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In dieser Arbeit wird die analytische Fortsetzung von Quantenfeldtheorien auf dem nichtkommutativen Euklidischen Moyal-Raum mit kommutativer Zeit zur entsprechenden Moyal-Minkowski Raumzeit (Wick Rotation) erarbeitet. Dabei sind diese Moyal-Räume durch eine konstante Nichtkommutativität gegeben. Einerseits wird die Wick Rotation im Kontext der algebraischen Quantenfeldtheorie, ausgehend von einer Arbeit von Schlingemann, hergeleitet. Von einem Netz Euklidischer Observablen wird die Lorentz'sche Theorie durch alle Bilder der fortgesetzten Poincaré Gruppenwirkung auf der Zeit-Null Schicht erhalten. Dabei wird gezeigt, dass die Vorgänge der nichtkommutativen Deformation und der Wick Rotation kommutieren. Andererseits ist so eine analytische Fortsetzung ebenfalls für Quantenfeldtheorien, die durch einen Satz von Schwingerfunktionen definiert ist, möglich. Durch die Gültigkeit einer Kombination aus Wachstumsbedinungen, die aus der Wick Rotation von Osterwalder und Schrader bekannt sind, kann der Übergang zu einer deformierten Wightman-Theorie gezeigt werden. Abschließend beinhaltet diese Arbeit ergänzende Resultate zu den physikalischen Eigenschaften der Kovarianz und der Lokalität.

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Chapter 1

Introduction

Quantum field theory was initiated as a generalization of quantum mechanics obeying to the fundamental principles of special relativity. It has become the prior instrument that is supposed to solve the main problems of modern physics. Due to its enormous increase both in specificity and versatility over the last 80 years, scientists were made to focus on certain branches in order to come to new conclusions. Since the beginning of modern science, sustainable physical theories had to pass the tests by the experiment. This is the main reason why quantum field theory has reached great popularity: for example, quantum electrodynamics came up with a prediction of the anomalous magnetic moment of the electron which agreed with the experimental data up to 10 significant digits [ea01]. Success like this was possible due to the progression of renormalization theory. The physically relevant models of quantum field theory, like quantum electrodynamics or quantum chromodynamics may so far only be treated perturbatively because of their intrinsic nonlinearity coming from self-interaction. The occurring integrals inhabit divergences, which superficially render the perturbation series - and therefore the theory - meaningless. This lead the founders of quantum field theory to think about alternative viewpoints and brought young scientist Hartland Snyder to publish the first model on a noncommutative space [Sny47]. Nevertheless, the systematic absorption of the brought up divergences into redefined physical constants, *i.e.*, renormalization, let theorists rapidly come back to the primary perturbative approach. In the second half of the twentieth century, the standard model of high energy physics was completed: a renormalized symbiosis of models describing the electromagnetic as well as the weak and the strong nuclear forces of nature.

Despite this great achievement, certainly not all problems found an appropriate solution. While the divergences can be kept under control at each perturbation order, not a single interacting model in four space-time dimensions could be shown to have a converging perturbation series. At first sight this could be classified as minor problem, but from a viewpoint closer related to philosophy of science, this spoils the ability of deriving physical predictions from a mathematical well-defined theory. Each of the perturbation series contained in the standard model is asymptotic and the consequences of this fact are subtle. In an asymptotic series it may happen that the first, say, four approximation orders make predictions very close to the experimental data, whereas the next ten orders differ dramatically from any physical measurement. In other words, converging perturbation series are desirable because they guarantee for approximations getting better order by order.

Parallel to the investigation of renormalization theory, mathematical physicists sought for a set of few striking properties that are fulfilled by every quantum field theory model in use. The so-called *Wightman(-Garding) axioms* [SW89] in brief demand the following of the correlation functions of a quantum field theory: structural relation to quantum mechanics, covariance with respect to the Poincaré group, relativistic causality and some technical analyticity and domain properties[1]. With the exception of the purely mathematical requirements, any of these not to hold would question our empirical picture drawn from thousands of experiments so far. This may give an idea of the desire for having a well-defined model that satisfies all the Wightman axioms in four dimensions.

A somewhat differently motivated approach comprises the school of R. Haag's *algebraic quantum field theory*; there, the structural relations of quantum field theoretic models are central, irrespective of any realization through vacuum expectation values. As will be explained in great detail in the following sections of this thesis, a physical theory is built up of C^*-algebras[2] associated to space-time regions. This net structure obeys the so-called *Haag-Kastler axioms* [HK64], which under some further conditions can be well related to the Wightman axioms [FH81].

The task of having an example at one's disposal - either obeying to the Wightman or to the Haag-Kastler axioms - marks the starting point of *constructive quantum field theory* and we are going to state the main achievements of this branch now. For a thorough introduction to the topic, see [Sum12]. Models featuring no self-interaction on Minkowski space-time were the first successfully constructed and soon the same was attempted for interacting models. Even at "toy

[1] We will come to a precise definition in Ch. 3 of this thesis.

[2] In fact, they are rather *-algebras comprising most properties of C^*-algebras but are unbounded in general.

models" which are restricted in both interaction complexity and space-time dimension, the arising technical difficulties were astonishing. While in the beginning the constructions were carried out in Minkowski space-time using operator algebraic and functional analytic methods, this techniques did not work out for a long time. By a theorem which is referred to as "Haag's theorem" [Haa55, RS75], an interacting quantum field theoretic model cannot be represented on Fock space. Rather, the underlying Hilbert space must have another form which is unknown in general.

A completely different approach resulted in more success. In the 1970's it was realized that going over to Euclidean space, one can consider the real scalar field as a generalized random process [Sym66, Nel73a]. Before we deal with the Euclidean framework in detail, we try to give a summary of its influence on the constructive program. Besides the interest of having proved existence of certain models, one clearly is interested in the physical significance of such. Not just more (low-dimensional) interacting models have been shown to exist, but the particle content and scattering properties have been analyzed to a much greater extent in the Euclidean framework. Space-times of dimension $d = 4$ are for many physicists the only "realistic" scene for a quantum field theory to take place. Unfortunately, it is exactly this case which has not been reached satisfactorily by constructive field theory up until today. Additionally we want to remark that there well have been models who were shown to exist in four (and potentially in higher) dimensions, but every single one of them proved to be trivial, *i.e.*, its correlation functions coincide with those of the corresponding free theory. As we will discuss in Sec. 1.2.5, the noncommutative approach to quantum field theory is likely to provide an improvement of the situation, and it is the Euclidean framework which is the method of choice again.

Euclidean quantum field theory started approximately when K. Symanzik realized that the passage to "imaginary time" used before by Schwinger and Dyson at loop integral calculations can be formalized more extensively[3] [Sym66]. Nelson made clear the connection of the correlation functions in the Euclidean formulation, the *Schwinger functions*, and stochastic processes. He showed that these Schwinger functions are in fact moments of a probability measure defining a Markov process [Nel73a, Nel73b]. Given the "field measure" $\mathrm{d}\mu(\phi)$, the Schwinger functions $\mathfrak{S}_n$ write

[3]Concerning the time scale involved: K. Osterwalder in [Ost73] attests the Euclidean framework “a long history”. That was almost 40 years ago...

$$\mathfrak{S}_n(x_1, \ldots, x_n) = \int \phi(x_1) \cdots \phi(x_n) \mathrm{d}\mu(\phi) \ .$$

The necessary generalization is the use of a continuous time parameter, opposed to discrete time steps, making the Euclidean scalar field a generalized stochastic process. In fact the integral representation we just wrote down is only valid at non-coinciding Euclidean points $x_k \in \mathbb{R}^4$, $k = 1, \ldots, n$, which can be cured by smearing with suitable test-functions, in the same way as in the Minkowski case. Pursuing the analogy with stochastics, one can introduce a generating functional from which any Schwinger function can be obtained via functional derivation, see for example [GJ87, Roe94]. These generating functionals are the structure reminiscent of the path integral of Feynman, with the slight difference that it can be given a well-defined mathematical meaning much more easily[4].

But all this keen model building will lead to nothing valuable as long as the way of passing through to Minkowski space-time is vague. Therefore the precise analytical continuation of Euclidean towards Lorentzian models is of high importance. Nelson gave a sufficient set of axioms a Markov process has to fulfill to be a field theory on Minkowski space-time. This very important step was further enhanced to theories which can be fully described by their set of n-point functions (which by Wightman's theorem is sufficient to completely determine the physical field and the underlying Hilbert space, *cf.* [Wig56]), but which do not fit into a description by a probability measure. These *Osterwalder-Schrader-Axioms* [OS73, OS75] are the key contribution to the field. The exact formulation of these and the similar set of axioms geared to the path integral formulation can be found in Appendix A and Ch. 3. In a mathematically rigorous way, they clarified the connection of given sets of Wightman functions and their corresponding Schwinger functions. It is one of the main purposes of the thesis at hand to investigate the generalizations of analytical continuations of this type to noncommutative theories.

The second unresolved problem of quantum field theory consists of the unification of general relativity, the still unchallenged theory of classical gravity, and the standard model. Regarding this point, the situation seems less clear than ever. There are both major conceptual and technical difficulties plaguing the development of a quantum theory of gravity that last for decades. In the ongoing twenty-first century, a few more or less large branches of research have formed. First of all,

[4]We do not want to ignore that using more subtle methods, the Feynman path integral can also be tamed [Alb08]

String Theory, having become vast both in the number of scientists involved and ideas uttered so far. This set of theories tries to unify the fundamental forces by considering the oscillation modes of closed or open space-time curves, the “strings”. These are regarded as the building blocks of the physical world; indeed, the elementary particles are produced by string oscillations, while inspired by quantum mechanics, their energy is proportional to the vibration frequency. We immediately see that String Theory relies on simple ideas of the space-time structure which circumvent some insistent problems of conventional quantum field theory such as intrinsic divergences. A contemporary introduction to the field is given in [BBS06]. Admittedly, we stressed that a physical theory must be experimentally tested, and this might be the biggest problem for all the candidates trying to achieve grand unification. And due to the comparably high proportion of ideas born, models proposed, etc. and concrete outcome, unique falsifiable predictions, etc., String Theory might be the framework suffering most from this obstruction. Having lost contact with reality or not, the mathematical theories pursued or even invented during the study of String Theory speak for themselves [CHSW85, Wit89, GPR94].

Secondly, we mention the theory of Loop Quantum Gravity. Ashtekar [Ash86] found variables for classical gravity that resemble the structure of canonical quantum field theory. People in this program have tried to benefit from the upcoming analogies with Yang-Mills theory to quantize gravity. The theory got its name from the loop-representation, decomposing the physical states into Wilson-loops, which are known from gauge theory. Surely, the elaboration of Loop Quantum Gravity is far from being complete, as is that of the other approaches to a unified theory. An introductory course including a summary of current objections and methods of resolution is given in [GP11]. Unfortunately, a more specified introduction to these areas of research cannot be given within the scope of this thesis.

Our investigations are aiming at the third approach, *noncommutative quantum field theory*. At the end of the 1980's, French mathematician Alain Connes caused a stir in the mathematics society by his invention of *noncommutative geometry* [Con94]. This theory relies on the strikingly close relationship between algebraic and geometric entities. This relationship becomes manifest by the theorems of Gelfand and Naimark [GN43], as well as Serre and Swan [Ser55, Swa62], for example. Actually, the former is the name for two theorems. The first one, called the “commutative Gelfand Naimark theorem“ states that a commutative C^*-algebra is isometrically *-isomorphic to the space of continuous functions on a locally compact Hausdorff space. The second one is the noncommutative generalization, implying that any C^*-algebra admits a faithful embedding into the space of bounded operators on a Hilbert space. The main focus of Connes' work lies on the geometrical consequences of equivalence theorems like these and culminates in statements about reconstructing geometries out of algebraic data. To be a bit more precise, given a so-called *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, consisting of a C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a self-adjoint, unbounded operator D, satisfying specific properties, one can reconstruct a compact spin manifold [Con96].

Let us keep the mathematical achievements of noncommutative geometry this fragmentary for now and come to the physical applications. As we have already mentioned, a noncommutative structure additional to that of the quantum operators was once introduced to overcome the divergences arising during the perturbative analysis of quantum field theory models. It have been semi-classical considerations of combining quantum mechanical rules with those of general relativity that lead to the revival of noncommutative position operators in physics. Nowadays, it seems like an improvement of the convergence crisis could be the first benefit thereof. But first things first. The semi-classical *gedanken experiment* we mentioned is attributed to J.A. Wheeler and says the following: in order to resolve space-time points that are very close to one another, we have to put in a high amount of energy. Assuming the validity of general relativity to very small length scales (and up until today there is no reason why one should doubt that) the energy density in this small space-time region will be high enough to form a black hole, preventing any accretion of knowledge about the region in focus. A physically well-motivated and very influential approach to quantum field theory on noncommutative space is due to Doplicher, Fredenhagen and Roberts [DFR95]. They introduced space-time commutation relations determined by a tensorial entity that is assumed to fulfill physical properties, making it possible to maintain Poincaré covariance despite deforming the space-time structure. Later in this thesis, we will contemplate a simpler model that has gained some attention, too.

1.1 Algebraic Quantum Field Theory

As has been said in the general introduction, the Wightman axioms arose out of the desire to have a short list of properties which any quantum field model fulfills. Mandatory from a mathematical point of view, this list shall be of importance for future model building, too. But clearly, it can only be meant provisionally: we simply cannot know which characteristics better developed theories of the future will possess. Going a bit more into detail, the Wightman axioms are aiming at n-point functions, expectation values of field products in a physical state. Most often this state is the physical vacuum and its realization is actually a part of the axioms: motivated by quantum mechanics, they shall be expressed via vectors on a Hilbert space $\mathcal{H}$. Moreover, the Poincaré group is to be unitarily represented on $\mathcal{H}$ and there exists a unique invariant state, which is the vacuum. Due to assumed Poincaré covariance the energy-momentum operator P^μ must have its spectrum contained in the forward light-cone. The Wightman functions are demanded to be sequences of tempered distributions, which also incorporate the theories' covariance. Being a vacuum expectation value, they are expected to be invariant under actions of the Poincaré group. Furthermore, the fundamental concept of locality is realized by making sure that the Wightman functions coincide whenever spatially separated fields in their evaluated products are interchanged. The stability of matter relies in large part on the positivity of energy, a physical concept which is included by a positive Hamiltonian on the one hand and by demanding that the Wightman functions represent a positive state on the field algebra on the other hand. The latter property is not straightforward to appreciate, but will be treated with greater rigor and detailedness. Finally, a cluster property is assumed to hold in order to have a unique vacuum. This last axiom will be the least important for our treatment.

In the late 1950's and the beginning 1960's, mathematical physicists added a further degree of abstraction. As one of the founders of algebraic quantum field theory, R. Haag, argues in his book [Haa92], the quantum field itself is just an auxiliary entity for physics. Its main purpose is to establish the concept of locality, while the physically accessible quantities are scattering cross-sections of elementary particles. So if one wants to capture the main structure of quantum physics, one should focus on the physical observables which can be measured in a certain region of space-time. An accurate inspection shows that the observable quantities form an algebra. The representation of its elements as operators on the Hilbert space makes the use of C^*-algebras manifest. This was first pointed out by Segal [Seg47]. Haag and Kastler [HK64] showed that the best-suited concept comprises nets of C^* or von Neumann algebras $\mathcal{A}(O)$, indexed by open, bounded subsets O of Minkowski space-time. We collect these subsets into $\mathscr{O}$ and occasionally call them *regions*. Since we are aiming at the Euclidean framework as well, we will be a bit more general in the upcoming definition of our observable net. It is important how to implement symmetries there. If a group G is realized on the quasi-local algebra $\mathcal{A}$ (the inductive limit of the algebra net) by automorphisms,

$$G \ni g \mapsto \alpha_g \in \operatorname{Aut} \mathcal{A} \ ,$$

where all the α_g leave $\mathscr{O}$ invariant, then G is called a *symmetry group*. In fact, it can be shown that Wigner's theorem on space-time symmetries being unitary operators on Hilbert space can be suitably generalized to this notion, *c.f.* [Ara99, chapter 4]. On a net of *-algebras concretely realized on a Hilbert space, these automorphisms will be adjoint actions of unitary operators.

Definition 1. Given a manifold M, a family $\mathscr{O}$ of subsets of M, and a symmetry group G of point transformations of M, a *G-covariant net* $(\mathcal{A}, \mathscr{O}, \alpha)$ on M is defined as the following structure. $\mathcal{A}$ is a map from $\mathscr{O}$ to C^*-algebras $\mathcal{A}(O)$ (respectively von Neumann algebras acting on a common Hilbert space $\mathcal{H}$), such that

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2) \ \text{ for } \ O_1 \subset O_2 \,. \tag{1.1}$$

The smallest C^*-algebra (respectively von Neumann algebra) containing all $\mathcal{A}(O)$, $O \in \mathscr{O}$, is also denoted $\mathcal{A}$, and α is an automorphic action of G on $\mathcal{A}$, such that

$$\alpha_g(\mathcal{A}(O)) = \mathcal{A}(gO) \,, \qquad g \in G \,,\ O \in \mathscr{O} \,. \tag{1.2}$$

Thus so far the axiom of covariance was included in a natural way into the algebraic context. Locality can be incorporated straightforwardly. To this end we specify the considerations to Minkowski space-time again and write $\mathcal{M}(O)$ for the corresponding $\mathcal{P}(d)$-covariant net. The reason for us to do so is that there will be no reference to any curved space-time in this thesis. We can easily incorporate the spectrum condition in the same way as in the Wightman case: we demand the spectrum of the translation generators to be subsets of the forward light-cone. Now we call a $\mathcal{P}(d)$-covariant net $\mathcal{M}(O)$ *local*, if the following equation is valid,

$$[\mathcal{M}(O_1), \mathcal{M}(O_2)] = \{0\} \qquad \text{for} \qquad O_1 \subset O_2' \,, \tag{1.3}$$

where O_2' denotes the causal complement of O_2 with respect to the Minkowski metric $\eta = \operatorname{diag}(+1, -1, ..., -1)$ on $\mathbb{R}^d$. By causal complement, we mean the following set: $y \in O_2'$ if $(x^\mu - y^\mu)\eta_{\mu\nu}(x^\nu - y^\nu) < 0$ for all $x \in O_2$.

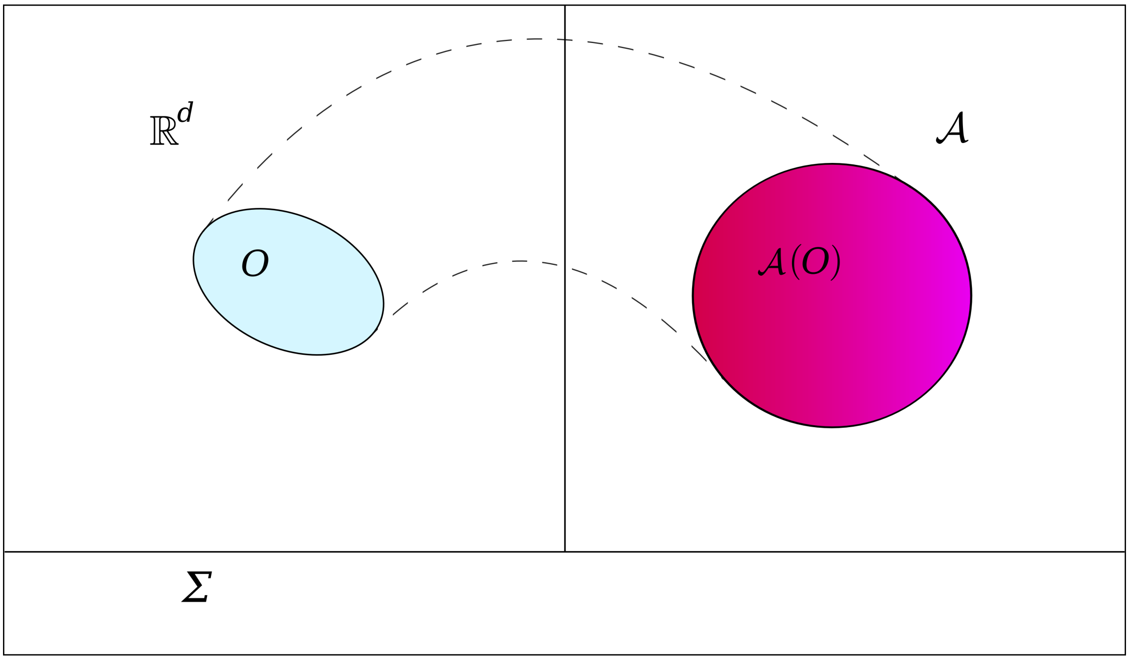

Figure 1.1: Schematic picture of assigning algebras to regions.

We have got almost all the main structures needed for an algebraic approach to quantum field theory. But the main notion for the quantum structure is still missing: the physical states. We adopt the nomenclature which has become standard in the literature.

Definition 2. Let $G^{\mathcal{M}} \subset \mathcal{P}(d)$ be a subgroup of the Poincaré group. Let furthermore $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}})$ be a $G^{\mathcal{M}}$-covariant net on $\mathbb{R}^d$. A *vacuum state* on $\mathcal{M}$ is a normalized, positive, linear functional $\omega : \mathcal{M} \to \mathbb{C}$ such that

- $G^{\mathcal{M}} \ni g \longmapsto \omega(A\alpha_g^{\mathcal{M}}(B))$ is continuous for all $A, B \in \mathcal{M}$,
- $\omega \circ \alpha_g^{\mathcal{M}} = \omega$ for all $g \in G^{\mathcal{M}}$,
- There is a weakly dense subset $\mathcal{D} \subset \mathcal{M}$ such that $-i\, \frac{d}{dt}\big|_{t=0} \omega(A^* \alpha_{t\cdot e,1}^{\mathcal{M}}(A))$ exists and is non-negative for all $A \in \mathcal{D}$.

A few explaining remarks are in order. The last property of ω is the algebraic version of the positive energy requirement. $\alpha_{t\cdot e,1}^{\mathcal{M}}$ denotes a translation of the amount t in the direction e, which we choose to be timelike. So implicitly, we have picked subgroups $G^{\mathcal{M}}$ of $\mathcal{P}(d)$ which contain these "time translations". The introduced notions are the building blocks of algebraic quantum field theory and can in principle incorporate the main structure of a variety of physical models. Nevertheless, as we have started to explain in the main introduction, such models still have not been found to exist in a mathematically rigorous sense. One therefore hopes that the purely mathematical consequences of the axioms and the properties of better and better models will support each other in the way towards a sound and predictive description of quantum field theory.

Before we go on we want to remark that it has shaped up to be advantageous to decouple the description from an underlying Hilbert space in some fields of study. Concerning thermodynamics, it happens that thermodynamic equilibrium states in the infinite volume limit cannot be represented as density matrices in the vacuum Hilbert space. Algebraic quantum field theory is able to well incorporate these states, as can be comprehended from [HHW67]. The situation appears more fundamental at *quantum field theory on curved space-times*, which has become a comprehensive area of research, see for example [BF09]. There, it is always sensible to start the theory from the algebraic structure alone as there is no vacuum state in generic space-times. One of the implications of this fact is cosmological particle creation, which is introduced in [Wal94].

Nonetheless, when contemplating quantum field structures, it is not restrictive to take the elements of the local algebras to be represented on a Hilbert space with the symmetry action given by the adjoint action of unitaries. In this case we will talk of a *covariant representation*. Indeed, by the GNS construction [GN43, Seg47], any C^*-algebra which possesses a state ω gives rise to a Hilbert space $\mathcal{H}_\omega$, a *-representation π_ω and a cyclic vector Ω_ω, such that all elements $A \in \mathcal{A}$ can be realized on $\mathcal{H}_\omega$ through the relation $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$.

As has been remarked in [BLS11, Above Sec. 2.1], any abstract algebra faithfully represented on a separable Hilbert space gives rise to a covariant representation [Ped79, Lemma 7.4.9, Prop. 7.4.7]. We will thus consider concretely realized algebras throughout this thesis.

Next we want to study the correspondence between the introduced algebraic framework and the one situated on Euclidean space. The measure theoretic approach to Euclidean field theory treated in [GJ87] amongst others delivers a local C^*-algebra by the closure of

$$\{\exp\{i\phi(f)\} \,|\, f \in \mathscr{S}(\mathbb{R}^d)\,,\, \mathrm{supp}\, f \subset O\}\ ,$$

where $\phi(f)$ arises as moment of the Euclidean measure $\mathrm{d}\mu(\phi)$. This measure defines an underlying generalized stochastic process. As a concrete realization, ϕ maps a Schwartz function to an operator on the Euclidean Hilbert space $L^2(\mathscr{S}'(\mathbb{R}^d) \to \mathbb{C}, \mathrm{d}\mu(\phi))$. The Schwinger functions can be obtained by functional derivatives of the generating functional $S\{f\}$, defined as the Fourier transform of the Euclidean measure.

Approaching a framework irrespective of an underlying measure, we again fix a unit vector $e \in \mathbb{R}^d$ and write $e^\perp \subset \mathbb{R}^d$ for the hyperplane orthogonal to e. Let r denote the e-reflection on $\mathbb{R}^d$: $r : (x_0, \underline{x}) \mapsto (-x_0, \underline{x})$ and write $\mathbb{R}^d_>$ for the subspace of $\mathbb{R}^d$ with positive components in e-direction, *i.e.*, $\mathbb{R}^d_> := \mathbb{R}_+ e + e^\perp$. We are just ready to give a definition of the notion analogous to Def. 2:

Definition 3. Let $G^{\mathcal{E}} \subset E(d)$ be a subgroup of the Euclidean group. Let furthermore $(\mathcal{E}, O, \alpha^{\mathcal{E}})$ be a $G^{\mathcal{E}}$-covariant net on $\mathbb{R}^d$. A *reflection positive functional* on $\mathcal{E}$ is a continuous normalized linear functional $\sigma : \mathcal{E} \to \mathbb{C}$ such that

1. $G^{\mathcal{E}} \ni g \longmapsto \sigma(A\alpha^{\mathcal{E}}_g(B))$ is continuous for all $A, B \in \mathcal{E}$,
2. $\sigma \circ \alpha^{\mathcal{E}}_g = \sigma$ for all $g \in G^{\mathcal{E}}$,
3. There exists an automorphism ι of $\mathcal{E}$ such that $\iota\alpha^{\mathcal{E}}_g\iota = \alpha^{\mathcal{E}}_{\gamma(g)}$ for all $g \in G^{\mathcal{E}}$ and $\gamma(x, R) := (rx, rRr)$. Furthermore, σ fulfills the following property,

$$\sigma(\iota(A^*)A) \geqslant 0 \ , \tag{1.4}$$

for all $A \in \mathcal{E}_> := \mathcal{E}(\mathbb{R}^d_>)$.

Inequality (1.4) is the important concept of reflection positivity. In principle, any Euclidean direction would be fine for this definition, as they are all equivalent. When it comes to the physical interpretation, e will be determined to be "Euclidean time", *i.e.*, the imaginary part of complex generalization of the time coordinate in Minkowski space-time. An $E(d)$-covariant net together with a reflection positive functional will be the starting point for our algebraic Wick rotation in Ch. 2.

1.2 Noncommutative Quantum Field Theory

1.2.1 Noncommutative Geometry

The notions introduced so far give an idea how important C^*-algebraic techniques are for modern quantum physics. The theorems by Gelfand and Naimark prepare a clear correspondence between the abstract notion of a C^*-algebra and that of bounded operators on Hilbert spaces, which are mandatory for quantum mechanics and quantum field theory. *Commutative* C^*-algebras are even isometrically *-isomorphic to the space of continuous functions on a locally compact Hausdorff space. This circumstance serves as a first example of fully describing geometrical properties by means of purely algebraic ones. Extending this consideration to noncommutative C^*-algebras was the starting point for A. Connes in developing *noncommutative geometry* [Con94]. There one most often starts by the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a manifold M and builds the corresponding *deformed* noncommutative algebra using a certain deformation scheme. One main achievement is the direct generalization of vector bundles; we will only sketch the main ideas. Given a vector bundle in which M is the base space, its space of sections is a vector space carrying a representation of $\mathcal{C}^\infty(M)$ through multiplication by smooth functions on M. Put differently, the section space is a module over the algebra of functions. By the theorem of Serre-Swan, any vector bundle corresponds canonically to a *finitely generated projective* module, *i.e.*, a direct summand of a free module which has a generating set of finite rank. This establishes a generalization of vector bundles in noncommutative geometry.

Constitutively, the differential calculus of operators was developed. The derivation comes through the spectral calculus and pseudo-differential operators. It is just impossible to do justice to this performance in an introductory section of a PhD thesis. What we can do is to present a small part of the "noncommutative dictionary", a collection of entities of geometric or topological nature and their according algebraic perceptions, linked via noncommutative geometric elaborations [GBVF00]:

TOPOLOGY	ALGEBRA
continuous proper maps	morphisms
homeomorphisms	automorphisms
compact	unital
Baire measure	positive functional
$\ddots$	$\ddots$

The main notion of Connes' noncommutative geometry is that of spectral triples:

Definition 4. A *spectral triple* $(\mathcal{A}, D, \mathcal{H})$ consists of a *-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator D (called the *Dirac operator*), which is chosen such that its resolvent operator is compact and the operators $[D, A]$ for $A \in \mathcal{A}$ are bounded on $\mathcal{H}$.

As a commutative example, the complex valued smooth functions on M form an algebra, and the space of square-integrable sections on the irreducible spinor bundle over M is a suitable Hilbert space. For D we may pick the usual Dirac operator $\not{D} = -i\gamma^\mu\partial_\mu$ with respect to the Euclidean metric there. This is sometimes being referred to as the "canonical (spectral) triple". Now from a general spectral triple one is able to extract an exterior algebra of forms with the derivation $[D, .]$ and that enables the treatment of connections on modules. One of the key results of the noncommutative geometry framework is Connes' reconstruction theorem [Con96]. It essentially says that given a spectral triple with $\mathcal{A} = \mathcal{C}^\infty(M)$ for a compact manifold M which satisfies a certain list of axioms, one can reconstruct the geometrical data of the corresponding spin manifold. There have been some generalizations of this theorem and we are going to introduce one of them in the following subsection. Furthermore, one can use the spectral triples to consider noncommutative generalizations of Yang-Mills theories. In fact, Connes and Chamseddine [CC07] have carried the noncommutative model building up until predictions for the Higgs mass in their "almost commutative" version of the standard model of particle physics. Indeed, the concrete prediction was precluded by experiments at Tevatron [B$^+$08], but this example shows that using more refinements could really bring the abstract mathematical generalizations to new physics some time.

Ending our general introduction to the growing branch of noncommutative geometry, we mention that most applications to physics are still handled at Euclidean signature. Anyhow, it is clear that becoming a true candidate for improved quantum field theory needs the Minkowski signature, especially when it comes to efforts concerning the inclusion of gravity. For a Lorentzian approach to spectral triples, one can have a look at [Bar07], for example.

1.2.2 Moyal Space

Seiberg and Witten [SW99] have found a connection between String Theory and a certain variant of a noncommutative space, so-called *Moyal space*. The defining relations there, for the moment just in a formal sense, are the famous commutation relations

$$[X_\mu, X_\nu] = i\theta_{\mu\nu} \quad , \quad \mu, \nu = 0, 1, \ldots, d-1 \tag{1.5}$$

between coordinate operators X_μ. Here, $\theta_{\mu\nu}$ are the components of a completely antisymmetric, d-dimensional matrix with real, constant entries. The Moyal commutator defines the simplest model which is still consistent with the demand of the X_μ being self-adjoint operators. Due to its plainness, it is the model most widely used in noncommutative quantum field theory and may serve as a first approximation to generalized and more predictive models in the future.

The matrix θ is sometimes referred to as "noncommutativity", as it gives an account of how far the coordinate operators are away from being commutative. The similarity to the fundamental commutator of quantum mechanics is obvious, and one may even draw at least some related conclusions. Indeed, if the commutation relations (1.5) are to hold on a space(-time) in a sense needed to be accurately exposed, the minimal observable distance is limited by the analog of Heisenberg's uncertainty relations. Without a deeper knowledge, one understands that the underlying space(-time) is divided into cells of magnitude $\sqrt{\vartheta}$, where ϑ denotes the non-zero entries of θ. Of course, it is not required *per se* that all the parameters ϑ in θ need to be the same. Summing up, we deduce the standard noncommutativity matrix for 4-dimensional Moyal theory:

$$\Theta_1 = \begin{pmatrix} 0 & \vartheta_e & 0 & 0 \\ -\vartheta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & \vartheta_m \\ 0 & 0 & -\vartheta_m & 0 \end{pmatrix} . \tag{1.6}$$

Clearly, the specific choice constituting which coordinates are non-commuting is completely arbitrary. However the simplicity of Moyal space(-time) brings along inapplicabilities: by again having a glance at (1.5) we directly infer the breaking of Lorentz covariance (but emphasize the persisting of translation covariance) of any physical theory defined thereon. Thus the incorporation of such a commutator changes physics in a rigid way. So, one prevalent philosophy is to see the institution of the Moyal commutator as a "deformation" of the underlying theory. In other words, sending all parameters ϑ to zero re-establishes the previous, *commutative* theory. This latter operation is called the *commutative limit* and is one of the most important parts of well-defining a theory on noncommutative spaces.

It was shown in [GGBI$^+$04] that Euclidean space with superimposed commutation relations (1.5), which is called the "Moyal plane", fits into the framework of spectral triples. In order to sustain more knowledge on how this is achieved and for later convenience, too, we introduce the so-called *star-product*:

Definition 5. Let $f, g \in \mathscr{S}(\mathbb{R}^d)$ and let Q be a skew-symmetric, invertible matrix. Then, the *Groenewold-Moyal star product* is defined in the following way,

$$(f \star_Q g)(x) := (2\pi)^{-d/2} \int \mathrm{d}^d k \, \mathrm{d}^d v \, \mathrm{e}^{-ikv} f(x + Qk/2) g(x + v) \ .$$

Let us add as a short remark that this product, as the whole Moyal plane, has a long history of development concerning quantum mechanics on the phase space [Gro46, Moy49]. In a technically challenging work it was possible to go over to a suitable unitization $\widetilde{\mathcal{A}}_\theta$ of the nonunital algebra $\mathcal{A}_\theta := (\mathscr{S}(\mathbb{R}^d), \star_\theta)$. Since we do not want to give a more detailed repetition of the spectral triple formalism, we refrain from formulating the exact theorem, but give an incomplete summary (again, D denotes the usual Euclidean Dirac operator): *The Moyal planes* $(\mathcal{A}_\theta, \widetilde{\mathcal{A}}_\theta, D, L^2(\mathbb{R}^d) \otimes \mathbb{C}^{2^N})$ *are connected real noncompact spectral triples.* In particular, the noncommutative position operators X_μ can be included into the involved algebras by left multiplication. We close the recapitulation of the Moyal plane by posing a lemma which tells about the different representations of the star product on Schwartz space.

Lemma 1. *For* $f, g \in \mathscr{S}(\mathbb{R}^d)$*, the following equations hold for* $f \star_Q g$*:*

$$
\begin{aligned}
(f \star_Q g)(x) &= \int \mathrm{d}^d y\, \mathrm{d}^d w\, \mathrm{e}^{-2i(Q^{-1}(y-x))(w-x)} f(y) g(w) \\
&= (2\pi) \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{e}^{i(p+q)x + ip(Qq)/2} \tilde{f}(p) \tilde{g}(q) \ .
\end{aligned}
$$

Proof. A straightforward calculation. □

1.2.3 Generalization to Operator Algebras

We are able to gladly observe that the simple deformation of Euclidean space given by the Moyal plane can be rigorously fit into the noncommutative geometry framework. As a consequence, the commutator (1.5) readily serves as a first order approximation of a general intrinsic noncommutativity of space-time to certain quantum field theoretic models. On the other hand, we straightforwardly comprehend that this still is not enough to act on algebraic quantum field theory for the sake of its intended generality. But also in this case, we can access an elaborate generalization. The monograph [Rie93b] has successfully adopted the deformed products to the C^*-algebra setting. We will now explain the cornerstones of this treatment.

Definition 6 ([Rud87])**.** A locally convex topological Hausdorff space is called a *Fréchet space*, if it is complete as a uniform space and its topology can be induced by a countable set of semi-norms $\| \ \|_m$. A topological algebra $\mathcal{A}$ is called a *Fréchet algebra*, if its vector space structure is that of a Fréchet space.

Let G denote a finite-dimensional Lie group with identity element $\mathbb{1}_G$, endowed with an inner product $(\ , \ )$ and α a strongly continuous, isometric action of G, realized as automorphisms of the Fréchet algebra $\mathcal{A}$. Like it is done in [Rie93b], we call a representation isometric, if it is isometric for all the semi-norms $\| \ \|_m$ on $\mathcal{A}$. Then, we define $\mathcal{A}^\infty$ to be the dense subalgebra of *smooth vectors* for α, *i.e.*, those A admitting the map $G \ni g \mapsto \alpha_g(A)$ to be smooth.

For the following, the subset of smooth vectors (or, *smooth elements*) is a very important notion. In order to explain this a bit further, we need an integral on a general Lie group. For achieving this aim, we follow mainly [Tay86]: If the dimension of our Lie group G is n, we can pick an n-form $\omega(\mathbb{1}_G) \in \bigwedge^n T^*_{\mathbb{1}_G}(G)$, which is determined up to a scalar factor. If we write $L_g : G \to G$ for the left multiplication with g, then we set $\omega(g) := \bigwedge^n (DL_{g^{-1}})^* \omega(\mathbb{1}_G)$, where D denotes the derivative on the tangent space here. We directly deduce that $\omega(g)$ is left multiplication invariant. Furthermore, it is different from zero on the whole of G and non-degenerate. Thus, it serves as an orientation form and the measure $\mathrm{d}g$ gained from ω is called *Haar measure.*

Lemma 2. *$\mathcal{A}^\infty$ is dense in $\mathcal{A}$.*

Proof. Let $f \in \mathscr{C}_\mathrm{c}^\infty(G)$ and $A \in \mathcal{A}$. Then define the entity

$$A(f) := \int_G f(g)\alpha_g(A)\mathrm{d}g \ ,$$

which is a bounded element of $\mathcal{A}$ due to the compact support of f. Given $g_1 \in G$, the important property is

$$\alpha_{g_1}A(f) = \int_G f(g)\alpha_{g_1 g}(A)\mathrm{d}g = \int_G f(g_1^{-1}g)\alpha_g(A)\mathrm{d}g \ ,$$

being valid for every $f \in \mathscr{C}_\mathrm{c}^\infty$. Hence we directly infer that the map $g \mapsto \alpha_g A(f)$ is a smooth map, in other words, $A(f) \in \mathcal{A}^\infty$. Now let $f_k \in \mathscr{C}_\mathrm{c}^\infty$ be a *delta sequence*, *i.e.*, a sequence of normalized functions weakly converging to the delta distribution. In this rather abstract setting, this means that $\int f_k(g)\mathrm{d}g \to 1$ for $k \to \infty$ and the f_k are supported on smaller and smaller compact neighborhoods of e. Then, by continuity,

$$\lim_{k\to\infty} A(f_k) = \lim_{k\to\infty} \int_G f_k(g)\alpha_g(A)\mathrm{d}g = A \ ,$$

which shows that an arbitrary algebra element $A \in \mathcal{A}$ can be approximated by smooth elements. $\square$

We are able to define a set of seminorms $\|A\|_{jk} := \sup_{m\leqslant j} \sum_{|\mu|\leqslant k} (\mu!)^{-1}\|\partial^\mu A\|_m$, where $\| \ \|_m$ denotes the seminorms inducing the topology of $\mathcal{A}$. Furthermore, we comprehend that $\mathcal{A}^\infty$ is a Fréchet algebra with seminorms $\| \ \|_{jj}$ and $\mathcal{A}^\infty$ is closed with respect to the $\| \ \|_{jj}$-isometric, differentiable action α. For any linear map $Q : G \to G$ and for any $A, B \in \mathcal{A}^\infty$, the function $\alpha_{Qu}(A)\alpha_v(B)$ is a smooth element and bounded w.r.t. the seminorms $\| \ \|_{jj}$. This allows for the well-definition of the following product:

Definition 7. With the notation as above, the product

$$A \times_Q B := (2\pi)^{-d/2} \iint \mathrm{d}u\, \mathrm{d}v\ \alpha_{Qu}(A)\alpha_v(B)\mathrm{e}^{i(u,v)}$$

is called *deformed product* or *Rieffel product.*

Next, we specify a list of properties permitting a reasonable implementation of the Rieffel product for (algebraic) quantum field theory.

Proposition 1 ([Rie93b])**.** *Let A, B, $C \in \mathcal{A}$ and $F \in \mathcal{A}^\infty$. The Rieffel product fulfills the following properties:*

1. *Continuity: for any j there is a k and a constant c such that*

$$\|A \times_Q B\|_{jj} \leqslant c\|A\|_k \|B\|_k \ . \tag{1.7}$$

2. *Let $M : G \to G$ be a linear map. Then*

$$\alpha_M(A \times_Q B) = A \times_{MQM^\intercal} B \tag{1.8}$$

3. *Associativity:*

$$A \times_Q (B \times_Q C) = (A \times_Q B) \times_Q C \ . \tag{1.9}$$

4. *Let $\mathcal{A}$ comprise a continuous involution and α be a *-automorphism. Then the involution on $\mathcal{A}$ is compatible with $\times_Q$:*

$$(A \times_Q B)^* = B^* \times_Q A^* \ . \tag{1.10}$$

5. *Let $\mathcal{A}$ be a C^*-algebra and Q be invertible. Then the left multiplication operator L_F, defined by $L_F B := F \times_Q B$, is bounded on the multiplier algebra. In particular,*

$$\|L_F\| \leqslant |\det(Q^{-1})| \|F\|_1 \ , \tag{1.11}$$

where $\| \ \|$ denotes the operator norm and $\| \ \|_1$ the usual L^1-norm.

The *-algebra given by the linear space $\mathcal{A}^\infty$ equipped with the product $\times_Q$ is denoted $\mathcal{A}_Q^\infty$. From the treatment in [Rie93b, chapter 4] it follows that there exists C^*-norm induced by $(\mathcal{A}^\infty, \times_Q)$, which we denote by $\| \, . \, \|_Q$. The C^*-algebra obtained by completing $(\mathcal{A}^\infty, \times_Q)$ in the norm $\| \cdot \|_Q$ is then named $\mathcal{A}_Q$.

Let us mention another property of the Rieffel product, which will gain more importance in the context of deformations with respect to degenerate matrices θ:

Lemma 3. *Denote the rank and the image of θ by $\operatorname{rk}\theta$ and $\operatorname{Im}\theta$, respectively. Then, for $A, B \in \mathcal{A}$, the Rieffel product fulfills*

$$A \times_\theta B = (2\pi)^{-\operatorname{rk}\theta/2} \int\limits_{\operatorname{Im}\theta \times \operatorname{Im}\theta} \mathrm{d}k \, \mathrm{d}v \, \mathrm{e}^{i(k,v)} \alpha_{\theta k}(A) \alpha_v(B)$$

Proof.

$$\begin{aligned} A \times_\theta B &:= (2\pi)^{-d/2} \int \mathrm{d}k \, \mathrm{d}v \, \mathrm{e}^{i(k,v)} \alpha_{\theta k}(A) \alpha_v(B) = (2\pi)^{-d} \int \mathrm{d}k \, \mathrm{d}v \, \mathrm{e}^{i(k,v)_{\ker\theta} + i(k,v)_{\operatorname{Im}\theta}} \alpha_{\theta k}(A) \alpha_v(B) \\ &= (2\pi)^{-d/2 + (\dim(\ker\theta))/2} \int\limits_{\operatorname{Im}\theta} \mathrm{d}k \int \mathrm{d}v \, \mathrm{e}^{i(k,v)_{\operatorname{Im}\theta}} \delta^{(\ker\theta)}(v) \alpha_{\theta k}(A) \alpha_v(B) \\ &= (2\pi)^{-\operatorname{rk}\theta/2} \int\limits_{\operatorname{Im}\theta \times \operatorname{Im}\theta} \mathrm{d}k \, \mathrm{d}v \, \mathrm{e}^{i(k,v)} \alpha_{\theta k}(A) \alpha_v(B) \ . \end{aligned}$$

□

From the latter property we infer that given the kernel of a certain noncommutativity matrix θ is nontrivial, the corresponding algebras of functions will only be deformed in a proper subspace of the underlying space-time. Naturally, this will lead to important conclusions in the "commutative-time" setting later on.

1.2.4 Warped Convolution

Until this point, we have worked with the noncommutative deformation of a commutative C^*-algebra in terms of the Rieffel product. D. Buchholz and S. Summers [BS08] have investigated a way of deforming the algebra itself and leaving the product unchanged. The so-called *warped convolutions* keep the main features of noncommutative quantum fields while they seem to be more applicable to the construction of such. In a following paper, the same authors together with G. Lechner [BLS11] have worked out the connection between usual C^*-algebraic dynamical systems equipped with the Rieffel product and these warped convolutions.

The deformation in terms of warped convolutions relies on a general setting that mimics the vacuum representation of quantum field theory. So, in the following, we will consider a concrete realization of the C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ again. We will denote the smooth elements represented on $\mathcal{H}$ by $\mathcal{C}^\infty$. The Moyal space setting is reached when the general Lie group G of Rieffel's treatment is chosen to be simply $\mathbb{R}^d$ and the automorphic action α is given by the adjoint action of the unitary operator U: $\alpha_x(A) = U(x)AU(x)^{-1}$ for $x \in \mathbb{R}^d$ and $A \in \mathcal{A}$. In this guise, the Rieffel product takes the following form

$$A \times_Q B = (2\pi)^{-d} \lim_{\epsilon \to 0} \int \mathrm{d}^d k \, \mathrm{d}^d v \, \chi(\epsilon k, \epsilon v) \, \alpha_{Qk}(A) \alpha_v(B) \, \mathrm{e}^{i(u,v)} \ , \tag{1.12}$$

where χ denotes a necessary, but irrelevant mollifier satisfying $\chi(0,0) = 1$. The main existence result of [BLS11] is the validity of the following expression:

Definition 8. Let Φ denote a smooth vector w.r.t. the action of U and let E be the spectral resolution of U. Then the *warped convolution* is defined as the following object:

$$A_Q \Phi := \int \alpha_{Qx}(A) \mathrm{d}E(x) \Phi \ .$$

A listing of some fundamental properties of this notion is in order.

Proposition 2 ([BLS11])**.** *Let $A, B \in \mathcal{C}^\infty$ and let Q, Q_1 be d-dimensional skew-symmetric matrices. Then we have*

1. $(A_Q)^* = (A^*)_Q$ *and* $1_Q = 1$.

2. $A_Q B_Q = (A \times_Q B)_Q$

3. $V\, A_Q\, V^{-1} = (V\, A\, V^{-1})_{\Lambda Q \Lambda^{\intercal}}$

   for V a unitary (or anti-unitary) operator on $\mathcal{H}$, such that $VU(x,1)V^{-1} = U(\Lambda x, 1)$.

4. $(A_Q)_{Q_1} = A_{Q+Q_1}$

5. *Let $\Omega \in \mathcal{H}$ be a U-invariant vector. Then $A_Q \Omega = A\Omega$.*

6. $A_0 = A$.

The form we will use during the remainder of this thesis has its origin in the concrete form of $\mathbb{R}^d$-translations implemented by α and reads,

$$A_Q \Psi = (2\pi)^{-d} \lim_{\epsilon \to 0} \int \mathrm{d}^d k \, \mathrm{d}^d v \; h(\epsilon k, \epsilon v) \, \mathrm{e}^{ikv} \, \alpha_{Qk}(A) U(v) \Psi \ , \tag{1.13}$$

for a smooth vector $\Psi \in \mathcal{D} \subset \mathcal{H}$ and h a mollifier which satisfies $h(0,0) = 1$. It has been shown in [BLS11] that A_Q is independent of the concrete choice of h.

Thus the fundamental technical concepts are available at this stage. Before we finally get started on adopting these methods in a combined way to noncommutative Wick rotation, we are going to give an impression on how advantageous new results thereon might be with respect to quantum field theory.

1.2.5 Advances in Field Theory

After its rebirth at the end of the twentieth century, noncommutative quantum field theory was studied intensively in terms of renormalization. The intuitive picture of space-time cells mentioned in Sec. 1.2.2 made most theorists involved believe that ultra-violet (UV) divergences appearing naturally in standard quantum field theory could be cured right from the start. To be more precise, the occurring of a "minimal length" was thought to prevent theories from inconsistencies coming from the formal limit of "distance going to zero", *i.e.* the UV-limit in momentum space. Feynman rules for Moyal-deformed scalar field theory introduced in [Fil96] were shown to inhibit a completely new phenomenon [MVRS00]: so-called *ultraviolet - infrared* (UV-IR) *mixing*. In a nutshell, it can be described as follows: in a field theoretic perturbation series, there are graphs containing integrated phases $e^{ip\theta q}$, which have been described in Sec. 1.2.2. When performing the loop integrations (involving integrations of *inner* momenta), these phase factors lead to terms proportional to $(\theta p)^{-1}$. These specific terms are responsible for divergences when the *outer* momenta are sent to zero. Eventually, the appearance of subgraphs leading to UV-IR mixing at any order of the perturbation series spoils any attempt of renormalization.

Surprisingly enough, the cure of the UV-IR mixing problem in Euclidean scalar field theory lead to positive results which are not contained in the corresponding commutative theories. In [GW05] it was shown that adding a harmonic oscillator term into the action of Moyal deformed scalar ϕ^4-theory leads to the disappearance of UV-IR mixing. Thus the oscillator-extended noncommutative ϕ^4 on four dimensional Euclidean space is renormalizable to all orders of perturbation theory. In addition to this success, the new action shows more interesting properties. In fact, it features a non-trivial renormalization group fixed point [DGMR07], which the commutative ϕ^4-theory does not exhibit. Investigations aiming at the constructability of this model are in progress, *c.f.* [Wan11, GW12]. These achievements concerning renormalization and presumably constructive field theory gave rise to new noncommutative models, containing scalar as well as gauge fields. All these models[5], including the Grosse-Wulkenhaar model, contain problematic constituent parts, either from a physical and mathematical point of view.

Roughly speaking, all these new invented field theory models are meant to build an intermediate step in understanding the consequences of quantum field theory on noncommutative spaces. If they are ever going to produce predictions falsifiable in physical experiments, they have to be re-arranged in a relativistic setting. This marks one of the central reasons why the Wick rotation of Moyal deformed theories in particular, and noncommutative quantum field theory in general is desirable. In addition, directly working out noncommutative theories on Minkowski space-time is even in the simplest cases highly non-trivial [LS02b, LS02a, BFG$^+$03].

[5]see for example [GMRT09, GVT10] for other renormalizable scalar field models, or [MST00, MSSW00] and references therein for implications on gauge fields

Chapter 2

Algebraic Wick Rotation

In this chapter, the analytic continuation of a Euclidean net of *-algebras towards the corresponding Minkowskian net of observables will keep us occupied. We are going to describe in detail the results of the paper [GLLV11], concerning noncommutative Wick rotation in terms of algebraic quantum field theory. To this end we will first recapitulate the known transition from "commutative" Euclidean nets satisfying full covariance with respect to the Euclidean group $E(d)$ towards Haag-Kastler nets. Afterwards, the generalization to theories covariant just with respect to a proper subgroup of $E(d)$ is obtained, without any reference to noncommutative geometry at this stage. The next step consists in the deformation in terms of warped convolutions, together with the analogous Wick rotation result for the noncommutative Euclidean net. Eventually, we are interested in the net evolving from the deformation of the commutative Haag-Kastler net which leads to the main finding that it is in fact isomorphic to the one gained before.

2.1 The Schlingemann Approach

Apparently detached from the known theorems about Wick rotation, most importantly those by Osterwalder and Schrader, we introduced the main notions of algebraic quantum field theory in the introduction. There are tight relations between theories described via local observable algebras and those defined through the sequence of n-point functions [FH81]. Nevertheless, it was Schlingemann [Sch99] in 1999 who gave a precise figure to algebraic Wick rotation. As an analogous road is taken when it comes to deformed theories, we will summarize his work in the following.

