Quantum Spacetime from QM and QFT

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In this work we study the realization of noncommutative spacetimes from quantum mechanics and quantum field theory.

Quantum Spacetime from QM and QFT

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Dedicated to my parents and my sister

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Chapter 1

Introduction

1.1 Motivation

The four fundamental interactions in the universe are the gravitational force, the electromagnetic-, as well as the weak- and the strong interaction. A major discovery in the 20th century was that all forces can be described by corresponding particles. In this description of nature, interactions are described by scattering processes which occur on a quantum level between particles. In the case of the electromagnetic interaction the photon mediates the electromagnetic force. While for the weak interaction the W and Z bosons, and in the strong interaction eight gluons are responsible for the interaction. The boson responsible for the gravitational force has still not been experimentally verified. This boson is known under the name of graviton.

Attempts to quantize the gravitational field in a quantum field theoretical context have failed, due to the lack of an important and very successful principle which leads to finite theories: renormalizability. By treating the gravitational field like the electromagnetic field and quantizing it in a similar fashion, one ends up with a quantum field theory (QFT) that is nonrenormalizable. This means that the quantum field theory by hand does not make any sense in sensible physical terms and has to be discarded.

Despite unsuccessful attempts to unify gravity with quantum field theory, efforts to unify the electromagnetic and the weak interactions were rewarded by the electroweak theory. Furthermore, by amplifying the electroweak theory with quantum chromodynamics, the theory that describes strong interactions, on was led to the formulation of the standard model.

Even among nonphysicists, it is agreed upon that the standard model displays one of the greatest intellectual achievements in human history. But despite major success on experimental and theoretical level, the standard model fails to explain many aspects of nature.

First of all, as already mentioned above, the graviton has still not been experimentally verified. Furthermore at energies where the particle is expected, the standard model will not hold anymore. Secondly, the most accurate description of gravitation is given by the theory of general relativity. In this theory, matter and spacetime are intertwined by the famous field equations of Einstein, and gravitation is understood as the curvature of spacetime induced by the matter by hand. Attempts to unify quantum field theory with general relativity have failed even in the linear approximation of gravitation. The reasons vary from renormalizability to the understanding of spacetime not as a stage where the dynamics takes place, but as an intrinsic property of the dynamics of the matter content.

Moreover, cosmological observations predict the existence of dark matter. The prediction of existence of dark matter can be deduced from its gravitational effects on atomic matter, radiation and the large-scale structure of the universe. It supposedly constitutes about 84% of the matter and energy content of the universe. Because the standard model only deals with atomic, i.e. visible particles, the major part of the matter contribution in the universe can not be explained or modeled by the standard model.

Another major philosophical problem in the search for a theory of quantum gravity concerns the picture of spacetime at small scales. According to an argument originating from John A. Wheeler the classical picture of spacetime should break down at very short distances of the order of the Planck- length $l_p = \sqrt{G\hbar/c^3}$. Around this scale the concept of spacetime as a continuum breaks down due to the quantum indeterminacy.

Wheeler argues that, if one wants to probe an event in the length scale of Planck length with a photon, by the uncertainty principle, the particle has to have roughly Planck energy. Now according to general relativity, a photon on such energy scales causes a gravitational collapse and therefore it does not yield any information of the event. The gravitational collapse is caused by the fact that the Schwarzschild radius of a particle with Planck energy is approximately equal to the Planck length. Consequently, due to the uncertainty principle and the Schwarzschild radius, the very measurement of an event in this length scale creates a black hole and no information about this event will emerge. The region with Planck-length of radius, therefore becomes in a sense noncontinuous, i.e. experimentally not accessible.

There are many approaches available in order to solve the problem of unifying quantum field theory with general relativity, and to solve the measurement problem on scales of order of the Planck-length there. In this introduction the author will try to give a brief review of popular approaches on the market and point out the shortcomings.

One idea to approach the problem of handling spacetime on a quantum level, is to quantize the geometry in a canonical fashion. The theory is well known under the name of loop quantum gravity (LQG). Quantization of the spacetime geometry in LQG leads to the picture of a granular space. Thus space itself becomes discretized. Another point usually stressed in the context of LQG, is the background independence. This in particular means, that for the theory by hand no assumption of a preexisting spacetime is needed. Thus one does not have to start with a classical background (spacetime) and try to quantize it. As appealing as LQG may sound, no semi-classical limit for recovering general relativity has been shown to exist. Therefore it is not clear if LQG describes spacetime on a quantum level at all. Furthermore, the theory is unfortunately far from describing matter and achieving similar successes as the standard model.

Another major field of research is string theory. In this theory, the point like objects referred to in the standard model as particles, are replaced by 1-dimensional oscillating lines called strings. It then depends on the oscillations of the string to give the particles their flavor, charge, mass and spin. At first, only the bosonic string was formulated. The investigation how to include fermions led to the use supersymmetry. Theories that include fermionic strings are known as superstring theories. Five different superstring theories were defined and were further shown to be equivalent, since in particular these theories are different limits to M-theory.

Up to this date, string theory was not able to give a mathematical precise answer why it should be the theory that unifies all fundamental forces. Furthermore, the theory requires additional dimensions that have still not been experimentally verified. Very high energies are needed to test or falsify the theory. Up to this date no precise falsifiable experimental prediction that can be measured with the accessible energies has been done to proof or disproof parts of the theory. The lack of experimental evidence is dangerous due to the fact that it places theories in corners that are out of reach for science. There is also the lack of background independence. String theory assumes a preexisting classical spacetime. Background independence is a property that is expected from a true theory of quantum gravity.

A more down to earth approach, is quantum field theory on curved spacetimes (QFTCST). In QFTCST, one tries to extend the definition of a QFT in flat Minkowski space to curved spacetimes. Scattering processes in QFT on flat spacetime can be calculated by the S-matrix. One very important assumption therein, is that the incoming and outgoing particles behave like free particles. In QFTCST, the notion of incoming and outgoing particles are only recovered in the situation of asymptotically flat spacetimes. But also in this special case, the particle number depends on the observers. This means that different observers may measure a different number of particles in such a spacetime. On CST, there is also an issue with the vacuum state. In QFT on flat spacetimes, the vacuum state is unique due to the condition of it being the only Poincaré invariant state. In QFTCST on the other hand, unless the metric of the curved spacetime has a global time-like Killing vector, there is no unique way of defining the vacuum.

Nevertheless, QFTCST seems to have many applications to cosmology, like Hawking's prediction of thermal radiating black holes and the prediction of the primordial density perturbation spectrum arising from cosmic inflation, just to name a few. Despite its problems and predictions, QFTCST can only be considered as a first approximation to quantum gravity, because the curved spacetime is always taken to be classical, i.e not quantized.

Another appealing approach that gained wide popularity among theoretical physicists is known under the name of noncommutative quantum field theories (NCQFT). Roughly speaking, in this approach one constructs a noncommutative spacetime and tries to define a reasonable QFT on it. The noncommutative spacetime is constructed in a similar fashion as the phase space in quantum mechanics. One simply replaces the coordinates of space and time by operators on a Hilbert space. Furthermore, one endows the operators with an noncommutative algebraic structure. The physical idea for this ansatz is the following: Due to the measurement problem on the scale of Planck lengths on NC spacetimes, the pointwise structure is replaced by some sort of cell structure. In measurement terms, this means that one cannot experimentally determine the spacetime coordinate of a spacetime event with arbitrary accuracy. In the next step a QFT can be defined on such a spacetime by replacing the pointwise product with a deformed product. By taking the limit of the deformation parameter to zero, one recovers the classical picture of spacetime. The deformation parameter which has to be determined by experiment, gives the strength of noncommutativity.

It is interesting to note that even among the string theory community NCQFT, gained popularity due to the observation that NCQFT can be obtained in a certain low energy limit from string theory, [SW99]. From a quantum field theoretical aspect it gained interest due to many reasons. First of all, it was thought that by the introduction of a fundamental length, renormalization ambiguities will disappear and many ultra violet divergences will cancel. It turned out that, the quantum field theoretical euclidean approach exhibited a new type of divergences, the so called UV-IR mixing. Nevertheless, in a series of papers [GW03, GW05a, GW05b], the authors proved renormalizablity of the ϕ^4 model on a NC space to all orders, by adding a term due to duality considerations. This was a great breakthrough, because it provided the way towards other renormalizable noncommutative field theories.

After remarkable progress has been made in understanding field theory on a fixed NC spacetime, the next step would be to try to formulate a dynamical structure of NC spacetime, to enable the incorporation of GR with QFT. A realization of this idea was proposed in a matrix model and has been published by a few authors [Ste07, Yan09]. The basic observation is that matrix models that define noncommutative (NC) gauge theory contains a specific version of gravity. This provides a dynamical theory for noncommutative spaces. The ideas of these matrix models were further investigated [Ste08, GSW08, Muc] and were moreover applied to cosmology, [BS10].

After much work was done on euclidean space, NCQFT was further developed on Minkowski space. Quantum field theory on a noncommutative Minkowski spacetime was rigorously realized in [DFR95]. The quantum field therein was defined on a tensor product space $\mathcal{V} \otimes \mathscr{H}$. Where $\mathscr{H}$ is the Bosonic Fock space and $\mathcal{V}$ is the representation space of the noncommuting coordinate operators $\hat{x}_\nu$, satisfying the Moyal-Weyl plane commutator relations, i.e. $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$. The matrix $\theta_{\mu\nu}$ is a constant, nondegenerate and skew-symmetric matrix.

There were serious questions posed about the unitarity of the S-matrix for scalar QFT defined on the Moyal-Weyl plane. Nevertheless, in a series of papers [LS02c, LS02b] it was shown that the unitarity of the S-matrix is not violated. The authors proved that the apparent violation of unitarity is due to the naive approach of taking the commutative Feynman rules for NCQFT. Instead, one has to be careful due to the fact that the time-ordering procedure does not commute with the star multiplication in the case of time space noncommutativity.

Many authors [ABJJ07, Gro79, GL07] succeeded in representing the scalar field on $\mathscr{H}$ instead of $\mathcal{V} \otimes \mathscr{H}$. In [GL07], this representation was used to construct a map from the set of skew-symmetric matrices, which describes the noncommutativity, to a set of wedges. The next step was to apply the construction to map the noncommutative scalar field to a scalar field living on a wedge. The respective model led to weakened locality and covariance properties of the field and to a nontrivial S-matrix. A result which is astonishing because notions of covariance and locality are usually lost on a noncommutative spacetime.

The method of deformation was further generalized in [BS08, BLS11, Lec12] and was made public under the name of warped convolutions. The method in [Lec12] was also successfully used to define deformations of a scalar massive Fermion, [Ala12]. It is interesting to note that the model formulated in [GL07] can be obtained from warped convolutions by using the momentum operator for the deformation. In fact any strongly continuous unitary representation of the group $\mathbb{R}^n$ can be used to deform the free scalar field.

1.2 Aim and overview of the thesis

The main focus of the present work is the construction of a **quantum spacetime** from theory. We take physical objects, i.e. operators of the underlying quantum theory and deform with those objects, to obtain noncommutative structures. The obtained noncommutativity will give us some insight about the physical nature of the deformation parameter. The general guideline of this thesis is to construct such quantum spacetimes in a purely algebraic way. This means that we use deformation procedures and try to extract from them in a sensible physical way, quantum spacetimes.

One advantage of the construction of those quantum spacetimes is their background independence. Although we take the flat Minkowski metric as our background, the quantum spacetime that we obtain is of purely algebraic nature and also obtained in a very different way from usual approaches. The commutative limit which in our approach is equivalent to the flat limit, can be obtained by setting the deformation parameter equal to zero.

The organization of the thesis is as follows: In Chapter 2 we lay out the fundamental building blocks of the deformation that we use throughout the entire work. This deformation method is called warped convolutions. To use this method in mathematical sensible way one is obliged to have a strongly continuous unitary representation. Such a unitary representation is taken in the third chapter by the exponential of the coordinate operator and the deformation is performed with the so defined group. In Chapter 3 quantum mechanical objects, as for example the Hamiltonian, are deformed. Many interesting physical phenomena follow and those physical theories are used to define a quantum space. We further show that this method can be generalized to the case of the scalar fields and commit ourselves to such a search in a quantum field theoretical context. Before doing so, we review in Chapter 4 some important developments that were achieved in the context of NCQFT on Minkowski space.

In Chapter 5 we construct for the massless and massive scalar field reasonable coordinate operators and show many paths of obtaining a quantum spacetime in this way. We further deform the scalar field with the constructed operators and show that they live on a nonconstant noncommutative momentum space. Moreover, we show that a QFT defined on such a space still fulfills reasonable properties. We prove that the field satisfies the Wightman properties, wedge-covariance under a subgroup the Poincaré group and wedge-locality.

In Chapter 6 we deform the free massless scalar quantum field with the special conformal operator. The proof of self-adjointness of the special conformal operators was given rigorously in [SV73] and is shortly sketched. The proof therein relies on the fact that the momentum operator and the special conformal operator are unitarily equivalent. Furthermore, we use the unitary equivalence to proof convergence of the deformation in the Hilbert space norm. The Wightman properties, transformation properties and wedge-locality of the deformed field are further proven. In the end of the Section we show how the deformation with the special conformal operators leads to a nonconstant noncommutative spacetime. The last chapter focuses on the conclusion and outlook.

Chapter 2

Warped Convolutions

The main focus of this work is the construction of an **emerging quantum spacetime**. In particular this means that we take physical objects, i.e. operators of the underlying quantum theory, and deform with those objects in order to obtain noncommutative structures. The obtained noncommutativity may give us some insight about the physical nature of the deformation parameter. The deformation technique that is used throughout the entire thesis is known by the name of warped convolutions, [BS08, BLS11].

The reason for our specific choice is owed to the fact that warped convolutions give a mathematical rigorous framework in which deformations of operators in QM and QFT can be considered. For easy reference we recapitulate this novel deformation procedure and state the most important definitions, lemmas and propositions in a form appropriate for the current work. For proofs of the lemmas and propositions introduced in this Section we refer the reader to the original papers [BS08, BLS11].

2.1 C^* algebra

To give a mathematical rigorous definition of the deformation of operators, one has to work in a C^* algebraic setting, where deformations of bounded operators are considered. For this purpose, let us first define a C^* algebra. We start by defining an algebra $\mathcal{A}$, [Lan97].

Definition 2.1. $\mathcal{A}$ is called an **algebra** over the vector space of complex numbers $\mathbb{C}$, if $\alpha a + \beta b$ with $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, are well defined. In addition, there is a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which is distributive over addition,

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc, \qquad \forall a, b, c \in \mathcal{A}.$$

The algebra $\mathcal{A}$ is called a **unital algebra** if it has a unit I.

Definition 2.2. An algebra $\mathcal{A}$ is called a ***-algebra** if it admits an involution $^* : \mathcal{A} \to \mathcal{A}$, such that

$$a^{**} = a, \qquad (ab)^* = b^* a^*, \qquad (\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$$

for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ and bar denoting the usual complex conjugation.

Definition 2.3. An algebra $\mathcal{A}$ with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ is called a **normed algebra** if the following properties are fulfilled,

$$\|a\| \geq 0, \|a\| = 0 \Leftrightarrow a = 0, \qquad \|\alpha a\| = |\alpha|\|a\|,$$

$$\|a + b\| \leq \|a\| + \|b\|, \qquad \|ab\| \leq \|a\|\|b\|$$

for any $a, b \in \mathcal{A}$ and $\alpha, \in \mathbb{C}$.

The topology defined by the norm is called the norm topology. The neighborhoods of any $a \in \mathcal{A}$ are given by

$$U(a)_\epsilon = \{b \in \mathcal{A} : ||a - b|| < \epsilon\}, \quad \epsilon > 0.$$

Definition 2.4. A **Banach algebra** is a normed algebra which is complete in the norm topology.

Definition 2.5. A **Banach *-algebra** is a normed *-algebra which is complete and satisfies the additional requirement

$$\|a^*\| = \|a\|, \qquad \forall a \in \mathcal{A}.$$

Definition 2.6. A C^***-algebra** is a Banach *-algebra whose norm satisfies the additional identity

$$\|a^* a\| = \|a\|^2, \qquad a \in \mathcal{A}.$$

2.2 Deformation with warped convolutions

Warped convolutions were introduced in the realm of algebraic quantum field theory as a new tool to construct non-trivial quantum field theories. It was further shown in [BLS11], that the deformation procedure is an isometric representation of Rieffel's strict deformations of C^*-dynamical systems in [Rie93]. In this Section we introduce the important definitions, lemmas and propositions of warped convolutions.

The authors in [BS08, BLS11] start by considering the C^*-algebra $\boldsymbol{\mathcal{C}}$ of all uniformly continuous bounded functions $\mathbf{A} : \mathbb{R}^n \to \mathcal{B}(\mathscr{H})$, where $\mathcal{B}(\mathscr{H})$ is the Hilbert space of bounded operators. The algebraic operations of the C^*-algebra $\boldsymbol{\mathcal{C}}$ are pointwise defined,

$$(\mathbf{A} + \mathbf{B})(x) = \mathbf{A}(x) + \mathbf{B}(x), \quad (\mathbf{AB})(x) = \mathbf{A}(x)\mathbf{B}(x), \qquad \mathbf{A}^*(x) = \mathbf{A}(x)^*, \qquad x \in \mathbb{R}^n.$$

The norm on $\boldsymbol{\mathcal{C}}$ is given by the supremum norm

$$\|\mathbf{A}\| = \sup_{x \in \mathbb{R}^n} \|\mathbf{A}(x)\|.$$

Elements $\mathbf{A} \in \boldsymbol{\mathcal{C}}$ that are considered for deformation belong to the subalgebra $\boldsymbol{\mathcal{C}}^\infty := \{\mathbf{A} \in \boldsymbol{\mathcal{C}} : \|\partial^\mu \mathbf{A}(x)\| \leq \infty\}$. In the proofs of the lemmas and propositions it is important to integrate the functions $x \mapsto \mathbf{A}(x)$ and in general these functions are not absolutely integrable, w.r.t. the Lebesgue measure. In order to have suitable decay properties, mollifiers $F_n : \mathbb{R}^n \to \mathbb{C}$ are introduced and a specific choice for $F_n \in L^1(\mathbb{R}^n)$ is given by

$$L_n(x) = (i + x_1 + \cdots + x_n)^{-1} \prod_{k=1,\cdots,n} (i + x_k)^{-1}, x \in \mathbb{R}^n.$$

The next lemma is of great technical importance for all further lemmas concerning the warped convolutions of operators.

Lemma 2.1. *Let* $\mathbf{A}$, $\mathbf{B} \in \boldsymbol{\mathcal{C}}$ *be* $n+1$ *times continuously differentiable and let* $f \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ *with* $f(0,0) = 1$.

(i) The norm limit of the Bochner integrals in $\mathcal{B}(\mathscr{H})$,

$$\lim_{\epsilon \to 0} (2\pi)^{-n} \iint dx\, dy\, f(\epsilon x, \epsilon y)\, e^{-ixy}\, \mathbf{A}(x)\, \mathbf{B}(y) \doteq \mathbf{A} \times \mathbf{B},$$

exists and does not depend on f. *Here* xy, $x, y \in \mathbb{R}^n$ *is any symmetric bilinear form on* $\mathbb{R}^n$ *with determinant* 1 *or* -1.

(ii) With L_n *as above, there exists a polynomial* $u, v \mapsto P_n(u, v)$ *on* $\mathbb{R}^n \times \mathbb{R}^n$ *of degree* $n+1$ *in the components of* u *and* v*, respectively, such that*

$$\mathbf{A} \times \mathbf{B} = (2\pi)^{-n} \iint dx\, dy\, e^{-ixy}\, P_n(\partial_x, \partial_y)\, L_n(x)\, \mathbf{A}(x)\, L_n(y)\, \mathbf{B}(y),$$

where the integral is defined as a Bochner integral in $\mathcal{B}(\mathscr{H})$.

(iii) $\|\mathbf{A} \times \mathbf{B}\| \leq c_n \|\mathbf{A}\|_{n+1} \|\mathbf{B}\|_{n+1}$, *for a universal constant* c_n.

(iv) Let $C \in \mathcal{B}(\mathscr{H})$. *Then*

$$(C\mathbf{A} \times \mathbf{B}) = C(\mathbf{A} \times \mathbf{B}), \quad (\mathbf{A} \times \mathbf{B}C) = (\mathbf{A} \times \mathbf{B})C, \quad (\mathbf{A}C \times \mathbf{B}) = (\mathbf{A} \times C\mathbf{B})$$

and the linear map

$$C \mapsto \mathbf{A} \times C\mathbf{B}$$

is continuous on the unit sphere of $\mathcal{B}(\mathscr{H})$ *in the strong operator topology.*

To define the deformation of operators belonging to a C^*-algebra $\mathcal{C}$, we consider elements belonging to the subalgebra $C^\infty \subset C^*$. The subalgebra C^∞ is defined to be the $*-$algebra of smooth elements with respect to α, which is the adjoint action of a weakly continuous unitary representation U of $\mathbb{R}^n$ given by

$$\alpha_x(A) = U(x)(A)U(x)^{-1}, \quad x \in \mathbb{R}^n.$$

By using Lemma 2.1 and the spectral calculus one can present the warped convolution of $A \in \mathcal{C}^\infty$ given by $\int \alpha_{\theta x}(A)dE(x)$ or $\int dE(x)\alpha_{\theta x}(A)$, on the dense domain $\mathcal{D} \subset \mathscr{H}$ of vectors smooth w.r.t. the action of U, in terms of strong limits

$$\int \alpha_{\theta x}(A)dE(x)\Phi = (2\pi)^{-n} \lim_{\epsilon \to 0} \int \int dxdy f(\epsilon x, \epsilon y)e^{-ixy} \alpha_{\theta x}(A)U(y)\Phi, \qquad \Phi \in \mathcal{D},$$

$$\int dE(x)\alpha_{\theta x}(A)\Phi = (2\pi)^{-n} \lim_{\epsilon \to 0} \int \int dxdy f(\epsilon x, \epsilon y)e^{-ixy} U(y)\alpha_{\theta x}(A)\Phi, \qquad \Phi \in \mathcal{D},$$

where E is the spectral resolution.

The following lemma shows first that the two different warped convolutions are equivalent. Second, it shows how the complex conjugation acts on the warped convoluted operator.

Lemma 2.2. *Let θ be a real skew symmetric matrix on $\mathbb{R}^n$ and let $A \in \mathcal{C}^\infty$. Then*

(i) $\int \alpha_{\theta x}(A) dE(x) = \int dE(x) \alpha_{\theta x}(A)$

(ii) $\left(\int \alpha_{\theta x}(A) dE(x)\right)^* \subset \int \alpha_{\theta x}(A^*) dE(x)$

The deformations of operators that we consider make use of the following definition.

Definition 2.7. Let θ be a real skew symmetric matrix on $\mathbb{R}^n$ and let $A \in \mathcal{C}^\infty$. The corresponding warped convolution A_θ of A is defined on the dense domain $\mathcal{D} \subset \mathscr{H}$ according to

$$A_\theta \Phi := \int dE(x) \alpha_{\theta x}(A) \Phi = \int \alpha_{\theta x}(A) dE(x) \Phi, \qquad \Phi \in \mathcal{D}. \tag{2.1}$$

In particular, $1_\theta = 1$.

In the following lemma we introduce the deformed product, known as the Rieffel product [Rie93], by using warped convolutions [Rie93]. The circumstance that the two are interrelated is due to the fact that the warped convolutions supply isometric representations of Rieffel's strict deformations of C^*-dynamical systems with actions of $\mathbb{R}^n$. The definition of the Rieffel product, given by warped convolutions, is used to calculate deformed commutators. For example, in 6 one can use the Rieffel product to argue that the deformation with the special conformal operator induces a nonconstant noncommutative spacetime.

Lemma 2.3. *Let θ be a real skew symmetric matrix on $\mathbb{R}^n$ and let $A, B \in \mathcal{C}^\infty$. Then*

$$A_\theta B_\theta \Phi = (A \times_\theta B)_\theta \Phi, \qquad \Phi \in \mathcal{D},$$

The deformed product $\times_\theta$ is known as the Rieffel product on $\mathcal{C}^\infty$ and is given by,

$$(A \times_\theta B)\Phi = (2\pi)^{-n} \lim_{\epsilon \to 0} \int \int dx dy f(\epsilon x, \epsilon y) e^{-ixy} \alpha_{\theta x}(A) \alpha_y(B) \Phi, \qquad \Phi \in \mathcal{D}. \tag{2.2}$$

The next proposition gives the transformation property of the warped convolution of an operator under the adjoint action of a unitary or antiunitary operator on $\mathscr{H}$. This is important in Chapters 5 and 6 when we examine the transformation properties of deformed operators under the Lorentz transformations.

Proposition 2.1. *Let V be a unitary or antiunitary operator on $\mathscr{H}$ such that $VU(x)V^{-1} = U(Mx)$, $x \in \mathbb{R}^n$, for some invertible matrix M. Then, for $A \in \mathcal{C}^\infty$,*

$$VA_\theta V^{-1} = (VAV^{-1})_{\sigma M \theta M^T},$$

where M^T is the transpose of M w.r.t the chosen bilinear form, $\sigma = 1$ if V is unitary and $\sigma = -1$ if V is antiunitary.

The transformation property of the deformed operator becomes important after we relate in the realm of quantum field theory, skew symmetric matrices θ to wedges $\mathcal{W}$

by using the homomorphism given in [GL07]. This in particular means that to each deformed operator with deformation matrix θ there is a corresponding wedge $\mathcal{W}$. The transformation behavior of the deformed operator given in proposition 2.1 corresponds to the transformation property of a wedge-covariant field.

The following proposition is crucial to prove that two deformed fields satisfy a weakened locality know as wedge locality.

Proposition 2.2. *Let $A, B \in \mathcal{C}^\infty$ be operators such that $[\alpha_{\theta x}(A), \alpha_{-\theta y}(B)] = 0$ for all $x, y \in spU$. Then*

$$[A_\theta, B_{-\theta}] = 0.$$

Due to the fact that in physics one usually works with unbounded operators, in the next Sections we are obliged to show that the deformation formulas still have a proper mathematical meaning.

Chapter 3

Deformations in QM

In this Chapter we study deformations of operators from a quantum mechanical point of view. The tool used to study deformations is the warped convolutions. At first, we study deformations of the simplest Hamiltonian of quantum mechanics, namely that of a free particle. The generators of a unitary representation, needed to define the warped convolutions, are chosen to be the coordinate operators. The idea behind deformations of QM with the coordinate operator is the intention to study similar deformations of a QFT. Thus, the main objective in this Section is gaining some insight into the physical effects appearing by a deformation with the coordinate operator and try to implement those ideas in the realm of QFT.

3.1 The canonical commutation relations

In quantum mechanics the momentum P_i and coordinate X_j are represented as **self-adjoint operators** on a Hilbert space $\mathscr{H}$. These operators do not commute mutually, but satisfy the canonical commutation relations instead. In this section, we give a functional analytic introduction to the representation of the coordinate and the momentum operator as self-adjoint operators. We start by giving the definition of the canonical commutation relations, [RS75a, Chapter VIII.5, Example 2].

Definition 3.1. A pair of self-adjoint operators (P_i, X_j) is said to satisfy the **canonical commutation relations** if

$$P_i X_j - X_j P_i = -i\delta_{ij} I_{\mathscr{H}}, \tag{3.1}$$

where $I_{\mathscr{H}}$ is the unity operator on $\mathscr{H}$ and δ_{ij} is the Kronecker delta in n-dimensions.

Operators satisfying the canonical commutation relations are unbounded operators and therefore one has further specify the domain of self-adjointness. In this context it is easier to work with bounded operators and use the fundamental theorems of von Neumann and Stone to obtain the essentially self-adjoint operators (P_i, X_j). Let us start by defining strongly continuous unitary representations of the additive group $\mathbb{R}^n$. This is done by using Stone's theorem, [RS75a, Theorem VIII.7].

Theorem 3.1. *Let Q_j be commuting self-adjoint operators, with $j = \{1, \cdots, n\}$, i.e. $[Q_j, Q_k] = 0$, and define $C(b) := e^{ib_k Q^k}$. Then*

1. *For each $b_k \in \mathbb{R}$, $C(b)$ is a unitary operator and $C(b)C(s) = C(b+s)$, for all $b_k, s_k \in \mathbb{R}$.*
2. *If $\phi \in L^2(\mathbb{R}^n)$ and $b_k \to b_k^0$, then $C(b)\phi \to C(b^0)\phi$.*

The next definitions are given in [RS75a, Chapter VIII].

Definition 3.2. An operator valued function $C(b)$ satisfying the properties stated in the last theorem is called a **strongly continuous unitary group**.

Definition 3.3. If $C(b)$ is a strongly continuous unitary group, then the self-adjoint operator Q_i with $C(b) = e^{ib_k Q^k}$ is called the **infinitesimal generator** of $C(b)$.

In the next step we define the strongly continuous unitary groups satisfying the so called Weyl relations.

Definition 3.4. Let $V(a)$ and $U(b)$, with $a, b \in \mathbb{R}^n$, be two continuous unitary groups on a separable Hilbert space $\mathscr{H}$ satisfying the following relations

$$U(a)V(b) = e^{-ia_i b^i} V(b)U(a). \tag{3.2}$$

Then the groups $V(a)$ and $U(b)$ are said to satisfy the **Weyl relations**. The unitary operators $V(a)$ and $U(b)$ are defined by using the canonical conjugate pair (X_j, P_k) in the following way

$$V(a) := e^{ia^j X_j}, \quad U(b) := e^{ib^j P_j}, \qquad a, b \in \mathbb{R}^n.$$

The corollary that follows shows that any strongly continuous unitary group fulfilling the Weyl relations (3.2), have infinitesimal generators satisfying the canonical commutation relations. Furthermore, the generators turn out to be essentially self-adjoint operators on a dense domain $D \subset \mathscr{H}$, [RS75a, Chapter VIII.5, page 275].

Corollary 3.1. *Let $V(a)$ and $U(b)$ be strongly continuous unitary groups satisfying the Weyl relations (3.2) on a separable Hilbert space $\mathscr{H}$. Let the operator P_i be the generator of $U(b)$ and the operator X_k be the generator of $V(a)$. Then, there is a dense domain $D \subset \mathscr{H}$ so that*

1. *$P_i : D \to D, X_k : D \to D$,*
2. *$P_i X_k \varphi - X_k P_i \varphi = -i\delta_{ik}\varphi, \qquad \forall \varphi \in D$,*
3. *P_i and X_k are essentially self-adjoint on D.*

The common domain of essential self-adjointness is chosen to be the Schwartz space $\mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, defined in (8.20). In the next section we use the fact that the essentially self-adjoint operators (P_i, X_j) define unitary representations of the additive group $\mathbb{R}^n$, in order to define warped convolutions.

3.2 Warped convolutions in QM

For a deformation of the Hamiltonian we choose to work in the standard realization of quantum mechanics, the so called **Schrödinger representation**, [BEH08, RS75a, Tes01]. In this representation the pair of operators (P_j, X_k) satisfying the canonical commutation relations are represented as essentially self-adjoint operators on the dense domain $\mathscr{S}(\mathbb{R}^n)$. Here P_j and X_k are the closures of $i\partial/\partial x^j$ and multiplication by x_k on $\mathscr{S}(\mathbb{R}^n)$, respectively. In this section we apply the definitions of a strongly continuous unitary group to define the warped convolutions formula (2.7) of a densely defined operator A.

First, we define the warped convolutions by using the coordinate operator.

Definition 3.5. Let B be a real skew-symmetric matrix on $\mathbb{R}^n$ and let $\chi \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. Then, the warped convolution of an operator A with the coordinate operator, denoted as $A_{B,X}$, is defined on the dense domain $\mathscr{S}(\mathbb{R}^n)$ as follows

$$A_{B,X}\Psi := (2\pi)^{-n} \lim_{\epsilon\to 0} \iint d^n y\, d^n k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k) V(k) \alpha_{By}(A)\Psi. \tag{3.3}$$

The automorphisms α are implemented by the adjoint action of the strongly continuous unitary representation $V(y)$ of $\mathbb{R}^n$ given by

$$\alpha_x(A) = V(x)\, A\, V(x)^{-1}, \quad x \in \mathbb{R}^n.$$

In the quantum mechanical case most of our considerations involve the deformation of operators using the coordinate operator.

However, for some arguments we also need warped convolutions defined with the momentum operator.

Definition 3.6. Let θ be a real skew-symmetric matrix on $\mathbb{R}^n$ and let $\chi \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. Then, the warped convolution of an operator A with the momentum operator, denoted as $A_{\theta,P}$, is defined on the dense domain $\mathscr{S}(\mathbb{R}^n)$ as follows

$$A_{\theta,P}\Psi := (2\pi)^{-n} \lim_{\epsilon\to 0} \iint d^n y\, d^n k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k) U(k) \alpha_{\theta y}(A)\Psi. \tag{3.4}$$

The automorphisms α are implemented by the adjoint action of the strongly continuous unitary representation $U(y)$ of $\mathbb{R}^n$ given by

$$\alpha_x(A) = U(x)\, A\, U(x)^{-1}, \quad x \in \mathbb{R}^n.$$

3.3 Deforming the Hamiltonian

To explore the physical consequences of deformation, we take the free Hamiltonian of quantum mechanics and deform it by using the novel tool of warped convolutions. Later on we solve the eigenvalue equation to obtain a deeper physical insight. The Hamiltonian of a free particle in quantum mechanics is given as follows

$$H_0 = -\frac{P_j P^j}{2m}. \tag{3.5}$$

This Hamiltonian describes a non-relativistic and non-interacting particle. For the following considerations, let us restrict the deformation to three space dimensions. A restriction obvious due to its physical relevance. Let us start the section with an important theorem concerning the domain of self-adjointness and the spectrum of the free undeformed Hamiltonian H_0, [Tes01, Theorem 7.8].

Theorem 3.2. *The free Schrödinger operator H_0 is self-adjoint on the domain $\mathcal{D}(H_0)$ given as*

$$\mathcal{D}(H_0) = H^2(\mathbb{R}^3) = \{\varphi \in L^2(\mathbb{R}^3) | |\boldsymbol{P}|^2\varphi \in L^2(\mathbb{R}^3)\},$$

and its spectrum is characterized by $\sigma(H_0) = [0, \infty)$.

Before proceeding with deformation, a mathematical problem arises at this point of our work. The deformation formula, given by warped convolutions, is only well-defined in the strong operator topology for a subset of **bounded operators** that are **smooth** w.r.t. unitary representation U of $\mathbb{R}^n$. In view of the fact that this thesis deals with unbounded operators we have to show that the deformation formula, given as an oscillatory integral, is well-defined. For the subsequent discussion let us introduce the notion of an oscillatory integral, ([Hör04], Section 7.8, Equation 7.8.1).

Definition 3.7. Let $X \subset \mathbb{R}^n$ be open and let Γ be an open cone on $X \times (\mathbb{R}^N \backslash \{0\})$ for some N. This means that Γ is invariant under multiplication by positive scalars of components in $\mathbb{R}^N$. We shall say that a function $\phi \in C^\infty(\Gamma)$ is a **phase function** in Γ if

- $\phi(x, ty) = t\phi(x, y)$ if $(x, y) \in \Gamma, t > 0$.
- $\mathrm{Im}\phi \geq 0$ in Γ,
- $\mathrm{d}\phi \neq 0$ in Γ.

Then an integral of the form

$$\int e^{i\phi(x,y)} b(x, y)\, dy,$$

is called an **oscillatory integral**.

Another important notion in our subsequent discussion is that of a symbol ([Hör04], Section 7.8, Definition 7.8.1).

Definition 3.8. Let m, ρ, δ, be real numbers with $0 < \rho \leq 1$ and $0 \leq \delta < 1$. Then we denote by $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$, the set of all $b \in C^\infty(X \times \mathbb{R}^n)$ such that for every compact set $K \subset X$ and all α, β the estimate

$$|\partial_x^\beta \partial_k^\alpha b(x, k)| \leq C_{\alpha,\beta,K}(1 + |k|)^{m-\rho|\alpha|+\delta|\beta|}, \qquad x \in K,\ k \in \mathbb{R}^n,$$

is valid for some constant $C_{\alpha,\beta,K}$. The elements $S^m_{\rho,\delta}$ are called symbols of order m and type ρ, δ.

By using the former definitions it can be shown [Hör04, Section 7.8, Theorem 7.8.2], ([LW11]), ([Jos99]), that if $m < -n + 1$ the oscillatory integral converges to a well-defined function. In the case $m \geq -n + 1$, the oscillatory integral can still be defined in a distributional manner. The cases considered in this thesis belong to the second class. Thus, the task throughout the thesis is to show that $b(x, y)$ belongs to a symbol class and therefore can be well-defined as a distribution.

To prove that the deformation formula (3.3) holds in the case of the unbounded operator H_0, let us consider the deformed free Hamiltonian $H_{B,X}$ as follows

$$\begin{aligned}\langle\Psi, H_{B,X}\Phi\rangle &= (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_l k^l}\,\chi(\epsilon y,\epsilon k)\langle\Psi, V(k)\alpha_{By}(H_0)\Phi\rangle\\ &=: (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_l k^l}\,\chi(\epsilon y,\epsilon k)\, b(k,y)\end{aligned}$$

for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$.

Remark 3.1. The scalar product $\langle\mathbf{y},\mathbf{k}\rangle$ satisfies the conditions of a phase function given in Definition 3.7. It is actually one of the most considered examples of a phase function, due to its close relation to the Fourier transformation.

As one can see $\langle\Psi, H_{B,X}\Phi\rangle$ is given in the form of an oscillatory integral. Thus to prove that the expression is well-defined, we show in the next lemma that $b(k,y)$ belongs to a symbol class.

Lemma 3.1. *Let the function $b(y,k)$ be given as the scalar product $\langle\Psi, V(k)\alpha_{By}(H_0)\Phi\rangle$. Then $b(y,k)\in S^2_{1,0}$, for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$ and therefore deformation with the coordinate operator, via warped convolution, of the free Hamiltonian H_0 is given as a well-defined oscillatory integral.*

Proof. For the proof we first calculate the adjoint action of $V(By)$ on H_0 given by,

$$\begin{aligned}\alpha_{By}(H_0) = V(By)H_0V(-By) &= -\frac{1}{2m}V(By)P_jP^jV(-By)\\ &= -\frac{1}{2m}V(By)P_jV(-By)V(By)P^jV(-By).\end{aligned}$$

To solve this expression we look at the adjoint action of $V(By)$ on the momentum operator,

$$\begin{aligned}e^{i(By)_kX^k}P_je^{-i(By)_kX^k} &= P_j + i(By)_k[X^k,P_j] + \underbrace{\frac{i^2}{2}(By)_l(By)_k[X^l,[X^k,P_j]] + \cdots}_{=0}\\ &= P_j + (By)_j. \end{aligned}\tag{3.6}$$

By taking equation (3.6) into account we obtain for the adjoint action of $V(y)$ on H_0

$$\begin{aligned}\alpha_{By}(H_0) &= -\frac{1}{2m}(P_j + B_{jk}y^k)(P^j + B^{jr}y_r)\\ &= H_0 - \frac{1}{2m}\left(2P^jB_{jk}y^k + B_{jk}y^kB^{jr}y_r\right).\end{aligned}$$

It is easily derived by using the canonical commutation relation and the Baker-Campbell-Hausdorff formula.

