < \left(\frac{2m+1}{2m-1}\right)^2.$$

So we in fact prove that this tiling satisfies condition (R) for the sequence $R_n = 2n + 1$.

Similar things happen to the tiling $(3, 12^2)$. First we check (3.5.2) for $i = 1, 2, 3, 4$. Then for $i \geq 4$, if $i = 4m$, $m \geq 1$, we have

$$\begin{aligned} \frac{|B_i|}{3} =& \sum_{l=1}^{m} (10l - 2) + \sum_{l=1}^{m-1} (9l + 3) \sum_{l=1}^{m-1} (8l + 6) + \sum_{l=1}^{m-1} (9l + 6) \\ =& 18m^2 + 5m - 1. \end{aligned}$$

Then we get

$$\frac{|B_i|}{3} = \begin{cases} 18m^2 + 5m - 1, & \text{if } i = 4m; \\ 18m^2 + 14m + 2, & \text{if } i = 4m + 1; \\ 18m^2 + 22m + 8, & \text{if } i = 4m + 2; \\ 18m^2 + 31m + 14, & \text{if } i = 4m + 3. \end{cases}$$

Now one can check (3.5.2) holds when $i = 4m, 4m + 1, 4m + 2$. But for $i = 4m + 3$,

$$\frac{|B_{i+1}|}{|B_i|} = \frac{18m^2 + 41m + 22}{18m^2 + 31m + 14} > \left(\frac{4m+4}{4m+3}\right)^2, \text{ for } m \geq 3.$$

However, the good thing is that when $i = 4m + 2$,

$$\frac{|B_i|}{|B_{i-3}|} = \frac{18m^2 + 22m + 8}{18m^2 - 5m + 1} < \left(\frac{4m+2}{4m-1}\right)^2.$$

So we in fact prove that this tiling satisfies condition (R) for the sequence $R_n = 4n + 2$.

4. For tiling $(4,6,12)$, the distance sequence formula is a little complicated. Let's first make it clear. Note for $i = 30k + j$, $2 \le j \le 31$, $k \ge 0$,

$$A_i = 72k + A_j.$$

And for $2 \le j \le 26$, $j = 31$,

$$\begin{aligned} A_j =& 3 + 2(i-1) + \{5; i\}_5 + 2\{6; i\}_5 + 2\left\lfloor \frac{1-2}{5} \right\rfloor \\ =& \begin{cases} 5 + 12m, & \text{if } j-1 = 5m+1; \\ 7 + 12m, & \text{if } j-1 = 5m+2; \\ 9 + 12m, & \text{if } j-1 = 5m+3; \\ 12 + 12m, & \text{if } j-1 = 5m+4; \\ 15 + 12m, & \text{if } j-1 = 5m+5. \end{cases} \end{aligned}$$

The remaining terms are

$$A_{27} = 64;\ A_{28} = 64;\ A_{29} = 66;\ A_{30} = 71.$$

Then we calculate the volume sequence. First for $2 \le i \le 26$, if $i - 1 = 5m + 1$, we have

$$\begin{aligned} \frac{|B_i|}{3} =& 4 + \sum_{k=2}^{i} (3 + 2(k-1)) + 2 \times \left(5 \sum_{l=1}^{m-1} l + m \right) + 3m \\ =& 30m^2 + 30m + 9. \end{aligned}$$

Furthermore, we have for $2 \le i \le 26$,

$$\frac{|B_i|}{3} = \begin{cases} 30m^2 + 30m + 9, & \text{if } i-1 = 5m+1; \\ 30m^2 + 42m + 16, & \text{if } i-1 = 5m+2; \\ 30m^2 + 54m + 25, & \text{if } i-1 = 5m+3; \\ 30m^2 + 66m + 37, & \text{if } i-1 = 5m+4; \\ 30m^2 + 78m + 52, & \text{if } i-1 = 5m+5. \end{cases}$$

and

$$\frac{|B_i|}{3} = \begin{cases} 908, & \text{if } i = 27; \\ 972, & \text{if } i = 28; \\ 1038, & \text{if } i = 29; \\ 1109, & \text{if } i = 30; \\ 1184, & \text{if } i = 31. \end{cases}$$

Now for general $i = 30k + j$, $2 \le j \le 31$, $k \ge 0$,

$$\begin{aligned} \frac{|B_1|}{3} =& \frac{|B_{31}| - |B_1|}{3} + \sum_{l=1}^{k-1} \left(\frac{|B_{31}| - |B_1|}{3} + 72 \times 30l \right) \\ &+ \frac{|B_j| - |B_1|}{3} + 72k(j-1) \\ =& 1080k^2 + 100k + 72k(j-1) + \frac{|B_j|}{3}. \end{aligned}$$

Now one can check for the cases $i = 30k + j$, $2 \le j \le 25$, $j - 1 = 5m + 1$, $5m + 2$, $5m + 3$, $5m + 4$, $5m + 5$ and $i = 30k + 26$, $30k + 27$, $30k + 28$, $30k + 29$, $30k + 30$, $30k + 31$ that (3.5.2) holds and then (3.5.3).

For tiling $(3^4, 6)$, we also first clarify the formula for the distance sequence. Similarly, for this tiling if $i = 120k + j$, $2 \le j \le 121$,

$$A_i = 576k + A_j.$$

Furthermore we have for $2 \le j \le 121$,

$$A_j = \begin{cases} 9 + 24m, & \text{if } j - 1 = 5m + 1; \\ 15 + 24m, & \text{if } j - 1 = 5m + 2; \\ 19 + 24m, & \text{if } j - 1 = 5m + 3; \\ 24 + 24m, & \text{if } j - 1 = 5m + 4; \\ 5 + 24m, & \text{if } j - 1 = 5m,\ m \ne 2, 5, 8, 11, 14, 17, 20, 23; \\ 4 + 24m, & \text{if } j - 1 = 5m,\ m = 2, 5, 8, 11, 14, 17, 20, 23. \end{cases}$$

Then we calculate the volume sequence. $|B_1| = 6$, $|B_{16}| = 653$ and for $2 \le i \le 121$, if $i - 1 = 15m + 1$,

$$\begin{aligned} \frac{|B_i|}{5} =& |B_1| + \sum_{l=0}^{k-1} \left(\frac{|B_{16}| - |B_1|}{5} + 72 \times 15l \right) + 9 + 72m \\ =& 540m^2 + 179m + 15. \end{aligned}$$

Furthermore, we get for $2 \le i \le 121$, the values of $\frac{|B_i|}{5}$ when $i - 1 = 15m+1, 15m+2, \ldots, 15m+15$ are $540m^2+179m+15$, $540m^2+1251m+30$, $540m^2 + 323m + 49$, $540m^2 + 395m + 73$, $540m^2 + 467m + 102$, $540m^2 + 539m+135$, $540m^2+611m+174$, $540m^2+683m+217$, $540m^2+755m+265$, $540m^2 + 827m + 317$, $540m^2 + 899m + 374$, $540m^2 + 971m + 437$, $540m^2 + 1043m + 504$, $540m^2 + 1115m + 576$, $540m^2 + 1187m + 653$ respectively.

For general $i = 120k + j$, $2 \le j \le 121$, we have

$$\begin{aligned} \frac{|B_i|}{5} =& \sum_{l=0}^{k-1} \left(\frac{|B_{121}| - |B_1|}{5} + 576 \times 120l \right) + 576k(j-1) + \frac{|B_j|}{5} \\ =& 34560k^2 + 280k + 576kj + \frac{|B_j|}{5}. \end{aligned}$$

Finally one can check that (3.5.2) holds when $i = 120k + 15m + 2, 120k + 15m + 3, \ldots, 120k + 15m + 16, 120k + 121$ and then (3.5.3).

In conclusion we get the following lemma.

Lemma 14. *All Archimedean tilings on a plane satisfies condition* (R).

3.6 Poincaré inequality

In this section, we shall prove the Poincaré inequality on a semiplanar graph with nonnegative curvature from the Poincaré inequality Theorem 9 on the regular polygonal surface.

Theorem 39. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then there exist two constants $C(D)$ and $C > 1$ such that for any $p \in G, R > 0, f : B_{CR}(p) \to \mathbb{R}$, we have*

$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 d_x \leq C(D) R^2 \sum_{x,y \in B_{CR}(p); x \sim y} (f(x) - f(y))^2, \tag{3.6.1}$$

where $f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x) d_x$.

For any function on G, $f : V \to \mathbb{R}$, we shall construct a local $W^{1,2}$ function, denoted by f_2, on $S(G)$ with controlled energy in two steps. Then by the Poincaré inequality (1.5.4) on $S(G)$, we obtain the Poincaré inequality on the graph G. In the first step, by linear interpolation, we extend f to a piecewise linear function on G_1, $f_1 : G_1 \to \mathbb{R}$. Then for an edge e with two incident vertices u and v, we have

$$\int_e f_1^2 = \int_0^1 (tf(u) + (1-t)f(v))^2 dt = \frac{1}{3}(f(u)^2 + f(u)f(v) + f(v)^2),$$

hence

$$\frac{1}{6}(f(u)^2 + f(v)^2) \leq \int_e f_1^2 \leq \frac{1}{2}(f(u)^2 + f(v)^2). \tag{3.6.2}$$

In addition,

$$\int_e (f_1')^2 = (f(u) - f(v))^2. \tag{3.6.3}$$

That is, the L^2 norms and the energies of f and f_1 control each other.