Let $E(d) = \mathbb{R}^d \rtimes SO(d)$ denote the d dimensional connected Euclidean group. The Euclidean quantum field theory one starts with is given by a $E(d)$-covariant net $\{\mathcal{E}(O)\}_{O\subset\mathscr{O}}$, where $\mathscr{O}$ is the set of open bounded regions in $\mathbb{R}^d$, together with a reflection positive functional σ of Def. 3. The net shall satisfy a specific condition in accordance with the physically demanded notion of locality. Namely, we demand that any two algebra elements $A \in \mathcal{E}(O_1)$ and $B \in \mathcal{E}(O_2)$ commute under the condition that $O_1 \cap O_2 = \varnothing$. We are able to write

$$O_1 \cap O_2 = \varnothing \Rightarrow [\mathcal{E}(O_1), \mathcal{E}(O_2)] = \{0\} \,. \tag{2.1}$$

Since on $\mathbb{R}^d$ with Euclidean metric there is no preferred direction, we choose some direction e to be "Euclidean time". Then its orthogonal complement $e^\perp$ will serve as the "time-zero-plane". As we will see, one fundamental concept of the algebraic Wick rotation is the time-zero content of the theory. Since on both the Euclidean and the Minkowski framework we will deal with them, we define the *time-zero algebras* for more general G-covariant nets $(\mathcal{A}, \mathscr{O}, \alpha)$ on $\mathbb{R}^d$:

$$\mathcal{A}_0(K) := \bigcap_{O\supset K} \mathcal{A}(O)\,, \qquad K \subset e^\perp\,. \tag{2.2}$$

Schlingemann's approach, as well as our noncommutative generalization coming up later in this chapter, largely relies on the following additional assumption. We say that a G-covariant net $(\mathcal{A}, \mathscr{O}, \alpha)$ satisfies the *time-zero condition*, if its time-zero algebras are enough to generate any local algebra $\mathcal{A}(O)$ when acted on with the full automorphic symmetry action, *i.e.*, if

$$\mathcal{A}(O) = \left(\bigcup_{K\subset e^\perp} \{\alpha_g(\mathcal{A}_0(K)) \,|\, g \in G,\, gK \subset O\} \right)'' \,, \qquad O \in \mathscr{O}\,, \tag{TZ}$$

where $\mathcal{B}''$ for $\mathcal{B} \subset \mathscr{B}(\mathcal{H})$ denotes the double commutant, which is equivalent to the weak closure in the von Neumann setting if $\mathcal{B}$ is a unital *-algebra. In a C^*-setting, one would prefer the norm-closure instead.

Let us explain this definition more extensively. Any *local algebra* $\mathcal{A}(O)$, O in a suitable index set, coming from a net of C^*-algebras satisfying the time-zero condition can be obtained in the following way:

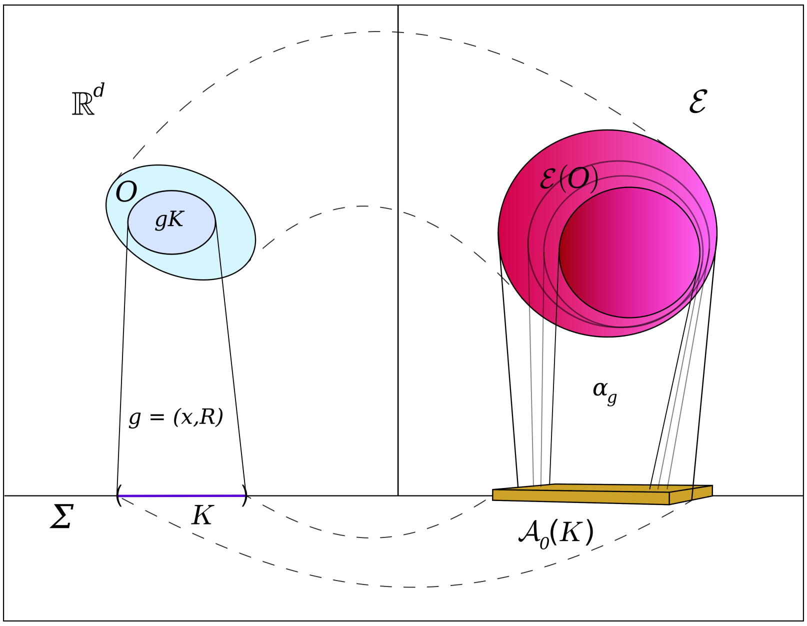

Figure 2.1: Time-zero condition: each local algebra $\mathcal{E}(O)$ is generated by the time-zero algebras $\mathcal{A}_0(K)$ via actions of the actions α_g.

- Pick a subset K of the *time-zero plane* $e^\perp$ and build the time-zero algebra $\mathcal{A}_0(K)$ in terms of (2.2) upon it.

- Act with all α_g on $\mathcal{A}_0(K)$, where g is an element of the symmetry group G, such that K is mapped into O under g.

- Form the algebraic closure: build the union over all $K \subset e^\perp$ of these symmetry group images $\alpha_g \mathcal{A}_0(K) \subset \mathcal{A}(gK) \subset \mathcal{A}(O)$ and afterwards, go over to the weak closure of this union.

Thus the time-zero condition assumes that the algebra closure at the end of this procedure gives in fact the whole local algebra $\mathcal{A}(O)$. The following definition summarizes the net content and implicitly the axioms a Euclidean theory shall fulfill in order to give sense to a corresponding analytical continuation:

Definition 9. A $E(d)$-covariant net $(\mathcal{E}, \mathscr{O}, \alpha)$ on $\mathbb{R}^d$ together with a reflection positive functional σ is called *Euclidean field theory*. If in addition $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ satisfies the locality property (2.1), it is said to be[1] *local*.

[1]Observe a somewhat unhandy nomenclature: a local *algebra* is the image $\mathcal{A}(O)$ of the net prescription and got

While the goal of the following recapitulation is a Haag-Kastler net obtained by Wick rotation of such a Euclidean field theory, its achievement goes through a number of important steps:

1. Define the "physical" Hilbert space $\mathcal{H}$ via the reflection-positive functional σ and properly represent the Euclidean algebra on $\mathcal{H}$.

2. Examine how the representation of the Euclidean group acts on this Hilbert space and analytically continue it to a unitary representation of the Poincaré group $\mathcal{P}(d)$.

3. Define the Minkowski net of observables $\mathcal{M}(O)$ as the image of the time-zero algebras under this $\mathcal{P}(d)$-representation.

4. Show that $\mathcal{M}(O)$ satisfies the Haag-Kastler axioms.

Since we are going to explain all these steps in full detail for deformed theories, we will not give a pronounced repetition at the scenario of full $E(d)$-covariant nets. We will see in the next section that deforming causes a restriction of the underlying symmetry group and this leads to our treatment being a generalization of step 2. here.

The biggest drawback, however, is the lack of locality in the noncommutative case. Having said that, it is only natural to demand such a loss of locality due to general noncommutativity of the algebras involved. Nevertheless, one is able to re-establish a remnant of locality here as well: so-called *wedge localiy*, which we will discuss in Ch. 4. So let us proceed to the noncommutative case after summarizing commutative algebraic Wick rotation in the following

Theorem 1 ([Sch99])**.** *Let $(\mathcal{E}, \mathscr{O}, \alpha)$ be a local Euclidean field theory satisfying the time zero condition. Further let $\alpha^{\mathcal{M}}$ denote the analytically continued representation of d-dimensional Poincaré group and π be the *-representation of $\mathcal{E}_0$ on the physical Hilbert space. Then, the net*

$$\begin{aligned} O &\mapsto \mathcal{A}(O) \\ \mathcal{M}(O) &:= \overline{\bigcup_{K \subset e^{\perp}} \left\{ \alpha_g^{\mathcal{M}}(\pi \mathcal{E}_0(K)) \,|\, g \in \mathcal{P}(d),\, gK \subset O \right\}}^{\|\cdot\|} \end{aligned}$$

is a $\mathcal{P}(d)$-covariant Haag-Kastler net represented on the Hilbert space $\mathcal{H}$.

Remark 1. By definition, the net $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}})$ of Thm. 1 satisfies the time-zero condition as well.

its name from containing those quantum observables measurable in the space-time region O. On the other hand, a *net* is said to be local it fulfills (a Euclidean version of) *Einstein locality*, *i.e.*, micro-causality.

2.2 Reduced Symmetry Groups

Since we are aiming at formulating a noncommutative generalization of a reconstruction theorem similar to the famous one of Osterwalder and Schrader, we will deal with Moyal space later on. A certain property of this deformation scheme is that the symmetry group gets restricted. In order to give an eligible definition of the noncommutative framework, we consider nets which are covariant with respect to a subgroup of the ordinary Euclidean or Poincaré group. We therefore introduce the reduced symmetry groups as follows:

Definition 10. The (connected) *reduced Euclidean and Poincaré groups* $E_\theta(d)$ and $\mathcal{P}_\theta(d)$ are defined to be

$$\begin{aligned} E_\theta(d) \quad &:= \{(x,R) \,:\, (x,R) \in E(d),\ \ R\theta = \theta R\} = \mathrm{SO}_\theta(d) \ltimes \mathbb{R}^d\,, &(2.3)\\ \mathcal{P}_\theta(d) \quad &:= \{(x,\Lambda) \,:\, (x,\Lambda) \in \mathcal{P}(d)^\uparrow_+,\ \ \Lambda\theta = \theta\Lambda\} = \mathcal{L}_\theta(d)^\uparrow_+ \ltimes \mathbb{R}^d\,, &(2.4)\end{aligned}$$

respectively.

As a small anticipation, we can easily legitimate definitions like the latter by demanding an automorphic symmetry action α which is at the same time a homomorphism with respect to the Rieffel product, see subsection 1.2.3. Take the invertible matrix M to be a realization of any symmetry group element $g = (a, M)$, then from property (1.8) we obtain

$$\alpha_g(A \times_\theta B) = \alpha_g A \times_{M\theta M^{-1}} \alpha_g B \tag{2.5}$$

which leaves invariant the noncommutativity if and only if $M\theta = \theta M$, just as in (2.3),(2.4).

Construction of the physical Hilbert space Consider the algebra $\mathcal{E}_> := \mathcal{E}(e^\perp + \mathbb{R}_+ e)$, which serves as our "positive-time environment". Furthermore, define a sesquilinear form on $\mathcal{E}_>$ as follows:

$$\mathcal{E}_> \times \mathcal{E}_> \ni (A, B) \mapsto \sigma(\iota(A^*)B)$$

This sesquilinear form is positive due to reflection positivity of σ and can be degenerate. Therefore, we build equivalence classes with respect to its null space

$$\mathcal{N}_\sigma := \{A \in \mathcal{E}_> \,|\, \sigma(\iota(A^*)A) = 0\} \ ,$$

in the standard way, $[\ ]_\sigma : A \mapsto A + \mathcal{N}_\sigma$, and observe that we can thus define a scalar product on the pre-Hilbert space $\mathcal{E}_>/\mathcal{N}_\sigma$ via

$$\langle [A]_\sigma, [B]_\sigma \rangle := \sigma(\iota(A^*)B) \ .$$

Finally, we define the completion $\mathcal{H} := (\overline{\mathcal{E}_>/\mathcal{N}_\sigma}, \langle ., . \rangle)$ with respect to the $\langle ., . \rangle$-induced norm to be our physical Hilbert space. For later convenience, we assert that the vector $\Omega := [1]_\sigma$ will play the role of the vacuum. We demanded of our Euclidean field theory in *2.* of Def. 3 that e-reflections act covariantly on the net $\mathcal{E}(O)$ and we easily comprehend that all the time-zero algebras $\mathcal{E}_0(K)$ are ι-invariant. Hence, for $B \in \mathcal{E}_0$, which is the same as $\mathcal{E}(e^\perp)$, the element $\iota(B^*)C$ lies again in $\mathcal{E}_>$ for all $C \in \mathcal{E}_>$. By the Cauchy-Schwarz inequality, we infer that $\mathcal{N}_\sigma = \{A \in \mathcal{E}_> \,|\, \sigma(\iota(A^*)C) = 0 \ \forall\, C \in \mathcal{E}_>\}$. These facts lead to the well-definition of the map $\pi : \mathcal{E}_0 \to \mathscr{B}(\mathcal{H})$, which is given in the following way,

$$\pi(B)[A]_\sigma := [BA]_\sigma \quad , \quad A \in \mathcal{E}_> \, , \, B \in \mathcal{E}_0 \ . \tag{2.6}$$

In other words, π is a well-defined GNS representation of $\mathcal{E}_0$ on $\mathcal{H}$. Indeed, let us take $A \in \mathcal{N}_\sigma$, then we have

$$\langle \pi(B)[A]_\sigma, [C]_\sigma \rangle \ = \ \sigma(\iota((BA)^*)C) = \sigma(\iota(A^*)\iota(B^*)C) = 0 \ ,$$

for all $C \in \mathcal{E}_>$. Hence $\pi(B)[A]_\sigma$ is again in $[\mathcal{N}_\sigma]_\sigma$ and π is well-defined.

At some stage in the current treatise we want to go over to a noncommutative deformation of the algebraic content. Keeping this in mind, we work out how the reduced symmetry groups $E_\theta(d)$ and $\mathcal{P}_\theta(d)$ act on the Hilbert space $\mathcal{H}$ in advance. Consider the involutive automorphism γ of *3* in Def. 3 on $E(d)$, which was defined by

$$\gamma : \ (x, R) \mapsto (rx, rRr) \ .$$

The e-reflection r acts on $E(d)$, and therefore on $E_\theta(d)$, via γ. It can be directly seen that γ is an involution, thus it possesses exactly the eigenvalues ± 1. The subgroup $E_\theta^e(d)$ of $E_\theta(d)$ is defined to consist of fixed points of γ and thus is called "spatial subgroup".

We could go on by directly adjusting the analytic continuation of the Euclidean group representation to a unitary representation of the Poincaré group as it was done in [KL82]. Though we consider it more appropriate to make the underlying mathematical structure visible for sake of generality. But this requires a little background, which we shall sketch, following mainly [Hel62]:

Definition 11. Let G be a Lie group, K be a closed subgroup of G and γ be an involutive automorphism. Furthermore, let K_γ be the set of fixed points in K with respect to γ and $(K_\gamma)_0$ its identity component. Then the triple (G, K, γ) is called a *symmetric space*, if

$$(K_\gamma)_0 \subseteq K \subseteq K_\gamma$$

The group G is sometimes called symmetric group.

Theorem 2 ([Hel62]). *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ is the Lie algebra of G, while $\mathfrak{k}$ and $\mathfrak{m}$ denote the eigenspaces of γ for the eigenvalues $+1$ and -1, respectively, then we have*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \ , \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \ , \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{2.7}$$

Definition 12. The dual symmetric Lie algebra $\mathfrak{g}^*$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{m}$$

and due to the relations (2.7) it is still a real Lie algebra. We call the symmetric space (G^*, K^*, γ) the *dual space* of (G, K, γ), where $\mathfrak{g}^*$ is the Lie algebra of G^* and called the *dual symmetric Lie algebra* of $\mathfrak{g}$.

In complete analogy to [FOS83] we distinguish between two cases of symmetric spaces in the following; the choice to do so will be justified in a moment when it comes to the interpretation of these abstract notions.

Definition 13. A symmetric space (G, K, γ) is defined to be of type (I) if K is compact. A symmetric space (G, K, γ) is defined to be of type (II) if

$$G = G_0 \ltimes T \ , \quad K = K_0 \ltimes P \ , \quad K_0 \overset{\text{cpt.}}{\subseteq} G_0 \ , \ P \subseteq T \ , \ T \text{ abelian}$$

Remark 2. For type (II) symmetric spaces it holds:

$$\mathfrak{g} \oplus \mathfrak{t} = (\mathfrak{k}_0 \oplus \mathfrak{m}_0) \oplus (\mathfrak{p} \oplus \mathfrak{h}) = \mathfrak{k} \oplus \mathfrak{m}$$

where the small German letters denote the Lie algebras of the corresponding capital letter Lie groups and $\mathfrak{h}$ is the Lie algebra of T/P. From this we can see

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subset \mathfrak{k}_0 \ , \quad [\mathfrak{k}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0 \ , \quad [\mathfrak{m}_0, \mathfrak{m}_0] \subset \mathfrak{k}_0 \ ,$$

as well as

$$[\mathfrak{k}_0, \mathfrak{p}] \subset \mathfrak{p} \ , \quad [\mathfrak{k}_0, \mathfrak{h}] \subset \mathfrak{h} \ ,$$

and

$$[\mathfrak{m}_0, \mathfrak{h}] \subset \mathfrak{p} \ , \quad [\mathfrak{m}_0, \mathfrak{p}] \subset \mathfrak{h} \ , \quad [\mathfrak{t}, \mathfrak{t}] = 0 \ .$$

Remark 3. Physical examples one has in mind are:

- Type (I):

$$G = SO(n) \ , \ K = SO(n-1) \Rightarrow G^* = \overline{SO(1, n-1)}$$

- Type (II):

$$\begin{aligned} G &= E(n) = \overline{SO(n)} \ltimes \mathbb{R}^n \ , \ K = E(n-1) = \overline{SO(n-1)} \ltimes \mathbb{R}^{n-1} \\ \Rightarrow G^* &= \overline{SO(1, n-1)} \ltimes \mathbb{R}^n \ . \end{aligned}$$

Coming back to the specific case at hand, the involution γ introduced before Def. 11 induces a decomposition of the Lie algebra $\mathfrak{e}_\theta(d)$ of $E_\theta(d)$ into corresponding eigenspaces with eigenvalues ± 1,

$$\mathfrak{e}_\theta(d) = \mathfrak{e}^e_\theta(d) \oplus \mathfrak{m}_\theta \,, \tag{2.8}$$

where $\mathfrak{e}^e_\theta(d)$ denotes the Lie algebra of $E^e_\theta(d)$. The pair $(\mathfrak{e}_\theta(d), \mathfrak{e}^e_\theta(d))$ in fact has the structure of a symmetric Lie algebra, *i.e.*,

$$[\mathfrak{e}^e_\theta, \mathfrak{e}^e_\theta] \subset \mathfrak{e}^e_\theta \ , \quad [\mathfrak{e}^e_\theta, \mathfrak{m}_\theta] \subset \mathfrak{m}_\theta \ , \quad [\mathfrak{m}_\theta, \mathfrak{m}_\theta] \subset \mathfrak{e}^e_\theta \tag{2.9}$$

Our next task consists of obtaining the correct representation of $E_\theta(d)$ on $\mathcal{H}$. For a given element of $E_\theta(d)$, we have to take care of it staying in the equivalence class of the positive-time algebra $\mathcal{E}_>$, because $\mathcal{H}$ was defined in this way. Hence it is necessary to consider a sufficiently small neighborhood $\mathscr{U}$ of the identity in $E_\theta(d)$ and define the representing operators $V(g)$ there for elements $A \in \mathcal{E}_>$ as follows:

$$V(g)[A]_\sigma := [\alpha_g(A)]_\sigma \,, \qquad g \in \mathscr{U} \,. \tag{2.10}$$

More precisely, for a given $g \in \mathscr{U}$, we consider all regions $O \subset \mathbb{R}^d_>$ such that both, gO and $\gamma(g)^{-1}gO$, are still contained in $\mathbb{R}^d_>$. For $A \in \mathcal{E}(O)$, the right hand side of (2.10) is then well-defined by covariance and isotony, $\alpha_g(A) \in \mathcal{E}(gO) \subset \mathcal{E}_>$. Furthermore, we have to check that the above assignment is well-defined, *i.e.*, independent of the choice of representative in $[A]_\sigma$. In fact, for $A \in \mathcal{N}_\sigma$ we can use the $E_\theta(d)$-invariance of σ to compute

$$\begin{aligned} \langle V(g)[A]_\sigma, V(g)[A]_\sigma\rangle &= \|[\alpha_g(A)]_\sigma\|^2 = \sigma(\iota(\alpha_g(A))^*\alpha_g(A)) = \sigma(\alpha_{\gamma(g)}(\iota(A))^*\alpha_g(A)) \\ &= \sigma(\iota(A)^*\alpha_{\gamma(g)^{-1}g}(A)) \,. \end{aligned} \tag{2.11}$$

According to our assumption on the region O, we have $\gamma(g)^{-1}gO \subset \mathbb{R}^d_>$, and hence $\alpha_{\gamma(g)^{-1}g}(A)$ is again an element of $\mathcal{E}_>$. Thus the Cauchy-Schwarz inequality yields $V(g)[A]_\sigma = 0$, which shows that $V(g)$ leaves invariant the null space $\mathcal{N}_\sigma$ of the product $\langle , \rangle$ and therefore is well-defined. The subspace of $\mathcal{H}$ which is spanned by all $[\mathcal{E}(O)]_\sigma$, where O runs over the described set of regions, will be taken as the domain $\mathrm{dom}V(g)$ of $V(g)$, *i.e.*,

$$\mathrm{dom}V(g) \;=\; \{[\mathcal{E}(O)]_\sigma \,|\, O \subset \mathbb{R}^d_+ \,,\, gO \subset \mathbb{R}^d_+ \,,\, \gamma(g)^{-1}gO \subset \mathbb{R}^d_+\} \subset \mathcal{H} \;.$$

In order to obtain a common domain for all operators $V(g)$, $g \in E_\theta(d)$, we choose a particular region.

In spite of the more general approach permitted by the usage of an arbitrary Euclidean direction serving as the time direction, it is oftentimes more convenient to go over to a concrete description of these spaces. This is the reason why we introduce orthonormal coordinates $(x_0, ..., x_{d-1})$ of $\mathbb{R}^d$, with the property $e = (1, 0, ..., 0)$. Whenever we consider it more convenient, we write $(x^0, \underline{x}) := (x_0, ..., x_{d-1})$. Since we are dealing with the Euclidean group, corresponding generators of translations are needed and denoted by $P_0, ..., P_{d-1}$. In addition we have operators M_{kl}, $k < l$, $k, l = 0, ..., d-1$, which implement the rotations in the x_k-x_l-plane. It is straightforward to comprehend that the Lie algebra $\mathfrak{e}^e_\theta(d)$ is spanned by $P_1, ..., P_{d-1}$ (spatial translations) and all linear combinations of M_{kl}, $k \geqslant 1$, which commute with θ.

Remark 4. At the physically interesting case of $d = 4$ the Lie algebra $\mathfrak{e}^e_\theta$ contains rotations with respect to exactly one axis. Therefore, $\mathfrak{e}^e_\theta$ generates $SO(2)$. The remaining linear space $\mathfrak{m}_\theta$ is spanned by P_0 and all linear combinations of M_{0k}, $k = 1, 2, 3$. But by the definition of $E_\theta(d)$ these M_{0k} must commute with θ. Hence $\mathfrak{m}_\theta$ generates $SO(2)$ as well. This makes $\mathfrak{e}_\theta(4)$ the Lie algebra of $\mathbb{R}^4 \rtimes (SO(2) \times SO(2))$.

The region serving as the common domain of our representation is defined to be

$$C := \{x \in \mathbb{R}^d \,:\, x_0 > 1 + (x_1^2 + ... + x_{d-1}^2)^{1/2}\} \,. \tag{2.12}$$

Clearly there exists a neighborhood $\mathscr{U} \subset E_\theta(d)$ of the identity such that gC and $\gamma(g)^{-1}gC$ are both contained in $\mathbb{R}^d_>$ for all $g \in \mathscr{U}$. Furthermore, we can choose $\mathscr{U}$ so large that it contains the full spatial subgroup $E^e_\theta(d)$ (Euclidean group elements commuting with the time reflection ι), since the latter cannot push C to a region of negative time: $gC \subset \mathbb{R}^d_>$ for all $g \in E^e_\theta(d)$. These considerations qualify the subspace

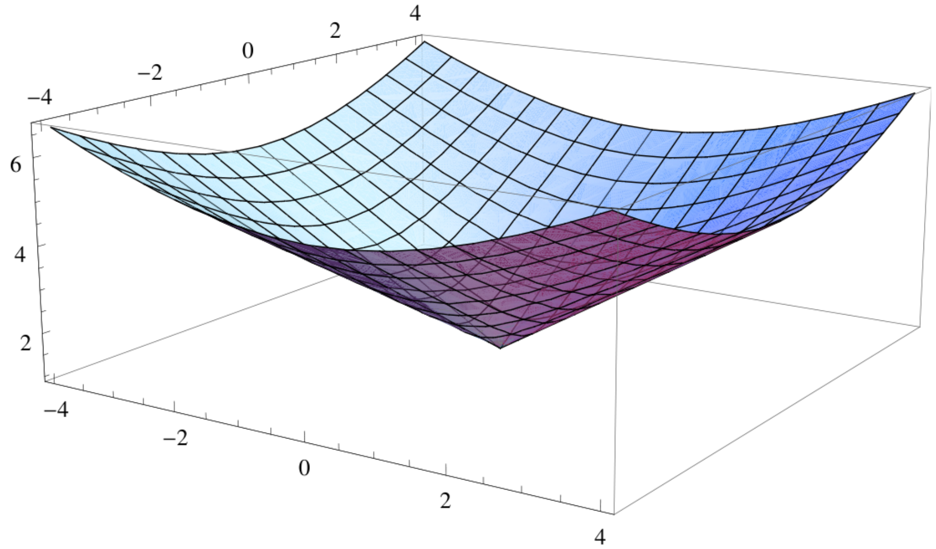

Figure 2.2: The boundary of the common domain C plotted for $d = 3$.

$$\mathcal{D}_0 := [\mathcal{E}(C)]_\sigma \subset \mathcal{H} \tag{2.13}$$

to be the domain for our representation. In the context of symmetric spaces, a theorem guaranteeing the existence of an analytic continuation of certain group representations is available [FOS83]. Before we reflect the exact statement, we define the notions the said theorem deals with and simultaneously prove that the representation we constructed fits into that framework:

Proposition 3. *The data $(\mathscr{U}, \mathcal{D}_0, V)$ form a* virtual representation [FOS83] *of $E_\theta(d)$, i.e.*

1. *$\mathcal{D}_0 \subset \mathcal{H}$ is dense and for all $g \in \mathscr{U}$, one has $\mathcal{D}_0 \subset \mathrm{dom} V(g)$.*
2. *If g_1, g_2 and $g_1 g_2$ lie in $\mathscr{U}$ and $\Psi \in \mathcal{D}_0$, then $V(g_2)\Psi \in \mathrm{dom} V(g_1)$ and*
$$V(g_1)V(g_2)\Psi = V(g_1 g_2)\Psi\,.$$
3. *For $g \in E_\theta^e(d)$, the operator $V(g)$ extends to a unitary on all of $\mathcal{H}$.*
4. *For $g = \exp M \in \mathscr{U}$ with $M \in \mathfrak{m}_\theta$, the operator $V(g)$ is hermitian.*

5. Let $\Psi \in \mathcal{D}_0$. Then $\mathscr{U} \ni g \mapsto V(g)\Psi$ is strongly continuous.

6. The translations in e-direction $\{V(\exp tP_0)\}_{t\geqslant 0}$ form a contraction semi-group.

Proof. 2. Let $g_1, g_2 \in \mathscr{U}$ such that also $g_1 g_2 \in \mathscr{U}$. It follows $g_1 g_2 C \subset \mathbb{R}^d_>$. $\mathscr{U}$ was chosen such that $\gamma(g)^{-1} g C \subset \mathbb{R}^d_+$ for all $g \in \mathscr{U}$, thus by $V(g_2)\mathcal{D}_0 = [\mathcal{E}(g_2 C)]_\sigma$ we have established $V(g_2)\mathcal{D}_0 \subset \mathrm{dom} V(g_1)$. The group law $V(g_1)V(g_2)\Psi = V(g_1 g_2)\Psi$, $\Psi \in \mathcal{D}_0$, follows from (2.10). Indeed, let $\Psi \in \mathcal{D}_0$, then it follows $\Psi = [A]_\sigma$ for an $A \in \mathcal{E}(C)$. Thus,

$$\begin{aligned} V(g_1)V(g_2)\Psi &= V(g_1)V(g_2)[A]_\sigma = V(g_1)[\alpha_{g_2}(A)]_\sigma = [\alpha_{g_1}(\alpha_{g_2}(A))]_\sigma \\ &= [\alpha_{g_1 g_2}(A)]_\sigma = V(g_1 g_2)[A]_\sigma = V(g_1 g_2)\Psi\ . \end{aligned}$$

To establish *3.* and *4.*, we compute with $A, B \in \mathcal{E}(C)$ and $g \in \mathscr{U}$

$$\begin{aligned} \langle V(g)[A]_\sigma,\, [B]_\sigma\rangle &= \sigma(\iota(\alpha_g(A^*))B) = \sigma(\alpha_{\gamma(g)}(\iota(A^*))B) \\ &= \sigma(\iota(A^*)\alpha_{\gamma(g)^{-1}}(B)) = \langle [A]_\sigma,\, V(\gamma(g)^{-1})[B]_\sigma\rangle\,, \end{aligned} \tag{2.14}$$

yielding $V(g)^* \supset V(\gamma(g)^{-1})$. This shows that for a γ-invariant group element $g \in E^e_\theta(d)$, we have $V(g)^* \supset V(g)^{-1}$, and once we have checked that $\mathcal{D}_0$ is dense (part *1.*), it is clear that such $V(g)$ extend to unitaries on all of $\mathcal{H}$. On the other hand, for $g = \exp M \in \mathscr{U}$, $M \in \mathfrak{m}_\theta$ (the "time-dependent" part of the Euclidean Lie algebra), we have $\gamma(g) = g^{-1}$, and thus the representing operator $V(g)$ is hermitian.

5. Let $g \in \mathscr{U}$ and $A \in \mathcal{E}(C)$. Then we compute as in (2.11),

$$\|V(g)[A]_\sigma - [A]_\sigma\|^2 = \sigma\left(\iota(A^*)\left(\alpha_{\gamma(g)^{-1}g}(A) - \alpha_{\gamma(g)^{-1}}(A) - \alpha_g(A) + A\right)\right)\ .$$

In the limit where g approaches the identity in $E_\theta(d)$, this norm difference vanishes because of the assumed continuity of σ (*1.* of Def. 3).

6. For positive parameters $t \geqslant 0$, the domain of the e-translations $V_1(t) := V(\exp tP_0)$ is the dense subspace $\mathrm{dom} V_1(t) = \mathcal{D} = \mathcal{E}_>/\mathcal{N}_\sigma$. As the generator P_0 of translations along e lies in the -1 eigenspace $\mathfrak{m}_\theta$ of γ, we find analogously to the proof of *4.* that $V_1(t)$ is a hermitian operator for any $t \geqslant 0$. As $\mathbb{R}_+ \ni t \mapsto V_1(t)$ is strongly continuous on $\mathcal{D}$ by the continuity of σ, it follows that V_1

is a symmetric local semi group [KL81]. In particular, there exists a self-adjoint operator H with $\mathcal{D} \subset \operatorname{dom} e^{-tH}$ for any $t \geqslant 0$, such that $V_1(t) = e^{-tH}$ on $\mathcal{D}$. For $A \in \mathcal{E}_>$, $t \geqslant 0$, we estimate

$$\|e^{-tH}[A]_\sigma\| = \|V_1(t)[A]_\sigma\| = \|[\alpha_{\exp tP_0}(A)]_\sigma\| \leqslant \|\alpha_{\exp tP_0}(A)\|_{\mathcal{E}} = \|A\|_{\mathcal{E}}\,.$$

Since this bound is uniform over all $t \geqslant 0$, and $\mathcal{D} \subset \mathcal{H}$ is dense, it follows that H is positive, *i.e.* V_1 is a contraction semi group.

1. With this information, we can now prove that $\mathcal{D}_0$ is a dense subspace of $\mathcal{H}$ as well, using a Reeh-Schlieder type argument (see also [Sch99]). For $\Psi \in \mathcal{D}_0^\perp$ and $A \in \mathcal{E}(O)$ with some bounded $O \subset \mathbb{R}^d_>$, consider the function $f : \mathbb{R} \to \mathbb{C}$,

$$f(s) := \langle \Psi, e^{-sH}[A]_\sigma \rangle = \langle \Psi, [\alpha_{\exp sP_0}(A)]_\sigma \rangle\,. \tag{2.15}$$

Since H is positive and V_1 is strongly continuous, f extends to a holomorphic function in the right half plane, with continuous boundary values. Moreover, as O is bounded, there exists $s_0 > 0$ such that $O + s\cdot e \subset C$ and thus $\alpha_{\exp sP_0}(A) \in \mathcal{E}(C)$ for all $s \geqslant s_0$. Hence $f(s) = 0$ for $s \geqslant s_0$, which by the analyticity of f implies $0 = f(0) = \langle \Psi, [A]_\sigma \rangle$, *i.e.*, $\Psi \perp [\mathcal{E}(O)]_\sigma$. But by definition of $\mathcal{E}_>$, the union of all $\mathcal{E}(O)$, where O runs over all bounded regions in $\mathbb{R}^d_>$, is norm-dense in $\mathcal{E}_>$, and by construction of the Hilbert space, $[\mathcal{E}_>]_\sigma$ is a dense subspace of $\mathcal{H}$. Hence $\Psi = 0$, which proves the density of $\mathcal{D}_0 \subset \mathcal{H}$. The fact that $\mathcal{D}_0 \subset \operatorname{dom} V(g)$ for all $g \in \mathscr{U}$ follows from the definitions of $\mathcal{D}_0$ and the domains $\operatorname{dom} V(g)$: $\mathcal{D}_0$ was defined to be $[\mathcal{E}(C)]_\sigma$ and we already noticed that the region C causes gC as well as $\gamma(g)^{-1}gC$ being elements of $\mathbb{R}^d_+$ for all $g \in \mathscr{U}$. The subalgebra generated by elements satisfying exactly these relations was defined to be $\operatorname{dom} V(g)$. $\square$

Now we are ready to perform the analytic continuation of the virtual representation established right now. For the full Euclidean group, it has been shown in miscellaneous guises by a number of authors that this is possible. For a thorough treatment in a more general group theoretic setting see [JO99]. Using the notions worked out so far, we can generalize their results to the case of reduced symmetry groups, which is of great importance when it comes to deformation.

We start by the assertion that we will obtain the group we are aiming at: given $\mathfrak{e}_\theta(d)$, its dual Lie algebra leads to the corresponding reduced version of the d-dimensional Poincaré group:

Lemma 4. *The connected, simply connected Lie group $E_\theta(d)^*$ with Lie algebra $\mathfrak{e}_\theta(d)^*$ is the universal covering group of the reduced Poincaré group,*

$$E_\theta(d)^* = \widetilde{\mathcal{P}}_\theta(d)\,. \tag{2.16}$$

Proof. For $\theta = 0$, this fact is well known [FOS83, JO99]. Now $\mathfrak{e}_\theta(d)^* \subset \mathfrak{e}(d)^*$ consists of all those elements of $\mathfrak{e}(d)^*$ which commute with θ, *i.e.*, $\mathfrak{e}_\theta(d)^*$ coincides with the Lie algebra of the reduced Poincaré group $\mathcal{P}_\theta(d)$, defined in (2.4). Hence $E_\theta(d)^*$ is the unique connected simply connected Lie group with the same Lie algebra as $\mathcal{P}_\theta(d)$, that is, the universal covering group $\widetilde{\mathcal{P}}_\theta(d)$. □

The adaptation of the framework of symmetric spaces and virtual representations reaches its aim when it comes to the application of the following theorem. We have shown that the representation of the reduced Euclidean group $E_\theta(d)$, given by the operators $V(g)$ on our physical Hilbert space $\mathcal{H}$ is "virtual" in the above sense. This makes us able to profit from the following:

Theorem 3 ([FOS83])**.** *Let $(G_0 \ltimes T, K_0 \ltimes P, \gamma)$ be a type (II) symmetric space and let π be a virtual representation of $(G_0 \ltimes T)$ such that there exists a basis $\{e_i\}_{i=1}^r$ of T/P with the property that $\{\pi(\mathrm{e}^{-te_i})\}_{t\leqslant 0}$ is a symmetric contraction semi-group for each $i = 1, \ldots r$. Then π can be analytically continued to a unitary representation π^* of $(G_0 \ltimes T)^*$.*

Combining Prop. 3, Lemma 4 and Thm. 3 we obtain that V can be analytically continued to a unitary representation $\tilde{U}$ of $E_\theta(d)^*$, which is nothing else than $\widetilde{\mathcal{P}}_\theta(d)$. Before we go on, we remark that in the concrete situation at hand, $\tilde{U}$ actually descends to a unitary representation U of the reduced Poincaré group $\mathcal{P}_\theta(d)$ itself: For $\theta = 0$, this follows from the analysis in [KL82], where the analytic continuation of V was carried out for $E_0(d) \equiv E(d)$ and shown to result in a unitary representation of $\mathcal{P}_0(d) \equiv \mathcal{P}(d)$ instead of the universal covering group. This feature then restricts to the reduced group $\mathcal{P}_\theta(d) \subset \mathcal{P}_0(d)$ for $\theta \neq 0$.
Furthermore, the vacuum vector $\Omega = [1]_\sigma \in \mathcal{D}_0$ is invariant under all $V(g)$, $g \in \mathscr{U}$, as can be seen from (2.10). As U is obtained from V by analytic continuation, Ω is invariant under the representation U as well.
We collect what we have obtained in the following theorem, preparing the ground for going over to Moyal space(-time):

Theorem 4. *There exists a strongly continuous unitary representation U of $\mathcal{P}_\theta(d)$ on $\mathcal{H}$ such that*

1. $U(g) = V(g)$ *for* $g \in E^e_\theta(d) \subset \mathcal{P}_\theta(d)$.

2. $U(g)\Omega = \Omega$ *for all* $g \in \mathcal{P}_\theta(d)$.

The representation U provides us in particular with a strongly continuous representation $x \mapsto U(x, 1)$ of the translation group $\mathbb{R}^d$, generated by $P_0 := H$ and the (Euclidean) momentum operators $P_1, ..., P_{d-1}$. It is important to note that the usual spectrum condition gets modified as well by the restriction to proper symmetry subgroups.

Proposition 4. *The joint spectrum* S *of the generators* $P_0, ..., P_{d-1}$ *of the translations* $U(x, 1) = e^{iP\cdot x}$ *is a* $\mathcal{P}_\theta(d)$*-invariant subset of* $\mathbb{R}^d$ *satisfying* $\{p_0 : (p_0, ..., p_{d-1}) \in S\} \subset \mathbb{R}_+$ *and* $0 \in S$.

Proof. As $U(x, 1)$ extends to a representation of $\mathcal{P}_\theta(d)$, the joint spectrum of its generators is a $\mathcal{P}_\theta(d)$-invariant subset of $\mathbb{R}^d$. Moreover, P_0 is positive by *6.* of Prop. 3, and hence

$$\{p_0 : (p_0, ..., p_{d-1}) \in S\} = \mathrm{spec} P_0 \subset \mathbb{R}_+ \ .$$

We have $0 \in S$ because Ω is translation invariant. □

Remark 5. Let us give more details on the explicit shape of the translation generators' spectrum. In terms of this "noncommutative" spectrum condition one can easily realize a physical consequence of the strict Moyal deformation prior to its actual implementation:

- For a theory satisfying full $E(d)$-covariance, we as usual gain a unitary $\mathcal{P}(d)$-representation, which leaves the spectrum of the $P_0, \ldots, P_{d-1}$ invariant due to the latter proposition. Together with the condition $\{p_0 : (p_0, ..., p_{d-1}) \in S\} \subset \mathbb{R}_+$ this means nothing else than S must be contained in the closed forward light-cone.

- If $d > 2$ is even and $\ker\theta = \{(p_0, p_1, 0, ..., 0) \ : \ p_0, p_1 \in \mathbb{R}\}$, just the boosts $\Lambda_1(\beta)$ in x_1-direction lie in $\mathcal{P}_\theta(d)$: indeed, we know from (2.5) that for any matrix M to be an element of $\mathcal{P}_\theta(d)$, it is necessary and sufficient that $(x, M) \in \mathcal{P}(d)$ for any $x \in \mathbb{R}^d$ and $M\theta M^{-1} = \theta$. Given an arbitrary boost realization $\Lambda_k(\beta)$ the only possibility for the latter equation to hold is $k = 1$, as one can easily realize while having a look at (1.6) and setting $\vartheta_e = 0$. It is immediately clear that $\Lambda_1(\beta)$ being the only boost in $\mathcal{P}_\theta(d)$ is more than ever correct if one does not set

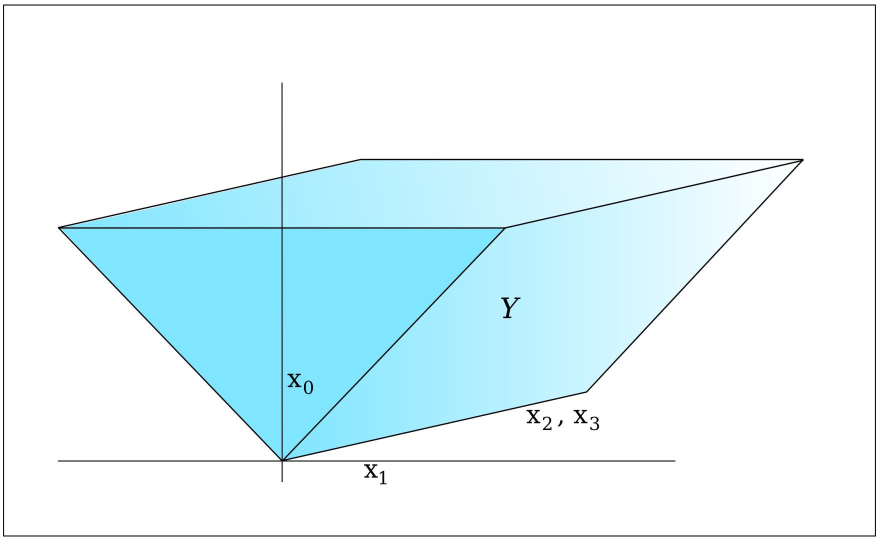

Figure 2.3: The light-wedge Y

ϑ_e equal to zero, *i.e.*, at a noncommutative-time scenario.
For an explicit form of the spectrum it is thus enough to consider $(\Lambda_1(\beta)p)_0$ and from

$$(\Lambda_1(\beta)p)_0 = \cosh(\beta)p_0 + \sinh(\beta)p_1$$

as well as from the positivity statement it follows that p_0 may not be smaller than $|p_1|$. Hence in this case, we only get $S \subset Y$, where Y denotes the closed *light-wedge*

$$Y := \{p \in \mathbb{R}^d \,:\, p_0 \geqslant |p_1|\}\,, \tag{2.17}$$

which is indicated in Fig. 2.3. Spectrum conditions like these have also been discussed elsewhere [AGVM03].

We are almost done with our Wick rotation of an algebraic Euclidean theory with a reduced symmetry group. The last step will comprise a meaningful definition of a theory on Minkowski space-time, fulfilling adapted Haag-Kastler axioms. As we will see in a moment, the analytical continuation of the symmetry group representation was already the most exhaustive part of achieving

this aim. The rest relies purely on the given algebraic properties of the Euclidean field theory and the time-zero condition.
At first we denote by $\alpha^{\mathcal{M}}$ the automorphic action of the unitary representation U of $\mathcal{P}_\theta(d)$,

$$\alpha_g^{\mathcal{M}}(A) := U(g)AU(g)^{-1} \quad , \quad g \in \mathcal{P}_\theta(d)\,,\ A \in \mathcal{E}_> \tag{2.18}$$

So we have reached the point where we have collected everything necessary to come to our Lorentzian theory: we define it to be the net $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}})$, where $\mathscr{O}$ is still the collection of open bounded regions of $\mathbb{R}^d$. We recall the representation π (2.6) of the time-zero algebra $\mathcal{E}_0 = \mathcal{E}(e^\perp)$ on $\mathcal{H}$ and remark that by the time-zero condition on the net $\mathcal{E}$, this algebra is non-trivial, or put differently, not equal to $\mathbb{C}1$. The next theorem includes a definition and finishes the commutative treatment of algebraic Wick rotation in this thesis:

Theorem 5. *1. The algebras $\mathcal{M}(O)$, defined as*

$$\mathcal{M}(O) := \left(\bigcup_{K \subset e^\perp} \left\{ \alpha_g^{\mathcal{M}}(\pi\mathcal{E}_0(K)) \,\middle|\, g \in \mathcal{P}_\theta(d)\,,\ gK \subset O \right\} \right)'' \tag{2.19}$$

for any open bounded region $O \subset \mathbb{R}^d$ and the action $\alpha^{\mathcal{M}}$ given in (2.18) *form a net $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}})$ of von Neumann algebras on Minkowski space-time which satisfies the following properties:*

(a) Isotony: $O_1 \subset O_2 \in \mathscr{O} \Rightarrow \mathcal{M}(O_1) \subset \mathcal{M}(O_2)$,

(b) Covariance: $h \in \mathcal{P}_\theta(d) \Rightarrow \alpha_h^{\mathcal{M}}\mathcal{M}(O) = \mathcal{M}(hO)\ \forall\, O \in \mathscr{O}$,

(c) Time-Zero Condition: (TZ) *holds with respect to $\mathcal{P}_\theta(d)$.*

2. The state $\omega(A) := \langle \Omega, A\Omega \rangle$ is a $\mathcal{P}_\theta(d)$-invariant vacuum state on $\mathcal{M}$.

Proof. 1. First of all, we make sure that equation (2.19) displays well-defined von Neumann algebras $\mathcal{M}(O)$ depending on $O \in \mathscr{O}$. This on the one hand is guaranteed by the weak closure applied to the rhs. of this very equation. On the other hand, $\alpha^{\mathcal{M}} = \mathrm{Ad}\, U$ is an automorphic $\mathcal{P}_\theta(d)$-action on $\mathscr{B}(\mathcal{H})$, which is part of Thm. 4. Thus, according to Def. 1, we now are able to show the claimed properties.

(b) Let $O \subset \mathbb{R}^d$ be open, $K \subset e^\perp$, $A \in \mathcal{E}_0(K)$, and $g \in \mathcal{P}_\theta(d)$ such that $gK \subset O$. Then, for an element $B \in \mathcal{M}(O)$ defined to be $B := \alpha_g^{\mathcal{M}}(\pi(A))$, and for any $h \in \mathcal{P}_\theta(d)$, we have $\alpha_h^{\mathcal{M}}(B) = \alpha_{hg}^{\mathcal{M}}(\pi(A))$. Since clearly $hgK \subset hO$, we have $\alpha_h^{\mathcal{M}}(B) \in \mathcal{M}(hO)$, and as $\mathcal{M}(O)$ is generated by operators B, also $\alpha_h^{\mathcal{M}}(\mathcal{M}(O)) \subset \mathcal{M}(hO)$. By also considering the inverse transformation h^{-1}, we arrive at the covariance property $\alpha_h^{\mathcal{M}}(\mathcal{M}(O)) = \mathcal{M}(hO)$.

(a) Let $O_1 \subset O_2$ be an inclusion of two regions in $\mathbb{R}^d$, and $K \subset e^\perp$ such that there exists $g \in \mathcal{P}_\theta(d)$ with $gK \subset O_1$. Then we obtain $gK \subset O_2$. Hence, by covariance we have that for any $A \in \mathcal{E}_0(K)$ it follows $\alpha_g^{\mathcal{M}}(\pi(A)) \in \mathcal{M}(O_1)$. By the first consideration, this is contained in $\mathcal{M}(O_2)$ as well. As these $\alpha_g^{\mathcal{M}}(\pi(A))$ generate $\mathcal{M}(O_1)$, isotony holds, *i.e.*, $\mathcal{M}(O_1) \subset \mathcal{M}(O_2)$.

(c) This property is satisfied by the very construction of the net $\{\mathcal{M}(O)\}_{O\in\mathscr{O}}$.

2. By definition, ω is a linear functional on $\mathcal{M}$, which is positive: on the one hand we have $\langle\Omega, A\Omega\rangle := \sigma(\iota(1)^* A1)$. On the other hand take $B \in \mathcal{M}$ to be a positive element of $\mathcal{M}$. Then there exists $A \in \mathcal{M}$ such that $B = A^*A$. Hence we have

$$\omega(B) = \omega(A^*A) = \langle\Omega, A^*A\Omega\rangle = \langle A\Omega, A\Omega\rangle \geqslant 0 \ .$$

Covariance follows similarly: Since Ω is a $\mathcal{P}_\theta(d)$-invariant unit vector, it follows for every $g \in \mathcal{P}_\theta(d)$ and every $A \in \mathscr{B}(\mathcal{H})$,

$$\omega(\alpha_g^{\mathcal{M}} A) = \langle\Omega, \alpha_g^{\mathcal{M}} A\Omega\rangle = \langle U(g)^{-1}\Omega, AU(g)^{-1}\Omega\rangle = \omega(A) \ ,$$

thus ω is an $\alpha^{\mathcal{M}}$-invariant state on $\mathscr{B}(\mathcal{H})$. This implies the claim of $\mathcal{P}_\theta(d)$-invariance on the subalgebra $\mathcal{M} \subset \mathscr{B}(\mathcal{H})$. Furthermore, Thm. 4 implies that U is strongly continuous, which means that

$$\mathcal{P}_\theta(d) \ni g \mapsto \omega(A\alpha_g^{\mathcal{M}}(B)) = \langle U(g)^{-1}A^*\Omega,\, B\,U(g)^{-1}\Omega\rangle$$

is continuous for all $A, B \in \mathcal{M}$, as required in Def. 3.

Finally, we show that ω fulfills positivity: $\mathcal{M}$ is weakly closed and invariant under the translations $\alpha^{\mathcal{M}}_{x,1}$, $x \in \mathbb{R}^d$. Moreover, we use that U is strongly continuous to deduce that $\mathcal{M}$ contains a strongly dense subalgebra $\mathcal{M}^\infty$ consisting of operators A for which $x \mapsto \alpha^{\mathcal{M}}_{x,1}(A)$ is smooth (see Lemma 2 in the introduction). In particular, the function $\mathbb{R} \ni t \mapsto \omega(A^*\alpha^{\mathcal{M}}_{t\cdot e,1}(A))$ is differentiable at $t = 0$ for such A, and $-i\frac{d}{dt}|_{t=0}\omega(A^*\alpha^{\mathcal{M}}_{t\cdot e,1}(A)) = \langle A\Omega, P_0 A\Omega\rangle \geqslant 0$ because P_0 is positive. This completes the list of properties of ω required in Def. 3. □

2.3 Noncommutative Algebraic Wick Rotation

In the present chapter we made some effort in preparing the ground for the application of a certain deformation scheme in order to get to a noncommutative algebraic quantum field theory. Nonetheless, not at any time we have relied on such, not even on a fairly general noncommutative theory up until now. In other words, all we have done in the ongoing chapter so far is valid for any algebraic Euclidean field theory, covariant with respect to a certain subgroup of the Euclidean group. At this point we are going to implement the procedures of 1.2.3 and 1.2.4 in our framework of Euclidean and Lorentzian nets of observables. Though our treatment happens to be quite abstract, we expect some practicability: all our assumptions with the time-zero condition as the only exception[2] are considered natural in the area of Moyal deformation.

Before we start going into detail, we explain the main ambition of the following considerations. At first, we work on the net of observables resulting from the deformation of a Euclidean field theory in the sense of Def. 9. We saw in (2.5) that choosing our degenerate noncommutativity θ for the general skew-symmetric matrix Q causes the appearance of the reduced symmetry group $E_\theta(d)$. The circumstances under which a Wick rotation of such deformed nets are still possible will be explained in detail. By taking a different point of view, we can first use the existing notions of Wick rotation to arrive at an undeformed Lorentzian net of observables from the usual Euclidean field theory. There is no reason why one should not use the warped convolutions framework to deform a theory situated on Minkowski space-time. Quite the contrary, this deformation scheme was designed to work on Lorentzian theories originally. So another main result of the following is the uniformity of the noncommutative net of observables on Minkowski space-time. What just has been sketched can apparently be visualized in Fig. 2.4.