In the next step we look at the expression

$$
\begin{aligned}
|\partial^{\alpha}_{k^i}\partial^{\beta}_{y^r} b(y,k)| &= |\langle \Psi, \left(\partial^{\alpha}_{k^i} V(k)\right) \partial^{\beta}_{y^r} \left(\alpha_{By}(H_0)\right) \Phi\rangle| \\
&\leq \underbrace{\|(-iX^{\alpha})\Psi\|}_{=:C_{1,\alpha}} \left\| \partial^{\beta}_{y^r} \left(H_0 - \frac{1}{m} P^j (By)_j - \frac{1}{2m}(By)_l (By)^l \right) \Phi \right\| \\
&\leq C_{1,\alpha} \left(\left\| \partial^{\beta}_{y^r} H_0 \Phi \right\| + \frac{1}{m} |\partial^{\beta}_{y^r}(By)^j| \, \|P_l \Phi\| + \frac{1}{2m} |\partial^{\beta}_{y^r}(By)^j|^2 \, \|\Phi\| \right),
\end{aligned}
$$

where the Cauchy-Schwarz inequality was used to obtain the last line.

Remark 3.2. In the first line we changed the order of integration and differentiation. This can be done due to the following relation,

$$
b(k,y) = \int d\mathbf{x} e^{-ik_l \mathbf{x}^l} \Psi(\mathbf{x}) \left(-\Delta_{\mathbf{x}} + i \frac{1}{m} (By)_j \frac{\partial}{\partial \mathbf{x}_j} + \frac{1}{2m} (By)_l (By)^l \right) \Phi(\mathbf{x}).
$$

By using the fact that $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$ and $e^{-ik_l \mathbf{x}^l}$ is continuous in k, we can interchange the order for k. Furthermore, the adjoint action of $V(By)$ on H_0 yields a second order polynomial in y at most and thus it is clear that we can interchange the order of differentiation and integration.

Without loss of generality, one can choose the skew-symmetric matrix B to have the form $B_{ij} = \epsilon_{ijk} B^k$, where ϵ_{ijk} is the three dimensional epsilon-tensor. Then the following inequality for the norm of vector $B_{ij} y^j$ holds,

$$
|B_{ij} y^j| \leq \sqrt{2} |\mathbf{B}| |\mathbf{y}|. \tag{3.7}
$$

This is easily seen by using the Cauchy-Schwarz inequality and the relation $|a| - |b| \leq |a| + |b|$.

$$
\begin{aligned}
|B_{ij} y^j|^2 &= (-B_{ij} y^j B^{is} y_s) \\
&= -\epsilon_{ijk} B^k y^j \epsilon^{isr} B_r y_s \\
&= -\left(\delta^s_j \delta^r_k - \delta^r_j \delta^s_k\right) B_r B^k y^j y_s \\
&= \left(B_r y^r B_k y^k - B_r B^r y^j y_j \right) \\
&= (\mathbf{B}\mathbf{y})^2 - (\mathbf{B})^2 \mathbf{y}^2 \leq (\mathbf{B})^2 \mathbf{y}^2 + (\mathbf{B}\mathbf{y})^2 \\
&\leq 2 |\mathbf{B}|^2 |\mathbf{y}|^2.
\end{aligned}
$$

Thus for $\underline{\beta = 0}$ and by choosing $B_{ij} = \epsilon_{ijk} B^k$, we have inequality

$$
\begin{aligned}
|\partial^{\alpha}_{k^i} b(y,k)| &\leq C_{1,\alpha} \left(\|H_0 \Phi\| + \frac{1}{m} |(By)^j| \, \|P_l \Phi\| + \frac{1}{2m} |(By)^j|^2 \, \|\Phi\| \right) \\
&\leq C_{1,\alpha} \left(\underbrace{\|H_0 \Phi\|}_{=:C_2} + 2|\mathbf{y}| \underbrace{\frac{|\mathbf{B}|}{\sqrt{2}m} \|P_l \Phi\|}_{=:C_3} + |\mathbf{y}|^2 \underbrace{\frac{|\mathbf{B}|^2}{m} \|\Phi\|}_{=:C_4} \right) \\
&\leq \underbrace{C_{1,\alpha} C_B}_{=:C_{1,\alpha,B}} (1 + |\mathbf{y}|)^2,
\end{aligned}
$$

where in the last lines a constant C_B satisfying

$$C_B \geq C_1, C_2, C_3. \tag{3.8}$$

has been chosen, which is possible since C_1, C_2 and C_3 are finite constants. Therefore $b(y,k)$ belongs to the symbol class $S^2_{1,0}$. Note that for $\beta = 1, 2$ the behavior of $b(y,k)$ for large y becomes even better.

Having shown that $b(k,y)$ belongs to a symbol class theorem [Hör04, Section 7.8, Theorem 7.8.2] can be applied and thus it follows that the oscillatory integral is defined in a distributional sense. □

In the subsequent discussion we drop the scalar product in order to clarify the physical outcome of the deformation. Nevertheless, throughout this work we show that the deformation formula of warped convolutions holds for the unbounded operators that are used.

After clearing the mathematical problem of well-definedness we turn to the explicit result of deformation. The resulting Hamiltonian obtained by deformation of the free Hamiltonian using warped convolutions is given in the next proposition.

Proposition 3.1. *Let the free Hamiltonian H_0 be given as in (3.5), then the deformed free Hamiltonian $H_{B,X}$, which is obtained by using warped convolutions as defined in (3.3) is given by*

$$H_{B,X}\Psi = -\frac{1}{2m}(P_j + B_{jk}X^k)(P^j + B^{jr}X_r)\Psi. \tag{3.9}$$

Proof. To solve the integral of deformation for H_0 we let $H_{B,X}$ act on wave functions $\Psi(\mathbf{q}) \in \mathscr{S}(\mathbb{R}^3)$, which are eigenfunctions of the coordinate operator.

$$\begin{aligned}
&(H_{B,X}\Psi)(\mathbf{q}) = \\
&= -(2\pi)^{-3} \lim_{\epsilon\to 0} \left(\int\int d^3k d^3y e^{-ik_r(y-q)^r} \chi(\epsilon k, \epsilon y)\, \alpha_{By}(H_0)\Psi \right)(\mathbf{q}) \\
&= -(2\pi)^{-3} \lim_{\epsilon_1\to 0} \left(\int d^3y \lim_{\epsilon_2\to 0} \left(\int d^3k e^{-ik_r(y-q)^r} \chi_2(\epsilon_2 k) \right) \chi_1(\epsilon_1 y)\, \alpha_{By}(H_0)\Psi \right)(\mathbf{q}) \\
&= -(2\pi)^{-3} \lim_{\epsilon_1\to 0} \left(\int d^3y\, (2\pi)^3\, \delta(\mathbf{y}-\mathbf{q})\, \chi(\epsilon_1 y)\, \alpha_{By}(H_0)\Psi \right)(\mathbf{q}) \\
&= -\frac{1}{2m}(P_j + B_{jk}X^k)(P^j + B^{jr}X_r)\Psi(\mathbf{q}).
\end{aligned}$$

Here we used the fact that the oscillatory integral does not depend on the cut-off function chosen. As in [Rie93] and [MM11], we choose $\chi(\epsilon k, \epsilon y) = \chi_2(\epsilon_2 k)\chi_1(\epsilon_1 y)$ with $\chi_l \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\chi_l(0,0) = 1$, $l = 1, 2$, and obtained the delta distribution $\delta(\mathbf{y}-\mathbf{q})$ in the limit $\epsilon_2 \to 0$, [Hör04, Section 7.8, Equation 7.8.5]. □

The proposition tells us that the deformation with the coordinate operator amounts to a non-constant shift of the momenta. In physics this is usually referred to as **minimal substitution**. Such a minimal substitution is in QM usually based on Galilei invariance and then implemented accordingly by an external electromagnetic field (see [JM67]), [Sib].

In our approach we obtain such a substitution by deformation. The connection between deformation and an external electromagnetic field is explored in the next sections.

For our next results the deformation of the momentum operator is necessary. Due to the unboundedness of P^j we are obliged to show that formula (3.3) can be defined as a well-defined oscillatory integral. A reader more interested in the result of the deformation can skip the following lemma.

As for the proof of Lemma 3.1 we take the scalar product of $P^j_{B,X}$,

$$\begin{aligned}\langle \Psi, P^j_{B,X}\Phi\rangle &= (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_l k^l}\,\chi(\epsilon y,\epsilon k)\langle\Psi, V(k)\alpha_{By}(P^j)\Phi\rangle\\ &=: (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_l k^l}\,\chi(\epsilon y,\epsilon k)\, b^j(k,y)\end{aligned}$$

for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$.

Lemma 3.2. *Let the function $b^j(y,k)$, for $j = 1, 2, 3$, be given as the scalar product $\langle\Psi, V(k)\alpha_{By}(P^j)\Phi\rangle$. Then $b^j(y,k)\in S^1_{1,0}$, for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$ and thus the deformation with the coordinate operator, via warped convolutions, of the momentum operator is given as a well-defined oscillatory integral.*

Proof. We start by looking at the following expression,

$$\begin{aligned}|\partial^\alpha_{k^i}\partial^\beta_{y^r} b^j(y,k)| &= |\langle\Psi, \left(\partial^\alpha_{k^i}V(k)\right)\partial^\beta_{y^r}\left(\alpha_{By}(P^j)\right)\Phi\rangle|\\ &\underbrace{\leq \|(-iX^\alpha)\Psi\|}_{=:C_{1,\alpha}}\left\|\partial^\beta_{y^r}\left(P^j+(By)^j\right)\Phi\right\|\\ &\leq C_{1,\alpha}\left(\left\|\partial^\beta_{y^r}P^j\Phi\right\| + |\partial^\beta_{y^r}(By)^j|\,\|\Phi\|\right),\end{aligned}$$

where in the last lines we used the adjoint action given in Equation (3.6). By using inequality (3.7) for $B_{ij}=\epsilon_{ijk}B^k$ and taking $\underline{\beta = 0}$ we obtain

$$\begin{aligned}|\partial^\alpha_{k^i} b^j(y,k)| &\leq C_{1,\alpha}\left(\underbrace{\left\|P^j\Phi\right\|}_{=:C_5} + |\mathbf{y}|\underbrace{\sqrt{2}|\mathbf{B}|\,\|\Phi\|}_{=:C_6}\right)\\ &\leq C_{1,\alpha}\left(C_5 + C_6|\mathbf{y}|\right)\\ &\leq \underbrace{C_{1,\alpha}C_D}_{=:C_{1,\alpha,B}}\left(1+|\mathbf{y}|\right),\end{aligned}$$

where in the last line a constant C_D satisfying

$$C_D \geq C_5, C_6, \tag{3.10}$$

has been chosen, which is possible since C_5 and C_6 are finite. Therefore $b^j(y,k)$ belongs to the symbol class $S^1_{1,0}$ for $i = 1,2, 3$. Note that for $\beta = 1$ the behavior of $b(y,k)$ for large y becomes even better.

Now again by the virtue of the theorem given in [Hör04, Section 7.8, Theorem 7.8.2] it follows that the oscillatory integral is well-defined in a distributional sense. ☐

Proposition 3.2. *Let the deformed Hamiltonian $H^{'}_{B,X}$ be defined by the deformed momentum operator $P^{j}_{B,X}$ as follows*

$$H^{'}_{B,X}\Psi := -\frac{1}{2m}P^{B,X}_{j}P^{j}_{B,X}\Psi, \qquad \Psi \in \mathscr{S}(\mathbb{R}^3). \tag{3.11}$$

Then, the operator $H^{'}_{B,X}$ is ***equal*** *to the deformed Hamiltonian $H_{B,X}$ given in (3.9), i.e. $H^{'}_{B,X} = H_{B,X}$.*

Proof. We begin by looking at the deformed momentum operator. As before, calculations are performed by using the action of the deformed operator on eigenfunctions of the coordinate operator.

$$\begin{aligned}
&\left(P^{j}_{B,X}\Psi\right)(\mathbf{q}) = \\
&= -(2\pi)^{-3}\lim_{\epsilon\to 0}\left(\int\int d^3k d^3y e^{-ik_r(y-q)^r}\chi(\epsilon k,\epsilon y)\,\alpha_{By}(P^j)\Psi\right)(\mathbf{q}) \\
&= -(2\pi)^{-3}\lim_{\epsilon_1\to 0}\left(\int d^3y \lim_{\epsilon_2\to 0}\left(\int d^3k e^{-ik_r(y-q)^r}\chi_2(\epsilon_2 k)\right)\chi_1(\epsilon_1 y)\,\alpha_{By}(P^j)\Psi\right)(\mathbf{q}) \\
&= -(2\pi)^{-3}\lim_{\epsilon_1\to 0}\left(\int d^3y\,(2\pi)^3\,\delta(\mathbf{y}-\mathbf{q})\,\chi(\epsilon_1 y)\,\alpha_{By}(P^j)\Psi\right)(\mathbf{q}) \\
&= (P^j + B^{jr}X_r)\Psi(\mathbf{q}).
\end{aligned}$$

Where again in the last lines we chose $\chi(\epsilon k,\epsilon y) = \chi_2(\epsilon_2 k)\chi_1(\epsilon_1 y)$ with $\chi_l \in \mathscr{S}(\mathbb{R}^3\times\mathbb{R}^3)$ and $\chi_l(0,0) = 1$, $l = 1, 2$, and obtained the delta distribution $\delta(\mathbf{y}-\mathbf{q})$ in the limit $\epsilon_2 \to 0$. Therefore, we obtain for the deformed momenta,

$$P^{j}_{B,X}\Psi = \left(P^j + B^{jr}X_r\right)\Psi. \tag{3.12}$$

□

The previous proposition settles the arbitrariness of deformation, because a deformed Hamiltonian could be defined as given in (3.11).

Another important result of this deformation is the fact that the deformed momentum operator **does not commute** along its components and is therefore an example of a noncommutative space. Therefore, we have the following result.

Lemma 3.3. *Let θ be a skew symmetric matrix of the Moyal-Weyl plane $\mathbb{R}^3_\theta$. Then, the deformed momentum operators $P^{B,X}_{j}$ given in (3.12), satisfy the commutator relations of a Moyal-Weyl plane with the skew symmetric matrix $\theta = -2B$,*

$$[P^{B,X}_{j}, P^{B,X}_{k}] = -2iB_{jk}. \tag{3.13}$$

Furthermore, the velocity operators of the system $\dot{X}^{B,X}_{j}$ satisfy the following commutation relations

$$[\dot{X}^{B,X}_{j}, \dot{X}^{B,X}_{k}] = -\frac{2i}{m^2}B_{jk}.$$

Proof. To calculate the commutator of the deformed momentum operator one makes use of the canonical commutation relations and the skew-symmetry of the deformation matrix B_{jk}, namely

$$[P_j^{B,X}, P_k^{B,X}] = [P_j + B_{jr}X^r, P_k + B_{kl}X^l] = B_{kl}[P_j, X^l] - B_{jl}[P_k, X^l]$$
$$= iB_{kj} - iB_{jk} = -2iB_{jk}.$$

The velocity operator of the system $\dot{X}_j^{B,X}$ is calculated by applying the Heisenberg equation,

$$i[H_{B,X}, X_j] = \dot{X}_j^{\ B,X} = \frac{1}{m}\Big(P_j + B_{jr}X^r\Big) = \frac{1}{m}P_j^{B,X}.$$

By using relation (3.13) the commutation relations of the velocity operators follow

$$[\dot{X}_j^{B,X}, \dot{X}_k^{B,X}] = -\frac{2i}{m^2}B_{jk}.$$

□

For solving the eigenvalue equation in the next section it is important to rewrite the Hamiltonian in terms of the velocity operator as follows,

$$H_{B,X} = -\frac{m}{2}\dot{X}_j^{B,X}\dot{X}^j_{B,X}.$$

3.3.1 Solving the eigenvalue problem

The eigenvalue equation of the deformed Hamiltonian $H_{B,X}$ is solved in order to gain some insight concerning the deformed energy properties of a free particle. Using standard quantum mechanical methods, we diagonalize the Hamiltonian and solve the eigenvalue equation. The solution of the eigenvalue equation is given in the following lemma.

Lemma 3.4. *Let the deformation matrix B_{ij} have the following form*

$$B_{ij} = -\frac{\kappa_B^2}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.14}$$

Then, the eigenvalues of the deformed Hamiltonian $H_{B,X}$ obtained in (3.9), are given by

$$E_{B,n} = \frac{p_1^2}{2m} + \left(n + \frac{1}{2}\right)\omega_B,$$

where $\omega_B := \kappa_B^2/m$ and $p_1 \in \mathbb{R}, n \in \mathbb{N}$.

Proof. In particular, it turns out that $H_{B,X}$ can be rewritten in terms of creation and annihilation operators similar to the Hamiltonian of an harmonic oscillator. The velocity operators $\dot{X}_2^{B,X}$ and $\dot{X}_3^{B,X}$ are **conjugate variables** satisfying the following commutation relation

$$[\dot{X}_2^{B,X}, \dot{X}_3^{B,X}] = i\frac{\kappa_B^2}{2}. \tag{3.15}$$

Since the velocity operators form a canonical pair we define deformed annihilation and creation operators a_B, a_B^* as follows

$$a_B := (\dot{X}_2^{B,X} + i\dot{X}_3^{B,X})\sqrt{\frac{m}{2\omega_B}}, \qquad a_B^* := (\dot{X}_2^{B,X} - i\dot{X}_3^{B,X})\sqrt{\frac{m}{2\omega_B}}.$$

By using Relation (3.15) one can easily see that the annihilation and creation operators satisfy

$$[a_B, a_B^*] = 1.$$

The deformed Hamiltonian is written in terms of the annihilation and creation operators and therefore takes the form of an harmonic oscillator, namely

$$H_{B,X} = \frac{P_1^2}{2m} + \left(a_B^* a_B + \frac{1}{2}\right)\omega_B.$$

Hence, the eigenvalue problem can therefore be solved by standard methods and we obtain quantized energy values

$$E_{B,n} = \frac{p_1^2}{2m} + \left(n + \frac{1}{2}\right)\omega_B, \qquad p_1 \in \mathbb{R}, n \in \mathbb{N},$$

for fixed p_1. □

This result implies that the deformation of the free Hamiltonian with the coordinate operator lead the continuous spectrum into a discrete spectrum. This already gives us a hint on the physical nature of the deformation. A well known example of quantum mechanical models where the transition of the energy from a continuous spectrum to a discrete spectrum occurs, involves the interaction of a magnetic field with a free particle. This will be explored in the next sections.

3.3.2 Langmann-Szabo Duality

An interesting duality appearing in noncommutative quantum field theory is the so called Langmann Szabo duality. This duality appears for noncommutative quantum field theories of charged bosons, described by Lagrangians and parameterized by some matrices that give the noncommutativity of spacetime and momentum space. The duality of these noncommutative quantum field theories preserves their form under Fourier transformation, [LS02a]. The same duality appears in the deformed model introduced above.

Lemma 3.5. *Let $\mathscr{F}(H_{B,X})$ be the Fourier transformation of the deformed Hamiltonian $H_{B,X}$. Then, the following duality holds*

$$H_{B,X}(\vec{x}) = \mathscr{F}(H_{B,X})(\vec{Bx}).$$

Proof. To see that this duality holds we apply the deformed Hamiltonian to an eigenfunction Ψ of the coordinate operator.

$$(H_{B,X}\Psi)(\vec{x}) = -\frac{1}{2m}\bigg((P_j + B_{jk}X^k)(P^j + B^{jr}X_r)\Psi\bigg)(\vec{x})$$

$$= -\frac{1}{2m}\left(i\frac{\partial}{\partial x^j} + B_{jk}x^k\right)\left(i\frac{\partial}{\partial x_j} + B^{jr}x_r\right)\Psi(\vec{x}).$$

In the next step we calculate the Fourier transformation of the deformed Hamilton operator to obtain the representation in momentum space.

$$\mathscr{F}(H_{B,X}\Psi)(\vec{p}) = -\frac{1}{2m}\left(p_j - iB_{jk}\frac{\partial}{\partial p_k}\right)\left(p^j - iB^{jl}\frac{\partial}{\partial p^l}\right)\Psi(\vec{p})$$

□

It is not just a fact of academic interest that this duality holds, but it also has a deeper physical meaning which is compatible with the result of the previous section. The duality appears in NCQFT due to a constant magnetic field B. We have a rather simpler model with the Langmann Szabo duality but nevertheless the physical interpretation of B can also be adopted for this simple model. Namely, the deformation matrix B plays the role of a **constant magnetic field**.

3.4 Physical models from deformation

One of the most important aspects of the interplay between mathematics and physics lies in the **physical dimensionality** of the physical constants. The main question that motivated this work is the following one: What is the physical meaning of the deformation parameter? In the simple model above there is an interesting answer.

3.4.1 Landau quantization

An example of a dynamical system interacting with a magnetic field in quantum mechanics, is given by the Landau quantization. It is also an important example of the appearance of quantum space in a physical context. The Landau quantization describes the dynamics of a system of nonrelativistic electrons **confined to a plane**, lets say in the $y-z$ plane ($\vec{A} = (0, y, z)$), in the presence of a homogeneous magnetic field in x-direction $\vec{B} = B(1, 0, 0)$. In the **symmetric gauge** the Hamiltonian of the Landau quantization is given by, [Eza, Equation 9.2.1],

$$H_L = -\frac{1}{2m}\left(P_i - eA_i\right)\left(P^i - eA^i\right),$$

with gauge field $A_i = -\epsilon_{ijk}B^k X^j$. The deformed Hamiltonian $H_{B,X}$ we introduced above describes almost the Landau model with the main difference that in our deformed model the deformation parameter κ_B^2 is **still variable**. If one sets this parameter equal to zero one obtains the Hamiltonian of a free particle. By setting κ_B^2 to be equal to a constant with physical dimension we obtain the following lemma.

Lemma 3.6. *Let the deformation parameter κ_B^2 of the matrix B_{ij} (3.14) be equal to eB, where B characterizes the constant of a homogeneous magnetic field and e the elementary charge. Then, the deformed free Hamiltonian $H_{B,X}$ becomes the Hamiltonian H_L of the Landau quantization.*

Proof. For the proof we consider the free deformed Hamiltonian $H_{B,X}$ which is given in Proposition 3.1.

$$H_{B,X} = -\frac{1}{2m}(P_j + B_{jk}X^k)(P^j + B^{jr}X_r).$$

In the next step we set the deformation parameter $\kappa_B^2 = eB$. Then B_{ij} is given as $B_{ij} = -e\epsilon_{ijk}B^k$, where B^k is the homogeneous magnetic field in the x-direction ($B^k = B(1,0,0)$). $\square$

Now this is truly an astonishing result. We started with the free Hamiltonian and deformed it with warped convolutions using the coordinate operator. By simply taking the deformation parameter to be equal to the physical constant eB we obtain the Landau problem. Therefore, deformation with the coordinate operator is physically of great importance. Note that our model is formulated in a general manner, and just for the specific choice of the deformation parameter $\kappa_B^2 = eB$ we obtained the Landau effect.

3.4.2 Zeeman effect from Deformation

The Hamiltonian of the hydrogen atom is given as follows, [Thi79, Equation 4.1.1]

$$H^A = -\frac{P_jP^j}{2m} + \frac{e^2}{(-X_kX^k)^{1/2}}. \tag{3.16}$$

By solving the stationary Schrödinger equation $H^A\psi = E\psi$ one obtains the energy spectrum of the hydrogen atom, the so called **Balmer series**, [Str08]. In the presence of a constant magnetic field, an interesting physical effect influences the spectral lines of the hydrogen atom. The spectral lines split into further spectral lines depending on the presence of the homogeneous magnetic field B_k. This phenomenon is called the **Zeeman effect** and is governed by the following Hamiltonian, [Thi79, Equation 4.2.1]

$$H^{AZ} = -\frac{1}{2m}(P_j - e\epsilon_{ijk}B^kX^k)(P^j - e\epsilon_{ijk}B^kX_r) + \frac{e^2}{(-X_kX^k)^{1/2}}. \tag{3.17}$$

In the last section we discovered that deformation with the coordinate operator induces a gauge field. Taking this as a lesson we perform a deformation on the Hamiltonian of the hydrogen atom to obtain the Hamiltonian of the Zeeman effect. As for the previous Hamiltonians we are obliged to show that for the (from above) unbounded operator H^A formula (3.3) can be defined as an oscillatory integral in a distributional sense. We first note that the Hamiltonian of the hydrogen atom is $H^A = H_0 + e^2/(-X_kX^k)^{1/2}$. Since we already proved in Lemma 3.1 that H_0 can be deformed by using the warped convolutions formula (3.3), it suffices to show it holds for the second term. So again as before, we take the scalar product involving $H^D := e^2/(-X_kX^k)^{1/2}$,

$$\begin{aligned}\langle\Psi, H^D_{B,X}\Phi\rangle &= (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_lk^l}\,\chi(\epsilon y,\epsilon k)\langle\Psi, V(k)\alpha_{By}(H^D)\Phi\rangle\\ &=: (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_lk^l}\,\chi(\epsilon y,\epsilon k)\, b^D(k,y)\end{aligned}$$

for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$.

Lemma 3.7. *Let the function $b^D(y,k)$, be given as the scalar product $\langle\Psi, V(k)\alpha_{By}(H^D)\Phi\rangle$. Then $b^D(y,k)\in S^0_{1,0}$, i.e. just a constant function for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$ and thus the deformation, via warped convolution, of H^A is given as a well-defined oscillatory integral.*

Proof.

$$\begin{aligned}|\partial^\alpha_{k^i}\partial^\beta_{y^r} b^D(y,k)| &= |\langle\Psi, \left(\partial^\alpha_{k^i}V(k)\right)\partial^\beta_{y^r}\left(\alpha_{By}(H^D)\right)\Phi\rangle| \\ &\leq \underbrace{\|(-iX^\alpha)\Psi\|}_{=:C_{1,\alpha}}\underbrace{\left\|\left(H^D\right)\Phi\right\|}_{=:C_D} \\ &\leq C_{1,\alpha}C_D,\end{aligned}$$

where in the last lines we used the commutation relation , $[X_i,X_j]=0$ to show that H^D is invariant under the adjoint action of $V(By)$. Moreover, since $C_{1,\alpha}$ and C_D are finite constants, $b^D(y,k)$ belongs to the symbol class $S^0_{1,0}$. □

In the next step we choose the deformation parameter to be the same as in the last section and therefore the deformation of H^A yields the following lemma.

Lemma 3.8. *Let the deformation parameter κ_B^2 of the matrix B_{ij} (3.14) be equal to eB, where B characterizes the constant of a homogeneous magnetic field and e the elementary charge. Then the deformed Hamiltonian of H^Z (3.16), denoted by $H^Z_{B,X}$, becomes the Hamiltonian of the Zeeman effect H^{AZ} (3.17).*

Proof. Due to the fact that the coordinate operator commutes with itself the only part of the Hamiltonian H^Z which is affected is the free part and therefore we obtain

$$\begin{aligned}H^Z_{B,X}\Psi &= \left(-\frac{1}{2m}(P_j+B_{jk}X^k)(P^j+B^{jr}X_r)+\frac{e^2}{(-X_kX^k)^{1/2}}\right)\Psi \\ &= \left(-\frac{1}{2m}P^{B,X}_j P^j_{B,X}+\frac{e^2}{(-X_kX^k)^{1/2}}\right)\Psi.\end{aligned}$$

By setting the deformation parameter $\kappa_B^2=eB$, the skew-symmetric matrix B_{ij} takes the same form as in the last section, $B_{ij}=-e\epsilon_{ijk}B^k$, and we obtain the Hamiltonian of the Zeeman effect for a homogeneous magnetic field in the x-direction. □

As in the case of the Landau quantization the deformation parameter plays the role of the magnetic field which leads to this important physical effect. Let us summarize the result. We deformed the Hamiltonian of the hydrogen atom using the coordinate operator. By setting the deformation parameter equal to the constant of a magnetic field we obtain the Zemann effect.

3.4.3 The Aharonov-Bohm effect

In the last sections we recognized the consequence of deformation with the coordinate operator. Warped convolutions with the coordinate operator induces a **gauge field**. Now since we work in a quantum mechanical setting we want to reproduce other physical effects where magnetic fields play a significant role. One of the most striking ones is the

Aharanov-Bohm (AB) effect. It takes place in a system in which the gauge field A_k influences the dynamics of a charged particle even in regions where the magnetic field B_k vanishes, [Ber96, Eza]. To be more precise, it tells one how to quantize if the configuration space is not simply connected. The gauge field $A_k(\mathbf{x})$ of the magnetic version of the AB effect for a homogeneous magnetic field in x-direction takes the following form

$$A_k = \frac{\phi_M}{2\pi(X_2^2 + X_3^2)} \epsilon_{ijk} e^k X^j,$$

where ϕ_M is the **magnetic flux** and e^k is the unit vector in x-direction. Moreover, from quantum mechanical considerations it follows that the **interference pattern** is the same for two values of fluxes ϕ_1 and ϕ_2 if only if

$$e(\phi_1 - \phi_2) = 2\pi n, \qquad n \in \mathbb{Z}. \tag{3.18}$$

In this section we take the free Hamiltonian of quantum mechanics and deform it with a vector-valued function of the coordinate operator and set the deformation parameter equal to a physical constant, namely that of a magnetic flux. The gauge field induced by this specific deformation is equal to the gauge field of the Aharonov-Bohm effect.

As before, prior to the deformation we prove that the formula of warped convolutions (3.3) is well-defined in the case of deformation with the operator $F_j(\mathbf{X}) := X_j/(-\sum_{s=2}^3 X_s X^s)^{1/2}$, of the unbounded operator H_0. Therefore we consider the deformed free Hamiltonian $H_{\phi_M,F(\mathbf{X})}$ in the scalar product, where the unitary operator is now defined by $V_F(y) := e^{iy^j F_j(\mathbf{X})}$. So we have

$$\begin{aligned}
\langle \Psi, H_{\phi_M,F(\mathbf{X})}\Phi\rangle &= (2\pi)^{-3} \lim_{\epsilon\to 0} \iint d^3y\, d^3k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k) \langle \Psi, V_F(k)\alpha_{By}(H_0)\Phi\rangle \\
&=: (2\pi)^{-3} \lim_{\epsilon\to 0} \iint d^3y\, d^3k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k)\, b^F(k,y)
\end{aligned}$$

for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$. As one can see $\langle \Psi, H_{\phi_M,F(\mathbf{X})}\Phi\rangle$ is given in the form of an oscillatory integral and thus to prove the expression is well-defined, we show in the next lemma that $b^F(k,y)$ belongs to a symbol class.

Lemma 3.9. *Let the function $b^F(y,k)$ be given as the scalar product $\langle \Psi, V_F(k)\alpha_{By}(H_0)\Phi\rangle$. Then, $b(y,k) \in S^2_{1,0}$, for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$ and thus the deformation, via warped convolution, of the free Hamiltonian, using $V_F(k)$, is given as a well-defined oscillatory integral.*

Proof. We first calculate the adjoint action of $V_F(By)$ on H_0 given by,

$$\begin{aligned}
\alpha_{By}(H_0) &= -\frac{1}{2m} V_F(By) P_j P^j V_F(-By) \\
&= -\frac{1}{2m} V_F(By) P_j V_F(-By) V_F(By) P^j V_F(-By).
\end{aligned}$$

To solve this expression one first has to calculate the adjoint action of $V(By)$ on the momentum operator P_j. This is done by using the Baker-Campbell-Hausdorff formula. So we have

$$
\begin{aligned}
V_F(By)P_jV_F(By) &= P_j - i(By)_k\left(i\eta_j^k/(-X_sX^s)^{1/2} + iX^kX_j/(-X_sX^s)^{3/2}\right)+\\
&\quad\underbrace{+\frac{i^2}{2}(By)_l(By)_k[X^l/(-X_sX^s)^{1/2},[X^k/(-X_rX^r)^{1/2},P_j]]+\dots}_{=0}\\
&= P_j + (By)_k\underbrace{\left(\eta_j^k/(-X_sX^s)^{1/2} + X^kX_j/(-X_sX^s)^{3/2}\right)}_{=:X_j^k}, \qquad (3.19)
\end{aligned}
$$

here in the last lines we used the CCR and the fact that the coordinate operator satisfies $[X_i, X_j] = 0$. Thus, the adjoint action w.r.t. $V_F(By)$ on H_0 is

$$
\begin{aligned}
\alpha_{By}(H_0) &= -\frac{1}{2m}(P_j + (By)_sX_j^s)(P^j + (By)^rX_r^j)\\
&= H_0 - (By)^r\underbrace{\frac{1}{2m}\left(P_jX_r^j + X_r^jP_j\right)}_{=:R_r} - (By)^r(By)_s\underbrace{\frac{1}{2m}X_j^sX_r^j}_{=:R_r^s}\\
&= H_0 - (By)^rR_r - (By)^r(By)_sR_r^s. \qquad (3.20)
\end{aligned}
$$

Remark 3.3. The term R_r^s is important in the following considerations and one should note that it has a structure which allows us to use the Cauchy-Schwarz inequality, namely

$$
\begin{aligned}
R_r^s &= \frac{1}{2m}\left(\eta_t^s/(-X_lX^l)^{1/2} + X^sX_t/(-X_rX^r)^{3/2}\right)\left(\eta_r^t/(-X_kX^k)^{1/2} + X^tX_r/(-X_zX^z)^{3/2}\right)\\
&= \left(\eta_r^s/(-X_lX^l) + X^sX_r/(-X_lX^l)^2\right). \qquad (3.21)
\end{aligned}
$$

Furthermore, the interchanging of the order of integration and differentiation is allowed due to the same arguments given in Remark (3.2).

In the next step we look at the expression

$$
\begin{aligned}
|\partial_{k^i}^{\alpha}\partial_{y^r}^{\beta}b^F(y,k)| &= |\langle\Psi, \left(\partial_{k^i}^{\alpha}V_F(k)\right)\partial_{y^r}^{\beta}\left(\alpha_{By}(H_0)\right)\Phi\rangle|\\
&\le \underbrace{\|(-iF(X)^{\alpha})\Psi\|}_{=:C_{1,\alpha}}\left\|\partial_{y^r}^{\beta}\left(H_0 - (By)^rR_r - (By)^r(By)_sR_r^s\right)\Phi\right\|\\
&\le C_{1,\alpha}\left(\left\|\partial_{y^r}^{\beta}H_0\Phi\right\| + |\partial_{y^r}^{\beta}(By)^j|\,\|R_l\Phi\| + |\left\|\partial_{y^r}^{\beta}(By)^j(By)^sR_t^l\Phi\right\|\right).
\end{aligned}
$$

Thus, for $\underline{\beta = 0}$, $B_{ij} = \epsilon_{ijk}B^k$ and by the form of R_r^s given in Equation (3.21) we have the following inequality

$$
\begin{aligned}
|\partial_{k^i}^{\alpha}b^F(y,k)| &\le C_{1,\alpha}\left(\|H_0\Phi\| + |(By)^j|\,\|R_l\Phi\| + 2|(By)^j|^2\left\|(-X_lX^l)^{-1}\Phi\right\|\right)\\
&\le C_{1,\alpha}\left(\underbrace{\|H_0\Phi\|}_{=:C_2} + 2|\mathbf{y}|\underbrace{\frac{|\mathbf{B}|}{\sqrt{2}}\|R_l\Phi\|}_{=:C_3} + |\mathbf{y}|^2\underbrace{\frac{4|\mathbf{B}|^2}{m}\left\|(-X_lX^l)^{-1}\Phi\right\|}_{=:C_4}\right)\\
&\le \underbrace{C_{1,\alpha}C_B}_{=:C_{1,\alpha,B}}(1 + |\mathbf{y}|)^2, \qquad (3.22)
\end{aligned}
$$

As in the proof of Lemma 3.1 a constant C_B obeying inequality (3.22) exists, because C_1, C_2 and C_3 are finite constants. Therefore, $b^F(y,k)$ belongs to the symbol class $S^2_{1,0}$ and it follows that the oscillatory integral is defined in a distributional sense. □

Let us now return to the result for deforming the free Hamiltonian with unitary operators $V_F(k)$.

Proposition 3.3. *Let the* ***deformation parameter*** κ_B^2 *of the matrix* B_{ij} *(3.14) be equal to* $-e\phi_M/2\pi$*, where* ϕ_M *characterizes the magnetic flux. Moreover, let* $H_{\phi_M,F(\mathbf{X})}$ *denote the free Hamiltonian (3.5) deformed with unitary representations* $V_F(y)$ *of* $\mathbb{R}^3$ *that are given as follows*

$$V_F(y) = e^{iy^j F_j(\mathbf{X})}, \qquad F_j(\mathbf{X}) := X_j/(-\sum_{s=2}^{3} X_s X^s)^{1/2}.$$

Then, the induced gauge field obtained from $H_{\phi_M,F(\boldsymbol{X})}$ *is equal to the gauge field of the magnetic version of the Aharonov-Bohm effect for a magnetic field in x-direction.*

Furthermore, if the deformation parameters of the Hamiltonians $H_{\phi_1,F(\mathbf{X})}$ *and* $H_{\phi_2,F(\mathbf{X})}$ *fulfill Equation (3.18), the physical systems described by the Hamiltonians have the same interference pattern.*

Proof. To prove this proposition we use the spectral measure representation of the deformation given in Definition 2.7. As shown in [BLS11] the deformation with the spectral measure is equivalent to the one given as an integral. Due to simplicity and readability reasons we use the integral representation for the current case. The deformation of H_0 is then given as follows,

$$\begin{aligned}
H_{\phi_M,F(\mathbf{X})}\Psi &= \int dE(y)\, \alpha_{By}\left(H_0\right)\Psi \\
&= \int dE(y) \left(-\frac{1}{2m}(P_j + (By)_s X^s_j)(P^j + (By)^r X^j_r)\right) \\
&= -\frac{1}{2m}(P_j + (BF(\mathbf{X}))_s X^s_j)(P^j + (BF(\mathbf{X}))^r X^j_r) \\
&= -\frac{1}{2m}\left(P_j + (BX)_j/(-\sum_{s=2}^{3} X_s X^s)\right)^2, \qquad (3.23)
\end{aligned}$$

where in the last lines we used Equation (3.20) and the skew-symmetry of B. Therefore, the deformed Hamiltonian $H_{\phi_M,F(\mathbf{X})}$ takes the following form

$$\begin{aligned}
H_{\phi_M,F(\mathbf{X})} &= -\frac{1}{2m}\left(P_i + B_{ik}X^k/(-X^s X_s)\right)^2 \\
&\quad -\frac{1}{2m}\left(P_i - e\frac{\phi_M}{2\pi(X_2^2 + X_3^2)}\epsilon_{ijk}e^k X^j\right)^2 \\
&= -\frac{1}{2m}\left(P_i - eA_i\right)^2.
\end{aligned}$$

The gauge field $A_i(\mathbf{x})$ which is induced by deformation is given as follows

$$A_i(\mathbf{x}) = \frac{\phi_M}{2\pi(X_2^2 + X_3^2)} \epsilon_{ijk} e^k X^j,$$

which is exactly the gauge field of the AB effect for a homogeneous magnetic field in x-direction. $\square$

This is an important result. We were able to induce the AB-gauge field by deforming the free Hamiltonian with a function of the essentially selfadjoint coordinate operator X_j. In this case the deformation parameter corresponds to the magnetic flux rather as in the previous cases to the magnetic field itself. This shows the extendability of this rather simple model. In particular one can induce an arbitrary (abelian) gauge field by deformation.