In the second step, we extend f_1 to each face of G and hope for similar controls. For any regular n-polygon $\triangle_n$ of side length one, there is a bi-Lipschitz map

$$L_n : \triangle_n \to B_{r_n},$$

where B_{r_n} is the circumscribed circle of $\triangle_n$ of radius $r_n = \frac{1}{2\sin\frac{\alpha_n}{2}}$ (for $\alpha_n = \frac{2\pi}{n}$). Without loss of generality, we may assume that the origin $\underline{o} = (0,0)$ of $\mathbb{R}^2$ is the barycenter of $\triangle_n$, the point $(x,y) = (r_n, 0) \in \mathbb{R}^2$ is a vertex of $\triangle_n$, and $B_{r_n} = B_{r_n}(\underline{o})$. Then in polar coordinates, L_n reads

$$L_n : \triangle_n \ni (r, \theta) \mapsto (\rho, \eta) \in B_{r_n}(\underline{o}),$$

where for $\theta \in [j\alpha_n, (j+1)\alpha_n]$, $j = 0, 1, \ldots, n-1$,

$$\begin{cases} \rho = \frac{r\cos\left(\theta - (2j+1)\frac{\alpha_n}{2}\right)}{\cos\frac{\alpha_n}{2}}, \\ \eta = \theta. \end{cases} .$$

It maps the boundary of $\triangle_n$ to the boundary of $B_{r_n}(\underline{o})$. Direct calculation shows that L_n is a bi-Lipschitz map, i.e. for any $x, y \in \triangle_n$ we have $C_1|x-y| \leq |L_n x - L_n y| \leq C_2|x-y|$, where C_1 and C_2 do not depend on n. Then for any $\sigma \in F$, we denote $n := \deg(\sigma)$. Let $g : B_{r_n}(\underline{o}) \to \mathbb{R}$ satisfy the following boundary value problem

$$\begin{cases} \Delta g = 0, & in\ \mathring{B}_{r_n}(\underline{o}) \\ g|_{\partial B_{r_n}(\underline{o})} = f_1 \circ L_n^{-1}, \end{cases} ,$$

where $\mathring{B}_{r_n}(\underline{o})$ is the open ball. Then we define $f_2 : S(G) \to \mathbb{R}$ as

$$f_2|_\sigma = g \circ L_n. \tag{3.6.4}$$

It can be shown that f_2 is a local $W^{1,2}$ function on $S(G)$, since the singular points of $S(G)$ are isolated (see Kuwae-Machigashira-Shioya [79]).

We need to control the energy of f_2 by its boundary values. The following lemma is standard. We denote by B_1 the closed unit disk in $\mathbb{R}^2$.

Lemma 15. *For any Lipschitz function $h : \partial B_1 \to \mathbb{R}$, let $g : B_1 \to \mathbb{R}$ satisfy the following boundary value problem*

$$\begin{cases} \Delta g = 0, & in\ \mathring{B}_1 \\ g|_{\partial B_1} = h. \end{cases} .$$

Then we have

$$\int_{B_1} |\nabla g|^2 \leq \int_{\partial B_1} h_\theta^2,$$

$$\int_{\partial B_1} h^2 \leq C\left(\int_{B_1} g^2 + \int_{\partial B_1} h_\theta^2\right),$$

where $h_\theta = \frac{\partial h}{\partial \theta}$.

Proof. Let $\frac{1}{\sqrt{2\pi}}, \frac{\sin n\theta}{\sqrt{\pi}}, \frac{\cos n\theta}{\sqrt{\pi}}$ (for $n = 1, 2, \cdots$) be the orthonormal basis of $L^2(\partial B_1)$. Then $h : \partial B_1 \to \mathbb{R}$ can be represented in $L^2(\partial B_1)$ by

$$h(\theta) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(a_n \frac{\cos n\theta}{\sqrt{\pi}} + b_n \frac{\sin n\theta}{\sqrt{\pi}}\right).$$

So the harmonic function g with boundary value h is

$$g(r, \theta) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(a_n r^n \frac{\cos n\theta}{\sqrt{\pi}} + b_n r^n \frac{\sin n\theta}{\sqrt{\pi}}\right).$$

Since $\Delta g = 0$, we have $\Delta g^2 = 2|\nabla g|^2$. Then

$$\int_{B_1} |\nabla g|^2 = \frac{1}{2}\int_{B_1} \Delta g^2 = \frac{1}{2}\int_{\partial B_1} \frac{\partial g^2}{\partial r},$$

follows from integration by parts. So that

$$\int_{B_1} |\nabla g|^2 = \int_{\partial B_1} g g_r = \sum_{n=1}^{\infty} n(a_n^2 + b_n^2).$$

In addition,

$$\int_{\partial B_1} h_\theta^2 = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).$$

Hence,

$$\int_{B_1} |\nabla g|^2 \leq \int_{\partial B_1} h_\theta^2. \tag{3.6.5}$$

The second part of the theorem follows from an integration by parts and the Hölder inequality.

$$\begin{aligned}
\int_{\partial B_1} h^2 &= \int_{\partial B_1} (h^2 x)\cdot x = \int_{B_1} \nabla\cdot(g^2 x) \\
&= 2\int_{B_1} g^2 + 2\int_{B_1} g\nabla g\cdot x \\
&\leq 2\int_{B_1} g^2 + 2(\int_{B_1} g^2)^{\frac{1}{2}}(\int_{B_1} |\nabla g|^2)^{\frac{1}{2}} \quad (by\ |x|\leq 1) \\
&\leq 3\int_{B_1} g^2 + \int_{B_1} |\nabla g|^2 \\
&\leq 3\int_{B_1} g^2 + \int_{\partial B_1} h_\theta^2. \qquad (by\ (3.6.5))
\end{aligned}$$

□

Note that for the semiplanar graph G with nonnegative curvature and any face $\sigma = \triangle_n$ of G, we have $3 \leq n \leq D$, $\frac{1}{\sqrt{3}} \leq r_n = \frac{1}{\sin\frac{\pi}{n}} \leq \frac{1}{2\sin\frac{\pi}{D}} = C(D)$. Then the scaled version of Lemma 15 reads

Lemma 16. *For $3 \leq n \leq D$, and any Lipschitz function $h : \partial B_{r_n} \to \mathbb{R}$, we denote by g the harmonic function satisfying the Dirichlet boundary value problem*

$$\begin{cases} \Delta g = 0, & in\ \mathring{B}_{r_n} \\ g|_{\partial B_{r_n}} = h. \end{cases}.$$

Then it holds that

$$\int_{B_{r_n}} |\nabla g|^2 \leq C(D)\int_{\partial B_{r_n}} h_T^2,$$

$$\int_{\partial B_{r_n}} h^2 \leq C(D)\left(\int_{B_{r_n}} g^2 + \int_{\partial B_{r_n}} h_T^2\right),$$

where $T = \frac{1}{r_n}\partial_\theta$ *is the unit tangent vector on the boundary* ∂B_{r_n} *and* h_T *is the directional derivative of* h *in* T.

The following lemma follows from the bi-Lipschitz property of the map $L_n : \triangle_n \to B_{r_n}$.

Lemma 17. *Let* G *be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Let* σ *be a face such that* $\sigma = \triangle_n$. *Let* $f_2|_\sigma$ *be constructed as (3.6.4), then we have*

$$\int_{\triangle_n} |\nabla f_2|^2 \leq C(D)\int_{\partial\triangle_n} (f_1)_{T_n}^2, \tag{3.6.6}$$

$$\int_{\partial\triangle_n} f_1^2 \leq C(D)\left(\int_{\triangle_n} f_2^2 + \int_{\partial\triangle_n} (f_1)_{T_n}^2\right), \tag{3.6.7}$$

where T_n *is the unit tangent vector on the boundary* $\partial\triangle_n$ *and* $(f_1)_{T_n}$ *is the directional derivative of* f_1 *in* T_n.

Now we arrive at the point to prove the Poincaré inequality.

Proof of Theorem 39. Let $B_R^{G_1} := B_R^{G_1}(p)$ denote the closed geodesic ball in G_1. We set the constant $c_R := f_{2,B^S_{R+1+C_3(D)}} = \frac{1}{|B^S_{R+1+C_3(D)}|}\int_{B^S_{R+1+C_3(D)}} f_2$. By (3.6.2), we have

$$\sum_{x\in B_R} (f(x) - c_R)^2 d_x \leq 6\int_{B_{R+1}^{G_1}} (f_1 - c_R)^2. \tag{3.6.8}$$

Let $W_{R+1} = \{\sigma \in F : \sigma \cap B_{R+1}^{G_1} \neq \emptyset\}$ and $\overline{W_{R+1}} = \bigcup_{\sigma\in W_{R+1}} \sigma$. Since $B_{R+1}^{G_1} \subset \bigcup_{\sigma\in W_{R+1}} \partial\sigma$, we have

$$\begin{aligned}\int_{B_{R+1}^{G_1}} (f_1 - c_R)^2 &\leq \sum_{\sigma\in W_{R+1}} \int_{\partial\sigma} (f_1 - c_R)^2 \\ &\leq C(D)\sum_{\sigma\in W_{R+1}}\left(\int_\sigma (f_2 - c_R)^2 + \int_{\partial\sigma} (f_1)_T^2\right),\end{aligned} \tag{3.6.9}$$

where the last inequality follows from (3.6.7). For any $y \in \overline{W_{R+1}}$, since diam $\sigma \leq C_3(D)$ for any $\sigma \in F$, we have $d(p, y) \leq R + 1 + C_3(D)$. This implies that $\overline{W_{R+1}} \subset B^S_{R+1+C_3(D)}$. Hence by (3.6.9)

$$\begin{aligned}\int_{B_{R+1}^{G_1}} (f_1 - c_R)^2 &\leq C(D)\int_{B^S_{R+1+C_3(D)}} (f_2 - c_R)^2 + C(D)\sum_{\sigma\in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2 \\ &\leq C(D)(R + 1 + C_3(D))^2 \int_{B^S_{R+1+C_3(D)}} |\nabla f_2|^2 \\ &\quad + C(D)\sum_{\sigma\in W_{R+1}} \int_{\partial\sigma} (f_1)_T^2,\end{aligned} \tag{3.6.10}$$

where we use the Poincaré inequality (1.5.4).