[2] In fact, this restriction is valid in many constructed models; we will come to that later.

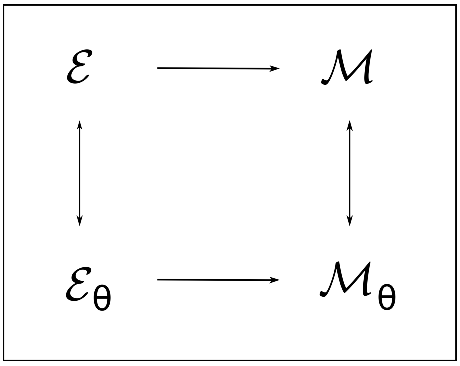

Figure 2.4: The apparent objective of algebraic Wick rotation

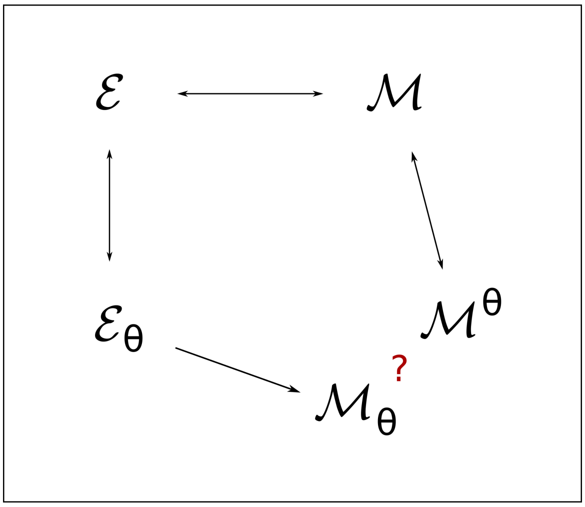

Figure 2.5: The complete task of our algebraic Wick rotation.

Taking a closer look, however, infers that one actually is left with two deformed nets on Minkowski space-time. One, denoted by $\mathcal{M}_\theta$, arises when figuring out the Wick rotation of section 2.2 for the deformed Euclidean net $\mathcal{E}_\theta$ and the other one is obtained by deforming the Wick rotated net $\mathcal{M}$. That latter net we are going to denote by $\mathcal{M}^\theta$. The task of working out whether these two deformed nets are equivalent is depicted in Fig. 2.5.

While in 1.2.3 we have prepared the main features of Rieffel's deformation of the C^*-algebra product, we present more such results useful for the algebraic framework now. Making a specific choice for the matrix Q of deformation parameters is the first mandatory step. From now on, we will deal with the following noncommutativity matrix in arbitrary $d = s + 2n$ dimensions:

$$\theta := 0_s \oplus \theta_1 \oplus \cdots \oplus \theta_n \tag{2.20}$$

where the 2×2 matrices θ_k are given by the standard Moyal form:

$$\theta_k := \begin{pmatrix} 0 & \vartheta_k \\ -\vartheta_k & 0 \end{pmatrix} , \quad \vartheta_k \in \mathbb{R} , \quad k = 1, \ldots, n$$

For maintaining the largest noncommutativity possible in presence of the "commutative-time" scenario (corresponding to $s > 0$ here) we will oftentimes choose $s = 2$ if d is even and $s = 1$ if d is odd. So, the most interesting case in our treatment will be $d = 2 + 2$, where θ assumes the form

$$\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vartheta \\ 0 & 0 & -\vartheta & 0 \end{pmatrix} , \tag{2.21}$$

which coincides with (1.6) when ϑ_e is set to zero and $\vartheta_m =: \vartheta$. Let us consider the dense subalgebra $\mathcal{A}^\infty \subset \mathcal{A}$ of the (for the moment still general) C^*-algebra $\mathcal{A}$ consisting of smooth elements. By Lemma 3, it suffices to perform the integration at the Rieffel product along a basis of $\operatorname{Im} \theta$. These we will sometimes call "noncommutative directions".

Next we are going to figure out some of the modifications of the warped convolution deformation when deciding to use such a noncommutativity matrix θ. All of the fundamental properties of Prop. 2 stay correct when we consider a skew-symmetric matrix θ of the degenerate Moyal type. The covariance property *3.* there in particular simplifies even. Indeed, let $A, B \in \mathcal{C}^\infty$. Then we get a simpler relation for V being a unitary operator on $\mathcal{H}$ such that $VU(x)V^* = U(Mx)$, $x \in \mathbb{R}^d$ and for some $M \in GL(d)$ satisfying $M^T = M^{-1}$ and $M\theta = \theta M$:

$$VA_\theta V^* = (VAV^*)_\theta \ . \tag{2.22}$$

In particular, $U(x)A_\theta U(-x) = (U(x)AU(-x))_\theta$ for all $x \in \mathbb{R}^d$.

We need one more result before we can finally appease our appetite for the Moyal deformation of local nets of observables. Namely, the fact that the Hilbert space representation of the original algebra can be directly continued to a representation of the noncommutative algebra:

Lemma 5 ([BLS11]). *Let $\mathcal{A}$ be a C^*-algebra with strongly continuous $\mathbb{R}^d$-action α and π an α-covariant representation of $\mathcal{A}$ on $\mathcal{H}$, i.e., $U(x)\pi(A)U(x)^{-1} = \pi(\alpha_x(A))$, $A \in \mathcal{A}$. Then*

$$\pi(\mathcal{A}^\infty) \subset \mathcal{C}^\infty$$

and the map

$$\pi_\theta(A) := \pi(A)_\theta \quad , \quad A \in \mathcal{A}^\infty$$

extends continuously to an α-covariant representation of the deformed C^*-algebra $\mathcal{A}_\theta$.

2.3.1 Deformation of a Euclidean Field Theory

Now that we have introduced all the notions and facts we are going to need in the algebraic framework, we can start to explain how the diagram 2.5 happens to commute in the case of degenerate Moyal deformation. To this end we start with a Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ as introduced in Def. 9. The first question that arises is how to obtain a local algebra via warped convolutions. The best way in order to achieve this may be considering the Rieffel-type deformation of the net $\mathcal{E}$ on Moyal space. What is schematically represented with the arrow $\mathcal{E} \to \mathcal{E}_\theta$ in the Figs. 2.4 and 2.5 actually consists in a distinct procedure which gets started by the following definition:

Definition 14. Given the global algebra $\mathcal{E}$ of a Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha)$, we define the *Rieffel deformed algebra* $\mathcal{E}_\theta$ to be $(\mathcal{E}^\infty, \times_\theta)$ with $\times_\theta$ being the Rieffel product of Def. 7 and the corresponding norm being $\| \, \|_\theta$.

The general inner product $(\, , \,)$ of Def. 7 is taken to be the Euclidean product from now on. and For two vectors $x, y \in \mathbb{R}^d$ it is oftentimes denoted by xy. We want to proceed to the largely equivalent, but more convenient point of view regarding warped convolutions. A question of high importance concerns the generalizability of the field theory structure from a given net of observables to a Rieffel-deformed one. The following lemma answers this question in large part and is for sake of generality formulated for C^*-algebras not restricted to the cases at hand:

Lemma 6 ([Rie93a, Rie93b]). *1. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be C^*-algebras with strongly continuous automorphic $\mathbb{R}^d$-actions α_1, α_2 and $\beta : \mathcal{A}_1 \to \mathcal{A}_2$ an isomorphism such that $\beta\alpha_{1,x} = \alpha_{2,Mx}\beta$ for all $x \in \mathbb{R}^d$ and some $M \in \mathrm{GL}(d)$ with $M^T = M^{-1}$ and $M\theta = \theta M$, where the transpose M^T refers to the bilinear form used in Def. 7. Then $\beta(\mathcal{A}_1^\infty) = \mathcal{A}_2^\infty$, and $\beta|_{\mathcal{A}_1^\infty}$ extends to an isomorphism $\beta^\theta : \mathcal{A}_{1,\theta} \to \mathcal{A}_{2,\theta}$ such that $\beta^\theta\alpha^\theta_{1,x} = \alpha^\theta_{2,Mx}\beta^\theta$ for all $x \in \mathbb{R}^d$.*

2. Let ν be an α-invariant linear continuous functional on $\mathcal{A}$. Then $\nu|_{\mathcal{A}^\infty}$ extends to a linear continuous functional ν^θ on $\mathcal{A}_\theta$, and there holds

$$\nu(A \times_\theta B) = \nu(AB)\,, \qquad A, B \in \mathcal{A}^\infty. \tag{2.23}$$

Proof. 1. This is a combination of Thm. 5.12 and Prop. 2.12 of [Rie93b].

Part *2.* of this lemma can be inferred from [Rie93a], where it was proven that a translationally invariant *state* ω on $\mathcal{E}$ satisfies $|\omega(A)| \leqslant c\|A\|_\theta$ for all $A \in \mathcal{E}^\infty$ and some constant $c > 0$. But following the proof of [Rie93a, Thm. 4.1] it becomes apparent that neither positivity nor normalization of ω are necessary for the argument. Hence also ν can be extended to a continuous functional on $\mathcal{A}_\theta$, which finishes the proof of the lemma. □

Remark 6. Up until this point, we could well have worked with a noncommutativity matrix of full rank, *i.e.* a noncommutative-time scenario. The only thing that would have changed is the reduced groups would no longer contain discrete transformations. Since we were concerned with the connected groups all along, we experience no modifications. All the lemmata and propositions above have been proved for general skew-symmetric matrices Q as deformation parameters in the corresponding original treatise.

Now that we realized that the constituents of C^*-algebras behave well under the deformation, the next step consists in bothering about the net structure. One possible choice to get to the respective noncommutative net of von Neumann algebras is built up of deforming each element of $\mathcal{E}(O)$, *i.e.*,

$$\mathcal{E}(O)_\theta := \{A_\theta \mid A \in \mathcal{E}(O)\}'' \quad , \quad O \in \mathscr{O}\ .$$

As one can see straightforwardly, $\mathcal{E}(O)_\theta$ is a well-defined algebra, generated by $\mathcal{E}(O)$. Anyhow, we will not work with this prescription in the thesis at hand. The reason for that is as follows:

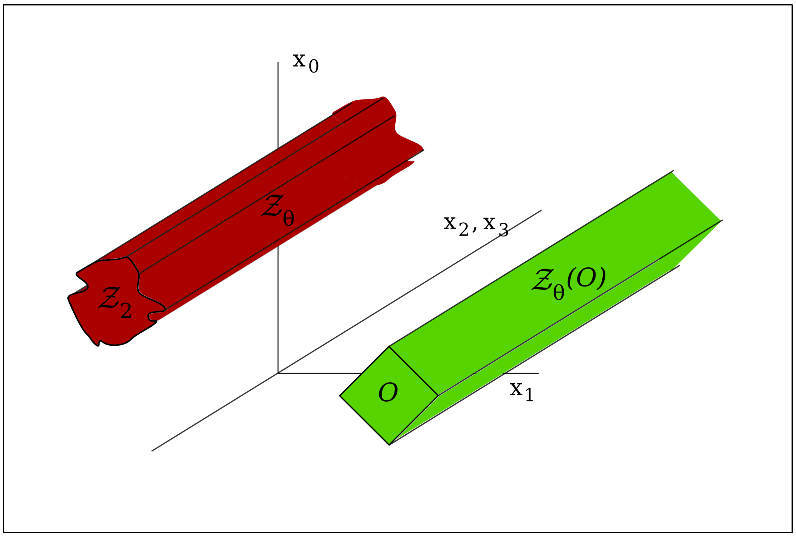

Figure 2.6: Cylindrical regions: $\mathcal{Z}_2$ is the x_0, x_1-projection of $\mathcal{Z}_\theta$ and $\mathcal{Z}_\theta(O)$ is the cylindrical region associated to O.

Let $O \subset \mathbb{R}^d$ be an open, bounded region. Then the set $\{A_\theta \mid A \in \mathcal{E}(O)\}$ is not a local algebra indexed by O w.r.t. the Rieffel product. This can be seen by examining Lemma 3 and *2.* of Prop. 2: indeed, the product of two warped convoluted algebra elements equals the warped convolution of their Rieffel product. But at the product, the elements are translated in the direction of $\operatorname{Im}\theta$ to arbitrary extent. So, any region bounded in this direction will be outranged by the smallest region containing the Rieffel product. These considerations illustrate the fact that the only reason $\mathcal{E}(O)_\theta$ is closed under the operator product is due to building the weak closure in its very definition. Put differently, together with the warped convolutions of each undeformed algebra element $A \in \mathcal{E}(O)$ one includes countable sums of products of such into the definition of $\mathcal{E}(O)_\theta$. But this abstract closure makes it unhandy for working out the changes the Euclidean net of algebras faces during the deformation.

So we prefer to establish an algebra of warped convolutions indexed by another family $\mathcal{Z}_\theta$ of $\mathbb{R}^d$-subsets. We demand that this family makes every local prescription $\mathcal{Z}_\theta \ni Z \mapsto \mathcal{E}_\theta(Z)$ a net of well-defined von Neumann algebras. We have just shown that the smallest sets applicable for building an algebra of warped convolutions are

$$\mathcal{Z}_\theta(O) := \{x + \theta k \,,\, x \in O, k \in \mathbb{R}^d\} \,, \tag{2.24}$$

which are called *cylindrical regions*. In other words, $(\mathcal{E}_\theta(\mathcal{Z}_\theta(O)), \times_\theta)$ is a local algebra. It is obvious that in terms of algebra nets one can forget about the explicit form of O, the region that "generated" $\mathcal{Z}_\theta(O)$. So, we give another definition

$$\mathcal{Z}_\theta := \{Z \subset \mathbb{R}^d \, , \, Z + \mathrm{Im}\theta = Z\} = \{\mathcal{Z}_\theta(O) \, , \, O \subset \mathbb{R}^d\} \ . \tag{2.25}$$

Some cylindrical regions are shown in Fig. 2.6. We appreciate from these definitions that $\mathcal{Z}_\theta$ still allows for E_θ and $\mathcal{P}_\theta$-covariance: $\mathrm{Im}\,\theta$ is left invariant with respect to x_2, x_3-rotations and any boost or rotation involving just the x_0, x_1-plane.

Remark 7. Concerning the generalization to a noncommutative-time scenario, *i.e.* contemplating a noncommutativity Θ which is (any $SO(d)$-rotation) of the form (1.6), we will no longer be able to argue the way we did to arrive at a local algebra of warped convolutions. This follows directly from having a look at (2.25) and the fact that $\mathrm{Im}\,\Theta = \mathbb{R}^d$, so the "full Moyal" cylinders would all be the whole space.

We find that the net $\{\mathcal{E}(\mathcal{Z}_\theta(O))\}_{O\in\mathscr{O}}$ was the correct choice for our purposes:

Proposition 5. *The data $\mathcal{E}_\theta(Z)$, $Z \in \mathcal{Z}_\theta$, and the automorphic symmetry action restricted to the reduced Euclidean group, $\alpha|_{E_\theta(d)}$, form a $E_\theta(d)$-covariant net $(\mathcal{E}_\theta, \mathcal{Z}_\theta, \alpha^\theta)$ of C^*-algebras.*

Proof. We have already seen that the net structure can be regained by going over to the set $\mathcal{Z}_\theta$. The remaining parts of the proposition follow mainly by the application of Lemma 6: At first we choose $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{E}$ and $\alpha_1 = \alpha_2 = \alpha|_{\mathbb{R}^d}$ there. This implies that the deformed C^*-algebra $\mathcal{E}_\theta$ carries an automorphic action α^θ of $E_\theta(d)$ which coincides with $\alpha|_{E_\theta(d)}$ on the smooth subalgebra $\mathcal{E}^\infty$. Furthermore, ι gives rise to an automorphism ι^θ representing the reflection r on $\mathcal{E}_\theta$.

By Def. 14, we see that $\mathcal{E}_\theta(Z)^\infty$ and $\mathcal{E}(Z)^\infty$ are equal as linear spaces, and thus

$$\mathcal{E}_\theta(Z_1)^\infty \subset \mathcal{E}_\theta(Z_2)^\infty \quad , \quad \forall\, Z_1, Z_2 \in \mathcal{Z}_\theta \ \text{ such that } \ Z_1 \subset Z_2 \ .$$

This inclusion remains valid after closing in $\|\cdot\|_\theta$, yielding a net $Z \mapsto \mathcal{E}_\theta(Z)$ indexed by cylindrical regions. Furthermore, we have made clear below the definition in ((2.25)) that the family $\mathcal{Z}_\theta$ is invariant under the reduced Euclidean group $E_\theta(d)$. Finally, we are making use of Lemma 6 again:

this time, we set $\mathcal{A}_1 = \mathcal{E}(Z)$, $\mathcal{A}_2 = \mathcal{E}(gZ)$, $g \in E_\theta(d)$, and α_1, α_2 both being the action of the translations along $\operatorname{Im}\theta$ on these algebras. We hence see that

$$\alpha_g^\theta(\mathcal{E}_\theta(Z)) = \mathcal{E}_\theta(gZ)\,, \qquad g \in E_\theta(d),\ Z \in \mathcal{Z}_\theta\,. \tag{2.26}$$

which brings the proof to an end.

□

Remark 8. Again we want to highlight the necessity of the special form we chose for θ in this context. The construction of the physical Hilbert space is based on the functional σ which fulfills reflection positivity on $\mathcal{E}_>$, the subalgebra generated by open bounded subsets of $e^\perp + e\mathbb{R}_+$. If we had used a noncommutativity Θ of full rank, due to property 2. of Prop. 2, $\mathcal{E}_>$ would have been unstable with respect to the Rieffel product.

2.3.2 From Deformed Euclidean to Deformed Lorentzian Theories

The upcoming treatment will rely on a generalization of the time-zero condition. To this end, let us denote the notion of a (time-zero) stripe S by

$$S \in \mathcal{S}_\theta := \{x + \theta k \mid x \in K \text{ (open, bounded)} \subset e^\perp\,,\ k \in \mathbb{R}^d\}\ .$$

Analogously to the case just before, $\mathcal{E}_0(S)$ is stable with respect to the Rieffel product $\times_\theta$, and hence $\mathcal{E}_{0,\theta}(S)$ is well-defined as a local algebra of warped convolutions.

The following lemma shows that for the warped net of algebras, the symmetry group $E_\theta(d)$ is sufficient to keep a generalized time-zero condition; since there is no fundamental difference to the Lorentzian case here, we prove both cases at once:

Lemma 7. *Let $S \in \mathcal{S}_\theta$ and $g \in E(d)$ (respectively $g \in \mathcal{P}(d)$) such that $gS \subset Z$ for some $Z \in \mathcal{Z}_\theta$. Then there exist $g_1 \in E_\theta(d)$, $g_2 \in E(d)$ (respectively $g_1 \in \mathcal{P}_\theta(d)$, $g_2 \in \mathcal{P}(d)$) such that $g = g_1 g_2$ and $g_2 S = S$.*

Proof. We write $g = (x, M)$ referring to the semidirect product structure of $E(d) = \mathbb{R}^d \rtimes \mathrm{SO}(d)$ respectively $\mathcal{P}(d) = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow(d)$. The set MS satisfies $MS + x = MS$ for $x \in M\operatorname{Im}\theta$. If

$M \operatorname{Im}\theta \not\subset \operatorname{Im}\theta$, it follows that MS is not bounded in projection to $\ker\theta$, and can thus not be contained in an element of the family $\mathcal{Z}_\theta$. Thus the assumption implies $M \operatorname{Im}\theta \subset \operatorname{Im}\theta$, and since M is invertible, $M \operatorname{Im}\theta = \operatorname{Im}\theta$ and $M^T \ker\theta = \ker\theta$. In the Euclidean case, $M^T = M^{-1}$, and we have $M \ker\theta = \ker\theta$. In the Lorentzian case, $M^T = \eta M^{-1}\eta$. But the metric η commutes with θ, such that also in this case we arrive at $M \ker\theta = \ker\theta$.
So for both signatures, M decomposes as a direct sum $M = M_1 \oplus M_2 \in \mathscr{B}(\ker\theta) \oplus \mathscr{B}(\mathrm{Im}\theta)$, and we define $g_1 := (x, M_1 \oplus 1)$, $g_2 := (0, 1 \oplus M_2)$. Then $g_1 g_2 = (x, M_1 \oplus M_2) = g$, and as the slice S contains the full image of θ, we have $g_2 S = S$ and consequently $g_1 S = g_1 g_2 S = gS$. Furthermore, $g_1 \in E_\theta(d)$ (respectively $g_1 \in \mathcal{P}_\theta(d)$) as $M_1 \oplus 1$ commutes with $\theta = 0_s \oplus \theta_1 \oplus \cdots \oplus \theta_n$. □

Remark 9. The preceding Lemma 7 is significant for the operability of noncommutative Wick rotation, but might seem a bit unhandy. So let us consider the special case most interesting from a physical point of view, namely that of $d = 4$. We are going to illustrate the proof of the latter lemma from a more geometric point of view in this case.

Let Z_2 denote the projection of $Z \in \mathcal{Z}_\theta$ to the subspace of commutative directions $\ker\theta \simeq \mathbb{R}^2$. Further let $d_1 := \max\limits_{v,w\in\partial Z_2} \{|v_1 - w_1| \mid v_0 = w_0\}$ be the maximum width of Z_2 (and thus of Z) in x_1-direction. Then, any line on the x_1-axis of length smaller than d_1 can be translated into Z_2, i.e., for any $S \in \mathcal{S}_{\theta,2}$ of the form

$$S_a^b = \{x \in \mathbb{R}^2 \mid x_0 = 0\,,\ a \leqslant x_1 \leqslant b\,,\ a, b \in \mathbb{R}\,,\ |b - a| < d_1\}\ ; \tag{2.27}$$

see Fig. 2.7 *a*.
If $d_m := \max\limits_{v,w\in\partial Z_2} \{\|v - w\,\|\}$ denotes the maximal diameter of Z in the x_0, x_1-plane (which is bounded by definition), we can always find stripes of width smaller than d_m, that can be rotated into Z via a specific x_0, x_1-rotation. This can bee seen in Fig. 2.7 *b*.

Let $Z \in \mathcal{Z}_\theta$ and take $g \in E(d)\backslash E_\theta(d)$. Then g has to have a non-vanishing component generated by one of B_2, B_3, R_1, R_2, where B_k (resp. R_k) denotes the generator of the x_0, x_k-rotation (resp. a spatial rotation). Without loss of generality we take S to be of the form (2.27). We then have

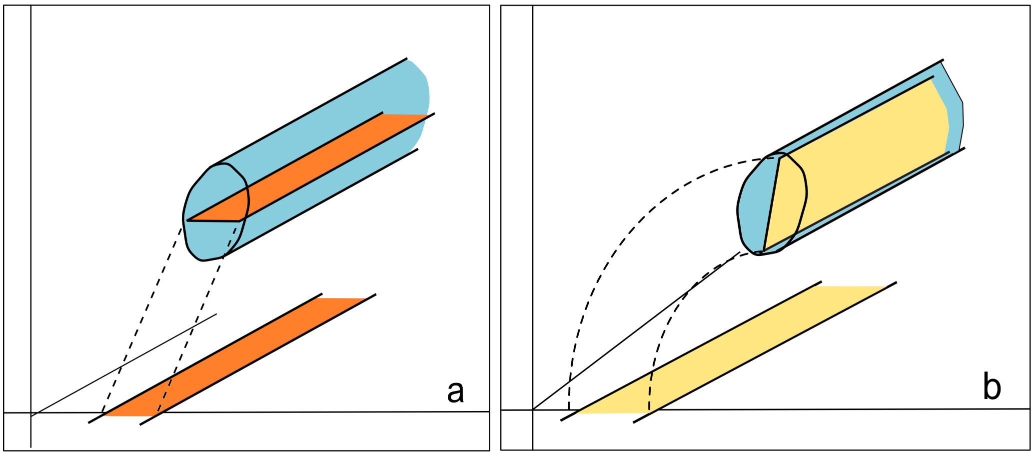

Figure 2.7: A stripe translated into a cylinder (a) and one rotated (b)

$$
\begin{aligned}
B_2(\beta)S &= \{x \in \mathbb{R}^d \mid x_0 \cos\beta + x_2 \sin\beta = 0\,,\ a \leqslant x_1 \leqslant b\} \\
B_3(\beta)S &= \{x \in \mathbb{R}^d \mid x_0 \cos\beta + x_3 \sin\beta = 0\,,\ a \leqslant x_1 \leqslant b\} \\
R_1(\beta)S &= \{x \in \mathbb{R}^d \mid x_0 = 0\,,\ a \leqslant x_1 \cos\beta + x_2 \sin\beta \leqslant b\} \\
R_2(\beta)S &= \{x \in \mathbb{R}^d \mid x_0 = 0\,,\ a \leqslant x_1 \cos\beta + x_3 \sin\beta \leqslant b\}\ .
\end{aligned}
$$

None of these rotated stripes can lie in any $Z \in \mathcal{Z}_\theta$, except for those resulting of rotations with $\beta = k\pi\ , k = 0, 1, \ldots$, which let S_a^b invariant or map it to[3] S_{-b}^{-a}. But the latter are elements of $E_\theta(d)$ and all the others are unbounded in a direction not contained in $\operatorname{Im}\theta$ and therefore cannot be bounded when projected on $\ker\theta$. Thus we conclude $gS \subset Z \Rightarrow g \in E_\theta(d)$.

These findings enforce us with the generalization of the time-zero condition to the case of deformed nets of reduced symmetry:

Proposition 6. *1. The net $(\mathcal{E}_\theta, \mathcal{Z}_\theta, \alpha^\theta)$ satisfies the time zero condition with the Rieffel-deformed time zero C^*-algebras*

$$\mathcal{E}_{\theta,0}(S) = \mathcal{E}_{0,\theta}(S)\ ,\ \ S \in \mathcal{S}_\theta\ .$$

[3]Concretely, $B_2(k\pi)$, $B_3(k\pi)$, $R_1(2l\pi)$, $R_2(2l\pi)$ leave S_a^b invariant while $R_1((2l+1)\pi)$, $R_2((2l+1)\pi)$ map it to S_{-b}^{-a}, $k, l = 1, 2, \ldots$

2. The restriction of σ to $\mathcal{E}^\infty$ extends to a reflection positive functional σ^θ on $\mathcal{E}_\theta$.

Proof. 1. As vector spaces, the smooth time zero algebras of the net $\mathcal{E}_\theta$ are, $S \in \mathcal{S}_\theta$,

$$\mathcal{E}_{\theta,0}(S)^\infty = \mathcal{E}^\infty \cap \bigcap_{\substack{Z\in\mathcal{Z}_\theta \\ Z\supset S}} \mathcal{E}_\theta(Z) = \bigcap_{\substack{Z\in\mathcal{Z}_\theta \\ Z\supset S}} \mathcal{E}_\theta(Z)^\infty = \bigcap_{\substack{Z\in\mathcal{Z}_\theta \\ Z\supset S}} \mathcal{E}(Z)^\infty = \mathcal{E}_0(S)^\infty\,,$$

where we have used that $\mathcal{E}_\theta(Z)^\infty = \mathcal{E}(Z)^\infty$ as vector spaces. Thus the closure in the norm $\|\cdot\|_\theta$ gives $\mathcal{E}_{\theta,0}(S) = \mathcal{E}_{0,\theta}(S)$. By assumption, the undeformed net $\mathcal{E}$ satisfies the time-zero condition, that is, the time zero algebras $\mathcal{E}_0(S)$ generate the cylinder algebras,

$$\mathcal{E}(Z) = \left(\bigcup_{S\in\mathcal{S}_\theta} \{\alpha_g(\mathcal{E}_0(S))\,|\,g\in E(d),\, gS\subset Z\}\right)''\,. \tag{2.28}$$

According to Lemma 7 in its Euclidean version, the transformations $g \in E(d)$, $gS \subset Z$, which appear here, split as $g = g_1 g_2$ with $g_1 \in E_\theta(d)$ and $g_2 S = S$. In view of the covariance of the undeformed net, this implies that α_{g_2} leaves $\mathcal{E}_0(S)$ invariant, *i.e.*, $\alpha_{g_2}(\mathcal{E}_0(S)) = \mathcal{E}_0(S)$. Thus (2.28) also holds if we restrict to $g \in E_\theta(d) \subset E(d)$.

To make the transition to the deformed C^*-algebras, we first contemplate the smooth time zero algebras $\mathcal{E}_0(S)^\infty \subset \mathcal{E}_0(S)$, and consider the *-algebra $\hat{\mathcal{E}}(Z)$ generated by all $\alpha_g(\mathcal{E}_0(S)^\infty)$, where S runs over $\mathcal{S}_\theta$ and g over $E_\theta(d)$ such that $gS \subset Z$. As vector spaces, $\mathcal{E}_{0,\theta}(S)^\infty = \mathcal{E}_0(S)^\infty$, and also the automorphisms α_g^θ and α_g, $g \in E_\theta(d)$, coincide on $\mathcal{E}^\infty$. Hence $\alpha_g^\theta(\mathcal{E}_{0,\theta}(S)^\infty) = \alpha_g(\mathcal{E}_0(S)^\infty)$, and as this algebra is $\|\cdot\|_\theta$-dense in $\alpha_g^\theta(\mathcal{E}_{0,\theta}(S))$, it follows that the $\|\cdot\|_\theta$-closure of $\hat{\mathcal{E}}(Z)$ coincides with $\mathcal{E}_\theta(Z)$. In particular, we have the claimed time zero property,

$$\mathcal{E}_\theta(Z) = \left(\bigcup_{S\in\mathcal{S}_\theta} \left\{\alpha_g^\theta(\mathcal{E}_{0,\theta}(S))\,|\,g\in E_\theta(d),\, gS\subset Z\right\}\right)''\,. \tag{TZ$_\theta$}$$

2. The restriction of the continuous linear translationally invariant functional σ to $\mathcal{E}^\infty$ extends to a $\|\cdot\|_\theta$-continuous functional σ^θ on $\mathcal{E}_\theta$ by *2.* of Lemma 6. Since σ is $E(d)$-invariant, it follows that this extension is invariant under the extension α^θ of $\alpha|_{E_\theta(d)}$ from $\mathcal{E}^\infty$ to $\mathcal{E}_\theta$. The continuity $E_\theta(d) \ni g \mapsto \sigma^\theta(A\alpha_g(B))$, $A, B \in \mathcal{E}_\theta$, is then clear.

It remains to check reflection positivity. By the translational invariance of σ, we have for smooth $A, B \in \mathcal{E}^\infty$ by *2.* of Lemma 6,

$$\sigma(\iota(A^*) \times_\theta B) = \sigma(\iota(A^*)B) \, ,$$

and hence in particular $\sigma(\iota(A^*) \times_\theta A) \geqslant 0$ for $A \in \mathcal{E}^\infty_>$. In view of the $\| \cdot \|_\theta$-continuity of σ^θ and ι^θ, this positivity extends to $\mathcal{E}_{>,\theta}$. □

Finally, let us collect what we have got before we write down the candidate for the $\mathcal{P}_\theta(d)$-covariant net of von Neumann algebras on Minkowski space-time. We realized that the deformed Euclidean field theory $(\mathcal{E}_\theta, \mathcal{Z}_\theta, \alpha^\theta, \sigma^\theta)$ satisfies the time-zero condition, if the corresponding undeformed theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ does. Detached from any deformation intent, we demonstrated the analytical continuation of reduced Euclidean group representations (with the groups forsightfully named $E_\theta(d)$) to unitary representations of the reduced Poincaré group. This directly admits an automorphic action $\alpha^{\theta,\mathcal{M}}$ of $\mathcal{P}_\theta(d)$, whose corresponding unitary representation we call U_θ. To be specific, $\alpha^{\theta,\mathcal{M}}$ is exactly the adjoint action of the unitary representation on the physical Hilbert space $\mathcal{H}_\theta$ gained from the virtual $E_\theta(d)$-representation by the methods of section 2.2. We remember that the construction of $\mathcal{H}_\theta$ comes with equivalence classes $[\ ]^\theta_\sigma$ and a $\mathcal{E}_{\theta,0}$-representation denoted by π_θ. The latter representation of the time-zero data acts in the following way:

$$B \in \mathcal{E}_{\theta,0} : \quad \pi_\theta(B)[A]^\theta_\sigma = [B \times_\theta A]^\theta_\sigma \quad \forall A \in \mathcal{E}_\theta \, .$$

Finally, we will combine these elaborations to conclude the noncommutative Wick rotation of an algebraically given Euclidean theory.

Definition 15. On $\mathcal{H}_\theta$, the noncommutative analogue of the physical Hilbert space in section 2.1, we define a deformed Lorentzian net prescription as follows:

$$\begin{aligned} \mathcal{Z}_\theta \ni Z \quad &\mapsto \quad \mathcal{M}_\theta(Z) \, , \\ \mathcal{M}_\theta(Z) \quad &:= \quad \left(\bigcup_{S \in \mathcal{S}_\theta} \{ \alpha^{\theta,\mathcal{M}}_g (\pi_\theta \mathcal{E}_{0,\theta}(S)) \,|\, g \in \mathcal{P}_\theta(d), \, gS \subset Z \} \right)'' \end{aligned}$$

Furthermore, we call the vector $\Omega_\theta := [1]^\theta_\sigma$ the *vacuum*.

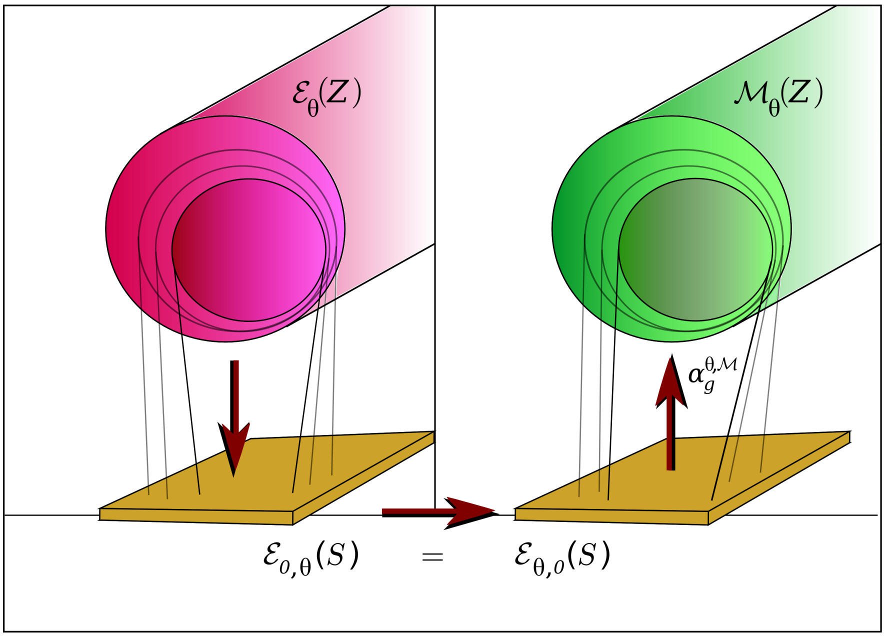

Figure 2.8: The noncommutative algebraic Wick rotation visualized.

Proposition 7. *The collection* $(\mathcal{M}_\theta, \mathcal{Z}_\theta, \alpha^{\theta,\mathcal{M}}, \Omega)$ *given in Def.* 15 *is a* $\mathcal{P}_\theta(d)$*-covariant net of von Neumann algebras on the Hilbert space* $\mathcal{H}_\theta$*, satisfying the time-zero condition and the modified spectrum condition of* (2.17)*,*

$$\operatorname{spec}\{P_0, P_1, \ldots, P_{d-1}\} \subset Y \ .$$

Moreover, the vacuum Ω_θ *is* $\mathcal{P}_\theta(d)$*-invariant and cyclic for the chosen representation on* $\mathcal{H}_\theta$*.*

Proof. We start by checking the validity of the time-zero condition for $\{\mathcal{M}_\theta(Z)\}_{Z\in\mathcal{Z}_\theta}$: by construction, $(\pi_\theta\mathcal{E}_{0,\theta}(S))''$ is the time-zero data of $\mathcal{M}_\theta(Z)$ for suitable $S \in \mathcal{S}_\theta,\ Z \in \mathcal{Z}_\theta$. Moreover, $\alpha^{\theta,\mathcal{M}}$ acts strongly continuously while π_θ is a continuous mapping. Therefore, the smooth time-zero content of the Lorentzian net takes the following form:

$$\mathcal{M}_{\theta,0}(S)^\infty = \pi_\theta(\mathcal{E}_{0,\theta}(S))^\infty = \pi_\theta(\mathcal{E}_0(S)^\infty) \ .$$

The last ingredient needed to appreciate the time-zero condition is the weak density of $\mathcal{C}^\infty \subset \mathscr{B}(\mathcal{H})$ which directly leads to the relation $\mathcal{M}_{\theta,0}(S) = (\mathcal{M}_{\theta,0}(S)^\infty)''$. The statement now follows

from the very definition of $\mathcal{M}_\theta$.
We know that $\alpha_g^{\theta,\mathcal{M}}$ is a homomorphism for $g \in \mathcal{P}_\theta(d)$ and that $gS \subset Z$ is equivalent to $kS \subset hZ$ for all $k = hg \in \mathcal{P}_\theta(d)$. It thus follows $\alpha_h^{\theta,\mathcal{M}}\mathcal{M}_\theta(Z) = \mathcal{M}_\theta(hZ)$ from $\mathcal{P}_\theta(d)\mathcal{S}_\theta = \mathcal{S}_\theta$, which proves $\mathcal{P}_\theta(d)$-covariance.
The net structure itself has been treated in detail in the paragraphs before Prop. 5 and stays unaffected by the change of the symmetry action $\alpha^\theta \to \alpha^{\theta,\mathcal{M}}$.
Similarly, the claimed version of the spectrum condition has already been proved for general $\mathcal{P}_\theta(d)$-covariant nets and for our choice of the underlying noncommutativity θ before and in Remark 5, respectively.
We finish the proof by observing the stated properties of the vacuum just by glancing at its definition. □

A schematic picture of this noncommutative algebraic Wick rotation can be seen in Fig. 2.8.

2.3.3 Deformation of a Lorentzian Theory

We will now follow another point of view in arriving at a noncommutative net of observables on Minkowski space-time. Sketched by our main diagram, there is always the possibility of first performing the Wick rotation $(\mathcal{E}, \mathscr{O}, \alpha, \sigma) \to (\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}}, \Omega)$ as we presented it in section 2.2. The resulting Lorentzian net $\mathcal{M}$ is equally well qualified for the deformation in the sense of warped convolutions. Before we start that deformation, we call attention to the connection between the two deformed nets $\mathcal{E}_\theta$ and $\mathcal{M}_\theta$ in terms of analyticity. In the same way as on Euclidean space, one can define Moyal-Minkowski space-time by imposing commutation relations

$$[X_\mu^{\mathcal{M}}, X_\nu^{\mathcal{M}}] = i\theta_{\mu\nu}^{\mathcal{M}}$$

on the Moyal space-time coordinates $X_0^{\mathcal{M}}, ..., X_{d-1}^{\mathcal{M}}$. A Wick rotation shall link a theory on analogously deformed Euclidean space involving Euclidean coordinates X_μ^θ and parameters $\theta_{\mu\nu}^{\mathcal{E}}$ with a theory on such Moyal-Minkowski space-time. Though Wick rotation is a complicated procedure, it schematically transforms the Euclidean time components into purely imaginary values, $X_0^{\mathcal{M}} = iX_0^{\mathcal{E}}$ which are then interpreted as "real" time. During this transformation, the spatial components stay unchanged. In the thesis at hand, we are always considering the case of commutative time, so Lemma 4 has the effect of coinciding deformation parameters, *i.e.*, $\theta^{\mathcal{E}} = \theta^{\mathcal{M}}$. This shows that the same noncommutativity θ can be used consistently for both the Euclidean and Lorentzian signature.

In spite of the usability of the warped convolutions on the Wick rotated Lorentzian observable net $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}}, \Omega)$ we risk a clash of notation. To avoid this, we emphasize on writing $\mathcal{M}^\theta$ for the net obtained in the way just explained. This $\mathcal{M}^\theta$ is built up of the elements A_θ, which for their part are obtained via

$$A_\theta \Psi = \lim_{\epsilon \to 0} \int \mathrm{d}k \, \mathrm{d}v \; h(\epsilon k, \epsilon v) \, \mathrm{e}^{-i(k,v)} \, \alpha_{\theta k}(A) U(v) \Psi \ , \quad \Psi \in \mathcal{D}, \tag{2.29}$$

where $(.,.)$ denotes the Minkowski space-time product and $\alpha^{\mathcal{M}}$ denotes the adjoint action of the unitary representation U gained from the virtual $E(d)$-representation V on $\mathcal{H}$. Hence the definition of the Lorentzian net we are using now writes

$$\mathcal{M}^\theta(Z) := \{A_\theta \,|\, A \in \mathcal{M}(Z)\}'' \quad , \quad Z \in \mathcal{Z}_\theta \ . \tag{2.30}$$

The $\mathcal{P}(d)$-action $\alpha^{\mathcal{M}}$ provides a corresponding $\mathcal{P}_\theta(d)$-action, which is called $\alpha^{\mathcal{M},\theta}$.

Remark 10. We admit that the notation might be confusing at this point. As a memory hook, $\alpha^{\theta,\mathcal{M}}$ was the reduced symmetry automorphism we introduced earlier on and corresponds to "first deforming, then Wick rotating". Regarding $\alpha^{\mathcal{M},\theta}$ we are treating right now, it's the other way round. And to the readers' alleviation, later results will imply that the distinction between the two is in fact redundant.

Applying the treatment around Prop. 5, we follow that $\mathcal{M}^\theta$ gives rise to a well-defined Lorentzian net as well:

Proposition 8. *The collection $(\mathcal{M}^\theta, \mathcal{Z}_\theta, \alpha^{\mathcal{M},\theta})$ is a $\mathcal{P}_\theta(d)$-covariant, isotonic net of von Neumann algebras satisfying the time-zero condition.*

Proof. It should be clear from the defining relation (2.30) that $\mathcal{Z}_\theta \ni Z \mapsto \mathcal{M}^\theta(Z)$ is a net of von Neumann algebras on $\mathcal{H}$. Due to property *3.* of Prop. 2, it holds that

$$U(g) A_\theta U(g)^{-1} = (U(g) A U(g)^{-1})_\theta$$

for $g \in \mathcal{P}_\theta(d)$ and $A \in \mathcal{M}^\infty$. Taking into account the covariance of the undeformed net $\mathcal{M}$ and remembering $\mathcal{M}^\infty = \mathcal{M}$ on the vector space level, this implies

$$U(g)\mathcal{M}^\theta(Z)^\infty U(g)^{-1} = \mathcal{M}^\theta(gZ)^\infty \,, \qquad Z \in \mathcal{Z}_\theta \,,\ g \in \mathcal{P}_\theta(d) \,.$$

Now the shape of the cylinder regions was designed to permit

$$A \times_\theta B \in \mathcal{M}(Z)^\infty \ , \quad A, B \in \mathcal{M}(Z)^\infty \ , \quad Z \in \mathcal{Z}_\theta \ .$$

In view of Lemma 3, property *1.* of Prop. 2 states that the warped convolution commutes with the algebra involution. Together with the product property *2.* of the same lemma, this implies that the set $\{A_\theta \,|\, A \in \mathcal{M}(Z)^\infty\}$ is a *-algebra. Hence the double commutant in (2.30) amounts to just taking the weak closure, and covariance of $\mathcal{M}^\theta(Z)$ follows.

The smooth time zero algebras of the net $\mathcal{M}^\theta$ are, $S \in \mathcal{S}_\theta$,

$$\mathcal{M}^\theta{}_0(S)^\infty = \bigcap_{\substack{Z\in\mathcal{Z}_\theta \\ Z\supset S}} \mathcal{M}^\theta(Z)^\infty = \mathcal{M}_0{}^\theta(S)^\infty \,.$$

By the same reasoning as in Prop. 7, the validity of the Wick rotation of the deformed Euclidean net, we have

$$\mathcal{M}_0(S) = \pi(\mathcal{E}_0(S))'' = (\pi(\mathcal{E}_0(S))^\infty)'' \ ,$$

while $\pi(\mathcal{E}_0(S))^\infty = \pi(\mathcal{E}_0(S)^\infty)$ implies that

$$\mathcal{M}^\theta{}_0(S) = \pi(\mathcal{E}_0(S)^\infty)_\theta{}'' \ .$$

As we know, the undeformed nets $\{\mathcal{M}(Z)\}_{Z\in\mathcal{Z}_\theta}$ fulfill the time zero condition. Having in mind $(\mathcal{M}_0(S)^\infty)'' = \mathcal{M}_0(S)$, we follow that $\mathcal{M}(Z)$ is the smallest von Neumann algebra containing the set

$$\{U(g)\mathcal{M}_0(S)^\infty U(g)^{-1} \,|\, g \in \mathcal{P}_0(d) \,,\ S \in \mathcal{S}_\theta \ : \ gS \subset Z\} \ .$$

As in the proof of Prop. 5, we can apply (the Lorentzian version of) Lemma 7 to conclude that restriction to $g \in \mathcal{P}_\theta(d) \subset \mathcal{P}(d)$ does not change the generated von Neumann algebra. After passing to warped convolution time zero algebras ${\mathcal{M}^\theta}_0(S)^\infty = \pi(\mathcal{E}_0(S)^\infty)_\theta$, we obtain

$$\begin{aligned} \mathcal{M}^\theta(Z) &= \left(\bigcup_{S \in \mathcal{S}_\theta} \left\{ \left(U(g)\pi(\mathcal{E}_0(S)^\infty)U(g)^{-1} \right)_\theta \mid g \in E_\theta(d), gS \subset Z \right\} \right)'' \\ &= \left(\bigcup_{S \in \mathcal{S}_\theta} \left\{ U(g){\mathcal{M}^\theta}_0(S)^\infty U(g)^{-1} \mid g \in E_\theta(d), gS \subset Z \right\} \right)'' , \end{aligned}$$

which is the claimed time zero condition. □

2.3.4 Interrelation Between the Lorentzian Nets

So far we have accomplished two types of Wick rotation, both starting from a Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$. The first one, elaborated on in the above part of the ongoing section, consists in the deformation of $\mathcal{E}$ and is followed by the analytical continuation of the reduced symmetry groups. The second type we have just worked out takes a different route and first performs this sort of Wick rotation on the undeformed Euclidean net. If we then deform according to the warped convolutions, the last proposition showed that we gain a deformed Lorentzian net, which might be quite detached from the one we obtained going the first way. From an operational point of view, this would only be constrictedly helpful since this could prefer the deformation of a distinct metric signature. Before we spread out the possible consequences of such an incompatibility, we are able to present the heart of the algebraic Wick rotation on degenerate Moyal space(-time):

Theorem 6. *The two nets* $(\mathcal{M}^\theta, \mathcal{Z}_\theta, \mathrm{Ad}U^{\mathcal{M}}|_{\mathcal{P}_\theta(d)})$ *and* $(\mathcal{M}_\theta, \mathcal{Z}_\theta, \mathrm{Ad}U^{\mathcal{M}}_\theta)$ *are isomorphic, i.e., there exists a unitary operator* $W : \mathcal{H} \to \mathcal{H}_\theta$ *such that*

$$W\Omega = \Omega_\theta \,, \tag{2.31}$$

$$WU(g)W^* = U_\theta(g) \,, \qquad g \in \mathcal{P}_\theta(d) \,, \tag{2.32}$$

$$W\mathcal{M}^\theta(Z)W^* = \mathcal{M}_\theta(Z) \,, \qquad Z \in \mathcal{Z}_\theta \,. \tag{2.33}$$

Proof. In order to prove the first statement, we inspect the connections between the two Hilbert spaces $\mathcal{H}_\theta$ and $\mathcal{H}$. The definition of the following map W_0 linking them is already the key issue of the proof:

$$\begin{aligned} W_0 : \mathcal{E}^\infty_>/(\mathcal{N}_\sigma \cap \mathcal{E}^\infty_>) &\to \mathcal{E}^\infty_>/(\mathcal{N}_{\sigma^\theta} \cap \mathcal{E}^\infty_>) \\ [A]_\sigma &\mapsto [A]^\theta_\sigma \,. \end{aligned}$$

For $A, B \in \mathcal{E}^\infty_>$, we have by the translational invariance of σ and point *2.* of Lemma 6,

$$\langle [A]_\sigma, [B]_\sigma \rangle_{\mathcal{H}} = \sigma(\iota(A^*)B) = \sigma^\theta(\iota(A^*) \times_\theta B) = \langle [A]^\theta_\sigma, [B]^\theta_\sigma \rangle_{\mathcal{H}_\theta} \,.$$

This shows that W_0 is well-defined and isometric. Since its domain and range lie dense in the corresponding Hilbert spaces, we can extend it to a unitary operator $W : \mathcal{H} \to \mathcal{H}_\theta$. Clearly W satisfies $W\Omega = W[1]_\sigma = [1]^\theta_\sigma = \Omega_\theta$.

Using this, relation (2.32) is not hard to realize. Strictly speaking, we have two virtual representations $g \mapsto V_\theta(g)$ and $g \mapsto WV_0(g)W^*$ of the reduced Euclidean group $E_\theta(d)$ on $\mathcal{H}_\theta$. For g in a sufficiently small neighborhood of the identity, these representations act according to (2.10), $A \in \mathcal{E}^\infty_>$,

$$\begin{aligned} V_\theta(g)[A]^\theta_\sigma &= [\alpha^{\mathcal{E}}_g(A)]^\theta_\sigma \,, \\ WV_0(g)W^*[A]^\theta_\sigma &= WV_0(g)[A]_\sigma = W[\alpha^{\mathcal{E}}_g(A)]_\sigma = [\alpha^{\mathcal{E}}_g(A)]^\theta_\sigma \,, \end{aligned}$$

where we use the same fact in both of these equations. That is, the two virtual representations coincide. After analytic continuation to unitary representations of $\mathcal{P}_\theta(d)$, this implies (2.32) again by $\mathcal{E}^\infty$ being dense in $\mathcal{E}$.

To show that W also intertwines the nets, it is sufficient to consider the time zero algebras. Indeed, both $\mathcal{M}_\theta$ and $\mathcal{M}^\theta$ satisfy the time zero condition and are generated from their time zero data by representations of $\mathcal{P}_\theta(d)$. We have just shown that these representations are in fact W-equivalent. So let us write down the two time-zero algebras at first:

$$\begin{aligned} \mathcal{M}_{\theta,0}(S)^\infty &= \pi_\theta(\mathcal{E}_0(S)^\infty) = \{\pi_\theta(A) \,:\, A \in \mathcal{E}_0(S)^\infty\} \,, \\ {\mathcal{M}^\theta}_0(S)^\infty &= \pi(\mathcal{E}_0(S)^\infty)_\theta = \{\pi(A)_\theta \,:\, A \in \mathcal{E}_0(S)^\infty\} \,. \end{aligned}$$

Comparing these, one fact becomes apparent: it is sufficient to show that W intertwines $\pi(A)_\theta$ and $\pi_\theta(A)$ for $A \in \mathcal{E}_0(S)^\infty$, as then (2.33) follows by continuity.