For our next results we deform the momentum operator with $F_j(\mathbf{X})$. Due to the unboundedness of P^j, we are obliged to show that deformation formula (3.3) is well-defined in this case. Analogously for the proof of Lemma 3.2, we take the scalar product of $P^j_{B,F(\mathbf{X})}$,

$$\begin{aligned}\langle \Psi, P^j_{B,F(\mathbf{X})} \Phi\rangle &= (2\pi)^{-3} \lim_{\epsilon\to 0} \iint d^3y\, d^3k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k) \langle \Psi, V_F(k)\alpha_{By}(P^j)\Phi\rangle \\ &=: (2\pi)^{-3} \lim_{\epsilon\to 0} \iint d^3y\, d^3k\, e^{-iy_l k^l}\, \chi(\epsilon y, \epsilon k)\, b^j(k,y)\end{aligned}$$

for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$.

Lemma 3.10. *Let the function $b^j(y,k)$, for $j = 1, 2, 3$, be given as the scalar product $\langle \Psi, V_F(k)\alpha_{By}(P^j)\Phi\rangle$. Then $b^j(y,k) \in S^1_{1,0}$, for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^3)$ and thus the deformation, via warped convolutions, of the momentum operator P^j is given as a well-defined oscillatory integral.*

Proof. We start by looking at the following expression,

$$\begin{aligned}|\partial^\alpha_{k^i}\partial^\beta_{y^r} b^j(y,k)| &= |\langle \Psi, \left(\partial^\alpha_{k^i} V_F(k)\right) \partial^\beta_{y^r} \left(\alpha_{By}(P^j)\right) \Phi\rangle| \\ &\le \underbrace{\|(-iF(X)^\alpha)\Psi\|}_{=:C_{1,\alpha}} \left\|\partial^\beta_{y^r}\left(P^j + (By)^r X^j_r\right)\Phi\right\| \\ &\le C_{1,\alpha}\left(\left\|\partial^\beta_{y^r} P^j \Phi\right\| + |\partial^\beta_{y^r}(By)^j| \left\|X^l_r \Phi\right\|\right),\end{aligned}$$

where in the last lines we used the adjoint action given in Equation (3.19). By using inequality (3.7) for $B_{ij} = \epsilon_{ijk}B^k$ and for $\underline{\beta = 0}$ we obtain

$$
\begin{aligned}
|\partial^\alpha_{k^i} b^j(y,k)| &\leq C_{1,\alpha}\left(\underbrace{\left\|P^j\Phi\right\|}_{=:C_5} + |\mathbf{y}| \underbrace{\sqrt{2}|\mathbf{B}| \left\|X^l_r\Phi\right\|}_{=:C_6}\right) \\
&\leq C_{1,\alpha}\left(C_5 + C_6|\mathbf{y}|\right) \\
&\leq \underbrace{C_{1,\alpha}C_D}_{=:C_{1,\alpha,B}}\left(1 + |\mathbf{y}|\right),
\end{aligned}
$$

A a constant C_D satisfying

$$C_D \geq C_5, C_6,$$

can be found, since C_5 and C_6 are finite constants. Therefore, $b^j(y,k)$ belongs to the symbol class $S^1_{1,0}$, for $i = 1,2,3$. As before by the virtue of the theorem given in [Hör04, Section 7.8, Theorem 7.8.2] it follows that the oscillatory integral is well-defined in a distributional sense. □

Analogously to Proposition 3.2 the following proposition holds.

Proposition 3.4. *Let the deformed Hamiltonian $H^{'}_{\phi_M,F(\boldsymbol{X})}$ be defined by the deformed momentum operator $P^j_{\phi_M,F(\boldsymbol{X})}$ as follows*

$$H^{'}_{\phi_M,F(\boldsymbol{X})}\Psi := -\frac{1}{2m}P^{\phi_M,F(\boldsymbol{X})}_j P^j_{\phi_M,F(\boldsymbol{X})}\Psi, \qquad \Psi \in \mathscr{S}(\mathbb{R}^3).$$

*Then, the operator $H^{'}_{\phi_M,F(\boldsymbol{X})}$ is **equal** to the deformed Hamiltonian $H_{\phi_M,F(\boldsymbol{X})}$ given in (3.23), i.e. $H^{'}_{B,X} = H_{B,X}$.*

Proof. As in the proof of Proposition 3.2 we calculate the deformation of the momentum operator using unitary representations of $\mathbb{R}^3$, defined by $X_j/(-\sum_{s=2}^3 X_sX^s)^{1/2}$.

$$
\begin{aligned}
P^j_{\phi_M,F(\mathbf{X})}\Psi &= \int dE(y)\,\alpha_{By}\left(P^j\right)\Psi \\
&= \int dE(y)\,\left(P^j + (By)^r X^j_r\right) \\
&= (P^j + (BF(\mathbf{X}))^r X^j_r) \\
&= \left(P^j + (BX)^j/(-\sum_{s=2}^{3} X_sX^s)\right),
\end{aligned}
$$

where we used the skew-symmetry of B and Equation (3.19). Thus the deformed momentum operator is the following

$$P^j_{\phi_M,F(\mathbf{X})}\Psi = P^j + B^{jk}X_k/(-X^sX_s)\Psi.$$

□

The last proposition settles, as in the last sections, the arbitrariness of defining the deformed Hamiltonian.

There are two ways to interpret these results. The first one lies in understanding deformation, in the case of QM, as the rightful minimal substitution. Thus the procedure sheds new light on quantum mechanical effects involving magnetic fields. The fields have to be understood as consequences of deformation with the coordinate operator.

The other way of understanding the result is the following. The coupling of an external magnetic field in QM is well understood and studied for various physical applications and models. Deformation on the other hand is a mathematical tool, rather than a procedure that generates physical effects. Hence, in these examples deformation of a QM system can be understood as the coupling of an external field. Thus, if the deformation goes hand in hand with Moyal-type spacetimes one sees in these examples that Moyal spacetimes correspond to ordinary spacetimes in the presence of an external field. By having this observation in mind it does not seem far fetched that certain deformations of spacetime correspond to gravitation.

3.4.4 Physical Moyal-Weyl plane from deformation

In this section we show that the **Moyal-Weyl plane** occurs in a limit of the deformed model, given in Proposition 3.1. To see the appearance of a noncommutative plane one has to rewrite the deformed free Hamiltonian in to a **Lagrangian**, which is done by a Legendre transformation.

$$\begin{aligned} L_{B,X} &= -P^i \dot{X}_i^{B,X} - H_{B,X} \\ &= -(m\dot{X}^i_{B,X} - B^{ik}X_k)\dot{X}_i^{B,X} + \frac{m}{2}\dot{X}_j^{B,X}\dot{X}^j_{B,X} \\ &= -\frac{m}{2}\dot{X}_j^{B,X}\dot{X}^j_{B,X} + B^{ik}X_k\dot{X}_i^{B,X} \end{aligned} \tag{3.24}$$

In the next step one imposes the quantization condition upon the conjugate momenta and the coordinate operators, and obtains in a particular limit the Moyal-Weyl plane.

Lemma 3.11. *By imposing the quantization condition between the conjugate momenta Π_i, obtained from the deformed Lagrangian $L_{B,X}$ (3.24), and a coordinate operator X_j one obtains for* ***large values*** *of the deformation parameter $\kappa_B^2 >> m^2$, the Moyal-Weyl plane $\mathbb{R}^3_\theta$*

$$[X_i, X_j] = i\theta_{ij}. \tag{3.25}$$

Proof. For large values of the deformation parameter where one can neglect the mass term, $\kappa_B^2 >> m^2$, the conjugate momenta Π_i, obtained from differentiating the Lagrangian w.r.t $\dot{X}_i^{B,X}$, is given as

$$\Pi_i := -\frac{\partial L_{B,X}(\mathbf{X}, \dot{\mathbf{X}}^{B,X})}{\partial \dot{X}_i^{B,X}} = -B^{ik}X_k.$$

By imposing the quantization condition that the conjugate momenta and the coordinates form a canonical conjugate pair, $[\Pi_i, X_j] = i\eta_{ij}$ one obtains, [Sza04]

$$[X_i, X_j] = i(B^{-1})_{ij}.$$

Now by defining $(B^{-1})_{ij}$ to be θ_{ij} one obtains the well-known three dimensional Moyal-Weyl plane $\mathbb{R}^3_\theta$. □

Remark 3.4. We use in this work the inverse of a skew-symmetric 3×3 matrix. This is in general singular. Thus the inverse of B_{ij} (see Equation 3.14), is given as

$$B^{-1}_{ij} = \frac{\kappa_B^{-2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The three dimensional Moyal-Weyl plane $\mathbb{R}^3_{B^{-1}}$ can also be obtained by defining the guiding center coordinates Q_i as in [Eza, Equation 9.2.4] by

$$Q_i := X_i + (B^{-1})_{ik}P^k. \tag{3.26}$$

This is usually done to describe the circular motion of an electron in the lowest Landau level, [Eza, Sza04].

For our next result we show by deforming the coordinate operator with the momentum operator using Definition 3.4, that the deformed coordinate operator corresponds to the guiding center coordinates.

As before, we first prove that the deformation formula (3.4) is well defined in this case. Thus, let us consider the deformed coordinate operator as follows

$$\begin{aligned}\langle\Psi, X^j_{\theta,P}\Phi\rangle &= (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_lk^l}\,\chi(\epsilon y,\epsilon k)\langle\Psi, U(k)\alpha_{\theta y}(X^j)\Phi\rangle\\ &=: (2\pi)^{-3}\lim_{\epsilon\to 0}\iint d^3y\, d^3k\, e^{-iy_lk^l}\,\chi(\epsilon y,\epsilon k)\, b^j(k,y)\end{aligned}$$

for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$.

Lemma 3.12. *Let the function $b^j(y,k)$, for $j = 1, 2, 3$, be given as the scalar product $\langle\Psi, U(k)\alpha_{\theta y}(X^j)\Phi\rangle$. Then, $b^j(y,k)\in S^1_{1,0}$, for $\Psi,\Phi\in\mathscr{S}(\mathbb{R}^3)$ and thus the deformation via warped convolutions of the coordinate operator X^j is given as a well-defined oscillatory integral.*

Proof. For the proof we first calculate the adjoint action of $U(y)$ on the coordinate operator.

$$\begin{aligned}\alpha_{\theta y}(X^j) &= U(\theta y)X^jU(-\theta y)\\ &= X^j - (\theta y)^j,\end{aligned}\tag{3.27}$$

where in the last line we used the canonical commutation relations and the Baker-Campbell-Hausdorff formula. Now let us look at the following

$$\begin{aligned}|\partial^\alpha_{k^i}\partial^\beta_{y^r}b^j(y,k)| &= |\langle\Psi, \left(\partial^\alpha_{k^i}U(k)\right)\partial^\beta_{y^r}\left(\alpha_{\theta y}(X^j)\right)\Phi\rangle|\\ &\le \underbrace{\|(-iP^\alpha)\Psi\|}_{=:C_{1,\alpha}}\left\|\partial^\beta_{y^r}\left(X^j-(\theta y)^j\right)\Phi\right\|\\ &\le C_{1,\alpha}\left(\left\|\partial^\beta_{y^r}X^j\Phi\right\| + |\partial^\beta_{y^r}(\theta y)^j|\,\|\Phi\|\right),\end{aligned}$$

where in the last lines we used Equation (3.27). By using Inequality (3.7), $\theta_{ij}=\epsilon_{ijk}\theta^k$ and $\underline{\beta = 0}$ we obtain

$$\begin{aligned}|\partial^\alpha_{k^i}b^j(y,k)| &\le C_{1,\alpha}\left(\underbrace{\left\|X^j\Phi\right\|}_{=:C_5} + |\mathbf{y}|\underbrace{\sqrt{2}|\boldsymbol{\theta}|\,\|\Phi\|}_{=:C_6}\right)\\ &\le C_{1,\alpha}\left(C_5 + C_6|\mathbf{y}|\right)\\ &\le \underbrace{C_{1,\alpha}C_D}_{=:C_{1,\alpha,\theta}}\left(1+|\mathbf{y}|\right),\end{aligned}$$

a constant C_D satisfying

$$C_D \ge C_5, C_6.$$

exists, because C_5 and C_6 are finite constants and therefore $b^j(y,k)$ belongs to the symbol class $S^1_{1,0}$ for i = 1,2,3. As before, by the theorem given in [Hör04, Section 7.8, Theorem 7.8.2] it follows that the oscillatory integral is well-defined in a distributional sense. □

Let us now turn to the explicit result of deformation given by the following lemma.

Lemma 3.13. *The deformation of the coordinate operator via warped convolutions using the momentum operator P^i yields*

$$X^j_{\theta,P} = X^j - \theta^{jr} P_r. \tag{3.28}$$

The operators $X^i_{\theta,P}$ satisfy the commutation relations of the Moyal-Weyl plane $\mathbb{R}^3_{-2\theta}$,

$$[X^i_{\theta,P}, X^j_{\theta,P}] = -2i\theta^{ij}. \tag{3.29}$$

Moreover, let $-\theta^{ij}$ be $(B^{-1})^{ij}$ then the deformed coordinate operators $X^i_{\theta,P}$ are equal to the guiding center coordinates given in Equation (3.26).

Proof. We start by calculating the action of the deformed operator on eigenfunctions of the momentum operator $\Psi(\mathbf{p}) \in \mathscr{S}(\mathbb{R}^3)$ and obtain

$$\begin{aligned}
&\left(X^j_{\theta,P}\Psi\right)(\mathbf{p}) = \\
&= -(2\pi)^{-3} \lim_{\epsilon\to 0} \left(\int\int d^3k d^3y e^{-ik_r(y-p)^r} \chi(\epsilon k, \epsilon y)\, \alpha_{\theta y}(X^j)\Psi\right)(\mathbf{p}) \\
&= -(2\pi)^{-3} \lim_{\epsilon_1\to 0} \left(\int d^3y \lim_{\epsilon_2\to 0}\left(\int d^3k e^{-ik_r(y-p)^r} \chi_2(\epsilon_2 k)\right) \chi_1(\epsilon_1 y)\, \alpha_{\theta y}(X^j)\Psi\right)(\mathbf{p}) \\
&= -(2\pi)^{-3} \lim_{\epsilon_1\to 0} \left(\int d^3y\, (2\pi)^3\, \delta(\mathbf{y}-\mathbf{p})\, \chi(\epsilon_1 y)\, \alpha_{\theta y}(X^j)\Psi\right)(\mathbf{p}) \\
&= (X^j - \theta^{jr}P_r)\Psi(\mathbf{p}),
\end{aligned}$$

here again in the last lines we chose $\chi(\epsilon k, \epsilon y) = \chi_2(\epsilon_2 k)\chi_1(\epsilon_1 y)$ with $\chi_l \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\chi_l(0,0) = 1$, $l = 1, 2$. Moreover, we obtained the delta distribution $\delta(\mathbf{y}-\mathbf{p})$ in the limit $\epsilon_2 \to 0$. The commutator of the deformed coordinate operator is calculated by using the canonical commutation relations (3.1) and the skew-symmetry of the deformation matrix θ_{jk}.

$$\begin{aligned}
[X^j_{\theta,P}, X^k_{\theta,P}] &= [X^j - \theta^{jr}P_r, X^k - \theta^{kl}P_l] \\
&= -\theta^{kl}[X^j, P_l] + \theta^{jl}[X^k, P_l] \\
&= -2i\theta^{jk}.
\end{aligned}$$

Although implied, it is important to emphasize that the deformation parameter used to define θ is equal to the inverse of κ_B^2. □

Lemma 3.13 gives a well-defined path to obtain an effective quantum plane by the deformation procedure of warped convolutions. As shown, the lemma results from well understood physical models and ideas, which are in circulation in condensed matter field theory for quite some time. In the example of the Landau problem one defines guiding center coordinates which satisfy the commutator relations of the Moyal-Weyl Plane. The reader is cautioned to notice that the effective quantum plane obtained by the Landau problem is not merely an abstract construct, but has the precise meaning that the coordinates cannot be measured simultaneously. A more precise mathematical way to obtain this Moyal-Weyl plane is introduced in this work. We receive the Landau problem by deforming the Hamiltonian of a free nonrelativistic particle. The deformation is performed using the coordinate operator for the deformation and by setting the deformation parameter equal to a certain physical value that corresponds to a magnetic field. Furthermore, we show that the non commuting coordinates referred to as the guiding coordinates are obtained by deforming the coordinate operator, using the momentum operator. In our opinion, this method can be further used in the quantum field theoretical approach to define an effective quantum plane.

3.5 Groups from Warped Convolution

This section is devoted to the mathematical explanation why the Landau problem appears from deformation with the coordinate operator.

3.5.1 Heisenberg-Weyl Group

The free Hamiltonian H_0 is defined by using the momentum operator P_k. Furthermore, the deformation of H_0 is performed by using the coordinate operator X_j. Therefore, the two operators involved in the deformation procedure form the canonical conjugate pair (P_i, X_k). The canonical conjugate pair generate continuous unitary groups which satisfy the Weyl relations. These operators define a representation of the Heisenberg-Weyl group. In the following we define the Heisenberg-Weyl group by using the groups $V(a)$ and $U(b)$, that satisfy the Weyl relations, [BEH08, Example 10.2.2].

Definition 3.9. Let $V(a)$ and $U(b)$ be strongly continuous unitary groups satisfying the Weyl relations (3.2). These operators give rise to a **true** representation of the **Heisenberg-Weyl group** defined as the $(2n+1)$-parameter set $G = \{g(s,t,u) : s, t \in \mathbb{R}^n, u \in \mathbb{R}\}$ with the binary operation

$$g(s,t,u)g(s',t',u') := g\left(s+s', t+t', u+u' + \frac{1}{2}(ts' - st')\right).$$

It is a $(2n+1) - \textit{dimensional}$ noncommutative Lie group. The corresponding Lie algebra is spanned by elements P_i, X_k, N, where the pair (P_i, X_k) satisfies the canonical commutation relations and N commutes with all other generators. Therefore, the Schrödinger representation of the CCR yields a representation of the generators of the Heisenberg-Weyl algebra.

3.5.2 Magnetic Translation Group

The magnetic translation group was first defined in [Zak64a], [Zak64b]. In the presence of a homogeneous magnetic field the group corresponds to an extension of the translation group generated by P_k. In this section we give representations of the magnetic translation group in the same functional analytic manner in which we defined the Heisenberg-Weyl group.

Definition 3.10. Let $W(a)$ be a strongly continuous unitary group on a separable Hilbert space $\mathscr{H}$ satisfying the following relations

$$W(a)W(b) = e^{ieF^{ij}a_i b_j} W(b)W(a), \qquad a, b \in \mathbb{R}^n.$$

Then, the group generated by the unitary operators $W(a)$ is referred to as the **magnetic translation group**.

Definition 3.11. Let π_i be the canonical momentum operator defined as follows

$$\pi_i := P_i - eA_i, \tag{3.30}$$

where P_i is the momentum operator and A_i is the vector potential of the magnetic field B_i. Furthermore, let the spatial part of the field strength tensor F_{ij} be defined as follows

$$eF_{ij} := -e[P_i, A_j] + e[P_j, A_i].$$

The infinitesimal generators of the n-dimensional magnetic translation group are the canonical momentum operators π_i defined by Equation (3.30). In particular the strongly continuous unitary operator $W(a)$ is defined as follows

$$W(a) := e^{ia^j \pi_j}, \qquad a \in \mathbb{R}^n.$$

From the definition of the the magnetic translation group it is obvious that the translation group follows by setting F^{ij} equal to zero.

3.5.3 Deformation of the Heisenberg-Weyl Operators

In this section we intend to show that from the deformed momentum operator of the Heisenberg-Weyl group, the infinitesimal generators of the magnetic translation group follow. As before, we use the method of warped convolutions for deformation.

Note that from a constant magnetic field a constant spatial field strength tensor follows. This is owed to the relation between the magnetic field and the field strength tensor given by $F_{ij} = -e\epsilon_{ijk}B^k$. In the symmetric gauge, (3.31), the vector potential A_i for a constant magnetic field is given by

$$eA_i = \frac{e}{2}F_{ij}X^j = -\frac{e}{2}\epsilon_{ijk}B^k X^j. \tag{3.31}$$

Lemma 3.14. *Let the deformation matrix B_{ij} be related to the field strength tensor in the following way $B_{ij} = -\frac{e}{2}F_{ij}$. Then the deformed momentum operator $P^j_{B,X}$ (3.12) becomes the canonical momentum operator of the magnetic translation group (3.30) in the symmetric gauge (3.31).*

Proof. For $B_{ij} = \frac{e}{2}F_{ij}$ the deformed momentum operator is given as

$$P^j_{B,X} = P^j - \frac{e}{2}F^{jk}X_k = P^j - eA^j.$$

This is exactly the definition of the canonical momentum operator π_i, (3.30). □

This is a very interesting result. We started with the Heisenberg-Weyl group and deformed the momentum operator. This deformation induced a shift given by the coordinate operator. By making a simple identification of the deformation matrix with the field strength tensor we obtained the magnetic translation group. Thus, in the spirit of deformation the magnetic translation group can be understood as a deformation of the Heisenberg-Weyl group. This gives a group theoretical explanation why the effects of a magnetic field appear, when we deform the underlying operators, i.e. the momentum operators, with the coordinate operator.

3.5.4 Noncommutative Torus from deformation of the HW group

Next, we show that the commutator relations of the noncommutative torus $\mathcal{T}^2_{0,B}$ follow from deformation with warped convolutions. The first index in $\mathcal{T}^2_{0,B}$ denotes the noncommutativity of the coordinate space while the other index denotes the noncommutativity of the momentum space. More precisely the noncommutative torus $\mathcal{T}^2_{\theta,\tilde{B}}$, is defined as follows, [MP01].

Definition 3.12. The associative **noncommutative torus** algebra $\mathcal{T}^2_{\theta,\tilde{B}}$ is generated by the operator P_i and the unitary operator $U_j := e^{iX_j}$, for $i,j = 2,3$ and $\theta \in \mathbb{R}$. (X_i, P_j) forms a canonical conjugate pair (3.1) and the algebra spanned by P_i and the unitary operator U_j is the following

$$U_2U_3 = e^{-i\theta}U_3U_2, \qquad P_iU_j = U_j(P_i + \delta_{ij}) \qquad [P_i, P_j] = i\tilde{B}\epsilon_{ij}.$$

Definition 3.13. Let the unitary operators V_2, V_3 satisfy the following algebra

$$V_2^BV_3^B = e^{2\pi im/n}V_3^BV_2^B,$$

for $n, m \in \mathbb{N}$. Then, the operators are said to satisfy the so called **clock and shift algebra**.

Furthermore, the authors in [MP01] proved that the torus algebra $\mathcal{T}^2_{\theta,\tilde{B}}$ is isomorphic to the torus algebra $\mathcal{T}^2_{0,B}$ iff $B = \tilde{B}/(1 - \tilde{B}\theta)$.

Lemma 3.15. *Let the deformation matrix be given as $B_{ij} = B\epsilon_{ij}$. Then, the noncommutative torus algebra $\mathcal{T}^2_{0,B}$ is spanned by the unitary operator $U_j^{B,X} := e^{iX_j^{B,X}}$ and the deformed momentum operator $P_j^{B,X}$ (3.12). Furthermore, if the deformation parameter is given by $B = -2\pi m/n$, for $n, m \in \mathbb{N}$, the unitary operators $V_i^{B,X} := e^{iP_i^{B,X}}$ satisfy the clock and shift algebra.*

Proof. The operators deformed with the coordinate operator satisfy the following relations,

$$[X_2^{B,X}, X_3^{B,X}] = 0, \quad [P_2^{B,X}, P_3^{B,X}] = iB, \quad [X_i^{B,X}, P_j^{B,X}] = i\delta_{ij}.$$

One can easily see that the deformed operators generate the torus algebra $\mathcal{T}^2_{0,B}$. In addition, the unitary operators V defined with the deformed momentum operator satisfy the following relations

$$V_j^BV_k^B = R_{jk}V_k^BV_j^B. \qquad R_{jk} = e^{-2iB_{jk}}.$$

Setting the deformation constant $B = -2\pi m/n$, yields that the unitary operators $V_i^B := e^{iP_j^{B,X}}$ satisfy the clock and shift algebra

$$V_2^BV_3^B = e^{2\pi im/n}V_3^BV_2^B.$$

□

Chapter 4

NCQFT on Minkowski space

In this chapter the author intends to give a brief review about well established mathematical building blocks of NCQFT on Minkowski space. The review is aiming in the direction of a better understanding of the the next chapters.

The starting point is [DFR95]. In this work major progress was achieved in defining a scalar quantum field on a NC spacetime. This achievement was obtained by constructing coordinate operators, obeying the relations of a Moyal-Weyl plane, from a set of axioms. Next, a quantum field on such a noncommutative spacetime was defined.

4.1 Quantum spacetime and QF

In [DFR95], the authors propose uncertainty relations for the spacetime coordinates in four dimensions. These relations are motivated by the fact, that localization with high accuracy at Planck scale causes a gravitational collapse. Thus spacetime, as a continuous entity has no meaning below the Planck scale. After imposing such uncertainty relations one replaces the commutative algebra of functions, for example the algebra given by $\mathscr{C}_0^\infty(M)$ over a commutative manifold M, by a noncommutative algebra describing a noncommutative spacetime, i.e. a quantum spacetime.

The uncertainty relations found by the exploration of localizing measurements, which are the outcome of the possibility of creating a black hole by concentration of energies, are the following

$$\begin{aligned}\Delta\hat{x}_0\left(\Delta\hat{x}_1+\Delta\hat{x}_2+\Delta\hat{x}_3\right)&\geq l_p^2,\\ \Delta\hat{x}_1\Delta\hat{x}_2+\Delta\hat{x}_1\Delta\hat{x}_3+\Delta\hat{x}_2\Delta\hat{x}_3&\geq l_p^2.\end{aligned}$$

The selfadjoint operators $\hat{x}_\mu$ implying those uncertainty relations satisfy certain conditions dubbed as **quantum conditions**. They are given by the following set of equations,

$$
\begin{aligned}
[[\hat{x}_\mu, \hat{x}_\nu], \hat{x}_\rho] &= 0, \\
[\hat{x}_\mu, \hat{x}_\nu][\hat{x}^\mu, \hat{x}^\nu] &= 0, \\
\left(\frac{1}{8}[\hat{x}_\mu, \hat{x}_\nu][\hat{x}_\rho, \hat{x}_\sigma]\epsilon^{\mu\nu\rho\sigma}\right)^2 &= l_p^8.
\end{aligned}
$$

The *-algebra which is generated by the selfadjoint operators $\hat{x}_\mu$ and the quantum conditions has a center given by the commutator relation

$$[\hat{x}_\mu, \hat{x}_\nu] = -i\theta_{\mu\nu}. \tag{4.1}$$

The tensor θ represents the noncommutative matrix, which contains the deformation parameter that gives the strength of noncommutativity. By setting the parameter to zero, the commutative case results.

In the next step, the authors in [DFR95] defined a free scalar quantum field on the noncommutative space that is generated by the commutation relations (4.1).

Before we give their realization, let us define free scalar field ϕ with mass m on the $(n+1)$-dimensional Minkowski spacetime as an operator-valued distribution acting on its domain in the Bosonic Fock space. Such a particle with momentum $\mathbf{p} \in \mathbb{R}^n$ has the energy given by $\omega_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$.

Definition 4.1. The **Bosonic Fock space** $\mathscr{H}^+$ is defined as in [Fre06, Sib93]:

$$\mathscr{H}^+ = \bigoplus_{k=0}^{\infty} \mathscr{H}_k^+$$

where the k-particle subspaces are given as

$$
\begin{aligned}
\mathscr{H}_k^+ = \{\Psi_k : H_m^+ \times \cdots \times H_m^+ \to \mathbb{C} \quad \text{symmetric}| \\
\|\Psi_k\|^2 = \int d^n\mu(\mathbf{p}_1) \ldots \int d^n\mu(\mathbf{p}_k) |\Psi_k(\mathbf{p}_1, \ldots, \mathbf{p}_k)|^2 < \infty\},
\end{aligned}
$$

with

$$H_m^+ := \{p \in \mathbb{R}^d | p^2 = m^2, p_0 > 0\}.$$

The particle annihilation and creation operators a, a^* for the massive Bosonic Fock space are defined in the following.

Definition 4.2. The **particle annihilation** and **creation operators** are defined by their action on k-particle wave functions,

$$(a(f)\Psi)_k(\mathbf{p}_1, \ldots, \mathbf{p}_k) = \sqrt{k+1} \int d^n\mu(\mathbf{p}) \overline{f(\mathbf{p})} \Psi_{k+1}(\mathbf{p}, \mathbf{p}_1, \ldots, \mathbf{p}_k)$$

$$(a(f)^*\Psi)_k(\mathbf{p}, \mathbf{p}_1, \ldots, \mathbf{p}_k) = \begin{cases} 0, & k = 0 \\ \frac{1}{\sqrt{k}} \sum_{i=1}^{k} f(\mathbf{p}_i) \Psi_{k-1}(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_k), & k > 0 \end{cases}$$

with $f \in \mathscr{H}_1$ and $\Psi_k \in \mathscr{H}_k^+$. The commutator relations of $a(f), a(f)^*$ follow immediately and are given as

$$[a(f), a(g)^*] = (f, g) = \int d^n\mu(\mathbf{p})\overline{f(\mathbf{p})}g(\mathbf{p}), \qquad [a(f), a(g)] = 0 = [a^*(f), a^*(g)].$$

Particle annihilation and creation operators with sharp momentum are introduced as operator valued distributions and are given by

$$a(f) = \int d^n\mu(\mathbf{p})\overline{f(\mathbf{p})}a(\mathbf{p}), \qquad a(f)^* = \int d^n\mu(\mathbf{p})f(\mathbf{p})a^*(\mathbf{p}),$$

where the particle annihilation and creation operators with sharp momentum satisfy the following commutator relations

$$[a(\mathbf{p}), a(\mathbf{q})^*] = 2\omega_{\mathbf{p}}\delta^n(\mathbf{p} - \mathbf{q}), \qquad [a(\mathbf{p}), a(\mathbf{q})] = 0 = [a^*(\mathbf{p}), a^*(\mathbf{q})]. \tag{4.2}$$

By using the former definitions of the particle creation and annihilation operators with sharp momentum, one writes the free field ϕ with mass m

$$\phi(x) = \int d^n\mu(\mathbf{p}) \left(e^{-ipx}a(\mathbf{p}) + e^{ipx}a^*(\mathbf{p})\right), \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in H_m^+.$$

Since we work with unbounded distribution valued operators, the massive free scalar field is smeared with test functions $f \in \mathscr{S}(\mathbb{R}^d)$,

$$\phi(f) = \int d^dx f(x)\phi(x) = a(\overline{f^-}) + a^*(f^+), \tag{4.3}$$

where the test functions $f^{\pm}$ are chosen as follows

$$f^{\pm}(\mathbf{p}) := \int d^dx f(x)e^{\pm ipx}, \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in H_m^+. \tag{4.4}$$

The particle number operator and the momentum operator are given in terms of particle creation and annihilation operators, and are defined in the following manner.

$$N = \int d^n\mu(\mathbf{p})a^*(\mathbf{p})a(\mathbf{p}), \qquad P_\mu = \int d^n\mu(\mathbf{p})p_\mu a^*(\mathbf{p})a(\mathbf{p}) \tag{4.5}$$

By using the former definitions we can give the realization of the free scalar field on the constant noncommutative spacetime.

Let the space $\mathcal{V}$ be the representation space of the selfadjoint coordinate operators $\hat{x}_\mu$ and $\mathscr{H}^+$ the Bosonic Fock space. Then, the free scalar field on noncommutative Minkowski spacetime can be realized on the tensor product space $\mathcal{V} \otimes \mathscr{H}^+$ as follows,

$$\phi_\otimes(x) := \int d\mu(\mathbf{p}) \left(e^{-ipx}a_\otimes(\theta, \mathbf{p}) + e^{ipx}a^*_\otimes(\theta, \mathbf{p})\right), \qquad p \in H_m^+,$$

with the following creation and annihilation operators

$$a_\otimes(\theta, \mathbf{p}) := e^{-ip\hat{x}} \otimes a(\mathbf{p}), \qquad a^*_\otimes(\theta, \mathbf{p}) := e^{ip\hat{x}} \otimes a^*(\mathbf{p}). \tag{4.6}$$

Now by taking the canonical commutation relations of the particle operators and the noncommutative algebra generated by (4.1) into account, one obtains for the operators given in (4.6) the following relations

$$\begin{aligned}
a_\otimes(\theta,\mathbf{p})a_\otimes(\theta,\mathbf{p}') &= e^{-ip\theta p'}a_\otimes(\theta,\mathbf{p}')a_\otimes(\theta,\mathbf{p}),\\
a^*_\otimes(\theta,\mathbf{p})a^*_\otimes(\theta,\mathbf{p}') &= e^{-ip\theta p'}a^*_\otimes(\theta,\mathbf{p}')a^*_\otimes(\theta,\mathbf{p})\\
a_\otimes(\theta,\mathbf{p})a^*_\otimes(\theta,\mathbf{p}') &= e^{+ip\theta p'}a^*_\otimes(\theta,\mathbf{p}')a_\otimes(\theta,\mathbf{p}) + 2\omega_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}')\mathrm{id}_{\mathcal{V}\otimes\mathscr{H}^+}.
\end{aligned}$$

In this review we adapted the notation of the authors in [GL07].

4.2 Wedge fields

A number of authors [ABJJ07, Gro79, GL07] succeeded representing the scalar field on $\mathscr{H}^+$ instead of $\mathcal{V}\otimes\mathscr{H}^+$. In [GL07] the representation of the scalar field on noncommutative Minkowski space was used to show that one should take a whole orbit of noncommutative matrices. This idea was the starting point to construct a map from the set of skew-symmetric matrices, i.e. the noncommutative matrices θ describing the noncommutativity, to a set of wedges. As next, we briefly review those results in order to relate the fields defined on a noncommutative spacetime, to fields defined on a wedge.

By using the momentum operator P_μ (see Equation 4.5), one is able to represent the distributions (4.6) on $\mathscr{H}^+$ as follows,

$$a(\theta,\mathbf{p}) = e^{i/2p\theta P}a(\mathbf{p}), \qquad a^*(\theta,\mathbf{p}) = e^{-i/2p\theta P}a^*(\mathbf{p}). \tag{4.7}$$

The corresponding field can be defined as

$$\phi(\theta,x) := \int d\mu(\mathbf{p})\left(e^{-ipx}a(\theta,\mathbf{p}) + e^{ipx}a^*(\theta,\mathbf{p})\right). \tag{4.8}$$

Now by considering the adjoint action of $U(0,\Lambda)$ on $a(\theta,\mathbf{p})$ and $a^*(\theta,\mathbf{p})$, the following transformation is induced upon θ

$$\theta \longmapsto \gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T; & \Lambda\in\mathcal{L}^\uparrow, \\ -\Lambda\theta\Lambda^T; & \Lambda\in\mathcal{L}^\downarrow. \end{cases},$$

where $\mathcal{L}^\uparrow$ and $\mathcal{L}^\downarrow$ denote the subgroups of orthochronous and antiorthochronous Lorentz transformations, respectively. The transformation is calculated by taking the Lorentz covariant behavior of the momentum operator into account.

Since the adjoint action of the unitary operator $U(0,\Lambda)$ also acts on the skew-symmetric matrix θ, it seems of importance to consider a whole family of fields given by the deformation parameters that characterize θ. This was the starting consideration to define a correspondence from the set of wedges to a set $\mathcal{Q}_0\subset\mathbb{R}^-_{d\times d}$ of skew-symmetric matrices.

Before we give this correspondence, we define the wedge.

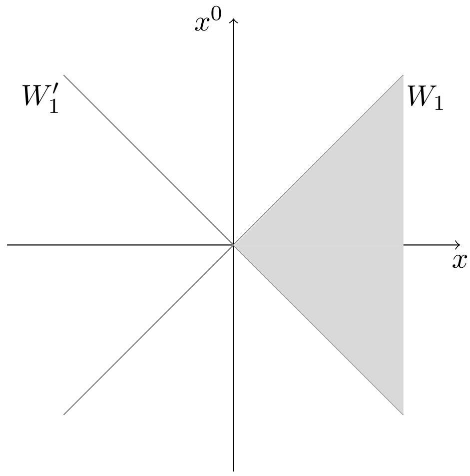

Figure 4.1: Right Wedge W_1

Definition 4.3. The reference wedge region most commonly used, is called the **right wedge** (see figure 4.1). It is defined to be given by following region,

$$W_1 := \{x \in \mathbb{R}^d : x_1 > |x_0|\}.$$

Under coordinate reflections, i.e. $j_\mu : x_\mu \mapsto -x_\mu$, the wedge region transforms in the following manner,

$$j_0 W_1 = W_1, \qquad j_1 W_1 = -W_1 = W_1', \qquad j_k W_1 = W_1, \quad k > 1, \tag{4.9}$$

where W_1' denotes the causal complement of W_1.