Let $U_{R+1} := \{\tau \in F : \tau \cap B^S_{R+1+C_3(D)} \neq \emptyset\}$. Since $B^{G_1}_{R+1} \subset B_{R+1} \subset B^S_{R+1+C_3(D)}$, we have $W_{R+1} \subset U_{R+1}$. By Lemma 9, it follows that

$$U_{R+1} \cap G_1 \subset B^{G_1}_{C^{-1}(R+1+C_3(D))+D}. \tag{3.6.11}$$

By (3.6.6), (3.6.3), (3.6.10) and (3.6.11), we obtain that

$$\begin{aligned}
\int_{B^{G_1}_{R+1}} (f_1 - c_R)^2 &\leq C(D)(R+1+C_3(D))^2 \sum_{\tau \in U_{R+1}} \int_\tau |\nabla f_2|^2 + \\
&\quad + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)^2_T \\
&\leq C(D)(R+1+C_3(D))^2 \sum_{\tau \in U_{R+1}} \int_{\partial\tau} (f_1)^2_T + \\
&\quad + C(D) \sum_{\sigma \in W_{R+1}} \int_{\partial\sigma} (f_1)^2_T \\
&\leq C(D)(R+1+C_3(D))^2 \sum_{\tau \in U_{R+1}} \int_{\partial\tau} (f_1)^2_T \\
&\leq C(D)(R+1+C_3(D))^2 \sum_{\substack{x,y \in B_{C^{-1}(R+1+C_3(D))+D} \\ x \sim y}} (f(x) - f(y))^2.
\end{aligned} \tag{3.6.12}$$

For $R \geq C^{-1}(C_3(D)+1) + D = r_0(D)$, we have $C^{-1}(R+1+C_3(D)) + D \leq (C^{-1}+1)R = C_1 R$ and $R+1+C_3(D) \leq 2R$. Let $f_{B_R} := \frac{1}{|B_R|} \sum_{x \in B_R} f(x) d_x$, then by (3.6.8) and (3.6.12) we obtain

$$\sum_{x \in B_R} (f(x) - f_{B_R})^2 d_x \leq \sum_{x \in B_R} (f(x) - c_R)^2 d_x \leq C(D) R^2 \sum_{\substack{x,y \in B_{C_1 R} \\ x \sim y}} (f(x) - f(y))^2. \tag{3.6.13}$$

For $1 \leq R \leq r_0(D)$, let $G^R = (V^R, E^R)$ be the subgraph induced by B_R. For any $x \in G^R$, we denote by d_{x,G^R} the degree of the vertex x in G^R. The volume of G^R is then defined as $\mathrm{vol} G^R = \sum_{x \in V^R} d_{x,G^R}$ and the diameter of G^R is defined as $\mathrm{diam} G^R = \sup_{x,y \in G^R} d^{G^R}(x,y)$. Let $\lambda_1(G^R)$ be the first nonzero eigenvalue of the normalized Laplace operator of G^R, then the Rayleigh formula (2.4.5) in this case reads

$$\lambda_1(G^R) = \inf_{f: G^R \to \mathbb{R}} \frac{\sum_{x,y \in G^R; x \sim y} (f(x) - f(y))^2}{\sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R}},$$

where $f_{G^R} := \frac{1}{\mathrm{vol} G^R} \sum_{x \in G^R} f(x) d_{x,G^R}$. We recall a lower bound estimate for $\lambda_1(G^R)$ by the diameter and volume of G^R (see Chung [24]),

$$\lambda_1(G^R) \geq \frac{1}{\mathrm{diam} G^R \cdot \mathrm{vol} G^R}.$$

Since $3 \leq d_x \leq 6$, we have $\frac{1}{6}d_x \leq d_{x,G^R} \leq d_x$. It is easy to see that $\mathrm{diam}G^R \leq 2R$ and $\mathrm{vol}G^R \leq |B_R| \leq C(D)R^2$ by (3.4.5). So that we have

$$\lambda_1(G^R) \geq \frac{1}{2R \cdot C(D)R^2} \geq \frac{1}{2r_0(D) \cdot C(D)r_0^2(D)} \geq C(D),$$

which implies that

$$\sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R} \leq C(D) \sum_{x,y \in G^R; x \sim y} (f(x) - f(y))^2,$$

for any $f : V^R \to \mathbb{R}$.

Hence we obtain that

$$\begin{aligned} \sum_{x \in B_R} (f(x) - f_{B_R})^2 d_x &\leq 6 \sum_{x \in G^R} (f(x) - f_{G^R})^2 d_{x,G^R} \\ &\leq C(D) \sum_{x,y \in G^R; x \sim y} (f(x) - f(y))^2 \\ &\leq C(D)R^2 \sum_{x,y \in B_R; x \sim y} (f(x) - f(y))^2. \quad (3.6.14) \end{aligned}$$

For $0 < R < 1$, the Poincaré inequality (3.6.1) is trivial. The theorem is proved by (3.6.13) and (3.6.14). □

3.7 Application of volume doubling property and Poincaré inequality

In this section, we shall study the analytic consequences of the volume doubling property and the Poincaré inequality.

In Riemannian manifolds, it is well known that the volume doubling property and the Poincaré inequality are sufficient for the running of the Nash-Moser iteration scheme which implies many analytic consequences (see Grigor'yan[57], Saloff-Coste [106]).

Let G be a graph. Recall the Definition 8 for the normalized graph Laplace operator Δ. In this unweighted case, for a function $f : V \to \mathbb{R}$, it is defined as

$$\Delta f(x) = \frac{1}{d_x} \sum_{y, y \sim x} (f(y) - f(x)).$$

Also recall that the analogue of the square norm of gradient $\Gamma(f, f)$ of f (see (2.1.11)) is defined as

$$\Gamma(f, f)(x) = \frac{1}{2d_x} \sum_{y, y \sim x} (f(y) - f(x))^2.$$

Given a subset $\Omega \subset V$, a function f is called harmonic (subharmonic, superharmonic) on Ω if $\Delta f(x) = 0 (\geq 0, \leq 0)$ for any $x \in \Omega$.

3.7.1 Harnack inequality and Liouville theorem

In the Riemannian case, the Nash-Moser iteration implies the Harnack inequality for positive harmonic functions. It was proved by Delmotte [44] and Holopainen-Soardi [63] independently that the Harnack inequality for positive harmonic functions holds on graphs satisfying the volume doubling property and the Poincaré inequality. Applying their results to our case, we obtain the following theorem.

Theorem 40 ([44, 63])**.** *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$*. Then there exist constants $C_1 > 1$, $C_2(D) < \infty$ such that for any $R > 0, p \in G$ and any positive harmonic function u on $B_{C_1R}(p)$, we have*

$$\max_{B_R(p)} u \leq C_2(D) \min_{B_R(p)} u. \tag{3.7.1}$$

Remark 24. In [45], Delmotte obtained the parabolic Harnack inequality and the Gaussian estimate for the heat kernel which is stronger than the elliptic one of the preceding theorem.

In the Nash-Moser iteration, the mean value inequality for nonnegative subharmonic functions is obtained (see Coulhon-Grigor'yan [41]). Since the square of a harmonic function is subharmonic, we obtain

Lemma 18. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$*. Then there exist two constants C_1 and $C_2(D)$ such that for any $R > 0, p \in G$ and any harmonic function u on $B_{C_1R}(p)$, we have*

$$u^2(p) \leq \frac{C_2(D)}{|B_{C_1R}(p)|} \sum_{x \in B_{C_1R}(p)} u^2(x) d_x. \tag{3.7.2}$$

The Liouville theorem for positive harmonic functions follows from the Harnack inequality (see e.g. Saloff-Coste[107]).

Theorem 41. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$*. Then any positive harmonic function on G must be constant.*

Proof. Since G is a semiplanar graph with $\mathrm{Sec}G \geq 0$, then $D_G < \infty$. Let u be a positive harmonic function on G. By the Harnack inequality (3.7.1), we obtain

$$\max_{B_R}(u - \inf_G u) \leq C_2(D_G) \min_{B_R}(u - \inf_G u), \tag{3.7.3}$$

for any $R > 0$. The right hand side of (3.7.3) tends to 0 if $R \to \infty$. Therefore,

$$u \equiv \inf_G u = const.$$

□

3.7.2 Parabolicity

A manifold or a graph is called parabolic if it does not admit any nontrivial positive superharmonic function. The parabolicity of a manifold has been extensively studied in the literature (see e.g. Grigor'yan [58], Holopainen-Koskela [64], Rigoli-Salvatori-Vignati [104]). Let $G = (V, E)$ be a graph, C_1 be a finite subset of V and $C_1 \subset C_2 \subset V$.

Definition 15. *The capacity of C_1 with respect to C_2 is defined as*

$$Cap(C_1, C_2) = \inf \left\{ \sum_{x \in V} \Gamma(u, u)(x) d_x : u \text{ is finitely supported in } C_2, u|_{C_1} \equiv 1 \right\},$$

where the function u is called an admissible function for (C_1, C_2).

For $C_2 = G$, we denote $Cap(C_1) = Cap(C_1, G)$.

The following lemma is a criterion of the parabolicity of a graph given by Kanai [73].

Lemma 19. *A graph G is parabolic if and only if $Cap(S) = 0$, for some nonempty finite subset $S \subset G$.*

The following theorem is standard in the Riemannian case, and we prove it in the graph setting. Readers are referred to Holopainen-Koskela [64], Grigor'yan [58].

Theorem 42. *A graph G is parabolic if*

$$\int_1^\infty \frac{t}{|B_t|} = \infty \quad (\textit{equivalently} \sum_{i=1}^\infty \frac{i}{|B_i|} = \infty), \tag{3.7.4}$$

where $|B_t| = \sum_{x \in B_t(p)} d_x$ for some $p \in G$.