With $A \in \mathcal{E}_0(S)^\infty$ and $B \in \mathcal{E}_>^\infty$, we compute, using property *5* stated in Prop. 2,

$$\begin{aligned} W\pi(A)_\theta W^*[B]_\sigma^\theta &= W\pi(A)_\theta[B]_\sigma = W\pi(A)_\theta\pi(B)\Omega \\ &= W\pi(A)_\theta\pi(B)_\theta\Omega \ . \end{aligned}$$

Note that the warped convolutions $\pi(A)_\theta$, $\pi(B)_\theta$ are built here with the representation U and the Minkowski inner product in the oscillatory integral (2.29). But as $\theta e = 0$, we have already seen in Lemma 3 and used several times that only spatial translations along $x \perp e$ enter. For $p, x \perp e$, the Euclidean and Minkowski inner products differ by a sign which is compensated by the definitions of the respective warped convolutions. Furthermore, for $x \perp e$, the unitaries $U^{\mathcal{M}}(x,1)$ implement $\alpha_x^{\mathcal{E}}$. So we can use Lemma 5 stating that $A \mapsto \pi(A)_\theta$ is an α-covariant representation of the Rieffel-deformed C^*-algebra $\mathcal{E}_{0,\theta}(S)$, and again *5.* of Prop. 2 to compute further

$$\begin{aligned} W\pi(A)_\theta W^*[B]_\sigma^\theta &= W\pi(A \times_\theta B)_\theta\Omega = W\pi(A \times_\theta B)\Omega = W[A \times_\theta B]_\sigma \\ &= [A \times_\theta B]_\sigma^\theta = \pi_\theta(A)[B]_\sigma^\theta \ . \end{aligned}$$

As all operators appearing here are bounded and $\{[B]_\sigma^\theta \ : \ B \in \mathcal{E}_>^\infty\} \subset \mathcal{H}_\theta$ is dense, we obtain $W\pi_\sigma(A)_\theta W^* = \pi_\sigma^\theta(A)$ by continuity. $\square$

Thm. 6 finally establishes the commuting diagram, given in Fig. 2.9, for the case of commutative time deformations of algebraic quantum field theory. One of the consequences is the independence of a noncommutative Lorentzian theory being obtained either by Wick rotation of a deformed Euclidean one or by deforming a commutative theory on Minkowski space-time. A more extensive discussion of the various consequences of our findings will be contained in Ch. 5. The more abstract the algebraic framework presents itself, the more beneficial it gets when it comes to applications in concrete models. In the next chapter, we will examine the model of the free scalar field, which on the one hand is the simplest thinkable model. On the other hand, it is the only existing (in the sense of rigorous constructed) physical model in four space-time dimensions, up to freedom in definition of the mass.

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & \mathcal{M} \arrow[d, leftrightarrow] \\
\mathcal{E}_\theta \arrow[r] & \mathcal{M}_\theta \arrow[r, phantom, "\cong"] & \mathcal{M}^\theta
\end{tikzcd}
$$

Figure 2.9: The finally established commutative diagram.

Since we are going to treat the free field in terms of its n-point functions, we will also discover properties that still hold for more general models.

Chapter 3

Correlation Functions

In the preceding chapter, we have shown that under certain assumptions which are restrictive to some extent, it is possible to perform a Wick rotation from a noncommutative Euclidean to a quantum field theory on Minkowski space-time in the algebraic setting. Due to its generality, the outcome of the algebraic approach can be applied to a variety of physically interesting models. A different approach is given by the definition of a physical theory through its set of correlation functions. According to Wightman's reconstruction theorem [Wig56], a physical theory given by properties of a quantum field is equivalent to the theory defined via its corresponding set of n-point functions. In the current chapter we will focus on this viewpoint.

There are strong interrelations between the algebraic framework and the one given by n-point functions [FH81, Haa92]. In the chapter at hand, these will be used to make visible the abstract Wick rotation explained in the previous part of this work on the example of the Euclidean free scalar field at first. The connections to the standard approaches concerning analytical continuation of Schwinger functions will then be investigated. During this treatment, new implications on the applicability of noncommutative Wick rotation for general theories in terms of n-point functions will occur.

3.1 The Euclidean Free Scalar Field

Let us consider Euclidean $\mathbb{R}^d$, endowed with the inner product $xy := x^0y^0 + x^1y^1 + \ldots + x^{d-1}y^{d-1}$ for $x, y \in \mathbb{R}^d$. We write $\mathbb{R}^d \ni x \equiv (x^0, \underline{x})$. Let V denote the unitary representation of the Euclidean group $E(d) := O(d) \ltimes \mathbb{R}^d$ on the underlying Euclidean Hilbert space $\mathcal{H}^{\mathcal{E}}$. Since a thorough description of $\mathcal{H}^{\mathcal{E}}$ would require to rigorously derive the theory of Euclidean free fields and since we are going to use the physical Hilbert space $\mathcal{H}$ instead, we omit such a detailed explanation. The Euclidean (commutative) free scalar field $\phi(\tau, \underline{x})$ is governed by the well-known action

$$S[\phi] := \frac{1}{2} \iint \mathrm{d}\tau \, \mathrm{d}^{d-1}x \; \phi(\tau, \underline{x}) \left(-\partial_\tau^2 - \Delta_{\underline{x}} + m^2 \right) \phi(\tau, \underline{x}) \; , \tag{3.1}$$

which incorporates the Helmholtz equation

$$\left(-\partial_\tau^2 - \Delta_{\underline{x}} + m^2 \right) \phi(\tau, \underline{x}) = 0 \tag{3.2}$$

as the equation of motion. As is also commonly known, the latter equation written down for imaginary values of τ is the Klein-Gordon equation. So, one directly sees that at least the fundamental field equations of the free field on Euclidean and Minkowski space(-time) respectively are linked by the mapping $\tau \to i\tau$.

We would like to see the exact connection between the corresponding n-point functions. In the usual undeformed scenario, the mainly complete answer was given in the 1970's, as has been indicated in the introduction. Regarding the noncommutative case, we want to see how the framework of Ch. 2 applies to the Euclidean free field. So, at first we shall identify the notions needed for the algebraic Wick rotation in terms of free field entities.
The Green function $C(x-y)$ of the differential operator of (3.2), written as $-\Delta + m^2$, fulfills

$$(-\Delta + m^2) C(x-y) = \delta(x-y) \; .$$

Inspired by the Gaussian measure setting of the free field (see for example [GJ87] and [Roe94] for thorough treatments), $C(x-y)$ is oftentimes referred to as "free covariance", while the quantum field theory perspective would rather favor the name "propagator". All these different designations mean the same object, namely the distribution on $\mathbb{R}^d \times \mathbb{R}^d$, which can be made a bilinear form on $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ and reads

$$C(x,y) = C(x-y) = \int \mathrm{d}^d p \frac{\mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)} \quad , \; x,y \in \mathbb{R}^d, \tag{3.3}$$

$$C(f_1,f_2) = \int \mathrm{d}^d p\, \mathrm{d}^d x\, \mathrm{d}^d y\, \frac{\mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)} f_1(x) f_2(y) \quad , \; f_1,f_2 \in \mathscr{S}(\mathbb{R}^d) \; , \tag{3.4}$$

where $p^2 := p_0^2 + \underline{p}^2$ denotes the Euclidean inner product of $\mathbb{R}^d$-vectors $p = (p_0, \underline{p})$, $p_0 \in \mathbb{R}$, $\underline{p} \in \mathbb{R}^{d-1}$. We remark that (3.3) can also be written in the following form,

$$C(x-y) = (2\pi)^{-d/2} \left(\frac{m}{|x-y|} \right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}(m|x-y|) \; , \tag{3.5}$$

where K_ν denotes the modified Bessel function of the second kind [GJ87, sec. 7.2]. Sometimes it is more convenient to write the free covariance in Fourier representation:

$$C(f_1,f_2) = \int \mathrm{d}^d p\, \frac{\widetilde{f}_1(-p)\widetilde{f}_2(p)}{p^2+m^2} \; .$$

In order to investigate the structural manifestations of our algebraic theory, we recall the main properties of the free covariance. Before we do so, we need some further notions: first of all, the implementation of the time reflection $r : (x^0, \underline{x}) \mapsto (-x^0, \underline{x})$ is again denoted ι on Schwartz space *i.e.*, $\iota f(x) = f(rx)$. This means that we choose the Euclidean time direction e to be $(1, \underline{0})$. The Euclidean group shall act on Schwartz functions via the pull-back, *i.e.*,

$$\begin{aligned} g = (a,R) \in E(d) \quad &, \quad f \in \mathscr{S}(\mathbb{R}^{dn}) \; , \\ f_{(g)}(x_1,\ldots,x_n) \; &:= \; (f \circ (g^{-1} \otimes \cdots \otimes g^{-1}))(x_1,\ldots,x_n) = \\ &= \; f(R^{-1}(x_1-a),\ldots,R^{-1}(x_n-a)) \; . \end{aligned}$$

In accordance with the literature (with respect to the general impact and the practicability for this thesis, we are geared mainly to [OS73]), we define

$$
\begin{aligned}
\mathscr{S}(\mathbb{R}^d_+) &:= \{f \in \mathscr{S}(\mathbb{R}^d) \,|\, \operatorname{supp} f \subset \{(x^0, \underline{x}) : x^0 \geqslant 0\}\} \ , && (3.6)\\
D^\alpha f(x_1, \ldots, x_n) &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f(x_1, \ldots, x_n) \ , \ |\alpha| := \alpha_1 + \ldots + \alpha_n \ , && (3.7)\\
\mathscr{S}_0(\mathbb{R}^{nd}) &:= \{f \in \mathscr{S}(\mathbb{R}^{nd}) \,|\, D^\alpha f(x_1, \ldots, x_n) = 0 \ \forall\, |\alpha| \geqslant 0 \text{ if } \exists\, j \neq k : \ x_j = x_k\} \ ,\\
\mathscr{S}_+(\mathbb{R}^{nd}) &:= \{f \in \mathscr{S}(\mathbb{R}^{nd}) \,|\, D^\alpha f(x_1, \ldots, x_n) = 0 \ \forall\, |\alpha| \geqslant 0 \text{ unless } 0 < x_1^0 < \cdots < x_n^0\} \ . && (3.8)
\end{aligned}
$$

Proposition 9. *The Euclidean scalar field propagator $C : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$ fulfills the following properties:*

1. *Continuity: The map $E(d) \ni g \mapsto C(f_{1,(g)}, f_2)$ is continuous for all $f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$.*
2. *Invariance: Let $g \in E(d)$. Then*
$$C(f_{1,(g)}, f_{2,(g)}) = C(f_1, f_2) \qquad \forall f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$$
3. *Reflection positivity:*
$$C(\iota f^*, f) \geqslant 0 \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^d_+)$$
4. *Symmetry:*
$$C(f_1, f_2) = C(f_2, f_1) \qquad \forall f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$$

Proof. 1. This is an easy consequence of Lebesgue's dominated convergence theorem. For the sake of completeness we prove it for the translations (the rotations follow by decomposing them into small angles). From the definition of $C(f_1, f_2)$ we infer

$$C(f_{1,(a,1)}, f_2) = \int \mathrm{d}^d p \, \mathrm{d}^d x \, \mathrm{d}^d y \ \frac{\mathrm{e}^{ip(x-y)}}{(2\pi)^d (p^2 + m^2)} f_1(x-a) f_2(y) = \int \mathrm{d}^d p \, \mathrm{e}^{ipa} \, \frac{\widetilde{f}_1(-p) \widetilde{f}_2(p)}{p^2 + m^2} \ .$$

Thus the translation just results in a phase. The function $e^{ipa}\widetilde{f}_1(-p)\widetilde{f}_2(p)(p^2+m^2)^{-1}$ is bounded by $\left|\widetilde{f}_1(-p)\widetilde{f}_2(p)(p^2+m^2)^{-1}\right|$ for all $a \in \mathbb{R}$ and therefore

$$\lim_{a\to 0} C(f_{1,(a,1)}, f_2) = \lim_{n\to\infty} C(f_{1,(a_n,1)}, f_2) = C(f_1, f_2)$$

by the dominated convergence theorem.

2. Follows straightforwardly from the invariance of the kernel $C(x-y)$: let $g = (x, R) \in E(d)$ where $x \in \mathbb{R}^d$ and $R \in \mathfrak{M}_{d\times d}(\mathbb{R})$ an orthogonal matrix. Then

$$\begin{aligned} C(f_{1,(x,R)}, f_{2,(x,R)}) &= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)}\, f_1(R^{-1}(x-a)) f_2(R^{-1}(y-a))\, \mathrm{d}^d x\, \mathrm{d}^d y \\ &= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ipR(x-y)}}{(2\pi)^d(p^2+m^2)}\, f_1(x) f_2(y)\, \mathrm{d}^d x\, \mathrm{d}^d y \\ &= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)}\, f_1(x) f_2(y)\, \mathrm{d}^d x\, \mathrm{d}^d y \\ &= C(f_1, f_2)\ . \end{aligned}$$

In the third equality, we have substituted p by Rp.

3. Define the spatial Fourier transform $\widehat{f}$ to be

$$\widehat{f}(x^0, \underline{p}) := (2\pi)^{-\frac{d-1}{2}} \int \mathrm{d}^{d-1}\underline{x}\ \mathrm{e}^{-i\underline{p}\underline{x}} f(x^0, \underline{x})\ , \tag{3.9}$$

then we have for $f \in \mathscr{S}(\mathbb{R}^d_+)$ and $\omega_{\underline{p}} := \sqrt{\underline{p}^2 + m^2}$,

$$
\begin{aligned}
C(\iota f^*, f) &= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)}\, \iota f^*(x) f(y)\, \mathrm{d}^d x\, \mathrm{d}^d y \\
&= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)}\, \overline{f(-x^0,\underline{x})} f(y^0,\underline{y})\, \mathrm{d}^d x\, \mathrm{d}^d y \\
&= \int\limits_{-\infty}^{0} \mathrm{d}x^0 \int\limits_{0}^{\infty} \mathrm{d}y^0 \int \frac{\mathrm{d}^d p\, \mathrm{e}^{ip^0(x^0-y^0)}}{(2\pi)((p^0)^2+\underline{p}^2+m^2)}\, \overline{\hat{f}(-x^0,\underline{p})} \hat{f}(y^0,\underline{p}) \\
&= \iint\limits_{0}^{\infty} \mathrm{d}x^0\, \mathrm{d}y^0 \int \frac{\mathrm{d}^d p\, \mathrm{e}^{-ip^0(x^0+y^0)}}{(2\pi)((p^0)^2+\underline{p}^2+m^2)}\, \overline{\hat{f}(x^0,\underline{p})} \hat{f}(y^0,\underline{p}) \\
&= \iint\limits_{0}^{\infty} \mathrm{d}x^0\, \mathrm{d}y^0 \int \frac{\mathrm{d}^{d-1}\underline{p}}{2\omega_{\underline{p}}}\, \mathrm{e}^{-\omega_{\underline{p}}(x^0+y^0)}\, \overline{\hat{f}(x^0,\underline{p})} \hat{f}(y^0,\underline{p}) \\
&= \int \frac{\mathrm{d}^{d-1}\underline{p}}{2\omega_{\underline{p}}} \left| \int\limits_{0}^{\infty} \mathrm{d}x^0\, \mathrm{e}^{-\omega_{\underline{p}}x^0} \hat{f}(x^0,\underline{p}) \right|^2 \geqslant 0 \;, \qquad (3.10)
\end{aligned}
$$

since $(2\omega_{\underline{p}})^{-1} \geqslant 0$. In the fifth equality, we have utilized the residue theorem in the form

$$
\int \frac{\mathrm{d}p^0}{p^2+m^2}\, \mathrm{e}^{ip^0 A} = \int \frac{\mathrm{d}p^0}{(p^0 - i\omega_{\underline{p}})(p^0 + i\omega_{\underline{p}})}\, \mathrm{e}^{ip^0 A} = \frac{2\pi}{\omega_{\underline{p}}}\, \mathrm{e}^{-|A|\,\omega_{\underline{p}}} \;. \qquad (3.11)
$$

4. Symmetry:

$$
\begin{aligned}
C(f_1, f_2) &= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(x-y)}}{(2\pi)^d(p^2+m^2)}\, f_1(x) f_2(y)\, \mathrm{d}^d x\, \mathrm{d}^d y \\
&= \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{-ip(y-x)}}{(2\pi)^d(p^2+m^2)}\, f_1(x) f_2(y)\, \mathrm{d}^d x\, \mathrm{d}^d y \\
&\overset{p\to -p}{=} \iint \frac{\mathrm{d}^d p\, \mathrm{e}^{ip(y-x)}}{(2\pi)^d(p^2+m^2)}\, f_2(y) f_1(x)\, \mathrm{d}^d x\, \mathrm{d}^d y = C(f_2, f_1) \;. \qquad (3.12)
\end{aligned}
$$

□

It happens that arbitrary n-point functions can be completely determined by the exact form of the free covariance. In other words, the vacuum representation of the free field provides a quasi-free state. Although we wrote $C(x-y)$ for the kernel of the two-point function $C(f_1, f_2)$ we choose to denote those of more general n-point functions in German letters ("fraktur"). That is to say, we write

$$\mathfrak{S}_n(f) = \int \mathrm{d}^{dn}x \; \mathfrak{S}_n(x_1, \ldots, x_n) f(x_1, \ldots, x_n) \; ,$$

where this association is distributional in general.

One can derive the reduction formula in the sense of quasi-freedom for the Euclidean free field n-point distributions $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^{dn})$ in various ways, but we are not going to recapitulate this to a large extent at this place. We rather mention that any theory built up of fields which for their part are sums of a creation and annihilation operator (which is valid for the free field) satisfies the preconditions of Wick's theorem. Therefore the vacuum expectation value of a product of fields cascades into sums of products of field contractions. The Euclidean case in particular can as well be derived by varying the generating functional with respect to a source function, which is not the philosophy we are into in the framework at hand. Whatever way one chooses to gain a reduction relation for a (quasi-)free theory, they all have the following form,

$$\begin{aligned}
\mathfrak{S}_1(x) &= 0 \\
\mathfrak{S}_2(x_1, x_2) &= C(x_1 - x_2) \\
\mathfrak{S}_n(x_1, \ldots, x_n) &= \sum_{k=1}^{n-1} \mathfrak{S}_{n-2}(x_1, \ldots, \widehat{x}_k, \ldots, x_{n-1}) C(x_k - x_n) \; , \; n > 2
\end{aligned} \tag{3.13}$$

where the variables $\widehat{x}_k$ are meant to be omitted.

3.2 Fock Space

After this short part intended to make ourselves familiar with the two-point function of the Euclidean free scalar field again, we are ready to catch the many particle aspect of this theory. This is needed for the Euclidean algebra of observables to take shape in the case at hand. A well established arena for countable particle excitations is the so-called *Borchers-Uhlmann algebra* [Bor62, Uhl62], and is defined to be the direct sum

$$\underline{\mathscr{S}} := \bigoplus_{n=0}^{\infty} \mathscr{S}(\mathbb{R}^{nd}) \quad , \quad \mathscr{S}(\mathbb{R}^0) := \mathbb{C} \ .$$

The elements of $\underline{\mathscr{S}}$ are terminating sequences of Schwartz space functions

$$\underline{\mathscr{S}} \ni \underline{f} := (f_0, f_1, f_2, \ldots, f_N, 0, \ldots) \ , \quad f_0 \in \mathbb{C} \ , \quad f_n \in \mathscr{S}(\mathbb{R}^{nd})$$

for an arbitrary $N \in \mathbb{N}$. Let α denote the action of $E(d)$ on this algebra,

$$\alpha_g \underline{f} = \alpha_g(f_0, f_1, f_2 \ldots) := (f_0, f_{1,(g)}, f_{2,(g)} \ldots) \ .$$

The tensor product

$$(\underline{f} \otimes \underline{g})_n(x_1, x_2, \ldots, x_n) := \sum_{k=0}^{n} f_k(x_1, x_2, \ldots, x_k) g_{n-k}(x_{k+1}, x_{k+2}, \ldots, x_n)$$

makes $\underline{\mathscr{S}}$ an algebra. One can as well define an involution and a Euclidean time reflection as

$$\begin{aligned} (\underline{f}^*)_n(x_1, x_2, \ldots, x_n) &:= \overline{f_n(x_n, x_{n-1}, \ldots, x_1)} \ , \\ (\iota \underline{f})_n(x_1, x_2, \ldots, x_n) &:= f_n(rx_1, rx_2, \ldots, rx_n) \ , \end{aligned}$$

respectively. Together with $\delta_{n,0}$, these definitions make the Borchers-Uhlmann algebra a unital *-algebra with automorphic $E(d)$-action α.

How are we going to see the connection to the Euclidean algebra of observables now? The Borchers-Uhlmann algebra is the correct framework to give a description of the field excitations, whatever exact field model one is considering. As in the one-particle case, the important entities are the functionals on the suitable function space.

In the following way we are able to define a net $\{\mathcal{E}(O)\}_{O \in \mathscr{O}}$ of Euclidean *-algebras using the Borchers-Uhlmann algebra

$$\mathscr{O} \ni O \mapsto \mathcal{E}(O) := \{\underline{f} \in \underline{\mathscr{S}} \,|\, \operatorname{supp} f_n \subset O \overbrace{\times \cdots \times}^{n \text{ times}} O\} = \{\underline{f} \in \underline{\mathscr{S}} \,|\, \operatorname{supp} f_n \subset O^{\times n}\} \ .$$

According to the investigations in Ch. 2 we denote the inductive limit of this net by $\mathcal{E} := \underline{\mathscr{S}}$. Nevertheless: defined in this way, the Euclidean algebra $\mathcal{E}$ is not a set of bounded operators on some Hilbert space. This causes no serious problems for our treatment, but we remark that we could have defined it as the set of all $\exp\{i\phi(f)\}$ (with the right support properties), which is more suitable to a C^*-algebra setting.

The next ingredient needed for the application of the algebraic Wick rotation is the reflection positive functional. Given a sequence of distributions $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^{nd})$ satisfying the specific customized properties of the Euclidean n-point functions (see section A.1 in the appendix), one defines the Euclidean functional σ to be

$$\begin{aligned} \sigma : \quad & \mathcal{E} \to \mathbb{C} \\ \sigma(\underline{f}) \quad := \quad & \sum_{n=0}^{\infty} \mathfrak{S}_n(f_n) \ . \end{aligned} \tag{3.14}$$

It remains to work out whether the Euclidean axioms as stated in Definition 3 are fulfilled for the Euclidean Borchers-Uhlmann functional σ. This is done next.

Lemma 8. *The linear functional σ on $\mathcal{E}$ defined in* (3.14) *satisfies the following properties:*

1. *Continuity: The map $E(d) \ni g \mapsto \sigma(\alpha_g \underline{f} \otimes \underline{h})$ is continuous for all $\underline{f}, \underline{h} \in \mathcal{E}$.*

2. *Invariance: $\sigma(\alpha_g \underline{f}) = \sigma(\underline{f})$ $\forall g \in E(d)$, $\forall \underline{f} \in \mathcal{E}$*

Proof. 1. This results from the map in question being a composition of continuous functions: The map $g \mapsto \alpha_g \underline{f}$ is continuous for all $\underline{f} \in \underline{\mathscr{S}}$ since α is an automorphic symmetry action. Furthermore, the tensor right multiplication $\mathsf{R}^{\otimes}_{\underline{h}} : \underline{\mathscr{S}} \ni \underline{f} \mapsto \underline{f} \otimes \underline{h}$ is continuous in the topology of $(\underline{\mathscr{S}}, \otimes)$. Since σ is a continuous functional on $\underline{\mathscr{S}}$, the map

$$g \mapsto (\sigma \circ \mathsf{R}^{\otimes}_{\underline{h}} \circ \alpha_{(.)} \underline{f})(g)$$

is continuous for all $\underline{f}, \underline{h} \in \underline{\mathscr{S}}$.

2. Follows from the definition of α on $\underline{\mathscr{S}}$ and the Euclidean invariance of the Schwinger functions given in $(E1)$ of Sec. A.1 in the appendix.

□

As we have seen in the last chapter, the Lorentzian quantum field theory can be obtained from the Euclidean one with the aid of the reflection positive linear functional σ. For this notion of reflection positivity and, as a consequence, for the construction of the physical Hilbert space, it is essential to remain on the "positive-time" algebra $\mathcal{E}_>$. At the Borchers-Uhlmann algebra this circumstance gets well visualized, as $\mathcal{E}_>$ is defined to be

$$\mathcal{E}_> := \underline{\mathscr{S}}_+ := \{\underline{f} \in \underline{\mathscr{S}} \,|\, \mathrm{supp}\, f_n \subset \mathbb{R}^{nd}_+\} \ .$$

Now let us define the following sesqui-linear product on $\mathcal{E}_>$:

$$\langle \underline{f}, \underline{g} \rangle := \sigma(\iota \underline{f}^* \otimes \underline{g}) \quad , \quad \underline{f}, \underline{g} \in \mathcal{E}_> \ .$$

In the algebraic framework we arrived at the physical Hilbert space $\mathcal{H}$ by performing a GNS-like construction with the aid of a general version of the product $\langle .,. \rangle$. Now we want to identify the concrete form of the corresponding Hilbert space for the free scalar field. To this end we first study the "one particle" content, described by vectors $\underline{f}_1 := (0, f_1, 0, \ldots)$. Recapitulating the proof of reflection positivity of the free covariance (*3.* of Prop. 9) and the spatial Fourier transform $\widehat{f}$, defined in equation (3.9), we are able to write

$$\langle f, f \rangle = C(\iota f^*, f) =: \int \frac{\mathrm{d}^{d-1}p}{2\omega_{\underline{p}}} \, \overline{f_\bullet(\underline{p})} f_\bullet(\underline{p}) =: \langle f_\bullet, f_\bullet \rangle_{\mathcal{H}} \ , \tag{3.15}$$

with the map

$$\begin{aligned} (.)_\bullet : \mathscr{S}(\mathbb{R}^d_+) \quad &\rightarrow \quad \mathscr{S}(\mathbb{R}^{d-1}) \\ f_\bullet(\underline{p}) \quad &= \quad \int \mathrm{d}^d x \, \mathrm{e}^{-i\underline{p}\underline{x} - \omega_{\underline{p}} x^0} \, f(x^0, \underline{x}) = (2\pi)^{\frac{d-1}{2}} \int_0^\infty \mathrm{d}x^0 \, \mathrm{e}^{-\omega_{\underline{p}} x^0} \, \widehat{f}(x^0, \underline{p}) \ , \end{aligned} \tag{3.16}$$

see also [Roe94, Sec. 7.5.2]. This map $(.)_\bullet$ is the Fourier-Laplace transform of a Schwartz function having support at positive time values. Thus $(.)_\bullet$ is an isometry from $(\mathscr{S}(\mathbb{R}^d_+), \langle .,. \rangle)$ to $(\mathscr{S}(\mathbb{R}^{d-1}), \langle .,. \rangle_{\mathcal{H}})$ endowed with the inner product

$$\langle s_1, s_2 \rangle_{\mathcal{H}} = \int \mathrm{d}\underline{\mu}\ \overline{s_1(\underline{p})} s_2(\underline{p}) \ ,$$

where we use the abbreviation $\mathrm{d}\underline{\mu} := \mathrm{d}^{d-1}p(2\omega_{\underline{p}})^{-1}$.

Lemma 9. *On the one-particle sector, the algebraic process of forming the quotient of* $(\mathscr{S}(\mathbb{R}^d_+), \langle\, , \rangle)$ *with the kernel* $\mathcal{N}_\sigma$ *of* $\langle\, , \rangle$ *reveals the pre-Hilbert space* $(\mathscr{S}(\mathbb{R}^{d-1}), \mathrm{d}\underline{\mu})$. *Thus,*

$$\mathcal{H}_1 = (L^2(\mathbb{R}^{d-1}), \mathrm{d}\underline{\mu}) \ .$$

Proof. The image of $(.)_\bullet$ is a dense subset of $(\mathscr{S}(\mathbb{R}^{d-1}), \mathrm{d}\underline{\mu})$, as one can see in the following way: in [OS75, Lemma 2.4], it was proven that the Fourier-Laplace transform is a continuous map from $\mathscr{S}(\mathbb{R}^{4n}_+)$ to $\mathscr{S}(\overline{\mathbb{R}^{4n}_+})$ with dense range and trivial kernel. This finding goes back to [OS73, Lemma 8.2]. Clearly, the generalization of this fact to $\mathscr{S}(\mathbb{R}^{dn}_+)$ is straightforward, as the Laplace transform stays one-dimensional. Hence the completion of this mapping's image coincides with the completion of $(\mathscr{S}(\mathbb{R}^{d-1}), \mathrm{d}\underline{\mu})$, which is nothing else than the one-particle Hilbert space $\mathcal{H}_1 = (L^2(\mathbb{R}^{d-1}), \mathrm{d}\underline{\mu})$.

Furthermore, let $g \in \ker\langle .,. \rangle$. Then we cannot deduce $g \equiv 0$, but from inequality (3.10) we may infer $g_\bullet = 0$. Hence $\ker\langle .,. \rangle_{\mathcal{H}} = \{0\}$. $\square$

The next step consists of the generalization of this procedure to arbitrary elements $\underline{f} = (f_0, f_1, f_2, \ldots)$ of the Borchers-Uhlmann algebra. Consider the Bosonic Fock space $\mathcal{F}$ built up on the Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}^{d-1}, \mathrm{d}\underline{\mu})$, that is

$$\begin{aligned} \mathcal{F} &= \bigoplus_{n \geqslant 0} P_+ \mathcal{H}^n \\ \mathcal{H}^n &:= (\mathcal{H}_1^{\otimes n}, \langle .,. \rangle_n) \ . \end{aligned}$$

Here, $P_+ := \frac{1}{n!} \sum\limits_{\pi \in \tau(n)} V(\pi)$ denotes the symmetrization operator on $\mathcal{H}_1^{\otimes n}$ and $V(\pi)$ is the implementation of the n-permutation $\pi \in \tau(n)$, while $\langle .,. \rangle_n$ denotes the fully symmetrized n-particle version of the scalar product $\langle .,. \rangle_{\mathcal{H}}$, *i.e.*,

$$\langle \Psi, \Phi \rangle_n := \langle P_+\Psi, P_+\Phi \rangle_{\mathcal{H}_1^{\otimes n}} \ .$$

The scalar product on $\mathcal{F}$ is given by $\langle\Psi,\Phi\rangle_{\mathcal{F}} := \sum_{n\geqslant 0}\langle\Psi_n,\Phi_n\rangle_n$ while in the usual way, we define the vacuum to be $\Omega := (1,0,0,\ldots)$. The creation and annihilation operators on $\mathcal{F}$ are essential for the generalization of the one-particle quotient map $(.)_\bullet$ to $\mathcal{E}_>$. Let $\chi\in\mathcal{H}_1$ and $\Psi\in\mathcal{F}$, then we identify

$$
\begin{aligned}
(a^\dagger(\chi)\Psi)_n(\underline{p}_1,\ldots,\underline{p}_n) &:= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\chi(\underline{p}_k)\Psi_{n-1}(\underline{p}_1,\ldots,\widehat{\underline{p}}_k,\ldots\underline{p}_n)\ ,\ (a^\dagger(\chi)\Psi)_0 = 0\ ,\\
(a(\chi)\Psi)_n(\underline{p}_1,\ldots,\underline{p}_n) &:= \sqrt{n+1}\int \mathrm{d}\underline{\mu}\ \overline{\chi(\underline{p})}\Psi_{n+1}(\underline{p},\underline{p}_1,\ldots,\underline{p}_n)\ .
\end{aligned}
$$

Furthermore, let us define another Fock space operator by

$$
\begin{aligned}
\phi:\ \mathcal{H}_1 &\rightarrow L(\mathcal{F})\\
\phi(\chi) &:= a(\mathsf{p}\chi^*) + a^\dagger(\chi)\ ,
\end{aligned}
\tag{3.17}
$$

which is our Euclidean field candidate ($L(\mathcal{F})$ denoting general linear operators on $\mathcal{F}$ here). In the latter formula, p denotes the parity operator, $\mathsf{p}s(\underline{x}) := s(-\underline{x})$, and its importance in the latter definition will become clear in a moment. One immediately comprehends that ϕ is compatible with the involution on $\mathcal{E}_>$, *i.e.*,

$$
\begin{aligned}
\phi(f_\bullet)^* &= a(\mathsf{p}(f_\bullet)^*)^* + a^\dagger(f_\bullet)^* \supseteq a^\dagger(\mathsf{p}(f_\bullet)^*) + a(f_\bullet) = a^\dagger((f^*)_\bullet) + a(\mathsf{p}^2 f_\bullet)\\
&= a^\dagger((f^*)_\bullet) + a(\mathsf{p}((f^*)_\bullet)^*) = \phi((f^*)_\bullet)\ ,
\end{aligned}
\tag{3.18}
$$

just by $(f_\bullet)^* = \mathsf{p}(f^*)_\bullet$.

Lemma 10. *σ can be continuously extended to a linear functional acting in a well-defined way on elements of the form $\delta\otimes\underline{s} := (s_0,\delta\otimes s_1,\delta\otimes s_2,\ldots)$, where $s_k\in\mathscr{S}(\mathbb{R}^{(d-1)k})$.*

Proof. We follow the standard approach and firstly verify this statement on the one-particle space $\mathscr{S}(\mathbb{R}^d)$. Take $(g_n)_{n\in\mathbb{N}}$ to be a delta-sequence in $\mathscr{S}(\mathbb{R}_+)$. Then we have

$$
\begin{aligned}
\lim_{n\to\infty} C(g_n\otimes s,f) &= \lim_{n\to\infty}\int \mathrm{d}^dp\,\mathrm{d}^dx\,\mathrm{d}^dy\,\frac{\mathrm{e}^{ip^0(x^0-y^0)}}{(2\pi)^d(p^2+m^2)}g_n(x^0)\widetilde{s}(-\underline{p})\widehat{f}(y^0,\underline{p})\\
&= \int\frac{\mathrm{d}^dp}{(2\pi)(p^2+m^2)}\int_0^\infty \mathrm{d}y^0\widetilde{s}_1(-\underline{p})\,\mathrm{e}^{-ip^0y^0}\,\widehat{f}(y^0,\underline{p}) = \int\frac{\mathrm{d}^dp}{(p^2+m^2)}\,\widetilde{s}_1(-\underline{p})\widetilde{f}(p)\ .
\end{aligned}
$$

In particular, for $(g_n^j)_{n\in\mathbb{N}}$ denoting two delta-sequences we obtain

$$\lim_{n\to\infty} C(g_n^1 \otimes s_1, g_n^2 \otimes s_2) \quad = \quad \int \mathrm{d}\underline{\mu}\, \widetilde{s}_1(-\underline{p})\widetilde{s}_2(\underline{p}) \ , \tag{3.19}$$

so we are able to define the numbers $C(\delta\otimes s, f)$ and $C(\delta\otimes s_1, \delta\otimes s_2)$ by the continuity of C. The vacuum representation of the free field in use is quasi-free, thus for $\underline{f} = (f_0, f_1, f_2, \ldots)$ consisting of Schwartz functions of product form, $f_n := f_n^{(1)} \otimes \cdots \otimes f_n^{(n)}$, $f_n^{(j)} \in \mathscr{S}(\mathbb{R}_+^d)$ for $j = 1, \ldots, n$ it holds

$$\sigma(\underline{f}) = \sum_{n\geqslant 0} \mathfrak{S}_n(f_n) = \sum_{n\geqslant 0}\sum_{k=1}^{n-1} \mathfrak{S}_{n-2}(f_n^{(1)}, \ldots, \widehat{f}_n^{(k)}, \ldots, f_n^{(n-1)}) C(f_n^{(k)}, f_n^{(n)}) \ ,$$

which means that we may insert functions supported "at sharp times", *i.e.*, those being of the form $\delta\otimes s$, since σ is a regular functional (see Lemma 8). This regularity together with the linearity of σ leads to the validity of this conclusion for arbitrary functions $f_n \in \mathscr{S}(\mathbb{R}^{nd})$. $\square$

From now on, operating with the generalized function $\delta \otimes s$ on $\mathcal{E}$ is meant in the way explained in the above lemma. Now it is time to introduce the field content on $\mathcal{F}$.

Definition 16. On the Fock space $\mathcal{F}$ defined via the reflection positive functional σ on the Euclidean Borchers-Uhlmann algebra $\mathcal{E}_>$, we prepare the following notions:

1. The time-zero algebra $\mathcal{E}_0$ is defined to be the Borchers-Uhlmann algebra generated by all generalized functions of the form $\delta \otimes s$ for s being a Schwartz function, *i.e.*,

$$\mathcal{E}_0 := \{\delta \otimes \underline{s} = (s_0, \delta \otimes s_1, \delta \otimes s_2, \ldots) \mid s_n \in \mathscr{S}(\mathbb{R}^{n(d-1)})\} \ .$$

2. The time-zero field $\varphi(s)$ is given by the representation of the generalized function $\delta \otimes s$:

$$\varphi(s)[\underline{g}]_\sigma := [\delta \otimes s \otimes \underline{g}]_\sigma \ .$$

Finally,

3. Let $f = \sum_{m \in \mathbb{N}} h^m \otimes s^m$ for $h^m \in \mathscr{S}(\mathbb{R}_+)$ and $s^m \in \mathscr{S}(\mathbb{R}^{d-1})$ for $m \in \mathbb{N}$. Then we define the Euclidean field $\phi^{\mathcal{E}}$ in the following way,

$$\phi^{\mathcal{E}}(f) := \sum_m \int_0^\infty \mathrm{d}t \, h^m(t) \alpha_t \varphi(s^m) \ .$$

The first two points of the latter definition are well-defined according to Lemma 10. The symmetry group shall act on these fields consistently with respect to the algebraic approach,

$$g \in E(d) : \quad \alpha_g \phi^{\mathcal{E}}(f) := V(g) \phi^{\mathcal{E}}(f) V(g)^{-1} \quad \forall f \in \mathscr{S}(\mathbb{R}^d_+) \ ,$$

and the same shall be true for the time-zero field φ.

Proposition 10. *1. The time-zero field fulfills the following relation,*

$$\varphi(s) = \phi((\delta \otimes s)_\bullet) \ \forall \ s \in \mathcal{E}_0 :$$

2. φ is a well-defined representation of $\mathcal{E}_0$ on $\mathcal{F}$.

3. The two-point function in terms of $\phi^{\mathcal{E}}$ equals the free covariance, i.e.,

$$\langle \phi^{\mathcal{E}}(f)\Omega, \phi^{\mathcal{E}}(g)\Omega \rangle_{\mathcal{F}} = C(f,g) \quad \forall f, g \in \mathscr{S}(\mathbb{R}^d) \ .$$

Proof. 1. We start by calculating the scalar product of two time-zero elements of the one-particle sector:

$$\begin{aligned} \langle (\delta \otimes s^*)_\bullet, (\delta \otimes u)_\bullet \rangle_{\mathcal{H}_1} &= \int \mathrm{d}\underline{\mu} \iint_0^\infty \mathrm{d}x^0 \, \mathrm{d}y^0 \ \delta(x^0) \overline{\widetilde{s^*}(\underline{p})} \, \mathrm{e}^{-\omega_{\underline{p}}(x^0+y^0)} \, \delta(y^0) \widetilde{u}(\underline{p}) \\ &= \int \mathrm{d}\underline{\mu} \ \widetilde{s}(-\underline{p}) \, \widetilde{u}(\underline{p}) = C(\delta \otimes s, \delta \otimes u) \ . \end{aligned} \tag{3.20}$$

The last equality follows from (3.19). Form the two-point function of the operator $\phi((.)_\bullet)$ to obtain

$$\begin{aligned}\langle \Omega, \phi((\delta\otimes s)_\bullet)\phi((\delta\otimes u)_\bullet)\Omega\rangle_{\mathcal{F}} &= \langle (\delta\otimes s^*)_\bullet, (\delta\otimes u)_\bullet\rangle_{\mathcal{H}_1} = C(\delta\otimes s, \delta\otimes u)\\ &= \sigma(\iota(1)^*\otimes(\delta\otimes s\otimes\delta\otimes u)) = \langle [1]_\sigma, \varphi(s)\varphi(u)[1]_\sigma\rangle_{\mathcal{F}}\ ,\end{aligned}$$

where the first equality is gained by the definition of ϕ and the second is just equation (3.20).

2. Let $s \in \mathscr{S}(\mathbb{R}^{d-1})$ and $\underline{g} \in \mathcal{E}_>$ such that $\langle [\underline{g}]_\sigma, [\underline{h}]_\sigma\rangle_{\mathcal{F}} = 0$ for all $\underline{h} \in \mathcal{E}_>$. Using *1.*, we observe,

$$\begin{aligned}\langle \varphi(s)[\underline{g}]_\sigma, [\underline{h}]_\sigma\rangle_{\mathcal{F}} &= \langle \phi((\delta\otimes s)_\bullet)[\underline{g}]_\sigma, [\underline{h}]_\sigma\rangle_{\mathcal{F}} = \langle [\underline{g}]_\sigma, \phi((\delta\otimes s)_\bullet)^*[\underline{h}]_\sigma\rangle_{\mathcal{F}}\\ &= \langle [\underline{g}]_\sigma, \phi((\delta\otimes s^*)_\bullet)[\underline{h}]_\sigma\rangle_{\mathcal{F}} = \langle [\underline{g}]_\sigma, \varphi(s^*)[\underline{h}]_\sigma\rangle_{\mathcal{F}}\\ &= \langle [\underline{g}]_\sigma, [\delta\otimes s^*\otimes\underline{h}]_\sigma\rangle_{\mathcal{F}} = 0\ ,\end{aligned}$$

where in the third equation we have used relation (3.18). Thus $\varphi(s)$ is independent of the concrete representative s.

3. Let $f, g \in \mathscr{S}(\mathbb{R}^d_+)$, such that $f = \sum_{m_1\in\mathbb{N}} h_1^{m_1}\otimes s_1^{m_1}$ and $g = \sum_{m_2\in\mathbb{N}} h_2^{m_2}\otimes s_2^{m_2}$ and such that $f\otimes g \in \mathscr{S}_>(\mathbb{R}^{2d})$. This latter condition reflects the demand on complete time-ordering. Then we infer,

$$\begin{aligned}\langle \phi^{\mathcal{E}}(f)\Omega, \phi^{\mathcal{E}}(g)\Omega\rangle_{\mathcal{F}} &= \sum_{m_1,m_2\in\mathbb{N}} \iint \mathrm{d}t_1\,\mathrm{d}t_2\; h_1^{m_1}(t_1)h_2^{m_2}(t_2)\langle \alpha_{t_1}\varphi(s_1^{m_1}), \alpha_{t_2}\varphi(s_2^{m_2})\rangle\\ &= \sum_{m_1,m_2\in\mathbb{N}} \iint \mathrm{d}^d x\,\mathrm{d}^d y\; h_1^{m_1}(x_0)h_2^{m_2}(y_0)\int \mathrm{d}\underline{\mu}\;\mathrm{e}^{-\omega_{\underline{p}}(y_0-x_0)} s_1^{m_1}(\underline{x}) s_2^{m_1}(\underline{y})\\ &= C(f,g)\ ,\end{aligned}$$

where in the second equality we have renamed t_1 and t_2 into x_0 and y_0, respectively. By time-ordering, $x_0 < y_0$ and the exponential is a damping. The last equality is then obtained by recollecting the Schwartz functions f and g, while drawing on the techniques of Sec. 3.1. The free covariance in its form (3.3) is seen to be independent of any time-ordering.

□

Proposition 11. *1. For the Euclidean free field the Fock space $\mathcal{F}$ over $\mathcal{H}_1$ is given by $\overline{\mathcal{E}_>/\mathcal{N}_\sigma}$. The abstract quotient map $[\,]_\sigma : \mathcal{E} \to \langle\, , \,\rangle/\mathcal{N}_\sigma$ given in Ch. 2 has the following form:*

$$
\begin{array}{rcl}
[.]_\sigma : \mathcal{E}_> & \to & \mathcal{F} \\
\underline{f} & \mapsto & \sum\limits_{\substack{n\geqslant 0\\ M\geqslant 0}} \prod\limits_{k=1}^{n} \phi(f_{n,\bullet}^{(k),M})\Omega \ .
\end{array}
$$

Here, $\lim\limits_{N\to\infty} \sum\limits_{M=0}^{N} \bigotimes\limits_{k=1}^{n} f_n^{(k),M}$ denotes the tensor product expansion of an arbitrary function f_n in $\mathscr{S}(\mathbb{R}^{dn})$.

2. The virtual representation V on $\mathcal{F}$ is given by $V(g)[\underline{f}]_\sigma := [\underline{f_{(g)}}]_\sigma$.

Proof. 1. The one-particle structure was shown in Lemma 9. For the arbitrary particle numbers we contemplate a Fock space vector Ψ having product form, *i.e.*,

$$
\Psi = (\psi_0, \psi_1^{(1)}, \psi_2^{(1)} \otimes \psi_2^{(2)}, \ldots, \psi_n^{(1)} \otimes \cdots \otimes \psi_n^{(n)}, \ldots) \ .
$$

The connection between $\mathcal{F}$ and the Euclidean free field given by σ on $\mathcal{E}_>$ will be realized by the following calculation with the help of relation (3.15): we calculate the scalar product of two vectors evolving from the action of a polynomial $\sum\limits_n \prod\limits_{k=1}^{n} \phi(\psi_n^{(k)})$ on the vacuum, *i.e.*,

$$
\begin{aligned}
&\left\langle \sum_{n=0}^{2} \prod_{k=1}^{n} \phi(\psi_n^{(k)})\Omega, \sum_{m=0}^{2} \prod_{l=1}^{m} \phi(\chi_m^{(l)})\Omega \right\rangle_{\mathcal{F}} = \\
= \ & \langle (\psi_0 + \phi(\psi_1^{(1)}) + \phi(\psi_2^{(1)})\phi(\psi_2^{(2)}))\Omega, (\chi_0 + \phi(\chi_1^{(1)}) + \phi(\chi_2^{(1)})\phi(\chi_2^{(2)}))\Omega \rangle \\
= \ & \overline{\psi_0}\chi_0 + 0 + 0 + \langle \psi_1^{(1)}, \chi_1^{(1)} \rangle + \overline{\psi_0} \langle \mathsf{p}\chi_2^{(1)*}, \chi_2^{(2)} \rangle + \chi_0 \langle \psi_2^{(2)}, \mathsf{p}\psi_2^{(1)*} \rangle + 0 + 0 \\
+ \ & (\langle \psi_2^{(2)}, \mathsf{p}\psi_2^{(1)*} \rangle \langle \mathsf{p}\chi_2^{(1)*}, \chi_2^{(2)} \rangle + \langle \psi_2^{(2)}, \chi_2^{(1)} \rangle \langle \psi_2^{(1)}, \chi_2^{(2)} \rangle + \langle \psi_2^{(1)}, \chi_2^{(2)} \rangle \langle \psi_2^{(2)}, \chi_2^{(1)} \rangle) \ .
\end{aligned}
$$

Now consider two functions $\delta \otimes \underline{s}, \delta \otimes \underline{u} \in \mathcal{E}_0$ of product form such that $(\delta \otimes \underline{s})_{n,\bullet}^{(k)} = \psi_n^{(k)}$ and $(\delta \otimes \underline{u})_{n,\bullet}^{(k)} = \chi_n^{(k)}$ for all $k, n \in \mathbb{N}$. Then from equation (3.15) we recognize that the last two lines of the latter calculation are equal to $\sum\limits_{n,m}^{2} \mathfrak{S}_{n+m}(\iota(\delta \otimes \underline{s})_n^* \otimes (\delta \otimes \underline{u})_m)$.

In order to identify the scalar product of general polynomials in ϕ with the sum of all Then, we assume the equation to hold for the sum over n until $N-1$ and m until $M-1$, respectively. Abbreviating $\Sigma_\psi^N := \sum\limits_{n=0}^{N} \prod\limits_{k=1}^{n} \phi(\psi_n^{(k)})\Omega$ we calculate,

$$
\begin{aligned}
\langle \Sigma_\psi^N, \Sigma_\phi^M \rangle_{\mathcal{F}} &= \langle \Sigma_\psi^{N-1}, \Sigma_\chi^{M-1} \rangle + \langle \prod_{k=1}^{N} \phi(\psi_N^{(k)})\Omega, \prod_{l=1}^{M} \phi(\chi_M^{(l)})\Omega \rangle_{\mathcal{F}} \\
&+ \langle \prod_{k=1}^{N} \phi(\psi_N^{(k)})\Omega, \Sigma_\chi^{M-1} \rangle_{\mathcal{F}} + \langle \Sigma_\psi^N, \prod_{l=1}^{M} \phi(\chi_M^{(l)})\Omega \rangle_{\mathcal{F}}
\end{aligned}
$$

Now we apply Wick's theorem. It states that the expectation value of any field monomial is equal to the sum of all possible contractions. The two-valent contractions are given by the free covariance C. In view of relation (3.13), any higher contraction is given by the suitable product of such free covariances. In a reduced notation leaving out the distributional arguments this leads to,

$$
\begin{aligned}
\langle \Sigma_\psi^N, \Sigma_\phi^M \rangle_{\mathcal{F}} &= \sum_{n,m=0}^{N-1,M-1} \mathfrak{S}_{n+m} + \mathfrak{S}_{N+M} + \sum_{m=0}^{M-1} \mathfrak{S}_{N+m} + \sum_{n=0}^{N-1} \mathfrak{S}_{n+M} \\
&= \sum_{n,m=0}^{N,M} \mathfrak{S}_{n+m} \ ,
\end{aligned}
$$

which proves the assertion. Thus, for $\mathcal{E}_0$-elements $\delta \otimes \underline{s}$ and $\delta \otimes \underline{u}$ having product form it holds

$$
\begin{aligned}
\sigma(\iota(\delta \otimes \underline{s})^* \otimes (\delta \otimes \underline{u})) &= \left\langle \sum_{n \geqslant 0} \prod_{k=1}^{n} \phi((\delta \otimes \underline{s})_{n,\bullet}^{(k)})\Omega, \sum_{m \geqslant 0} \prod_{l=1}^{m} \phi((\delta \otimes \underline{u})_{m,\bullet}^{(l)})\Omega \right\rangle_{\mathcal{F}} \\
&= \left\langle \sum_{n \geqslant 0} \prod_{k=1}^{n} \varphi(s_n^{(k)})\Omega, \sum_{m \geqslant 0} \prod_{l=1}^{m} \varphi(u_m^{(l)})\Omega \right\rangle_{\mathcal{F}} \ . \qquad (3.21)
\end{aligned}
$$

For an arbitrary element $f_n \in \mathscr{S}(\mathbb{R}^{dn})$ we have $f_n = \lim\limits_{N\to\infty} \sum\limits_{M=1}^{N} f^{(1),M} \otimes \cdots \otimes f^{(n),M}$, where $f^{(l),M} \in \mathscr{S}(\mathbb{R}^d)$ for all $M = 1, \ldots, N$ and $l = 1, \ldots, n$. Moreover, ϕ and therefore also φ is a linear operator on Fock space. Hence we can pull out the sum of this latter expansion to conclude

$$\sigma(\iota \underline{f}^* \otimes \underline{g}) \quad = \quad \left\langle \sum_{\substack{n \geqslant 0 \\ M \geqslant 0}} \prod_{k=1}^{n} \varphi(s_n^{(k),M})\Omega, \sum_{\substack{m \geqslant 0 \\ M' \geqslant 0}} \prod_{l=1}^{m} \varphi(u_m^{(l),M'})\Omega \right\rangle_{\mathcal{F}} , \tag{3.22}$$

for general elements $\underline{f} := \delta \otimes \underline{s}$ and $\underline{g} := \delta \otimes \underline{u}$ of $\mathcal{E}_0$.