Definition 4.4. The set $\mathcal{W}$ of all wedges in $\mathbb{R}^d$ is defined as the set of all Poincaré transforms of W_1, given as follows

$$\mathcal{W} := \mathcal{P} W_1,$$

where $\mathcal{P}$ is the Poincaré group. In this work we mainly work with a subgroup $\mathcal{W}_0 \subset \mathcal{W}$ consisting only of the Lorentz transforms of W_1,

$$\mathcal{W}_0 := \mathcal{L}_+^\uparrow W_1, \qquad (d > 2).$$

Now in order to give the map between wedges and skew-symmetric matrices, we specify two homogeneous spaces for the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ and the subgroup $\hat{\mathcal{L}} \subset \mathcal{L}$, generated by the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ and spacetime reflections j_μ defined in (4.9). The first space is $(\mathcal{W}_0, \tau)$ endowed with the action τ defined by

$$\tau_\Lambda(W) := \Lambda W,$$

which corresponds to the set of wedges. The second space is an $\hat{\mathcal{L}}$-homogeneous space $(\mathcal{Q}_0, \gamma)$, corresponding to the skew-symmetric matrices, defined as

$$\mathcal{Q}_0 := \{\gamma_\Lambda(\theta_1) : \Lambda \in \hat{\mathcal{L}}\},$$

with action γ defined by

$$\theta \longmapsto \gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T; & \Lambda \in \mathcal{L}_+^\uparrow, \\ -\Lambda\theta\Lambda^T; & \Lambda \in j\mathcal{L}_+^\downarrow. \end{cases} .$$

As next, we show that the two defined homogeneous spaces $(\mathcal{Q}_0, \gamma)$ and $(\mathcal{W}_0, \tau)$ are isomorphic. Matrix θ_1 has to have a specific form such that the map Q,

$$Q : \mathcal{W}_0 \longmapsto \mathcal{Q}_0, \qquad Q(\Lambda W_1) := \gamma_\Lambda(\theta_1). \tag{4.10}$$

is well-defined. The specific form of θ_1 is the subject of the following lemma ([GL07], Lemma 3.1).

Lemma 4.1. *(i) The mapping Q (4.10) is a homomorphism of the $\hat{\mathcal{L}}$-homogeneous spaces $(\mathcal{W}_0, \tau)$ and $(\mathcal{Q}_0, \gamma)$ if and only if θ_1 has in d dimensions the following form*

$$\begin{pmatrix} 0 & \lambda & 0 & \cdots & 0 \\ \lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \lambda \geq 0. \tag{4.11}$$

and for the physical most interesting case of four dimensions the skew-symmetric matrix θ has a more general form

$$\begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}, \qquad \lambda \geq 0, \eta \in \mathbb{R}. \tag{4.12}$$

If θ_1 is not equal to zero, the map Q is an isomorphism.

(ii) If θ_1 has the form (4.11, 4.12), its $\hat{\mathcal{L}}$-orbit $\mathcal{Q}_0$ is

$$\begin{aligned} \mathcal{Q}_0 = \quad & \{\theta_1, -\theta_1\}, && (d = 2), \\ \mathcal{Q}_0 = \quad & \{\Lambda\theta_1\Lambda^T : \Lambda \in \Lambda \in \mathcal{L}_+^\uparrow\}, && (d > 2), \end{aligned} \tag{4.13}$$

and for any $W \in \mathcal{W}_0$, there holds

$$Q(W') = -Q(W) = \gamma_j(Q(W)).$$

By using the map Q (4.10), one can relate the scalar fields on NC Minkowski to wedges. In the next step one can show that these scalar fields satisfy concepts of locality and covariance in a wedge setting. Therefore, in what follows we give the definitions of a wedge-covariant and a wedge-local field, ([GL07], Definition 3.2).

Definition 4.5. Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions of the Wightman axioms. Then, the field ϕ is defined to be a wedge-local quantum field transforming covariantly if the following two condition are satisfied:

- **Covariance:** For any $W \in \mathcal{W}_0$ and $f \in \mathscr{S}(\mathbb{R}^d)$ the following holds

$$U(y,\Lambda)\phi_W(f)U(y,\Lambda)^{-1} = \phi_{\Lambda W}(f \circ (y,\Lambda)^{-1}), \qquad (y,\Lambda) \in \mathcal{P}_+^\uparrow,$$
$$U(0,j)\phi_W(f)VU(0,j)^{-1} = \phi_{jW}(\overline{f} \circ j)^{-1}.$$

- **Wedge-locality:** Let $W, \tilde{W} \in \mathcal{W}_0$ and $f \in \mathscr{S}(\mathbb{R}^2)$. If

$$\overline{W + \text{supp } f} \subset (\tilde{W} + \text{supp } g)',$$

  then

$$[\phi_W(f), \phi_{\tilde{W}}(g)]\Psi = 0, \quad \Psi \in D.$$

The last definition can be given in a simpler form due to the geometrical properties of the wedges. This is the subject of the following lemma, ([GL07], Lemma 3.3).

Lemma 4.2. *Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions stated in Definition 4.5. Then ϕ is wedge-local if and only if*

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \quad \Psi \in D,$$

for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$.

It was shown that the collection of fields ϕ_W, given by the deformed field (4.8) and the map Q, satisfies the Wightman properties and is a wedge-local field that transforms wedge covariantly under the unitary representation $U(a,\Lambda)$ of $\hat{P} := \hat{L} \rtimes \mathbb{R}^d$.

Furthermore, a relation between the model defined by the collection of fields ϕ_W and completely integrable models was found. The calculation of the two-particle S-matrix elements leads to the description of non-trivial interactions.

4.3 DFR-Model from Deformation

In this section we obtain the DFR-model represented on $\mathscr{H}^+$ given in (4.8), by deformation with the momentum operator using warped convolutions. To do so we use the formula of deformation given in Section 2 (see Definition 2.7), on the creation and annihilation operators of the free scalar field.

Before deforming we have to concern ourselves with the issue of convergence for the Formula 2.1 applied to the free field. One possibility as before would be to show that the scalar product of the adjoint action acting on the free field is a symbol. Another approach is to show that the expression $\langle\Psi, \phi_{\theta,P}(f)\Phi\rangle$ converges by using the bounds of a free field and thus conclude that deformation with the momentum operator is well-defined. Since in the next sections we deform the free field with operators unitarily equivalent to the momentum operator, it seems wise to choose the second path. The result is given in the following lemma.

Lemma 4.3. *The deformed field $\phi_{\theta,P}(f)$ for f as given in Equation (4.4) fulfills in the scalar product with $\Psi_l, \Phi_l \in \mathscr{S}(\mathbb{R}^{n\times l})$ the following inequality*

$$\langle\Psi_l, \phi_{\theta,P}(f)\Phi_l\rangle \leq 2\sqrt{l+1}\|\Psi_l\|\|\Phi_l\| \left(\int d^n\mu(\mathbf{p})|f^{\pm}(\mathbf{p})|^2\right)^{1/2} < \infty. \tag{4.14}$$

Therefore, the deformation of the free scalar field $\phi(f)$ with the momentum operator P^μ is well-defined.

Proof. Let us show that Inequality (4.14) holds.

$$\begin{aligned}\langle\Psi_l, \phi_{\theta,P}(f)\Phi_l\rangle &\leq \|\Psi_l\|\|\phi_{\theta,P}(f)\Phi_l\| \\ &\leq 2\sqrt{l+1}\|\Psi_l\|\|\Phi_l\| \left(\int d^n\mu(\mathbf{p})|f^{\pm}(\mathbf{p})|^2\right)^{1/2} \\ &< \infty,\end{aligned}$$

where in the last lines we used Cauchy-Schwartz and the inequality given in [GL07, Prop 2.2, b)]. Thus, by showing that the integral of deformation is bounded and for wave functions $\in \mathscr{S}(\mathbb{R}^{n\times l})$ it converges, the Formula (2.1) is well-defined. □

In the next step we calculate the deformation of the free scalar field with the momentum operator. For the convenience of the reader we use the notation $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N)$ and we define $u^\mu := \sum_{l=1}^{N} q_l^\mu$. To solve the integral of deformation we let the operators act on wave functions $\Psi_N(\mathbf{q}) \in \mathscr{S}(\mathbb{R}^{3N})$ that are eigenfunctions of the momentum operator. Let us first apply the deformation on the annihilation operator,

$$
\begin{aligned}
&(a_{\theta,P}(\mathbf{p})\Psi_N)(\mathbf{q}) = \\
&= (2\pi)^{-d} \lim_{\epsilon\to 0} \left(\int\int d^d k d^d y e^{-ik(y-u)} \chi(\epsilon k, \epsilon y)\, \alpha_{\theta y}(a(\mathbf{p}))\Psi_N \right)(\mathbf{q}) \\
&= (2\pi)^{-d} \lim_{\epsilon_1\to 0} \left(\int d^d y \lim_{\epsilon_2\to 0} \left(\int d^d k e^{-ik(y-u)} \chi_2(\epsilon_2 k) \right) \chi_1(\epsilon_1 y)\; e^{-ip\theta y} a(\mathbf{p})\Psi_N \right)(\mathbf{q}) \\
&= (2\pi)^{-d} \lim_{\epsilon_1\to 0} \left(\int d^d y\, (2\pi)^d\, \delta(y-u)\, \chi(\epsilon_1 y)\, e^{-ip\theta y} a(\mathbf{p})\Psi_N \right)(\mathbf{q}) \\
&= e^{-ip\theta P} a(\mathbf{p})\Psi_N(\mathbf{q}).
\end{aligned}
$$

where in the last lines we calculated the adjoint action of $U(y)$ on $a(\mathbf{p})$ by using the Baker-Campbell-Hausdorff formula, and as in the former proofs we chose $\chi(\epsilon k, \epsilon y) = \chi_2(\epsilon_2 k)\chi_1(\epsilon_1 y)$ with $\chi_l \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\chi_l(0,0) = 1$, $l = 1,\ 2$, to obtain the delta distribution $\delta(y-u)$ in the limit $\epsilon_2 \to 0$. The same calculation can be done for the creation operator $a^*(\mathbf{p})$ and thus one obtains the following,

$$
a_{\theta,P}(\mathbf{p}) = e^{-ip\theta P} a(\mathbf{p}), \qquad a^*_{\theta,P}(\mathbf{p}) = e^{ip\theta P} a^*(\mathbf{p}).
$$

As one can easily see the deformed creation and annihilation operator are equal to the ones given in Equation. (4.7) as the representation of the DFR field on $\mathscr{H}^+$. Note that they differ by a constant. This means that these operators correspond to a field on NC Minkowski obeying the commutator relations $[\hat{x}_\mu, \hat{x}_\nu] = -2i\theta_{\mu\nu}$. This is merely a question of convention.

Next we show how to obtain the underlying NC spacetime in the framework of warped convolutions. A deformed associative product is defined in the following way.

Definition 4.6. An associative deformed product $\times_\theta$ of A, B is defined as

$$
A \times_\theta B = (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} \alpha_{\theta v}(A)\alpha_u(B).
$$

Furthermore, the deformed commutator $[A \overset{\times_\theta}{,} B]$ of A, B is defined in the following way

$$
[A \overset{\times_\theta}{,} B] := A \times_\theta B - B \times_\theta A. \tag{4.15}
$$

The deformed product can be used to calculate the commutator of the coordinates. In the case of deformation with the momentum operator P_μ one obtains the following lemma.

Lemma 4.4. *Let the deformed product given in Definition 4.6 be defined by the generator of translations P_μ. Then the deformed commutator (4.15) of the coordinates gives the Moyal-Weyl plane (see figure 4.2),*

$$
[x_\mu \overset{\times_\theta}{,} x_\nu] := x_\mu \times_\theta x_\nu - x_\nu \times_\theta x_\mu = -2i\theta_{\mu\nu}.
$$

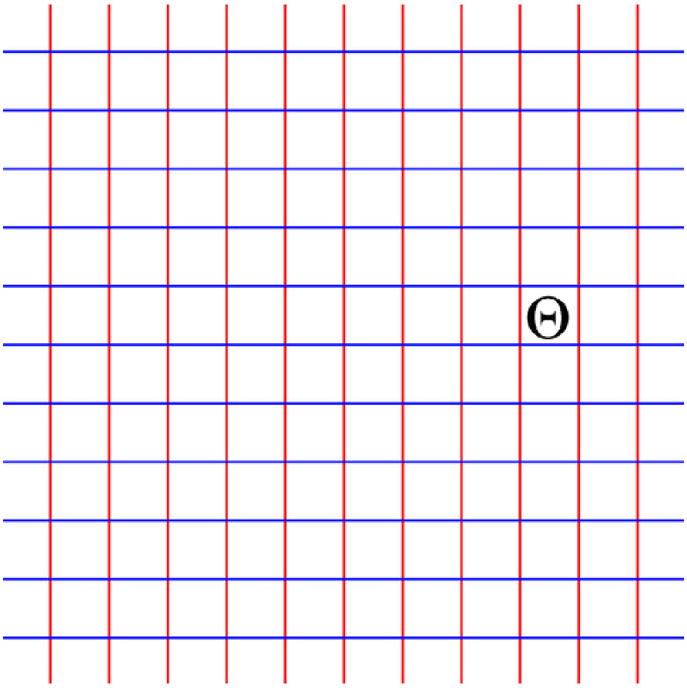

Figure 4.2: Example of a Moyal-Weyl plane $\mathbb{R}^2_\theta$

Proof. We first calculate the deformed product of the coordinates using Definition 4.6.

$$\begin{aligned} x_\mu \times_\theta x_\nu &= (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} \alpha_{\theta v}(x_\mu)\alpha_u(x_\nu) \\ &= (2\pi)^{-d} \iint d^d v d^d u e^{-ivu} (x_\mu + (\theta v)_\mu)(x_\nu + u_\nu) = x_\mu x_\nu - i\theta_{\mu\nu} \end{aligned}$$

In the last lines, we applied the adjoint action of the momentum operator P_μ on the coordinates, which induces a translation. The next step consists in calculating the deformed commutator of the coordinates. Due to the skew-symmetry of the deformation matrix θ, one obtains for the deformed commutator the Moyal-Weyl plane. □

This result is not surprising. As already mentioned, in [GL07] a quantum field was defined on the Moyal-Weyl plane, which also can obtained by using the momentum operator for deformation via warped convolutions. Therefore, it is only natural that the Moyal-Weyl plane appears for the deformed commutator of the coordinates. In the next chapters we calculate the commutator of the coordinates by using the deformed product induced by the operators that are used for deformation.

Thus one option to obtain a quantum spacetime, is to deform the underlying field theory with certain operators and to examine the wedge-covariance and wedge-locality properties of the deformed theory. Now by letting wedge-locality be our guiding principle we pick those deformed fields that satisfy it, and calculate the spacetime that they induce. Thus we have a deformed field that satisfies certain locality properties and in addition it lives on a NC spacetime that is induced upon from deformation. This is a very different approach from the usual path taken. Rather than starting on some given noncommutative spacetime, we rather first demand that the deformation satisfies certain covariance and locality properties. Then, we take those deformed field that satisfy such relations and calculate the underlying noncommutative spacetime. Thus in this spirit it is appropriate to say that the approach is background independent. This is due to the fact that we do not start a priori with a certain spacetime, but it rather emerges by the conditions demanded from the QF.

Chapter 5

Coordinate operator in QFT

"One can see the world with the p-eye or one can see the world with the x-eye, if one opens both eyes at once, one becomes crazy."
— Pauli to Heisenberg, *Letter 19. October 1926*

"I opened both eyes."
— Anonymous, *Leipzig, 07. March 2013*

In this Chapter we review the main results of [SS09]. The authors constructed operators X_μ by requiring them to fulfill a symplectic structure with the momentum operator, i.e. $[P_\mu, X_\nu] = i\eta_{\mu\nu}N$, for $\mu = \nu$, where N is the particle number operator. The operators are constructed in a quantum field theoretical manner by using building blocks of the Fock space, the creation and annihilation operators.

The same methods are used in this thesis to define coordinate operators in the massive case. The massive temporal part of the coordinate operator was constructed in [SS09] and in this work we construct the spatial part of the coordinate operator.

The operators X_μ obtained by construction do not commute and therefore we show that even on a level without deformation it is possible to obtain the Moyal-Weyl plane. This is done by calculating the expectation value of the operator that captures the noncommutative structure.

Furthermore a noncommutative spacetime is also obtained from the coordinate operator by applying an idea found in the QM context (see Lemma 3.13). An interpretation of the so obtained deformation parameter is given. It is important to note that in the nonrelativistic limit the QFT Moyal-Weyl becomes the one found in the QM chapter.

Since the construction of the operators X_μ is not unique we give further arguments why the constructed position X_j fulfills the requirements of a spatial coordinate. One of the main arguments is the essential self-adjointness of X_j, which is proven by using the unitary equivalence, via Fourier transformation, to the spatial part of the momentum operator P_j. After proving the essential self-adjointness of X_j we investigate the unitary transformations of the free scalar field under the adjoint action of the coordinate operators.

In the final section, the coordinate operator is used to deform the underlying quantum field theory. We further investigate the Wightman properties, the transformation properties under the Poincaré group and the locality properties of the deformed scalar field.

5.1 Coordinate operators for the massless case

Before we give explicit expressions for the operators X_μ, we define in this section the Bosonic Fock space for a free massless scalar field.

The free scalar field ϕ with mass $m = 0$ is defined on the $n+1$-dimensional Minkowski spacetime as an operator-valued distribution. Such a particle with momentum $\mathbf{p} \in \mathbb{R}^n$ has energy $\omega_{\mathbf{p}}$ defined by $\omega_{\mathbf{p}} = |\mathbf{p}|$. Furthermore, we use for the following definitions the well known Lorentz-invariant measure $d^n\mu(\mathbf{p}) := d^n\mathbf{p}/2\omega_{\mathbf{p}}$.

Definition 5.1. The **massless Bosonic Fock space** $\mathscr{H}_0^+$ is defined as in [Fre06, Sib93]:

$$\mathscr{H}_0^+ = \bigoplus_{k=0}^{\infty} \mathscr{H}_{0,k}^+$$

where the m particle subspaces are given as

$$\begin{aligned}\mathscr{H}_{0,k}^+ = \{\Psi_k : \partial V_+ \times \cdots \times \partial V_+ \to \mathbb{C} \quad \text{symmetric}| \\ \|\Psi_k\|^2 = \int d^n\mu(\mathbf{p}_1)\ldots\int d^n\mu(\mathbf{p}_k)|\Psi_k(\mathbf{p}_1,\ldots,\mathbf{p}_k)|^2 < \infty\},\end{aligned}$$

with

$$\partial V_+ := \{p \in \mathbb{R}^d | p^2 = 0, p_0 > 0\}.$$

In the next step we define the so called particle annihilation and creation operators a, a^*. With the help of these operators one can go from a k-particle subspace $\mathscr{H}_{0,k}^+$ to a $k+1$-particle subspace $\mathscr{H}_{0,k+1}^+$ or to a $k-1$- particle subspace $\mathscr{H}_{0,k-1}^+$, respectively. Furthermore, with the particle operators one can define the particle number, the momentum and the energy of the k- particle system. Another important property of the annihilation operator is that the vacuum vector $\Omega = (1, 0, \cdots)$ can be characterized by the equation $a\Omega = 0$.

Definition 5.2. The **particle annihilation** and **creation operators** are defined by their action on k-particle wave functions

$$(a(f)\Psi)_k(\mathbf{p}_1,\ldots,\mathbf{p}_k) = \sqrt{k+1}\int d^n\mu(\mathbf{p})\overline{f(\mathbf{p})}\Psi_{k+1}(\mathbf{p},\mathbf{p}_1,\ldots,\mathbf{p}_k)$$

$$(a(f)^*\Psi)_k(\mathbf{p},\mathbf{p}_1,\ldots,\mathbf{p}_k) = \begin{cases} 0, & k=0 \\ \frac{1}{\sqrt{k}}\sum_{i=1}^{k} f(\mathbf{p}_i)\Psi_{k-1}(\mathbf{p}_1,\ldots,\mathbf{p}_{i-1},\mathbf{p}_{i+1},\ldots,\mathbf{p_k}), & k>0 \end{cases}$$

with $f \in \mathscr{H}_1$ and $\Psi_k \in \mathscr{H}^+_{0,k}$. The commutator relations of $a(f), a(f)^*$ follow immediately and are given as follows

$$[a(f), a(g)^*] = (f, g) = \int d^n\mu(\mathbf{p})\overline{f(\mathbf{p})}g(\mathbf{p}), \qquad [a(f), a(g)] = 0 = [a^*(f), a^*(g)].$$

Particle annihilation and creation operators with sharp momentum are introduced as operator valued distributions and are given as follows

$$a(f) = \int d^n\mu(\mathbf{p})\overline{f(\mathbf{p})}a(\mathbf{p}), \qquad a(f)^* = \int d^n\mu(\mathbf{p})f(\mathbf{p})a^*(\mathbf{p}),$$

where particle annihilation and creation operators with sharp momentum satisfy the following commutator relations

$$[a(\mathbf{p}), a(\mathbf{q})^*] = 2\omega_{\mathbf{p}}\delta^n(\mathbf{p}-\mathbf{q}), \qquad [a(\mathbf{p}), a(\mathbf{q})] = 0 = [a^*(\mathbf{p}), a^*(\mathbf{q})]. \tag{5.1}$$

By using the former definitions of particle creation and annihilation operators with sharp momentum, one writes a free massless field ϕ as follows

$$\phi(x) = \int d\mu(\mathbf{p}) \left(e^{-ipx}a(\mathbf{p}) + e^{ipx}a^*(\mathbf{p})\right), \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in \partial V_+.$$

Since we work with operator-valued distributions, the massless free field has to be smeared with test functions $f \in \mathscr{S}(\mathbb{R}^d)$,

$$\phi(f) = \int d^dx f(x)\phi(x) = a(\overline{f^-}) + a^*(f^+), \tag{5.2}$$

where the test functions $f^\pm$ are chosen as follows

$$f^\pm(\mathbf{p}) := \int d^dx f(x)e^{\pm ipx}, \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in \partial V_+.$$

The particle number operator and the momentum operator are given in terms of particle creation and annihilation operators, and are defined in the following manner.

$$N = \int d^n\mu(\mathbf{p})a^*(\mathbf{p})a(\mathbf{p}), \qquad P_\mu = \int d^n\mu(\mathbf{p})p_\mu a^*(\mathbf{p})a(\mathbf{p}) \tag{5.3}$$

As already mentioned, the authors in [SS09] constructed the coordinate operators X_μ by demanding the following relation

$$[X_\mu, P_\nu] = -i\eta_{\mu\nu}N, \tag{5.4}$$

for $\mu = \nu$. In the next section we give an explanation to the occurrence of off-diagonal terms. Now by imposing the commutator relations the authors in [SS09] obtained for the temporal part an operator which is not hermitian. The use of a standard trick, i.e. $A^{herm} = 1/2(A+A^*)$, turns it into a hermitian operator and for X_0, we have the following,

$$X_0 = -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{1}{\omega_{\mathbf{p}}}\left(\frac{n}{2} - 1 + p^j\frac{\partial}{\partial p^j}\right)a(\mathbf{p}).$$

Note that we choose to work with the covariant normalization. The operator X_μ is given in n-dimensions, which is a generalization of the coordinate operator given in [SS09]. For the spatial part one obtains

$$X_j = -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\left(\frac{p_j}{2\omega^2_{\mathbf{p}}} + \frac{\partial}{\partial p^j}\right)a(\mathbf{p}).$$

5.1.1 Algebra of massless coordinate operators

By using commutation relations (5.1) we calculate the algebra of X_μ with generators of the Poincaré group. We first give the commutator relations of the operator X_μ and the momentum operator P_ν. As already mentioned the diagonal terms satisfy the canonical commutation relations which were required and are motivated by quantum mechanics.

$$[X_0, P_0] = -iN, \qquad [X_i, P_j] = -i\eta_{ij}N$$

The off-diagonal terms give the following relations

$$[X_0, P_j] = -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{p_j}{\omega_{\mathbf{p}}}a(\mathbf{p}),$$

$$[X_j, P_0] = -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{p_j}{\omega_{\mathbf{p}}}a(\mathbf{p}). \tag{5.5}$$

The non-vanishing of the commutator between the spatial coordinate operator and the Hamiltonian is not a mere construction failure but points out an important fact that is given in the following lemma.

Lemma 5.1. *The operator V_j obtained by commutator relation (5.5) is the Fockspace equivalent of a relativistic velocity for massless particles.*

Proof. This commutator is equivalent to the Heisenberg equation, i.e. $\dot{A} = i[A, P_0]$. The equation describes the time derivative of the coordinate operator, which for a dynamical system should not be equal to zero. The time derivative of the spatial coordinate operator is the velocity operator denoted as V_j. That this operator, represents the velocity of a relativistic system can easily be seen, by calculating the action of V_j on a wave function $\varphi \in \mathscr{H}_1$ as follows

$$\left(V_j\varphi\right)(\mathbf{p}) = \frac{p_j}{\omega_{\mathbf{p}}}\varphi(\mathbf{p}).$$

This is the velocity for a relativistic particle. Furthermore, the operator V_j acting on an n-particle wave function is the following,

$$\left(V_j\varphi\right)(\mathbf{p^1}, \ldots, \mathbf{p^k}) = \sum_{l=1}^{k}\frac{p_j^l}{\omega_{\mathbf{p^l}}}\varphi(\mathbf{p^1}, \ldots, \mathbf{p^k}) = \sum_{l=1}^{k}V_j(\mathbf{p^l})\varphi(\mathbf{p^1}, \ldots, \mathbf{p^k})$$

Thus the operator V_j acts as a velocity operator for an n-particle system. □

To calculate the transformation of the coordinate operator under Lorentz transformations it is important to know the algebra of Lorentz generators and operator X_μ. The calculation of these relations requires the representation of the Lorentz generators in terms of creation and annihilation operators a, a^*. This was done in [IZ80, Equation 3.54] and in this context see also [SS09, Appendix]. The generator of the Lorentz boosts has the following form

$$M_{0j} = i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\left(\omega_{\mathbf{p}}\frac{\partial}{\partial p^j}\right)a(\mathbf{p}), \tag{5.6}$$

and the Lorentz generator of rotations in the (i,k)-plane is given as

$$M_{ik} = i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\left(p_i\frac{\partial}{\partial p^k} - p_k\frac{\partial}{\partial p^i}\right)a(\mathbf{p}).$$

Again, by using relations (5.1) we calculate the commutator of the Lorentz generators with the coordinate operators. The temporal operator X_0 satisfies the following relations.

$$[X_0, M_{jk}] = 0, \qquad [X_0, M_{0j}] = -\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{p_j}{\omega_{\mathbf{p}}^2}\left(\frac{n}{2} - 1 + p_k\frac{\partial}{\partial p^k}\right)a(\mathbf{p}).$$

As one can see the second commutator relation destroys the right transformation property of the operator under Lorentz transformations, since it does not produce the Lorentz covariant relation $[X_0, M_{0j}] = iX_j$. Although the coordinate operator X_μ does not transform as a four-vector, the spatial component transforms covariantly under rotations in the (i,k)-plane due to the following commutator relations

$$[X_i, M_{jk}] = i\left(\delta_{ij}X_k - \delta_{ik}X_j\right).$$

The spatial coordinate operator fulfills the following commutator relation with the generator of Lorentz boosts

$$[X_i, M_{0j}] = -\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{1}{2\omega_{\mathbf{p}}}\left(\delta_{ij} + 2\frac{p_ip_j}{\omega_{\mathbf{p}}^2} + 2p_i\frac{\partial}{\partial p^j}\right)a(\mathbf{p}).$$

In the next step we give the result of the commutator of the coordinate operators. Before doing so, we define the commutator as

$$[X_\mu, X_\nu] =: i\hat{\theta}_{\mu\nu}.$$

The skew-symmetric operator valued matrix $\hat{\theta}$ has the following components,

$$\begin{aligned}\hat{\theta}_{0j} &= -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{1}{\omega_{\mathbf{p}}}\left((n-1)\frac{p_j}{2\omega_{\mathbf{p}}^2} + \left(\delta_j^k + \frac{p^kp_j}{\omega_{\mathbf{p}}^2}\right)\frac{\partial}{\partial p^k}\right)a(\mathbf{p}), \qquad (5.7)\\ \hat{\theta}_{ij} &= 0.\end{aligned}$$

As one can see from the algebra, the spatial components commute, which means that they have a common eigenvector. This fact will be further explored in the next sections.

5.2 Coordinate operators in the massive case

Analogously to the last sections, we construct the coordinate operators for a massive scalar field.

The authors in [SS09] constructed a temporal coordinate operator X_0 for the massive case as well and obtained

$$X_0 = i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{\omega_{\mathbf{p}}}{p_ip^i}\left(\frac{n}{2} - 1 + p^j\frac{\partial}{\partial p^j}\right)a(\mathbf{p}).$$

The massive temporal operator was in addition to requirement (5.4) constructed in such a way, that the massless operator follows from the massive one by taking the massless limit, i.e. $m \to 0$. During this thesis we constructed the spatial coordinate operator for the massive case. After some cumbersome calculations, we obtained the following operator for the spatial part of the coordinate operator.

$$X_j = -i \int d^n\mu(\mathbf{p}) a^*(\mathbf{p}) \left(\frac{p_j}{2\omega_{\mathbf{p}}^2} + \frac{\partial}{\partial p^j} \right) a(\mathbf{p}). \tag{5.8}$$

It is not accidental that the spatial part of the coordinate operator in the massive case, is identical in form to the spatial part in the massless case. This reason is explored in the next sections.

5.2.1 Algebra of massive coordinate operators

In this section we calculate the algebra of the massive coordinate operator with generators of the Poincaré group. Let us start by considering the algebra of the operator X_μ with P_ν. As before the on-diagonal terms fulfill the requirement (5.4),

$$[X_0, P_0] = -iN, \qquad [X_i, P_j] = -i\eta_{ij} N$$

The non-vanishing off-diagonal terms in the massive case are given by the following relations,

$$\begin{aligned} i\tilde{V}_j &:= [X_0, P_j] = i \int d^n\mu(\mathbf{p}) a^*(\mathbf{p}) \frac{\omega_{\mathbf{p}} p_j}{p_i p^i} a(\mathbf{p}), \\ -iV_j &:= [X_j, P_0] = -i \int d^n\mu(\mathbf{p}) a^*(\mathbf{p}) \frac{p_j}{\omega_{\mathbf{p}}} a(\mathbf{p}). \end{aligned} \tag{5.9}$$

The second commutator relation is the subject of the following lemma.

Lemma 5.2. *The operator V_j obtained by commutator relation (5.9) is the Fockspace equivalent of a relativistic velocity for massive particles.*

Proof. The proof can be done along the same lines as the proof of Lemma 5.5 in the massless case. □

In the next step we give the commutator relations of the coordinate operator with the Lorentz generators. We start by giving the commutator between the massive temporal operator and the Lorentz generators.

$$[X_0, M_{jk}] = 0,$$

$$[X_0, M_{0j}] = -\int d^n\mu(\mathbf{p}) a^*(\mathbf{p}) \left(\left(\frac{m^2}{p_i p^i} \right) \frac{\partial}{\partial p^j} - \frac{p_j}{p_i p^i} \left(1 + \frac{2\omega_{\mathbf{p}}^2}{p_r p^r} \right) \left(\frac{n}{2} - 1 + p^k \frac{\partial}{\partial p^k} \right) \right) a(\mathbf{p}).$$

As in the massless case, the second commutator relation destroys the right transformation property of the operator under Lorentz transformations, since it does not produce the Lorentz covariant relation $[X_0, M_{0j}] = iX_j$. Although the coordinate operator X_μ does not transform as a four vector, the spatial component transforms covariantly under rotations in the (i, k)-plane due to the following commutator relations

$$[X_i, M_{jk}] = i\left(\delta_{ij}X_k - \delta_{ik}X_j\right). \tag{5.10}$$

The massive spatial coordinate operator fulfills the following commutator relation with the generator of Lorentz boosts

$$[X_i, M_{0j}] = -\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{1}{2\omega_{\mathbf{p}}}\left(\delta_{ij} + 2\frac{p_ip_j}{\omega_{\mathbf{p}}^2} + 2p_i\frac{\partial}{\partial p^j}\right)a(\mathbf{p}).$$

As one can see, the spatial part of the massive coordinate operator satisfies the same relations as in the massless case. The constructed massive coordinate operators do not commute in the $(0j)$-component. Before giving the noncommutativity of the operators, we define the massive noncommutative matrix as follows

$$[X_\mu, X_\nu] =: i\hat{\theta}^m_{\mu\nu}.$$

The algebra can be calculated as before and one obtains for the skew-symmetric operator valued matrix $\hat{\theta}^m$ the following

$$\begin{aligned}
\hat{\theta}^m_{0j} &= -i\int d^n\mu(\mathbf{p})a^*(\mathbf{p})\frac{1}{p_ip^i\omega_{\mathbf{p}}}\Bigg(\left(\frac{n-1}{2} + \frac{p^kp_k}{\omega_{\mathbf{p}}^2} + (n-2)\frac{\omega_{\mathbf{p}}^2}{p^kp_k}\right)p_j \\
&\qquad + \left(1 + \frac{2\omega_{\mathbf{p}}^2}{p^kp_k}\right)p_jp^r\frac{\partial}{\partial p^r} - \omega_{\mathbf{p}}^2\frac{\partial}{\partial p^j}\Bigg)a(\mathbf{p}), \\
\hat{\theta}^m_{ij} &= 0.
\end{aligned} \tag{5.11}$$

As in the massless case, the components of the spatial coordinate operator commute and therefore have a common eigenvector.

5.3 Expectation value of massless coordinate operators

The constructed coordinate operators X_μ do not commute in the $(0j)$-component and thus we are interested in the physical interpretation of this noncommutativity. To obtain a physical interpretation for the operator-valued matrices given in Equations (5.7) and (5.11), we calculate the expectation value. Furthermore, all calculations in this section are done for $n = 3$. For the calculation of the expectation values we choose different states $\psi \in L^2(\mathbb{R}^3)$. The following lemma simplifies the calculations of the expectation values.

Remark 5.1. For simplicity reasons we work during certain parts of this thesis with the non-covariant representation of particle creation and annihilation operators. The non-covariant representation is realized by

$$\tilde{a}(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a(\mathbf{p}), \qquad \tilde{a}^*(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a^*(\mathbf{p}).$$

Lemma 5.3. *Let $\psi \in L^2(\mathbb{R}^3)$ and let the massless operator-valued matrix, (5.7) be given in the following form*

$$i\hat{\theta}_{0j} = \int d^3\mathbf{p}\tilde{a}^*(\mathbf{p})\left(F_{0j}(\mathbf{p}) + F^l_{0j}(\mathbf{p})\frac{\partial}{\partial p^l}\right)\tilde{a}(\mathbf{p}),$$

where the function valued matrices F_{0j} and F^l_{0j} are given by

$$F_{0j}(\mathbf{p}) = \frac{p_j}{\omega^3_{\mathbf{p}}}, \qquad F^l_{0j}(\mathbf{p}) = \left(\frac{\delta^l_j}{\omega_{\mathbf{p}}} + \frac{p_j p^l}{\omega^3_{\mathbf{p}}}\right).$$

Then, the expectation value of $i\hat{\theta}_{0j}$, w.r.t. a state ψ is given as follows,

$$\langle\psi|i\hat{\theta}_{0j}|\psi\rangle = \int \left(\left(F_{0j}(\mathbf{p}) - \partial^{\mathbf{p}}_l F^l_{0j}(\mathbf{p})\right)|\psi(\mathbf{p})|^2 - F^l_{0j}(\mathbf{p})\psi(\mathbf{p})\frac{\partial}{\partial p^l}\psi^*(\mathbf{p})\right) d^3p.$$

Proof. For the proof we split $i\hat{\theta}_{0j}$ in two parts

$$i\hat{\theta}^1_{0j} := \int d^3\mathbf{p}\tilde{a}^*(\mathbf{p})F_{0j}(\mathbf{p})\tilde{a}(\mathbf{p}), \qquad i\hat{\theta}^2_{0j} := \int d^3\mathbf{p}\tilde{a}^*(\mathbf{p})F^l_{0j}(\mathbf{p})\frac{\partial}{\partial p^l}\tilde{a}(\mathbf{p}),$$

We begin calculating the expectation value of the first part $i\hat{\theta}^1_{0j}$.

$$\begin{aligned}\langle\psi|i\hat{\theta}^1_{0j}|\psi\rangle &= \int d^3p \int d^3q \int d^3m \psi^*(\mathbf{q})F_{0j}(\mathbf{m})\psi(\mathbf{p})\langle\mathbf{q}|a^\dagger(\mathbf{m})a(\mathbf{m})|\mathbf{p}\rangle \\ &= \int d^3p F_{0j}(\mathbf{p})|\psi(\mathbf{p})|^2.\end{aligned}$$

We used the momentum representation of the Hilbert space $L^2(\mathbb{R}^3)$ to carry out the calculations. The second part $i\hat{\theta}^2_{0j}$ of the expectation value of the noncommutative matrix is

$$\langle\psi|i\hat{\theta}^2_{0j}|\psi\rangle = -\int d^3p\left(F^l_{0j}(\mathbf{p})\psi(\mathbf{p})\partial^{\mathbf{p}}_l\psi^*(\mathbf{p}) + \partial^{\mathbf{p}}_l F^l_{0j}(\mathbf{p})|\psi(\mathbf{p})|^2\right).$$

We add the expectation values of the two parts of $i\hat{\theta}^2_{0j}$ and obtain

$$\langle\psi|i\hat{\theta}_{0j}|\psi\rangle = \int \left(\left(F_{0j}(\mathbf{p}) - \partial^{\mathbf{p}}_l F^l_{0j}(\mathbf{p})\right)|\psi(\mathbf{p})|^2 - F^l_{0j}(\mathbf{p})\psi(\mathbf{p})\frac{\partial}{\partial p^l}\psi^*(\mathbf{p})\right) d^3p.$$

□

In the next step we give a lemma showing how one obtains the constant matrix of the Moyal-Weyl plane, by the calculation of the expectation value of $i\hat{\theta}_{\mu\nu}$ with a simple wave packet.

Lemma 5.4. *Let the wave function $\Psi_a(\mathbf{p}) \in L^2(\mathbb{R}^3)$ be given as*

$$\Psi_a(\mathbf{p}) = e^{ia_k p^k} f(\mathbf{p}), \qquad f(\mathbf{p}) = \left(\frac{\alpha}{\pi}\right)^{3/4} e^{-\frac{\alpha}{2}|\mathbf{p}|^2}, \tag{5.12}$$

where the L^2-norm of Ψ_a is equal to one, i.e. $||\Psi_a||^2 = 1$. Then, the expectation value of $i\hat{\theta}_{\mu\nu}$ given in (5.7), w.r.t. Ψ_a is given as follows

$$\langle\psi_a|i\hat{\theta}_{\mu\nu}|\psi_a\rangle = i\frac{8}{3}\sqrt{\frac{\alpha}{\pi}}\begin{pmatrix} 0 & a^j \\ -a_j & \mathbf{0}_{3\times 3}\end{pmatrix}. \tag{5.13}$$

Furthermore, for the following choice of the dimensional physical constant α and the dimensional physical constant vector a_j

$$\alpha = \pi\beta^2\theta \qquad a_j = \sqrt{\theta}(\tfrac{3}{8\beta})(1, \quad 0, \quad 0) \ , \qquad \beta \in \mathbb{R}^+,$$

one obtains the Moyal-Weyl plane $\mathbb{R}^4_{\hat{\theta}}$, for noncommutative space-time and commuting space-space. Moreover the plane is characterized by the deformation parameter θ.