Proof. Let $B_k = B_{2^{2k}}(p)$, $C_k = B_{2^{2k+1}}(p)$, for $k = 0, 1, \ldots$. It is obvious that $B_0 \subset C_0 \subset B_1 \subset C_1 \subset \cdots$. We claim that

$$Cap(B_0) \leq \left(\sum_{k=0}^\infty Cap^{-1}(B_k, C_k) \right)^{-1}. \tag{3.7.5}$$

For any $\epsilon > 0$, by definition of capacity, there exist admissible functions u_k for (B_k, C_k) such that $u_k|_{B_k} \equiv 1, u_k|_{G - C_k} \equiv 0$, and

$$\sum_{x \in V} \Gamma(u, u)(x) d_x \leq Cap(B_k, C_k) + \epsilon.$$

Let $N \in \mathbb{N}$, $a_k \geq 0$ for $k = 0, 1, \ldots, N$, and $\sum_{k=0}^{N} a_k = 1$. We define a function $v = \sum_{k=0}^{N} a_k u_k$ which is finitely supported and $v|_{B_0} \equiv 1$. Then by the definition of $Cap(B_0)$, we have

$$\begin{aligned}
Cap(B_0) &\leq \sum_{x\in V} \Gamma(v,v)(x)d_x = \sum_{x\in V} \frac{1}{2} \sum_{y\sim x} \left[\sum_{k=0}^{N} a_k(u_k(y) - u_k(x))\right]^2 \\
&= \sum_{x\in V} \frac{1}{2} \sum_{y\sim x} \sum_{k=0}^{N} a_k^2 (u_k(y) - u_k(x))^2 \\
&= \sum_{k=0}^{N} a_k^2 \sum_{x\in V} \Gamma(u_k, u_k)(x)d_x \leq \sum_{k=0}^{N} a_k^2 Cap(B_k, C_k) + \sum_{k=0}^{N} a_k^2 \epsilon \\
&\leq \sum_{k=0}^{N} a_k^2 Cap(B_k, C_k) + \sum_{k=0}^{N} a_k \epsilon \\
&= \sum_{k=0}^{N} a_k^2 Cap(B_k, C_k) + \epsilon,
\end{aligned}$$

where we use in the second line $(u_k(y) - u_k(x))(u_l(y) - u_l(y)) = 0$, for any $k \neq l$ and $x \sim y$. We choose $a_k = \frac{Cap^{-1}(B_k,C_k)}{\sum_{k=0}^{N} Cap^{-1}(B_k,C_k)}$. Then we let $\epsilon \to 0$ to get

$$Cap(B_0) \leq \left(\sum_{k=0}^{N} Cap^{-1}(B_k, C_k)\right)^{-1}.$$

We prove the claim by letting $N \to \infty$.

Next, we estimate the capacity $Cap(B_k, C_k)$ of (B_k, C_k) for $k = 0, 1, \ldots$. Let

$$w_k(x) = \begin{cases} 1, & d^G(p,x) < 2^{2k}, \\ \frac{2^{2k+1} - d^G(p,x)}{2^{2k}}, & 2^{2k} \leq d^G(p,x) < 2^{2k+1}, \\ 0, & 2^{2k+1} \leq d^G(p,x). \end{cases}$$

It is easy to check that w_k is an admissible function for (B_k, C_k). Furthermore, $\Gamma(w_k, w_k)(x) \leq \frac{1}{2^{4k+1}}$ for any $x \in G$ and is supported in $C_k = B_{2^{2k+1}}$. Therefore we have

$$Cap(B_k, C_k) \leq \sum_{x\in V} \Gamma(w_k, w_k)(x)d_x \leq \frac{|B_{2^{2k+1}}|}{2^{4k+1}}.$$

By the claim (3.7.5), we have

$$Cap(B_0) \leq \left(\sum_{k=0}^{\infty} \frac{2^{4k+1}}{|B_{2^{2k+1}}|)}\right)^{-1}. \tag{3.7.6}$$

By the assumption (3.7.4), we have

$$\begin{aligned}\infty = \int_2^\infty \frac{t}{|B_t|} &= \sum_{k=0}^{\infty} \sum_{i=2^{2k+1}}^{2^{2k+3}} \frac{i}{|B_i|} \\ &\leq \sum_{k=0}^{\infty} \frac{2^{2k+3}}{|B_{2^{2k+1}}|}(2^{2k+3} - 2^{2k+1} + 1) \\ &= C \sum_{k=0}^{\infty} \frac{2^{4k+1}}{|B_{2^{2k+1}}|}.\end{aligned}$$

Hence by (3.7.6), we obtain that $Cap(B_0) = 0$, which implies that G is parabolic by Lemma 19.

□

Corollary 10. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then G is parabolic.*

Proof. Since $D_G < \infty$, we have the quadratic volume growth property (3.4.5),

$$|B_R(p)| \leq C(D_G)R^2.$$

Hence

$$\int_1^\infty \frac{t}{|B_t|} = \infty,$$

which implies that G is parabolic by Theorem 42. □

3.7.3 Polynomial growth harmonic functions theorem

In the last part of this section, we investigate the polynomial growth harmonic function theorem on graphs. Let $H^d(G) := \{u : \Delta u \equiv 0, |u(x)| \leq C(d^G(p,x) + 1)^d\}$ be the space of polynomial growth harmonic functions of growth degree d on G.

On Riemannian manifolds, the polynomial growth harmonic function theorem was proved by Colding and Minicozzi in [38]. And the proof was then simplified by Li [81]. By assuming the volume doubling property (3.4.12) and the Poincaré inequality (3.6.1) on the graph, Delmotte [43] proved the polynomial growth harmonic function theorem with the dimension estimate in our case

$$\dim H^d(G) \leq C(D)d^{v(D)},$$

where $C(D)$ and $v(D)$ depend on the maximal facial degree D of the semiplanar graph G with nonnegative curvature. In fact

$$v(D) = \frac{\ln(4C_{rel}(D))}{\ln 2} = 2 + \frac{\ln C_{rel}(D)}{\ln 2},$$

where $C_{rel}(D)$ is the constant in (3.4.1). We improve Delmotte's dimension estimate of $H^d(G)$ by using the relative volume comparison (3.4.1) instead of the volume doubling property (3.4.12).

Theorem 43. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$. *Then*

$$\dim H^d(G) \leq C(D)d^2, \tag{3.7.7}$$

for any $d \geq 1$.

Our proof of the theorem follows Li's argument by the mean value inequality. From now on, we fix some vertex $p \in V$, and denote $B_R = B_R(p)$ for short. We need the following lemmas.

Lemma 20. *For any finite dimensional subspace $K \subset H^d(G)$, there exists a constant $R_0(K)$ depending on K such that for any $R \geq R_0(K)$, $u, v \in K$,*

$$A_R(u,v) := \sum_{x \in B_R} u(x)v(x)d_x$$

is an inner product on K.

Proof. We only need to prove that if $A_R(u,u) = 0$, then $u \equiv 0$ not only on B_R but also on the whole graph. This can be easily proved by a contradiction argument as in Hua [66]. □

Lemma 21. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$, *K be a k-dimensional subspace of $H^d(G)$. Given $\beta > 1, \delta > 0$, for any $R_1 \geq R_0(K)$ there exists $R > R_1$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $A_{\beta R}$, then*

$$\sum_{i=1}^k A_R(u_i, u_i) \geq k\beta^{-(2d+2+\delta)}.$$

Proof. Note that $\sum_{i=1}^R A_{\beta R}(u_i, u_i) = k$. The proof is the same as Li [82], Delmotte [43], Hua [66]. We omit the details here and only remind the reader here that the polynomial growth of harmonic functions and volume growth estimate (3.4.5) are used in the proof to estimate

$$A_R(u_i, u_i) = \sum_{x \in B_R} u_i^2(x)d_x \leq C(D)R^{2d+2}.$$

□

Lemma 22. *Let G be a semiplanar graph with* $\mathrm{Sec}G \geq 0$, *K be a k-dimensional subspace of $H^d(G)$. Then there exists a constant $C(D)$ such that for any basis of K, $\{u_i\}_{i=1}^k$, $R > 0, 0 < \epsilon < \frac{1}{2}$, we have*

$$\sum_{i=1}^k A_R(u_i, u_i) \leq C(D)\epsilon^{-2} \sup_{u \in <A,U>} \sum_{y \in B_{(1+\epsilon)R}} u^2(y)d_y,$$

where $< A, U >:= \{w = \sum_{i=1}^k a_i u_i : \sum_{i=1}^k a_i^2 = 1\}$.

Proof. For any $x \in B_R$, we set $K_x = \{u \in K : u(x) = 0\}$. It is easy to see that $\dim K/K_x \leq 1$. Hence there exists an orthonormal linear transformation $\phi : K \to K$, which maps $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$ such that $v_i \in K_x$, for $i \geq 2$. The mean value inequality (3.7.2) implies that

$$\begin{aligned}\sum_{i=1}^k u_i^2(x) &= \sum_{i=1}^k v_i^2(x) = v_1^2(x)\\ &\leq C(D)|B_{(1+\epsilon)R-r(x)}|^{-1} \sum_{y\in B_{(1+\epsilon)R-r(x)}(x)} v_1^2(y)d_y\\ &\leq C(D)|B_{(1+\epsilon)R-r(x)}|^{-1} \sup_{u\in<A,U>} \sum_{y\in B_{(1+\epsilon)R}} u^2(y)d_y,\end{aligned}$$

where $r(x) = d^G(p, x)$.

By the relative volume comparison (3.4.1), we have

$$\begin{aligned}|B_{(1+\epsilon)R-r(x)}| &\geq C(D)\left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^2 |B_{2R}(x)|\\ &\geq C(D)\left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^2 |B_R(p)|.\end{aligned}$$

Then we arrive at

$$\sum_{i=1}^k \sum_{x\in B_R} u_i^2(x)d_x \leq \frac{C(D)}{|B_R|} \sum_{x\in B_R} \left(1+\epsilon - \frac{r(x)}{R}\right)^{-2} d_x \sup_{u\in<A,U>} \sum_{y\in B_{(1+\epsilon)R}} u^2(y)d_y.$$

An estimate of the summation

$$\sum_{x\in B_R} \left(1+\epsilon - \frac{r(x)}{R}\right)^{-2} d_x \leq \epsilon^{-2}|B_R| \tag{3.7.8}$$

completes the proof of this lemma. □

Proof of Theorem 43. For any k-dimensional subspace $K \subset H^d(G)$, we set $\beta = 1+\epsilon$. By Lemma 21, there exists $R > R_0(K)$ such that for any orthonormal basis $\{u_i\}_{i=1}^k$ of K with respect to $A_{(1+\epsilon)R}$, we have

$$\sum_{i=1}^k A_R(u_i, u_i) \geq k(1+\epsilon)^{-(2d+2+\delta)}.$$

Lemma 22 implies that

$$\sum_{i=1}^k A_R(u_i, u_i) \leq C(D)\epsilon^{-2}.$$

Setting $\epsilon = \frac{1}{2d}$, and letting $\delta \to 0$, we obtain

$$k \leq C(D)\left(\frac{1}{2d}\right)^{-2}\left(1+\frac{1}{2d}\right)^{2d+2+\delta} \leq C(D)d^2. \tag{3.7.9}$$

□

The dimension estimate in (3.7.7) is not satisfactory since in Riemannian geometry the constant $C(D)$ depends only on the dimension of the manifold rather than the maximal facial degree of G. Note that Theorem 32 shows that the semiplanar graph G with $\mathrm{Sec}G \geq 0$ and $D_G \geq 43$ has a special structure, i.e. the one-side cylinder structure of linear volume growth. In Riemannian geometry, Sormani [108] used Yau's gradient estimate and the nice behavior of the Busemann function on a one-end Riemannian manifold with nonnegative Ricci curvature of linear volume growth to show that it does not admit any nontrivial polynomial growth harmonic function. Inspired by the work [108] and the special structure of semiplanar graphs with nonnegative curvature and large face degree, we shall prove the following theorem.