We thus observe that due to relation (3.22) the map $[\,]_\sigma$ maps a Borchers-Uhlmann vector $\underline{g} \in \mathcal{E}_0$ to the polynomial $\sum\limits_{\substack{n \geqslant 0 \\ M \geqslant 0}} \prod\limits_{k=1}^{n} \phi(g_{n,\bullet}^{(k)})\Omega$ in the admittedly not very clear version using one-particle contributions. One could just as well define a corresponding n-valent functional Φ by $\Phi(f_n) := \sum_m \prod_{k=1}^n \phi(f_n^{(k),m})$, which would simplify the relation above to $[\underline{f}]_\sigma := \sum_n \Phi(f_{n,\bullet})$, where $f_{n,\bullet}$ for $n = 1$ is given in (3.16) while for arbitrary n, the generalization is obvious,

$$f_{n,\bullet}(\underline{p}_1, \ldots, \underline{p}_n) = (2\pi)^{-\frac{n(d-1)}{2}} \int_0^\infty \mathrm{d}^n x^0 \, \mathrm{e}^{-\sum\limits_{k=1}^{n} \omega_{\underline{p}_k} x_k^0} \, \widehat{f}_n((x_1^0, \underline{p}_1), \ldots, (x_n^0, \underline{p}_n)) \, .$$

It remains to show that the quotient map is of this form for arbitrary elements in $\mathcal{E}_>$, and not just for those of $\mathcal{E}_0$. But this is a straightforward implementation of *3.* of Prop. 10. Indeed, the sum structure stays untouched by the action of the Euclidean time-translations. Thus, the two-point functions of d-dimensional Schwartz functions gained through the procedure above will result in the corresponding free covariances, which proves the identity

$$\begin{aligned} \sigma(\iota(\underline{f}^*) \otimes \underline{g}) \quad &= \quad \left\langle \sum_{\substack{n \geqslant 0 \\ M \geqslant 0}} \prod_{k=1}^{n} \phi^{\mathcal{E}}(f_n^{(k),M})\Omega, \sum_{\substack{m \geqslant 0 \\ M' \geqslant 0}} \prod_{l=1}^{m} \phi^{\mathcal{E}}(g_m^{(l),M'})\Omega \right\rangle_{\mathcal{F}} \\ &=: \quad \left\langle \sum_{n \geqslant 0} \Phi^{\mathcal{E}}(f_n)\Omega, \sum_{m \geqslant 0} \Phi^{\mathcal{E}}(g_m)\Omega \right\rangle_{\mathcal{F}} \, . \end{aligned} \tag{3.23}$$

So, we have assured ourselves that the Hilbert space obtained by the completion of the quotient space of $(\mathcal{E}_>, \langle .,. \rangle)/\ker \langle .,. \rangle$ is given by the Bosonic Fock space $\mathcal{F}$.

2. Now it is straightforward to identify the "virtual" representation of the Euclidean group on $\mathcal{H}_1$ and, consequently, also on $\mathcal{F}$. Again, everything follows from the one-particle extent, where we have

$$\begin{aligned} V_1(t) f_\bullet(\underline{p}) &:= (f_{((t,0),1)})_\bullet(\underline{p}) = (2\pi)^{-\frac{d-1}{2}} \int \mathrm{d}x^0 \, \mathrm{e}^{-\omega_{\underline{p}} x^0} \widehat{f}(x_0 - t, \underline{p}) \\ &= (2\pi)^{-\frac{d-1}{2}} \mathrm{e}^{-\omega_{\underline{p}} t} \int \mathrm{d}x^0 \, \mathrm{e}^{-\omega_{\underline{p}} x^0} \widehat{f}(x_0, \underline{p}) = \mathrm{e}^{-\omega_{\underline{p}} t} f_\bullet(\underline{p}) \ . \end{aligned}$$

$V_1(t)$ is a symmetric operator on $\mathcal{H}_1$, generating a strongly continuous contraction semi-group. Hence we indeed have $V_1(t) = \mathrm{e}^{-tH}$ for the positive generator H, fulfilling

$$\begin{aligned} H f_\bullet(\underline{p}) &= \omega_{\underline{p}} f_\bullet(\underline{p}) \ , \ \forall f \in \mathscr{S}(\mathbb{R}^d_+) \ , \\ (H\Psi)_n(\underline{p}_1, \ldots, \underline{p}_n) &= \left(\sum_{k=1}^{m} \omega_{\underline{p}_k} \right) \Psi_n(\underline{p}_1, \ldots, \underline{p}_n) \ . \end{aligned}$$

The Euclidean subgroup commuting with the time reflection, which we have termed $E_e(d)$ in the algebraic setting defines a unitary operator

$$g \in E_e(d): \quad U_{0,1}(g) f_\bullet := (f_{(g)})_\bullet \ ,$$

while the boosts have to be defined on an invariant, dense subspace of $\mathcal{H}_1$, as we have done in Ch. 2. The second quantization of these operators is standard and writing $V(g)$ for the implementation of $E(d)$ on $\mathcal{F}$ we have

$$\begin{aligned} \alpha_g \phi(\chi) &:= V(g)\phi(\chi)V(g)^{-1} \\ g \in E(d): \ V(g)\Psi &= V(g)(\psi_0, \psi_1, \psi_2, \ldots) := (\psi_0, V_1(g)\psi_1, V_\otimes(g)\psi_2, \ldots) \ , \end{aligned}$$

where

$$V_\otimes(g)(\psi_n^{(1)} \otimes \cdots \otimes \psi_n^{(n)}) := (V_1 \otimes \cdots \otimes V_1)(g)(\psi_n^{(1)} \otimes \cdots \otimes \psi_n^{(n)}) = (V_1(g)\psi_n^{(1)} \otimes \cdots \otimes V_1(g)\psi_n^{(n)}) \ .$$

□

A consequence of this finding is the following

Corollary 1. *The Euclidean functional σ fulfills reflection positivity,* i.e.,

$$\sigma(\iota \underline{f}^* \otimes \underline{f}) \geqslant 0 \ \forall \underline{f} \in \mathcal{E}_> \ .$$

Proof. We rely on relation (3.23). By continuity of σ, we obtain

$$\sigma(\iota \underline{f}^* \otimes \underline{f}) \;=\; \sum_{n,m} S_{n+m}(\iota f_n^* \otimes f_m) = \sum_{M \geqslant 0} \left\langle \sum_{n \geqslant 0} \prod_{k=1}^{n} \phi^{\mathcal{E}}(f_n^{(k),M})\Omega, \sum_{m \geqslant 0} \prod_{l=1}^{m} \phi^{\mathcal{E}}(f_m^{(l),M})\Omega \right\rangle_{\mathcal{F}} \geqslant 0 \ .$$

□

At this stage we have built up enough to verify the consistency of our Fock space operator ϕ with the Euclidean field $\phi^{\mathcal{E}}$. So far we have identified the structure $(\mathcal{E}, \mathscr{O}, \alpha)$ as well as the time-zero data $\mathcal{E}_0$ at the special case of the free field, so let us list the relevant notions now,

$$\begin{aligned} \mathcal{E}(O) := \{ \underline{f} \in \underline{\mathscr{L}} \mid \operatorname{supp} f_n \subset O^{\times n} \} \quad &, \quad \sigma(\underline{f}) := \sum_{n \geqslant 0} \mathfrak{S}_n(f_n) \\ \mathcal{F} = \bigoplus_{n \geqslant 0} P_+ \mathcal{H}_1^{\otimes n} \ , \ \mathcal{H}_1 = L^2(\mathbb{R}^{d-1}, \mathrm{d}\underline{\mu}) \quad &, \quad \varphi(s) = \phi((\delta \otimes s)_\bullet) \\ g = ((a_0, \underline{a}), R) \in E(d) : \ \alpha_{((0,\underline{a}),\underline{R})} = \mathrm{ad}[U_0(\underline{a}, \underline{R})] \quad &, \quad V_1(a_0) f_\bullet(\underline{p}) = \mathrm{e}^{-\omega_{\underline{p}} a_0} f_\bullet(\underline{p}) \\ (V(g)\Psi)_n(\underline{p}_1, \ldots, \underline{p}_n) = (V_\otimes(g)\psi_n)(\underline{p}_1, \ldots, \underline{p}_n) \quad &, \quad \Psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F} \ . \end{aligned} \tag{3.24}$$

Since the basic framework is set up, we can move on to extract the Minkowskian theory out of our free field. During the next section, we will stick to the findings of section 2.2. This is well-grounded, since we provided the net of Euclidean algebras defined by the free field and it is evident that the action of $V(g)$, $g \in E_\theta(d)$ is an automorphic symmetry action thereon.

3.3 Reconstruction of Quantum Fields

We have shown in Ch. 2 that the representation $V(g)$ of the Euclidean group $E(d)$ on the physical Hilbert space fits into the framework of "virtual representations" [FOS83]. This physical Hilbert space in fact is the Bosonic Fock space $\mathcal{F}$ in the free scalar field scenario. By analytic continuation of V it follows that we are left with a unitary representation $U(g)$ of the Poincaré group $\mathcal{P}(d)$ on $\mathcal{F}$. We denote the automorphic action on the Fock space operators by $\alpha^{\mathcal{M}}$, leading to the symmetry action on the Fock space,

$$A : \mathcal{F} \to \mathcal{F} \ , \ g \in \mathcal{P}(d) : \ \ \alpha^{\mathcal{M}}_g(A) = U(g) A U(g)^{-1} \ . \tag{3.25}$$

The translations are most important for the ongoing treatment. Their representation $U(t)$ is given by the analytical continuation of $V(t)$ to purely imaginary values of t, thus it acts on the Hilbert space in the following way,

$$U(t) f_\bullet(\underline{p}) = \mathrm{e}^{i\omega_{\underline{p}} t} f_\bullet(\underline{p}) \ ,$$

and correspondingly on $\mathcal{F}$. Moreover, let us abbreviate the action $\alpha^{\mathcal{M}}_{((t,0),1)}$ by $\alpha^{\mathcal{M}}_t$. Relying on Proposition 2, we are lead to

Definition 17. 1. We define the Minkowski field $\phi^{\mathcal{M}}(f)$ as the image of the time-zero field φ under the Poincaré representation $U(t)$, smeared with a time-dependent Schwartz function, that is

$$\phi^{\mathcal{M}}(f) := \sum_m \int \mathrm{d}t \, h^m(t) \alpha^{\mathcal{M}}_t \varphi(s^m) \ , \quad f = \sum_{m \in \mathbb{N}} h^m \otimes s^m \in \mathscr{S}(\mathbb{R}^d) \ .$$

2. To the space-time region $O \subset \mathbb{R}^d$ we associate the Lorentzian algebra $\mathcal{M}(O)$, defined in the following way:

$$\mathcal{M}(O) := \bigvee_{\substack{K \subset \mathbb{R}^{d-1} \\ K+t \subset O}} \{\alpha^{\mathcal{M}}_t(\delta \otimes \underline{s}) \mid \underline{s} = (s_0, s_1, \ldots) \, , \, s_n \in \mathscr{S}(\mathbb{R}^{n(d-1)}) \, , \, \mathrm{supp}\, s_n \subset K\} \ .$$

Remark 11. For the Minkowski field we do no longer need the restriction to positive values of the time coordinate. Therefore, we can plug in any Schwartz function on $\mathbb{R}^d$ and integrate over the whole of $\mathbb{R}$ in the definition of $\phi^{\mathcal{M}}$.

So, in the same way as in the algebraic part, $\mathcal{M}(O)$ defines a von Neumann algebra and is obtained by the application of the Poincaré action on the time-zero algebra $\mathcal{E}_0$. In fact, $\{\mathcal{M}(O)\}_{O\in\mathscr{O}}$ defines an isotonous and $\mathcal{P}(d)$-covariant net. The proof goes along the lines of that given for the more generally defined Lorentzian net given in Ch. 2, but using the full Poincaré group instead of a proper subgroup. This causes no more difficulty.

Let us check if our definition of the free field on Minkowski space-time gives rise to the correct Wightman functions. To this end, we appreciate the behavior under Minkowski-time translations as they are given by analytic continuation,

$$(U(t)\Psi)_n(\underline{p}_1,\ldots,\underline{p}_n) = (\mathrm{e}^{iHt}\Psi)_n(\underline{p}_1,\ldots,\underline{p}_n) = \mathrm{e}^{i\sum_{k=1}^{n}\omega_{\underline{p}_k}t}\,\Psi_n(\underline{p}_1,\ldots,\underline{p}_n)\ ,$$

for all Fock space vectors Ψ. For the expectation value of two Minkowski fields $f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$ we calculate

$$\begin{aligned}
\langle\Omega,\phi^{\mathcal{M}}(f_1)\phi^{\mathcal{M}}(f_2)\Omega\rangle \quad &:= \quad \sum_{m,m'}\langle\Omega,\int \mathrm{d}t_1\, h_1^m(t_1)\alpha_{t_1}^{\mathcal{M}}\varphi(s_1^m)\int \mathrm{d}t_2\, h_2^{m'}(t_2)\alpha_{t_2}^{\mathcal{M}}\varphi(s_2^{m'})\Omega\rangle \\
&\overset{t_j=:x_j^0}{=} \quad \sum_{m,m'}\iint \mathrm{d}^dx_1\,\mathrm{d}^dx_2\int \mathrm{d}\underline{\mu}\ h_1^m(x_1^0)h_2^{m'}(x_2^0)\widetilde{s_1^m}(-\underline{p})\widetilde{s_2^{m'}}(\underline{p})\mathrm{e}^{-i\omega_{\underline{p}}(x_1^0-x_2^0)} \\
&= \quad \lim_{\epsilon\to 0}\int \mathrm{d}^dx_1\,\mathrm{d}^dx_2\int\frac{\mathrm{d}^dp}{(2\pi)^d}\ \frac{\mathrm{e}^{-ip_0(x_1^0-x_2^0)}\mathrm{e}^{i\underline{p}(\underline{x}_1-\underline{x}_2)}}{p_0^2-\underline{p}^2-m^2+i\epsilon}f_1(x_1)f_2(x_2)\ , \quad (3.26)
\end{aligned}$$

which equals the Feynman propagator. Defining the Wightman functions in the canonical way, *i.e.*,

$$\mathfrak{W}_n(f_1,\ldots,f_n) := \langle\Omega,\phi^{\mathcal{M}}(f_1)\cdots\phi^{\mathcal{M}}(f_n)\Omega\rangle\ ,$$

the calculation above shows that we obtain the correct Minkowski two-point function and, as we have stressed continuously, all the Wightman functions by quasi-freedom.

3.4 Deforming the Theory

We intentionally made some effort to restate the main facts about the Euclidean free field in terms of the Borchers Uhlmann algebra in the last sections. Also, the time-zero content of the theory was introduced with care, in order to make the application of the deformation scheme more visible.

Since we operate on Schwartz space (of several variables), the warped convolutions and Rieffel product deformations are equivalent to the utilization of the Moyal product. In this context it is essential that the product of Def. 5 incorporates the evaluation at a single point, which is not appropriate for the tensor structure of $\mathscr{S}(\mathbb{R}^d)\overbrace{\times\cdots\times}^{n}\mathscr{S}(\mathbb{R}^d)\simeq\mathscr{S}(\mathbb{R}^{dn})$. The *Moyal tensor product* for functions $f,g\in\mathscr{S}(\mathbb{R}^d)$ is more suitable for these purposes [GL08],

$$(f\otimes_\theta g)(x,y):=(2\pi)^{-d/2}\iint \mathrm{d}^d u\,\mathrm{d}^d w\;\mathrm{e}^{iuw}f(x+\theta u/2)g(y+w)\ ,$$

and for coinciding arguments $y\to x$ passes on to the Groenewold-Moyal product. For more general $f_n\in\mathscr{S}(\mathbb{R}^{nd})$, $g_m\in\mathscr{S}(\mathbb{R}^{md})$ it is best written down using the Fourier transformation:

$$(\widetilde{f_n\otimes_\theta g_m})(p_1,\ldots,p_n,q_1,\ldots,q_m)=\prod_{k=1}^{n}\prod_{l=1}^{m}\mathrm{e}^{-\frac{i}{2}p_k\theta p_l}\tilde{f}_n(p_1,\ldots,p_n)\tilde{g}_m(q_1,\ldots,q_m)\ ,$$

where we use the notation $p\theta q:=\sum\limits_{k,l=0}^{d}p^k\theta_{kl}q^l$. Due to the complete antisymmetry and the degeneracy of θ just two terms remain in this sum at the probably most interesting case of $d=4$: $p^2\theta_{23}q^3+p^3\theta_{32}q^2$ which result in $p^2\vartheta q^3-p^3\vartheta q^2$, where $\pm\vartheta$ are the only nonzero entries of the matrix θ. The n-fold product of $\mathscr{S}(\mathbb{R}^d)$-functions can be written in the following way

$$\mathcal{F}_\theta\left(\bigotimes_{k=1}^{n}f_k\right)(\underline{p}_1,\ldots,\underline{p}_n)=\prod_{j=1}^{n}\tilde{f}_j(\underline{p}_j)\prod_{1\leqslant l<m\leqslant n}\mathrm{e}^{-\frac{i}{2}p_l\theta p_m} \tag{3.27}$$

where $\mathcal{F}_\theta\left(\bigotimes_{k=1}^{2}f_k\right):=(\widetilde{f_1\otimes_\theta f_2})$. For later convenience, let us abbreviate the Moyal phase factor $\prod\limits_{1\leqslant l<m\leqslant n}\mathrm{e}^{-\frac{i}{2}p_l\theta p_m}$ by $\mathfrak{D}^\theta_n(\underline{p})$. In opposition to the "point-wise Moyal star product" the Moyal tensor product is not plagued by convolutions when the Fourier transformation is applied. For the product of two functions we calculate

$$
\begin{aligned}
(\widetilde{f} \otimes_\theta \widetilde{g})(p,q) &= (2\pi)^{-d/2} \iint \mathrm{d}^d k\, \mathrm{d}^d v\ \mathrm{e}^{ikv} \widetilde{f}_{(\theta k/2,1)}(p) \widetilde{g}_{(v,1)}(q) \\
&= (2\pi)^{-d/2} \iint \mathrm{d}^d k\, \mathrm{d}^d v\ \mathrm{e}^{ikv + \frac{ip\theta k}{2} + iqv} \widetilde{f}(p) \widetilde{g}(q) \\
&= \int \mathrm{d}^d k\ \delta(k+v)\, \mathrm{e}^{\frac{ip\theta k}{2}} \widetilde{f}(p) \widetilde{g}(q) = \widetilde{f}(p)\, \widetilde{g}(q)\, \mathrm{e}^{-\frac{ip\theta q}{2}} \\
&= (\widetilde{f \otimes_\theta g})(p,q) \ ,
\end{aligned}
$$

where the last equality is valid according to formula (3.27). This relation holds for arbitrary n Moyal factors by induction. Moreover, we have got a "cyclic formula":

$$
\begin{aligned}
\int \mathrm{d}^d y (f \otimes_\theta g)(x,y) &= (2\pi)^{-d/2} \int \mathrm{d}^d y\, \mathrm{d}^d k\, \mathrm{d}^d v\, \mathrm{e}^{-ikv}\, f(x+\theta k/2) g(y+v) \\
&= (2\pi)^{d/2} \int \mathrm{d}^d y\, \mathrm{d}^d k\, \mathrm{d}^d q\, \delta(q-k)\, f(x+\theta k/2) \widetilde{g}(q)\, \mathrm{e}^{iqy} \\
&= (2\pi)^{d} \int \mathrm{d}^d k\, f(x+\theta k/2) \widetilde{g}(k) \delta(k) = (2\pi)^{d/2} \int \mathrm{d}^d y f(x) g(y) \ . \qquad (3.28)
\end{aligned}
$$

Back at our free field model, we realize that for spatial $\underline{x} \in \mathbb{R}^{d-1}$, we have $\alpha_{\underline{x}} \varphi(s) = \varphi(s_{((0,-\underline{x}),1)})$ in the Euclidean and the Minkowski case. Thus we can define a "tensorial Rieffel product" for (time-zero) fields in the same way we did for Schwartz functions. For convenience, we write down its definition:

$$
\begin{aligned}
(\varphi \times_\theta \varphi)(s_1, s_2) &:= (2\pi)^{-d/2} \int \mathrm{d}^d k\, \mathrm{d}^d v\, \mathrm{e}^{-ikv} \alpha_{\theta k/2} \varphi(s_1) \alpha_v \varphi(s_2) \\
&= (2\pi)^{-d/2} \int \mathrm{d}^d k\, \mathrm{d}^d v\, \mathrm{e}^{-ikv} \varphi(s_{1,(\theta k/2)}) \varphi(s_{2,(v)}) \ .
\end{aligned}
$$

The time-zero field φ is a linear operator on the Fock space $\mathcal{F}$, thus its warped convolution with respect to the skew-symmetric matrix θ is well-defined and denoted by φ_θ, *c.f.* Sec. 1.2.4.

3.4.1 Noncommutative Four Point Function

Given that we just deform the action, i.e. use star products for every commutative product there, the bilinear terms stay unchanged due to the cyclicity of the star product ("leaving out one star"), see equation (3.28) for coinciding arguments. So the deformed free field theory would then be exactly the commutative one. Using the deformation to a greater extent at least lets the bilinear part stay the same and therefore also the propagator. This implies that from $n \geqslant 4$, the correlation functions differ from their commutative counterparts. Now let us go over to this first nontrivial noncommutative correlation function, the deformed time-zero 4-point function. It reads

$$\begin{aligned}
\mathfrak{S}_4^\theta(s_1, s_2, s_3, s_4) &= \langle \Omega, \varphi_\theta(s_1)\varphi_\theta(s_2)\varphi_\theta(s_3)\varphi_\theta(s_4)\Omega \rangle \\
&= \langle \Omega, ((\varphi \times_\theta \varphi \times_\theta \varphi \times_\theta \varphi)(s_1, s_2, s_3, s_4))_\theta \, \Omega \rangle \\
&= \langle \Omega, (\varphi \times_\theta \varphi \times_\theta \varphi \times_\theta \varphi)(s_1, s_2, s_3, s_4)\Omega \rangle \, ,
\end{aligned}$$

due to the product property $(\varphi_1)_\theta(\varphi_2)_\theta = (\varphi_1 \times_\theta \varphi_2)_\theta$ and the translational invariance of the vacuum. We will give an account of its derivation in appendix C, while for now, we go on with the result,

$$\begin{aligned}
\mathfrak{S}_4^\theta(s_1, s_2, s_3, s_4) &= C(s_1, s_2)C(s_3, s_4) + C(s_1, s_4)C(s_2, s_3) + \\
&+ \int \mathrm{d}\mu(\underline{p})\, \mathrm{d}\mu(\underline{q})\; \tilde{s}_1(-\underline{q})\tilde{s}_2(-\underline{p})\tilde{s}_3(\underline{q})\tilde{s}_4(\underline{p})\mathrm{e}^{iq\theta p} \, . \qquad (3.29)
\end{aligned}$$

This specific form makes us learn a few things right from looking at it:

1. For every $n \geqslant 4$, the expansion (3.13) gets modified by an integrated Moyal phase.
2. Symmetry cannot hold for arbitrary noncommutative n-point functions. In fact, $\mathfrak{S}_2^\theta = \mathfrak{S}_2$ is the only one fulfilling permutation symmetry of arguments.
3. We know that we have to shrink the symmetry group to E_θ or $\mathcal{P}_\theta$ respectively in order to gain a well-defined theory. All deformed n-point functions will then satisfy the corresponding invariance.

Giving an intermediate summary using shortened notation, we have so far

$$\begin{aligned}
s_k \in \mathscr{S}(\mathbb{R}^{d-1}) \quad &, \quad f_k \in \mathscr{S}(\mathbb{R}^d_+)\,,\ k \in \mathbb{N} \\
\mathfrak{S}_2^\theta(s_1, s_2) = \mathfrak{S}_2(s_1, s_2) \quad &, \quad C(f_1, f_2) = \left\langle \int h_1\, \alpha\varphi_1 \int h_2\, \alpha\varphi_2 \right\rangle \\
\mathfrak{W}_2(f_1, f_2) = \left\langle \int h_1\, \alpha^{\mathcal{M}}\varphi_1 \int h_2\, \alpha^{\mathcal{M}}\varphi_2 \right\rangle \quad &\Rightarrow \quad \mathfrak{W}_2^\theta(f_1, f_2) = \mathfrak{W}_2(f_1, f_2) \, . \qquad (3.30)
\end{aligned}$$

Assuming the time-ordering $t_1 < t_3$, $t_2 < t_4$, we can write the full Euclidean deformed 4-point function in the following way,

$$
\begin{aligned}
&\mathfrak{S}_4^\theta(f_1,f_2,f_3,f_4) = C(f_1,f_2)C(f_3,f_4) + C(f_1,f_4)C(f_2,f_3) + \Delta_4^\theta, \\
\Delta_4^\theta \; &= \int \mathrm{d}t_1\, h_1(t_1)\cdots \mathrm{d}t_4\, h_4(t_4) \int \mathrm{d}\mu(\underline{p})\, \mathrm{d}\mu(\underline{q})\; \mathrm{e}^{\omega_{\underline{p}}(t_2-t_4)+\omega_{\underline{q}}(t_1-t_3)+iq\theta p}\tilde{s}_1(-\underline{q})\tilde{s}_2(-\underline{p})\tilde{s}_3(\underline{q})\tilde{s}_4(\underline{p})\Big|_{t_1<t_3,t_2<t_4} \\
&= \int \mathrm{d}t_1\, h_1(t_1)\cdots \mathrm{d}t_4\, h_4(t_4) \int \frac{\mathrm{d}^d p\, \mathrm{d}^d q\; \mathrm{e}^{-ip_0(t_2-t_4)-iq_0(t_1-t_3)+iq\theta p}}{(p^2+m^2)(q^2+m^2)} \tilde{s}_1(-\underline{q})\tilde{s}_2(-\underline{p})\tilde{s}_3(\underline{q})\tilde{s}_4(\underline{p}) \\
&= \int \frac{\mathrm{d}^d p\, \mathrm{d}^d q\; \mathrm{e}^{iq\theta p}}{(p^2+m^2)(q^2+m^2)} \tilde{h}_1(-q_0)\tilde{h}_2(-p_0)\tilde{h}_3(q_0)\tilde{h}_4(p_0)\tilde{s}_1(-\underline{q})\tilde{s}_2(-\underline{p})\tilde{s}_3(\underline{q})\tilde{s}_4(\underline{p}) \\
&= \int \frac{\mathrm{d}^d p\, \mathrm{d}^d q}{(p^2+m^2)(q^2+m^2)} \tilde{f}_1(-q)\tilde{f}_2(-p)\tilde{f}_3(q)\tilde{f}_4(p)\, \mathrm{e}^{iq\theta p}\;, \qquad (3.31)
\end{aligned}
$$

as one can check by inserting suitable $V(t)\tilde{s}$ for $\tilde{s}$ and applying the residue theorem "in the other direction". The time-ordering condition has to be satisfied in order to make possible a representation by means of exponential factors $\mathrm{e}^{-\omega_{\underline{p}}\Delta t}$. After the reverse application of the residue theorem, one can drop any condition on time-ordering again. Equivalently, one could define the particular n-point function as limit of a finite time-integration, since the transition to the familiar form of $\mathfrak{S}_4^\theta$ is done via a p_0-integration only.

In complete analogy to the Euclidean case, one finds for the deformed Minkowski 4-point function

$$
\begin{aligned}
&\mathfrak{W}_4^\theta(f_1,f_2,f_3,f_4) = D(f_1,f_2)D(f_3,f_4) + D(f_1,f_4)D(f_2,f_3) + D_4^\theta, \\
D_4^\theta \; &= \lim_{\epsilon\to 0}\int \frac{i\,\mathrm{d}^d p\, \mathrm{d}^d q}{(p_0^2-\underline{p}^2-m^2)(q_0^2-\underline{q}^2-m^2)+i\epsilon} \tilde{f}_1(-q)\tilde{f}_2(-p)\tilde{f}_3(q)\tilde{f}_4(p)\, \mathrm{e}^{iq\theta p}\;. \qquad (3.32)
\end{aligned}
$$

3.4.2 Moving on to Higher Orders

Each (Minkowski or Euclidean) n-point function can be treated via its corresponding (time-zero) distributions, i.e.

$$
\mathfrak{S}_n(s_n) \;=\; \int d^{d-1}x_1\cdots d^{d-1}x_n\, \mathfrak{S}_n(\underline{x}_1,...,\underline{x}_n)\, s_n(\underline{x}_1,...,\underline{x}_n)\;,\; s_n \in \mathscr{S}(\mathbb{R}^{(d-1)n})
$$

Note that by the kernel theorem, it suffices to work with tensor products of $\mathscr{S}(\mathbb{R}^{d-1})$-functions on the time-zero hyperplane, which makes it possible to directly define the deformed time-zero distributions as

$$\mathfrak{S}_n^\theta(s_1,\ldots,s_n) \;:=\; \int \mathrm{d}^{d-1}x_1\cdots\mathrm{d}^{d-1}x_n\,\mathfrak{S}_n(\underline{x}_1,...,\underline{x}_n)\,(s_1\otimes_\theta\cdots\otimes_\theta s_n)(\underline{x}_1,...,\underline{x}_n)\;. \tag{3.33}$$

We can make use of the simple form (3.27) of the Moyal tensor product in momentum space

$$\mathfrak{S}_n^\theta(s_1,\ldots,s_n) \;=\; \int d^{d-1}p_1\cdots d^{d-1}p_n\,\widetilde{\mathfrak{S}}_n(-\underline{p}_1,...,-\underline{p}_n)\,\widetilde{s}_1(\underline{p}_1)\cdots\widetilde{s}_n(\underline{p}_n)\prod_{1\leqslant l<m\leqslant n}\mathrm{e}^{\frac{i}{2}p_l\theta p_m} \tag{3.34}$$

to deduce a surprisingly simple relation between the commutative and the deformed n-point functions,

$$\widetilde{\mathfrak{S}}_n^\theta(\underline{p}_1,...,\underline{p}_n)=\widetilde{\mathfrak{S}}_n(\underline{p}_1,...,\underline{p}_n)\prod_{1\leqslant l<m\leqslant n}\mathrm{e}^{\frac{i}{2}p_l\theta p_m}\;. \tag{3.35}$$

This general form leads to another way of deriving the noncommutative four-point correlator. Let us write C_0 for the free covariance restricted to the time-zero plane. Bearing in mind that the Fourier transformed free covariance is $\widetilde{C}_0(\underline{p}_1,\underline{p}_2)=\frac{\delta(\underline{p}_1+\underline{p}_2)}{2\omega_{\underline{p}_1}}$, we obtain by setting $n=4$ in formula (3.35):

$$\widetilde{\mathfrak{S}}_4^\theta(\underline{p}_1,...,\underline{p}_4)=\left(\frac{\delta(\underline{p}_1+\underline{p}_2)\delta(\underline{p}_3+\underline{p}_4)}{4\omega_{\underline{p}_1}\omega_{\underline{p}_3}}+\frac{\delta(\underline{p}_1+\underline{p}_3)\delta(\underline{p}_2+\underline{p}_4)}{4\omega_{\underline{p}_1}\omega_{\underline{p}_2}}+\frac{\delta(\underline{p}_1+\underline{p}_4)\delta(\underline{p}_2+\underline{p}_3)}{4\omega_{\underline{p}_1}\omega_{\underline{p}_2}}\right)\mathfrak{D}_4^\theta(\underline{p})\;.$$

Due to the specific combinations of the delta-distributions, all Moyal phases vanish except for two which sum up in the second term. Thus one is left with the distributional kernel of the concrete form (3.29).

Returning to the case of general n-point functions we just have to build the Fourier transform of (3.13) to obtain

$$\widetilde{\mathfrak{S}}_n^\theta(\underline{p}_1,...,\underline{p}_n)=\sum_{k=1}^{n-1}\widetilde{\mathfrak{S}}_{n-2}(\underline{p}_1,\ldots,\widehat{\underline{p}}_k,\ldots,\underline{p}_{n-1})\widetilde{C}_0(\underline{p}_k,\underline{p}_n)\prod_{1\leqslant l<m\leqslant n}\mathrm{e}^{\frac{i}{2}p_l\theta p_m}\;, \tag{3.36}$$

which happens to be a comfortably simple formula linking deformed and undeformed Schwinger functions in momentum space. Still we have to admit that the phase factors of the form $\mathrm{e}^{ip\theta q}$ cannot be absorbed into the corresponding two-point functions of the reduction formula. This means that the vacuum representation of the Moyal deformed theory does no longer incorporate a quasi-free state.

3.5 Application of the Algebraic Framework

In this section, we will have a look at the consequences of the algebraic setting treated in Ch. 2 when applied to the deformed Euclidean scalar field. We have already worked out all the necessary relations, so we just have to bring things together.

Let us collect what we have done with the noncommutative free field two- and four-point functions up until now. Starting from Fock space $\mathcal{F}$ we constructed the correlation functions in terms of Schwartz functions on the "time-zero hyperplane" $\mathbb{R}^{d-1}$. Preparing the ground for a specific representation α of the Euclidean group $E(d) = \mathbb{R}^d \rtimes O(d)$, we gained the correct Euclidean two- and four-point functions by building the time-zero correlation function of the "dynamically integrated" field $\phi^{\mathcal{E}}(f) = \sum_{m\in\mathbb{N}} \int \mathrm{d}t\, h^m(t)\alpha_t\varphi(s^m)$ for Schwartz functions h^m defined such that $f = \sum_m h^m \otimes s^m$. And, most importantly, in the same manner we obtained the Minkowski propagator and four-point function by time-zero correlations of fields acted on with the analytically continued group action $\alpha^{\mathcal{M}}$ and tested with suitable time-dependent functions. So, starting with the time-slice, we get everything we need by acting with the dynamics. These results correspond to the free field manifestation of the arrow $\mathcal{E} \to \mathcal{M}$ in Fig. 2.9.

Lemma 11. *For the Euclidean free scalar field, given by the operator $\phi^{\mathcal{E}}$ on $\mathcal{F}$, the Wick rotation of Schlingemann, c.f.* [Sch99]*, can be explicitly performed and results in the free field $\phi^{\mathcal{M}}$ on Minkowski space-time.*

Proof. According to *2.* of Prop. 10, φ is a well-defined representation of the time-zero algebra $\mathcal{E}_0$ on $\mathcal{F}$. Starting from the free Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ as given in Sec. 3.1, we are able to obtain the time-zero algebra $\mathcal{E}_0$ by performing the limit

$$\mathscr{S}(\mathbb{R}^d) \ni f = \sum_{m\in M} h^m \otimes s^m \to \sum_{m\in M} \delta \otimes s^m = \delta \otimes s \ ,$$

understood to take place as in Lemma 10. By (3.26), the two-point function of the Minkowski field $\phi^{\mathcal{M}}$, introduced upon $\mathcal{E}_0$ in Def. 17, equals the Feynman propagator. The state ω on $\mathcal{F}$ is given by $\omega(\underline{f}) := \langle [1]_\sigma, \underline{f}\,[1]_\sigma \rangle$ for $\underline{f} \in \underline{\mathscr{S}}$ and is quasi-free. Thus, each n-point function can be exactly determined from the propagator by an expansion of the form (3.13). The set of correlation functions in turn characterizes the free field on Minkowski space-time. $\square$

So, the free scalar field fits well into our framework. This was mainly the content of Secs. 3.1, 3.2 and 3.3. If we deform the theory by a degenerate skew-symmetric matrix θ, we are able to draw the analog conclusion:

Proposition 12. *The Euclidean free scalar field may be deformed in terms of warped convolutions. The resulting theory, given by the operators $\phi^{\mathcal{E}}_\theta$ on $\mathcal{F}$, where θ is of the form* (2.20)*, may be analytically continued to the deformed free field on Minkowski space-time.*

Proof. Given a commutative Euclidean n-point function $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^d)$, we can deform it in terms of warped convolutions, is equivalent to the deformation in terms of Moyal tensor products. We have discussed this in Sec. 3.4. Due to relations (3.27), the deformed n-point functions differ by integrated Moyal phases from their commutative counterparts.

In the proof of Lemma 11 we referenced the continuation of the two-point function. As Moyal deformed propagators coincide with the corresponding commutative ones, the conclusion carries over to the case at hand. Next we check the covariance properties. From the explicit form of the two-point function,

$$\mathfrak{W}^\theta_2(x_1, x_2) = \mathfrak{W}_2(x_1, x_2) = \int \frac{\mathrm{d}^{d-1}p}{2\omega_{\underline{p}}}\, \mathrm{e}^{i\omega_{\underline{p}}(x_1^0 - x_2^0) - i\underline{p}(\underline{x}_1 - \underline{x}_2)}\ , \tag{3.37}$$

one can read off the complete Poincaré invariance. The direct allocation to the Lorentzian case of the expansions (3.13) and (3.36) reveals the $\mathcal{P}_\theta(d)$-invariance of arbitrary $\mathfrak{W}^\theta_n$, the restriction becoming necessary due to the emergence of the noncommutative phase factors. In almost the same manner, the spectrum condition and positivity of the noncommutative n-point functions follow on Minkowski space-time. Indeed, the fact that noncommutativity restricts the symmetry group, the spectrum of the four momentum operators has to be generalized. The single remaining boost direction together with full translational invariance leads to

$$\operatorname{spec} U \subset \{p \in \mathbb{R}^d \mid p_0 > |p_1|\} =: Y \ , \tag{3.38}$$

where U denotes the analytically continued translation operator,

$$U : \mathbb{R}^d \ni (t, \underline{x}) \mapsto \mathrm{e}^{itH} \mathrm{e}^{i\underline{x}\,\underline{P}} \ .$$

The adjoint action of $\mathcal{P}_\theta(d)$ is denoted by $\alpha^{\theta,\mathcal{M}}$ and acts on $\mathcal{F}$ as described in (3.25). The deformed field and observable net are defined in complete analogy to Def. 17, with φ exchanged by φ_θ and $\alpha^{\mathcal{M}}$ by $\alpha^{\theta,\mathcal{M}}$. Thus, in terms of n-point functions, we obtain the deformed Lorentzian theory $(\mathcal{M}_\theta, \mathscr{O}, \alpha^{\theta,\mathcal{M}}, \omega^\theta)$ from the deformed Euclidean field theory $(\mathcal{E}_\theta, \mathscr{O}, \alpha^\theta, \sigma^\theta)$. $\square$

A lesson from the free field The operative point is that we can apply the unitary representation of the reduced Poincaré group on the time-zero correlators. That is to say, the Wick rotation of theories fulfilling the time-zero condition largely relies on the analytic continuation of their symmetry group representations. In (3.26) we showed the consistency of this approach when it comes to the application on the Euclidean free field. Clearly, the deformation is unaffected by the action of the translation group; in particular, the phase factors $\mathfrak{D}_n^\theta$ are.

We are ready to give an interim result. Given a noncommutativity matrix θ, degenerate according to the commutative-time case, the Moyal deformation effects the rise of phase factors $\mathfrak{D}_n^\theta$ in the n-point functions of any theory. Furthermore, if a certain theory is known to allow time-zero fields, we obtain the general time-zero correlator

$$\underline{\mathfrak{S}}_n^\theta(s_n^{(1)}, \ldots, s_n^{(n)}) \;=\; \int \mathrm{d}^{d-1}p_1 \cdots \mathrm{d}^{d-1}p_n \, \tilde{\underline{\mathfrak{S}}}_n(-\underline{p}_1, ..., -\underline{p}_n) \, \tilde{s}_n^{(1)}(\underline{p}_1) \cdots \tilde{s}_n^{(n)}(\underline{p}_n) \prod_{1 \leqslant l < m \leqslant n} \mathrm{e}^{\frac{i}{2} p_l \theta p_m} \ .$$

By "allowing time-zero fields" the well-defined evaluation of the field distributions on elements of the form $(f_0, \delta \otimes s_1, (\delta \otimes s_2^{(1)}) \otimes (\delta \otimes s_2^{(2)}), \ldots)$ is meant, see Lemma 10. The set of distributions $\underline{\mathfrak{S}}_n^\theta \in \mathscr{S}'(\mathbb{R}^{(d-1)n})$ also determines the time-zero content of the corresponding Lorentzian theory. Since the residual symmetry group commutes with θ (see (2.4)), the time-dependent part responsible for the reconstruction also will. Moreover, it will keep this property after analytical continuation. Though it cannot directly be adducted in terms of n-point functions, Prop. 6 states the persistence of the time-zero condition after deformation. Thus, also in more general theories, the image of the time-zero content under the analytically continued symmetry group representation will be a good candidate for the theory on Minkowski space-time.

3.6 Relation to the Standard Approach

Momentarily leaving aside the advantages of directly visualizing the algebraic Wick rotation at the free field scenario, the setting just described is not absolutely practical. One is rather contemplating a full Euclidean or Lorentzian theory and highly interested in implications on the analytic continuation without the intermediate step of restricting to the time-zero plane.

Nevertheless, we want to point out that many constructed quantum field theory models allow for such a restriction. The $P(\phi)_2$ models [GJ87] and the ϕ^4-model in 3 dimensions [DF77] can be analyzed in this way, for example. For general theories defined by its set of Schwinger functions, heavy use is made of the full Euclidean or Minkowski invariance and microcausality, usually. We do not wish to use theses axioms as they will not survive the deformation.
The two famous papers by Osterwalder and Schrader were the first to give necessary and sufficient conditions for a set of Schwinger functions to define a Wightman theory. The linear growth condition there was designed to be a substitute for demanding the existence of a holomorphic Fourier transform directly. As a consequence, it is easier to validate in concrete models, but still requires some effort.
Before finishing the axioms for Euclidean Green's functions, lecture notes of Osterwalder [Ost73] concerning this subject have been published. There, Schwinger functions satisfying a somewhat stronger growth condition are smeared out in the spatial variables and analytically continued to Wightman functions just in terms of the time-coordinates. We can adjust this approach to our setting, mainly because $SO(d)$-invariance is not explicitly used to perform the continuation when starting from a Euclidean theory. Before we treat the Euclidean n-point functions in higher generality, we will visualize this alternative approach in terms of the free field.
Nevertheless, we are assuming that *time components are completely ordered*. This means that we contemplate Schwartz functions out of $\mathscr{S}_+(\mathbb{R}^{nd})$ to be our test functions, see (3.6). In doing so, we procure the various representations of n-point distributions.

Continuation of the propagator Our starting point is again the (commutative as well as noncommutative) two-point function of the free field, written in the difference variable ξ:

$$S_1(\xi) := \int \frac{\mathrm{d}^d p}{(2\pi)^d (p^2 + m^2)} \mathrm{e}^{ip_0\xi_0 + i\underline{p}\underline{\xi}} = \int \mathrm{d}\underline{\mu}\, \mathrm{e}^{-|\xi_0|\omega_{\underline{p}} + i\underline{p}\underline{\xi}} \ .$$

We define this distribution on $\mathscr{S}_+(\mathbb{R}^d) \equiv \mathscr{S}(\mathbb{R}^d_+)$, so we can drop the modulus in the exponent. Furthermore, we write down the candidate for the analytical continuation of the distribution S_1:

$$S_1(\zeta^0|s) = \int \mathrm{d}\underline{\mu}\, \mathrm{e}^{-\zeta^0\omega_{\underline{p}}}\, \widetilde{s}(\underline{p}) \ , \ \zeta^0 \in \mathbb{C} \ , \tag{3.39}$$

and this notation emphasizes the spatial smearing with the Schwartz function $s \in \mathscr{S}(\mathbb{R}^{d-1})$.

Lemma 12. *For fixed $s \in \mathscr{S}(\mathbb{R}^{d-1})$, the distribution $S_1(\xi^0|s)$ is the restriction to the positive real half axis of a complex function $S_1(\zeta^0|s)$, analytic in $\mathbb{C}_+$. The Fourier-Laplace transform $\widetilde{W}_1$ of S_1 exists and is the Fourier transform of the Wightman two-point function $\mathfrak{W}_2$.*

Proof. One realizes that the domain of analyticity is indeed $\mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\}$. Hence, given $s \in \mathscr{S}(\mathbb{R}^{d-1})$, $\zeta^0 \mapsto S_1(\zeta^0|s)$ is an analytic function in $\mathbb{C}_+$. What remains to be checked are suitable estimates on S_1. In our simple case, writing $\zeta^0 = \tau + it$, we are left with

$$\begin{aligned}
\sup_{\tau>0} \int |S_1(\tau + it|s)|^2\, \mathrm{d}t \ &= \ \sup_{\tau>0} \int \left(\iint \mathrm{d}\underline{\mu}(p)\, \mathrm{d}\underline{\mu}(q)\ \mathrm{e}^{-\tau(\omega_{\underline{p}}+\omega_{\underline{q}}) + it(-\omega_{\underline{p}}+\omega_{\underline{q}})}\ \widetilde{s}(\underline{p})\widetilde{s}^*(\underline{q}) \right) \mathrm{d}t \\
&\leqslant \ \int \left(\iint \mathrm{d}\underline{\mu}(p)\, \mathrm{d}\underline{\mu}(q)\ \mathrm{e}^{it(-\omega_{\underline{p}}+\omega_{\underline{q}})}\ \widetilde{s}(\underline{p})\widetilde{s}^*(\underline{q}) \right) \mathrm{d}t \\
&= \ \iint \left| \mathrm{d}\underline{\mu}(p)\, \mathrm{d}\underline{\mu}(q)\ \delta(-\omega_{\underline{p}} + \omega_{\underline{q}})\ \widetilde{s}(\underline{p})\widetilde{s}^*(\underline{q}) \right| \ .
\end{aligned}$$

The evaluation of the delta distribution causes the absolute value of the vectors $\underline{p}$ and $\underline{q}$ to coincide. Thus, they are related by an orthogonal matrix $R \in SO(d-1)$. This further implies,

$$\sup_{\tau>0} \int |S_1(\tau + it|s)|^2\, \mathrm{d}t \ \leqslant \ \int \left| \frac{\mathrm{d}\underline{\mu}(p)}{2\omega_{\underline{p}}}\, \widetilde{s}(\underline{p})\widetilde{s}^*(\underline{Rp}) \right| =: \int \left| \frac{\mathrm{d}\underline{\mu}(p)}{2\omega_{\underline{p}}}\, \widetilde{s}(\underline{p})\widetilde{r}^*(\underline{p}) \right| \ ,$$

for $\widetilde{r}(\underline{p}) := \widetilde{s}(\underline{Rp})$. By the Cauchy-Schwarz inequality, we finally deduce

$$\begin{aligned}
\sup_{\tau>0} \int |S_1(\tau+it|s)|^2 \, \mathrm{d}t &\leqslant \int \left| \frac{\mathrm{d}^{d-1}\underline{p}}{4\omega_{\underline{p}}^2} \, \widetilde{s}(\underline{p}) \widetilde{r}^*(\underline{p}) \right| \\
&\leqslant \left(\int \mathrm{d}^{d-1}\underline{p} \left| \frac{\widetilde{s}(\underline{p})}{2\omega_{\underline{p}}} \right|^2 \right)^{1/2} \left(\int \mathrm{d}^{d-1}\underline{q} \left| \frac{\widetilde{r}(\underline{q})}{2\omega_{\underline{q}}} \right|^2 \right)^{1/2} \\
&= \int \frac{\mathrm{d}^{d-1}\underline{p}}{4\omega_{\underline{p}}^2} \, |\widetilde{s}(\underline{p})|^2 \ ,
\end{aligned}$$

which is square of the L^2-norm of the function $\underline{p} \mapsto \widetilde{s}(\underline{p})/(2\omega_{\underline{p}})$. Hence by the Paley-Wiener Theorem 10 (in the appendix) it follows that there is a $\widetilde{W}_1(q^0|s) \in L^2(\mathbb{R}_+)$ such that

$$S_1(\zeta^0|s) = \int_0^\infty \widetilde{W}_1(q^0|s) \, \mathrm{e}^{-q^0\zeta^0} \, \mathrm{d}q^0 = \iint_{\mathbb{R}_+^d} \mathrm{d}^d q \, \widetilde{W}_1(q^0,\underline{q}) \widetilde{s}(\underline{q}) \, \mathrm{e}^{-q^0\zeta^0} \ .$$

The latter equation follows from the concrete form (3.39) of S_1. As $\mathrm{e}^{-q^0\zeta^0}$ for $\zeta^0 \in \mathbb{C}_+$ is an exponential damping times a phase, it leaves $\mathscr{S}(\mathbb{R}_+)$ invariant. This shows the existence of a distribution $\widetilde{W}_1 \in \mathscr{S}'(\mathbb{R}_+^d)$, such that we can remove the smearing in the spatial coordinates to see that

$$S_1(\xi) = \int \mathrm{d}^d q \, \mathrm{e}^{-\xi^0 q^0 - i\underline{\xi}\underline{q}} \, \widetilde{W}_1(q) \ . \tag{3.40}$$

The corresponding Wightman distribution is then defined to be

$$\mathfrak{W}_2(x,y) := \int \mathrm{d}^d q \, \mathrm{e}^{iq^0(x^0-y^0)+i\underline{q}(\underline{x}-\underline{y})} \, \widetilde{W}_1(q) \ .$$

Again, in this special case, the explicit forms of the functions involved are obvious: $\widetilde{W}_1(q)$ is nothing else than $(q_0^2 - \underline{q}^2 - m^2 + i\epsilon)^{-1}$, as one can verify by integrating over q_0 in (3.40). □

So, as a next step we have to check whether the approach described and calculated for the two-point function generalizes to correlation functions of higher order in the Rieffel deformed free field case.

Writing the expansion (3.13) in difference variables, we obtain

$$\begin{aligned} S_n(\xi_1,\ldots,\xi_n) &= S_{n-2}(\xi_2,\ldots,\xi_{n-1})S_1(\xi_1+\ldots+\xi_n) + S_{n-2}(\xi_1,\ldots,\xi_{n-2})S_1(\xi_n) \\ &+ \sum_{k=2}^{n-1} S_{n-2}(\xi_1,\ldots,\xi_{k-2},\xi_{k-1}+\xi_k,\xi_{k+1},\ldots,\xi_{n-1})S_1(\xi_k+\ldots+\xi_n)\ , \end{aligned} \tag{3.41}$$

so the effect of leaving out one variable x_k is to contract ξ_{k-1} and ξ_k into a sum in the resulting distribution in difference variables. Furthermore, we observe that the noncommutative deformation has the consequence of inserting Moyal phases of the form $\mathrm{e}^{iq\theta p}$ into the distributions' integral representations. More generally, we have the following

Lemma 13. *Let $f \in \mathscr{S}(\mathbb{R}^{dn})$ and let $\mathfrak{D}_n^\theta(p) := \prod_{1\leqslant k<l}^{n} e^{ip_k\theta p_l}$. Then $\widetilde{f}_\theta := \widetilde{f}\mathfrak{D}_n^\theta \in \mathscr{S}(\mathbb{R}^{dn})$ and, using $f_\theta := \mathcal{F}^{-1}[\widetilde{f}_\theta]$, for the Schwinger functions $\mathfrak{S}_n$ it follows*

$$\mathfrak{S}_n^\theta(f) = \mathfrak{S}_n(f_\theta)\ . \tag{3.42}$$

Proof. It is well-known that Schwartz space is invariant under multiplication with phase factors, so $f_\theta \in \mathscr{S}(\mathbb{R}^{dn})$ for any $f \in \mathscr{S}(\mathbb{R}^{dn})$. For the second part we calculate

$$\begin{aligned} (\phi^{\mathcal{E}} \times_\theta \phi^{\mathcal{E}})(f_1\otimes f_2) &= \int \mathrm{d}v\,\mathrm{d}k\ \mathrm{e}^{ikv}\alpha_{\theta k/2}\phi^{\mathcal{E}}(f_1)\alpha_v\phi^{\mathcal{E}}(f_2) \\ = \quad & \int \mathrm{d}v\,\mathrm{d}k\,\mathrm{d}x\,\mathrm{d}y\ \mathrm{e}^{ikv}\phi^{\mathcal{E}}(x)f_1(x-\theta k/2)\phi^{\mathcal{E}}(y)f_2(y-v) \\ = \quad & \int \mathrm{d}x\,\mathrm{d}y\ \phi^{\mathcal{E}}(x)\phi^{\mathcal{E}}(y)(f_1\otimes_\theta f_2)(x,y) = (\phi^{\mathcal{E}}\otimes\phi^{\mathcal{E}})(f_1\otimes_\theta f_2)\ . \end{aligned}$$

Inductively, this is correct for arbitrary Schwartz functions of tensor form. We can take advantage of Prop. 11 now. Indeed, by linearity of $\phi^{\mathcal{E}}$ and continuity of the functional σ, it therefore follows that $\langle\Omega,(\phi^{\mathcal{E}}\times_\theta\cdots\times_\theta\phi^{\mathcal{E}})(f)\Omega\rangle = \langle\Omega,(\phi^{\mathcal{E}}\otimes\cdots\otimes\phi^{\mathcal{E}})(f_\theta)\Omega\rangle$ holds for arbitrary $f \in \mathscr{S}(\mathbb{R}^{dn})$. By the definition of $\mathfrak{S}_n^\theta(f)$, relation (3.42) comes about.