Proof. In this proof we only look at the components $i\hat{\theta}_{0j}$, due to the fact that $\hat{\theta}_{ij} = 0$. By using Lemma 5.3 and the form of the wave functions that we chose, we obtain for the expectation values of $i\hat{\theta}_{0j}$

$$\begin{aligned}
\langle\psi|i\hat{\theta}_{0j}|\psi\rangle &= \int d^3p \left(\left(F_{0j}(\mathbf{p}) - \partial_l^{\mathbf{p}} F^l_{0j}(\mathbf{p}) \right) |\psi(\mathbf{p})|^2 - F^l_{0j}(\mathbf{p})\psi(\mathbf{p})\partial_l^{\mathbf{p}}\psi^*(\mathbf{p}) \right) \\
&= \int d^3p \left(\left(F_{0j}(\mathbf{p}) - \partial_l^{\mathbf{p}} F^l_{0j}(\mathbf{p}) \right) |f(\mathbf{p})|^2 - F^l_{0j}(\mathbf{p}) \left(-ia_l|f(p)|^2 + \frac{1}{2}\partial_l f(p)^2 \right) \right)
\end{aligned}$$

Note that a simplification occurs due to partial integration and from $F_{0j}(\mathbf{p}) = \frac{1}{2}\partial_l^{\mathbf{p}} F^l_{0j}(\mathbf{p})$. Hence we obtain,

$$\begin{aligned}
\langle\psi|i\hat{\theta}_{0j}|\psi\rangle &= +ia_l \int F^l_{0j}(\mathbf{p})|f(p)|^2 d^3p \\
&= ia_l \left(2\pi\delta^l_j \sqrt{\frac{\alpha}{\pi}} + \int d^3p \frac{p_j p^l}{\omega^3_{\mathbf{p}}} |f(\mathbf{p})|^2 \right) \\
&= ia_l \left(2\delta^l_j \sqrt{\frac{\alpha}{\pi}} + \frac{2}{3}\delta^l_j \sqrt{\frac{\alpha}{\pi}} \right) \\
&= ia_j \frac{8}{3}\sqrt{\frac{\alpha}{\pi}}
\end{aligned}$$

After using spherical coordinates for the second term one can easily see that the terms $l \neq j$ disappear. This is a consequence of the rotational invariance of the wave packet chosen.

Next we take a closer look at the matrix $\bar{\theta}_{\mu\nu}$ where the bar denotes the mean value of the Fock space operator $\hat{\theta}_{\mu\nu}$.

$$\bar{\theta}_{\mu\nu} = \frac{8}{3}\sqrt{\frac{\alpha}{\pi}} \begin{pmatrix} 0 & a^j \\ -a_j & \mathbf{0}_{3\times 3} \end{pmatrix}.$$

With the specific choice for α and a_j

$$\alpha = \pi\beta^2\theta \qquad a_j = \sqrt{\theta}(\tfrac{3}{8\beta})(1, \quad 0, \quad 0) \ , \qquad \beta \in \mathbb{R}^+,$$

we obtain the matrix of the Moyal-Weyl plane $\mathbb{R}^4_{\theta}$.

$$\bar{\theta}_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix $\bar{\theta}_{\mu\nu}$ represents the noncommutative matrix of the Moyal-Weyl case for space-time noncommutativity with space-space commutativity. $\square$

Thus we showed, that the Moyal-Weyl plane $\mathbb{R}^4_{\bar{\theta}}$ follows from an expectation value of the operator valued matrix $\hat{\theta}$. The conclusion was reached by adjusting the dimensional constants a^j and α in a particular way. To understand their physical nature one has to treat the wave packet (5.12) in the position space. This is done by calculating the Fourier transformation of the wave packet.

$$\begin{aligned}\mathscr{F}(\psi)(x) &= \int d^3p e^{-ix_kp^k} e^{ia_kp^k} f(\mathbf{p}) \\ &= \left(\frac{\alpha}{4\pi}\right)^{-3/4} e^{-\frac{1}{2\alpha}(\mathbf{x}-\mathbf{a})^2}\end{aligned}$$

The wave function in position space is a Gaussian wave packet, where the vector $\mathbf{a}$ characterizes the location of the peak of this Gaussian and the square root of the constant α denotes the standard deviation (or the width) of the wave packet, [Cla97]. From the expectation value (5.13) it is clear that the more the packet disperses, the bigger the uncertainty becomes.

5.4 Expectation value of massive coordinate operators

Lemma 5.5. *Let $\psi \in L^2(\mathbb{R}^3)$ and let the massive operator-valued matrix, (5.11) be given in the following form*

$$i\hat{\theta}^m_{0j} = \int d^3\mathbf{p}\tilde{a}^*(\mathbf{p}) \left(F_{0j}(\mathbf{p},m) + F^l_{0j}(\mathbf{p},m)\frac{\partial}{\partial p^l}\right)\tilde{a}(\mathbf{p}),$$

where the function valued matrices $F_{0j}(\mathbf{p},m)$ and $F^l_{0j}(\mathbf{p},m)$ are given by

$$\begin{aligned}F_{0j}(\mathbf{p},m) &= \frac{p_j}{2}\left(\frac{1}{\omega^3_{\boldsymbol{p}}} + \frac{1}{\omega_{\boldsymbol{p}} p^i p_i} + \frac{2\omega_{\boldsymbol{p}}}{(p^ip_i)^2}\right), \\ F^l_{0j}(\mathbf{p},m) &= \frac{\omega_{\boldsymbol{p}}}{p^kp_k}\delta^l_j - \frac{p_jp^l}{\omega_{\boldsymbol{p}} p^kp_k} - 2\frac{\omega_{\boldsymbol{p}} p_jp^l}{(p^kp_k)^2}.\end{aligned}$$

Then, the expectation value of $i\hat{\theta}^m_{0j}$, w.r.t. the state ψ is given as follows,

$$\langle\psi|i\hat{\theta}^m_{0j}|\psi\rangle = \int\left(\left(F_{0j}(\mathbf{p},m) - \partial^{\mathbf{p}}_l F^l_{0j}(\mathbf{p},m)\right)|\psi(\mathbf{p})|^2 - F^l_{0j}(\mathbf{p},m)\psi(\mathbf{p})\frac{\partial}{\partial p^l}\psi^*(\mathbf{p})\right)d^3p.$$

Proof. The proof is equivalent to the proof of lemma 5.3. $\square$

In the next step we give a lemma about the expectation value of the massive noncommutative matrix valued operator $i\hat{\theta}^m_{\mu\nu}$ with the same wave function we used in Lemma 5.4.

Lemma 5.6. *Let the wave function $\Psi_a(\mathbf{p}) \in L^2(\mathbb{R}^3)$ be given as*

$$\Psi_a(\mathbf{p}) = e^{ia_k p^k} f(\mathbf{p}), \qquad f(\mathbf{p}) = (\frac{\alpha}{\pi})^{3/4} e^{-\frac{\alpha}{2}|\mathbf{p}|^2},$$

where the L^2-norm of Ψ_a is equal to one, i.e. $||\Psi_a||^2 = 1$. Then, the expectation value of $i\hat{\theta}^m_{\mu\nu}$ given in (5.11), w.r.t. Ψ_a in the limit $\alpha \to \frac{1}{m^2}$ is given as follows

$$\langle\psi_a|i\hat{\theta}^m_{\mu\nu}|\psi_a\rangle = i\frac{\rho}{3m}\begin{pmatrix} 0 & a^j \\ -a_j & \mathbf{0}_{3\times 3}\end{pmatrix}, \qquad \rho \in \mathbb{R}^+.$$

Furthermore, for the following choice of the dimensional physical constant vector a_j

$$a_j = \sqrt{\theta}(\tfrac{3}{\rho})(1, \quad 0, \quad 0)\ , \qquad \beta \in \mathbb{R}^+,$$

one obtains the Moyal-Weyl plane $\mathbb{R}^4_\theta$, where it is characterized by the deformation parameter $\sqrt{\theta}/m$.

Proof. By using Lemma 5.5 the mean value of $i\hat{\theta}^m_{0j}$ is given by

$$\langle\psi|i\hat{\theta}^m_{0j}|\psi\rangle = +ia_l \int F^l_j(\mathbf{p})|f(p)|^2 d^3p$$

Let us look at the first part of $i\hat{\theta}^m_{0j}$

$$\begin{aligned}\langle\psi|i\hat{\theta}^{m,1}_{0j}|\psi\rangle &= +ia_l\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\delta^l_j\int d^3p\frac{\omega_{\mathbf{p}}}{p^k p_k}e^{-\alpha|\mathbf{p}|^2}\\ &= \frac{2}{m}ia_j\gamma,\end{aligned}$$

where in the last lines spherical coordinates were used to simplify the calculation and we took the limit $\alpha \to \frac{1}{m^2}$. Note that γ is some finite constant. The second part of the expectation value of $i\hat{\theta}^m_{0j}$ is given as follows

$$\begin{aligned}\langle\psi|i\hat{\theta}^{m,2}_{0j}|\psi\rangle &= -ia_l\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int d^3p\frac{p_j p^l}{\omega_{\mathbf{p}} p^k p_k}e^{-\alpha|\mathbf{p}|^2}\\ &= ia_j\alpha^{3/2}\frac{m^2}{3}\mathrm{HypgeoU}[3/2,2,\beta], \qquad \beta = \frac{\alpha}{m^2}\\ &= ia_j\frac{g}{3m}.\end{aligned}$$

Due to the rotational invariance all the terms where $j \neq l$ are equal to zero. Moreover, in the last step we took the limit $\alpha \to \frac{1}{m^2}$. Note that g is a finite constant.

The last part of the expectation value of $i\hat{\theta}^m_{0j}$ is given as

$$\begin{aligned}\langle\psi|i\hat{\theta}^{m,3}_{0j}|\psi\rangle &= -2ia_l\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int d^3p\frac{\omega_{\mathbf{p}} p_j p^l}{(p^k p_k)^2}exp^{-\alpha|\mathbf{p}|^2}\\ &= ia_j m^2(\alpha)^{\frac{3}{2}}\frac{4}{3}\mathrm{HypgeoU}[1/2,2,\beta]\\ &= ia_j\frac{4}{3m}h.\end{aligned}$$

As before due to rotation invariance all terms where $j \neq l$ are equal to zero. Furthermore, in the last step we took the limit $\alpha \to \frac{1}{m^2}$. Again h is a finite constant.

Picking up all terms the matrix $i\hat{\theta}^m_{0j}$ takes the following form

$$\langle\psi|i\hat{\theta}^m_{0j}|\psi\rangle = \langle\psi|i\hat{\theta}^{m,1}_{0j}|\psi\rangle + \langle\psi|i\hat{\theta}^{m,2}_{0j}|\psi\rangle + \langle\psi|i\hat{\theta}^{m,3}_{0j}|\psi\rangle = ia_j\frac{\rho}{3m}.$$

$\square$

As one can see, the main difference in the expectation value of the noncommutative matrix valued operator $\hat{\theta}_{\mu\nu}$, lies in the deformation parameter when compared to a Moyal-Weyl plane. In the massive case the deformation parameter depends on the mass. As in the massless case the physical interpretation of the dimensional constant vector **a** and the constant m can be deduced by calculating the wave function in position space. Again, it is a Gaussian wave packet, where the vector **a** characterizes the location of the peak of this Gaussian and the square root of the mass m denotes the width of the wave packet, [Cla97].

5.5 Deformation of coordinate operator

In the context of QM (see Section 3.4.4) we deformed the coordinate operator with the momentum operator and obtained a NC quantum plane. This plane can be well understood in view of the fact, that coordinate operators describing the Landau problem are exactly the coordinate operators obtained by deformation with the momentum operators. In the present section we follow the idea found in QM (see Lemma 3.13) and deform our quantum field theoretical coordinate operators with the momentum. Hence, we calculate the commutator of the deformed coordinate operators and call the resulting quantum plane, the **QFT-Moyal-Weyl**.

Before deforming we calculate the unitary transformation of the coordinate operators under translations. This is done on an operator-valued distributional level, by considering the unitary transformation of the translations on particle creation and annihilation operators. Let us first define the unitary operator of translations as follows,

$$U(\beta) := e^{i\beta_\mu P^\mu}. \tag{5.14}$$

The unitary operator $U(\beta)$ with $\beta \in \mathbb{R}^d$ transforms the particle creation and annihilation operators $\tilde{a}, \tilde{a}^*$ in the following way, [IZ80, Sib93]

$$U(\beta)\tilde{a}(\mathbf{p})U(\beta)^{-1} = e^{-i\beta_\mu p^\mu}\tilde{a}(\mathbf{p}), \qquad U(\beta)\tilde{a}^*(\mathbf{p})U(\beta)^{-1} = e^{i\beta_\mu p^\mu}\tilde{a}(\mathbf{p}). \tag{5.15}$$

The transformation property (5.15) is used in the next lemma to calculate the unitary transformations of the coordinate operator.

Lemma 5.7. *Under the adjoint action of the unitary transformation $U(\beta) := e^{i\beta_\mu P^\mu}$, with $\beta \in \mathbb{R}^d$, the massless spatial coordinate operator transforms as follows,*

$$\alpha_\beta\left(X_j\right) = e^{i\beta_\mu P^\mu} X_j e^{-i\beta_\mu P^\mu} = X_j + \beta_0 V_j - \beta_j N,$$

and the massless temporal coordinate operator X_0 transforms in the following way,

$$\alpha_\beta\left(X_0\right) = X_0 - \beta_0 N - \beta_j V^j.$$

Proof. By using the transformation property (see Equation (5.15)) of the creation and annihilation operator under the unitary operator $U(\beta)$, we obtain for the spatial component of the coordinate operators the following,

$$\begin{aligned}
e^{i\beta_\mu P^\mu} X_j e^{-i\beta_\mu P^\mu} &= -i\int d^n\mathbf{p} e^{i\beta_\mu P^\mu}\tilde{a}^*(\mathbf{p})e^{-i\beta_\mu P^\mu}\partial_j\left(e^{i\beta_\mu P^\mu}\tilde{a}(\mathbf{p})e^{-i\beta_\mu P^\mu}\right)\\
&= -i\int d^n\mathbf{p} e^{i\beta_\mu p^\mu}\tilde{a}^*(\mathbf{p})\partial_j\left(e^{-i\beta_\mu p^\mu}\tilde{a}(\mathbf{p})\right)\\
&= X_j - \beta_\mu\int d^n\mathbf{p}(\partial_j p^\mu)\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p})\\
&= X_j - \beta_0\int d^n\mathbf{p}(-\frac{p_j}{\omega_\mathbf{p}})\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) - \beta_k\int d^n\mathbf{p}\delta_j^k\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p})\\
&= X_j + \beta_0 V_j - \beta_j N.
\end{aligned}$$

As in the proof for the spatial components we use equation (5.15) and obtain for the temporal operator the following

$$\begin{aligned}
e^{i\beta_\mu P^\mu} X_0 e^{-i\beta_\mu P^\mu} &= -i \int d^n\mathbf{p} \left(\frac{n-1}{2\omega_{\mathbf{p}}} \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) + e^{i\beta_\mu p^\mu} \tilde{a}^*(\mathbf{p}) \frac{p^j}{\omega_{\mathbf{p}}} \partial_j \left(e^{-i\beta_\mu p^\mu} \tilde{a}(\mathbf{p}) \right) \right) \\
&= X_0 - \beta_\mu \int d^n\mathbf{p} \left(\frac{p^j}{\omega_{\mathbf{p}}} \partial_j p^\mu \right) \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= X_0 - \beta_0 \int d^n\mathbf{p} (-\frac{p_j p^j}{\omega_{\mathbf{p}}^2}) \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) - \beta_j \int d^n\mathbf{p} \frac{p^j}{\omega_{\mathbf{p}}} \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= X_0 - \beta_0 N - \beta_j V^j .
\end{aligned}$$

□

The next lemma gives the transformation of the massive coordinate operators under the adjoint action of translations.

Lemma 5.8. *Under the adjoint action of the unitary transformation* $U(\beta) := e^{i\beta_\mu P^\mu}$*, with* $\beta \in \mathbb{R}^d$*, the massive spatial coordinate operator transforms as in the massless case,*

$$\alpha_\beta (X_j) = e^{i\beta_\mu P^\mu} X_j e^{-i\beta_\mu P^\mu} = X_j + \beta_0 V_j - \beta_j N.$$

The massive temporal coordinate operator X_0 *transforms in the following manner,*

$$\alpha_\beta (X_0) = e^{i\beta_\mu P^\mu} X_0 e^{-i\beta_\mu P^\mu} = X_0 - \beta_0 N + \beta_j \tilde{V}^j .$$

Proof. The proof for the massive spatial part of the coordinate operator is equivalent to the massless case. For the massive temporal operator we use equation (5.15) and obtain the following

$$\begin{aligned}
e^{i\beta_\mu P^\mu} X_0 e^{-i\beta_\mu P^\mu} &= X_0 + \beta_\mu \int d^n\mathbf{p} \left(\frac{p^j \omega_{\mathbf{p}}}{p_i p^i} \partial_j p^\mu \right) \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= X_0 - \beta_0 \int d^n\mathbf{p} (\frac{p_j p^j}{p_i p^i}) \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) + \beta_j \int d^n\mathbf{p} \frac{p^j \omega_{\mathbf{p}}}{p_i p^i} \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= X_0 - \beta_0 N + \beta_j \tilde{V}^j .
\end{aligned}$$

□

5.5.1 QFT-Moyal-Weyl from deformation

As for the deformation in QM, in QFT-case we are also obliged to prove that warped convolutions of the coordinate operator with the momentum operator is well-defined. Thus let us consider the deformed coordinate operator as follows

$$\begin{aligned}
\langle \Psi_l, X^\mu_{\theta,P} \Phi_l \rangle &= (2\pi)^{-4} \lim_{\epsilon\to 0} \iint d^4y\, d^4k\, e^{-iyk}\, \chi(\epsilon y, \epsilon k) \langle \Psi_l, U(k)\alpha_{\theta y}(X^\mu)\Phi_l \rangle \\
&=: (2\pi)^{-4} \lim_{\epsilon\to 0} \iint d^4y\, d^4k\, e^{-iyk}\, \chi(\epsilon y, \epsilon k)\, b^\mu(k, y)
\end{aligned}$$

for $\Psi, \Phi \in \mathscr{S}(\mathbb{R}^{3l})$.

Lemma 5.9. *Let the function $b^{\mu}(y,k)$, for $\mu = 0,\ 1,\ 2,\ 3,$ be given as the scalar product $\langle \Psi, U(k)\alpha_{\theta y}(X^j)\Phi\rangle$. Then $b^{\mu}(y,k) \in S^1_{1,0}$, for $\Psi_l, \Phi_l \in \mathscr{S}(\mathbb{R}^{3l})$ and thus the deformation via warped convolutions of the coordinate operator is given as a well-defined oscillatory integral.*

Proof. Let us first look at the function b^{μ} for $\mu = 0$,

$$\begin{aligned}
|\partial^{\alpha}_{k^i}\partial^{\beta}_{y^r} b^0(y,k)| &= |\langle \Psi_l, (\partial^{\alpha}_k U(k))\, \partial^{\beta}_y \left(\alpha_{\theta y}(X^0)\right)\Phi_l\rangle| \\
&\leq \underbrace{\|(-iP^{\alpha})\Psi_l\|}_{=:C_{l,\alpha}} \left\|\partial^{\beta}_y \left(X_0 - (\theta y)_0 N + (\theta y)_j \tilde{V}^j\right)\Phi_l\right\| \\
&\leq C_{l,\alpha}\left(\left\|\partial^{\beta}_{y^r} X^0 \Phi_l\right\| + |\partial^{\beta}_y (\theta y)_0|\, \|N\Phi_l\| + |\partial^{\beta}_y(\theta y)_j|\left\|\tilde{V}^j\Phi_l\right\|\right),
\end{aligned}$$

where in the last lines we used the adjoint action given in Lemma 5.8. By using inequality (3.7) for $\theta_{ij} = \epsilon_{ijk}\theta^k$ and for $\underline{\beta = 0}$ we obtain

$$\begin{aligned}
|\partial^{\alpha}_{k^i} b^0(y,k)| &\leq C_{l,\alpha}\left(\underbrace{\|X^0\Phi_l\|}_{=:C_0} + |\mathbf{y}| \underbrace{\left(|\theta_{0j}|\,\|N\Phi_l\| + \sqrt{2}|\boldsymbol{\theta}|\left\|\tilde{V}^j\Phi_l\right\|\right)}_{=:C_6} + |y_0|\underbrace{|\theta_{0r}|\left\|\tilde{V}^j\Phi_l\right\|}_{=:C_7}\right) \\
&\leq C_{l,\alpha}\left(C_0 + C_6|\mathbf{y}| + C_7|y_0|\right) \\
&\leq \underbrace{C_{l,\alpha}C_D}_{=:C_{l,\alpha,\theta}}\left(1 + |\mathbf{y}| + |y_0|\right),
\end{aligned}$$

where in the last lines we assumed there exists a constant C_D satisfying

$$C_D \geq C_0, C_6, C_7. \tag{5.16}$$

Due to the fact that C_0, C_6 and C_7 are finite constants, the existence of a constant satisfying inequality (5.16) is justified and therefore $b^0(y,k)$ belongs to the symbol class $S^1_{1,0}$.

Next we show that function $b^{\mu} \in S^1_{1,0}$ for $\mu = j = 1,2,3$,

$$\begin{aligned}
|\partial^{\alpha}_{k^i}\partial^{\beta}_{y^r} b^j(y,k)| &= |\langle \Psi_l, (\partial^{\alpha}_k U(k))\, \partial^{\beta}_y \left(\alpha_{\theta y}(X^i)\right)\Phi_l\rangle| \\
&\leq \underbrace{\|(-iP^{\alpha})\Psi_l\|}_{=:C_{l,\alpha}} \left\|\partial^{\beta}_y \left(X^j + (\theta y)_0 V^j - (\theta y)^j N\right)\Phi_l\right\| \\
&\leq C_{l,\alpha}\left(\left\|\partial^{\beta}_{y^r} X^j \Phi_l\right\| + |\partial^{\beta}_y (\theta y)_0|\, \left\|V^j\Phi_l\right\| + |\partial^{\beta}_y(\theta y)^j|\,\|N\Phi_l\|\right),
\end{aligned}$$

where in the last lines we used the adjoint action given in Lemma 5.8. By using inequality

(3.7) for $\theta_{ij} = \epsilon_{ijk}\theta^k$ and for $\underline{\beta = 0}$ we obtain

$$\begin{aligned}
|\partial^{\alpha}_{k^i} b^0(y,k)| &\leq C_{l,\alpha} \left(\underbrace{\left\|X^j\Phi_l\right\|}_{=:C_2} + |\mathbf{y}| \underbrace{\left(|\theta_{0k}| \left\|V^j\Phi_l\right\| + \sqrt{2}|\boldsymbol{\theta}| \left\|N\Phi_l\right\|\right)}_{=:C_6} + |y_0| \underbrace{|\theta^{0j}| \left\|N\Phi_l\right\|}_{=:C_7} \right) \\
&\leq C_{l,\alpha} \left(C_2 + C_6|\mathbf{y}| + C_7|y_0|\right) \\
&\leq \underbrace{C_{l,\alpha}C_D}_{=:C_{l,\alpha,\theta}} \left(1 + |\mathbf{y}| + |y_0|\right),
\end{aligned}$$

where in the last lines we the existence of a constant C_D satisfying

$$C_D \geq C_0, C_6, C_7.$$

is given by fact that C_0, C_6 and C_7 are finite constants and therefore $b^\mu(y,k)$ belongs to the symbol class $S^1_{1,0}$. As before, by the virtue of the theorem given in [Hör04, Theorem 7.8.2] it follows that the oscillatory integral is well-defined in a distributional sense.

□

Next we turn to the deformation of the constructed massless coordinate operators with the momentum operator.

Lemma 5.10. *The deformed massless coordinate operators $X^0_{\theta,P}$ and $X^j_{\theta,P}$, obtained by warped convolutions, are given by*

$$X^0_{\theta,P} = X^0 - (\theta P)^0 \, N - (\theta P)^j \, V_j, \qquad X^j_{\theta,P} = X^j + (\theta P)^0 \, V^j - (\theta P)^j \, N.$$

Proof. The proof makes use of defining equation (2.1) and Lemma 5.7. Let $\mathcal{D}$ be a dense domain of vectors smooth w.r.t. the action of U. By equation (2.1) the warped convolutions of X_0 is given on $\Psi \in \mathcal{D}$ as follows,

$$\begin{aligned}
X^0_{\theta,P}\Psi &= \int \alpha_{\theta k}(X^0) dE(k)\Psi \\
&= \int \left(X^0 - (\theta k)^0 N - (\theta k)^j V_j\right) dE(k)\Psi \\
&= (X^0 - (\theta P)^0 \, N - (\theta P)^j \, V_j)\Psi,
\end{aligned}$$

where in the last lines we used Lemma 5.7 and integrated over the projection valued measure. The same calculation can be performed for the spatial coordinate operator X_j on $\Psi \in \mathcal{D}$ as follows,

$$\begin{aligned}
X^j_{\theta,P}\Psi &= \int \alpha_{\theta k}(X^j) dE(k)\Psi \\
&= \int \left(X^j + \beta_0 V^j - \beta^j N\right) dE(k)\Psi \\
&= \left(X^j + (\theta P)^0 \, V^j - (\theta P)^j \, N\right) \Psi,
\end{aligned}$$

□

Lemma 5.11. *The deformed massive coordinate operators $X^0_{\theta,P}$ and $X^j_{\theta,P}$, obtained by warped convolutions, are given by*

$$X^0_{\theta,P} = X^0 - (\theta P)^0 N + (\theta P)^j \tilde{V}_j,$$

and the deformed spatial operator is given as follows

$$X^j_{\theta,P} = X^j + (\theta P)^0 V^j - (\theta P)^j N.$$

Proof. The proof for the spatial part is equivalent to the proof of the spatial part in Lemma 5.10. As before, let $\mathcal{D}$ be a dense domain of vectors smooth w.r.t. the action of U. Then, the warped convolutions of massive temporal operator is given on $\Psi \in \mathcal{D}$ as follows,

$$\begin{aligned} X^0_{\theta,P}\Psi &= \int \alpha_{\theta k}(X^0) dE(k)\Psi \\ &= \int \left(X^0 - (\theta k)^0 N - (\theta k)^j \int d^n\mathbf{p} \frac{p_j \omega_{\mathbf{p}}}{p_i p^i} \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \right) dE(k)\Psi \\ &= \left(X^0 - (\theta P)^0 N + (\theta P)^j \tilde{V}_j \right) \Psi, \end{aligned}$$

□

We first take the massless deformed coordinate operators $X^\mu_{\theta,P}$ and calculate the commutator.

Lemma 5.12. *The commutator between the deformed temporal and spatial coordinate operator, which were given in Lemma 5.10, is the following*

$$[X^0_{\theta,P}, X^j_{\theta,P}] = i\hat{\theta}^{0j} - 2i\theta^{0j}N^2 - 2i\theta^{0k}V^j V_k + 2i\theta^{jk}V_k N - i\,(\theta P)^k\, P^j_k,$$

where the operator P^j_k is given in terms of creation and annihilation operators as

$$P^j_k = \int d^n\mathbf{p} \frac{1}{\omega_{\boldsymbol{p}}} \left(\eta^j_k + \frac{p^j p_k}{\omega^2_{\boldsymbol{p}}} \right) \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}).$$

The commutator between the components of the deformed spatial coordinate operator is given in the following,

$$[X^j_{\theta,P}, X^k_{\theta,P}] = -2i \left(\theta^{0j} V^k - \theta^{0k} V^j \right) N + 2i\theta^{jk} N^2$$

Proof. We use the algebra of the massless coordinate operators, the algebra of the momentum operators and the fact that the particle number operator N commutes with X^μ and

P^ν to calculate the commutator.

$$\begin{aligned}
[X^0_{\theta,P}, X^j_{\theta,P}] &= [X^0 - (\theta P)^0\, N - (\theta P)^k\, V_k, X^j + (\theta P)^0\, V^j - (\theta P)^j\, N] \\
&= [X^0, X^j] - [(\theta P)^0\, N, X^j] - [(\theta P)^k\, V_k, X^j] + [X^0, (\theta P)^0\, V^j] - [X^0, (\theta P)^j\, N] \\
&= i\hat{\theta}^{0j} - \theta^{0k}[P_k, X^j]N - [(\theta P)^k\,, X^j]V_k - (\theta P)^k \underbrace{[V_k, X^j]}_{=:iP^j_k} + (\theta P)^0 \underbrace{[X^0, V^j]}_{=0} + \\
&+ \theta^{0k}[X^0, P_k]V^j - \theta^{j0}[X^0, P_0]N - \theta^{jk}[X^0, P_k]N \\
&= i\hat{\theta}^{0j} - i\theta^{0k}\eta^j_k N^2 - \theta^{k0}[P_0, X^j]V_k - \theta^{kr}[P_r, X^j]V_k - i\,(\theta P)^k\, P^j_k - \\
&- i\theta^{0k}V_kV^j + i\theta^{j0}N^2 + i\theta^{jk}V_kN \\
&= i\hat{\theta}^{0j} - i\theta^{0j}N^2 - i\theta^{k0}V^jV_k - i\theta^{kr}\eta^j_r V_kN - i\,(\theta P)^k\, P^j_k - \\
&- i\theta^{0k}V_kV^j + i\theta^{j0}N^2 + i\theta^{jk}V_kN \\
&= i\hat{\theta}^{0j} - 2i\theta^{0j}N^2 - 2i\theta^{0k}V^jV_k + 2i\theta^{jk}V_kN - i\,(\theta P)^k\, P^j_k
\end{aligned}$$

In the last lines we used the skew-symmetry of θ with respect to the Minkowski metric, i.e. $\theta_{0j} = \theta_{j0}$, $\theta_{kj} = -\theta_{jk}$. Next, we calculate the commutator of the deformed spatial operator.

$$\begin{aligned}
[X^j_{\theta,P}, X^k_{\theta,P}] &= [X^j + (\theta P)^0\, V^j - (\theta P)^j\, N, X^k + (\theta P)^0\, V^k - (\theta P)^k\, N] \\
&= [X^j, (\theta P)^0\, V^k] - [X^j, (\theta P)^k\, N] - j \leftrightarrow k \\
&= (\theta P)^0\, [X^j, V^k] + [X^j, (\theta P)^0]V^k - [X^j, (\theta P)^k]N - j \leftrightarrow k \\
&= -i\,(\theta P)^0\, P^{jk} + \theta^{0r}[X^j, P_r]V^k - \theta^{k0}[X^j, P_0]N - \theta^{kr}[X^j, P_r]N - j \leftrightarrow k \\
&= -i\theta^{0j}V^kN + i\theta^{k0}V^jN - i\theta^{kj}N - j \leftrightarrow k \\
&= -2i\left(\theta^{0j}V^k - \theta^{0k}V^j\right)N + 2i\theta^{jk}N^2
\end{aligned}$$

In the last lines we used the symmetry of P^{jk} and the skew-symmetry of θ w.r.t the Minkowski metric. □

For the massive deformed coordinate operator the commutator is calculated in an analogous manner and it is subject of the following lemma.

Lemma 5.13. *Let $X^\mu_{\theta,P}$ be the massive deformed coordinate operator given in Lemma 5.11. Then, the commutator between the deformed massive temporal and spatial coordinate operator is given as*

$$[X^0_{\theta,P}, X^j_{\theta,P}] = i\hat{\theta}^{0j} - 2i\theta^{0j}N^2 + 2i\theta^{0k}\tilde{V}_kV^j - 2i\theta^{jk}\tilde{V}_kN - i\,(\theta P)^k\,\tilde{P}^j_k + i\,(\theta P)^0\, R^j,$$

where the operator $\tilde{P}^j_k$ and R^j are given in terms of creation and annihilation operators as

$$\tilde{P}^j_k = \int d^n\mathbf{p}\left(\eta^j_k\frac{\omega_{\boldsymbol{p}}}{p^ip_i} - \frac{p^jp_k}{\omega_{\boldsymbol{p}}p^ip_i} - \frac{2\omega_{\boldsymbol{p}}p^jp_k}{(p^ip_i)^2}\right)\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}),$$

$$R^j = \int d^n\mathbf{p}\frac{p^jm^2}{\omega^2_{\boldsymbol{p}}p_ip^i}\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}).$$

The commutator between the components of the deformed spatial coordinate operator is the following one,

$$[X^j_{\theta,P}, X^k_{\theta,P}] = -2i\left(\theta^{0j}V^k - \theta^{0k}V^j\right)N + 2i\theta^{jk}N^2$$

Proof.

$$\begin{aligned}
[X^0_{\theta,P}, X^j_{\theta,P}] &= [X^0 - (\theta P)^0 N - (\theta P)^k \tilde{V}_k, X^j + (\theta P)^0 V^j - (\theta P)^j N] \\
&= [X^0, X^j] - [(\theta P)^0 N, X^j] - [(\theta P)^k \tilde{V}_k, X^j] + [X^0, (\theta P)^0 V^j] - [X^0, (\theta P)^j N] \\
&= i\hat{\theta}^{0j} - \theta^{0k}[P_k, X^j]N - [(\theta P)^k, X^j]\tilde{V}_k - (\theta P)^k \underbrace{[\tilde{V}_k, X^j]}_{=:i\tilde{P}^j_k} + (\theta P)^0 \underbrace{[X^0, V^j]}_{=:iR^j} + \\
&\quad + \theta^{0k}[X^0, P_k]V^j - \theta^{j0}[X^0, P_0]N - \theta^{jk}[X^0, P_k]N \\
&= i\hat{\theta}^{0j} - i\theta^{0j}N^2 - \theta^{k0}[P_0, X^j]\tilde{V}_k - \theta^{kr}[P_r, X^j]\tilde{V}_k - i(\theta P)^k \tilde{P}^j_k + \\
&\quad + i(\theta P)^0 R^j + i\theta^{0k}\tilde{V}_k V^j + i\theta^{j0}N^2 - i\theta^{jk}\tilde{V}_k N
\end{aligned}$$

$$\begin{aligned}
&= i\hat{\theta}^{0j} - i\theta^{0j}N^2 - i\theta^{k0}V^j\tilde{V}_k - i\theta^{kj}\tilde{V}_k N - i(\theta P)^k \tilde{P}^j_k + \\
&\quad + i(\theta P)^0 R^j + i\theta^{0k}\tilde{V}_k V^j + i\theta^{j0}N^2 - i\theta^{jk}\tilde{V}_k N \\
&= i\hat{\theta}^{0j} - 2i\theta^{0j}N^2 + 2i\theta^{0k}\tilde{V}_k V^j - 2i\theta^{jk}\tilde{V}_k N - i(\theta P)^k \tilde{P}^j_k + i(\theta P)^0 R^j
\end{aligned}$$

The proof for the spatial components is equivalent to the proof of Lemma 5.12, due to the fact that the commutation relations between X_j and the momentum operator do not change in the massive case. □

In this section we obtained a quantum plane by deforming the coordinate operators, constructed by demanding a canonical conjugate structure. Thus, following the same idea obtained in the QM context (see Section 3.4.4), the commutator gives us a quantum plane that is obtained by the deformed coordinate operators of QFT.

This is the main difference to other works on NCQFT. In this work we constructed coordinate operators with the underlying QFT by demanding the canonical commutation relations to be fulfilled. In the next step, we deform these constructed operators and calculate the commutator in order to obtain the quantum plane. The deformation is justified by the fact that in the QM case for the example of the Landau problem, we obtained the coordinate operators of the Landau problem by deforming the coordinate operators of a free particle using the momentum operator.

5.6 Self-adjointness of the position operator

In this section we study the essential self-adjointness of the spatial coordinate operator X_j. By using the Fourier transformation we show that the momentum operator is unitarily equivalent to the coordinate operator. The operator P_j is an essential self-adjoint operator on a dense domain and thus by unitarily equivalence it follows that the coordinate operator is an essential self-adjoint operator as well. This section is written in such a general manner that the considerations hold for the massless and massive case.

For the proof of self-adjointness of the coordinate operator, we work in the non-covariant representation of the particle creation and annihilation operators. The operator pair (P_k, X_l) satisfying (5.4) take in the non-covariant representation the following form

$$P_j = \int d^n\mathbf{p}\, p_j \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}), \qquad X_j = -i\int d^n\mathbf{p}\,\tilde{a}^*(\mathbf{p})\frac{\partial}{\partial p^j}\tilde{a}(\mathbf{p}). \tag{5.17}$$

Now in the following steps we first define the domain of essential self-adjointness of the momentum operator. On this dense domain we show that the momentum operator P_j is unitarily equivalent to the coordinate operator for the one particle subspace $\mathscr{H}_1 = L^2(\mathbb{R}^n)$. Further on, we generalize the lemma of unitarily equivalence to the k-particle subspace $\mathscr{H}_k$.

The following lemma gives the domain of essential self-adjointness of the momentum operator, [BEH08, RS75a].

Lemma 5.14. *The momentum operator P_k is an essential self-adjoint operator on the dense domain $\mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.*

Proof. The proof makes use of Corollary 3.1. □

The unitary equivalence of the operator pair (P_j, X_l) is subject of the following lemma.

Lemma 5.15. *Let $U_{\mathscr{F}}$ be the unitary operator of the Fourier transform (see Appendix (8.5)) and let the momentum operator P_j and some operator Q_l be acting on a function $\varphi(\boldsymbol{p}) \in \mathscr{S}(\mathbb{R}^n)$ as follows*

$$(P_j\varphi)(\boldsymbol{p}) = p_j\varphi(\boldsymbol{p}), \qquad (Q_l\varphi)(\boldsymbol{p}) = -i\frac{\partial}{\partial p^l}\varphi(\boldsymbol{p}). \tag{5.18}$$

Then, the operator Q_j is unitarily equivalent to the operator P_j and the equivalence is given by the following equation

$$Q_j = U_{\mathscr{F}}^{-1} P_j U_{\mathscr{F}}. \tag{5.19}$$

Furthermore, the operator Q_j is an essentially self-adjoint operator on the dense domain $\mathscr{S}(\mathbb{R}^n)$.