Theorem 44. *Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$ and $D_G \geq 43$. Then for any $d > 0$,*

$$\dim H^d(G) = 1.$$

To prove the theorem, we need a weak version of the gradient estimate given by Lin-Xi [87]. We recall the Cacciappoli inequality for harmonic functions on the graph G.

Theorem 45. *Let G be a graph and $d_m = \sup_{x\in G} d_x$. For any harmonic function u on B_{6r}, $r \geq 1$, we have*

$$\sum_{x\in B_r} \Gamma(u,u)(x)d_x \leq \frac{C(d_m)}{r^2} \sum_{y\in B_{2r}} u^2(y)d_y,$$

Moreover for any $x \in B_r$,

$$\Gamma(u,u)(x)d_x \leq \frac{C(d_m)}{r^2} \sum_{y\in B_{2r}} u^2(y)d_y. \tag{3.7.10}$$

Corollary 11. *Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$ and $D_G \geq 43$. For any harmonic function u on G, we have*

$$\sqrt{\Gamma(u,u)(x)d_x} \leq \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{2r}(x)} u, \tag{3.7.11}$$

where $\mathrm{osc}_{B_{2r}(x)} u = \max_{B_{2r}(x)} u - \min_{B_{2r}(x)} u$.

Proof. By Theorem 32, the regular polygonal surface $S(G)$ has linear volume growth. As in the proof of (3.4.5) in Theorem 38, we obtain that for any $x \in G$ and $r \geq 1$,

$$|B_r(x)| \leq C(D)r. \tag{3.7.12}$$

By (3.7.10) in Theorem 45 and $d_m \leq 6$, we have

$$\Gamma(u,u)(x)d_x \leq \frac{C}{r^2} \sum_{y\in B_{2r}(x)} u^2(y)d_y \leq \frac{C}{r^2} |B_{2r}(x)| \max_{B_{2r}(x)} |u|^2. \tag{3.7.13}$$

Replacing u by $u - \min_{B_{2r}(x)} u$ in (3.7.13) and noting (3.7.12), we obtain

$$\sqrt{\Gamma(u,u)(x)d_x} \leq \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{2r}(x)} u.$$

□

Remark 25. We call (3.7.10) the weak version of the gradient estimate since its scaling is not as usual, but it suffices for our application.

Proof of Theorem 44. Let G be a semiplanar graph with $\mathrm{Sec}\, G \geq 0$ and $D_G \geq 43$. Let σ be the largest face with $\deg(\sigma) = D_G = D \geq 43$. By Theorem 32, either G looks like $\sigma, L_1, L_2, \cdots, L_m, \cdots$ where each L_m has the same type of faces (triangle or square), i.e. $G = \sigma \cup \bigcup_{m=1}^{\infty} L_m$, or $G = T^{-1}(\sigma \cup \bigcup_{m=1}^{\infty} L_m)$, where T^{-1} is the graph operation defined in section 3.1.2. Denote by $A = \sigma \cap G$ the set of vertices incident to σ, by $d^G(x, A) = \min_{y \in A} d^G(x, y)$ the distance function of A in G. Let $B_r(A) = \{x \in G : d^G(x, A) \leq r\}$ and $\partial B_r(A) = \{x \in G : d^G(x, A) = r\}$. By the construction of G, for any $x, y \in \partial B_r(A)$, there is a path joining x and y in $B_r(A)$ with length less than or equal to $5D$. In addition, for any $q \in A$, we have

$$\partial B_r(A) \subset B_{r+D}(q) \setminus B_{r-D}(q). \tag{3.7.14}$$

Let $u \in H^d(G)$ and $M(r) = \mathrm{osc}_{\partial B_r(A)} u = \max_{\partial B_r(A)} u - \min_{\partial B_r(A)} u$. By the maximal principle which is a direct consequence of the definition of the harmonic function, we have $\max_{\partial B_r(A)} u = \max_{B_r(A)} u$ and $\min_{\partial B_r(A)} u = \min_{B_r(A)} u$, so that $M(r)$ is nondecreasing in r. To prove the theorem, it suffices to show that $M(r) = 0$ for any large r. Let $y_r, x_r \in \partial B_r(A)$ satisfy $u(y_r) = \max_{\partial B_r(A)} u$ and $u(x_r) = \min_{\partial B_r(A)} u$. Then there exists a path in $B_r(A)$ such that

$$y_r = z_0 \sim z_1 \sim \cdots \sim z_l = x_r,$$

where $z_i \in B_r(A)$ for $0 \leq i \leq l$ and $l \leq 5D$. Hence

$$\begin{aligned} M(r) &= u(y_r) - u(x_r) \leq \sum_{i=0}^{l-1} \sqrt{2\Gamma(u,u)(z_i)d_{z_i}} \\ &\leq C \sum_{i=0}^{l-1} \frac{C(D)}{\sqrt{r}} \mathrm{osc}_{B_{2r}(z_i)} u \\ &\leq C(D) \frac{\mathrm{osc}_{B_{3r+D}(q)} u}{\sqrt{r}} \cdot (5D) \\ &\leq C(D) \frac{\mathrm{osc}_{B_{3r+2D}(A)} u}{\sqrt{r}} \\ &\leq C(D) \frac{M(5r)}{\sqrt{r}} \quad for \;\; r \geq D, \end{aligned} \tag{3.7.15}$$

where we use (3.7.11) in Corollary 11, $B_{2r}(z_i) \subset B_{r+D+2r}(q)$ and (3.7.14).

Let $r \geq R_0(D,\delta) = (\frac{C(D)}{\delta})^2$, for $\delta < 1$ which will be chosen later. Then we have $\frac{C(D)}{\sqrt{r}} \leq \delta < 1$. By (3.7.15), for any $r \geq R_0(D,\delta)$, we obtain that for $k \geq 1$,

$$M(r) \leq \delta M(5r) \leq \delta^k M(5^k r).$$

Since $u \in H^d(G)$, we have

$$M(r) \leq 2 \max_{B_{r+D}(q)} |u| \leq 2C\big((r+D)^d + 1\big).$$

Hence

$$M(r) \leq C\delta^k\big((5^k r + D)^d + 1\big) \leq C2^{d+1}\delta^k(5^k r)^d = C(d)\left(\frac{1}{2}\right)^k r^d,$$

if we choose $\delta = \frac{1}{2\cdot 5^d}$. Then for any $r \geq R_0(D,\delta) = (C(D)2\cdot 5^d)^2$, we have

$$M(r) \leq C(d)\left(\frac{1}{2}\right)^k r^d.$$

Letting $k \to \infty$, we obtain $M(r) = 0$, which proves the theorem. □

Combining Theorem 43 with Theorem 44, we obtain a dimension estimate that does not depend on the maximal facial degree D_G.

Theorem 46. *Let G be a semiplanar graph with Sec $G \geq 0$. Then for any $d \geq 1$,*

$$\dim H^d(G) \leq Cd^2,$$

where C is an absolute constant.

Chapter 4

Conclusions

In this chapter we give a summary of the results we got in this thesis and propose some interesting open problems related to previous discussions for future research.

4.1 Summary

In this thesis, we have studied several synthetic curvature notions on graphs: Ollivier-Ricci curvature and the curvature dimension inequality, the combinatorial curvature and the generalized curvature bounds in the sense of Alexandrov.

In Chapter 2, we studied the Ollivier-Ricci curvature on locally finite graphs, its relation with the curvature dimension inequality, which is a generalization of Bochner's formula in Riemannian geometry, and its applications in the spectrum estimates of the normalized Laplace operator on finite graphs. We first got the following lower and upper bound estimates of Ollivier-Ricci curvature κ (Theorem 13, 14 in Section 2.2).

- On $G = (V, E)$, we have for any pair of neighbors $x, y \in V$,

$$\begin{aligned}\kappa(x,y) \geq k(x,y) := &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \vee \frac{w_{x_1y}}{d_y}\right)_+ \\ &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}\right)_+ \\ &+ \sum_{x_1 \in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y} + \frac{w_{xx}}{d_x} + \frac{w_{yy}}{d_y}.\end{aligned}$$

  Moreover, this inequality is sharp for certain graphs.

- On $G = (V, E)$, we have for every pair of neighbors x, y,

$$\kappa(x,y) \leq \sum_{x_1 \in \{x\} \cup \{y\} \cup N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}.$$

Therefore the Ollivier-Ricci curvature of neighboring vertices x, y is related to the number of triangles $\sharp(x, y)$ including x, y and the number of self-loops $s(x)$, $s(y)$ at x, y. In particular we got (Corollary 2 in Section 2.2)

- If there exists two vertices $x \sim y$ in G such that $\sharp(x, y) = s(x) = s(y) = 0$ then $\kappa(x, y) \leq 0$.

Furthermore, on simple locally finite graphs, the lower and upper bound estimates imply relations between a scalar curvature $\kappa(x)$ (which ia an average of Ollivier-Ricci curvature $\kappa(x, y)$ over all neighbors of x) and the Watts-Strogatz's local clustering coefficient $c(x)$ (see Subsection 2.2.3).

We then studied the curvature dimension inequality on locally finite graphs. We established the following inequality (Theroem 19 in Section 2.3).