□

Thus, by defining a new smearing function smearing function $\widetilde{s}_\theta(\underline{p}_1,\ldots,\underline{p}_n)$ we can go on like in the commutative case when we are focusing on the time-dependence.

Remark 12. The absorption of noncommutative phase factors into the spatial smearing function is only possible due to the time-independence of the noncommutativity matrix θ. Thus this distinct way of analytically continuing just the time-dependent part of the Schwinger distributions will get massively changed when considering space-time noncommutativity.
On the other hand, this approach is introduced for its greater simplicity and it may happen that the simultaneous continuation of the distributions in all their components is still possible.

In order to tackle the next step, we realize that

$$\widetilde{S}_n(q_1, \ldots, q_n) = \widetilde{\mathfrak{S}}_{n+1}(p_1, \ldots, p_n, p_{n+1}) \ , \ q_k := \sum_{j=1}^{k} p_j \ .$$

Implementing this in (3.42) and introducing the following notion for $g \in \mathscr{S}'(\mathbb{R}^{(d-1)(n+m-1)})$,

$$S_{n+m-1}(\xi_1^0, \ldots, \xi_n^0 | g) := \int S_{n+m-1}(\xi_1^0, \ldots, \xi_n^0) g(\underline{\xi}_1, \ldots, \underline{\xi}_n) \mathrm{d}^{(d-1)n} \xi \ , \tag{3.43}$$

we obtain,

$$S_n^\theta(\xi_1^0, \ldots, \xi_n^0 | g) = S_n(\xi_1^0, \ldots, \xi_n^0 | g D_n^\theta) \ , \ D_n^\theta(q) := \prod_{j=1}^{n-1} \mathrm{e}^{i q_j \theta q_{j+1}} \ .$$

Proposition 13. *The set $\{\mathfrak{S}_n^\theta\}_{n \in \mathbb{N}}$ of Schwinger distributions of the (commutative-time) deformed Euclidean free field can be analytically continued to distributions $\{\mathfrak{W}_n^\theta\}_{n \in \mathbb{N}}$, satisfying $\mathcal{P}_\theta(d)$-invariance and positivity.*

Proof. We find that $(\zeta_1^0, \ldots, \zeta_n^0) \mapsto S_n^\theta(\zeta_1^0, \ldots, \zeta_n^0 | g)$ defines a complex function analytic in $\mathbb{C}_+^n$ and a distribution in the spatial variables. This can be appreciated inductively: We saw that S_1 defines an analytic function in $\mathbb{C}_+$. Now assume that S_{n-2} is a complex function analytic in $\mathbb{C}_+^{n-2}$. From the first summand in (3.41), we deduce $\zeta_k^0 > 0$ for $k = 2, \ldots, n-1$ by the latter assumption. Moreover, the second summand has the consequence of $\zeta_n^0 > 0$. The appearing of S_1 in the first term finally delivers $\sum_{j=1}^{n} \zeta_j^0 > 0$, from which it follows $\zeta_1^0 > 0$. Hence, we really have analyticity of S_n on $\mathbb{C}_+^n$.

The Schwinger function $S_n^\theta(\xi_1^0,\ldots,\xi_n^0|g)$ decomposes into a sum of $n!!$ products of propagators and each of these products consists of $(n+1)/2$ factors. We deal with complex functions in time variables and smeared out in the spatial variables.

The absolute square of an N-fold product of functions S_1 will be

$$\begin{aligned}
\int \left| \prod_{l=1}^{N} S_1\left(i\sum_{k_l=1}^{n} v_{k_l}t_{k_l}\right)\right|^2 \mathrm{d}^n t &= \int \prod_{l=1}^{N} \frac{\mathrm{d}^{(d-1)n}p_l\,\mathrm{d}^{(d-1)n}p_l}{(2\omega_{\underline{p}_l})(2\omega_{\underline{q}_l})} \exp\left(-i\sum_{k_l=1}^{n} v_{k_l}t_{k_l}(\omega_{\underline{p}_l}-\omega_{\underline{q}_l})\right)\mathrm{d}^n t \\
&= \iint \mathrm{d}^N\mu(p)\,\mathrm{d}^N\mu(q)\,\exp\left(-itA\nu\right)\mathrm{d}^n t\ ,
\end{aligned}$$

where $t=(t_1,\ldots,t_n)$, $\nu^\mathsf{T}=(\omega_{\underline{p}_1}-\omega_{\underline{q}_1},\ldots,\omega_{\underline{p}_N}-\omega_{\underline{q}_N})$, whereas A is the corresponding coefficient matrix, consisting just of entries equal to zero or one. If the product is part of a Schwinger function S_n, then n delta-functions face $2N=n+1$ momentum integrations. So, there remains exactly one integral of $(f(\omega_{\underline{p}}))^{-1}$ for one $\underline{p}\in\mathbb{R}^{d-1}$ and a non-vanishing function f. Thus, smeared with a Schwartz function $g\in\mathscr{S}(\mathbb{R}^{n(d-1)})$, the absolute square of the product will fulfill

$$\exists\, C>0:\quad \iint \left| \prod_{l=1}^{N} S_1\left(i\sum_{k_l=1}^{n} v_{k_l}t_{k_l}\right)\right|^2 g(p)\mathrm{d}^{(d-1)n}p\,\mathrm{d}^n t \leqslant C\|g\|_\infty \quad \forall g\in\mathscr{S}(\mathbb{R}^{n(d-1)})\ .$$

For the mixed terms we can argue analogously to infer the validity of the following estimate:

$$\begin{aligned}
\exists K>0:\qquad & \sup_{\substack{\tau_k>0\\ k=1,\ldots,n}} \int \left|S_n^\theta(\tau_1+it_1,\ldots,\tau_n+it_n|g)\right|^2 \mathrm{d}^n t \leqslant \int \left|S_n^\theta(it_1,\ldots,it_n|g)\right|^2 \mathrm{d}^n t \\
&\leqslant \frac{(n+1)(n!!)^2}{2} K\|g\|_\infty \quad \forall g\in\mathscr{S}(\mathbb{R}^{n(d-1)})\ .
\end{aligned}$$

This assures by the Paley-Wiener Theorem that for any noncommutative Schwinger function S_n^θ there exist distributions $\widetilde{W}_n^\theta\in\mathscr{S}'(\overline{\mathbb{R}}_+^{dn})$ being their holomorphic Fourier transforms,

$$S_n^\theta(\xi_1,\ldots,\xi_n) = \int \mathrm{d}^{dn}q\ \mathrm{e}^{-\sum\limits_{k=1}^{n}(\xi_k^0 q_k^0 + i\underline{\xi}_k\underline{q}_k)}\,\widetilde{W}_n^\theta(q_1,\ldots,q_n)\ .$$

Even for more general Schwinger functions we have the following at our disposal:

Lemma 14. *Let $S_n \in \mathscr{S}'(\mathbb{R}^{dn})$ be the n-point Schwinger distributions in difference variables of a Euclidean quantum field theory and let $\widetilde{W}_n$ denote its holomorphic Fourier transform. Then, for the holomorphic Fourier transform $\widetilde{W}^\theta_n$ of the deformed Schwinger function S^θ_n we have,*

$$\widetilde{W}^\theta_n = \widetilde{W}_n D^\theta_n \quad \forall\, n \in \mathbb{N} \ .$$

Proof of the Lemma. On the one hand, using the spatial Fourier transform defined in (3.9), we have $\widehat{S}^\theta_n = \widehat{S}_n D^\theta_n$, as D^θ_n depends only on the spatial variables. Hence we calculate,

$$\begin{aligned}
\widetilde{W}^\theta_n(q_1,\ldots,q_n) &:= (2\pi)^{-(d-1)n/2} \int \mathrm{d}^{dn}\xi \ \mathrm{e}^{-q^0\xi^0 - i\underline{q}\underline{\xi}}\, S^\theta_n(\xi_1,\ldots,\xi_n) \\
&= \int \mathrm{d}^n\xi^0 \ \mathrm{e}^{-q^0\xi^0}\, \widehat{S}^\theta_n((\xi^0_1,\underline{q}_1),\ldots,(\xi^0_n,\underline{q}_n)) = \\
&= \int \mathrm{d}^n\xi^0 \ \mathrm{e}^{-q^0\xi^0}\, \widehat{S}_n((\xi^0_1,\underline{q}_1),\ldots,(\xi^0_n,\underline{q}_n))\, \mathrm{e}^{i\sum_{k=1}^{n-1} q_k\theta q_{k+1}} \\
&= \widetilde{W}_n(q_1,\ldots,q_n)\, \mathrm{e}^{i\sum_{k=1}^{n-1} q_k\theta q_{k+1}} = \widetilde{W}_n(q_1,\ldots,q_n) D^\theta_n(q) \ ,
\end{aligned}$$

and the claim is proved. □

Continuing the proof of Prop. 13, we recognize that by the latter lemma, we have

$$S^\theta_n(\xi_1,\ldots,\xi_n) = \int \mathrm{d}^{dn}q \ \mathrm{e}^{-\sum_{k=1}^{n}(\xi^0_k q^0_k + i\underline{\xi}_k\underline{q}_k)}\ \mathrm{e}^{\frac{i}{2}\sum_{j=1}^{n-1} q_j\theta q_{j+1}}\ \widetilde{W}_n(q_1,\ldots,q_n) \ . \tag{3.44}$$

Consistently with the literature, we define the Wightman distributions to be

$$\begin{aligned}
\mathfrak{W}^\theta_n(x_1,\ldots,x_n) &:= \int \mathrm{d}^{d(n-1)}q \ \mathrm{e}^{i\sum_{k=1}^{n-1} q_k(x_{k+1}-x_k)}\ \widetilde{W}^\theta_{n-1}(q_1,\ldots,q_{n-1}) \\
&= \int \mathrm{d}^{d(n-1)}q \ \mathrm{e}^{i\sum_{k=1}^{n-1} q_k(x_{k+1}-x_k)}\ \mathrm{e}^{\frac{i}{2}\sum_{j=1}^{n-2} q_j\theta q_{j+1}}\ \widetilde{W}_{n-1}(q_1,\ldots,q_{n-1}) \ .
\end{aligned} \tag{3.45}$$

We are now in the position to inspect the validity of (deformed) Wightman axioms. Nevertheless, we postpone this consideration to the next section, where we treat consequences of this approach to Schwinger functions of more general theories. □

3.7 Implications for General Schwinger Functions

On the basis of the free field we saw in the preceding sections that the analytic continuation of Schwinger functions belonging to the noncommutative free field can be arranged. Now the task is to investigate the generalization to arbitrary Euclidean theories. What we should learn from these considerations at least is that obtaining a Minkowski theory does not rely on full Euclidean invariance or a certain notion of locality. It is exactly this need which confuses the issue of generalizing the continuation in the other direction, *i.e.* gaining a Euclidean version of a deformed theory on Minkowski space-time.

There are several analyticity properties of Euclidean n-point functions sufficient for the well-definition of a Wightman theory [OS75]. The one which together with all the other axioms gives necessary and sufficient conditions for the existence of a Wightman theory is the following:

($E0'$) There exist $s \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, such that for all $f \in \mathscr{S}_0(\mathbb{R}^{dn})$ and $n \in \mathbb{N}$ we have

$$|\mathfrak{S}_n(f)| \leqslant \alpha(n!)^\beta \, |f|_{sn} \ , \tag{3.46}$$

where $|f|_m$ for $m \in \mathbb{N}$ denotes the Schwartz norm

$$\begin{aligned} |f|_m \ &:= \sup_{\substack{x\in\mathbb{R}^{dn} \\ |\alpha|\leqslant m}} \left|(1+x^2)^{m/2}(D^\alpha f)(x)\right| \\ &= \sup_{\substack{x\in\mathbb{R}^{dn} \\ |\alpha|\leqslant m}} \left|(1+(x_1)^2+\ldots+(x_n)^2)^{m/2} D^\alpha f(x_1,\ldots,x_n)\right| \ . \end{aligned} \tag{3.47}$$

This is indeed a norm, since the supremum is taken over all multi-indices of order smaller or equal to m. The sort of nomenclature has its origin in [OS75], where ($E0$) was the designation for the Schwinger functions to be just tempered distributions. It has been shown that demanding ($E0$) is not enough to guarantee for the existence of Wightman functions. As a remark, we mention another condition: that of the Schwinger functions being in the dual of Schwartz space with the topology given by the semi-norms $|f|\check{}_m := |f_\bullet|_m$, where $f_\bullet$ denotes the holomorphic Fourier transform of f on $\mathbb{R}^{dn}_+$. This axiom, called ($\check{E}0$), was shown to be equivalent to the Wightman axioms with the

least effort, but at the same time seems to be most difficult to verify.
For the upcoming considerations we are interested in $(E0')$ for our noncommutative Schwinger functions at first. An equation like (3.34) for time-dependent Schwartz functions can easily be written down

$$\mathfrak{S}_n^\theta(f) = \int \mathrm{d}^{dn}p\, \widetilde{\mathfrak{S}}_n(-p_1, \ldots, -p_n) \widetilde{f}(p_1, \ldots, p_n) \mathfrak{D}_n^\theta(p) \quad , \quad f \in \mathscr{S}(\mathbb{R}^{dn}) \; , \tag{3.48}$$

and draws our attention to the noncommutative phase factors $\mathfrak{D}_n^\theta$.

For the upcoming part of this section, we are going to refer to the following

Lemma 15. *Let $N \in \mathbb{N}$ and let $M(\mathbb{R}^N) := M(\mathbb{R}^N \to \mathbb{C})$ denote the measurable functions on $\mathbb{R}^N$. Furthermore, let $\psi : M(\mathbb{R}^N) \to [0, \infty)$ be a sublinear functional,* i.e., *$\psi(f+g) \leqslant \psi(f) + \psi(g)$ and $\psi(af) = |a|\psi(f)$ for all $a \in \mathbb{C}$ and all $f, g \in L(\mathbb{R}^N)$. Then, given $m \in \mathbb{N}$, we have*

$$\psi\left(\sum_{|\alpha| \leqslant m} c_\alpha f_\alpha\right) \leqslant (m+1)^N \max_{|\alpha| \leqslant m} |c_\alpha| \sup_{|\beta| \leqslant m} \psi(f_\beta) \;\forall\, c_\alpha \in \mathbb{C}\,,\; \forall\, f_\beta \in L(\mathbb{R}^N)\,,\; \alpha, \beta \text{ multi indices}\,.$$

Proof.

$$\begin{aligned}
\psi\left(\sum_{|\alpha| \leqslant m} c_\alpha f_\alpha\right) &= \psi(c_{(0,\ldots,0)} f_{(0,\ldots,0)} + c_{(1,0,\ldots,0)} f_{(1,0,\ldots,0)} + \ldots + c_{(0,\ldots,0,m)} f_{(0,\ldots,0,m)}) \\
&\leqslant |c_{(0,\ldots,0)}| \psi(f_{(0,\ldots,0)}) + |c_{(1,0,\ldots,0)}| \psi(f_{(1,0,\ldots,0)}) + \ldots + |c_{(0,\ldots,0,m)}| \psi(f_{(0,\ldots,0,m)}) \\
&\leqslant \max_{|\alpha| \leqslant m} |c_\alpha| (\psi(f_{(0,\ldots,0)}) + \psi(f_{(1,0,\ldots,0)}) + \ldots + \psi(f_{(0,\ldots,0,m)})) \;.
\end{aligned}$$

Now, the inequality $\psi(f_\beta) \leqslant \sup_{|\alpha| \leqslant m} \psi(f_\alpha)$ is valid for every set $\{f_\alpha\}_{|\alpha| \leqslant m}$ of functions in $M(\mathbb{R}^N)$. As the sum in the upper estimate contains $(m+1)^N$ terms, this finishes the proof.

□

Suppose that a field theory satisfying the Osterwalder-Schrader axioms including (3.46) is given. We want to know if the corresponding Moyal-deformed Schwinger functions also fulfill such an estimate. Relying on (3.42), we examine the behavior of the norms $|\,.\,|_m$ under Fourier transformation for being able to gain an estimate of $\mathfrak{S}_n^\theta(f)$ in terms of the Schwartz function f directly. To this end, we need the following notion: we say that a sequence $\{\sigma_m\}_{m \in \mathbb{N}}$ of positive numbers is *of factorial growth*, if there exist constants α and β, such that $|\sigma_m| \leqslant \alpha(m!)^\beta$ for all $m \in \mathbb{N}$.

Lemma 16. *For all $f \in \mathscr{S}(\mathbb{R}^{dn})$ and for all $m' \in \mathbb{N}$ there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of factorial growth and $s' \in \mathbb{N}$ such that*

$$|f|_{m'} \leqslant \sigma_{m'} |\widetilde{f}|_{m'+s'} \ ,$$

Proof. We start by writing the function f in Fourier representation and by pulling the derivative and the polynomial under the integral. Then, for even $m' = 2m$, $m \in \mathbb{N}$, we obtain by partial integration,

$$\begin{aligned} |f|_{2m} &= \sup_{\substack{x\in\mathbb{R}^{dn} \\ |\alpha|\leqslant 2m}} \left|(1+x^2)^m (D^\alpha f)(x)\right| = \sup_{\substack{x\in\mathbb{R}^{dn} \\ |\alpha|\leqslant 2m}} \left|\int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}}\, \mathrm{e}^{ipx} i^{|\alpha|} (1-D_p^2)^m (p^\alpha \widetilde{f}(p))\right| \\ &\leqslant \sup_{|\alpha|\leqslant 2m} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \left|(1-D_p^2)^m (p^\alpha \widetilde{f}(p))\right| \ , \end{aligned}$$

as the factor $\mathrm{e}^{ipx}(i)^{|\alpha|}$ is bounded by one. Due to the Leibniz rule, we are left with the supremum taken over arbitrary derivatives and powers of order smaller or equal $2m$. Put differently, we are left with the differential operator

$$\begin{aligned} \left|(1-D_p^2)^m (p^\alpha \widetilde{f}(p))\right| &= \left|\sum_{k=0}^{m} \sum_{|\beta|=k} (-1)^{m-k} \tbinom{m}{k}\tbinom{k}{\beta} D^{2\beta}(p^\alpha \widetilde{f}(p))\right| \\ &= \left|\sum_{k=0}^{m} \sum_{|\beta|=k} (-1)^{m-k} \tbinom{m}{k}\tbinom{k}{\beta} \sum_{\gamma\leqslant 2\beta} \tbinom{2\beta}{\gamma} D^\gamma p^\alpha (D^{2\beta-\gamma}\widetilde{f})(p)\right| \\ &= \left|\sum_{k=0}^{m} \sum_{|\beta|=k} \sum_{\gamma\leqslant \min\{2\beta,\alpha\}} (-1)^{m-k} \tbinom{m}{k}\tbinom{k}{\beta}\tbinom{2\beta}{\gamma} \frac{\gamma!}{(\alpha-\gamma)!} p^{\alpha-\gamma} (D^{2\beta-\gamma}\widetilde{f})(p)\right| \\ &\leqslant \sum_{k=0}^{m} \sum_{|\beta|=k} \sum_{\gamma\leqslant \min\{2\beta,\alpha\}} \tbinom{m}{k}\tbinom{k}{\beta}\tbinom{2\beta}{\gamma} \frac{\gamma!}{(\alpha-\gamma)!} \left|p^{\alpha-\gamma} (D^{2\beta-\gamma}\widetilde{f})(p)\right| \ , \end{aligned}$$

where we use the following notation: $\binom{m}{\beta}$ is the multinomial coefficient $\frac{m!}{\beta_1!\cdots\beta_{dn}!}$ whereas $\binom{2\beta}{\gamma}$ denotes the product of binomial coefficients $\binom{2\beta_1}{\gamma_1}\cdots\binom{2\beta_{dn}}{\gamma_{dn}}$. Additionally, when we write $\gamma \leqslant 2\beta$, we mean $\gamma_1 \leqslant 2\beta_1, \ldots, \gamma_{dn} \leqslant 2\beta_{dn}$. In the last step we used the triangle inequality.

We may now go on by estimating the remaining supremum over the sum of multi-derivatives. To this end, we may take advantage of Lemma 15, where we set $\psi(f_\gamma) := \int |f_\gamma(p)| \mathrm{d}^{dn}p$ and $f_\gamma := D^{2\beta-\gamma} f$ for $f \in \mathscr{S}(\mathbb{R}^{dn})$. By the triangle inequality, this indeed makes ψ a sublinear functional on $\mathscr{S}(\mathbb{R}^{dn})$. So, we obtain

$$
\begin{aligned}
|f|_{2m} &\leqslant \sup_{|\alpha|\leqslant 2m} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \sum_{k=0}^{m} \sum_{|\beta|=k} \sum_{\gamma\leqslant\min\{2\beta,\alpha\}} \tbinom{m}{k}\tbinom{k}{\beta}\tbinom{2\beta}{\gamma} \frac{\gamma!}{(\alpha-\gamma)!} \left| p^{\alpha-\gamma}(D^{2\beta-\gamma}\widetilde{f})(p) \right| \\
&\leqslant \sup_{|\alpha|\leqslant 2m} \sum_{k=0}^{m} \sum_{|\beta|=k} \tbinom{m}{k}\tbinom{k}{\beta} c_1(\alpha,\beta) N_1(\alpha,\beta) \sup_{\gamma\leqslant\min\{2\beta,\alpha\}} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \left| p^{\alpha}(D^{2\beta-\gamma}\widetilde{f})(p) \right| \\
&\leqslant \sup_{|\alpha|\leqslant 2m} \sum_{k=0}^{m} \tbinom{m}{k} c_2(\alpha,k) N_2(k) \sup_{|\beta|\leqslant k} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \left| p^{\alpha}(D^{2\beta}\widetilde{f})(p) \right| \ ,
\end{aligned}
$$

where $c_2(\alpha,k) := \max\limits_{|\tilde\beta|=k} \left\{ \max\limits_{\tilde\gamma\leqslant\alpha} \{ \tbinom{2\tilde\beta}{\tilde\gamma} \frac{\tilde\gamma!}{(\alpha-\tilde\gamma)!} \} \tbinom{k}{\tilde\beta} N_1(\alpha,\tilde\beta) \right\}$. The remaining sum over k can again be estimated by the maximal contribution times the number of upcoming terms. Thus, we obtain,

$$
|f|_{2m} \leqslant (m+1) c_3(m) \sup_{\substack{|\alpha|\leqslant 2m \\ |\beta|\leqslant m}} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \left| p^{\alpha}(D^{2\beta}\widetilde{f})(p) \right| \ ,
$$

where $c_3(m)$ denotes a cascade of three maxima. We will deal with this coefficient below. Now we can further estimate $|f|_{2m}$ by utilizing the fact that for all multi-monomials we have the inequality $p^{2\alpha} \leqslant (1+p^2)^{|\alpha|}$ by the multinomial formula, revealing

$$
|f|_{2m} \leqslant A(m) \sup_{|\beta|\leqslant 2m} \int \frac{\mathrm{d}^{dn}p}{(2\pi)^{dn/2}} \left| (1+p^2)^m (D^{\beta}\widetilde{f})(p) \right| \ ,
$$

with $A(m) := (m+1)c_3(m)$. Now we can insert a factor one, written as $\frac{(1+p^2)^{m+dn}}{(1+p^2)^{m+dn}}$ to move on,

$$
|f|_{2m} \leqslant A(m) \sup_{|\beta|\leqslant 2m} \int \mathrm{d}^{dn}p \frac{(1+p^2)^m}{(2\pi)^{dn/2}(1+p^2)^{m+dn}} \left| (1+p^2)^{m+dn} D^{\beta}\widetilde{f}(p) \right| \ .
$$

As the first factor is integrable (giving the number c_4), we can estimate the whole expression a bit generously by

$$|f|_{2m} \;\leqslant\; A(m)\, c_4 \sup_{\substack{p\in\mathbb{R}^{dn}\\ |\beta|\leqslant 2m}} \left|(1+p^2)^{m+dn} D^\beta \widetilde{f}(p)\right| \leqslant A(m)\, c_4 |\widetilde{f}|_{2(m+dn)} \ ,$$

which finishes the estimate for $\sigma'_m := A(m')c_4$, $m' = 2m$ and $s' := 2dn$.

For odd $m' = 2m-1$, $m \in \mathbb{N}$, we make use of the fact that $|f|_{m'} \leqslant |f|_{m'+1}$ for all $f \in \mathscr{S}(\mathbb{R}^{dn})$. The preceding estimate then implies

$$|f|_{m'} = |f|_{2m-1} \leqslant |f|_{2m} \leqslant c(m')|\widetilde{f}|_{m'+s'} \ , \ s' := 2dn+1 \ .$$

Now let us verify that the overall coefficient c is of factorial growth in m.

We start by the maximal coefficient $c_1(m)$ of $\sum_{\gamma\leqslant\alpha} \binom{2\beta}{\gamma}\frac{\alpha!}{(\alpha-\gamma)!}$, where $|\beta| = m$ and $|\alpha| \leqslant 2m$. For the extremal case where one $\gamma_k = 2m$ and all others vanish, we have a contribution of $2m!$. In any case, this marks an upper bound for the asymptotic behavior of the "true" binomial coefficient c_1, which grows exponentially. The number of summands N_1 and N_2 are both bounded by the number dn^{2m}, since this constitutes the total number of configurations when coinciding summands are not incorporated. What remains is the maximal multinomial coefficient $c_2(m)$ of $\binom{m}{\beta}$, which is increasing exponentially in m. It results that $A(m)$ is of the order $(2m)!\,(dn)^{4m}\exp(m)$, thus A defines a sequence in $2m$ of factorial growth, while c_4 is just the constant $2^{-dn/2}\frac{\Gamma(dn/2)}{\Gamma(dn)}$.

□

Remark 13. It is widely known that the Fourier transform acts continuously on Schwartz spaces of arbitrary dimension, but the methods used to arrive at the estimate of Lemma 16 in terms of the special norms $|\,.\,|_m$ will be useful in the following and are stronger than just continuity.

Due to this result we can get to an estimate on the functions $f_\theta = \mathcal{F}^{-1}[\widetilde{f}_\theta]$.

Lemma 17. *For all $f \in \mathscr{S}(\mathbb{R}^{dn})$ and for all $m \in \mathbb{N}$ there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ of factorial growth such that*

$$|\widetilde{f}_\theta|_m \;\leqslant\; b_m|\widetilde{f}|_{3m} \ .$$

Proof. Working in Fourier representation leads to the examination of the following norm,

$$|\widetilde{f}_\theta|_m \;=\; \sup_{\substack{p\in\mathbb{R}^{dn}\\|\alpha|\leqslant m}}\left|(1+p^2)^{m/2}D_p^\alpha(\widetilde{f}\,\mathfrak{D}_n^\theta)(p)\right| = \sup_{\substack{p\in\mathbb{R}^{dn}\\|\alpha|\leqslant m}}\left|(1+p^2)^{m/2}D_p^\alpha\left(\widetilde{f}(p)\,\mathrm{e}^{\frac{i}{2}\sum\limits_{k<r}p_k\theta p_r}\right)\right| \;.$$

Let us contemplate the multi-derivative on the product of the Fourier transform and the phase factors in more detail. In full length, it is given by a polynomial containing multi-derivatives of $\widetilde{f}$ and $\mathfrak{D}_n^\theta$.

$$\left|D_p^\alpha(\mathfrak{D}_n^\theta\,\widetilde{f}(p))\right| \;=\; \left|\sum_{\beta\leqslant\alpha}\tbinom{\alpha}{\beta}D_p^\beta\mathfrak{D}_n^\theta(p)D_p^{\alpha-\beta}\widetilde{f}(p)\right| = \left|\mathfrak{D}_n^\theta(p)\sum_{\beta\leqslant\alpha}\tbinom{\alpha}{\beta}\mathscr{P}^{(|\beta|)}(p)D_p^{\alpha-\beta}\widetilde{f}(p)\right| \;,$$

where $\mathscr{P}^{(|\beta|)}(p)$ is a polynomial in p of degree $|\beta|$, because every derivative with respect to any momentum component delivers exactly one momentum power times $-i\vartheta/2$ and the terms coming from the Leibniz rule do not increase the degree. For an arbitrary polynomial $\sum\limits_{j=1}^{N}c_jp^{\beta_j}$ we may infer

$$\left|\sum_{j=1}^{N}c_jp^{\beta_j}\right| \;\leqslant\; \sum_{j=1}^{N}|c_j|(1+p^2)^{\beta_j} \leqslant \sum_{j=1}^{N}|c_j|(1+p^2)^{\max\limits_k\beta_k} \leqslant N\max_j|c_j|(1+p^2)^{\max\limits_k\beta_k} \;,$$

which in our case leads to

$$\left|D_p^\alpha(\mathfrak{D}_n^\theta\,\widetilde{f}(p))\right| \leqslant b(|\alpha|)(1+p^2)^{|\alpha|}\sum_{\beta\leqslant\alpha}\left|D^{\alpha-\beta}\widetilde{f}(p)\right| \;,$$

with b denoting the maximal modulus of $\mathscr{P}^{(|\beta|)}(p)$-coefficients. If we re-insert this into our initial expression we obtain

$$\begin{aligned}|\widetilde{f}_\theta|_m \;&\leqslant\; \sup_{\substack{p\in\mathbb{R}^{dn}\\|\alpha|\leqslant m}}\left\{(1+p^2)^{m/2}b(|\alpha|)(1+p^2)^{|\alpha|}\sum_{\beta\leqslant\alpha}\left|D^{\alpha-\beta}\widetilde{f}(p)\right|\right\}\\ &=\; \sup_{\substack{p\in\mathbb{R}^{dn}\\|\alpha|\leqslant m}}\left\{(1+p^2)^{3m/2}b(|\alpha|)\sum_{\beta\leqslant\alpha}\left|D^{\alpha-\beta}\widetilde{f}(p)\right|\right\} \;.\end{aligned}$$

Like in the proof of Lemma 16 we can estimate the sum of derivatives by the maximal derivative times the number of upcoming terms. Indeed, choose $\psi(f_\beta) := \sup\limits_{p\in\mathbb{R}^{dn}} |f_\beta(p)|$ in Lemma 15, as well as $f_\beta := D^{\alpha-\beta}\widetilde{f}$. Thence, we collect all constants into b_m and deduce,

$$
\begin{aligned}
|\widetilde{f}_\theta|_m &\leqslant \sup_{|\alpha|\leqslant m} |b(|\alpha|)| N(\alpha) \sup_{\substack{p\in\mathbb{R}^{dn}\\ \beta\leqslant\alpha}} \left\{(1+p^2)^{3m/2}\left|D^{\alpha-\beta}\widetilde{f}(p)\right|\right\} \\
&\leqslant b_m \sup_{\substack{p\in\mathbb{R}^{dn}\\ |\alpha|\leqslant m}} \left\{(1+p^2)^{3m/2}\left|D^{\alpha}\widetilde{f}(p)\right|\right\} \leqslant b_m |\widetilde{f}|_{3m} \ .
\end{aligned}
$$

Let us determine the asymptotic behavior with respect to m again. At first, we can estimate the number $N(\alpha)$ of terms evolving in the sum over $\beta \leqslant \alpha$ by $(|\alpha|+1)^{dn} \leqslant (m+1)^{dn}$, as this is the maximal number of combinations for the multi-index β. Furthermore, we have to consider the maximal modulus $|b(\alpha)|$ of $\mathscr{P}^{|\beta|}(p)$-coefficients. These are binomial coefficients with the maximal possible contribution[1] of $\binom{m}{[m/2]}$ and irrelevant powers of ϑ. Hence, we conclude that $\{b_m\}_{m\in\mathbb{N}}$ is a sequence of factorial growth, which we consider apposite for our purposes.

□

Obviously, Lemma 16 for $g := \widetilde{f}$ serves as an estimate for the inverse Fourier transform. We use this fact when we formulate our findings in the following

Proposition 14. *Let $s \in \mathbb{N}$. Then, there exist $s' \in \mathbb{N}$ and a sequence $\{\sigma_m\}_{m\in\mathbb{N}}$ of factorial growth such that for all $n \in \mathbb{N}$ and $f \in \mathscr{S}(\mathbb{R}^{dn})$ the estimate*

$$
|f_\theta|_{sn} \leqslant \sigma_n \, |f|_{s'n} \tag{3.49}
$$

is valid.

Proof. For the norm $|f_\theta|_{sn}$ of the function $f_\theta \in \mathscr{S}(\mathbb{R}^{dn})$, the application of Lemma 16 entails the inequality $|f_\theta|_{sn} \leqslant c(n)|\widetilde{f}_\theta|_{n(s+d)}$ if sn is even and $|f_\theta|_{sn} \leqslant c(n)|\widetilde{f}_\theta|_{n(s+2d)+1}$ if sn is odd. Lemma 17 then leads to the following inequalities,

$$
|f_\theta|_{sn} \leqslant \begin{cases} b_{n(s+d)}|\widetilde{f}|_{3(s+d)n} \ , & sn \text{ even} \ , \\ b_{n(s+2d)+1}|\widetilde{f}|_{3(s+2d)n+3} \ , & sn \text{ odd} \ . \end{cases}
$$

[1] $[m/2]$ denotes the Gaussian bracket, *i.e.*, the largest integer which is smaller or equal to $m/2$.

We observe that $3(s+d)n = 3sn + 3dn$ for sn even can be both even and odd, depending on the dimension d. On the other hand, $3(s+2d)n+3 = 3sn+6dn+3$ is always even if sn is odd. Thus, a final implementation of Lemma 16 lets us end up with

$$|f_\theta|_{sn} \leqslant \begin{cases} \sigma_n^{(1)} |f|_{(3s+4d)n} \, , & sn \text{ even} \, , \quad 3(s+d)n \text{ even} \, , \\ \sigma_n^{(2)} |f|_{(3s+5d)n+1} \, , & sn \text{ even} \, , \quad 3(s+d)n \text{ odd} \, , \\ \sigma_n^{(3)} |f|_{(3s+7d)n+3} \, , & sn \text{ odd} \, . \end{cases}$$

This means that in (3.49), we observe $|f_\theta|_{sn} \leqslant c(n)|f|_{t_1 n + t_2}$ for either $t_1 = 3s+4d$, $3s+5d$, or $3s+7d$, as well as $t_2 = 0$, 1 or 3, correspondingly. In each of these cases, the claimed estimate follows from $|f|_{t_1 n + t_2} \leqslant |f|_{(t_1+t_2)n}$ for $s' := t_1 + t_2$. $\square$

A few remarks about the applicability of existing proofs concerning the analytical continuation of Schwinger functions are in order. The paper [OS75] has deservedly become a classic in mathematical physics over the last decades. The subtle details which at first spoiled the correctness of [OS73] are of importance when trying to modify the fundamental axioms, as becomes important while applying the Moyal deformation. As the authors explain with great care, the analytic continuation happens at the time-components of n-point distributions and it highly matters how to treat the spatial components. By means of the free field, we have shown in the preceding subsection that smearing out the Schwinger functions in the spatial variables (*i.e.*, treating them distributional) has the drawback of demanding a stronger temperedness condition and the benefit of full Euclidean covariance never entering the continuation. It is the restoring of the Wightman axioms for the Minkowski space-time n-point functions when Euclidean covariance is needed for the first time.

The second method consists of showing that the Schwinger functions can be continued to analytic functions in all the coordinates, including the spatial ones. Unfortunately, full $SO(d)$-covariance must be assumed to get to this result. Going a bit more into detail, we present a few parts of the proof residing in [OS75]: given a cone $C_\beta := \{x \in \mathbb{R}^d \,|\, x^0 > |\underline{x}| \tan\beta\}$ for $0 < \beta < \pi/4$ it is argued that any d linearly independent vectors contained in the dual cone $C_{\pi/2-\beta}$ can be mapped to the vector $(1,0,0,\ldots)$ by a suitable set of rotations. Clearly, these cannot be taken just out of $O(2) \times SO(d-2)$. To guarantee the sole usage of the latter group there is the need to go over to the light wedge Y_β. But there are still many vectors of the dual wedge (which would be the analog of the dual cone) $Y_{\pi/2-\beta}$ which cannot be mapped into $(1,0,0,\ldots)$ just by using x_0, x_1-rotations!

Showing analyticity in the spatial variables without using full $SO(d)$-covariance therefore must, if possible, be done in a different manner

Next we are going to use the stronger condition ($E0''$) for (certain) Moyal-deformed theories and examine the corresponding Wick rotation. We prefer to give a formulation for this condition for the set $\{S_n\}_n$ of Schwinger functions written in difference variables:

($E0''$) There exists $s \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and for all $f_k \in \mathscr{S}(\mathbb{R}^d_+)$, $k = 1, \ldots, n$ it holds

$$|S_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n)| \leqslant \alpha (n!)^\beta \prod_{k=1}^{n} |f_k|_s \ . \tag{3.50}$$

In [OS75, appendix], it has been worked out that this condition really implies ($E0'$). The free field does fulfill the latter condition. We elaborated on a distributional version in the preceding subsection, so we check the condition in terms of Schwartz norms now.

$$\begin{aligned} |S_1(f)| &= \left| \int \frac{\mathrm{d}^d p}{(2\pi)^d (p^2 + m^2)} \tilde{f}(p) \right| \leqslant \int \left| \frac{\mathrm{d}^d p}{(2\pi)^d (p^2 + m^2)^{d+1}} \right| \left| (p^2 + m^2)^d \tilde{f}(p) \right| \\ &\leqslant c_2 \sup_{p \in \mathbb{R}^d} \left| (p^2 + m^2)^d \tilde{f}(p) \right| \leqslant c_2 |\tilde{f}|_d \leqslant c_3 |f|_{3d} \ . \end{aligned}$$

In the last line we have used Lemma 16 for the function and its Fourier transform interchanged. This shows (3.50) for $n = 1$. Similarly as in the proof of Prop. 13, we may decompose S_n into a sum of products of the propagators S_1. Applying the triangle inequality, we infer that there is a constant $c(n)$ such that $|S_n(f_1 \otimes \cdots \otimes f_n)| \leqslant c(n) |f_1|_{3d} \cdots |f_n|_{3d}$. Being a bit more specific, we have $c(n) = n!! c_3^{(n+1)/2}$, thus it defines a sequence of factorial growth.

Now it is time for a noncommutative generalization of the Osterwalder approach to the analytical continuation of n-point functions satisfying ($E0''$).

Let $S_n \in \mathscr{S}'(\mathbb{R}^{dn}_+)$ be a sequence of Schwinger functions which satisfies (3.50) and again consider the norms $|\,.\,|_m$ for $m \in \mathbb{N}$ we have introduced in (3.47). Given that ($E0''$) holds for all Schwartz functions in $\mathscr{S}_+(\mathbb{R}^{dn})$, we choose those of tensor form between the time and space components: $f_k := h_k \otimes s_k$ for $h_k \in \mathscr{S}(\mathbb{R}_+)$, $s_k \in \mathscr{S}(\mathbb{R}^{d-1})$, $k = 1, \ldots, n$. We estimate the tensor product of Schwartz functions in the norms $|\,.\,|_m$ in the following. Let $a \in \mathscr{S}(\mathbb{R}^r)$, $b \in \mathscr{S}(\mathbb{R}^t)$. We then have

$$
\begin{aligned}
|a \otimes b|_m \quad &:= \sup_{\substack{(x,y)\in\mathbb{R}^{r+t} \\ |\alpha|\leqslant m}} \left|(1+x^2+y^2)^{m/2} D^\alpha (a\otimes b)(x,y)\right| \\
&\leqslant \sup_{\substack{(x,y)\in\mathbb{R}^{r+t} \\ |\alpha'|+|\alpha''|\leqslant m}} \left|(1+x^2)^{m/2}(D^{\alpha'}a)(x)(1+y^2)^{m/2}(D^{\alpha''}b)(y)\right| \\
&\leqslant \sup_{\substack{x\in\mathbb{R}^{r} \\ |\alpha'|\leqslant m}} \left|(1+x^2)^{m/2}(D^{\alpha'}a)(x)\right| \sup_{\substack{x\in\mathbb{R}^{t} \\ |\alpha''|\leqslant m}} \left|(1+y^2)^{m/2}(D^{\alpha''}b)(y)\right| \\
&= \quad |a|_m |b|_m \ .
\end{aligned} \tag{3.51}
$$

From this inequality it now follows that $|f_k|_m = |h_k \otimes s_k|_m \leqslant |h_k|_m |s_k|_m$. As a next step we define a distribution $\gamma_n^{\mathbf{h}}$ by its action on a tensor product

$$
\gamma_n^{\mathbf{h}}(s_1 \otimes \cdots \otimes s_n) := S_n(h_1 \otimes s_1 \otimes \cdots \otimes h_n \otimes s_n) \ ,
$$

which is in $\mathscr{S}'(\mathbb{R}_+^{(d-1)n})$ for every fixed combination $\mathbf{h} := (h_1, \ldots, h_n) \in \mathscr{S}(\mathbb{R}_+) \times \cdots \times \mathscr{S}(\mathbb{R}_+)$. Thus we obtain a useful relation from $(E0'')$ and (3.51):

$$
|\gamma_n^{\mathbf{h}}(s_1 \otimes \cdots \otimes s_n)| \leqslant \rho_n \prod_{k=1}^{n} |h_k|_m \prod_{j=1}^{n} |s_j|_m =: \rho_n^{\mathbf{h}} \prod_{j=1}^{n} |s_j|_m \tag{3.52}
$$

where $\rho_n^{\mathbf{h}}$ is obviously an s_j-independent sequence of factorial growth in n. We now take advantage of condition $(E0'')$ implying the estimate $(E0')$, as has been shown in [OS75, appendix]. Inequality (3.52) means that the requirements are fulfilled for $\gamma_n^{\mathbf{h}}$ as well, which implies that there exist $r \in \mathbb{N}$ and a constant c such that

$$
|\gamma_n^{\mathbf{h}}(s)| \leqslant c^n \rho_n^{\mathbf{h}} |s|_{nr} \ ,
$$

for all $s \in \mathscr{S}(\mathbb{R}^{(d-1)n})$.

We are going to show the applicability of the Wick rotation without demanding full $SO(d)$-covariance of the theory now. To this end, let us go over to the Moyal deformed theory defined by the Schwinger functions S^θ. These are obtained by the use of the Moyal tensor product, which due to our chosen scenario only affects the spatial components. We infer

$$
\begin{aligned}
|S_n^\theta(h_1 \otimes s_1 \otimes \cdots \otimes h_n \otimes s_n)| &= \left| \int S_n(\xi_1, \ldots, \xi_n)((h_1 \otimes s_1) \otimes_\theta \cdots \otimes_\theta (h_n \otimes s_n))(\xi_1, \ldots, \xi_n) \mathrm{d}^{dn}\xi \right| \\
&= \left| \int S_n(\xi_1, \ldots, \xi_n) h_1(\xi_1^0) \cdots h_n(\xi_n^0)(s_1 \otimes_\theta \cdots \otimes_\theta s_n)(\underline{\xi}_1, \ldots, \underline{\xi}_n) \mathrm{d}^{dn}\xi \right|
\end{aligned}
$$

Let $s_\theta := s_1 \otimes_\theta \cdots \otimes_\theta s_n$. Implementing Prop. 14, we are able to write down the best suited estimate for our noncommutative Schwinger functions then:

$(E0'')_\theta$:

$$
\begin{aligned}
|S_n^\theta(h_1 \otimes s_1 \otimes \cdots h_n \otimes s_n)| &\leqslant c^n \rho_n \prod_{k=1}^n |h_k|_m |s_\theta|_{nr} \\
&\leqslant c^n \rho_n \sigma_n \prod_{k=1}^n |h_k|_m |s|_{nr'} \\
&=: \hat{\sigma}_n \prod_{k=1}^n |h_k|_m |s|_{nr'} \ .
\end{aligned} \tag{3.53}
$$

The sequence defined through $\hat{\sigma}_n := c^n \rho_n \sigma_n$ is of factorial growth, as each factor is.

The remainder of this subsection is organized as follows: first of all we are going to generalize the analytical continuation of [Ost73] to general Moyal theories of commutative time. We have already checked its validity for the special case of the free field in 3.6. Next we will show that (3.53) leads to the existence of the holomorphic Fourier transform of the noncommutative Schwinger functions. Finally we are going to establish the Wightman functions and verify the axioms for them.

We utilized Hilbert space vectors made out of $\phi^{\mathcal{E}}$-monomials applied to Ω to represent the Schwinger functions as scalar products in the free field case. Nonetheless, such a representation is also valid for general Euclidean theories. Indeed, we have shown the construction of the physical Hilbert space $\mathcal{H}$ in chapter 2 in the algebraic context. We as well did this in the framework of the Borchers-Uhlmann algebra by dealing with the inner product

$$
\langle \underline{f}, \underline{g} \rangle := \sum_{n,m \geqslant 0} \mathfrak{S}_{n+m}(\iota f_n^* \otimes g_m) \ ,
$$

leading to the fact that $\mathcal{H}$ shapes up as being the Fock space $\mathcal{F}$ for the free field. We want to stress again that none of the preceding formulae are influenced by the Moyal deformation defined by θ. Hence it makes no difference so far if we contemplate the scalar product representation of the deformed Schwinger functions, because this results in smearing with different spatial Schwartz functions; those multiplied with the Moyal phase factors $\mathfrak{D}_n^\theta$ in Fourier space.

Osterwalder-Schrader Reconstruction We will show the validity of the Wick rotation of [Ost73] for the Moyal-deformed Schwinger functions of commutative time now.

By assumption, the distributions $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^{dn})$ are translation invariant. Due to the important rule of deformation given in (3.42) and the definition of the Moyal tensor product, the noncommutative distributions will keep that property. From translational invariance of the noncommutative Schwinger functions, we may infer that for each $\mathfrak{S}_n^\theta \in \mathscr{S}'(\mathbb{R}^{dn})$, there exists a $S_{n-1} \in \mathscr{S}'(\mathbb{R}^{d(n-1)})$, such that

$$\mathfrak{S}_n^\theta(f) = \int S_{n-1}(x_2 - x_1, \dots, x_n - x_{n-1}) f_\theta(x_1, \dots, x_n) \mathrm{d}^{dn}x \ ,$$

or, written distributionally, $\mathfrak{S}_n^\theta(x_1, \dots, x_n) = S_{n-1}^\theta(\xi_1, \dots, \xi_{n-1})$ for $\xi_k := x_{k+1} - x_k$. From the definition of our scalar product $\langle \, , \, \rangle$ on $\mathcal{H}$,

$$\langle \underline{f}, \underline{g} \rangle = \sum_{n,m} \int \overline{f_n(rx_n, \dots, rx_1)} g_m(y_1, \dots, y_m) \mathfrak{S}_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m) \ , \ \forall \underline{f}, \underline{g} \in \underline{\mathscr{S}} \ ,$$

we infer,

$$\begin{aligned} & \langle \underline{f}, \mathrm{e}^{-tH} \underline{g} \rangle = \\ = & \sum_{n,m} \int \overline{f_n(rx_n, \dots, rx_1)} g_m(y_1, \dots, y_m) S_{n+m-1}(\xi_1, \dots, \xi_{n-1}, y_1 - x_n + t, \xi'_1, \dots, \xi'_{m-1}) \\ & \mathrm{d}^{d(n-1)}\xi \, \mathrm{d}^{d(n-1)}\xi' \, \mathrm{d}^d y_1 \, \mathrm{d}^d x_n \ , \end{aligned}$$

where $\xi'_k := y_{k+1} - y_k$.

By (3.42) and the form (2.20) of θ belonging to the commutative-time scenario, together with the notation defined in (3.43), we conclude

$$S_{n+m-1}^\theta(\xi^0, x^0, \xi'^0 | g) \quad = \quad S_{n+m-1}(\xi^0, x^0, \xi'^0 | g_\theta) \ . \tag{3.54}$$

In the following, we omit the reference to the concrete smearing function g or g_θ. For arbitrary $h \in \mathscr{S}(\mathbb{R})$, the map

$$\begin{aligned} \mathscr{S}_<(\mathbb{R}^{d(n+m)}) \otimes \mathscr{S}(\mathbb{R}) &\rightarrow \mathbb{C} \\ (\iota f_n^* \otimes g_m, h) &\mapsto \int \langle [f_n]_\sigma, \mathrm{e}^{-(t+is)H}[g_m]_\sigma \rangle_{\mathcal{H}} h(s) \mathrm{d}s \end{aligned}$$

defines a continuous linear functional. Thus, we may write

$$\int \langle [f_n]_\sigma, \mathrm{e}^{-(t+is)H}[g_m]_\sigma \rangle_{\mathcal{H}} h(s) \mathrm{d}s = \int S_{n+m-1}(\xi^0, y_1^0 - x_n^0 + t, \xi'^0; s) \overline{h_1(-{}^0\xi)} h_2(\xi'^0) h(s)\, \mathrm{d}s\, \mathrm{d}^{n-1}\xi^0\, \mathrm{d}^{m-1}\xi' \ ,$$

for a distribution $S_{n+m-1}(\xi^0, y_1^0 - x_n^0 + t, \xi'^0; s) \in (\mathscr{S}(\mathbb{R}^{n+m-1}{}_+) \otimes \mathscr{S}(\mathbb{R}))'$ and $({}^0\xi) := (\xi_{n-1}^0, \dots, \xi_1^0)$. Different smearing lets us end up with the following distribution $F \in (\mathscr{S}(\mathbb{R}_+) \otimes \mathscr{S}(\mathbb{R}))'$,

$$F(t,s) := \int S_{n+m-1}(\xi_1^0, \dots, \xi_{n+m-1}^0; s) \chi(\xi_1^0, \dots, \widehat{\xi_n^0}, \xi_1'^0, \dots, \xi_{n+m-1}^0) \mathrm{d}\xi^0 \cdots \widehat{\mathrm{d}\xi_n^0} \cdots \mathrm{d}\xi_{n+m-1}^0 \ .$$

This distribution $(t,s) \mapsto F(t,s)$ fulfills the Cauchy-Riemann differential equations and therefore,

$$S_{n+m-1}(\xi^0, y_1^0 - x_n^0 + t, \xi'^0; s) = S_{n+m-1}(\xi^0, y_1^0 - x_n^0 + \tau, \xi'^0) \ , \ \tau := t + is \ ,$$

defines a distribution in ξ^0 and ξ'^0 as well as a function in $z := y_1^0 - x_n^0 + \tau$, analytic in the right half plane $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\}$.

Proposition 15. *Consider a fixed smearing function $g \in \mathscr{S}'(\mathbb{R}^{(d-1)n})$. For $k := n + m - 1$ fixed and $m = 0, \dots, k$, the distributions $S_k^\theta(\xi_1^0, \dots, \xi_k^0|g)$ can be analytically continued to functions $S_k^\theta(\zeta_1^0, \dots, \zeta_n^0)$, analytic in $\mathbb{C}_+^k$.*

Proof. We have already stressed in (3.54) that $S_k^\theta(\xi_1^0, \dots, \xi_k^0|g) = S_k(\xi_1^0, \dots, \xi_k^0|g_\theta)$. The statement thus follows from the proof of [Ost73, Thm. 11] with the following notational identifications:

$$\begin{aligned} \widehat{T}_t &:= \alpha_{((t,\underline{0}),1)} \\ \Phi(\underline{f}) &:= [\underline{f}]_\sigma \\ S_{n+m-1}(\underline{\xi}, x + x' + t, \underline{\xi}'|s) &:= S_{n+m-1}(\xi^0, y_1^0 - x_n^0 + t, \xi'^0; s) \ , \end{aligned}$$

and finally Ψ denotes the distributional kernel of $[\underline{f}]_\sigma$, written in difference variables. □

At this point we can profit by the estimate given in (3.53). For the sake of clarity, let us write the suppressed spatial smearing functions again.