Proof. This lemma can be found as an example in [BEH08]. Due to the fact that in the next step we extend it to the k-particle subspace, it is important to consider the proof for a one particle subspace. First note that the proof of unitarity for the Fourier transform can be found in [BEH08, RS75a, RS75b, SW89, SV73]. To show that unitarily equivalence (5.19) holds, it suffices to show

$$U_{\mathscr{F}}\left(Q_j\varphi\right)(\mathbf{p}) = \left(P_j U_{\mathscr{F}}\varphi\right)(\mathbf{p}).$$

This can be done by making use of Equation (8.1).

$$U_{\mathscr{F}}\left(Q_j\varphi\right)(\mathbf{p}) = U_{\mathscr{F}}\left(-i\partial_j\varphi\right)(\mathbf{p}) = p_j\left(U_{\mathscr{F}}\varphi\right)(\mathbf{p}) = \left(P_j U_{\mathscr{F}}\varphi\right)(\mathbf{p})$$

By applying on both sides the inverse operator $U_{\mathscr{F}}^{-1}$ one obtains the equivalence (5.19). By the virtue of Proposition 8.1, from the unitary equivalence the essential self-adjointness of Q_j follows. The domain of operator Q_j is the same as the domain of P_k, i.e. $D(\mathbf{P}) = D(\mathbf{Q}) = \mathscr{S}(\mathbb{R}^n)$. The statement holds, since the Fourier transform is a bijective operator which maps $\mathscr{S}(\mathbb{R}^n)$ into itself, (see 8.1, 8.5). Thus by Proposition 8.1 for the domains we have,

$$D(\mathbf{Q}) = U_{\mathscr{F}}^{-1}D(\mathbf{P}) = U_{\mathscr{F}}^{-1}\mathscr{S}(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n).$$

□

We intend to extend the essentially self-adjointness of the operator Q_j to the entire Bosonic Fock space. To do so, we first define the operator of second quantization and give a lemma concerning its domain of essential self-adjointness, [RS75a, Chapter VIII.10, Example 2] and [RS75b, Chapter X.7].

Definition 5.3. Let $\mathscr{H}^+$ be the Fock space over $\mathscr{H}_1$ and suppose that A is a self-adjoint operator on $\mathscr{H}_1$ with domain of essential self-adjointness $D(A)$. Let the following operator $A^{(n)} = A \otimes I \cdots \otimes I + \cdots + I \otimes I \cdots \otimes A$ be defined on $D(A)_{\otimes_k} := \bigotimes_{i=1}^k D(A)$ and define $A^{(0)} = 0$. Then, the operator $\mathbf{d\Gamma(A)}$ on $\mathscr{H}^+$ defined as

$$d\Gamma(A) := \sum_{k=0}^{\infty} A^{(k)},$$

is called the **second quantization** of A. Furthermore, if U is a unitary operator on $\mathscr{H}_1$ we define $\Gamma(U)$ to be second quantization of U, i.e. the unitary operator on $\mathscr{H}^+$ which equals $\bigotimes_{i=1}^k U$ when restricted to $\mathscr{H}_n$ for $n > 0$ and equal to identity on $\mathscr{H}_0$.

Next, we define the domain on which the second quantization of the essential self-adjoint operator $d\Gamma(A)$ is symmetric.

Definition 5.4. Let A be an essential self-adjoint operator with dense domain $D(A) \subset \mathscr{H}_1$. Then, the domain of the second quantization operator $d\Gamma(A)$ of A, denoted as $D(A)_{\otimes} \subset \mathscr{H}^+$, is defined to be the set of $\psi = \{\psi_0, \psi_1, \cdots\}$ such that $\psi_k = 0$ for k large enough and $\psi_k \in \bigotimes_{i=1}^k D(A)$ for each k.

The domain of the second quantized operator $D(A)_{\otimes}$ is dense in $\mathscr{H}^+$, since $D(A)$ is dense in $\mathscr{H}_1$. By using the definition of the extended domain $D(A)_{\otimes}$, one obtains the following lemma.

Lemma 5.16. *Let A be an essential self-adjoint operator with domain $D(A)$. Then, the second quantized operator $d\Gamma(A)$ is an essential self-adjoint operator on the domain $D(A)_{\otimes}$, as given in Definition 5.4.*

Proof. The lemma can be proven using Theorem 8.3. □

In the former definitions and lemma we did not mention the projections onto the symmetric and antisymmetric Fock spaces. This is because the second quantization operator $d\Gamma(A)$ commutes with the projections and thus it is not important for the proof of essential self-adjointness.

The importance of Theorem 8.3 and Lemma 5.16 for our work is the following. Let A be an essential self-adjoint operator on a dense subspace of the one particle Hilbert space, then by Definition 5.3 one can define the second quantized version of the operator A on the dense domain $D(A)_\otimes$ of the Fock space. Furthermore, one can prove that on the domain $D(A)_\otimes$ the second quantized operator A is an essential self-adjoint operator. By using Definitions 5.3, 5.4 and Lemma 5.16, we obtain the following theorem.

Theorem 5.1. *The* ***second quantization*** *of Q_j given by the operator $d\Gamma(Q_j)$, (see Definition 5.3), is an* ***essential self-adjoint operator*** *on the dense domain $D(Q_j)_\otimes$ (see Definition 5.4). Furthermore, the* ***second quantized operator*** *$d\Gamma(Q_j)$ is on the Bosonic Fock space* ***equivalent*** *to the position operator X_j given in Equation (5.17). Therefore, on the dense domain $D(Q_j)_\otimes$, the position operator X_j is an essentially self-adjoint operator.*

Proof. That the second quantization operator $d\Gamma(Q_j)$ of Q_j is an essentially self-adjoint operator on the dense domain $D(Q_j)_\otimes$ follows directly from Lemma 5.16, where the domain of essential self-adjointness of Q_j is given as $D(Q_j) = \mathscr{S}(\mathbb{R}^n)$. To prove that $d\Gamma(Q_j)$ is equal to X_j, we first give the second quantized operator of the Fourier transformation $\Gamma(U_\mathscr{F})$. This is done by using the Definition 5.3 of the second quantized version of a unitary operator. As mentioned before, the second quantization commutes with the projections to the symmetric Fock spaces so that the unitary equivalence (see Equation 5.19) can be extended to the second quantized version given as,

$$d\Gamma(Q_j) = \Gamma(U_\mathscr{F}^{-1})P_j\Gamma(U_\mathscr{F}). \tag{5.20}$$

In the next step we calculate the action of $d\Gamma(Q_j)$ on the functions $\varphi, \psi \in \mathscr{S}(\mathbb{R}^{n\times k})$.

$$\begin{aligned}
&\langle\psi, d\Gamma(Q_j)\varphi\rangle \\
&= \langle\psi, \Gamma(U_\mathscr{F}^{-1})P_j\Gamma(U_\mathscr{F})\varphi\rangle = \langle\Gamma(U_\mathcal{F})\psi, P_j\Gamma(U_\mathscr{F})\varphi\rangle \\
&= \int d^n\mathbf{p}^1\cdots\int d^n\mathbf{p}^k\,(\Gamma(U_\mathscr{F})\psi)(\mathbf{p}^1,\cdots,\mathbf{p}^k)\,(P_j\Gamma(U_\mathscr{F})\varphi)(\mathbf{p}^1,\cdots,\mathbf{p}^k) \\
&= \int d^n\mathbf{p}^1\cdots\int d^n\mathbf{p}^k\,(\Gamma(U_\mathscr{F})\psi)(\mathbf{p}^1,\cdots,\mathbf{p}^k)\sum_{l=1}^k p_j^l\,(\Gamma(U_\mathscr{F})\varphi)(\mathbf{p}^1,\cdots,\mathbf{p}^k) \\
&= \int d^n\mathbf{p}^1\cdots\int d^n\mathbf{p}^k\,(\Gamma(U_\mathscr{F})\psi)(\mathbf{p}^1,\cdots,\mathbf{p}^k)\left(\Gamma(U_\mathscr{F})\sum_{l=1}^k\left(-i\partial_j^l\right)\varphi\right)(\mathbf{p}^1,\cdots,\mathbf{p}^k) \\
&= \int d^n\mathbf{p}^1\cdots\int d^n\mathbf{p}^k\psi(\mathbf{p}^1,\cdots,\mathbf{p}^k)\sum_{l=1}^k\left(-i\frac{\partial}{\partial p^{l,j}}\right)\varphi(\mathbf{p}^1,\cdots,\mathbf{p}^k) = \langle\psi, X_j\varphi\rangle.
\end{aligned}$$

In the last lines we first used the unitarity of the second quantized Fourier transformation and the action of operator P_k, given in (5.17) on k-particle wave functions. In the next step we used equation (8.6) for multiple particles and the action of the operator X_k, given in (5.17) on k-particle wave functions. The equivalence of $d\Gamma(Q_j)$ and the operator X_j holds for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^{n\times k})$. Thus the second quantized operator $d\Gamma(Q_j)$, is equivalent to X_j and therefore the position operator is an essential self-adjoint operator. $\square$

The essential statement of Theorem 5.1 is that the position operator X_j is unique. In particular, the coordinate operator defined in a quantum field theoretical setting is the generalization of the coordinate operator in quantum mechanics. We were able to show this fact, by defining the coordinate operator in the quantum mechanical case as a unitary equivalent operator to the momentum operator. In the next step, we second quantized this definition and were able to prove that the operator we obtain is equivalent to the coordinate operator given by [SS09]. In the next chapter we will compare this coordinate operator with the Newton-Wigner-Pryce operator.

5.7 The Newton-Wigner-Pryce operator

In the context of relativistic particles the Newton-Wigner-Pryce (NWP) operator is usually mentioned as the rightful position or center-of-mass operator. To obtain the position operator the authors in [NW49] imposed certain physical requirements on localized states. Where in [Pry48] the author obtained the same operator by generalizing the Newtonian definition of the mass-center to the relativistic case. The NWP operator can be given as the product of generators of the Poincaré group in the following way, [Bac93, Ber65, Can65, Jor80, Ste05]

$$X_j^{NWP} = \frac{1}{2P_0}M_{0j} + M_{0j}\frac{1}{2P_0}.$$

On a one particle wave function the position operator defined in this way is given as, [Sch61, Chapter 3c, Equation 35]

$$(X_j^{NWP}\varphi)(\mathbf{p}) = -i\left(\frac{p_j}{2\omega_{\mathbf{p}}^2} + \frac{\partial}{\partial p^j}\right)\varphi(\mathbf{p}).$$

Theorem 5.2. *The second quantization of the Newton-Wigner-Pryce operator is equivalent to the position operator X_j given in Equation (5.8).*

Proof. The second quantization of X_j^{NWP} is given by the operator $d\Gamma(X_j^{NWP})$ (defined in 5.3). To prove equivalence between the two operators we first calculate the action of $d\Gamma(X_j^{NWP})$ on a symmetric function $\Psi \in \mathscr{S}(\mathbb{R}^{n\times k})$.

$$\begin{aligned}\left(d\Gamma(X_j^{NWP})\Psi\right)(\mathbf{p^1},\dots,\mathbf{p^k}) &= \sum_{l=1}^{k} X_j^{NWP}(\mathbf{p^l})\Psi(\mathbf{p^1},\dots,\mathbf{p^k})\\ &= -i\sum_{l=1}^{k}\left(\frac{p_j^l}{2\omega_{\mathbf{p^l}}^2} + \partial_{\mathbf{p^l_j}}\right)\Psi(\mathbf{p^1},\dots,\mathbf{p^k})\\ &= \left(X_j\Psi\right)(\mathbf{p^1},\dots,\mathbf{p^k})\end{aligned}$$

In the last line we used the action of the position operator X_j, given in the covariant normalization (5.8), on a function $\Psi \in \mathscr{S}(\mathbb{R}^{n\times k})$. $\square$

A few comments are in order. First, the Newton-Wigner-Pryce operator is often given as the product of generators of the Poincaré group. This representation is only true for one-particle states and not for n-particle states. This is because the product of second quantized operators is not equal to the second quantized product of the operators, i.e. $d\Gamma(M_{0j}P_0^{-1}) \neq d\Gamma(M_{0j})d\Gamma(P_0^{-1})$. Therefore, the representation of the NWP-operator as the product of the boost operator and the inverse of the Hamiltonian must be discarded for an n-particle system.

Second, from Theorem 5.2 it follows that the Newton-Wigner-Pryce operator is equivalent to the position operator that we also obtained by second quantization of the position operator in quantum mechanical setting. This is a major result concerning the uniqueness of the relativistic position operator. The reason why this fact may have not been obvious, is owed to the representation of the operator. In second quantization from QM it is given in a non-covariant fashion and as the NWP-operator it is given in a covariant fashion. Of course, the difference of representation is merely a normalization feature and thus for the physical interpretation, and specially for the second quantization not relevant.

5.8 Unitary transformations of the scalar field

In this section we study unitary transformations defined by coordinate operators on the scalar field. The investigation leads to the conclusion that they act as translation operators in momentum space. Next, we define scalar fields with the help of the unitary transformations and investigate the locality properties of such transformed scalar fields.

5.8.1 Unitary transformations generated by position operator

By using Theorem 3.1 and the essential self-adjointness of the position operator shown in Theorem 5.1, we define a strongly continuous unitary group as follows

$$V(\mathbf{b}) := e^{ib_j X^j}, \qquad b_j \in \mathbb{R}^n. \tag{5.21}$$

Next, we calculate the adjoint action of the unitary operator $V(\mathbf{b})$ on the free massless scalar field. This can be done by calculating the adjoint action of $V(\mathbf{b})$ on the particle creation and annihilation operators. The following lemma gives the unitary transformation of $\tilde{a}, \tilde{a}^*$.

Lemma 5.17. *The particle creation and annihilation operators $\tilde{a}(\mathbf{p}), \tilde{a}^*(\mathbf{p})$ transform under the adjoint action of the unitary operator $V(b)$, defined in (5.21), in the following manner.*

$$V(\mathbf{b})\tilde{a}(\mathbf{p})V^{-1}(\mathbf{b}) = \tilde{a}(\mathbf{p}-\mathbf{b}), \qquad V(\mathbf{b})\tilde{a}^*(\mathbf{p})V^{-1}(\mathbf{b}) = \tilde{a}^*(\mathbf{p}-\mathbf{b}).$$

Proof. We will give three proofs for this lemma. The last two proofs help us to calculate the adjoint action of the creation and annihilation operators under the temporal operator X_0. Let us start with the first proof.

The proof starts with calculating the commutator of the spatial coordinate operator with creation and annihilation operators. By using the non-covariant canonical commutation relations between $\tilde{a}$ and $\tilde{a}^*$, we obtain

$$[X^r, \tilde{a}(\mathbf{p})] = +i\frac{\partial}{\partial p_r}\tilde{a}(\mathbf{p}), \qquad [X^r, \tilde{a}^*(\mathbf{p})] = +i\frac{\partial}{\partial p_r}\tilde{a}^*(\mathbf{p}).$$

It continues with exponentiating and yields for the annihilation operator $\tilde{a}(\mathbf{p})$

$$V(\mathbf{b})\tilde{a}(\mathbf{p})V^{-1}(\mathbf{b}) = e^{-b^r\frac{\partial}{\partial p^r}}a(\mathbf{p}) = \tilde{a}(\mathbf{p}-\mathbf{b}),$$

analogously for the creation operator $\tilde{a}^*(\mathbf{p})$,

$$V(\mathbf{b})\tilde{a}^*(\mathbf{p})V^{-1}(\mathbf{b}) = e^{-b^r\frac{\partial}{\partial p^r}}a^*(\mathbf{p}) = \tilde{a}^*(\mathbf{p}-\mathbf{b}).$$

Note that in the last lines we used the well know equation $e^{-b^r\frac{\partial}{\partial p^r}}f(\mathbf{p}) = f(\mathbf{p}-\mathbf{b})$, which is a simple example of the Taylor series. Since the temporal operator is not a well known differential operator we will give two other proofs, which will help us in the case of transformation with X_0. □

Proof. In the second we express $V(\mathbf{b})\tilde{a}(\mathbf{p})V^{-1}(\mathbf{b})$ through investigating the effect of operators $V(\mathbf{b})^{-1}$ when applied to eigenstates $|\vec{p}\rangle$ of the vector operator P_j. The vector $V(\mathbf{b})^{-1}|\vec{p}\rangle$

$$\begin{aligned}
P_j V(\mathbf{b})^{-1}|\vec{p}\rangle &= V(\mathbf{b})^{-1}V(\mathbf{b})P_jV(\mathbf{b})^{-1}|\vec{p}\rangle \\
&= V(\mathbf{b})^{-1}\left(P_j + ib^i[X_i,P_j] + \frac{(i)^2b^ib^k}{2!}[X_i,[X_k,P_j]] + \dots\right)|\vec{p}\rangle \\
&= V(\mathbf{b})^{-1}\left(P_j - i^2b^iN\delta_{ij}\right)|\vec{p}\rangle = V(\mathbf{b})^{-1}\big(p_j+b_j\big)|\vec{p}\rangle \\
&= \big(p_j+b_j\big)V(\mathbf{b})^{-1}|\vec{p}\rangle
\end{aligned} \tag{5.22}$$

is an eigenvector of P_j for eigenvalue $\vec{p}+\vec{b}$, thus contained in the eigenspace $\mathscr{H}_{\vec{p}+\vec{b}}$ of P_j. This unitary transformation amounts to a constant shift in the momentum Hilbert space. Note that in the last equation we used the fact that X_j commutes with N. Let us now calculate the adjoint action of the unitary operator $V(\mathbf{b})$ on the momentum operator given by its representation on the Fock space, (5.17). We use the following ansatz

$$V(\mathbf{b})\tilde{a}(\mathbf{p})V^{-1}(\mathbf{b}) = \tilde{a}(\mathbf{p}-\mathbf{b}), \qquad V(\mathbf{b})\tilde{a}^*(\mathbf{p})V^{-1}(\mathbf{b}) = \tilde{a}^*(\mathbf{p}-\mathbf{b}). \tag{5.23}$$

It is deduced from the action of $V(\mathbf{b})^{-1}$ on the eigenspace $\mathscr{H}_{\vec{p}+\vec{b}}$ of P_j, [Sex01]. Next we look at the full expression.

$$\begin{aligned}
V(\mathbf{b})P_jV^{-1}(\mathbf{b}) &= \int d^n\mathbf{p}\, p_j\, \tilde{a}^*(\mathbf{p}-\mathbf{b})\tilde{a}(\mathbf{p}-\mathbf{b}) \\
&= \int d^n\mathbf{p}\, (p_j+b_j)\, \tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= P_j + b_jN.
\end{aligned}$$

This expression, obtained by ansatz (5.23) is the same one as obtained by the calculation in equation (5.22). Therefore, it follows that ansatz (5.23) is correct. We control our ansatz by calculating the following expression

$$\begin{aligned} V(\mathbf{b})NV^{-1}(\mathbf{b}) &= N + ib^i[X_i, N] + \frac{(i)^2 b^i b^j}{2!}[X_i,[X_j,N]] + ... \\ &= N. \end{aligned}$$

Where in the last line we used the fact the particle number operator commutes with the spatial operator X_j. Next, we apply the unitary transformation $V(\mathbf{b})$ on the particle creation and annihilation operators and obtain

$$\begin{aligned} V(\mathbf{b})NV^{-1}(\mathbf{b}) &= \int d^n\mathbf{p}\tilde{a}^*(\mathbf{p}-\mathbf{b})\tilde{a}(\mathbf{p}-\mathbf{b}) \\ &= \int d^n\mathbf{p}\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) \\ &= N, \end{aligned}$$

where in the last line we shifted the integral variable by b_i. As one can see, the unitary transformation calculated on an operational level, i.e. by using the commutation relations of operators X_j and N, is the same as the unitary transformation calculated on particle creation and annihilation operators. This of course has to be the case and is subsequently used.

In the next step we perform the unitary transformation on X_k. Since we have the commutator relation $[X_i, X_j] = 0$, the operator X_j remains invariant under this transformation.

$$V(\mathbf{b})X_jV^{-1}(\mathbf{b}) = X_j.$$

We apply the transformation on the particle creation and annihilation operators.

$$\begin{aligned} V(\mathbf{b})X_jV^{-1}(\mathbf{b}) &= -i\int d^n\mathbf{p}\tilde{a}^*(\mathbf{p}-\mathbf{b})\partial_j\tilde{a}(\mathbf{p}-\mathbf{b}) \\ &= -i\int d^n\mathbf{p}\tilde{a}^*(\mathbf{p})\partial_j\tilde{a}(\mathbf{p}) \\ &= X_j \end{aligned}$$

In the last line we shifted the integral variable by b_j. The derivative stays invariant under this shift so we obtain X_j again. □

Proof. The third proof proceeds along the lines presented in ([Ste05], Chapter VII). We also use the ansatz (5.23) obtained from (5.22). We find the following expression for the unitary operator $V(\mathbf{b})$ on an eigenvector $|\mathbf{p}\rangle$,

$$e^{ib_jX^j}|\mathbf{p}\rangle = |\mathbf{p}-\mathbf{b}\rangle. \tag{5.24}$$

This means we can write the unitary operator $V(\mathbf{b})$ in terms of creation and annihilation operators,

$$e^{ib_jX^j} = \int d^n\mathbf{p}\tilde{a}^*(\mathbf{p}-\mathbf{b})\tilde{a}(\mathbf{p}).$$

This representation of the unitary operator makes perfect sense due to the fact that it satisfies (5.24). Next, we calculate the infinitesimal generator of the unitary operator $V(\mathbf{b})$ by the following equation

$$\begin{aligned}
X_j &= -i \lim_{\mathbf{b}\to 0} \frac{\partial}{\partial b^j} e^{ib_k X^k} \\
&= -i \lim_{\mathbf{b}\to 0} \int d^n\mathbf{p} \left(\frac{\partial}{\partial b^j} \tilde{a}^*(\mathbf{p}-\mathbf{b}) \right) \tilde{a}(\mathbf{p}) \\
&= i \lim_{\mathbf{b}\to 0} \int d^n\mathbf{p} \left(\frac{\partial}{\partial p^j} \tilde{a}^*(\mathbf{p}-\mathbf{b}) \right) \tilde{a}(\mathbf{p}) \\
&= -i \int d^n\mathbf{p} \tilde{a}^*(\mathbf{p}) \left(\frac{\partial}{\partial p^j} \tilde{a}(\mathbf{p}) \right),
\end{aligned}$$

where in the last line we performed a partial integration. This is exactly the representation of the spatial coordinate operator in terms of creation and annihilation operators. Let us summarize. We obtained the ansatz (5.23) from equation (5.22). In the next step we wrote the unitary operator in terms of $\tilde{a}, \tilde{a}^*$ and calculated the infinitesimal generator, which turns out to be the operator we started with. □

In the following lemma we give the transformation of the free massless scalar field, performed with the unitary operator $V(\mathbf{b})$.

Lemma 5.18. *Under the adjoint action of the unitary operator $V(\mathbf{b})$, defined in (5.21), the free scalar field $\phi(f)$ (see Definitions in 4.3, 5.2), transforms as follows*

$$\begin{aligned}
\beta_{\mathbf{b}}\left(\phi(f)\right) :&= V(\mathbf{b})\phi(f)V(\mathbf{b})^{-1} \\
&= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}+\mathbf{b}}}} \left(f^-(\mathbf{p}+\mathbf{b})\tilde{a}(\mathbf{p}) + f^+(\mathbf{p}+\mathbf{b})\tilde{a}^*(\mathbf{p}) \right). \qquad (5.25)
\end{aligned}$$

Proof. To prove the lemma let us look at the following expression,

$$\begin{aligned}
V(\mathbf{b})\phi(f)V(\mathbf{b})^{-1} &= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left(f^-(\mathbf{p})V(\mathbf{b})\tilde{a}(\mathbf{p})V(\mathbf{b})^{-1} + f^+(\mathbf{p})V(\mathbf{b})\tilde{a}^*(\mathbf{p})V(\mathbf{b})^{-1} \right) \\
&= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left(f^-(\mathbf{p})\tilde{a}(\mathbf{p}-\mathbf{b}) + f^+(\mathbf{p})\tilde{a}^*(\mathbf{p}-\mathbf{b}) \right) \\
&= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}+\mathbf{b}}}} \left(f^-(\mathbf{p}+\mathbf{b})\tilde{a}(\mathbf{p}) + f^+(\mathbf{p}+\mathbf{b})\tilde{a}^*(\mathbf{p}) \right), \qquad (5.26)
\end{aligned}$$

where in the last lines we used Lemma 5.20 and we shifted the integral variable by b_j. □

Next, we examine the locality properties of the unitarily transformed free scalar field given in Lemma 5.18.

Lemma 5.19. *The free transformed scalar fields $\beta_{\mathbf{b}}(\phi(f))$ and $\beta_{\mathbf{b}'}(\phi(g))$ defined by the transformation given in Lemma 5.18, commute, if the support of f is space-like separated to the support of g and $\mathbf{b} = \mathbf{b}'$.*

Proof. The free transformed scalar fields $\beta_{\mathbf{b}}(\phi(f))$ and $\beta_{\mathbf{b}'}(\phi(f)\phi(g))$ are calculated by using Lemma 5.18 and the commutator for the two different vectors $\mathbf{b}$ and $\mathbf{b}'$ is given as follows

$$[\beta_{\mathbf{b}}(\phi(f)), \beta_{\mathbf{b}'}(\phi(g))] = \tag{5.27}$$
$$= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}+\mathbf{b}}}\sqrt{2\omega_{\mathbf{p}+\mathbf{b}'}}}\Big(f^-(\mathbf{p}+\mathbf{b})g^+(\mathbf{p}+\mathbf{b}') - f^+(\mathbf{p}+\mathbf{b})g^-(\mathbf{p}+\mathbf{b}')\Big)$$
$$= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}+\mathbf{b}-\mathbf{b}'}}\sqrt{2\omega_{\mathbf{p}}}}\Big(f^-(\mathbf{p}+\mathbf{b}-\mathbf{b}')g^+(\mathbf{p}) - f^+(\mathbf{p}+\mathbf{b}-\mathbf{b}')g^-(\mathbf{p})\Big), \tag{5.28}$$

where in the last line we shifted the integration variable $\mathbf{p} \to \mathbf{p} - \mathbf{b}'$. The commutator vanishes if $\mathbf{b} = \mathbf{b}'$ and the support of f is space-like to the support of g. This is due to the fact that for $\mathbf{b} = \mathbf{b}'$ the commutator of the transformed fields becomes the commutator of the free scalar fields $[\phi(f), \phi(g)]$, which vanishes if the supports of f and g are space-like to each other, [Fre06, Sib93]. □

In the next sections we find out that this expression fulfills another locality property, the so called wedge-locality. Moreover, the expression proves to be important, since we are able to show that the difference $|\mathbf{b} - \mathbf{b}'|$ acts exactly like a deformation parameter.

5.8.2 Transformations with the temporal operator

Calculations of transformations induced by the temporal operator have a complicated form, thus for clarity we work during this section in three space dimensions. As in the last subsection we define the following operator

$$V_0(\alpha) = e^{i\alpha X_0}, \qquad \alpha \in \mathbb{R} \tag{5.29}$$

Next, we calculate the adjoint action of the operator $V_0(\alpha)$ on the free massless scalar field. This is done as in the last subsection by deriving the adjoint action of $V_0(\alpha)$ on particle creation and annihilation operators. This will be the subject of the following lemma.

Lemma 5.20. *The particle creation and annihilation operators $\tilde{a}(\mathbf{p}), \tilde{a}^*(\mathbf{p})$ transform under the adjoint action of the operator $V_0(\alpha)$ defined in Equation (5.29), in the following manner.*

$$V_0(\alpha)\tilde{a}(\mathbf{p})V_0^{-1}(\alpha) = \left(1 - \frac{\alpha}{\omega_{\boldsymbol{p}}}\right)\tilde{a}(\mathbf{p}(1 - \frac{\alpha}{\omega_{\boldsymbol{p}}})),$$
$$V_0(\alpha)\tilde{a}^*(\mathbf{p})V_0^{-1}(\alpha) = \left(1 - \frac{\alpha}{\omega_{\boldsymbol{p}}}\right)\tilde{a}^*(\mathbf{p}(1 - \frac{\alpha}{\omega_{\boldsymbol{p}}}))$$

Proof. The first derivation follows along the same lines as the second proof for Lemma 5.20. An explicit expression for $V_0(\alpha)\tilde{a}(\mathbf{p})V_0^{-1}(\alpha)$ can be deduced from the investigation of the effect of the operators $V_0^{-1}(\alpha)$ when applied to the eigenstates $|\vec{p}\rangle$ of the vector operator P_j. The vector $V_0^{-1}(\alpha)|\vec{p}\rangle$ is given as

$$\begin{aligned}
P_j V_0(\alpha)^{-1}|\vec{p}\rangle &= V_0(\alpha)^{-1} V_0(\alpha) P_j V_0(\alpha)^{-1}|\vec{p}\rangle \\
&= V_0(\alpha)^{-1}\left(P_j + i\alpha[X_0, P_j] + \frac{(i)^2\alpha^2}{2!}[X_0,[X_0,P_j]] + ...\right)|\vec{p}\rangle \\
&= V_0(\alpha)^{-1}\left(P_j - i^2\alpha\int \frac{k_j}{\omega_{\mathbf{k}}} a^*(\mathbf{k})a(\mathbf{k})d^3k\right)|\vec{p}\rangle \\
&= V_0(\alpha)^{-1}\left(p_j + \alpha\frac{p_j}{\omega_{\mathbf{p}}}\right)|\vec{p}\rangle. \qquad (5.30)
\end{aligned}$$

In the last lines we used the fact that in the massless case the following expression

$$[X_0,[X_0,P_j]] = 0$$

vanishes. Therefore, $P_j V_0(\alpha)^{-1}|\vec{p}\rangle$ is an eigenvector of P_j for the eigenvalue $\vec{p}\left(1+\frac{\alpha}{\omega_{\mathbf{p}}}\right)$, thus contained in the eigenspace $\mathcal{H}_{\vec{p}\left(1+\frac{\alpha}{\omega_{\mathbf{p}}}\right)}$ of P_j. This transformation is a momentum-dependent dilatation in the momentum Hilbert space.

The second derivation consists in calculating adjoint action of operator $V_0(\alpha)$ on the momentum operator, given by its representation on the Fock space (5.17). To ease readability let us define the following function,

$$\alpha_{\mathbf{p}} := 1 - \frac{\alpha}{\omega_{\mathbf{p}}}$$

We use the following ansatz

$$\begin{aligned}
V_0(\alpha)\tilde{a}(\mathbf{p})V_0^{-1}(\alpha) &= \alpha_{\mathbf{p}}\tilde{a}(\mathbf{p}\alpha_{\mathbf{p}}), \qquad (5.31)\\
V_0(\alpha)\tilde{a}^*(\mathbf{p})V_0^{-1}(\alpha) &= \alpha_{\mathbf{p}}\tilde{a}^*(\mathbf{p}\alpha_{\mathbf{p}}),
\end{aligned}$$

which is deduced from the action of $V_0^{-1}(\alpha)$ on the eigenspace $\mathcal{H}_{\vec{p}\left(1+\frac{\alpha}{\omega_{\mathbf{p}}}\right)}$ of P_j, [Sex01]. We thus obtain,

$$\begin{aligned}
V_0(\alpha)P_jV_0^{-1}(\alpha) &= P_j - i^2\alpha\int\frac{p_j}{\omega_{\mathbf{p}}}a^*(\mathbf{p})a(\mathbf{p})d^3p \\
&= \int p_j\left(1+\frac{\alpha}{\omega_{\mathbf{p}}}\right)a^*(\mathbf{p})a(\mathbf{p})d^3p.
\end{aligned}$$

We can rewrite this term as follows

$$\int p_jV_0(\alpha)a^*(\mathbf{p})a(\mathbf{p})V_0^{-1}(\alpha)d^3p = \int p_jV_0(\alpha)a^*(\mathbf{p})V_0^{-1}(\alpha)V_0(\alpha)a(\mathbf{p})V_0^{-1}(\alpha)d^3p.$$

By using the ansatz (5.31) one is led to the same solution, as the transformation performed on an operator level.

$$\begin{aligned}
V_0(\alpha)P_jV_0^{-1}(\alpha) &= \int p_jV_0(\alpha)a^*(\mathbf{p})V_0^{-1}(\alpha)V_0(\alpha)a(\mathbf{p})V_0^{-1}(\alpha)d^3p \\
&= \int p_ja^*(\mathbf{p}\alpha_{\mathbf{p}})a(\mathbf{p}\alpha_{\mathbf{p}})(\alpha_{\mathbf{p}})^2d^3p \\
&= \int(1+\frac{\alpha}{\omega_{\mathbf{k}}})k_ja^\dagger(\mathbf{k})a^*(\mathbf{k})d^3k.
\end{aligned}$$

For obtaining the last line we made the following variable substitution

$$k_j = p_j(1 - \frac{\alpha}{\omega_{\mathbf{p}}}), \qquad p_j = (1 + \frac{\alpha}{\omega_{\mathbf{k}}})k_j \tag{5.32}$$

$$d^3k = \det(\delta_{ij}(1 - \frac{\alpha}{\omega_{\mathbf{p}}}) - \frac{p_i p_j}{\omega_{\mathbf{p}}^3})d^3p = \alpha_{\mathbf{p}}^2 d^3p,$$

We control our ansatz by calculating the effect of transformation on P_0 directly by using the algebra between X_0 and P_0,

$$\begin{aligned} V_0(\alpha)P_0V_0^{-1}(\alpha) &= P_0 + i\alpha[X_0, P_0] + \frac{(i)^2\alpha^2}{2!}[X_0, [X_0, P_0]] + ... \\ &= P_0 - i^2\alpha N. \end{aligned}$$

In the last line we used the commutator relations

$$[X_0, P_0] = -iN, \qquad [X_0, [X_0, P_0]] = 0.$$

By applying the transformation to the particle creation and annihilation operators individually, we obtain

$$\begin{aligned} V_0(\alpha)P_0V_0^{-1}(\alpha) &= \int \omega_{\mathbf{p}} V_0(\alpha)a^*(\mathbf{p})V_0^{-1}(\alpha)V_0(\alpha)a(\mathbf{p})V_0^{-1}(\alpha)d^3p \\ &= \int \omega_{\mathbf{p}}\alpha_{\mathbf{p}}^2 a^*(\mathbf{p}\alpha_{\mathbf{p}})a(\mathbf{p}\alpha_{\mathbf{p}})d^3p \\ &= \int \Big(\omega_{\mathbf{k}} + \alpha\Big)a^*(\mathbf{k})a(\mathbf{k})d^3k \\ &= P_0 + \alpha N, \end{aligned}$$

where in the last line we used the following variable substitution and relations

$$k_j = p_j(1 - \frac{\alpha}{\omega_{\mathbf{p}}}), \qquad d^3k = (1 - \frac{\alpha}{\omega_{\mathbf{p}}})^2 d^3p, \qquad \omega_{\mathbf{k}(1+\frac{\alpha}{\omega_{\mathbf{k}}})} = \omega_{\mathbf{k}} + \alpha.$$

We can control our ansatz by calculating expression

$$V_0(\alpha)NV_0^{-1}(\alpha) = N + i\alpha[X_0, N] + \frac{(i)^2\alpha^2}{2!}[X_0, [X_0, N]] + \dots.$$

The particle number operator commutes with all other operators thus

$$V_0(\alpha)NV_0^{-1}(\alpha) = N.$$

In the next step we apply the transformation on the particle creation and annihilation operators.

$$\begin{aligned} V_0(\alpha)NV_0^{-1}(\alpha) &= \int V_0(\alpha)a^*(\mathbf{p})V_0^{-1}(\alpha)V_0(\alpha)a(\mathbf{p})V_0^{-1}(\alpha)d^3p \\ &= \int \alpha_{\mathbf{p}}^2 a^*(\mathbf{p}\alpha_{\mathbf{p}})a(\mathbf{p}\alpha_{\mathbf{p}})d^3p \\ &= \int a^*(\mathbf{k})a(\mathbf{k})d^3k \\ &= N, \end{aligned}$$

where in the last line we used the variable substitutions given in (5.32).

Next, we perform the transformation on X_0. Due to the fact that the operator X_0 has to commute with itself, the operator has to remain invariant under the transformation.

$$V_0(\alpha)X_0V_0^{-1}(\alpha) = X_0.$$

Thus by applying the transformation on particle creation and annihilation operators the operator X_0 should remain invariant.

$$\begin{aligned}
V_0(\alpha)X_0V_0^{-1}(\alpha) &= -i\int\Big(\frac{1}{\omega_{\mathbf{p}}}V_0(\alpha)a^*(\mathbf{p})a(\mathbf{p})V_0^{-1}(\alpha) + \frac{p^r}{\omega_{\mathbf{p}}}V_0(\alpha)a^*(\mathbf{p})\partial_r a(\mathbf{p})V_0^{-1}(\alpha)\Big)d^3p \\
&= -i\int\Big(\frac{1}{\omega_{\mathbf{p}}}\alpha_{\mathbf{p}}^2 a^*(\mathbf{p}\alpha_{\mathbf{p}})a(\mathbf{p}\alpha_{\mathbf{p}}) + \frac{p^r}{\omega_{\mathbf{p}}}\alpha_{\mathbf{p}}a^*(\mathbf{p}\alpha_{\mathbf{p}})\partial_r\Big(\alpha_{\mathbf{p}}a(\mathbf{p}\alpha_{\mathbf{p}})\Big)\Big)d^3p \\
&= -i\int\Big(\frac{1}{\omega_{\mathbf{k}}}a^*(\mathbf{k})a(\mathbf{k})d^3k\Big) - i\int\Big(\frac{k^r}{\omega_{\mathbf{k}}}a^*(\mathbf{k})\frac{\partial k^n}{\partial p^r}\frac{\partial}{\partial k^n}a(\mathbf{k})d^3k\Big),
\end{aligned}$$

where in the last line we used the variable substitution (5.32) and the following relations

$$k^r\frac{\partial k^n}{\partial p^r} = k^n, \qquad \frac{1}{\omega_{\mathbf{k}}(1+\frac{\alpha}{\omega_{\mathbf{k}}})} + \frac{\alpha}{\omega_{\mathbf{k}}^2(1+\frac{\alpha}{\omega_{\mathbf{k}}})} = \frac{1}{\omega_{\mathbf{k}}},$$

thus we obtain

$$V_0(\alpha)X_0V_0^{-1}(\alpha) = -i\int\Big(\frac{1}{\omega_{\mathbf{k}}}a^*(\mathbf{k})a(\mathbf{k}) + \frac{k^n}{\omega_{\mathbf{k}}}a^*(\mathbf{k})\frac{\partial}{\partial k^n}a(\mathbf{k})d^3k\Big) = X_0$$

□

Proof. The second derivation proceeds along the same lines as the author in ([Ste05], Chapter VII). We use the ansatz (5.31) and obtain the following expression for the operator $V_0(\alpha)$ on an eigenvector $|\mathbf{p}\rangle$,

$$e^{i\alpha X^0}|\mathbf{p}\rangle = \alpha_{\mathbf{p}}|\mathbf{p}\alpha_{\mathbf{p}}\rangle. \tag{5.33}$$

This means that we can write the operator $V_0(\alpha)$ in terms of creation and annihilation operators,

$$e^{i\alpha X^0} = \int d^3\mathbf{p}\alpha_{\mathbf{p}}\tilde{a}^*(\mathbf{p}\alpha_{\mathbf{p}})\tilde{a}(\mathbf{p}).$$

As one can easily see this representation of the operator satisfies (5.33). In the next step, we calculate the infinitesimal generator of the operator $V_0(\alpha)$ by the following equation

$$\begin{aligned}
X_0 &= -i\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}e^{i\alpha X^0} \\
&= -i\lim_{\alpha\to 0}\int d^3\mathbf{p}\frac{\partial}{\partial\alpha}\left(\alpha_{\mathbf{p}}\tilde{a}^*(\mathbf{p}\alpha_{\mathbf{p}})\right)\tilde{a}(\mathbf{p}) \\
&= i\int d^3\mathbf{p}\left(\frac{1}{\omega_{\mathbf{p}}}\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p})\right) - i\lim_{\alpha\to 0}\int d^3\mathbf{p}\alpha_{\mathbf{p}}\frac{\partial}{\partial\alpha}\tilde{a}^*(\mathbf{p}\alpha_{\mathbf{p}})\tilde{a}(\mathbf{p}) \\
&= i\int d^3\mathbf{p}\left(\frac{1}{\omega_{\mathbf{p}}}\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p}) + \frac{p^j}{\omega_{\mathbf{p}}}\frac{\partial}{\partial p^j}\tilde{a}^*(\mathbf{p})\tilde{a}(\mathbf{p})\right) \\
&= X_0
\end{aligned}$$

where in the last line we performed a partial integration and replaced the derivative w.r.t. α on the creation operator as follows,

$$\frac{\partial}{\partial\alpha}\tilde{a}^*(\mathbf{p}'_\alpha) = \frac{\partial p'^j_\alpha}{\partial\alpha}\frac{\partial}{\partial p'^j_\alpha}\tilde{a}^*(\mathbf{p}'_\alpha) = -\frac{p^j}{\omega_\mathbf{p}}\frac{\partial}{\partial p'^j_\alpha}\tilde{a}^*(\mathbf{p}'_\alpha),$$

$$\mathbf{p}'_\alpha := \mathbf{p}\alpha_\mathbf{p},$$

in the limit $\alpha \to 0$ the vector $\mathbf{p}'_\alpha$ goes to the momentum $\mathbf{p}$.