- On (V, d, m), the Laplace operator satisfies

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}t_w(x) - 1\right)\Gamma(f, f)(x),$$

  where

$$t_w(x) := \min_{y, y\sim x}\left\{\frac{4w_{xy} + w_{yy}}{d_y} + \sum_{x_1, x_1\sim x, x_1\sim y}\left(\frac{w_{xy}}{d_y} \wedge \frac{w_{xx_1}}{d_{x_1}}\right)\frac{w_{x_1y}}{w_{xy}}\right\}.$$

The curvature function in the above inequality is also related to the number of triangles and the number of self-loops. Then, on simple locally finite graphs, supposing that Ollivier-Ricci curvature has a positive lower bound, we got the following curvature dimension inequality (Corollary 5 in Section 2.3).

- On a simple locally finite graph $G = (V, E)$, if $\kappa(x, y) \geq k > 0$, then we have

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}\min_{y, y\sim x}\left\{\frac{4}{d_y} + \frac{\lceil kd_x \vee d_y\rceil}{D(x)}\right\} - 1\right)\Gamma(f, f)(x).$$

For a complete graph we further got (Proposition 7 in Section 2.3)

- On a complete graph $\mathcal{K}_N$ $(N \geq 2)$ with N vertices, the Laplace operator Δ satisfies for $n \in [1, +\infty]$,

$$\Gamma_2(f, f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + \left(\frac{4 - N}{2(N - 1)} + \frac{n - 2}{n}\right)\Gamma(f, f)(x).$$

  Moreover, for every fixed dimension parameter n, the curvature term is optimal.

The interesting point is that when n of $\mathcal{K}_N$ is chosen to be $N-1$, the curvature function is exactly $\frac{1}{2}$ of the Ollivier-Ricci curvature of $\mathcal{K}_N$.

In Section 2.4, we studied the estimates of the spectrum of the normalized Laplacian operator on finite graphs and got the following result (Theorem 22).

- Let $k[t]$ be a lower bound of Ollivier-Ricci curvature of the neighborhood graph $G[t]$. Then for all $t \geq 1$ the eigenvalues of Δ on G satisfy

$$1-(1-k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1+(1-k[t])^{\frac{1}{t}}.$$

  Moreover, if G is not bipartite, then there exists a $t' \geq 1$ such that for all $t \geq t'$ the eigenvalues of Δ on G satisfy

$$0 < 1-(1-k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1+(1-k[t])^{\frac{1}{t}} < 2.$$

It turns out that curvature on neighborhood graphs also reflects the number of cycles with length larger than 3.

In Chapter 3, we studied the nonnegative combinatorial curvature on infinite semiplanar graphs. Unlike the previous Gauss-Bonnet formula approach, we adopted an Alexandrov approach. Given a semiplanar graph $G=(V,E,F)$, we have three metric spaces (V,d^G), (G_1,d^{G_1}) and $(S(G),d)$, where G_1 is the corresponding 1-dimensional simplicial complex and $S(G)$ is the regular polygonal surface of G. The restriction of d^{G_1} on V is the same as d^G. We proved that the restriction of intrinsic metric d of $S(G)$ on G_1 is bi-Lipschitz equivalent with d^{G_1} (Lemma 9 in Section 3.2).

- Let G be a semiplanar graph. Then there exists a constant C such that for any $x,y \in G_1$,

$$Cd^{G_1}(x,y) \leq d(x,y) \leq d^{G_1}(x,y).$$

Another basic observation was (Corollary 8 in Section 3.2)

- Let G be a semiplanar graph. Then G has nonnegative curvature everywhere if and only if $S(G)$ is an Alexandrov space with nonnegative curvature.

Applying the Cheeger-Gromoll splitting theorem on $S(G)$, we proved (Theorem 34 in Section 3.3)

- Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$. If $N(G_1) \geq 2$, then $S(G)$ is isometric to a cylinder without boundary.

Moreover, we got the metric classification of infinite semiplanar graphs with nonnegative curvature (Theorem 35 in Section 3.3).

- Let G be an infinite semiplanar graph with nonnegative curvature. If G has positive curvature somewhere, then $S(G)$ is isometric to a cap which is homeomorphic but not isometric to the plane. If G has vanishing curvature everywhere, then $S(G)$ is isometric to a plane, or a cylinder without boundary if it is orientable, and $S(G)$ is isometric to a metric space obtained from gluing the boundary of $[0, a] \times \mathbb{R}$ by a glide reflection, $\iota = T_{a,L} \circ F_L$, where L is perpendicular to the cylinder, if it is nonorientable.

At the end of Section 3.3, we discussed the construction of infinite semiplanar graphs with nonnegative curvature everywhere which can be embedded into the projective plane minus one point.

We proved two basic inequalities, relative volume comparison and Poincaré inequality on semiplanar graphs with nonnegative curvature (Theorem 38 in Section 3.4 and Theorem 39 in Section 3.6).

- Let $G = (V, E, F)$ be a semiplanar graph with $\mathrm{Sec}G \geq 0$. Then there exists a constant $C_{rel}(D)$ depending on D, such that for any $p \in G$ and $0 < r < R$, we have
$$\frac{|B_R(p)|}{|B_r(p)|} \leq C_{rel}(D) \left(\frac{R}{r}\right)^2.$$

This implies the volume doubling property. And we discussed the constant $C_{rel}(D)$ further on Archimedean tilings on a plane using Preá's distance sequence formula (with some modifications) in Section 3.5.

- Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$. Then there exist two constants $C(D)$ and $C > 1$ such that for any $p \in G, R > 0, f : B_{CR}(p) \to \mathbb{R}$, we have
$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 d_x \leq C(D) R^2 \sum_{x,y \in B_{CR}(p); x \sim y} (f(x) - f(y))^2,$$
where $f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x) d_x$.

We also studied several analytic consequences of these two basic inequalities. On infinite semiplanar graphs with nonnegative combinatorial curvature, we proved the Harnack inequality for positive harmonic functions and then the following Liouville theorem (Theorem 41 in Section 3.7).

- Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$. Then any positive harmonic function on G must be constant.

We proved the parabolicity (Corollary 10 in Section 3.7)

- Let G be a semiplanar graph with $\mathrm{Sec}G \geq 0$. Then G is parabolic in the sense that it does not support any nontrivial positive superharmonic function (equivalently, Brownian motion is recurrent).

Finally, we proved the polynomial growth harmonic functions theorem (Theorem 46 in Section 3.7). Let $H^d(G) := \{u : \Delta u \equiv 0, |u(x)| \leq C(d^G(p,x)+1)^d\}$ be the space of polynomial growth harmonic functions of growth degree d on G.

- Let G be a semiplanar graph with Sec $G \geq 0$. Then for any $d \geq 1$,

$$\dim H^d(G) \leq Cd^2,$$

  where C is an absolute constant.

4.2 Future research

In this final section, we propose the following problems for future research.

- In Subsection 2.2.3, we studied the relation between the scalar curvature $\kappa(x)$ and the Watts-Strogatz local clustering coefficient $c(x)$. We got Corollary 3 and in Remark 12 more precise relations in some special cases. Those precise relations has a rather simple form for d-regular graphs. We are interested in exploring those precise relations further on more general graphs.

- In Section 2.3, supposing the Ollivier-Ricci curvature has a positive lower bound k, we proved in Corollary 5 a curvature dimension inequality with dimension parameter 2. But the curvature function of the inequality is not exactly the lower Ollivier-Ricci curvature bound k as in Riemannian case. On complete graphs, by choosing a proper dimension parameter, we got a curvature dimension inequality with the curvature function equals to the Ollivier-Ricci curvature (up to a scalar $\frac{1}{2}$). Can one choose proper dimension parameter for general graphs and do similar things?

- In Section 2.4, we studied estimates of spectrum of the normalized Laplacian operator on finite graphs in terms of Ollivier-Ricci curvature. Note that the estimates of Neumann eigenvalues and Dirichlet eigenvalues of subgraphs and their relation with random walks have also been studied in the literature (see Chapter 8 in Chung [24] and the references therein). We are interested in exploring the estimates of Neumann or Dirichlet eigenvalues of subgraphs in terms of Ollivier-Ricci curvature.

- In Chapter 3, we studied the infinite semiplanar graphs with nonnegative combinatorial curvature, i.e. the 2-dimensional case of combinatorial curvature (the synthetic curvature defined by deficiency of angles). We know in the work of Regge [102], Stone [109], there are also curvature notions defined in higher dimension cases. Can one generalize the arguments in Chapter 3 to the higher dimension case?

Bibliography

[1] Z. Bai, Y. Shen, N. Shui, X. Guo, Liman jihe chubu (Chinese. Chinese summary) [The beginnings of Riemannian geometry], Gaodeng Jiaoyu Chubanshe (Higher Education Press), Beijing, 1992.

[2] D. Bakry, Functional inequalities for Markov semigroups, in: S. G. Dani and P. Graczyk (Eds.), Probability measures on groups: recent directions and trends, Tata Inst. Fund. Res., Mumbai, 2006, pp. 91-147.

[3] D. Bakry and M. Émery, Diffusions hypercontractives (French) [Hypercontractive diffusions], in: J. Azéma and M. Yor (Eds.), Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math. 1123, Springer, Berlin, 1985, pp. 177-206.

[4] D. Bakry, and M. Émery, Hypercontractivité de semi-groupes de diffusion (French. English summary) [Hypercontractivity for diffusion semigroups], C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 15, 775-778.

[5] A. Banerjee and J. Jost, On the spectrum of the normalized graph Laplacian, Linear Algebra Appl. 428 (2008), 3015-3022.

[6] M. Barlow, T. Coulhon and A. Grigor'yan, Manifolds and graphs with slow heat kernel decay, Invent. Math. 144 (2001), no. 3, 609-649.

[7] F. Bauer and J. Jost, Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian, http://arxiv.org/abs/0910.3118v3, to appear in Comm. Anal. Geom..

[8] F. Bauer, J. Jost and S. Liu, Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator, to appear in Math. Res. Lett..

[9] O. Baues and N. Peyerimhoff, Curvature and geometry of tessellating plane graphs, Disctete Comput. Geom. 25 (2001), 141-159.