Proposition 16. *For all $k \in \mathbb{N}$ and all $\zeta_k^0 \in \mathbb{C}_+^k$ there exist integers a, b and s, and a constant $c > 0$, such that*

$$\left|S_k^\theta(\zeta_1^0, \ldots, \zeta_n^0|g)\right| \leqslant c(1 + |(\zeta_1^0, \ldots, \zeta_k^0)|)^a (1 + \min_j \{\mathrm{Re}\, \zeta_j^0\}^{-1})^b |g|_{sk} \ . \tag{3.55}$$

Proof. Follows from (3.53), (3.54) and [Ost73, (15) in Thm. 11], relying on the notational identifications in the proof of Prop. 15. □

Remark 14. 1. The existence of constants such that this inequality holds means that we allow for any polynomial growth and singularity of each time component separately when the deformed Schwinger functions are restricted to hyperplanes of constant (*e.g.* zero) time.

2. In particular, we do not demand a time-zero condition from our deformed theory here.

As a next step, we implement a combination of two facts about analytic functions in several variables to our analytic continuations S_k of the Schwinger functions in difference variables:

Proposition 17. *1. Let S_n^θ be a holomorphic function in time and a distribution in spatial components. The existence of constants a, b, $s \in \mathbb{N}$ and $c > 0$, such that inequality* (3.55) *holds is sufficient for the holomorphic Fourier transform $\widetilde{W}_n^\theta$ of S_n^θ to exist. Moreover, $\widetilde{W}_n^\theta$ is then supported in $\overline{\mathbb{R}_+^{dn}}$ such that the boundary distributions restricted to the n-fold product of positive real axes in time fulfill the relation*

$$S_n^\theta(\xi_1, \ldots, \xi_n) = \int \mathrm{d}^{dn} q \ \mathrm{e}^{-\sum_{k=1}^{n} (\xi_k^0 q_k^0 + i \underline{\xi}_k \underline{q}_k)} \widetilde{W}_n^\theta(q_1, \ldots, q_n) \ .$$

2. This holomorphic Fourier transform $\widetilde{W}_n^\theta$ equals the deformed one of the corresponding commutative theory, i.e.,

$$\widetilde{W}_n^\theta = \widetilde{W}_n D_n^\theta \quad \forall\, n \in \mathbb{N} \ .$$

Proof. 1. First of all we notice that $\mathbb{C}^n_+ = \mathbb{R}^n_+ + i\mathbb{R}^n$ is a tubular domain. Any function f that is analytic on such and satisfies an estimate

$$|f(z_1,\ldots,z_n)| \leqslant M\left(1+\sqrt{(\operatorname{Re} z_1)^2+\ldots(\operatorname{Re} z_n)^2}\right)^m \;,\; (z_1,\ldots,z_n)\in\mathbb{C}^n_+ \;,$$

for suitable constants M and m is the holomorphic Fourier transform of a distribution due to [Vla66, section 26.1., eq. (1)]. The supposed stronger estimate (3.55) suffices for the existence of a unique boundary distribution which is independent of the concrete restricting sequence, *cf.* [Vla66, section 26.3., eq. (13)].

2. This was shown in Lemma 14. □

Since this well-suited growth condition indeed holds for the sequence $\{S^\theta_n\}_{n\in\mathbb{N}}$, Prop. 17 allows for the introduction of the noncommutative Wightman function candidates

$$\mathfrak{W}^\theta_{n+1}(x_1,\ldots,x_{n+1}) \;=\; \int \mathrm{d}^{dn}q\;\mathrm{e}^{i\sum\limits_{k=1}^{n} q_k(x_{k+1}-x_k)}\widetilde{W}^\theta_n(q_1,\ldots,q_n)\;.$$

2. of Prop. 17 now leads to the following form:

$$\mathfrak{W}^\theta_{n+1}(x_1,\ldots,x_{n+1}) \;=\; \int \mathrm{d}^{dn}q\;\mathrm{e}^{i\sum\limits_{k=1}^{n} q_k(x_{k+1}-x_k)}\widetilde{W}_n(q_1,\ldots,q_n)\,\mathrm{e}^{i\sum\limits_{k=1}^{n-1} q_k\theta q_{k+1}}\;.$$

Finally, let us verify that these deserve to be called Wightman functions:

Obviously, the $\mathfrak{W}^\theta_n$ are Schwartz distributions by definition and the translational invariance can just be read off there. The invariance with respect to the spatial rotations in $SO(d-2)$ follows from the assumed invariance of the spatial Fourier transformed part of the noncommutative Schwinger functions. A bit more but no more than is known from the literature has to be done for the verification of x_1-boost covariance. Due to the Euclidean x_0, x_1-rotational invariance of the Schwinger functions, we act with L_1, the infinitesimal generator of x_0, x_1-rotations, on S^θ_n to infer

$$
\begin{aligned}
0 &= L_1 S_n^\theta(\xi_1,\ldots,\xi_n) = \sum_k \left(\xi_k^0 \frac{\partial}{\partial \xi_k^1} - \xi_k^1 \frac{\partial}{\partial \xi_k^0}\right) S_n^\theta(\xi_1,\ldots,\xi_n) \\
&= -i \int \mathrm{d}^{dn} q \; \mathrm{e}^{-\sum_{l=1}^{n}(q_l^0 \xi_l^0 + i \underline{q}_l \underline{\xi}_l)} \sum_k \left(q_k^0 \frac{\partial}{\partial q_k^1} + q_k^1 \frac{\partial}{\partial q_k^0}\right) \widetilde{W}_n^\theta(q_1,\ldots,q_n) \\
&= -i \int \mathrm{d}^{dn} q \; \mathrm{e}^{-\sum_{l=1}^{n}(q_l^0 \xi_l^0 + i \underline{q}_l \underline{\xi}_l)} B_1 \widetilde{W}_n^\theta(q_1,\ldots,q_n) \;,
\end{aligned}
$$

where B_1 denotes the infinitesimal generator of x_1-boosts. Now we use that the kernel of the holomorphic Fourier transform is zero, *cf.* [OS73, Lemma 8.2.], which leads to the vanishing of $B_1 \widetilde{W}_n^\theta$. This implements the invariance of the $\mathfrak{W}_n^\theta$ under these boosts. As $\mathcal{P}_\theta(d) = \mathbb{R}^d \rtimes (SO(1,1) \times SO(d-2))$, $\mathfrak{W}_n^\theta$ is invariant under the whole of $\mathcal{P}_\theta(d)$. The spectrum condition follows directly from the invariance of $\widetilde{W}_n^\theta$ and the fact that $\operatorname{supp} \widetilde{W}_n^\theta \subset \overline{\mathbb{R}_+^{dn}}$. Remember that the physical Hilbert space was constructed using the Euclidean time inversion. Thus the demanded reflection positivity of the Schwinger functions (untouched by the deformation) restates the positivity of the scalar product in $\mathcal{H}$. Since the Wightman functions are the boundary values of the analytically continued Schwinger functions at purely imaginary time components, they fulfill the classical positivity condition. Using our gained relations for the Wightman and Schwinger functions, this reads

$$
\begin{aligned}
0 &\leqslant \sum_{n,m} S_{n+m}^\theta(\iota f_n^* \otimes f_m) = \sum_{n,m} \int \mathrm{d}\xi \, \mathrm{d}x \int \mathrm{d}q \; \mathrm{e}^{-\sum_{k=1}^{n+m-1}(\xi_k^0 q_k^0 + i \underline{\xi}_k \underline{q}_k)} \widetilde{W}_{n+m-1}^\theta(q) \iota f_n^*(-x,-\xi) f_m(x,\xi) \\
&= \sum_{n,m} \int \mathrm{d}q \; \widetilde{W}_{n+m-1}^\theta(q) f_{n,\bullet}^*(q) f_{m,\bullet}(q) \;.
\end{aligned}
$$

Here, $(.)_\bullet$ denotes the Fourier-Laplace transform of Schwartz functions of Sec. 3.2. Since this is a continuous mapping from $\mathscr{S}(\mathbb{R}_+^d)$ onto a dense subset of $\mathscr{S}(\mathbb{R}_+^d)$ with trivial kernel (see [OS75, Lemma 2.4]), the latter inequality establishes positivity for the noncommutative Wightman functions.

We summarize our findings in the following

Theorem 7. *Given a sequence $\{\mathfrak{S}_n\}_{n\in\mathbb{N}}$ of Schwartz distributions satisfying the Osterwalder-Schrader axioms of $E(d)$-invariance, reflection positivity and the growth condition $(E0'')$, the corresponding Moyal deformed set of distributions $\mathfrak{S}_n^\theta$ in the commutative-time scenario can be analyti-*

cally continued to tempered distributions $\mathfrak{W}_n^\theta$ fulfilling $\mathcal{P}_\theta(d)$-invariance, positivity and the deformed spectrum condition.

These are equal to the deformed n-point functions of the Wightman theory obtained by the Wick rotation of Osterwalder and Schrader, c.f. 2. of Prop. 17.

Remark 15. In order to achieve this analytic continuation, we neither need full $E(d)$-covariance nor permutation symmetry. These further axioms would be necessary for either requiring just the slightly weaker condition ($E0'$) or restoring locality. In fact, broadening the definition of the noncommutative field can lead to a full $E(d)$ or $\mathcal{P}(d)$-covariant theory, see [GL08]. Moreover, there are implications on conditions on Euclidean space leading to certain remnants of locality. Finally, we did not work on any cluster property and did not need such for our treatment. We would like to cover some of these open problems in a continuative work. See chapter 5 for more future perspectives.

Chapter 4

Covariance and Locality

Wick rotation based on the algebraic as well as the correlation functions setting of Chs. 2 and 3 possess the utilization of reduced symmetry groups as a common ground. From a physical perspective, this is not satisfactory. Even if one accepted the breaking of the fundamental property of full Poincaré invariance in the proximity of the Planck scale (or the "noncommutative regime", wherever it may start to be influential), there is no reason why a specific choice of boost- and rotation direction should be preferred. It is thus desirable to rebuild the covariance with respect to $E(d)$ and $\mathcal{P}(d)$ from that of $E_\theta(d)$ and $\mathcal{P}_\theta(d)$, respectively. By deforming the von Neumann algebras with respect to different parameters depending on the indexing region, we are going to do so in the upcoming parts of this chapter.

Before we start doing so, we recall Θ_1, the standard skew-symmetric noncommutativity matrix, initially given in (1.6):

$$\Theta_1 := \begin{pmatrix} 0 & \vartheta_e & & \\ -\vartheta_e & 0 & & \\ & & 0 & \vartheta_m \\ & & -\vartheta_m & 0 \end{pmatrix}. \tag{4.1}$$

4.1 Covariantization

The way of deforming the free scalar field invented in [GL07] was generalized in [GL08] to deform theories given by the field polynomial algebra. In the latter paper, the restricted covariance of the deformed theory coming from the Moyal noncommutativity matrix θ is enlarged by the technique of covariantization. Inspired by the influential paper [DFR95] and the latter approaches, we collect the possible noncommutativity matrices θ into the set

$$\Sigma := \{\Theta \in \mathfrak{M}_{-}^{4\times 4}(\mathbb{R}) \,|\, \mathrm{tr}(\Theta \cdot \Theta) = -2(\vartheta_e^2 + \vartheta_m^2)\,,\, \epsilon_{\mu\nu\rho\sigma}\Theta^{\mu\nu}\Theta^{\rho\sigma} = -8\vartheta_e\vartheta_m\} \ .$$

This means that Θ_1 is an element of Σ. In addition, this Σ is a Lorentz orbit of the standard noncommutativity on four dimensional Minkowski space-time, *i.e.*,

$$\Sigma = \{\Lambda\Theta_1\Lambda^{\mathsf{T}}\,,\ \Lambda \in \mathcal{L}_+^{\uparrow}(4)\} \ .$$

Therefore, it is a homogeneous space for proper orthochronous Lorentz transformations $\Lambda \in \mathcal{L}_+^{\uparrow}$ with respect to the prescription $\Sigma \ni \Theta \mapsto \Lambda\Theta\Lambda^{\mathsf{T}}$. From the Euclidean point of view, we realize that Σ is also homogeneous for the group $SO(4)$ of 4-dimensional rotations, *i.e.*,

$$\Sigma = \{R\Theta_1 R^{\mathsf{T}}\,,\ R \in SO(4)\} \ .$$

Now we use property *3.* of Prop. 2 to infer that

$$\alpha_g^{(\mathcal{M})}(A_\Theta) = (\alpha_g^{(\mathcal{M})}A)_{M\Theta M^{\mathsf{T}}} \ \forall\, g = (a, M)\,, \forall\, \Theta \in \Sigma \ , \tag{4.2}$$

where M is either a rotation matrix or a Lorentz transformation and $a \in \mathbb{R}^4$. Relying on this, we can define the following assignment,

$$\mathscr{O} \ni O \mapsto \check{\mathcal{E}}_\Sigma(O) := (\{A_\Theta \,|\, A \in \mathcal{E}(O)\,,\ \Theta \in \Sigma\})'' \ . \tag{4.3}$$

Lemma 18. *Let $\{\mathcal{E}(O)\}_{O\in\mathscr{O}}$ be an isotonous, $E(d)$-covariant net of von Neumann algebras. Then $\{\check{\mathcal{E}}_\Sigma(O)\}_{O\in\mathscr{O}}$ is also an isotonous, $E(d)$-covariant net of von Neumann algebras.*

Proof.

- Von Neumann algebras:

  Due to the algebraic closure taken in the defining relation (4.3), $\check{\mathcal{E}}_\Sigma(O)$ is a (very large) von Neumann algebra for each $O \in \mathscr{O}$.

- Covariance:

  Let $O \in \mathscr{O}$ and $g \in E(d)$. By equation (4.2) and the covariance of $\{\mathcal{E}(O)\}_{O\in\mathscr{O}}$ we deduce,

$$\begin{aligned} \alpha_g \check{\mathcal{E}}_\Sigma(O) &= (\{A_{R\Theta R^\intercal} \mid A \in \mathcal{E}(gO)\,,\ \Theta \in \Sigma\})'' \\ &= \big(\{A_\Theta \mid A \in \mathcal{E}(gO)\,,\ R^\intercal \Theta R \in \Sigma\}\big)''\ . \end{aligned}$$

  As Σ is homogeneous for $SO(d)$ by the action $\Sigma \ni \Theta \mapsto R\Theta R^\intercal$, we conclude $\alpha_g \check{\mathcal{E}}_\Sigma(O) = \check{\mathcal{E}}_\Sigma(gO)$ for all $O \in \mathscr{O}$.

- Isotony:

  Let $O_1 \subset O_2 \in \mathscr{O}$. From the isotony of $\{\mathcal{E}(O)\}_{O\in\mathscr{O}}$, we directly conclude,

$$\{A_\Theta \mid A \in \mathcal{E}(O_1)\,,\ \Theta \in \Sigma\} \subset \{A_\Theta \mid A \in \mathcal{E}(O_2)\,,\ \Theta \in \Sigma\}\ ,$$

  and the claim is proven by performing the algebraic closure.

$\square$

Hence, we discover that if we do not contemplate a net of von Neumann algebras deformed just by a single skew-symmetric matrix Θ, but the set of all such nets, we obtain a theory which is covariant with respect to the complete symmetry group. Anticipating some of the results coming up, we though need a somewhat tighter notion for our $E(d)$-covariant noncommutative nets. That is because taking *all* possible deformation parameters $\Theta \in \Sigma$ at each indexing set $O \in \mathscr{O}$ is too much for a predictive theory. In other words, $\check{\mathcal{E}}_\Xi$ is just an auxiliary net for exemplifying the enlargement of the symmetry group. We are going to contemplate advanced notions of covariantization in the following.

Commutative Time We want to include considerations like these into our framework, which deals with degenerate noncommutativities as a matter of principle. Therefore, we start on d-dimensional Euclidean space, where $d = s + 2n$, $s, n \in \mathbb{N}$. Analogously to Σ, we define the set

$$\Xi := \{\theta \in \mathfrak{M}_{-}^{d\times d}(\mathbb{R}) \,|\, \mathrm{tr}(\theta \cdot \theta) = -2\sum_{k=1}^{n} \vartheta_k^2 \,,\ \epsilon_{\mu\nu\rho\sigma}\theta_{\mu\nu}\theta_{\rho\sigma} = 0\} \ ,$$

for ϑ_k denotes a real number for $k = 1, \ldots, n$. In the same way as above, Ξ is a homogeneous space with respect to the action $\theta \mapsto R\theta R^{\mathsf{T}}$ for $R \in SO(d)$. For $d = 4$, Ξ contains the matrix θ_1, given in (2.21).

Given a Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ (see Def. 9), we may deform it with respect to the realization of θ given in (2.20). Thus we obtain the noncommutative net $(\mathcal{E}_\theta, \mathcal{Z}_\theta, \alpha^\theta, \sigma^\theta)$. This net is only $E_\theta(d)$-covariant, but we will show how to arrive at a Euclidean quantum field theory of full $E(d)$-covariance now.

Cylindrical Regions As the considerations of Ch. 2 have shown, the cylindrical regions $\mathcal{Z}_\theta := \{\mathcal{Z}_\theta(O)\,,\ O \in \mathscr{O}\}$ arise naturally when deforming a Euclidean field theory with respect to a degenerate noncommutativity matrix $\theta \in \Xi$. Let us introduce the notion $\mathcal{Z} := \{\mathcal{Z}_\theta\,,\ \theta \in \Xi\}$. By the definition given in (2.24), any such $Z \in \mathcal{Z}$ can be written as $Z = \mathcal{Z}_\theta(O)$ for a $\theta \in \Xi$ and an $O \in \mathscr{O}$. Now we associate a noncommutativity θ to any given $Z \in \mathcal{Z}$ in the following way,

$$Z = \mathcal{Z}_\theta(O) \mapsto \theta = \theta(Z) \ .$$

In other words, $\theta(Z) = \theta(\mathcal{Z}_\theta(O)) := \theta\ ,\ O \in \mathscr{O}$.

Immediately, it becomes clear that this specific assignment is some kind of problematic. The reason for this is the fact that $\mathcal{Z}_\theta(O)$ is equal to $\mathcal{Z}_{-\theta}(O)$ for all $O \in \mathscr{O}$. Therefore, the map $\mathcal{Z}_\theta(O) \mapsto \theta$ is not uniquely defined. According to [GL08], one axiomatically would have

1. $Z_1 := \{x + \theta_1 y\,,\ x \in O\,,\ y \in \mathbb{R}^d\} \mapsto \theta_1$ for all $O \in \mathscr{O}$.

2. $\mathcal{Z} \ni Z = RZ_1 \mapsto R\theta_1 R^{\mathsf{T}}$

But *1.* and the proof of covariance below imply that in the case of cylinders, the two attributions are equivalent. Due to the isotony property in Prop. 18 stated in a little while, the attribution $\mathcal{Z}_\theta \mapsto \pm\theta$ is the only disambiguity here.

Definition 18. Let $\mathscr{O} \ni O \mapsto \mathcal{E}(O)$ designate a Euclidean net of observables (local algebras) and $\mathcal{Z}_\theta \ni Z \mapsto \mathcal{E}_\theta(Z)$ its corresponding θ-deformed observable algebra, $\theta \in \Xi$. Then the *covariantization* of $\{\mathcal{E}_\theta(Z)\}_{Z\in\mathcal{Z}_\theta}$ is defined as follows,

$$\mathcal{Z} \ni Z \mapsto \mathcal{E}_\Xi(Z) := \big(\{A_{\theta(Z)}\,,\ A \in \mathcal{E}(Z)\} \cup \{A_{-\theta(Z)}\,,\ A \in \mathcal{E}(Z)\}\big)''\ ,$$

where $\theta(Z)$ is chosen in the way described above.

Examining the well-definition of this notion, we state the following

Lemma 19. *The physical Hilbert space $\mathcal{H}$ is left unchanged by the deformation.*

Proof. Clearly, each deformation parameter $\theta \in \Xi$ induces a primarily distinct noncommutative extension σ^θ of the Euclidean functional σ. All these σ^θ are reflection positive with respect to the Euclidean direction e in use, as σ is $E(d)$-invariant in particular. In order to verify the latter property, we appreciate that reflection positivity with respect to the direction e writes $\sigma(\iota_e(A^*)A) \geqslant 0$ for all $A \in \mathcal{E}^e_>$, where ι_e implements the e-reflection $r_e : x \mapsto x - 2(x,e)e$ and $\mathcal{E}^e_>$ was defined to be the Euclidean algebra generated by all open, bounded subregions of $\mathbb{R}_+e + e^\perp$. The reflection with respect to another direction $e' = Re$ therefore fulfills $\iota_{Re} = \alpha_{(0,R)} \circ \iota_e \circ \alpha_{(0,R^\intercal)}$ while we have $\mathcal{E}^{Re}_> = \alpha_{(0,R)}\mathcal{E}^e_>$. So, for the Euclidean functional we are able to deduce,

$$\sigma(\iota_{Re}(B^*)B) = \sigma(\iota_{Re}(\alpha_{(0,R)}A^*)\alpha_{(0,R)}A) = \sigma(\alpha_{(0,R)}(\iota_e(A^*)A)) = \sigma(\iota_e(A^*)A) \geqslant 0 \quad \forall\, B \in \mathcal{E}^{Re}_>\ ,$$

by Euclidean invariance. Really, all the σ^θ give rise to Hilbert spaces $\mathcal{H}_\theta$ for $\theta \in \Xi$ and by Thm. 6 they all are isomorphic to the one gained from the undeformed theory.

□

Proposition 18. *The action of α_g is well-defined on $\mathcal{E}_\Xi$ for all $g \in E(d)$. Furthermore, $\mathcal{E}_\Xi(Z)$ defines an isotonous, $E(d)$-covariant net of von Neumann algebras on $\mathbb{R}^d$.*

Proof. It is clear from the considerations in Secs. 1.2.3, 1.2.4 and 2.3 that given $Z \in \mathcal{Z}$, $\mathcal{E}_{\theta(Z)}(Z)$ is a von Neumann algebra.

- Isotony: Let $Z_1 \subset Z_2 \in \mathcal{Z}$. Then we have $\theta(Z_1) = \pm\theta(Z_2)$. Assume the contrary, $\theta(Z_1) \neq \pm\theta(Z_2)$ which can only be valid in two cases. Firstly, $\operatorname{Im}\theta(Z_1) \neq \operatorname{Im}\theta(Z_2)$ from which we infer that Z_1 cannot be bounded in projection to $\ker\theta(Z_2)$. In other words, $Z_1 \not\subset Z_2$. The second possibility incorporates $\theta(Z_1) = \lambda\theta(Z_2)$ for a real number $\lambda \neq \pm 1$. But as Ξ forms an $SO(d)$-orbit of the standard noncommutativity θ_1, $\theta(Z_1)$ could not be an element of Ξ then. So we really have $\theta(Z_1) = \pm\theta(Z_2)$.

  Writing $\mathcal{E}_{\theta(Z)}(Z) := \{A_{\theta(Z)}\,,\ A \in \mathcal{E}(Z)\}$, it follows

$$\mathcal{E}_{\theta(Z_1)}(Z_1) \cup \mathcal{E}_{-\theta(Z_1)}(Z_1) \quad = \quad \mathcal{E}_{\pm\theta(Z_2)}(Z_1) \cup \mathcal{E}_{\mp\theta(Z_2)}(Z_1) \subset \mathcal{E}_{\theta(Z_2)}(Z_2) \cup \mathcal{E}_{-\theta(Z_2)}(Z_2)\ ,$$

  because of the net structure of $Z \mapsto \mathcal{E}_\theta(Z)$ proved in Prop. 5. This reveals the claim after performing the algebraic closure.

- Covariance & well-definition:

  Let $g = (a, R) \in E(d)$. According to (2.22) and, more recently, (4.2), acting with the symmetry group on the warped element has the consequence of $\alpha_{(a,R)}(A_\theta) = (\alpha_{(a,R)}A)_{R\theta R^\intercal}$. The cylindrical regions react in the following way:

$$\begin{aligned} gZ &= g\mathcal{Z}_\theta(O) = \{Rx + R\theta k + a \,|\, x \in O\,,\ k \in \mathbb{R}^d\} = \{x + R\theta R^\intercal k \,|\, x \in RO + a\,,\ k \in \mathbb{R}^d\} \\ &= \mathcal{Z}_{R\theta R^\intercal}(gO)\ . \end{aligned}$$

  In fact, the latter equality implies $\theta(RZ + a) = R\theta(Z)R^\intercal$, because the "germ" $O \in \mathscr{O}$ has no influence on the assignment of θ.

  Let $g \in E(d)$, $Z \in \mathcal{Z}$ and $\mathfrak{p}$ be an arbitrary polynomial in $\mathcal{E}_\Xi(Z)$. Then

$$\mathfrak{p} = \sum_{k,j=1}^{N} c_{k,j} A^k_{k,\theta(Z)} B^j_{j,-\theta(Z)}\ ,$$

  where $c_{k,j} \in \mathbb{C}$ and A_k and B_j are elements of $\mathcal{E}(Z)$ for $k, j = 1, \ldots, N$. As α_g extends to a homomorphism on $\mathcal{E}_\theta$ for every $\theta \in \Xi$, it follows,

$$
\begin{aligned}
\alpha_g \mathfrak{p} &= \sum_{k,j=1}^{N} c_{k,j} \alpha_g (A^k_{k,\theta(Z)} B^j_{j,-\theta(Z)}) \\
&= \sum_{k,j=1}^{N} c_{k,j} \alpha_g (A^k_{k,\theta(Z)}) \alpha_g (B^j_{j,-\theta(Z)}) \\
&= \sum_{k,j=1}^{N} c_{k,j} ((\alpha_g A_k)_{R\theta(Z)R^{\intercal}})^k ((\alpha_g B_j)_{R(-\theta(Z))R^{\intercal}})^j \ ,
\end{aligned}
$$

where we have used that α is the adjoint action of unitaries and inserted $\mathbb{1}_{\mathcal{H}}$ in the second equality. As $R(-\theta)R^{\intercal} = -R\theta R^{\intercal}$ for all $R \in SO(d)$ and $\theta \in \Xi$, we arrive at,

$$
\alpha_g \mathfrak{p} = \sum_{k,j=1}^{N} c_{k,j} ((\alpha_g A_k)_{\theta(gZ)})^k ((\alpha_g B_j)_{-\theta(gZ)})^j \ ,
$$

because $R\theta R^{\intercal} = \theta(RZ) = \theta(gZ)$. Thus, $\alpha_g \mathfrak{p}$ generates an element in $\mathcal{E}_{\Xi}(gZ)$, as $\{\mathcal{E}(O)\}_{O \in \mathscr{O}}$ was assumed to be $E(d)$-covariant in Def. 18. By applying the same arguments once more, one directly infers that α_g acts as a homomorphism on the elements of $\mathcal{E}_{\Xi}$. So, α_g is indeed well-defined on $\mathcal{E}_{\Xi}$.

By carrying out the algebraic closure of $\mathcal{E}_{\theta(gZ)}(gZ) \cup \mathcal{E}_{-\theta(gZ)}(gZ)$, the covariance of $\{\mathcal{E}_{\Xi}(Z)\}_{Z \in \mathcal{Z}}$ is proven.

□

Remark 16. Let the net $\{\mathcal{E}(O)\}_{O \in \mathscr{O}}$ be realized via a continuous representation on a Hilbert space. Then the product of algebra elements deformed with respect to different noncommutativities is well-defined as juxtaposition of Hilbert space operators due to Lemma 19.

Lorentzian Covariantization The covariantization procedure we have specified is not bound to the Euclidean case. In order to give the corresponding definition, let us covariantize not before the utilization of the commutative-time Wick rotation. In particular, the starting point would then be a commutative Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$. Deforming it with respect to the warped convolutions formalism and defining a noncommutative Lorentzian theory $(\mathcal{M}_\theta, \mathcal{Z}_\theta, \alpha^{\theta,\mathcal{M}}, \omega)$ there upon leads to a $\mathcal{P}_\theta(d)$-covariant theory. Applying our covariantization procedure to this Lorentzian net, the symmetry group can be enlarged again. In this case,

$$\mathcal{M}_\Xi(Z) := \big(\{A_{\theta(Z)}\,,\ A \in \mathcal{M}(Z)\} \cup \{A_{-\theta(Z)}\,,\ A \in \mathcal{M}(Z)\}\big)''\ ,$$

is the covariantization of the Wick rotated net $\{\mathcal{M}_\theta(O)\}_{O\in\mathscr{O}}$.

Once we have appreciated the well-definition of $\mathcal{M}_\Xi(Z)$, the proof comprising $\mathcal{P}(d)$-covariance and isotony of $\mathcal{M}_\Xi$ is provided along the lines of that belonging to Prop. 18.

Adopting the complete result of Thm. 6, we appreciate that the deformation $(\mathcal{M}^\theta, \mathcal{Z}_\theta, \alpha^{\mathcal{M}}, \omega)$ of the Haag-Kastler net $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}}, \omega)$ leads to a covariantization isomorphic to $\{\mathcal{M}_\Xi(Z)\}_{Z\in\mathcal{Z}}$, *i.e.*, $\mathcal{M}^\Xi(Z) \simeq \mathcal{M}_\Xi(Z)$ for all $Z \in \mathcal{Z}$. This is a straightforward consequence of $\mathcal{M}^{\theta(Z)}(Z)$ being isomorphic to $\mathcal{M}_{\theta(Z)}(Z)$ for all $Z \in \mathcal{Z}$.

In Remark 5, we have stressed that in a situation of full $\mathcal{P}(d)$-covariance, the joint spectrum of the translation generators in the deformed theory is contained in the forward light-cone according to Prop. 4. This means that by reconstituting full Poincaré covariance, we get back the original spectrum condition.

An extensive inspection of fully covariant Wick rotation raises the question of whether we can directly continue the net $\mathcal{E}_\Xi$. There are two canonical ways of defining the time-zero content of a covariantized Euclidean theory:

$$\mathcal{E}_{\Xi,0}(S) := \bigcap_{\substack{Z\supset S\\ Z\in\mathcal{Z}}} \mathcal{E}_\Xi(Z) \quad , \quad \mathcal{E}_{0,\Xi}(S) := \big(\{B_{\theta(S)} \mid B \in \mathcal{E}_0(S)\} \cup \{B_{-\theta(S)} \mid B \in \mathcal{E}_0(S)\}\big)''$$

Lemma 20. *These two notions of covariantized time-zero algebras coincide,* i.e.,

$$\mathcal{E}_{\Xi,0}(S) = \mathcal{E}_{0,\Xi}(S) \quad \forall\, S \in \mathcal{S}\ ,$$

where $\mathcal{S} := \{\mathcal{S}_\theta\,,\ \theta \in \Xi\}$.

Proof. Choose certain $\theta \in \Xi$ and let $S \in \mathcal{S}_\theta$. Then

$$
\begin{aligned}
\mathcal{E}_{\Xi,0}(S) &= \bigcap_{\substack{Z\supset S\\ Z\in\mathcal{Z}_\theta}} \mathcal{E}_\Xi(Z) = \bigcap_{\substack{Z\supset S\\ Z\in\mathcal{Z}_\theta}} \left(\{A_{\theta(Z)} \mid A\in\mathcal{E}(Z)\} \cup \{A_{-\theta(Z)} \mid A\in\mathcal{E}(Z)\}\right)'' \\
&= (\mathcal{E}_{\theta,0}(S) \cup \mathcal{E}_{-\theta,0}(S))'' \ .
\end{aligned}
$$

By *1.* of Prop. 6, $\mathcal{E}_{\theta,0}(S)$ equals $\mathcal{E}_{0,\theta}(S)$ for a specific choice of θ. But this conclusion is true for any other deformation parameter, as they all induce well-defined C^*-norms $\|.\|_\theta$, Rieffel products and noncommutative algebras $(\mathcal{E}_\theta, \times_\theta)$. $\mathcal{S}$ was defined to be the set of all time-zero stripes. Thus, for each $S\in\mathcal{S}$ there is a $\theta\in\Xi$ such that $S\in\mathcal{S}_\theta$. As we have discussed above, the subspace of infinite extent up to a sign determines the corresponding deformation parameter $\theta(Z)$ of a cylinder $Z\in\mathcal{Z}$. Thus to a time-zero stripe $S\in\mathcal{S}_\theta$ the same noncommutativity is associated, up to a sign, as to any cylinder $Z\in\mathcal{Z}_\theta$. In other words, $\theta(S)=\pm\theta(Z)$ for all $S\in\mathcal{S}_\theta$, $Z\in\mathcal{Z}_\theta$. Therefore,

$$
\mathcal{E}_{\Xi,0}(S) = (\mathcal{E}_{0,\theta}(S) \cup \mathcal{E}_{0,-\theta}(S))'' = \mathcal{E}_{0,\Xi}(S) \ \forall\, S\in\mathcal{S}_\theta \ .
$$

As the concrete value $\theta\in\Xi$ was arbitrary, the conclusion holds for all $S\in\mathcal{S}$ as well. □

This leads to our definition of the Wick rotated covariantization:

$$
\widetilde{\mathcal{M}}_\Xi(Z) := \left(\bigcup_{S\in\mathcal{S}} \{\alpha_g^{\mathcal{M}} \pi\mathcal{E}_{0,\Xi}(S) \mid g\in\mathcal{P}(d):\ gS\subset Z\}\right)''
$$

The action of $\alpha^{\mathcal{M}}$ on $\mathcal{E}_{0,\Xi}(S)$ is well-defined, as one can now see from

$$
\begin{aligned}
g\in\mathcal{P}(d):\qquad & \alpha_g^{\mathcal{M}}\{\pi B_{\theta(S)} \mid B\in\mathcal{E}_0(S)\} = \{(\alpha_g^{\mathcal{M}}\pi B)_{\theta(gS)} \mid B\in\mathcal{E}_0(S)\} \\
&= \{A_{\theta(S)} \mid A\in\alpha_g^{\mathcal{M}}\pi\mathcal{E}_0(S)\} \ ,
\end{aligned}
$$

and the fact that $\alpha_g^{\mathcal{M}}\pi\mathcal{E}_0(S)$ generates $\mathcal{M}(Z)$. Isotony and covariance follow straightforwardly from this definition and the corresponding result of Prop. 7.

So we are able to write down a proposition summing up our results of noncommutative covariantization:

Proposition 19. *The Wick rotation* $\widetilde{\mathcal{M}}_\Xi$ *of the covariantization* $\mathcal{E}_\Xi$ *equals the covariantization* $\mathcal{M}_\Xi$ *of the Wick rotated net* $\mathcal{M}_\theta$*. In precise terms, we have,*

$$\widetilde{\mathcal{M}}_\Xi(Z) = \mathcal{M}_\Xi(Z) \quad \forall Z \in \mathcal{Z} \ .$$

In other words: In the commutative-time scenario, the procedures of Wick rotation and covariantization of quantum field theories on Moyal space commute. The situation is depicted in Fig. 4.1*.*

Proof. Let $g \in \mathcal{P}_{\theta(Z)}(d)$, and $S \in \mathcal{S}_{\theta(Z)}$, such that $gS \subset Z \in \mathcal{Z}_{\theta(Z)}$. Then we infer,

$$\begin{aligned} & (\alpha_g^{\mathcal{M}}\mathcal{E}_0)_{\theta(Z)}(S) \cup (\alpha_g^{\mathcal{M}}\mathcal{E}_0)_{-\theta(Z)}(S) = \alpha_g^{\mathcal{M}}\mathcal{E}_{0,\theta(g^{-1}Z)}(S) \cup \alpha_g^{\mathcal{M}}\mathcal{E}_{0,-\theta(g^{-1}Z)}(S) \\ = \ & \alpha_g^{\mathcal{M}}\mathcal{E}_{0,\theta(S)}(S) \cup \alpha_g^{\mathcal{M}}\mathcal{E}_{0,-\theta(S)}(S) = \alpha_g^{\mathcal{M}}(\mathcal{E}_{0,\theta(S)}(S) \cup \mathcal{E}_{0,-\theta(S)}(S)) \ , \end{aligned} \tag{4.4}$$

because $\theta(g^{-1}Z) = \pm\theta(S)$, as $S \subset g^{-1}Z$. According to Prop. 7, $\mathcal{M}_\theta(Z)$ fulfills the time-zero condition and this proof is again independent of the concrete parameter $\theta \in \Xi$. Next we realize,

$$\{(\alpha_g^{\mathcal{M}}\pi\mathcal{E}_0)_{\theta(Z)}(S) \mid g \in \mathcal{P}_{\theta(Z)}(d), \ gS \subset Z\} = \big(\{\alpha_g^{\mathcal{M}}\pi\mathcal{E}_0(S) \mid g \in \mathcal{P}_{\theta(Z)}(d), \ gS \subset Z\}\big)_{\theta(Z)} .$$

So, applying the latter relation to (4.4) and forming the algebraic closure indeed has the consequence,

$$\mathcal{M}_\Xi(Z) = \widetilde{\mathcal{M}}_\Xi(Z) \, , \ \forall \, Z \in \mathcal{Z}_{\theta(Z)} \ .$$

Finally, by the full $\mathcal{P}(d)$-covariance of both $\mathcal{M}_\Xi$ and $\widetilde{\mathcal{M}}_\Xi$, this equality also holds for arbitrary $Z \in \mathcal{Z}$.

□

Other Index Sets In the thesis at hand, we are concerned with noncommutative Wick rotation. This undertaking at the moment is only operable on noncommutative spaces featuring degenerate deformation matrices. Moreover, the cylindrical regions we considered are favored by these spaces.

$$
\begin{tikzcd}
\mathcal{E}_\theta \arrow[rr, leftrightarrow] \arrow[d] & & \mathcal{M}_\theta \arrow[d] \\
\mathcal{E}_\Xi \arrow[r, leftrightarrow] & \widetilde{\mathcal{M}}_\Xi \arrow[r, phantom, "\cong"] & \mathcal{M}_\Xi
\end{tikzcd}
$$

Figure 4.1: Another commutative diagram

Nonetheless, deformed nets indexed with respect to another set of space-time regions might be interesting as well, especially when it comes to (restricted notions of) locality. With regard to the *wedge-local quantum fields* [GL08], we pose the question of a Euclidean analog. Before we elaborate on the answer, we will give a short summary of the wedge and its connection to noncommutative quantum field theory.

The *standard wedge* is defined to be $W_1 := \{x \in \mathbb{R}^d \mid x_1 > |x_0|\}$ and the set $\mathcal{W}_0$ of all wedges of interest is given by the $\mathcal{L}_+^\uparrow$-orbit of W_1. Concerning Moyal-Minkowski space, we stick to the notation of [BS08, BLS11] for the sake of confirmability. These papers feature the standard noncommutativity Q_1 of the form

$$
Q_1 := \begin{pmatrix} 0 & \kappa_e & & \\ \kappa_e & 0 & & \\ & & 0 & \kappa_m \\ & & -\kappa_m & 0 \end{pmatrix} . \tag{4.5}
$$

In the following, we are going to consider the d-dimensional generalizations of the matrices Q_1 and Θ_1, *i.e.*,

$$
\begin{aligned}
Q_1 &:= K_e \oplus K_1 \oplus \cdots \oplus K_n \ , \ K_j := \begin{pmatrix} 0 & \kappa_j \\ -\kappa_j & 0 \end{pmatrix} \\
\Theta_1 &:= \theta_e \oplus \theta_1 \oplus \cdots \oplus \theta_n \ , \ \theta_j := \begin{pmatrix} 0 & \vartheta_j \\ -\vartheta_j & 0 \end{pmatrix} \ , \ j = 1, \ldots, n \ ,
\end{aligned}
$$

on $\mathbb{R}^d$. Q_1 is skew-symmetric w.r.t. the scalar product of Minkowski space-time and we have $Q_1 = \eta\Theta_1$ for $\kappa_e = \vartheta_e$ and $\kappa_m = -\vartheta_m$ and η denoting the Minkowski metric. The properties of $\mathcal{W}_0$ and the warped convolutions combine in an amazing way according to the requirements of quantum field theory, as one can see from the following list of properties:

1. $Q(W_1) = Q_1$,
2. $Q(\Lambda W_1) = \Lambda Q_1 \Lambda^{\mathsf{T}}$ for all $\Lambda \in \mathcal{L}_+^{\uparrow}$,
3. $Q_1 V_+ \subset W_1$ for V_+ denoting the forward light-cone ,
4. If $\exists\, (a, \Lambda) \in \mathcal{P}$ such that $\Lambda W_1 + a \subset W_1$ then $\Lambda Q_1 \Lambda^{\mathsf{T}} = Q_1$,
5. If $\exists\, (a, \Lambda) \in \mathcal{P}$ such that $\Lambda W_1 + a \subset W_1'$ then $\Lambda Q_1 \Lambda^{\mathsf{T}} = -Q_1$,

where $O' := \{x \in \mathbb{R}^d \,|\, (x - y, x - y) < 0 \ \forall\, y \in O\}$ denotes[1] the causal complement of the set O. Now by *wedge-locality* we mean the following property,

$$
[A_{Q_1}, B_{-Q_1}] = 0 \qquad \forall\, A \in \mathcal{M}(W_1) \, , \ B \in \mathcal{M}(W_1') \ .
$$

Let us write $\mathcal{W}$ for the Poincaré orbit of the standard wedge W_1. Then the important result states: the collection $(\mathcal{M}_\Sigma, \mathcal{W}, \alpha^{\mathcal{M}}, \omega)$ is an isotonous, $\mathcal{P}(d)$-covariant and wedge-local net of von Neumann algebras on Minkowski space-time, *cf.* [BLS11]. We will explain this in more detail in the next subsection.

In the following, we are going to show that there can be no subset of Euclidean space fulfilling a canonical modification of these properties 1. - 5.

[1]Reminder: $(.,.)$ was our symbol for the scalar product on Minkowski space-time

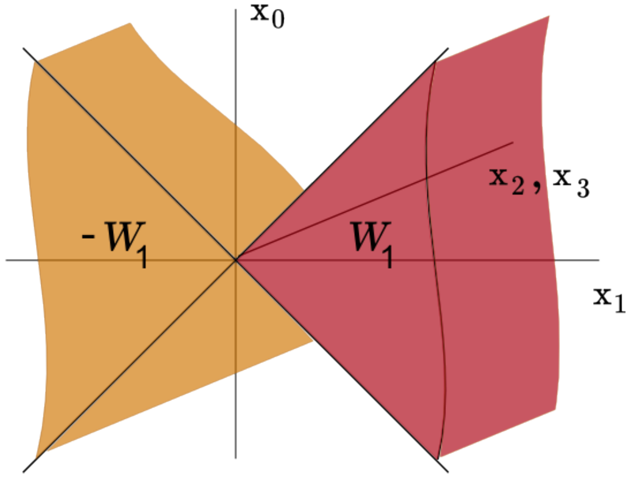

Figure 4.2: The standard wedge W_1 and its causal complement $-W_1$.

Definition 19. A subset $\mathcal{U} \subset \mathcal{O}$ of the open, bounded regions in $\mathbb{R}^d$ is said to consist of *Euclidean wedges*, if it fulfills the following list of properties:

1. $\exists\, U_1 \in \mathcal{U} : \theta(U_1) = \Theta_1$,
2. $\theta(R\Theta_1) = R\Theta_1 R^{-1} \;\forall\, R \in SO(d)$,
3. $\Theta_1 \mathbb{R}^d_+ \subset U_1$,
4. If $\exists\,(a, R) \in E(d)$ such that $RU_1 + a \subset U_1$ then $R\Theta_1 R^{-1} = \Theta_1$,
5. If $\exists\,(a, R) \in E(d)$ such that $RU_1 + a \subset U_1^c$ then $R\Theta_1 R^{-1} = -\Theta_1$.

Remark 17. The purpose of relations 1. and 2. is to covariantly associate skew-symmetric matrices to space-time regions. These postulations will *per se* not cause a problem for any of those regions. The example of cylindrical regions treated in the preceding paragraph illustrates this. The situation changes when the assignment $U \mapsto \theta(U)$ shall be uniquely determined, as we are going to see later. Coming to the third property, it is obvious that Euclidean space lacks the concept of a light-cone. In other words, Euclidean (time-)translations are not bounded to a proper subspace of $\mathbb{R}^d$, unless they are represented on the physical Hilbert space $\mathcal{H}$ in view of a subsequent Wick rotation. In that case, the analog of the forward light-cone V_+ would be $\mathbb{R}^d_+$.

Lemma 21. *There is no set of Euclidean wedges. More precisely, we have,*

1. *The only realization of $\mathcal{U}$ satisfying* 3. *of Def.* 19 *is given by the $SO(d)$-orbit of*

$$\mathbb{H}_1 := \{x \in \mathbb{R}^d \mid x_1 > 0\} \ .$$

   However, points 4. *and* 5. *are violated in this case.*

2. *If $\mathcal{U} \subset \mathcal{Z}$ and each $U \in \mathcal{U}$ is generated by a sphere centered at the origin, properties* 1., 2. *and* 4. *of Def.* 19 *are fulfilled whereas* 3. *and* 5. *are not. In case of commutative time (using θ from* (2.20) *instead of Θ_1),* 3. *holds.*

3. *Allowing for dilatations, properties* 1., 2. *and* 4. *of Def.* 19 *are valid also for the cylinders,*

$$Z_a^b := \{x \in \mathbb{R}^4 \mid a < x_0^2 + x_1^2 < b\} \quad , \quad 0 < a < b \ .$$

   Moreover, we can find a map $\hat{D}_\rho$ such that $\hat{D}_\rho U_1 \subset U_1^{\mathrm{c}}$ and $\hat{D}_\rho \Theta_1 \hat{D}_\rho^{-1} = -\Theta_1$, but point 5. *is still violated here.*

Proof. 1. In fact, $\Theta_1 \mathbb{R}_+^d = -\mathbb{H}_1$, so the set spanned by $SO(d)$-rotations of this half-space is really the only possible choice to fulfill *3.* of Def. 19. However, while Θ_1 is left invariant by arbitrary x_0, x_1-rotations, neither $\mathbb{H}_1$ nor $-\mathbb{H}_1$ is. So, property 4. is violated. As the same rotation about an angle of π inverts $\mathbb{H}_1$, but does not transfer Θ_1 into $-\Theta_1$, a predicate modifying 5. would also fail to be fulfilled.

2. For U_1 every cylinder in $\mathcal{Z}_{\theta_1}$ can be chosen. Furthermore, in the proof of covariance in Prop. 18, we have shown the validity of the second request of Def. 19 for all elements of $\mathcal{Z}$. It remains to show property 4. and to discuss 5. Take an element $R \in SO(2) \times SO(d-2)$ of the stabilizer group $E_{\Theta_1}(d)$ of Θ_1. Now, let B denote a sphere of arbitrary radius centered at the origin. By the spherical symmetry, $B + \operatorname{Im}\theta$ for θ of the form (2.20) will be invariant under x_0, x_1-rotations. As the cylinders are infinitely extended in the remaining directions, the $1 \times SO(d-2)$-invariance follows. This finishes the proof of feature 4. Yet the only Euclidean transformations moving $B + \operatorname{Im}\theta$ to its complement are translations which are known to have no effect on any element of Ξ.

3. For all $a, b > 0$, the set Z_a^b is an element of $\mathcal{Z}_\theta$. Its germ is given by an annulus of thickness $b-a$ and centered at the origin. Thus, it is straightforward to comprehend that properties 1. - 4. of Def. 19 are valid for cylinders of this kind. If one allows for space dilatations additionally to the Euclidean group, one can specify the map given by $\hat{D}_\rho := \mathrm{diag}\{\rho, -\rho, 1, -1\}$ for suitable values of $\rho > 0$. Indeed, we have

$$\hat{D}_\rho Z_a^b = \{x \in \mathbb{R}^d \mid \rho^2 a < x_0^2 + x_1^2 < \rho^2 b\} ,$$

which causes $\hat{D}_\rho Z_a^b \subset (Z_a^b)^{\mathrm{c}}$ for either $\rho \geqslant \sqrt{\frac{b}{a}}$ or $\rho \leqslant \sqrt{\frac{a}{b}}$. Furthermore, $\hat{D}_\rho \Theta_1 \hat{D}_\rho^{-1} = -\Theta_1$ holds according to $\hat{D}_\rho^{-1} = \hat{D}_{1/\rho}$. Clearly, $D_\rho := \mathrm{diag}\{\rho, \rho, 1, 1\}$ also maps Z_a^b to its complement under the above conditions, but does not map Θ_1 into $-\Theta_1$. So 5. is still not fulfilled completely, even when adding the dilatations.

□

Now assume that the Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ satisfies Euclidean locality (2.1), *i.e.*,

$$O_1, O_2 \in \mathscr{O} : \quad O_1 \cap O_2 = \varnothing \quad \Rightarrow [\mathcal{E}(O_1), \mathcal{E}(O_2)] = \{0\} . \tag{4.6}$$

Surely, the half-spaces $\pm\mathbb{H}_1$ of *1.* in the latter lemma would comply with a deformed version of locality similar to wedge-locality. According to (4.6), the Euclidean causal complement of an open bounded region in $\mathbb{R}^d$ would be its set-theoretic complement. Concerning the half-spaces, we have $(\mathbb{H}_1)^{\mathrm{c}} = -\mathbb{H}_1$. This in turn permits $[\alpha_{\Theta_1 x}(A), \alpha_{-\Theta_1 x}(B)] = 0$ for all $x, y \in \mathbb{R}_+^d$ and therefore $[A_{\Theta_1}, B_{-\Theta_1}] = 0$ for $A \in \mathbb{H}_1$ and $B \in -\mathbb{H}_1$.

4.2 Remnants of Locality

Assuming (4.6) to hold, it follows from [Sch99, Thm. 3.7] that the corresponding commutative Lorentzian theory $(\mathcal{M}, \mathscr{O}, \alpha^{\mathcal{M}}, \omega)$ satisfies micro-causality (locality), *i.e.*,

$$O_1, O_2 \in \mathscr{O} : \quad O_1 \subset O_2' \quad \Rightarrow [\mathcal{M}(O_1), \mathcal{M}(O_2)] = \{0\} .$$

We can take advantage of this fact and combine it with the results of the warped convolutions framework. Next, we define

$$\mathcal{U} \ni U \mapsto \widehat{\mathcal{E}}_\Xi(U) := \left(\{A_{\theta(U)}\,,\ A \in \mathcal{E}(U)\}\right)'' \ ,$$

and the corresponding notion $\widehat{\mathcal{M}}_\Xi$ for the Lorentzian nets. This new notion is necessary as the inclusion of both the deformations w.r.t. θ and $-\theta$ into the covariantization spoils wedge-locality right from the start. Preparing the next result, we contemplate regions $\mathcal{U}$ infinitely extended in the image of a deformation parameter $\theta \in \Xi$ of a commutative-time scenario. That is to say, we consider $\mathcal{U} \subset \mathcal{Z}$.

Lemma 22. *1. Let $\mathcal{U} \subset \mathcal{Z}$, such that the prescription $\mathcal{U} \ni U \mapsto \theta(U)$ is uniquely determined. Then $\{\widehat{\mathcal{E}}_\Xi(U)\}_{U\in\mathcal{U}}$ and $\{\widehat{\mathcal{M}}_\Xi(U)\}_{U\in\mathcal{U}}$ are isotonous nets of von Neumann algebras.*

2. The quantum field theory $\left\{\begin{array}{c} (\widehat{\mathcal{E}}_\Xi, \mathcal{U}, \alpha, \sigma) \\ (\widehat{\mathcal{M}}_\Xi, \mathcal{U}, \alpha^{\mathcal{M}}, \omega) \end{array}\right. \ \textit{is} \ \left.\begin{array}{c} E(d) \\ \mathcal{P}(d) \end{array}\right\}$*-covariant.*

Proof. 1. By the uniqueness of the prescription $\mathcal{U} \ni U \mapsto \theta(U)$, it follows that given $U_1 \subset U_2 \in \mathcal{U}$, we have $\theta(U_1) = \theta(U_2)$. The proof is analogous to that of isotony in Prop. 18 for the nets $\{\widehat{\mathcal{E}}_\Xi(U)\}_{U\in\mathcal{U}}$, because $\mathcal{U} \subset \mathcal{Z}$.