Thus the outcome is exactly the representation of the temporal coordinate operator in terms of creation and annihilation operators. Let us summarize the result. We obtained the ansatz (5.31) from equation (5.30). In the next step we gave the operator of transformations in terms of $\tilde{a}, \tilde{a}^*$ and calculated the infinitesimal generator, which turns out to be the operator we started with. $\square$

5.9 Deforming the scalar quantum field

In this section, we investigate the effect of deformation directly on a free scalar field. The unitary group used for deformation, is given by the position operator and moreover, in accordance with relativistic covariance, with a defined zero component of the coordinate operator. The operator, denoted by X_0, that we define in this section differs from the one obtained in [SS09]. It is defined by unitary equivalence to the zero component of the momentum operator, i.e. the energy. Due to the unitary equivalence, the vector operator $X_\mu = (X_0, X_k)$ is an essential self-adjoint operator on a dense domain and therefore defines a strongly continuous unitary group that we denote by $V(b) := e^{ib_\mu X^\mu}$. Furthermore, by using this abelian group, an adjoint action can be defined and used for the deformation in the framework of warped convolutions, [BS08, BLS11].

Definition 5.5. The operator X_μ is defined by the unitary equivalence to the momentum operator as follows,

$$X_\mu = \Gamma(U_{\mathscr{F}}^{-1}) P_\mu \Gamma(U_{\mathscr{F}}), \tag{5.34}$$

where the operator $\Gamma(U_{\mathscr{F}}) := \bigotimes_{i=1}^{k} U_{\mathscr{F}}$ is the second quantization of the unitary Fourier operator.

The zero component is defined in the same manner as the spatial coordinate operator. This definition can be considered as the relativistic generalization of the position operator in quantum mechanics to a quantum field theoretical context.

Proposition 5.1. *The operator X_μ defined by unitarily equivalence (see Definition 5.5) is an essentially self-adjoint operator on the domain $\mathscr{S}(\mathbb{R}^{d\times k})$ satisfying the following commutator relations,*

$$[X_\mu, X_\nu] = 0.$$

Therefore, the following unitary operator

$$V(p) = e^{ip_\mu X^\mu}, \tag{5.35}$$

defines a strongly continuous group for all $p \in \mathbb{R}^d$.

Proof. Essential self-adjointness for the spatial part of X_μ was shown in Theorem 5.1. The proof of essential self-adjointness for X_0 is done in an analogous manner, since the zero component of the momentum operator P_0 is an essentially self-adjoint operator. Hence, by unitary equivalence essential self-adjointness follows. Furthermore, we use the unitary equivalence in order to show the commutation relations between the different components of the operator X_μ.

$$\begin{aligned}
[X_\mu, X_\nu] &= [\Gamma(U_{\mathscr{F}}^{-1}) P_\mu \Gamma(U_{\mathscr{F}}), \Gamma(U_{\mathscr{F}}^{-1}) P_\mu \Gamma(U_{\mathscr{F}})] \\
&= \Gamma(U_{\mathscr{F}}^{-1}) [P_\mu, P_\mu] \Gamma(U_{\mathscr{F}}) \\
&= 0,
\end{aligned}$$

where in the last line we used the fact that the momentum operators commute. By applying Stone's theorem (see 3.1), it follows that $V(p)$ defines a strongly continuous unitary group. $\square$

Definition 5.6. Let θ be a real skew-symmetric matrix w.r.t. the Lorentzian scalar-product on $\mathbb{R}^d$ and let $\chi \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Furthermore, let $\phi(f)$ be the massive free scalar field smeared out with functions $f \in \mathscr{S}(\mathbb{R}^d)$. Then, the operator valued distribution $\phi(f)$ deformed with the coordinate operator X_μ (see Definition 5.5), denoted as $\phi_{\theta,X}(f)$, is defined on vectors of the dense domain $\mathscr{S}(\mathbb{R}^{n\times k})$ as follows

$$\begin{aligned}
\phi_{\theta,X}(f)\Psi_k :&= (2\pi)^{-d} \lim_{\epsilon\to 0} \iint dy\, du\, e^{-iyu}\, \chi(\epsilon y, \epsilon u) \beta_{\theta y}(\phi(f)) V(u)\Psi_k \\
&= (2\pi)^{-d} \lim_{\epsilon\to 0} \iint dy\, du\, e^{-iyu}\, \chi(\epsilon y, \epsilon u) \beta_{\theta y}\left(a(\overline{f^-}) + a^*(f^+)\right) V(u)\Psi_k \\
&=: \left(a_{\theta,X}(\overline{f^-}) + a^*_{\theta,X}(f^+)\right)\Psi_k.
\end{aligned} \tag{5.36}$$

The automorphism β is defined by the adjoint action of the unitary operator $V(y)$ and the test functions $f^\pm(\mathbf{p})$ in momentum space are defined as follows

$$f^\pm(\mathbf{p}) := \int d^d x f(x) e^{\pm ipx}, \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in H_m^+.$$

The integral (5.36) has to be understood as an integral in oscillatory sense, [Rie93]. The unboundedness of the operator X_μ questions the existence of the integral since we are dealing with unbounded operator valued distributions. To show that the integral (5.36) converges we use the unitary equivalence of the coordinate operator with the momentum operator.

The following lemma is about the existence of a unitary transformation connecting the warped convolutions of a free scalar field using the momentum operator, and the warped convolutions of a free scalar field using the coordinate operator.

Lemma 5.21. *For $f \in \mathscr{S}(\mathbb{R}^d)$ and $\Psi_k \in \mathscr{S}(\mathbb{R}^{n\times k})$, a transformation exists that maps the field deformed with the momentum operator $\phi_{\theta,P}(f)$ to the field deformed with the coordinate operator $\phi_{\theta,X}(f)$. This transformation is given as follows*

$$\phi_{\theta,X}(f)\Psi_k = \Gamma(U_{\mathscr{F}})^{-1} \left(\Gamma(U_{\mathscr{F}})\phi(f)\Gamma(U_{\mathscr{F}})^{-1}\right)_{\theta,P} \Gamma(U_{\mathscr{F}})\Psi_k.$$

Proof. By using the unitary equivalence given in Equation (5.34), the lemma is easily proven

$$\begin{aligned}
\phi_{\theta,X}(f)\Psi_k &= (2\pi)^{-d} \lim_{\epsilon\to 0} \iint dy\, du\, e^{-iyu}\, \chi(\epsilon y, \epsilon u)\, V(\theta y)\phi(f) V(-\theta y + u)\Psi_k \\
&= (2\pi)^{-d} \lim_{\epsilon\to 0} \iint dy\, du\, e^{-iyu}\, \chi(\epsilon y, \epsilon u)\, \Gamma(U_{\mathscr{F}})^{-1} T(\theta y) \Gamma(U_{\mathscr{F}})\phi(f)\Gamma(U_{\mathscr{F}})^{-1} \\
&\qquad\qquad \times T(-\theta y + u)\Gamma(U_{\mathscr{F}})\Psi_k \\
&= \Gamma(U_{\mathscr{F}})^{-1} \left(\Gamma(U_{\mathscr{F}})\phi(f)\Gamma(U_{\mathscr{F}})^{-1}\right)_{\theta,P} \Gamma(U_{\mathscr{F}})\Psi_k.
\end{aligned}$$

□

Lemma 5.22. *For $\Phi_m \in \mathscr{S}(\mathbb{R}^{n\times k})$ the familiar bounds of the free field hold for the deformed field $\phi_{\theta,X}(f)$ and therefore the deformation with operator X_μ is well-defined.*

Proof. By using Lemma 5.21 one obtains the familiar bounds for a free scalar field. For $\Phi_k \in \mathscr{S}(\mathbb{R}^{n\times k})$ there exists a $\Psi_k \in \mathscr{S}(\mathbb{R}^{n\times k})$ such that the following holds

$$\begin{aligned}
\|\phi_{\theta,X}(f)\Phi_k\| &= \|\phi_{\theta,X}(f)\Gamma(U_{\mathscr{F}})\Psi_k\| \\
&= \left\|(\Gamma(U_{\mathscr{F}})\phi(f)\Gamma(U_{\mathscr{F}})^{-1})_{\theta,P}\Psi_k\right\| = \|(\phi(U_{\mathscr{F}}f))_{\theta,P}\Psi_k\| \\
&\leq \left\|(a(\overline{U_{\mathscr{F}}f^-}))_{\theta,P}\Psi_k\right\| + \left\|(a^*(U_{\mathscr{F}}f^+))_{\theta,P}\Psi_k\right\| \\
&\leq \left\|U_{\mathscr{F}}f^+\right\| \left\|(N+1)^{1/2}\Psi_k\right\| + \left\|U_{\mathscr{F}}f^-\right\| \left\|(N+1)^{1/2}\Psi_k\right\| \\
&= \left\|f^+\right\| \left\|(N+1)^{1/2}\Psi_k\right\| + \left\|f^-\right\| \left\|(N+1)^{1/2}\Psi_k\right\|.
\end{aligned}$$

where in the last lines we used the triangle inequality, the Cauchy-Schwarz inequality, the bounds given in [GL07] and the fact that $U_{\mathscr{F}}$ is equal to one w.r.t. the Lorentz-invariant measure, (see Remark 5.2).

The obtained bounds are exactly the bounds of the free scalar field. Thus by the same arguments given in Lemma 4.3, concerning the deformed field $\phi_{\theta,P}$, it follows that the field deformed with the coordinate operator X_μ is well-defined.

Remark 5.2. During the calculations of the last inequality we used the fact that operator $U_{\mathscr{F}}$ is unitary w.r.t. the Lorentz-invariant measure. That this statement is holds, can be shown by a short calculation, which is important to do, since $U_{\mathscr{F}}$ is only been shown to be unitary w.r.t. the measure $d^n\mathbf{p}$.

Hence, we first establish that the norm of the unitary operator $U_{\mathscr{F}}$ acting on a function $f \in \mathscr{H}_1$ is equal to the norm of f. Note that the transformation $U_{\mathscr{F}}$ was constructed in the non-covariant representation. Thus, to calculate the action of the unitary operator on the function f in a covariant fashion we consider the following expression in the non-covariant representation and at the end switch to the covariant representation,

$$\begin{aligned}
(\Gamma(U_{\mathscr{F}})a(\overline{f})\Gamma(U_{\mathscr{F}})^{-1}) &= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p})(\Gamma(U_{\mathscr{F}})\tilde{a}(\mathbf{p})\Gamma(U_{\mathscr{F}})^{-1}) \\
&= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p})\,(U_{\mathscr{F}}\tilde{a})\,(\mathbf{p}) \\
&= \int \frac{d^n\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} \left(\sqrt{2\omega}U_{\mathscr{F}}\frac{1}{\sqrt{2\omega}}f\right)(\mathbf{p})\tilde{a}(\mathbf{p}) \\
&= \int d^n\mu(\mathbf{p}) \left(\sqrt{2\omega}U_{\mathscr{F}}\frac{1}{\sqrt{2\omega}}f\right)(\mathbf{p})a(\mathbf{p}) = a(\overline{U_{\mathscr{F}}f}).
\end{aligned}$$

By using the action of the unitary operator $U_{\mathscr{F}}$ on the function f, we can calculate the norm of $U_{\mathscr{F}}f$,

$$
\begin{aligned}
\|U_{\mathscr{F}} f\|^2 &= \int d^n\mu(\mathbf{p}) \overline{\left(\sqrt{2\omega} U_{\mathscr{F}} \frac{1}{\sqrt{2\omega}} f\right)(\mathbf{p})} \left(\sqrt{2\omega} U_{\mathscr{F}} \frac{1}{\sqrt{2\omega}} f\right)(\mathbf{p}) \\
&= \int d^n\mathbf{p} \left(U_{\mathscr{F}} \overline{\left(U_{\mathscr{F}} \frac{1}{\sqrt{2\omega}} f\right)}\right)(\mathbf{p}) \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) \\
&= \int d^n\mathbf{p} \left(U_{\mathscr{F}} U_{\mathscr{F}}^{-1} \overline{\left(\frac{1}{\sqrt{2\omega}} f\right)}\right)(\mathbf{p}) \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) \\
&= \int d^n\mathbf{p} \overline{\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p})} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) \\
&= \int d^n\mu(\mathbf{p}) |f(\mathbf{p})|^2 = \|f\|^2 ,
\end{aligned}
$$

where in the last line we use the fact that the conjugate of the transformation $U_{\mathscr{F}}$ is equal to the inverse of the operator.

□

5.9.1 Wightman properties of the deformed QF

It is important to note that due to the unitary equivalence we can show that the deformed field $\phi_{\theta,X}$ satisfies the Wightman properties with the exception of covariance and locality. This is the subject of the following proposition. Note that in the next proposition we use the symbol $\mathscr{H}$ for the massless and massive Bosonic Fockspace.

Proposition 5.2. *Let θ be a real skew-symmetric matrix w.r.t. the Lorentzian scalar-product on $\mathbb{R}^d$ and $f \in \mathscr{S}(\mathbb{R}^d)$.*

a) *The dense subspace $\mathcal{D}$ of vectors of finite particle number is contained in the domain $\mathcal{D}^{\theta,X} = \{\Psi \in \mathscr{H} | \, \|\phi_{\theta,X}(f)\Psi\|^2 < \infty\}$ of any $\phi_{\theta,X}(f)$. Moreover, $\phi_{\theta,X}(f)\mathcal{D} \subset \mathcal{D}$ and $\phi_{\theta,X}(f)\Omega = \phi(f)\Omega$.*

b) *For scalar fields deformed via warped convolutions and $\Psi \in \mathcal{D}$,*

$$f \longmapsto \phi_{\theta,X}(f)\Psi$$

is a vector valued tempered distribution.

c) *For $\Psi \in \mathcal{D}$ and $\phi_{\theta,X}(f)$ the following holds*

$$\phi_{\theta,X}(f)^*\Psi = \phi_{\theta,X}(\overline{f})\Psi.$$

For real $f \in \mathscr{S}(\mathbb{R}^d)$, the deformed field $\phi_{\theta,X}(f)$ is essentially self-adjoint on $\mathcal{D}$.

d) *The Reeh-Schlieder property holds: Given an open set of spacetime $\mathcal{O} \subset \mathbb{R}^d$ then*

$$\mathcal{D}_{\theta,X}(\mathcal{O}) := span\{\phi_{\theta,X}(f_1)\dots\phi_{\theta,X}(f_k)\Omega : k \in \mathbb{N}, f_1 \dots f_k \in \mathscr{S}(\mathcal{O})\}$$

is dense in $\mathscr{H}$.

Proof. a) The fact that $\mathcal{D} \subset \mathcal{D}^{\theta,X}$, follows immediately from Lemma 5.22 , since the deformed scalar field satisfies the same bounds as a free field. The fact that the deformed field acting on the vacuum is the same as the free field acting on Ω, can be easily shown due to the property of the unitary operators $V(b)\Omega = \Omega$.

b) By using Lemma 5.22 one can see that the right hand side depends continuously on the function f, hence the temperateness of $f \longmapsto \phi_{\theta,X}(f)\Psi$, $\Psi \in \mathcal{D}$ follows.

c) First, we prove hermiticity of the deformed field $\phi_{\theta,X}(f)$. This is done along the same lines as the proof of Lemma 2.2, demonstrating hermiticity of a deformed operator if the undeformed one is hermitian.

$$
\begin{aligned}
\phi_{\theta,X}(f)^*\Psi &= (2\pi)^{-d}\left(\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\chi(\epsilon y,\epsilon u)\,\beta_{\theta y}(\phi(f))V(u)\right)^*\Psi\\
&= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\overline{\chi(\epsilon y,-\epsilon u)}\,V(u)\beta_{\theta y}(\phi(f))^*\Psi\\
&= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\overline{\chi(\epsilon(y+\theta^{-1}u),-\epsilon u)}\,\beta_{\theta y}(\phi(\overline{f}))V(u)\Psi\\
&= \phi_{\theta,X}(\overline{f})\Psi.
\end{aligned}
$$

In the last lines we performed a variable substitution $(u_\mu \to -u_\mu)$ and $(y_\mu \to y_\mu + (\theta^{-1}u)_\mu)$.

For real f we can prove the essential self-adjointness of the hermitian deformed field $\phi_{\theta,X}(f)$. The first step consists in showing that the deformed field has a dense set of analytic vectors, (see Definition 8.19). Next, by Nelson's analytic vector theorem 8.4, it follows that the deformed field $\phi_{\theta,X}(f)$ is essentially self-adjoint on this dense set of analytic vectors, (for similar proof see [BR96, Chapter I, Proposition 5.2.3]).

For $\Psi_k \in \mathscr{H}_k$ the estimates of the l-power of the deformed field $\phi_{\theta,X}(f)$, are given in the following

$$
\left\|\phi_{\theta,X}(f)^l\Psi_k\right\| \leq 2^{l/2}(k+l)^{1/2}(k+l-1)^{1/2}\cdots(k+1)^{1/2}\,\|f\|^l\,\|\Psi_k\|,
$$

where in the last lines we used Lemma 5.22 for the estimates of the deformed field. Finally, we can write the sum

$$
\sum_{l\geq 0}\frac{|t|^l}{l!}\left\|\phi(f)^l\Psi_k\right\| \leq \sum_{l\geq 0}\frac{(\sqrt{2}|t|)^l}{l!}\left(\frac{(k+l)!}{k!}\right)^{1/2}\|f\|^l\,\|\Psi_k\| < \infty
$$

for all $t \in \mathbb{C}$. It follows that each $\Psi \in \mathcal{D}$ is an analytic vector for the deformed field $\phi_{\theta,X}(f)$. Since the set $\mathcal{D}$ is dense in $\mathscr{H}$, Nelson's analytic vector theorem implies that $\phi_{\theta,X}(f)$ is essentially self-adjoint on $\mathcal{D}$.

d) For the proof of the Reeh-Schlieder property we use the unitary equivalence given in Definition (5.5). First note that the spectral properties of the unitary operator $V(y)$, are the same as for the unitary operator $T(y)$ of translations. This leads to the application of the standard Reeh-Schlieder argument [SW89] which states that that

$\mathcal{D}_\theta(\mathcal{O})$ is dense in $\mathscr{H}$ if and only if $\mathcal{D}_\theta(\mathbb{R}^d)$ is dense in $\mathscr{H}$. We choose the functions $f_1, \ldots, f_k \in \mathscr{S}(\mathbb{R}^d)$ such that the Fourier transforms of the functions do not intersect the lower mass shell and therefore the domain $\mathcal{D}_\theta(\mathbb{R}^d)$ consists of the following vectors

$$\begin{aligned}\Gamma(U_{\mathscr{F}})\phi_{\theta,X}(f_1)\ldots\phi_{\theta,X}(f_k)\Omega &= \Gamma(U_{\mathscr{F}})a^*_{\theta,X}(f_1^+)\ldots a^*_{\theta,X}(f_k^+)\Omega \\ &= \Gamma(U_{\mathscr{F}})\Gamma(U_{\mathscr{F}})^{-1}a^*_{\theta,P}(U_{\mathscr{F}}f_1^+)\ldots a^*_{\theta,P}(U_{\mathscr{F}}f_k^+)\Gamma(U_{\mathscr{F}})^{-1}\Omega \\ &= a^*_{\theta,P}(U_{\mathscr{F}}f_1^+)\ldots a^*_{\theta,P}(U_{\mathscr{F}}f_k^+)\Omega \\ &= \sqrt{m!}P_m(S_m(U_{\mathscr{F}}f_1^+\otimes\cdots\otimes U_{\mathscr{F}}f_k^+)),\end{aligned}$$

where P_k denotes the orthogonal projection from $\mathscr{H}_1^{\otimes k}$ onto its totally symmetric subspace $\mathscr{H}_k$, and $S_k \in \mathscr{B}(\mathscr{H}_1^{\otimes k})$ is the multiplication operator given as

$$S_k(p_1, \ldots, p_k) = \prod_{1\leq l<j\leq k} e^{ip_l\theta p_j}.$$

Since the operator $U_{\mathscr{F}}$ is unitary and maps Schwartz functions into Schwartz functions we have, $U_{\mathscr{F}}f_k^+ \in \mathscr{S}(\mathbb{R}^d)$ for $f_k^+ \in \mathscr{S}(\mathbb{R}^d)$. This in particular means that $U_{\mathscr{F}}f_k^+$ will give rise to dense sets of functions in $\mathscr{H}_1$. Following the same arguments as in [GL07] the density of $\mathcal{D}_\theta(\mathbb{R}^d)$ in $\mathscr{H}$ follows. Note that we proved the density for vectors $\Gamma(U_{\mathscr{F}})\phi_{\theta,X}(f_1)\ldots\phi_{\theta,X}(f_k)\Omega$ and not for the vectors without the application of $\Gamma(U_{\mathscr{F}})$ as stated in the proposition. We use the unitary of $\Gamma(U_{\mathscr{F}})$ to argue that vectors dense in $\mathscr{H}$ stay dense after the application of a unitary operator. □

5.9.2 Wedge-covariant fields

The authors in [GL07] constructed a map $Q : W \mapsto Q(W)$ from a set $\mathcal{W}_0 := \mathcal{L}_+^\uparrow W_1$ of wedges, where $W_1 := \{x \in \mathbb{R}^d : x_1 > |x_0|\}$ to a set $\mathcal{Q}_0 \subset \mathbb{R}^-_{d\times d}$ of skew-symmetric matrices. In the next step they considered the corresponding fields $\phi_W(x) := \phi(Q(W), x)$. The meaning of the correspondence is that the field $\phi(Q(W), x)$ is a scalar field living on a NC spacetime which can be equivalently realized as a field defined on the wedge. To examine the covariance properties of the free scalar field deformed with the coordinate operator, we use the the homomorphism $Q : W \mapsto Q(W)$ to relate the deformed scalar field $\phi_{\theta,X}$ to a wedge-covariant field. Let us first define the following map.

Definition 5.7. Let θ be a real skew-symmetric matrix on $\mathbb{R}^d$ then the map $\gamma_\Lambda(\theta)$ is defined as follows

$$\gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T, & \Lambda \in \mathcal{L}^\uparrow, \\ -\Lambda\theta\Lambda^T, & \Lambda \in \mathcal{L}^\downarrow. \end{cases} \tag{5.37}$$

Furthermore, we need the transformation properties of the deformed field under the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ to examine the wedge-covariance of our field. It turns out that the deformed field $\phi_{\theta,X}$ only transforms covariant under a subgroup of $\mathcal{P}_+^\uparrow$. The following lemma gives the transformation property of the deformed scalar field under the action of a subgroup of the unitary operators of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$.

Lemma 5.23. *Let a subgroup of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ denoted as $\mathcal{L}_R$, be defined in the following way*

$$\mathcal{L}_R := \left\{ \Lambda = \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & R \end{pmatrix}, R \in SO(3) \right\} \subset \mathcal{L}_+^\uparrow.$$

Then, the deformed particle annihilation and creation operator given in Definition (5.36) transform under the adjoint action of $U(0,\Lambda)$ in the following manner

$$\begin{aligned} U(0,\Lambda) a_{\theta,X}(\mathbf{p}) U(0,\Lambda)^{-1} &= a_{\gamma_\Lambda(\theta),X}(\pm\Lambda\mathbf{p}), \\ U(0,\Lambda) a^*_{\theta,X}(\mathbf{p}) U(0,\Lambda)^{-1} &= a^*_{\gamma_\Lambda(\theta),X}(\pm\Lambda\mathbf{p}), \end{aligned}$$

where the first sign is for $\Lambda \in \mathcal{L}_R$ and the second sign is for $\Lambda \in \mathcal{L}^\downarrow$. Hence, the deformed field $\phi_{\theta,X}(x)$ transforms under the adjoint action $U(0,\Lambda)$ as follows,

$$U(0,\Lambda)\phi_{\theta,X}(x)U(0,\Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),X}(\Lambda x).$$

Proof. The proof is done along the lines of ([BLS11], Proposition 2.9). To follow the proof we first calculate the Lorentz transformation of the unitary operator $V(p)$ given in (5.35). This can be done by considering the following relation,

$$U(0,\Lambda)P_\mu U(0,\Lambda)^{-1} = (\Lambda^{-1})_\mu^{\ \rho} P_\rho.$$

Due to the unitary equivalence given in (5.34) from the Lorentz covariance of the momentum operator we have

$$U(0,\Lambda)\Gamma(U_{\mathscr{F}})X_\mu\Gamma(U_{\mathscr{F}}^{-1})U(0,\Lambda)^{-1} = (\Lambda^{-1})_\mu^{\ \rho}\Gamma(U_{\mathscr{F}})X_\rho\Gamma(U_{\mathscr{F}}^{-1}). \tag{5.38}$$

Now by applying $\Gamma(U_{\mathscr{F}}^{-1})$ on the right side and $\Gamma(U_{\mathscr{F}})$ on the left side of the equation we obtain

$$\Gamma(U_{\mathscr{F}}^{-1})U(0,\Lambda)\Gamma(U_{\mathscr{F}})X_\mu\Gamma(U_{\mathscr{F}}^{-1})U(0,\Lambda)^{-1}\Gamma(U_{\mathscr{F}}) = (\Lambda^{-1})_\mu^{\ \rho}X_\rho. \tag{5.39}$$

From commutator relation (5.10) the following transformation property of the spatial part of X_μ follows,

$$U(0,\Lambda)X_jU^{-1}(0,\Lambda) = (\Lambda^{-1})_j^{\ k}X_k, \qquad \Lambda \in \mathcal{L}_R.$$

By using this transformation property we are able to show that operators $U(0,\Lambda)$ and $\Gamma(U_{\mathscr{F}})$ commute for $\Lambda \in \mathcal{L}_R$,

$$\begin{aligned} \Gamma(U_{\mathscr{F}}^{-1})U(0,\Lambda)\Gamma(U_{\mathscr{F}})X_j\Gamma(U_{\mathscr{F}}^{-1})U(0,\Lambda)^{-1}\Gamma(U_{\mathscr{F}}) &= (\Lambda^{-1})_j^{\ \rho}X_\rho \\ &= U(0,\Lambda)X_jU^{-1}(0,\Lambda). \end{aligned}$$

Hence, from the commutativity of $U(\Lambda)$ and $\Gamma(U_{\mathscr{F}})$ it follows that the operator X_μ transforms in a Lorentz covariant manner under the rotational subgroup of the proper orthochronous Lorentz group,

$$U(0,\Lambda)X_\mu U^{-1}(0,\Lambda) = (\Lambda^{-1})_\mu^{\ \rho}X_\rho, \qquad \Lambda \in \mathcal{L}_R. \tag{5.40}$$

From equation (5.40) the unitary Lorentz transformation of $V(k)$ follows,

$$U(0,\Lambda)V(k)U(0,\Lambda)^{-1} = V(\Lambda k), \qquad \Lambda \in \mathcal{L}_R. \tag{5.41}$$

Next, we study the adjoint action of the antiunitary operator of time reversal $U(i_t)$ and the unitary operator of space inversion $U(i_s)$, acting on the operator X_μ. To proceed, we first give the transformation of the particle creation and annihilation operators under time reversal $U(i_t)$ and space inversion $U(i_s)$, [Sch61, Chapter 7c, Equation (118 a,b)]

$$U(i_{t,s})a(\mathbf{p})U^{-1}(i_{t,s}) = \eta_p a(-\mathbf{p}), \qquad U(i_{t,s})a^*(\mathbf{p})U^{-1}(i_{t,s}) = \eta_p a^*(-\mathbf{p}),$$

where $\eta_p = \pm 1$. From the action of the reversal operators on the particle creation and annihilation operators we can calculate the transformation of X_j under time reversal and space inversion and it is given in the following,

$$\begin{aligned} U(i_{t,s})X_jU^{-1}(i_{t,s}) &= -i\int d^3\mu(\mathbf{p})U(i_{t,s})a^*(\mathbf{p})\left(\frac{p_j}{2\omega_{\mathbf{p}}^2} + \frac{\partial}{\partial p^j}\right)a(\mathbf{p})U^{-1}(i_{t,s}) \\ &= -i\int d^3\mu(\mathbf{p})\eta_p^2 a^*(-\mathbf{p})\left(\frac{p_j}{2\omega_{\mathbf{p}}^2} + \frac{\partial}{\partial p^j}\right)a(-\mathbf{p}) \\ &= +i\int d^3\mu(\mathbf{p})a^*(\mathbf{p})\left(\frac{p_j}{2\omega_{\mathbf{p}}^2} + \frac{\partial}{\partial p^j}\right)a(\mathbf{p}) \\ &= -X_j, \end{aligned}$$

where in the last lines we used the fact that $\eta_p^2 = 1$ and shifted the integration variable $\mathbf{p} \to -\mathbf{p}$.

The position operator has the same transformation properties under time reversal and space inversion as the momentum operator. Hence, it follows from equations (5.38) and (5.39) that the time reversal operator commute with $\Gamma(U_{\mathscr{F}})$ and therefore

$$U(0,\Lambda)X_\mu U^{-1}(0,\Lambda) = (\Lambda^{-1})_\mu^\rho X_\rho, \qquad \Lambda \in \mathcal{L}^{\downarrow}. \tag{5.42}$$

Therefore, for the adjoint action of the antiunitary operator $U(0,\Lambda)$ for $\Lambda \in \mathcal{L}^{\downarrow}$ on the unitary operator $V(k)$ is

$$U(0,\Lambda)V(k)U(0,\Lambda)^{-1} = V(-\Lambda k). \tag{5.43}$$

Moreover, we use in the following proof the adjoint action of the unitary operators $U(0,\Lambda)$ on particle annihilation and creation operators given as follows, [Sch61, Chapter 7c, Equation (57), (58)]

$$U(0,\Lambda)a(\mathbf{p})U(0,\Lambda)^{-1} = a(\pm\mathbf{\Lambda p}), \qquad U(0,\Lambda)a^*(\mathbf{p})U(0,\Lambda)^{-1} = a^*(\pm\mathbf{\Lambda p}), \tag{5.44}$$

where the first sign is for $\Lambda \in \mathcal{L}_R$ and the second sign is for $\Lambda \in \mathcal{L}^{\downarrow}$. Therefore, by using (5.41), (5.43) and (5.44) we obtain that the deformed scalar field $\phi_{\theta,X}$ transforms under the adjoint action of the subgroup $\mathcal{L}_R$ and $\mathcal{L}^{\downarrow}$ in the following way:

$$\begin{aligned} &(2\pi)^{-d}U(0,\Lambda)\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\chi(\epsilon y,\epsilon u)\beta_{\theta y}(\phi(x))V(u)U(0,\Lambda)^{-1} \\ &\quad= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\chi(\epsilon y,\epsilon u)\beta_{\Lambda\theta y}(U(0,\Lambda)\phi(x)U(0,\Lambda)^{-1})V(\Lambda u) \\ &\quad= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,du\,e^{-iyu}\,\chi(\epsilon\sigma\Lambda^T y,\epsilon\Lambda^{-1}u)\,\beta_{\gamma_\Lambda(\theta)y}(\phi(\Lambda x))V(u) = \phi_{\gamma_\Lambda(\theta),X}(\Lambda x), \end{aligned}$$

where σ is $+1$ if $\Lambda \in \mathcal{L}_R$ and -1 if $\Lambda \in \mathcal{L}^{\downarrow}$. Moreover in the last lines the integration variable substitutions $(y,u) \to (\sigma\Lambda^T y, \Lambda^{-1}u)$ were performed. $\square$

Next, we use the homomorphism (5.37) to map the deformed field $\phi_{\theta,X}$ to a field defined on a wedge. Furthermore, we show that a field deformed with the spatial part of the coordinate operator is a wedge-covariant quantum field which transforms covariantly under the adjoint action of a subgroup of the Lorentz group. For this purpose let us first introduce the notion of a wedge-covariant quantum field as given in [GL07].

Definition 5.8. Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions of the Wightman axioms. Then, the field ϕ is defined to be a wedge Lorentz-covariant quantum field if the following condition is satisfied:

- For any $W \in \mathcal{W}_0$ and $f \in \mathscr{S}(\mathbb{R}^d)$ the following holds

$$\begin{aligned} U(\Lambda)\phi_W(f)U(\Lambda)^{-1} &= \phi_{\Lambda W}(f \circ (\Lambda)^{-1}), \qquad \Lambda \in \mathcal{L}_+^{\uparrow}, \\ U(j)\phi_W(f)VU(j)^{-1} &= \phi_{jW}(\overline{f} \circ j)^{-1}. \end{aligned}$$

To define the deformed fields $\phi_{\theta,X}$ as quantum fields defined on the wedge, we use the homomorphism $Q : W \mapsto Q(W)$, this is done in the following way

$$\phi_W(f) := \phi(Q(W), f) = \phi_{\theta,X}(f). \tag{5.45}$$

Proposition 5.3. *The family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ defined by deformation with the operator X_μ are wedge Lorentz-covariant quantum fields on the Bosonic Fock space, w.r.t. the unitary representation $U(0,\Lambda)$ of the subgroup $\mathcal{L}_R$ and the antiunitary representation $U(0,\Lambda)$ of $\mathcal{L}^{\downarrow}$.*

Proof. Following Lemma 5.23, the deformed field $\phi_{\theta,X}(x)$ transforms under the adjoint action $U(0,\Lambda)$ of the subgroup $\mathcal{L}_R$ and $\mathcal{L}^{\downarrow}$ in the following way

$$U(0,\Lambda)\phi_W(x)U(0,\Lambda)^{-1} = U(0,\Lambda)\phi_{\theta,X}(x)U(0,\Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),X}(\Lambda x) = \phi_{\Lambda W}(\Lambda x),$$

where in the last lines we applied the map $Q(\Lambda W) = \gamma_\Lambda(Q(W)) = \gamma_\Lambda(\theta)$. Therefore, one obtains the wedge Lorentz-covariance property of the scalar field under the subgroup $\mathcal{L}_R$ and $\mathcal{L}^{\downarrow}$. $\square$

5.9.3 Wedge-locality of the deformed field

The locality that we prove in this section is the so called wedge-locality. It seems to be the appropriate locality on noncommutative spacetimes. This is due to the fact that the notion of point-wise locality will not hold on NC spacetimes due to the loss of the notion of a point.

To show that the family of fields ϕ_W defined in (5.45) is a family of wedge-covariant quantum fields, we first give the notion of the wedge-local field as given in [GL07].

Definition 5.9. The fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are said to be wedge-local if the following commutator relation is satisfied

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \qquad \Psi \in \mathcal{D},$$

for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$.

By using the former definition the following proposition concerning the deformed field $\phi_{\theta,X}$ follows.

Proposition 5.4. *The family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ defined by $\phi_W(f) := \phi(Q(W), f) = \phi_{\theta,X}(f)$ are wedge-local fields on the Bosonic Fockspace $\mathscr{H}^+$.*

Proof. For the proof we use Proposition 2.2, the unitary equivalence given in Lemma 5.21 and the proof that the free scalar field deformed with the momentum operator is wedge local, [GL07]. To use Proposition 2.2, we have to show that the following commutator vanishes for $f \in C_0^\infty(W_1)$ and $g \in C_0^\infty(-W_1)$,

$$[\beta_{\theta x}(\phi(f)), \beta_{-\theta y}(\phi(g))] = [\beta_{\theta x}(a(\overline{f^-})), \beta_{-\theta y}(a^*(g^+)] - [\beta_{-\theta y}(a(\overline{g^-})), \beta_{\theta x}(a^*(f^+)],$$

where all other terms are equal to zero. Let us first take a look at the first expression of the commutator,

$$\begin{aligned}
[\beta_{\theta x}(a(\overline{f^-})), \beta_{-\theta y}(a^*(g^+)] &= \Gamma(U_{\mathscr{F}}^{-1})[\alpha_{\theta x}\big(\Gamma(U_{\mathscr{F}})a(\overline{f^-})\Gamma(U_{\mathscr{F}}^{-1})\big), \alpha_{-\theta y}\big(\Gamma(U_{\mathscr{F}})a^*(g^+)\Gamma(U_{\mathscr{F}}^{-1})\big)] \\
&\qquad \times \Gamma(U_{\mathscr{F}}) \\
&= \Gamma(U_{\mathscr{F}}^{-1})\Big(\int d^3\mu(\mathbf{p}) \int d^3\mu(\mathbf{k}) f^-(\mathbf{p}) g^+(\mathbf{k}) e^{-ip\theta x} e^{-ik\theta y} \\
&\qquad \times \Gamma(U_{\mathscr{F}})[a(\mathbf{p}), a^*(\mathbf{k})]\Gamma(U_{\mathscr{F}}^{-1})\Big)\Gamma(U_{\mathscr{F}}) \\
&= \Gamma(U_{\mathscr{F}}^{-1}) \left(\int d^3\mu(\mathbf{p}) f^-(\mathbf{p}) g^+(\mathbf{p}) e^{-ip\theta(x+y)}\right) \Gamma(U_{\mathscr{F}}) \\
&= \Gamma(U_{\mathscr{F}}^{-1}) \left(\int d^3\mu(\mathbf{p}) f^+(\mathbf{p}) g^-(\mathbf{p}) e^{ip\theta(x+y)}\right) \Gamma(U_{\mathscr{F}}),
\end{aligned}$$

where in the last lines we used the unitary equivalence (5.34), the commutation relations of the Fourier transformed particle operators, which are the same as the commutation relations of the untransformed particle operators. Furthermore the proof of Lemma 3.3 in [GL07] was used to change the signs of f and g. Finally, we look at the second expression of the commutator and obtain the following,

$$\begin{aligned}
[\beta_{-\theta y}(a(\overline{g^-})), \beta_{\theta x}(a^*(f^+)] &= \Gamma(U_{\mathscr{F}}^{-1})[\alpha_{-\theta y}\big(\Gamma(U_{\mathscr{F}})a(\overline{g^-})\Gamma(U_{\mathscr{F}}^{-1})\big), \alpha_{\theta x}\big(\Gamma(U_{\mathscr{F}})a^*(f^+)\Gamma(U_{\mathscr{F}}^{-1})\big)] \\
&\qquad \times \Gamma(U_{\mathscr{F}}) \\
&= \Gamma(U_{\mathscr{F}}^{-1})\Big(\int d^3\mu(\mathbf{p}) \int d^3\mu(\mathbf{k}) f^+(\mathbf{p}) g^-(\mathbf{k}) e^{ip\theta x} e^{ik\theta y} \\
&\qquad \times \Gamma(U_{\mathscr{F}})[a(\mathbf{k}), a^*(\mathbf{p})]\Gamma(U_{\mathscr{F}}^{-1})\Big)\Gamma(U_{\mathscr{F}}) \\
&= \Gamma(U_{\mathscr{F}}^{-1}) \left(\int d^3\mu(\mathbf{p}) f^+(\mathbf{p}) g^-(\mathbf{p}) e^{ip\theta(x+y)}\right) \Gamma(U_{\mathscr{F}}).
\end{aligned}$$

Since the second expression of the commutator $[\beta_{\theta x}(\phi(f)), \beta_{-\theta y}(\phi(g))]$ is equal to the first one with a sign difference, the commutator vanishes. Hence, the fields ϕ_W are wedge-local. □

Note that from the wedge-locality of the deformed field the locality of expression (5.26) trivially ($\theta_{0j} = 0$) follows. Thus the term given in Equation (5.26) is a wedge-local expression.