[10] O. Baues and N. Peyerimhoff, Geodesics in non-positively curved plane tessellations, Adv. Geom. 6 (2006), no. 2, 243-263.

[11] A.-I. Bonciocat and K.-T. Sturm, Mass transportation and rough curvature bounds for discrete spaces, J. Funct. Anal. 256 (2009), 2944-2966.

[12] R. Bubley and M. E. Dyer, Path coupling: A technique for proving rapid mixing in Markov chains, Proceedings of the 38th Annual Symposium on Foundations of Computer Science (FOCS '97), IEEE Comput. Soc., Washington, DC, 1997, pp. 223-231.

[13] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, Graduate Studies in Mathematics 33, American Mathematical Society, Providence, RI, 2001.

[14] Y. Burago, M. Gromov and G. Perelman, A. D. Aleksandrov spaces with curvatures bounded below, Russ. Math. Surv. 47 (1992), 1–58.

[15] P. Buser, A note on the isoperimetric constant, Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 2, 213-230.

[16] I. Chavel, Eigenvalues in Riemannian geometry, including a chapter by B. Randol, with an appendix by J. Dodziuk, Pure and Applied Mathematics 115. Academic Press, Inc., Orlando, FL, 1984.

[17] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999), no. 3, 428–517.

[18] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Diff. Geom. 6 (1971/72), 119-128.

[19] B. Chen, The Gauss-Bonnet formula of polytopal manifolds and the characterization of embedded graphs with nonnegative curvature, Proc. Amer. Math. Soc. 137 (2009), no. 5, 1601-1611.

[20] B. Chen and G. Chen, Gauss-Bonnet formula, finiteness condition, and characterizations of graphs embedded in surfaces, Graphs Combin. 24 (2008), no. 3, 159-183.

[21] M. Chen, Trilogy of couplings and general formulas for lower bound of spectral gap, in: L. Accardi and C. C. Heyde (Eds.), Probability towards 2000 (New York, 1995), Lecture Notes in Statist. 128, Springer, New York, 1998, pp. 123-136.

[22] M. Chen, Coupling, spectral gap and related topics I, Chinese Sci. Bull. 42 (1997), no. 16, 1321-1327.

[23] M. Chen and F. Wang, Application of coupling method to the first eigenvalue on manifold, Sci. China Ser. A 37 (1994), no. 1, 1-14.

[24] F. R. K. Chung, Spectral graph theory, CBMS Regional Conference Series in Mathematics 92, American Mathematical Society, Providence, RI, 1997.

[25] F. R. K. Chung, A. Grigor'yan and S. T. Yau, Upper bounds for eigenvalues of the discrete and continuous Laplace operators, Adv. Math. 117 (1996), no. 2, 165-178.

[26] F. R. K. Chung, A. Grigor'yan and S. T. Yau, Eigenvalues and diameters for manifolds and graphs, in: S. T. Yau (Ed.), Tsing Hua lectures on geometry & analysis (Hsinchu, 1990-1991), International Press, Cambridge, MA, 1997, pp. 79-105.

[27] F. R. K. Chung, A. Grigor'yan and S. T. Yau, Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs, Comm. Anal. Geom. 8 (2000), no. 5, 969-1026.

[28] F. R. K. Chung and S. T. Yau, A Harnack inequality for homogeneous graphs and subgraphs, Comm. Anal. Geom. 2 (1994), no. 4, 627-640.

[29] F. R. K. Chung and S. T. Yau, Eigenvalues of graphs and Sobolev inequalities, Combin. Probab. Comput. 4 (1995), no. 1, 11-25.

[30] F. R. K. Chung and S. T. Yau, A Harnack inequality for homogeneous graphs and subgraphs, Turkish J. Math. 19 (1995), no. 2, 119-129.

[31] F. R. K. Chung and S. T. Yau, Logarithmic Harnack inequalities, Math. Res. Lett. 3 (1996), 793-812.

[32] F. R. K. Chung and S. T. Yau, A combinatorial trace formula, in: S. T. Yau (Ed.), Tsing Hua lectures on geometry & analysis (Hsinchu, 1990-1991), International Press, Cambridge, MA, 1997, pp. 107-116.

[33] F. R. K. Chung and S. T. Yau, Eigenvalue inequalities for graphs and convex subgraphs, Comm. Anal. Geom. 5 (1997), no. 4, 575-623.

[34] F. R. K. Chung and S. T. Yau, Coverings, heat kernels and spanning trees, Electron. J. Combin. 6 (1999), Research Paper 12, 21 pp. (electronic).

[35] F. R. K. Chung and S. T. Yau, A Harnack inequality for Dirichlet eigenvalues, J. Graph Theory 34 (2000), no. 4, 247-257.

[36] F. R. K. Chung and S. T. Yau, Discrete Green's functions, In memory of Gian-Carlo Rota, J. Combin. Theory Ser. A 91 (2000), no. 1-2, 191-214.

[37] T. H. Colding and W. P. Minicozzi II, Harmonic functions with polynomial growth, J. Diff. Geom. 46 (1997), no. 1, 1-77.

[38] T. H. Colding and W. P. Minicozzi II, Harmonic functions on manifolds, Ann. of Math. (2) 146 (1997), no. 3, 725–747.

[39] T. H. Colding and W. P. Minicozzi II, Weyl type bounds for harmonic functions, Invent. Math. 131 (1998), no. 2, 257-298.

[40] T. H. Colding and W. P. Minicozzi II, Liouville theorems for harmonic sections and applications, Comm. Pure Appl. Math. 51 (1998), no. 2, 113-138.

[41] T. Coulhon and A. Grigor'yan, Random walks on graphs with regular volume growth, Geom. Funct. Anal. 8 (1998), no. 4, 656-701.

[42] T. Coulhon and L. Saloff-Coste, Variétés riemanniennes isométriques l'infini (French) [Isometric Riemannian manifolds at infinity], Rev. Mat. Iberoamericana 11 (1995), no. 3, 687-726.

[43] T. Delmotte, Harnack inequalities on graphs, Séminaire de Théorie Spectrale et Géométrie 16, Année 1997-1998, 217-228.

[44] T. Delmotte, Inégalité de Harnack elliptique sur les graphes (French) [Elliptic Harnack inequality on graphs], Colloq. Math. 72 (1997), no. 1, 19-37.

[45] T. Delmotte, Parabolic Harnack inequality and estimates of Markov chains on graphs, Rev. Mat. Iberoamericana 15 (1999), no. 1, 181-232.

[46] M. DeVos and B. Mohar, An analogue of the Descartes-Euler formula for infinite graphs and Higuchi's conjecture, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3287-3300.

[47] P. Diaconis and D. Stroock, Geometric bounds for eigenvalues of Markov chains, Ann. Appl. Probab. 1 (1991), 36-61.

[48] P. Diaconis and L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains, Ann. Appl. Probab. 6 (1996), 695-750.

[49] P. Diaconis, From shuffling cards to walking around the building: an introduction to modern Markov chain theory, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), Doc. Math., 1998, Extra Vol. I, 187-204.

[50] P. Diaconis, The Markov chain Monte Carlo revolution, Bull. Amer. Math. Soc. (N.S.) 46 (2009), 179-205.

[51] J. Dodziuk and L. Karp, Spectral and function theory for combinatorial Laplacians, in: R. Durrett and M. A. Pinsky (Eds.), Geometry of random motion (Ithaca, N.Y., 1987), Contemp. Math., 73, Amer. Math. Soc., Providence, RI, 1988, pp. 25-40.

[52] M. Erbar and J. Maas, Ricci curvature of finite Markov chains via convexity of the entropy, http://arxiv.org/abs/1111.2687.

[53] P. Erdös, F. Harary and W. T. Tutte, On the dimension of a graph, Mathematika 12 (1965), 118-122.

[54] L. C. Evans, Partial differential equations and Monge-Kantorovich mass transfer, in: R. Bott, A. Jaffe, D. Jerison, G. Lusztig and S. T. Yau (Eds.), Current developments in mathematics, 1997 (Cambridge, MA), International Press, Boston, MA, 1999, pp.65-126.

[55] R. Forman, Bochner's method for cell complexes and combinatorial Ricci curvature, Discrete Comput. Geom. 29 (2003), 323-374.

[56] S. Gallot, D. Hulin, J. Lafontaine, Riemannian geometry, Third edition, Universitext, Springer-Verlag, Berlin, 2004.

[57] A. Grigor'yan, The heat equation on noncompact Riemannian manifolds, (Russian) Mat. Sb. 182 (1991), no. 1, 55–87; translation in Math. USSR-Sb. 72 (1992), no. 1, 47-77.

[58] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 2, 135-249.

[59] A. Grigor'yan, Analysis on graphs, Lecture notes University of Bielefeld, 2009.

[60] M. Gromov, Hyperbolic groups, in: S. M. Gersten, (Ed.) Essays in group theory, M. S. R. I. Publ. 8, Springer, 1987, pp.75-263.

[61] B. Grünbaum and G. C. Shephard, Tilings and patterns, W. H. Freeman and Company, New York, 1987.

[62] Y. Higuchi, Combinatorial curvature for planar graphs, J. Graph Theory 38 (2001), no. 4, 220-229.

[63] I. Holopainen and P. M. Soardi, A strong Liouville theorem for p-harmonic functions on graphs, Ann. Acad. Sci. Fenn. Math. 22 (1997), no. 1, 205-226.

[64] I. Holopainen and P. Koskela, Volume growth and parabolicity, Proc. Amer. Math. Soc. 129 (2001), no. 11, 3425-3435.

[65] B. Hua, Generalized Liouville theorem in nonnegatively curved Alexandrov spaces, Chin. Ann. Math. Ser. B 30 (2009), no. 2, 111-128.

[66] B. Hua, Harmonic functions of polynomial growth on singular spaces with nonnegative Ricci curvature, Proc. Amer. Math. Soc. 139 (2011), 2191-2205.

[67] B. Hua, J. Jost and S. Liu, Geometric analysis aspects of infinite semiplanar graphs with nonnegative curvature, submitted.

[68] M. Ishida, Pserdo-curvature of a graph, lecture at "Workshop on topological graph theory", Yokohama National University, 1990.