2. The covariance w.r.t. the full symmetry groups is obtained in the same way as in the proof of covariance in Prop. 18, as this very proof does not depend on the concrete set of cylinders.

□

Remark 18. As we have seen, $\mathcal{Z}$ does not induce a uniquely determined map $\mathcal{Z} \to \Xi$. Examples for families of regions $\mathcal{U}$ that do allow such a map are the wedges $\mathcal{W}$, but also the $SO(d)$-orbits of the half spaces $\mathbb{H}_1$ and $e_+^\perp$, which we are going to consider below.

Now, a such deformed Wightman theory satisfies wedge-locality:

Proposition 20. *For a Euclidean field theory $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ the assumption of Euclidean locality* (4.6) *is sufficient for the associated noncommutative Lorentzian quantum field theory $(\widehat{\mathcal{M}}_\Sigma, \mathcal{W}, \alpha^{\mathcal{M}}, \omega)$ to fulfill wedge-locality.*

Proof. Having deformed the Wick rotated Lorentzian net $(\mathcal{M}, \mathscr{O})$ according to Q_1, given in (4.5), we suppose the symmetry group to be enlarged by the covariantization of $\mathcal{M}^{Q_1}$. Consequently, we can allow for all values of $\Theta \in \Sigma$. Then for every such Θ, there exists a Lorentz transformation

Λ effecting $\Lambda\Theta\Lambda^\intercal = -\Theta$. Thus we may consider warped elements with respect to $-Q_1$. As we have stressed, Q_1 maps the closed forward light-cone $\overline{V_+}$ into the wedge W_1, whereas $-Q_1$ transfers $\overline{V_+}$ to the causal complement $(W_1)' = -W_1$. Wedges are cones in particular, therefore it follows $y \pm Q_1 x \in \pm W_1$ for all $y \in \pm W_1$ and $x \in \overline{V_+}$. Hence by micro-causality we realize that $O_1 \subset O_2'$ for $O_1, O_2 \in \mathscr{O}$ implies

$$[\alpha^{\mathcal{M}}_{(Q_1 x,1)}(A), \alpha^{\mathcal{M}}_{(-Q_1 y,1)}(B)] = 0 \quad \forall A \in O_1,\ B \in O_2, \quad \forall x, y \in \overline{V_+}\ .$$

Now the operative point is [BLS11, Prop. 2.10]: it states that the latter relation implies

$$[A_{Q_1}, B_{-Q_1}] = 0\ . \tag{4.7}$$

The prescription $\mathcal{W} \ni W \mapsto \theta(W)$ is unique, as has been worked out in [GL08], so the collection $(\widehat{\mathcal{M}}_\Sigma, \mathcal{W}, \alpha^{\mathcal{M}}, \omega)$ really is a $\mathcal{P}(d)$-covariant and wedge-local, noncommutative quantum field theory. □

Let us apply these findings to our commutative-time scenario now. Instead of Q_1, we use the degenerate matrix θ, given in (2.20), as a reference. Of course, this θ and all images $\Lambda\theta\Lambda^\intercal$ are skew-symmetric, hence all the facts we have recapitulated still hold. It is Thm. 6 which says that we could well have started with the θ-deformed Euclidean net $(\mathcal{E}_\theta, \mathcal{Z}_\theta)$ and continued it to $(\mathcal{M}_\theta, \mathcal{Z}_\theta)$, because it is isomorphic to $(\mathcal{M}^\theta, \mathcal{Z}_\theta)$.

Both θ and $-\theta$ of this form map $\overline{V_+}$ to $\{0\} \times \{0\} \times \mathbb{R}^{2n} = \partial W_1$, and an arbitrary $\theta \in \Xi$ maps $\overline{V_+}$ to $\Lambda_\theta \partial W_1 \Lambda_\theta^\intercal$. Due to the specific form of (2.20) we have $[\alpha^{\mathcal{M}}_{(\theta x,1)}(A), \alpha^{\mathcal{M}}_{(-\theta y,1)}(B)] = 0$ for all $x, y \in \mathbb{R}^d$ (and not just for elements of the forward light-cone) if A and B lie in algebras of spacelike separated cylinders. In particular, we have $[\alpha^{\mathcal{M}}_{(\theta x,1)}(A), \alpha^{\mathcal{M}}_{(\theta y,1)}(B)] = 0\ \forall\, x, y \in \mathbb{R}^d$. It is important to note that in general this does not lead to the vanishing of a commutator of elements warped w.r.t. the same matrix.

In addition to these observations, we are in the position to improve the situation a bit more. In fact, the assumption of Euclidean locality is not necessary for this sort of locality. Now we specify a weaker condition on a Euclidean field theory causing the Wick rotated quantum field theory to satisfy wedge-locality. To this end, denote $e_\pm^\perp := \{x \in \mathbb{R}^d \mid x_0 = 0\,,\ \pm x_1 > 0\}$.

Lemma 23. *Let $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ be a Euclidean field theory fulfilling the time-zero condition and pick a Euclidean time direction e. If $[\mathcal{E}_0(e_+^\perp), \mathcal{E}_0(e_-^\perp)] = \{0\}$, then the deformed Wick rotated theory $(\widehat{\mathcal{M}}_\Sigma, \mathcal{W}, \alpha, \omega)$ fulfills wedge-locality.*

Proof. We can perform the Wick rotation of $(\mathcal{E}, \mathscr{O}, \alpha, \sigma)$ to obtain the Lorentzian field theory $(\mathcal{M}, \mathscr{O}, \alpha, \omega)$ by the methods of Sec. 2.1. Now we deform the Euclidean net with respect to the standard noncommutativity Θ_1 corresponding to e, which was defined in (4.1). Particularly in Remark 1, we have stressed that the time-zero content of the commutative nets coincide, *i.e.*,

$$\mathcal{M}_0(K) = \mathcal{E}_0(K) \quad \forall\, K \subset e^\perp \ .$$

From the time-zero condition given in (TZ) for general open bounded regions of $\mathbb{R}^d$ we obtain the respective ones for the wedges:

$$\begin{aligned}
\mathcal{M}(W_1) &= \{\alpha_g^{\mathcal{M}} \mathcal{M}_0(e_+^\perp) \mid g \in \mathcal{P}(d)\, ,\, g\, e_+^\perp \subset W_1\}'' \ , \\
\mathcal{M}(-W_1) &= \{\alpha_h^{\mathcal{M}} \mathcal{M}_0(e_+^\perp) \mid h \in \mathcal{P}(d)\, ,\, h\, e_-^\perp \subset -W_1\}'' \ .
\end{aligned}$$

The reasons for this simple form are the relation $e^\perp = e_-^\perp \cup e_0^\perp \cup e_+^\perp$ for $e_0^\perp := \{x \in \mathbb{R}^d \mid x_0 = x_1 = 0\}$ on the one hand and the nonexistence of $\mathcal{P}(d)$-elements mapping $e_-^\perp$ $(e_+^\perp)$ into W_1 $(-W_1)$ on the other hand. Due to the definition of the warped convolution on Minkowski space-time, we infer the extension of indexing space-time regions from half-planes to wedges, *i.e.*,

$$A \in \mathcal{M}_0(e_\pm^\perp) \quad \Rightarrow \quad A_{\pm Q_1} \in \mathcal{M}_{\pm Q_1}(\pm W_1) \ .$$

This is the analog to trading cylinders for arbitrary open, bounded regions and is caused by the fact that Q_1 maps the spectrum of space-time translations into the right wedge W_1, while it is mapped into the causal complement $-W_1$ by $-Q_1$. By assumption, the time-zero algebras generated by the half planes $e_+^\perp$ and $e_-^\perp$ commute. In the above paragraphs, we have stressed that this is enough to deduce for all $x, y \in \overline{V}_+$:

$$A \in \mathcal{M}_0(e_+^\perp)\, ,\ B \in \mathcal{M}_0(e_-^\perp) \ \ \Rightarrow [\alpha_{(Q_1 x, 1)}(A), \alpha_{(-Q_1 y, 1)}(B)] = 0 \ \ \Rightarrow [A_{Q_1}, B_{-Q_1}] = 0 \ .$$

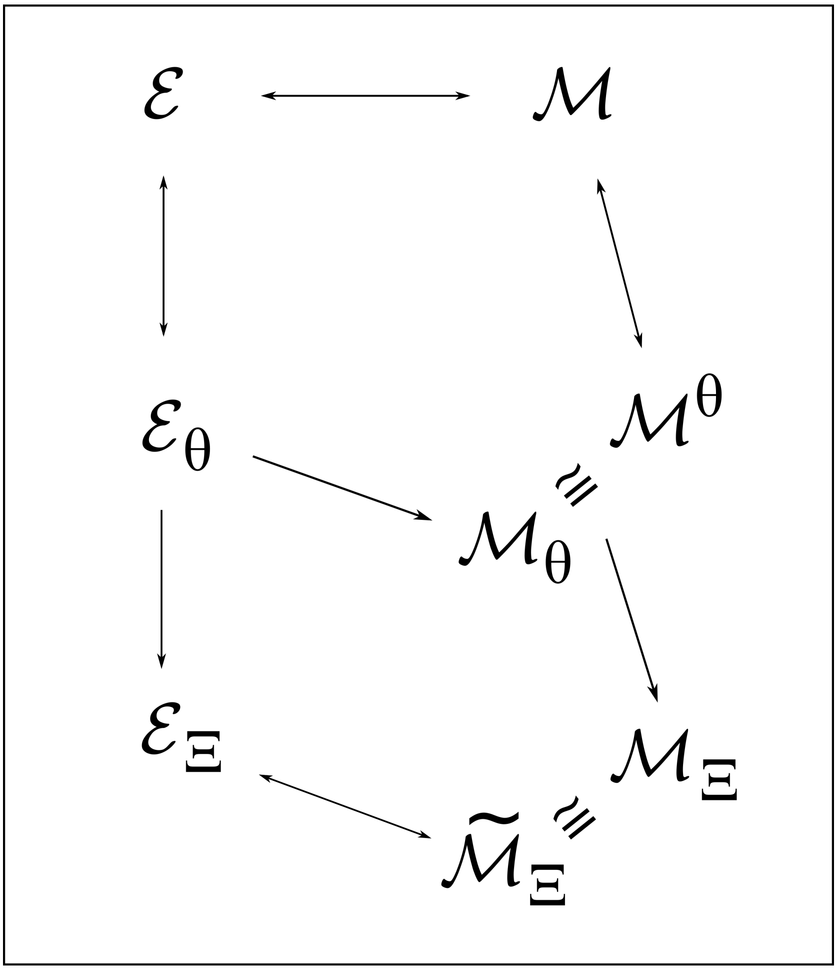

Figure 4.3: Variants of nc. Wick rotation & covariantization

Hence $(\mathcal{M}_{Q_1}, \mathcal{W}, \alpha^{\mathcal{M}}, \omega)$ fulfills wedge-locality. Now it is straightforward to consider the set of all theories indexed by Σ: any Poincaré transformation (a, Λ) which maps Q_1 to another element $\Theta \in \Sigma$ has the effect of mapping the net as well as the appropriate wedge W to $\mathcal{M}_{\Lambda Q_1 \Lambda^\intercal} = \mathcal{M}_\Theta$ and $\Lambda W + a$, respectively. Wedge-locality persists in each such "inertial system", and the assertion is proved. $\square$

Finally, we close this chapter by Fig. 4.3, which illustrates the various ways of Wick rotating a Euclidean field theory $\mathcal{E}$ towards a $\mathcal{P}(d)$-covariant and wedge-local quantum field theory $\mathcal{M}_\Xi$ in the commutative-time scenario.

Chapter 5

Discussion

Summary In this thesis, the Wick rotation of quantum field theories on noncommutative spaces is worked out in detail. The elaboration has been structured in two main perspectives: the first, located in Ch. 2, comprises the algebraic approach to quantum field theory, while the second, which is considered in Ch. 3, focuses on the theory given by its set of correlation functions. Furthermore, Ch. 4 consists of results concerning the recovering of the full symmetry group and conditions on restricted notions of locality.

Quantum field theories on noncommutative spaces are considered a possible way to improve the knowledge about high energy regimes of space-time, especially when it comes to the quantum nature of gravity [DFR95]. Furthermore, there are results using Euclidean metric indicating that noncommutative methods lead to better asymptotic behavior of individual models [GW05, DGMR07, GMRT09, Wan11]. However, the exact predictive power of such results has remained unclear until the last years. A rigorous way to generate such a noncommutative quantum field theory is to apply the framework of warped convolutions [BS08, BLS11]. It can both be applied to algebraic and "standard" quantum field theory, *i.e.*, given by the set of n-point functions.

The first result on Wick rotation in algebraic quantum field theory was given in [Sch99], where a list of axioms for a Euclidean net of C^*-algebras resulted in the analytical continuation towards a Haag-Kastler net. The main idea was not to prepare the feasibility of continuing the local algebras themselves, but rather to arrive at a unitary representation of the Poincaré group from the given Euclidean group representation. As a next step, the image of the time-zero net under this unitary representation was shown to fulfill the Haag-Kastler axioms. To accomplish that, the

C^*-algebras associated to regions of no extent in a chosen "Euclidean time direction" were obliged to be nontrivial. This additional requirement was called "time-zero condition".

In [GLLV11] we were able to show that this "commutative" Wick rotation can be generalized to theories located on Moyal space in a commutative-time setting. In order to achieve this aim, in Section 2.2 the result on Euclidean nets of observables was generalized to theories covariant with respect to subgroups of the Euclidean group, suitable for the noncommutative deformation. Constitutively, the Euclidean net deformed in terms of warped convolutions could be analytically continued to a noncommutative Lorentzian theory in Section 2.3. The proof carries through a deep analysis of the deformed observable nets and the connection with the time-zero condition written out. The cornerstones of this path are the following:

- Our deformation demands the usage of reduced symmetry groups which allow for an algebraic Wick rotation
- Also, the expansion of open bounded regions to cylindrical subsets of $\mathbb{R}^d$ is caused by this deformation
- These cylindrical regions are situated perfectly to assure the existence of a noncommutative time-zero condition despite the use of a proper symmetry subgroup

A crucial result of this chapter is Thm. 6, stating that the deformation of a Lorentzian theory is isomorphic to the analytical continuation of a theory on noncommutative Euclidean space. The outcome of the algebraic Wick rotation gets visualized in the commuting diagram given in Figure 2.9.

We gave a successive illustration of our noncommutative Wick rotation in terms of the free scalar field in Ch. 3. The quite abstract notions and results of the preceding part become more instructive when applied to this well-known example. Afterwards, it was shown that the noncommutative free field can be directly continued to its Lorentzian counterpart without the use of a time-zero condition.

Next, it was demonstrated that Wick rotation of more general theories is possible as well. The methods of analytically continuing given in [Ost73] were successfully generalized to sets of Moyal deformed Schwinger functions. A sequence of specific estimates were necessary to arrive at this result:

- Estimates needed for a linear growth condition $(E0')$ were shown to be fulfilled by the noncommutative Schwinger functions

- The Euclidean axiom ($E0''$), which is sufficient but not necessary for the Wick rotation, still holds for the time-dependent part of the commutative-time deformed theory
- As a consequence, the holomorphic Fourier transforms of the Schwinger functions exist and lead to noncommutative Wightman functions
- These are nothing else than the deformations of the corresponding commutative Wightman theory

In particular, there is no need for a time-zero condition when performing the Wick rotation of deformed Schwinger functions in a setting of commutative time.

The successful description of the symmetry group enlargement is done in Ch. 4. Considering the algebraic framework, we have presented the covariantization of both a Euclidean and a Lorentzian net there. It is defined to be the net of von Neumann algebras generated by deformation parameters dependent on the certain space-time region and makes possible the utilization of full Euclidean and Poincaré group, respectively. This is done by considering the symmetry group as a transformation between local algebras deformed with respect to different matrices.

- The noncommutative Wick rotation described in this thesis commutes with this kind of covariantization
- The axiom of Euclidean locality causes wedge-locality for nondegenerate deformation parameters and commutation of wedge-like separated cylinders for the commutative-time scenario
- Locality with respect to the left and right time-zero plane is sufficient for wedge-locality of the noncommutative-time deformation of the Wick rotated net

Conclusion The methods developed in this thesis and partly published in [GLLV11] to the author's knowledge mark the first rigorous treatment of Wick rotation on noncommutative spaces, both in the algebraic as well as in the n-point function setting. Finding physically relevant models on noncommutative Minkowski space-time is one of the actual objectives. As the majority of candidates have been situated on Euclidean Moyal space, the treatise at hand may be seen as a first step towards this goal.

Of course, it is more desirable to possess a Wick rotation of theories deformed with respect to a nondegenerate skew-symmetric matrix, *i.e.*, a theory of noncommutative time. At several points

(Remarks 7, 8 and 12) we have drawn the attention to the inconsistency of such a setting with the approach at hand. Nevertheless, the treatment shows that Wick rotation in a commutative-time setting is far from being trivial.

The outcome of Ch. 3 shows that we can well-define a quantum field theory on noncommutative Minkowski space-time whenever the corresponding Schwinger functions satisfy the Euclidean axioms including condition $(E0'')$. We do not need to assume any sort of time-zero condition. This condition $(E0'')$ mainly says that an n-point Schwinger function evaluated at an n-fold tensor product can be estimated by the n-independent norms of the particular tensor factors. It is not much more restrictive than the usual linear growth condition $(E0')$, as it is satisfied by practically all constructed models known so far [OS75, Remark 2. above IV.2.].

The symmetry group is reduced to a proper subgroup by the implementation of the theory on Moyal space(-time). Still, we are able to regain covariance w.r.t. the full group by the method of covariantization. Thus, starting from a Euclidean field theory, we are able to obtain the corresponding deformed Lorentzian theory without the abandonment of the full Poincaré group. Moreover, it is a matter of taste which path one wants to follow to arrive at this Lorentzian theory, as the results of the different paths were shown to be equivalent.

Last, but not least, we gave preconditions on a deformed Euclidean net of von Neumann algebras that lead to wedge-locality of the corresponding Lorentzian net. In order to give a necessary condition, one would have to perform the Wick rotation starting from a Lorentzian net, but the result of Lemma 23 looks minimal.

Outlook Theories built up of a degenerate noncommutativity matrix arise naturally in situations of three space-time dimensions. Indeed, utilizing the Moyal deformation there is attended by a completely anti-symmetric 3×3-matrix θ with constant real entries. This circumstance forces θ to be of rank 2. Although it is still possible to arrange one spatial coordinate to commute with all the others, time is picked to have this property instead. In physical situations, this is seen to be in line with the distinction of time.

Many three dimensional models have been suggested so far and it seems impossible to deal with a reasonable part of them on noncommutative spaces. We will just mention a few specific examples.

It is well-known that the Euclidean scalar ϕ^4-model in $d \leqslant 3$ dimensions can be rigorously constructed and fulfills all the Euclidean axioms necessary for analytic continuation to a Wightman or Haag-Kastler theory [FO76, MS77, BFS83, GJ87]. Therefore, it seems natural to desire a noncommutative deformation of the latter model and apply the Wick rotation of this thesis to observe a constructable and nontrivial model on noncommutative Moyal-Minkowski space-time. This would mark a complete novelty.

Up until now, the biggest success in giving precise meaning to noncommutative quantum field models uses the deformation of the corresponding actions. The one due to Grosse and Wulkenhaar [GW05] is characterized by the insertion of an oscillator potential and has a three dimensional specification [GVT10]. To achieve renormalizability, another nonlocal term was added inside the action there,

$$S_{\mathrm{GV}} := S_{\phi^4} + \int \mathrm{d}^4x \frac{\Omega^2}{\vartheta^2}(x_2^2 + x_3^2)\phi(x)^2 + \int \mathrm{d}^4x\, \mathrm{d}^2y\ \frac{\kappa^2}{\vartheta^2}\,\phi(x)\,\phi(x_0, x_1, y_2, y_3)\ ,$$

where S_{ϕ^4} denotes the action of a scalar field with a ϕ^4-interaction. Clearly, as the oscillator breaks the translational invariance in two spatial directions, the model as it stands cannot be considered physical yet. In the same way as in the Grosse-Wulkenhaar model, renormalizability is the main criterion here, as it is necessary for the self-consistency of a physical model. The physical consequences of insertions dependent on the parameters Ω and κ can be investigated once the Wick rotation of the model has been performed.

Finally, roughly touching the scope of low dimensional models in noncommutative quantum field theory, let us mention noncommutative gauge models. These have been used in three space-time dimensions to qualitatively describe the integer and fractional quantum Hall effect [Sus01, Pol01, HVR01].

Appendix A

Euclidean Axioms

We list two sets of axioms for a Euclidean quantum field theory to guarantee the analytic continuation to a Wightman quantum field theory.

A.1 Osterwalder-Schrader Axioms

The following list of axioms for the sequence $\{\mathfrak{S}_n\}_{n\in\mathbb{N}}$ of Schwinger functions is taken from [OS73, OS75] and adapted to our notation convention:

($E0$) *Temperedness / Analyticity*: see Section 3.7

($E1$) *Invariance*: $\mathfrak{S}_n(f) = \mathfrak{S}_n(f_{(g)})$ for all $g \in E(d)$, $f \in \mathscr{S}_0(\mathbb{R}^{dn})$.

($E2$) *Reflection Positivity*: $\sum\limits_{n,m} \mathfrak{S}_{n+m}(\iota f_n^* \otimes f_m) \geqslant 0$ for all $\underline{f} = (f_0, f_1, f_2, \ldots) \in \mathcal{E}_>$.

($E3$) *Permutation Symmetry*: $\mathfrak{S}_n(f) = \mathfrak{S}_n(f_{(\pi)})$ for all[1] $\pi \in \tau(n)$, $f \in \mathscr{S}_0(\mathbb{R}^{dn})$.

($E4$) *Cluster Property*:

$$\lim_{\lambda\to\infty} \sum_{n,m} \left\{ \mathfrak{S}_{n+m}(\iota f_n^* \otimes g_{m,(\lambda(0,\underline{a}),1)}) - \mathfrak{S}_n(f_n)\mathfrak{S}_m(g_m) \right\} = 0 \ ,$$

for $\underline{a} \in \mathbb{R}^{d-1}$, $|\underline{a}| = 1$.

[1] $\tau(n)$ denotes the set of all n-valent permutations, see the paragraphs below equation (3.16)

A.2 Path Integral Axioms

Let $\mathrm{d}\mu(\phi)$ be a Borel probability measure on the dual space of $\mathscr{C}_\mathrm{c}^\infty(\mathbb{R}^d)$. Define the *generating functional* S to be

$$S\{f\} := \int \mathrm{e}^{i\phi(f)}\, \mathrm{d}\mu(\phi) \quad , \quad f \in \mathscr{C}_\mathrm{c}^\infty(\mathbb{R}^d) \ .$$

Then the axioms for a Euclidean quantum field theory in the path integral formalism [GJ87] are the following:

$(EP0)$ *Analyticity*: for all $N \in \mathbb{N}$ and $f_k \in \mathscr{C}_\mathrm{c}^\infty(\mathbb{R}^d)$, $k = 1, \ldots, N$ and for all $z = \{z_1, \ldots, z_N\} \in \mathbb{C}^N$ the function

$$z \mapsto S\left\{\sum_{k=1}^{N} z_k f_k\right\}$$

is entire on $\mathbb{C}^N$.

$(EP1)$ *Regularity*: for some $1 \leqslant p \leqslant 2$, some $c \in \mathbb{R}$ and for all $f \in \mathscr{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ we have

$$|S\{f\}| \leqslant \mathrm{e}^{c\left(\|f\|_1 + \|f\|_p^p\right)} \ .$$

$(EP2)$ *Invariance*: let $g \in E(d)$. Then it follows $S\{f_{(g)}\} = S\{f\}$.

$(EP3)$ *Reflection Positivity*: consider the set

$$\mathcal{A}_+ := \left\{ A(\phi) = \sum_{k=1}^{N} c_k \mathrm{e}^{i\phi(f_k)} \mid c_k \in \mathbb{C}\,,\ f_k \in \mathscr{C}_\mathrm{c}^\infty(\mathbb{R}_+^d) \right\} \ ,$$

then $\int (\iota A)^* A\, \mathrm{d}\mu(\phi) \geqslant 0$ for all $A \in \mathcal{A}_+$.

$(EP4)$ *Ergodicity*: $\lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha_{s,1}(A)\, \mathrm{d}s = \int A(\phi)\, \mathrm{d}\mu(\phi)$.

Appendix B

Analytical Continuation of Distributions

There are several important results of complex analysis which are serviceable for the analytic continuation of n-point functions. In the following, we specify some of them.

Theorem 8 (Malgrange-Zerner-Kunze-Stein, [Eps66]). *Let*

$$f_1(z_1, x_2, \ldots, x_n)\,,\ f_2(x_1, z_2, x_3, \ldots, x_n)\,,\ \ldots, f_n(x_1, x_2, \ldots, z_n)$$

be functions of the real variables $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, *and* $z_k = x_k + iy_k$ *with the following properties*

1. $f_k(x_1, \ldots, x_k, \ldots, x_n)$ *is a smooth function of* $x_1, x_2, \ldots, x_n$, y_k *whenever* $0 \leqslant y_k \leqslant 1$ *and* $f_k(x_1, \ldots, z_k, \ldots, x_n)$ *is a holomorphic function of* z_k *for* $0 < y_k < 1$
2. *For every real* $x = (x_1, x_2, \ldots, x_n)$,

$$f_1(x_1, x_2, \ldots, x_n) = \ldots = f_n(x_1, x_2, \ldots, x_n) \tag{B.1}$$

Then there is a function $F(z_1, \ldots, z_n)$, *holomorphic in the domain*

$$H = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid 0 < y_k < 1,\, 1 \leqslant k \leqslant n,\, \sum_{j=1}^{n} y_j < 1 \right\} \tag{B.2}$$

and smooth in the closure of this domain. F coincides with $f_k(x_1, \ldots, x_{k-1}, z_k, x_{k+1}, \ldots, x_n)$ for $y_j = 0$ with $j \neq k$, $0 \leqslant y_k \leqslant 1$, $1 \leqslant k \leqslant n$.

Theorem 9 (Paley-Wiener, exponential type; [RS75], IX.11)**.** *An entire analytic function $g(\zeta)$ of d complex variables is the Fourier transform of a $\mathscr{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ function with support in the ball $\{x \,|\, |x| \leqslant R\}$ if and only if for each N there is a C_N such that*

$$|g(\zeta)| \leqslant \frac{C_N\, \mathrm{e}^{R|\mathrm{Im}\zeta|}}{(1+|\zeta|)^N} \qquad \forall \zeta \in \mathbb{C}^d \,. \tag{B.3}$$

Theorem 10 (Paley-Wiener, L^2 version; [Rud87], 19.2)**.** *Let f be an analytic function on $\Pi_+ := \{x+iy \in \mathbb{C} \,|\, y > 0\}$ and $\sup\limits_{0<y<\infty} \frac{1}{2\pi} \int |f(x+iy)|^2 \,\mathrm{d}x = C < \infty$. Then there exists a function $F \in L^2(\mathbb{R}_+)$ such that*

$$f(z) = \int_0^\infty F(t)\, \mathrm{e}^{itz} \mathrm{d}t \quad , \quad z \in \Pi_+ \tag{B.4}$$

and we have $\int_0^\infty |F(t)|^2 \,\mathrm{d}t = C$.

Appendix C

Moyal Calculations

The noncommutative four-point correlator of the free scalar field can be obtained by calculating the expectation value of the four-fold Moyal product, *i.e.*,

$$\mathfrak{S}^4_\theta(s_1, s_2, s_3, s_4) = \langle \Omega, (\varphi \otimes_\theta \varphi \otimes_\theta \varphi \otimes_\theta \varphi)(s_1, s_2, s_3, s_4)\Omega \rangle \ .$$

We remember that $\varphi(s) = a(\mathsf{p}\widetilde{s}^*) + a^\dagger(\widetilde{s})$, which reveals two terms that are non-vanishing: symbolically written they are $a_1 a_2 a_3^\dagger a_4^\dagger$ and $a_1 a_2^\dagger a_3 a_4^\dagger$. The latter one reduces to the usual product of commutative free covariances $C(s_1, s_2)C(s_3, s_4)$ since lowering directly after raising lets us end up with the commutative scalar product of functions due to the calculational rules for the Moyal tensor product such as (3.28). The first term $a_1 a_2 a_3^\dagger a_4^\dagger$ in fact provides two distinct contributions to the 4-point function; one of them differs significantly from the corresponding commutative parts. One starts by acting twice with a generator on the Fock vacuum. The second component is the only one different from zero there,

$$\big((a^\dagger \otimes_\theta a^\dagger)(\widetilde{s}_3, \widetilde{s}_4)\Omega\big)_n (\underline{p}_1, \underline{p}_2) = \delta_{n2} \frac{1}{\sqrt{2}} \Big(\mathrm{e}^{ip_1\theta p_2}\, \widetilde{s}_3(\underline{p}_1)\widetilde{s}_4(\underline{p}_2) + \mathrm{e}^{-ip_1\theta p_2}\, \widetilde{s}_4(\underline{p}_1)\widetilde{s}_3(\underline{p}_2)\Big) \ ,$$

while the application of the first of two annihilators is responsible for the two resulting terms:

$$
\begin{aligned}
&\Big((a\otimes_\theta a^\dagger\otimes_\theta a^\dagger)(\mathsf{p}\widetilde{s}_2^*,\widetilde{s}_3,\widetilde{s}_4)\Omega\Big)_1(\underline{p}_1)\\
=\;&\int \mathrm{D}_{kv}\mathrm{e}^{-ikv}a(U(\theta k)\mathsf{p}\widetilde{s}_2^*)\alpha_v\,\big((a^\dagger\otimes_\theta a^\dagger)(\widetilde{s}_3,\widetilde{s}_4)\Omega\big)_1(\underline{p}_1)\\
=\;&\delta_{n1}\int \mathrm{D}_{kv}\int \mathrm{d}\underline{\mu}\,\mathrm{e}^{-ikv-ip\theta k+i(p+p_1)v}\mathsf{p}\widetilde{s}_2(\underline{p})\,\Big(\mathrm{e}^{ip\theta p_1}\widetilde{s}_3(\underline{p})\widetilde{s}_4(\underline{p}_1)+\mathrm{e}^{-ip\theta p_1}\widetilde{s}_4(\underline{p})\widetilde{s}_3(\underline{p}_1)\Big)\\
=\;&\delta_{n1}\int \mathrm{d}k\int \mathrm{d}\underline{\mu}\,\delta(\underline{p}+\underline{p}_1-\underline{k})\mathrm{e}^{-ip\theta k}\mathsf{p}\widetilde{s}_2(\underline{p})\,\Big(\mathrm{e}^{ip\theta p_1}\widetilde{s}_3(\underline{p})\widetilde{s}_4(\underline{p}_1)+\mathrm{e}^{-ip\theta p_1}\widetilde{s}_4(\underline{p})\widetilde{s}_3(\underline{p}_1)\Big)\\
=\;&\delta_{n1}(\mathsf{p}\widetilde{s}_2^*,\widetilde{s}_3)\widetilde{s}_4(\underline{p}_1)+\delta_{n1}\int \mathrm{d}\underline{\mu}\,\mathrm{e}^{-ip\theta p_1}\mathsf{p}\widetilde{s}_2(\underline{p})\widetilde{s}_4(\underline{p})\widetilde{s}_3(\underline{p}_1)\;. \qquad \text{(C.1)}
\end{aligned}
$$

Finally, acting with the second annihilator brings about the evaluation of $\big\langle\Omega,(a\otimes_\theta a\otimes_\theta a^\dagger\otimes_\theta a^\dagger)(\mathsf{p}\widetilde{s}_1^*,\mathsf{p}\widetilde{s}_2^*,\widetilde{s}_3,\widetilde{s}_4)\Omega\big\rangle$, which can be calculated to deliver two expressions from $a_1a_2a_3^\dagger a_4^\dagger$. On the one hand, we get $C(s_2,s_3)C(s_1,s_4)$, which is obtained by building the Moyal tensor product of the last annihilator with the first term of equation (C.1) and using the cyclic property. On the other hand, acting in this way on the second term we recognize the first evidence for the deformed theory being different from the commutative one:

$$
\begin{aligned}
&\int \mathrm{D}_{kv}^{pq}\,\mathrm{e}^{-ikv-ip\theta q}\mathsf{p}\widetilde{s}_2(\underline{p})\widetilde{s}_4(\underline{p})U(\underline{\theta k})\mathsf{p}\widetilde{s}_1(\underline{q})U(\underline{v})\widetilde{s}_3(\underline{q})\\
=\;&\int \mathrm{d}\mu(\underline{p})\int \mathrm{d}\mu(\underline{q})\,\mathrm{e}^{iq\theta p}\mathsf{p}\widetilde{s}_2(\underline{p})\widetilde{s}_4(\underline{p})\mathsf{p}\widetilde{s}_1(\underline{q})\widetilde{s}_3(\underline{q})
\end{aligned}
$$

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Symbols & Abbreviations

Symbol		Meaning	Def./occurance
xy	...	Euclidean inner product $x^0y^0 + x^1y^1 + \dots x^{d-1}y^{d-1}$	Below Def. 14
(x, y)	...	Minkowski inner product $x^0y^0 - x^1y^1 - \dots x^{d-1}y^{d-1}$	Below (2.29)
$\otimes_\theta$	...	Moyal tensor product	Sec. 3.4
$\times_Q$	...	Rieffel product	Def. 7
$\lvert\,.\,\rvert_m$	...	Schwartz norm	(3.47)
$\mathcal{A}$	...	General *-algebra	Sec. 1.1
α	...	Automorphic (Euclidean) symmetry action	Sec. 1.1
$\alpha^{\mathcal{M}}$	...	Automorphic Poincaré action	(2.18)
C	...	Common domain of virtual representation	(2.12)
$C(x, y)$	...	Free covariance	(3.3)
$\mathcal{C}^\infty(M)$	...	Smooth elements (on M)	Sec. 1.2.1
$\mathscr{C}_{\mathrm{c}}^\infty(\mathbb{R}^d)$	...	Smooth functions of compact support on $\mathbb{R}^d$	Lemma 2
$\mathrm{d}\mu(\phi)$	...	Euclidean measure	Ch. 1
$\mathcal{E}$	...	Inductive limit of $\{\mathcal{E}(O)\}_{O\in\mathscr{O}}$ or Euclidean theory	Def. 3
$\mathcal{E}_>$	...	Positive-time algebra	Below (2.5)
$E_\theta(d)$	...	Restricted Euclidean group $(SO(2) \times SO(d-2)) \ltimes \mathbb{R}^d$	(2.3)
$\check{\mathcal{E}}_\Xi$	...	Auxiliary covariantization for demonstration	(4.3)
$\mathcal{E}_\Xi$	...	Euclidean covariantization w.r.t. cylinders	Def. 18
$\hat{\mathcal{E}}_\Xi$	...	Euclidean covariantization for unique maps	Sec. 4.2
$\eta_{\mu\nu}$	...	Minkowski metric	Sec. 1.1
$\hat{f}$	...	Spatial Fourier transformation of $f \in \mathscr{S}(\mathbb{R}^d)$	(3.9)
$f_\bullet$	...	Free field projection map	(3.16)
$\mathcal{M}$	...	Inductive limit of $\{\mathcal{M}(O)\}_{O\in\mathscr{O}}$ or Lorentzian theory	Sec. 1.1, (2.19)
$\mathcal{M}_\theta$, $\mathcal{M}^\theta$	...	Deformed Lorentzian nets	Def. 15, (2.30)
$\widetilde{\mathcal{M}}_\Xi$	...	Wick rotation of the covariantization	Below Lemma 20
$\mathfrak{M}_{m\times n}(\mathbb{K})$	...	Space of all $\mathbb{K}$-valued $m \times n$-matrices	*2.* of Prop. 9
$\mathcal{N}_\sigma$	...	Null space of σ	Below (2.5)
$\mathscr{O}$	...	Set of open bounded space-time regions	Sec. 1.1

Symbol		Meaning	Def./occurance
P_k	...	Generators of spatial translations, momentum operators	Above Remark 4
p	...	Parity operator: $f(x_0, \underline{x}) \mapsto f(x_0, -\underline{x})$	(3.17)
$\mathcal{P}(d)$	...	(Proper Orthochronous) d-dimensional Poincaré group	Sec. 1.1
$\mathcal{P}_\theta$	...	Restricted Poincaré group	(2.4)
π	...	Time-zero representation	(2.6)
ϕ	...	(Free) scalar field	(3.17)
$\phi^{\mathcal{E}}$	...	Euclidean free field	Def. 16
$\phi^{\mathcal{M}}$	...	Minkowski free field	Def. 17
φ	...	Time-zero field	Def. 16
s	...	Element of $\mathscr{S}(\mathbb{R}^{d-1})$	Lemma 10
$\mathcal{S}_\theta$	...	Time-zero stripes	Sec. 2.3.2
$\mathscr{S}(\mathbb{R}^d)$	...	Schwartz space of rapidly decreasing functions $f : \mathbb{R}^d \to \mathbb{C}$	Above Def. 3
$\mathfrak{S}_n$	...	Time-zero n-point distribution	Ch. 1
$\mathfrak{S}_n^\theta$	...	Deformed Schwinger function	Sec. 3.4.1
S_n	...	Schwinger function in difference variables	(3.48)
$\underline{\mathscr{S}}$	...	Borchers-Uhlmann algebra	Sec. 3.2
σ	...	Euclidean reflection positive functional	Def. 3, (3.14)
Σ	...	Space of noncommutativities	Sec. 4.1
θ	...	Standard commutative-time noncommutativity	(2.20)
$\mathscr{U}$	...	Suitable neighborhood of unity	Below (2.12)
W	...	Unitary intertwiner of deformed nets	Thm. 6
$\mathfrak{W}_n$	...	n-point Wightman function	Below (3.26)
$\mathfrak{W}_n^\theta$	...	Deformed Wightman function	Sec. 3.4.1
ω	...	Vacuum state	Def. 2
X_μ	...	Noncommutative position operator	(1.5)
Ξ	...	Space of commutative-time noncommutativities	Sec. 4.1
Y	...	Light wedge	(2.17)
$\mathcal{Z}_\theta$	...	Cylindrical regions	(2.24), (2.25)

Bibliography

[AGVM03] L. Alvarez-Gaume and M.A. Vazquez-Mozo. General Properties of Noncommutative Field Theories. *Nucl.Phys.*, B668:293–321, 2003.

[Alb08] S. Albeverio. *Mathematical Theory of Feynman Path Integrals: An Introduction*, volume 523 of *Lec.Not.Math.* Springer, 2008.

[Ara99] H. Araki. *Mathematical Theory of Quantum Fields.* Oxford University Press, 1999.

[Ash86] A. Ashtekar. New Variables for Classical and Quantum Gravity. *Phys.Rev.Lett.*, 57:2244–2247, 1986.

[B+08] G. Bernardi et al. Combined CDF and D0 Upper Limits on Standard Model Higgs Boson Production at High Mass ($155 - 200 - GeV/c^{2)}$ with 3 fb^{-1} of data. *Preprint*, 2008.

[Bar07] J.W. Barrett. A Lorentzian version of the non-commutative geometry of the standard model of particle physics. *J.Math.Phys.*, 48:012303, 2007.

[BBS06] K. Becker, M. Becker, and H. Schwarz. *String Theory and M-Theory.* Cambridge University Press, 2006.

[BF09] R. Brunetti and K. Fredenhagen. Quantum Field Theory on Curved Backgrounds. 2009.

[BFG+03] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, and R. Wulkenhaar. Space-Time Noncommutative Field Theories and Causality. *Eur.Phys.J.*, C29:133–141, 2003.

[BFS83] D.C. Brydges, J. Fröhlich, and A.D. Sokal. A New Proof of the Existence and Nontriviality of the Continuum Phi**4 in Two-Dimensions and Phi**4 in Three-Dimensions Quantum Field Theories. *Commun.Math.Phys.*, 91:141–186, 1983.

[BLS11] D. Buchholz, G. Lechner, and S.J. Summers. Warped Convolutions, Rieffel Deformations and the Construction of Quantum Field Theories. *Commun.Math.Phys.*, 304:95–123, 2011.

[Bor62] H. Borchers. On Structure of the Algebra of Field Operators. *Il Nuovo Cimento*, 24:214–236, 1962. 10.1007/BF02745645.

[BS08] D. Buchholz and S.J. Summers. Warped Convolutions: A Novel Tool in the Construction of Quantum Field Theories. In E. Seiler and K. Sibold, editors, *Quantum Field Theory and Beyond: Essays in Honor of Wolfhart Zimmermann*, pages 107–121. World Scientific, 2008.

[CC07] A.H. Chamseddine and A. Connes. Conceptual Explanation for the Algebra in the Noncommutative Approach to the Standard Model. *Phys.Rev.Lett.*, 99:191601, Nov 2007.

[CHSW85] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. Vacuum Configurations for Superstrings. *Nucl.Phys.*, B258:46–74, 1985.

[Con94] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, CA, 1994.

[Con96] A. Connes. Gravity coupled with matter and foundation of noncommutative geometry. *Commun.Math.Phys.*, 182:155–176, 1996.

[DF77] W. Driessler and J. Fröhlich. The Reconstruction of Local Observable Algebras from the Euclidean Green's Functions of a Relativistic Quantum Field Theory. *Annales Poincare Phys.Theor.*, 27:221–236, 1977.

[DFR95] S. Doplicher, K. Fredenhagen, and J.E. Roberts. The Quantum structure of space-time at the Planck scale and quantum fields. *Commun.Math.Phys.*, 172:187–220, 1995.

[DGMR07] M. Disertori, R. Gurau, J. Magnen, and V. Rivasseau. Vanishing of Beta Function of Non Commutative Phi**4(4) Theory to all orders. *Phys.Lett.*, B649:95–102, 2007.

[ea01] Brown et al. Precise measurement of the positive muon anomalous magnetic moment. *Phys.Rev.Lett.*, 86:2227–2231, Mar 2001.

[Eps66] H. Epstein. Some Analytic Properties of Scattering Amplitudes in Quantum Field Theory. In M. Chrétien & S. Deser, editor, *Axiomatic Field Theory, Volume 1*, page 1, 1966.

[FH81] K. Fredenhagen and J. Hertel. Local Algebras of Observables and Point-Like Localized Fields. *Commun.Math.Phys.*, 80:555, 1981.

[Fil96] T. Filk. Divergencies in a Field Theory on Quantum Space. *Phys.Lett.*, B376:53–58, 1996.

[FO76] J.S. Feldman and K. Osterwalder. The Wightman Axioms and the Mass Gap for Weakly Coupled (phi**4) in Three-Dimensions Quantum Field Theories. *Annals Phys.*, 97:80–135, 1976.

[FOS83] J. Fröhlich, K. Osterwalder, and E. Seiler. On Virtual representations of symmetric spaces and their analytic continuation. *Annals Math.*, 118:461–489, 1983.

[GBVF00] J.M. Gracia-Bondia, J.C. Varilly, and H. Figueroa. *Elements of Noncommutative Geometry*. Birkhäuser Advanced Texts. Birkhäuser Boston, 2000.

[GGBI+04] V. Gayral, J.M. Gracia-Bondia, B. Iochum, T. Schücker, and J. C. Varilly. Moyal Planes are Spectral Triples. *Commun.Math.Phys.*, 246:569–623, 2004.

[GJ87] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer-Verlag, 1987.

[GL07] H. Grosse and G. Lechner. Wedge-Local Quantum Fields and Noncommutative Minkowski Space. *JHEP*, 11:012, 2007.

[GL08] H. Grosse and G. Lechner. Noncommutative Deformations of Wightman Quantum Field Theories. *JHEP*, 09:131, 2008.

[GLLV11] H. Grosse, G. Lechner, T. Ludwig, and R. Verch. Wick Rotation for Quantum Field Theories on Degenerate Moyal Space(-Time). *Preprint*, 2011.

[GMRT09] R. Gurau, J. Magnen, V. Rivasseau, and A. Tanasa. A Translation-invariant Renormalizable Non-Commutative Scalar Model. *Commun.Math.Phys.*, 287:275–290, 2009.

[GN43] I. Gelfand and M. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec.Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[GP11] R. Gambini and J. Pullin. *A First Course in Loop Quantum Gravity*. Oxford University Press, 2011.

[GPR94] A. Giveon, M. Porrati, and E. Rabinovici. Target space duality in string theory. *Physics Reports*, 244(2-3):77 – 202, 1994.

[Gro46] H.J. Groenewold. On the Principles of Elementary Quantum Mechanics. *Physica*, XII(7):405–460, 1946.

[GVT10] H. Grosse and F. Vignes-Tourneret. Quantum Field Theory on the Degenerate Moyal Space. *J.Noncommut.Geom.*, 4:555–576, 2010.

[GW05] H. Grosse and R. Wulkenhaar. Renormalization of phi**4 Theory on Noncommutative R**4 in the Matrix Base. *Commun.Math.Phys.*, 256:305–374, 2005.

[GW12] H. Grosse and R. Wulkenhaar. Self-Dual Noncommutative ϕ^4-Theory in Four Dimensions is a Non-Perturbatively Solvable and Non-Trivial Quantum Field Theory. *Preprint*, 2012.

[Haa55] R. Haag. On Quantum Field Theories. *Matematisk-fysiske Meddelelser*, 29(12), 1955.

[Haa92] R. Haag. *Local Quantum Physics: Fields, Particles, Algebras*. Springer Verlag, 1992.

[Hel62] S. Helgason. *Differential Geometry and Symmetric Spaces*. Academic Press, 1962.

[HHW67] R. Haag, N.M. Hugenholtz, and M. Winnink. On the Equilibrium states in quantum statistical mechanics. *Commun.Math.Phys.*, 5:215–236, 1967.

[HK64] R. Haag and D. Kastler. An Algebraic Approach to Quantum Field Theory. *J.Math.Phys.*, 5:848–861, July 1964.

[HVR01] S. Hellerman and M. Van Raamsdonk. Quantum Hall physics equals noncommutative field theory. *JHEP*, 0110:039, 2001.

[JO99] P. E. T. Jorgensen and G. Olafsson. Unitary Representations and Osterwalder-Schrader Duality. *Proc.Sympos.Pure Math.*, 68:333–401, 1999.

[KL81] A. Klein and L.J. Landau. Construction Of A Unique Self-Adjoint Generator for a Symmetric Local Semigroup. *J.Funct.Anal.*, 44:121, 1981.

[KL82] A. Klein and L.J. Landau. From the Euclidean group to the Poincaré group via Osterwalder-Schrader positivity. *Commun.Math.Phys.*, 87:469–484, 1982.

[LS02a] Y. Liao and K. Sibold. Time Ordered Perturbation Theory on Noncommutative Spacetime. 2. Unitarity. *Eur.Phys.J.*, C25:479–486, 2002.

[LS02b] Y. Liao and K. Sibold. Time Ordered Perturbation Theory on Noncommutative Space-Time: Basic Rules. *Eur.Phys.J.*, C25:469–477, 2002.

[Moy49] J.E. Moyal. Quantum Mechanics as a Statistical Theory. *Proc.Camb.Phil.Soc.*, 45:99–124, 1949.

[MS77] J. Magnen and R. Seneor. Phase Space Cell Expansion and Borel Summability for the Euclidean phi**4 in Three-Dimensions Theory. *Commun.Math.Phys.*, 56:237, 1977.

[MSSW00] J. Madore, S. Schraml, P. Schupp, and J. Wess. Gauge Theory on Noncommutative Spaces. *Eur.Phys.J.*, C16:161–167, 2000.

[MST00] A. Matusis, L. Susskind, and N. Toumbas. The IR/UV Connection in the Non-Commutative Gauge Theories. *JHEP*, 12:2, December 2000.

[MVRS00] S. Minwalla, M. Van Raamsdonk, and N. Seiberg. Noncommutative Perturbative Dynamics. *JHEP*, 0002:020, 2000.

[Nel73a] E. Nelson. Construction of Quantum Fields from Markoff Fields. *J.Funct.Anal.*, 12(1):97–112, 1973.

[Nel73b] E. Nelson. The free markoff field. *J.Funct.Anal.*, 12(2):211–227, 1973.

[OS73] K. Osterwalder and R. Schrader. Axioms for Euclidean Green's Functions. *Commun.Math.Phys.*, 31:83–112, 1973.

[OS75] K. Osterwalder and R. Schrader. Axioms for Euclidean Green's Functions. 2. *Commun.Math.Phys.*, 42:281, 1975.

[Ost73] K. Osterwalder. Euclidean Green's Functions and Wightman Distributions. *Lect.Not.Phys.*, 25:71–93, 1973.

[Ped79] G.K. Pedersen. *C*-algebras and their automorphism groups.* L.M.S. monographs. Academic Press, 1979.

[Pol01] A.P. Polychronakos. Quantum Hall states as matrix Chern-Simons theory. *JHEP*, 0104:011, 2001.

[Rie93a] M. A. Rieffel. Compact Quantum Groups Associated with Toral Subgroups. *Cont.Math.*, 145, 1993.

[Rie93b] M. A. Rieffel. *Deformation Quantization for Actions of* $\mathbb{R}^d$, volume 106. American Mathematical Society, 1993.

[Roe94] G. Roepstorff. *Path Integral Approach to Quantum Physics: An Introduction.* Texts and Monographs in Physics. Springer, 1994.

[RS75] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II - Fourier Analysis.* Academic Press, 1975.

[Rud87] W. Rudin. *Real and Complex Analysis.* Mathematics series. McGraw-Hill, 1987.

[Sch99] D. Schlingemann. From Euclidean Field Theory to Quantum Field Theory. *Rev.Math.Phys.*, 11:1151–1178, 1999.

[Seg47] I.E. Segal. Irreducible Representations of Operator Algebras. *Bull.Am.Math.Soc.*, 53:73–88, 1947.

[Ser55] J.-P. Serre. Faisceaux Algebriques Coherents. *Annals Math.*, 61(2):pp. 197–278, 1955.

[Sny47] H. S. Snyder. Quantized Space-Time. *Phys.Rev.*, 71:38–41, Jan 1947.

[Sum12] S.J. Summers. A Perspective on Constructive Quantum Field Theory. *Preprint*, 2012.

[Sus01] L. Susskind. The Quantum Hall fluid and noncommutative Chern-Simons theory. *Preprint*, 2001.

[SW89] R.F. Streater and A.S. Wightman. *PCT, spin and statistics, and all that.* Advanced Book Classics. Addison-Wesley, Reading, Mass., 1989.

[SW99] N. Seiberg and E. Witten. String Theory and Noncommutative Geometry. *JHEP*, 9909:032, 1999.

[Swa62] R. G. Swan. Vector Bundles and Projective Modules. *Trans.Amer.Math.Soc.*, 105(2):pp. 264–277, 1962.

[Sym66] K. Symanzik. Euclidean Quantum Field Theory I. Equations for a Scalar Model. *J.Math.Phys.*, 7(3):510–525, September 1966.

[Tay86] M. Taylor. *Noncommutative Harmonic Analysis*, volume 22 of *Math. Surveys and Monogr.* Amer. Math. Soc., 1986.

[Uhl62] A. Uhlmann. Über die Definition der Quantenfelder nach Wightman und Haag. *Wiss. Z. Karl-Marx-Univ. Leipzig*, 11:213–217, 1962.

[Vla66] V. Vladimirov. *Methods of the Theory of Functions of Many Complex Variables.* The M.I.T. Press, 1966.

[Wal94] Robert M. Wald. Quantum field theory in curved space-time and black hole thermodynamics. *Chicago Lec. Phys.*, 1994.

[Wan11] Z. Wang. Construction of 2-dimensional Grosse-Wulkenhaar Model. *Preprint*, 2011.

[Wig56] A. S. Wightman. Quantum Field Theory in Terms of Vacuum Expectation Values. *Phys.Rev.*, 101:860–866, Jan 1956.

[Wit89] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun.Math.Phys.*, 121:351, 1989.