[69] W. Jäger, H. Kaul, Uniqueness and stability of harmonic maps and their Jacobi fields. Manuscripta Math. 28 (1979), no. 1-3, 269C291.

[70] J. Jost, Riemannian geometry and goemetric analysis, Fourth edition, Universitext, Springer, Heidelberg, 2011.

[71] J. Jost and S. Liu, Ollivier's Ricci curvature, local clustering and curvature dimension inequalities on graphs, submitted.

[72] A. Joulin and Y. Ollivier, Curvature, concentration and error estimates for Markov chain Monte Carlo, Ann. Probab. 38 (2010), no. 6, 2418-2442.

[73] M. Kanai, Rough isometries and the parabolicity of Riemannian manifolds, J. Math. Soc. Japan 38 (1986), no. 2, 227-238.

[74] M. Keller, Essential spectrum of the Laplacian on rapidly branching tessellations, Math. Ann. 346 (2010), no.1, 51-66.

[75] M. Keller, Curvature, geometry and spectral properties of planar graphs, Discrete Comput. Geom. 46 (2011), no. 3, 500-525.

[76] M. Keller and N. Peyerimhoff, Cheeger constants, growth and spectrum of locally tessellating planar graphs, Math. Z. 268 (2011), no. 3-4, 871-886.

[77] S. Klassert, D. Lenz, N. Peyerimhoff and P. Stollmann, Elliptic operators on planar graphs: unique continuation for eigenfunctions and nonpositive curvature, Proc. Amer. Math. Soc. 134 (2006), no. 5, 1549-1559.

[78] B. Kleiner, A new proof of Gromov's theorem on groups of polynomial growth, J. Amer. Math. Soc. 23 (2010), no. 3, 815-829.

[79] K. Kuwae, Y. Machigashira and T. Shioya, Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces, Math. Z. 238 (2001), no. 2, 269-316.

[80] D. A. Levin, Y. Peres and E. L. Wilmer, Markov chains and mixing times, With a chapter by J. G. Propp and D. B. Wilson, American Mathematical Society, Providence, RI, 2009.

[81] P. Li, Harmonic sections of polynomial growth, Math. Res. Lett. 4 (1997), no. 1, 35-44.

[82] P. Li, Harmonic functions and applications to complete manifolds, Lecture notes, preprint.

[83] P. Li and R. Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds. Acta Math. 153 (1984), no. 3-4, 279-301.

[84] P. Li and S. T. Yau, Estimates of eigenvalues of a compact Riemannian manifold, in: R. Osserman and A. Weinstein (Eds.), Geometry of the Laplace operator, Proc. of Symposia in Pure Mathematics 36, American Mathematical Society, Providence, R.I., 1980, pp. 205-240.

[85] Y. Lin, Ricci curvature on graphs, 2010 John H. Barrett Memorial Lecture, http://www.math.utk.edu/barrett/2010/talks/YongLinRicci.pdf.

[86] Y. Lin, L. Lu and S. T. Yau, Ricci curvature of graphs, Tohoku Math. J. 63 (2011), 605-627.

[87] Y. Lin and L. Xi, Lipschitz property of harmonic function on graphs, J. Math. Anal. Appl. 366 (2010), no. 2, 673-678.

[88] Y. Lin and S. T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, Math. Res. Lett. 17 (2010), 343-356.

[89] J. Lott and C. Villani, Ricci curvature for metric measure spaces via optimal transport, Ann. of Math. (2) 169 (2009), 903-991.

[90] S. B. Myers, Riemnannian manifolds with positive mean curvature, Duke Math. J. 8 (1941), no. 2, 401-404.

[91] S.-I. Ohta, On the measure contraction property of metric measure spaces, Comment. Math. Helv. 82 (2007), 805-828.

[92] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, J. Funct. Anal. 256 (2009), 810-864.

[93] Y. Ollivier, A survey of Ricci curvature for metric spaces and Markov chains, in: M. Kotani, M. Hino and T. Kumagai (Eds.), Probabilistic approach to geometry, Adv. Stud. Pure Math. 57, Math. Soc. Japan, Tokyo, 2010, pp. 343-381.

[94] Y. Ollivier and C. Villani, A curved Brunn–Minkowski inequality on the discrete hypercube, http://arxiv.org/abs/1011.4779v3.

[95] Y. Otsu and T. Shioya, The Riemannian structure of Alexandrov spaces, J. Differential Geom. 39 (1994), no. 3, 629-658.

[96] S.-H. Paeng, Volume, diameter of a graph and Ollivier's Ricci curvature, preprint.

[97] Y. Peres, Mixing for Markov chains and spin systems, Lecture notes, http://www.stat.berkeley.edu/ peres/ubc.pdf, 2005.

[98] P. Préa, Distance sequences in infinite regular tessellations, Discrete Mathematics **146**(1995), 211-233.

[99] P. Préa, Distance sequences and percolation thresholds in Archimedean tilings, Research Report ♯ 19, LIM, (1994).

[100] P. Préa, Distance sequence and percolation thresholds in Archimedean tilings, Mathl. Comput. Modelling, 26 (1997), 317-320.

[101] Z. Qian, H. C. Zhang and X. P. Zhu, Sharp spectral gap and Li-Yau's estimate on Alexandrov spaces, http://arxiv.org/abs/1102.4159v1.

[102] T. Regge, General relativity without coordinates, Nuovo Cimento (10) 19 (1961), 558-571.

[103] M.-K. von Renesse and K.-T. Sturm, Transport inequalities, gradient estimates, entropy, and Ricci curvature, Comm. Pure Appl. Math. 58 (2005), no. 7, 923C940.

[104] M. Rigoli, M. Salvatori and M. Vignati, Subharmonic functions on graphs, Israel J. Math. 99 (1997), 1-27.

[105] P. J. Ryan, Euclidean and non-Euclidean geometry. An analytical approach. Cambridge University Press, Cambridge, 1986.

[106] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differential Geom. 36 (1992), no. 2, 417-450.

[107] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series 289, Cambridge University Press, Cambridge, 2002.

[108] C. Sormani, Harmonic functions on manifolds with nonnegative Ricci curvature and linear volume growth, Pacific J. Math. 192 (2000), no. 1, 183-189.

[109] D. A. Stone, A combinatorial analogue of a theorem of Myers, Illinois J. Math. 20 (1976), 12-21; Correction, Illinois J. Math. 20 (1976), 551-554.

[110] K.-T. Sturm, On the geometry of metric measure spaces. I., Acta Math. 196 (2006), 65-131.

[111] K.-T. Sturm, On the geometry of metric measure spaces. II., Acta Math. 196 (2006), 133-177.

[112] L. Sun and X. Yu, Positively curved cubic plane graphs are finite, J. Graph Theory 47 (2004), 241-274.

[113] C. Villani, Topics in optimal transportation, Graduate Studies in Mathematics 58. American Mathematical Society, Providence, RI, 2003.

[114] C. Villani, Optimal transport, Old and new, Grundlehren der Mathematischen Wissenschaften 338. Springer, Berlin, 2009.

[115] G. Wang and C. Xia, An optimal anisotropic Poincaré inequality for covex domains, http://arxiv.org/abs/1112.4398v1.

[116] G. Wang and C. Xia, A sharp lower bound for the first eigenvalue on Finsler manifolds, http://arxiv.org/abs/1112.4401v1.

[117] D. J. Watts and S. H. Strogatz, Collective dynamics of 'small-world' networks. Nature 393 (1998), 440-442.

[118] W. Woess, A note on tilings and strong isoperimetric inequality, Math. Proc. Camb. Philos. Soc. 124 (1998), 385-393.

[119] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975), 201-228.

[120] S. T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Math. J. 25(1976), 659-670.

[121] S. T. Yau, Nonlinear analysis in geometry, Enseign. Math. (2) 33 (1987), no. 1-2, 109-158.

[122] S. T. Yau, Differential Geometry: Partial Differential Equations on Manifolds, in: R. Greene and S. T. Yau (Eds.), Proc. of Symposia in Pure Mathematics 54, Part 1, American Mathematical Society, Providence, RI, 1993.

[123] S. Zhu, The comparison geometry of Ricci curvature, in: K. Grove and P. Petersen (Eds.), Comparison geometry (Berkeley, CA, 1993-944), M. S. R. I. Publ. 30, Cambridge University Press, Cambridge, 1997, pp. 221-262.

[124] A. Żuk, On the norms of the random walks on planar graphs, Ann. Inst. Fourier (Grenoble) 47 (1997), no. 5, 1463-1490.

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- S. Liu, *Gradient Estimates for Solutions of the Heat Equation under Ricci Flow. Pacific Journal of Mathematics,* Volume 243, No.1, 2009, pp 165–180.

Artikel in Begutachtung

- B. Hua, J. Jost, S. Liu, *Geometric Analysis Aspects of Infinite Semiplanar Graphs with Nonegative Curvature,* Submitted.
- F. Bauer, J. Jost, S. Liu, *Ollivier-Ricci Curvature and the Spectrum of the Normalized Graph Laplace Operator,* to appear in Mathematical Research Letters.
- J. Jost, S. Liu, *Ollivier's Ricci Curvature, Local Clustering and Curvature Dimension Inequalities on Graphs,* Submitted.

Vorträge

- *Infinite semiplanar graphs with nonnegative combinatorial curvature,* talk at School of Mathematics and Statistics, Wuhan University, March 17, 2012.
- *Ollivier-Ricci curvature on neighborhood graphs,* talk at Oberseminar Analysis, Geometrie und Stochastik, Friedrich-Schiller-Universität Jena, November 16, 2011.
- *Ricci curvature and local clustering of graphs,* talk at Graduate student symposium at the TU Chemnitz within the International Summer School on Graphs and Spectra, July 18-23, 2011.
- *Curvature, random walks, and the spectrum of the graph Laplace operator,* joint talk with Dr. F. Bauer on the Spring School on Limits of Finite Graphs in Leipzig, April 26-30, 2011.
- *Some analytic problems relating to Ricci curvature notations on metric measure spaces,* IMPRS Workshop, December 9-11, 2010.
- *Ricci curvature notation on metric measure spaces,* talk at International Max Planck Research School (IMPRS) Workshop, February 5-6, 2010.

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