5.10 NC Momentum Plane

For some readers it may seem that the warped convolutions performed on a free scalar field with the coordinate operator defined in (5.34), is simply the deformation done by [GL07] but in momentum space, i.e. spanning a quantum plane in momentum space $[p_\mu, p_\nu] = iB_{\mu\nu}$. This is not the case. From warped convolutions with the coordinate operator one obtains a more physical quantum plane, since the deformation is done on the relativistic physical momentum. This argument is made more precise by calculating the deformed commutator of the momentum, which is defined by using the deformed product (2.2) induced by the position operator X_j. Let us first define the deformed commutator of two momentum vectors as the following

$$[p_\mu \overset{\times_B}{,} p_\nu] = (2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \left(\beta_{\mathbf{Bv}}(p_\mu)\beta_{\mathbf{u}}(p_\nu) - \beta_{\mathbf{Bv}}(p_\nu)\beta_{\mathbf{u}}(p_\mu)\right). \tag{5.46}$$

Lemma 5.24. *Let the deformed product (2.2) be defined by the position operator X_j. Then the deformed commutator (5.46) of the momentum four-vectors is given as*

$$[p_\mu \overset{\times_B}{,} p_\nu] = -2iB^{\mathbf{p}}_{\mu\nu},$$

where the noncommutative matrix $B^{\mathbf{p}}_{\mu\nu}$ is non-constant but momentum dependent and is given as follows

$$B^{\mathbf{p}}_{\mu\nu} = \begin{pmatrix} 0 & 2B^{kr}p_r/\omega_{\mathbf{p}} \\ -2B_{kl}p^l/\omega_{\mathbf{p}} & B_{jr} \end{pmatrix}. \tag{5.47}$$

Proof. Since the position operator generates a translation in momentum space, the adjoint action of the unitary transformation on a momentum vector is for the zero component given as,

$$\beta_{\mathbf{k}}(p_0) = \beta_{\mathbf{k}}(\omega_{\mathbf{p}}) = \omega_{\mathbf{p}+\mathbf{k}},$$

and on the spatial part as follows,

$$\beta_{\mathbf{k}}(p_r) = p_r + k_r.$$

In the next step we calculate the deformed commutator of the zero component and the spatial components,

$$\begin{aligned}
[p_0 \overset{\times_B}{,} p_j] &= (2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \left(\beta_{\mathbf{Bv}}(p_0)\beta_{\mathbf{u}}(p_j) - \beta_{\mathbf{Bv}}(p_j)\beta_{\mathbf{u}}(p_0)\right) \\
&= (2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \Big(\omega_{\mathbf{p}+\mathbf{Bv}}(p_j + u_j) - (p_j + (Bv)_j)\omega_{\mathbf{p}+\mathbf{u}}\Big)
\end{aligned}$$

Let us look at the first part,

$$\begin{aligned}
&(2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \omega_{\mathbf{p}+\mathbf{Bv}}(p_j + u_j) \\
&= (2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \omega_{\mathbf{p}+\mathbf{Bv}} p_j + (2\pi)^{-n} \iint d^n v d^n u \left(i\frac{\partial}{\partial v^j} e^{-iv_k u^k}\right) \omega_{\mathbf{p}+\mathbf{Bv}} \\
&= \omega_{\mathbf{p}} p_j - (2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \left(i\frac{\partial}{\partial v^j} \omega_{\mathbf{p}+\mathbf{Bv}}\right) \\
&= \omega_{\mathbf{p}} p_j - i(2\pi)^{-n} \iint d^n v d^n u e^{-iv_k u^k} \left(\frac{2(Bp)_j}{\omega_{\mathbf{p}+\mathbf{Bv}}} + \frac{2B_{jk}(Bv)^k}{\omega_{\mathbf{p}+\mathbf{Bv}}}\right) \\
&= \omega_{\mathbf{p}} p_j - 2i(Bp)_j/\omega_{\mathbf{p}}.
\end{aligned}$$

The second part of the integral gives the following

$$
\begin{aligned}
&-(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}(p_j+(Bv)_j)\omega_{\mathbf{p}+\mathbf{u}}\\
&=-(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}p_j\omega_{\mathbf{p}+\mathbf{u}}-(2\pi)^nB_{jk}\iint d^nvd^nu\left(i\frac{\partial}{\partial u_k}e^{-iv_ku^k}\right)\omega_{\mathbf{p}+\mathbf{u}}\\
&=-p_j\omega_{\mathbf{p}}+(2\pi)^{-n}B_{jk}\iint d^nvd^nue^{-iv_ku^k}\left(i\frac{\partial}{\partial u_k}\omega_{\mathbf{p}+\mathbf{u}}\right)\\
&=-p_j\omega_{\mathbf{p}}-(2\pi)^{-n}2iB_{jk}\iint d^nvd^nue^{-iv_ku^k}\frac{(p^k+u^k)}{\omega_{\mathbf{p}+\mathbf{u}}}\\
&=-p_j\omega_{\mathbf{p}}-2i(Bp)_j/\omega_{\mathbf{p}}.
\end{aligned}
$$

By summing the two parts of the integral we obtain for the deformed commutator the following

$$[p_0 \overset{\times_B}{,} p_j]=-4i(Bp)_j/\omega_{\mathbf{p}}.$$

The deformed commutator of the spatial parts is derived in the same manner

$$
\begin{aligned}
[p_k \overset{\times_B}{,} p_j]&=(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}\left(\beta_{\mathbf{Bv}}(p_k)\beta_{\mathbf{u}}(p_j)-\beta_{\mathbf{Bv}}(p_j)\beta_{\mathbf{u}}(p_k)\right)\\
&=(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}\left((p_k+(Bv)_k)(p_j+u_j)-k\leftrightarrow j\right)\\
&=\left(p_kp_j+(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}(Bv)_ku_j\right)-k\leftrightarrow j\\
&=(2\pi)^{-n}\iint d^nvd^nu\left(i\frac{\partial}{\partial v^j}e^{-iv_ku^k}\right)(Bv)_k-k\leftrightarrow j\\
&=-(2\pi)^{-n}\iint d^nvd^nue^{-iv_ku^k}\left(i\frac{\partial}{\partial v^j}(Bv)_k\right)-k\leftrightarrow j\\
&=-2iB_{kj}
\end{aligned}
$$

□

The noncommutative momentum plane, spanned by the deformed product, is nonconstant. It depends on the relativistic velocity, i.e. $p_j/\omega_{\mathbf{p}}$. The relativistic velocity appears in a non-trivial manner and this is owed to the fact that we have a condition on the zero component of the momentum. Thus the physical on-mass shell condition forces nontrivial commutation relations on the momentum plane.

Moreover, in th QM chapter (see Lemma 3.13), the NC coordinates describing the Landau quantization are given by a minimal substitution of the position operator, $Q_i = X_i + B^{-1}_{ik}P^k$. Let us compare in the QFT context the approach of minimal substitution, i.e.

$$X_i^B := X_i - \theta_{i\rho}P^{\rho}, \tag{5.48}$$

with the deformation of the coordinate operators. By having the QM case in mind, the matrix θ must be in the spatial components, i.e. θ_{ij}, equal to the inverse of the spatial part of the noncommutative matrix B (see Equation 5.47). Furthermore, we choose the $(0j)$-component equal to $\theta_{j0} = -2\theta_{jk}V^k$ and obtain the following lemma,

Lemma 5.25. *Let the minimal substituted coordinates X_i^B be given by Equation (5.48). Then, on a one-particle level X_i^B satisfies the commutator relations*

$$[X_i^B, X_j^B] = -2i\,(\theta_{0i}V_j - \theta_{0j}V_i) + 2i\theta_{ij},$$

These are the relations of the QFT-Moyal-Weyl (see Lemmas 5.12 and 5.13).

Proof. For the proof we take a look at the following commutator

$$\begin{aligned}
[X_i^B, X_j^B] &= [X_i - \theta_{i\rho}P^\rho, X_j - \theta_{j\sigma}P^\sigma] \\
&= -[X_i, \theta_{j\sigma}P^\sigma] - i \leftrightarrow j \\
&= -[X_i, \theta_{j\sigma}]P^\sigma - \theta_{j\sigma}[X_i, P^\sigma] - i \leftrightarrow j \\
&= -[X_i, \theta_{j0}]P^0 - \theta_{j0}[X_i, P^0] - \theta_{jk}[X_i, P^k] - i \leftrightarrow j \\
&= +2\theta_{jk}[X_i, V^k]P^0 + i\theta_{j0}V_i + i\theta_{ji} - i \leftrightarrow j \\
&= -2i\theta_{jk}(\eta_i^k + V_iV^k) + i\theta_{j0}V_i - i\theta_{ij} - i \leftrightarrow j \\
&= 2i\theta_{j0}V_i + i\theta_{ij} - i \leftrightarrow j \\
&= -2i\,(\theta_{i0}V_j - \theta_{jo}V_i) + 2i\theta_{ij},
\end{aligned}$$

where in the last lines we used the commutator relations of X_i and the momentum and the skew-symmetry of θ w.r.t. the Minkowski metric, i.e. $\theta_{0j} = \theta_{j0}$, $\theta_{kj} = -\theta_{jk}$.

□

Hence the QFT-Moyal-Weyl emerges by following the same steps as the Landau quantization in QM. Let us summarize, the deformation with the coordinate operator in a QF-theoretical context gives a wedge-local QF on a NC momentum space, that can equivalently be interpreted as a QF on a NC space. This is similar to the QM case, where we obtained by deformation with the coordinate operator, the Hamiltonian of a particle in the presence of a magnetic field. By performing a minimal substitution given by the deformation matrix (magnetic field), one obtains the guiding center coordinates, which give the right operator description of the underlying space.

In QM we had a physical interpretation for the deformation parameter. The correspondence was given by the constant of a homogeneous magnetic field. For the QFT case such an interpretation is still missing and will be the subject of the next section.

5.11 Deformation Parameter from Unitary Transformations

To obtain an interpretation for the deformation parameter we compare in this section expression $[\phi_{\theta,\mathbf{X}}(f), \phi_{-\theta,\mathbf{X}}(g)]$ with the commutator of two unitarily transformed fields given in Equation (5.28). Where the latter expression was obtained by calculating the commutator of two unitarily transformed scalar fields, using the unitary operators $V(\mathbf{b})$ and $V(\mathbf{b}')$. The investigation shows that the difference of the vectors $\mathbf{b}$ and $\mathbf{b}'$ can be used to obtain a deformation parameter.

Lemma 5.26. *By comparing the commutator of the unitary transformed scalar fields $\beta_{\mathbf{b}}(\phi(f))$ and $\beta_{\mathbf{b}'}(\phi(g))$ with the commutator of two deformed fields $\phi_{\theta,\mathbf{X}}(f)$ and $\phi_{-\theta,\mathbf{X}}(g)$, the square root of the deformation parameter η of the skew-symmetric matrix θ (see Equation (4.12)), can be understood as the difference of the vectors $\mathbf{b}$ and $\mathbf{b}'$.*

Proof. Let us first calculate the commutator of two deformed fields $\phi_{\theta,\mathbf{X}}(f)$ and $\phi_{-\theta,\mathbf{X}}(g)$ on eigenfunctions of the position operator and for the sake of clarity we define the multi variable $(\mathbf{x}) := (\mathbf{x}_1, \cdots \mathbf{x}_k)$,

$$\begin{aligned}
\left([\phi_{\theta,\mathbf{X}}(f), \phi_{-\theta,\mathbf{X}}(g)]\Psi_k\right)(\mathbf{x}) &= \left([a_{\theta,\mathbf{X}}(\overline{f^-}), a^*_{-\theta,\mathbf{X}}(g^+)] + [a^*_{\theta,\mathbf{X}}(f^+), a_{-\theta,\mathbf{X}}(\overline{g^-})]\Psi_k\right)(\mathbf{x}) \\
&= \int \frac{d^3\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}\sqrt{2\omega_{\mathbf{p}+\mathbf{s}}}} \left(f^-(\mathbf{p}+\mathbf{s})g^+(\mathbf{p}) - f^+(\mathbf{p}+\mathbf{s})g^-(\mathbf{p})\right)\Psi_k(\mathbf{x}),
\end{aligned}$$

where in the last line to ease readability, we defined $s_k := (0, 2(\theta \sum_{l=1}^{k} x_l)_r)$. By comparing the last line with Equation (5.28) one can make the obvious identification

$$b_k - b'_k \equiv s_k$$

Let us first write b and b' in the following way

$$b_k = b\beta_k, \quad b'_k = b'\beta_k, \quad |\beta_k| = 1$$

the difference becomes

$$b_k - b'_k = (b - b')\beta_k.$$

Let us set $\beta_1 = 0$ for the moment. We will now turn our attention to s_r.

$$\begin{aligned}
s_r = 2\theta_{re}\sum_{m=1}^{k} x^e_m = 2\begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}\begin{pmatrix} \sum_{m=1}^{k} x^1_m \\ \sum_{m=1}^{k} x^2_m \end{pmatrix} &= 2\sqrt{\eta}\underbrace{\begin{pmatrix} 0 & \sqrt{\eta} \\ -\sqrt{\eta} & 0 \end{pmatrix}\begin{pmatrix} \sum_{m=1}^{k} x^1_m \\ \sum_{m=1}^{k} x^2_m \end{pmatrix}}_{=dimensionless} \\
&:= 2\sqrt{\eta}\beta_r
\end{aligned}$$

In the last line we identified the dimensionless vector $\theta_{re}\sum_{m=1}^{k} x^e_m$ with β_k. From this simple considerations it follows that the deformation parameter $\sqrt{\eta}$ can be deduced from the difference $\mathbf{b} - \mathbf{b}'$.

$$|b_k - b'_k| = |b - b'| = 2\sqrt{\eta}$$

□

This is an interesting result. We first calculated the unitary transformations $V(\mathbf{b})$ on the free scalar field. Next, we derived the commutator for two different vectors $\mathbf{b}, \mathbf{b}'$ and identified the result with the commutator of the warped convoluted fields $[\phi_{\theta,\mathbf{X}}(f), \phi_{\theta,\mathbf{X}}(g)]$. We obtained as a consequence that the length of the difference of the vectors $\mathbf{b}, \mathbf{b}'$ is the deformation parameter. How can this result be understood?

To understand this let us analyze the action of the unitary transformations acting on the field. They act as translations in momenta space of the free scalar field. This means that the translation in momenta space in b_j direction and then the translation in momenta space in b'_k direction is not the same as the translation in momenta space in b'_k direction and then the translation in momentum space in b_j direction. The difference of the action of translation results in momenta space as a vector which is not equal to zero, but denotes a vector who's length is equal to the square root of deformation parameter η.

Thus by comparing two wedge-local quantities namely $[\phi_{\theta,\mathbf{X}}(f), \phi_{-\theta,\mathbf{X}}(g)]$ and expression (5.26), one deduces the meaning of the deformation parameter.

Chapter 6

Deformations with conformal operator

In the last chapter we obtained a physical Moyal-Weyl plane by deforming with the coordinate operator. The operator was chosen by requiring the fulfillment of a symplectic structure with the momentum operator. Moreover, the investigation of deformation showed that the deformed field satisfies certain properties, as for example wedge-locality.

In this chapter we deform a massless QFT by using the generators of the special conformal transformations. The choice comes from the fact that the special conformal operator fulfills a symplectic structure with the momentum operator and in addition transforms Lorentz-covariant.

Prior to deformation, we define the conformal group. Next, we recollect some basic results obtained in [SV73] concerning the self-adjointness of the special conformal operator and the transformation of the free scalar field under special conformal transformations. The investigation of deformation shows that the obtained field is wedge-local. Furthermore, we are able to obtain a space-time, induced by deformation with the special conformal operators, that is nonconstant and noncommutative. This leads to a new QFT model which on one hand can be interpreted as a QF on a nonconstant noncommutative spacetime, and on the other hand as a wedge-local QFT model.

This chapter is an extension of the paper published by the author, [Alb12].

6.1 Generators of the conformal group

We start discussing about the conformal group by giving a definition about the conformal transformations, ([DMS97], Chapter IV, Section 1).

Definition 6.1. A **conformal transformation** of the coordinates is defined to be an invertible mapping $\mathbf{x}' \to \mathbf{x}$, which leaves the d-dimensional metric g invariant up to a scale factor,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{6.1}$$

The set of all conformal transformations forms a group. This group is denoted as the **conformal group**.

It is obvious from the defining equation (6.1), that in the case $\Lambda(x) = 1$ one obtains the Poincaré group. Thus the conformal group has the Poincaré group as a subgroup.

By exploring the consequences of the conformal transformations on the coordinates one deduces that the finite transformations fulfilling equation (6.1) are the following.

(i) **Translations**,
$$x'^{\mu} = x^{\mu} + a^{\mu}, \qquad a \in \mathbb{R}^d.$$

(ii) **Dilatations**,
$$x'^{\mu} = \alpha x^{\mu}, \qquad \alpha \in \mathbb{R}^+.$$

(iii) **Lorentz transformations**,
$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \qquad \Lambda \in O(1, d-1).$$

(iv) **Special conformal transformations**,
$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b^{\nu} x_{\nu} + b^2 x^2}, \qquad b \in \mathbb{R}^d,$$

where $x^2 = x^{\mu} x_{\mu}$.

In the case of special conformal transformations, the scale factor given in the defining equation (6.1) depends on the coordinates. For example in the four dimensional case it is given as,
$$\Lambda(x) = \left(1 - 2b_{\mu} x^{\mu} + b^2 x^2\right)^2.$$
In the subsequent discussion on deformations of the free scalar field the sign of the scale factor becomes important. Furthermore, an important fact in the context of the special conformal transformations is the following. Any special conformal transformation can be expressed as a translation followed by an inversion $x^{\mu} \to x^{\mu}/x^2$. This condition is a fundamental building block of the construction of an essential self-adjoint operator of the special conformal operators in [SV73].

The generators of infinitesimal transformations of the conformal group are given by the (pseudo-)differential operators that generate

(i) **translations**,
$$P_{\mu} = -i\partial_{\mu},$$

(ii) **dilatations**,
$$D = -ix^{\mu}\partial_{\mu},$$

(iii) **Lorentz transformations**,
$$L_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right),$$

(iv) **special conformal transformations**,

$$K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right).$$

The generators $(P_\mu, K_\nu, L_{\rho\sigma}, D)$ define the conformal algebra, which is given by the following commutation relations,

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\left(\eta_{\mu\sigma} L_{\nu\rho} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}\right), \tag{6.2}$$

$$[P_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right), \tag{6.3}$$

$$[K_\rho, L_{\mu\nu}] = i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right), \tag{6.4}$$

$$[P_\rho, D] = iP_\rho, \tag{6.5}$$

$$[K_\rho, D] = -iK_\rho, \tag{6.6}$$

$$[P_\rho, K_\mu] = 2i\left(\eta_{\rho\mu} D - L_{\rho\mu}\right), \tag{6.7}$$

with all other commutators being equal to 0.

6.2 Isomorphism between the conformal group and $SO(2, d)$

To see the isomorphism between the conformal group in d dimensions and the pseudo-orthogonal group $SO(2, d)$, one introduces the following definitions:

$$\begin{gathered}
J_{4,\mu} := \frac{1}{2}\left(P_\mu - K_\mu\right), \qquad J_{5,\mu} := \frac{1}{2}\left(P_\mu + K_\mu\right), \\
J^{\pm}_\mu := J_{5,\mu} \pm J_{4,\mu} \qquad J_{-1,0} := D, \qquad J_{\mu\nu} := L_{\mu\nu}, \\
J_{ab} = -J_{ba}, \qquad a, b = 0, 1, \ldots, d, d+1.
\end{gathered}$$

The defined generators J_{ab} obey the algebra of $SO(2, d)$ with the following commutator relations:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\right),$$

where the diagonal metric has the following form

$$\eta_{aa} = (+1, \underbrace{-1, .., -1}_{d}, +1).$$

This shows the isomorphism between the conformal group and $SO(2, d)$. One can easily see in the context of the commutator relations that the full conformal group contains the Poincaré group as a subgroup.

6.3 Self-adjointness of the special conformal operators

To proceed with deforming via warped convolutions, it is necessary to prove self-adjointness of the special conformal operator K_μ. The proof was given in [SV73] and relies on the following definition.

Definition 6.2. The special conformal generator K_μ can be defined as an operator, which is unitary equivalent to the momentum operator

$$K_\mu := U_R P_\mu U_R, \tag{6.8}$$

where the unitary equivalence is given by U_R, the inversion operator.

As already mentioned, the reason for the definition is that any special conformal transformation

$$K(b)x^\mu = \frac{x_\mu - b_\mu x^2}{1 - 2b_\nu x^\nu + b^2 x^2},$$

can be written as a product

$$K(b) = RT(b)R,$$

of a translation $T(b)x^\mu = x^\mu + b^\mu$ and inversions $Rx^\mu = -x^\mu/x^2$. Another important reason for the definition (6.8) in a quantum field theoretical context, is the fact that the momentum operator P_μ is well known to be an essential self-adjoint operator on a dense subspace of the one particle Hilbert space. Thus the first step is to construct a self-adjoint unitary representation $U(R)$ of the inversion in the Hilbert space $L^2(d^n\mu(\mathbf{p})) := \{f : \int d^n\mathbf{p}\,(2|\mathbf{p}|)^{-1}\,|f(\mathbf{p})|^2 < \infty\}$. This is done by first constructing a symmetric sesquilinear form $R(g,f)$, on the dense domain

$$D^n := \{f(\mathbf{p}) \in L^2(d^n\mu(\mathbf{p})) \cap L^1(d^n\mu(\mathbf{p})) : \left(\mathbf{p}^2\right)^r \frac{\partial^k}{\prod_{i=1}^n \left(\partial \mathbf{p}^i\right)^{k_i}} f(\mathbf{p}) \in L^1(d^n\mu(\mathbf{p})),$$
$$\forall 0 \le r \le n, 0 \le k = \sum_{i=1}^n k_i \le 2n\}.$$

Next it is shown that the symmetric sesquilinear form $R(g,f)$, defines a self-adjoint unitary operator $U(R)$ in $L^2(d^n\mu(\mathbf{p}))$ for $n \ge 1$ acting on functions $f(\mathbf{p}) \in D^n$ as follows,

$$\left(U(R)f\right)(\mathbf{p}) = \frac{2}{(2\pi)^n} \int d^n x (\mathbf{x}^2)^{-\frac{n+1}{2}} \int d^n\mu(\mathbf{q}) |\mathbf{q}| e^{-i\left(\mathbf{p}\mathbf{x} - \frac{\mathbf{q}\mathbf{x}}{\mathbf{x}^2}\right)} f(\mathbf{q}).$$

Furthermore, by using the unitary operator $U(R)$ one constructs the essential self-adjoint operator K^μ for $\mu = \{0, 1, \dots, n\}$ of the special conformal group on the dense domain $\triangle^n(R) := U(R)\triangle^n(P)$. Where a dense domain $\triangle^n(P)$ is given as follows

$$\triangle^n(P) := \{f(\mathbf{p}) \in L^2(d^n\mu(\mathbf{p})) : |\left(\mathbf{p}^2\right)^r f(\mathbf{p})| \le c_r(f) < \infty; r = 0, 1, 2, \dots\}.$$

$\triangle^n(P)$ is contained in the domain of the essentially self-adjoint operators P_μ for $\mu = \{0, 1, \dots, n\}$ acting on functions $f(\mathbf{p}) \in \triangle^n(P)$ as

$$\left(P_\mu f\right)(\mathbf{p}) = p_\mu f(\mathbf{p}), \qquad p^0 := |\mathbf{p}|.$$

Moreover the dense domain $\triangle^n(P)$ is stable under the application of the translation operator P_μ:

$$P_\mu \triangle^n(P) \subseteq \triangle^n(P)$$

Finally, on the dense domain $\triangle^n(R)$ one defines the operators K^μ by

$$K_\mu := U(R) P_\mu U(R),$$

The self-adjointness of the operators K^μ follows from the fact that the operators P_μ are essentially self-adjoint and the inversion operator $U(R)$ is unitary. Moreover, it follows that P_μ and K_μ have the same spectrum. Thus, by choosing the irreducible representation of the Poincaré group to be in the closed forward cone, the spectrum of the operators P_μ and K_μ is contained in the closed forward cone

$$\overline{V_0^+} := \{p^\mu : p^\mu p_\mu \geq 0, p_0 \geq 0\}.$$

From the definition of the dense domain $\triangle^n(R)$ it follows,

$$K_\mu \triangle^n(R) \subseteq \triangle^n(R).$$

In the last step the authors show that the constructed essentially self-adjoint operator K^μ is identical to the special conformal operator $\hat{K}^\mu$ defined by,

$$(\hat{K}^\mu \tilde{f})(x) := i\left((n-1)x^\mu + 2x^\mu x^\nu \frac{\partial}{\partial x^\nu} - x^2 \frac{\partial}{\partial x_\mu}\right)\tilde{f}(x).$$

This is done by calculating the action of $\hat{K}^\mu$ in the scalar product of the Hilbert space $L^2(d^n\mu(\mathbf{p}))$ for all f from the dense subset $\mathscr{S}(\mathbb{R}^n) \subset L^2(d^n\mu(\mathbf{p}))$ and all $g \in L^2(d^n\mu(\mathbf{p}))$:

$$(g, \hat{K}^0 f) = -\int d^n\mu(\mathbf{p})\overline{g(\mathbf{p})}|\mathbf{p}|\frac{\partial}{\partial p^k}\frac{\partial}{\partial p_k} f(\mathbf{p})$$

$$(g, \hat{K}^i f) = \int d^n\mu(\mathbf{p})\overline{g(\mathbf{p})}\left((n-1)\frac{\partial}{\partial p_i} + 2p^k \frac{\partial}{\partial p^k}\frac{\partial}{\partial p_i} - p^i \frac{\partial}{\partial p^k}\frac{\partial}{\partial p_k}\right) f(\mathbf{p}).$$

The calculation is then compared with the result done with the operator K^μ and it follows

$$(K^\mu f)(\mathbf{p}) = (\hat{K}^\mu f)(\mathbf{p}), \qquad \forall f \in \mathscr{S}(\mathbb{R}^n).$$

6.4 Special conformal transformations

Since in the context of the present paper we need the transformation of the free scalar field under the special conformal group, we shall also briefly summarize those results obtained in [SV73].

For $n = 1$ the existence of a unitary representation for the whole conformal group was proven. The special conformal operator transforms the free scalar field $\phi(x)$ in the following manner

$$\alpha_b(\phi(x)) := e^{ib_\mu K^\mu}\phi(x)e^{-ib_\mu K^\mu} = \phi(x_b) - \phi(-\frac{b}{b^\mu b_\mu}),$$

where

$$x_b^\mu := \frac{x_\mu - b_\mu x^2}{1 - 2b_\nu x^\nu + b^2 x^2}.$$

In atwo dimensional spacetime test functions $f \in \mathscr{S}(\mathbb{R}^2)$, used to smear the distribution valued operator $\phi(x)$, are chosen to satisfy $\int d^2x f(x) = 0$. The reason for this specific choice is to circumvent IR-divergences and it is used throughout the entire work.

Now if $n = 2l + 1$ for $l \in \mathbb{N}$ one obtains the following result

$$\alpha_b(\phi(x)) = \sigma_b(x)^{\frac{1-n}{2}}\phi(x_b), \tag{6.9}$$

where

$$\sigma_b(x) := 1 - 2b_\mu x^\mu + b^2x^2. \tag{6.10}$$

It was further proven that one only obtains a unitary representation for the whole conformal group if $n = 4l+1$ for $l \in \mathbb{N}$. Thus, the formal transformation law (6.9) is only compatible with the correct transformation law of the scalar field under the action of the special conformal transformations if $n = 4l + 1$ for $l = \{0, 1, 2, \dots\}$. This is due to the following argument. For odd $n > 1$ the following transformation law holds,

$$e^{ib_\mu K^\mu}\phi(0)e^{-ib_\mu K^\mu} = \phi(0).$$

If one has a unitary representation of the whole conformal group for $n > 1$ in [HSS72] it is shown that the following relation holds

$$e^{ib_\mu K^\mu}e^{ix_\mu P^\mu} = TU\Lambda V,$$

with

$$T(x_b) = e^{ix_b^\mu P_\mu}, \quad V(\log|\sigma_b(x)|) = e^{i\log|\sigma_b(x)|D}.$$

This relation is proved by making use of the canonical decomposition. Now by using the relations given above the transformation of the scalar field under the adjoint action of the special conformal operator can be written as

$$e^{ib_\mu K^\mu}\phi(x)e^{-ib_\mu K^\mu} = e^{ib_\mu K^\mu}e^{ix^\mu P_\mu}\phi(0)e^{ix^\mu P_\mu}e^{-ib_\mu K^\mu} = TU\Lambda V\phi(0)V^{-1}\Lambda^{-1}U^{-1}T^{-1}.$$

By using the invariance of the scalar field under special conformal and Lorentz transformations in the origin, one obtains

$$TU\Lambda V\phi(0)V^{-1}\Lambda^{-1}U^{-1}T^{-1} = (|\sigma_b(x)|)^{\frac{1-n}{2}}\phi(x_b).$$

As one can easily see this result only coincides with the correct transformation property of the field under special conformal transformations if $n = 4l + 1$ for $l = \{0, 1, 2, \dots\}$, because in that case the absolute value does not play any role. Thus the reason for the non-existence of a unitary representation for the whole conformal group lies in the non positivity of the scale factor $\sigma_b(x)$.

For this work it becomes important, since we intend to work in four spacetime dimensions and therefore need a unitary representation of the whole conformal group. We prove that the scale factor $\sigma_b(x)$ is positive for the scalar field localized in the wedge and therefore argue that the deformation with the special conformal operator is well defined.

In the cases of even n one has to deal with representations of the covering of the conformal group. The subject of even n is not treated in this thesis.

6.5 Deforming the QF with special conformal operators

In this section we deform a massless scalar field with the special conformal operators using the framework warped convolutions. To proceed with the deformation, we use the definitions of the massless Bosonic Fock space $\mathscr{H}_0^+$, (see Definition 5.1). The undeformed free scalar field ϕ with mass $m = 0$ on the $(n+1)$-dimensional Minkowski spacetime is defined as an operator-valued distribution acting on its domain in the Bosonic Fock space $\mathscr{H}_0^+$. Such a particle with momentum $\mathbf{p} \in \mathbb{R}^n$ has the energy defined by $\omega_{\mathbf{p}} = |\mathbf{p}|$.

To define the warped convolutions of the free scalar field we use the essential self-adjointness of the generators K_μ which in turn define a unitary operator $U(b) := e^{ib_\mu K^\mu}$. The definition of the operator valued function $U(b)$ leads to a strongly continuous unitary representation of $\mathbb{R}^d$, for each $b_\mu \in \mathbb{R}^d$. This can be proven by using Stone's theorem (see Theorem 3.1).

For the deformation in a quantum field theoretical context we are obliged to give the second quantization of the special conformal operator. This is done in the following definition.

Definition 6.3. Let the second quantization of the momentum P_μ and the special conformal operator be denoted by the same symbol. Then, the operator K_μ can be given as the second quantization of the defining equation (6.8) as follows,

$$K_\mu = \Gamma(U_R) P_\mu \Gamma(U_R), \tag{6.11}$$

where $\Gamma(U_R) := \bigotimes_{i=1}^k U_R$ is the second quantization of unitary operator of the inversions on $\mathscr{H}_{0,k}^+$.

The former definition is used in the following proposition concerning the essential self-adjointness of the second quantized version of the special conformal operator.

Proposition 6.1. *Let the second quantized operator P_μ be an essential self-adjoint operator on the extended dense domain $\Delta_k^n(P)$ defined as $\Delta_k^n(P) := \bigotimes_{i=1}^k \Delta^n(P)$. Then, the second quantized operator K_μ given by unitary equivalence, is an essential self-adjoint operator on the extended dense domain $\Delta_k^n(R) = \Gamma(U_R)\Delta_k^n(P)$. Therefore, the unitary operator $U(p) = e^{ip_\mu K^\mu}$ defines a strongly continuous group for all $p \in \mathbb{R}^d$.*

Proof. According to Lemma (5.16), if an operator is essentially self-adjoint on a dense domain of the one-particle Hilbert space, then the second quantized operator is essentially self-adjoint on the tensor product of the dense domain. Thus, the the essential self-adjointness of the momentum operator follows. Due to the unitary equivalence we use Proposition 8.1 and it follows that the second quantized operator K_μ given in Definition 6.11 is essentially self-adjoint. □

Definition 6.4. Let θ be a real skew-symmetric matrix w.r.t. the Minkowski scalar-product on $\mathbb{R}^d$ and let $\chi \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Furthermore, let $\phi(f)$ be the massless free scalar field smeared out with functions $f \in \mathscr{S}(\mathbb{R}^d)$. Then the operator valued distribution $\phi(f)$ deformed with the second quantized special conformal operator K_μ, denoted as $\phi_{\theta,K}(f)$, is defined on vectors of the dense domain $\Delta_k^n(R)$ as follows

$$\begin{aligned}\phi_{\theta,K}(f)\Psi_k :&= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)\alpha_{\theta y}(\phi(f))U(v)\Psi_k\\ &= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)\alpha_{\theta y}\left(a(\overline{f^-})+a^*(f^+)\right)U(v)\Psi_k\\ &=: \left(a_{\theta,K}(\overline{f^-})+a^*_{\theta,K}(f^+)\right)\Psi_k,\end{aligned} \tag{6.12}$$

where $a(f)$ and $a^*(f)$ are the massless particle creation and annihilation operator given in definition 5.2 and the test functions $f^{\pm}(\mathbf{p})$ in momentum space are defined as follows

$$f^{\pm}(\mathbf{p}) := \int d^dx f(x)e^{\pm ipx}, \qquad p=(\omega_{\mathbf{p}},\mathbf{p})\in\partial V_+.$$

As in the last chapter the integral (6.12) has to be understood as an integral in oscillatory sense, [Rie93]. The unboundedness of operator K_μ and the undeformed unbounded operator-valued distribution ϕ imposes the question of existence of the integral. Hence, as before we show that integral (6.12) converges due to the unitary equivalence of the special conformal operator with the momentum operator.

For this purpose, we give the following transformation.

Lemma 6.1. *For $f\in\mathscr{S}(\mathbb{R}^d)$ and $\Psi_k\in\Delta_k^n(R)$, a transformation exists that maps the field deformed with the momentum operator $\phi_{\theta,P}(f)$ to the field deformed with the special conformal operator $\phi_{\theta,K}(f)$. This transformation is given as follows*

$$\phi_{\theta,K}(f)\Psi_k = \Gamma(U_R)\left(\Gamma(U_R)\phi(f)\Gamma(U_R)\right)_{\theta,P}\Gamma(U_R)\Psi_k.$$

Proof. By using the unitary equivalence (6.8) the lemma is easily proven

$$\begin{aligned}\phi_{\theta,K}(f)\Psi_k &= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)U(\theta y)\phi(f)U(-\theta y+v)\Psi_k\\ &= (2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)\Gamma(U_R)T(\theta y)\Gamma(U_R)\phi(f)\Gamma(U_R)\\ &\qquad\qquad\qquad\qquad\times T(-\theta y+v)\Gamma(U_R)\Psi_k\\ &= \Gamma(U_R)\left(\Gamma(U_R)\phi(f)\Gamma(U_R)\right)_{\theta,P}\Gamma(U_R)\Psi_k.\end{aligned}$$

□

Lemma 6.2. *For $\Phi_k\in\Delta_k^n(R)\subset\mathscr{H}_k^+$ the familiar bounds of the free field hold for the deformed field $\phi_{\theta,K}(f)$ and therefore the deformation with the special conformal operators is well-defined.*

Proof. By using lemma 6.1 one obtains the familiar bounds for a free scalar field. For $\Phi_k\in\Delta_k^n(R)$ there exists a $\Psi_k\in\Delta_k^n(P)$ such that the following holds

$$\begin{aligned}&\|\phi_{\theta,K}(f)\Phi_k\| = \|\phi_{\theta,K}(f)\Gamma(U_R)\Psi_k\| = \|(\Gamma(U_R)\phi(f)\Gamma(U_R))_{\theta,P}\Psi_k\| = \|(\phi(U_Rf))_{\theta,P}\Psi_k\|\\ &\leq \left\|(a(U_R\overline{f^-}))_{\theta,P}\Psi_k\right\| + \left\|(a^*(U_Rf^+))_{\theta,P}\Psi_k\right\| \leq \left\|U_Rf^+\right\|^2\left\|(N+1)^{1/2}\Psi_k\right\|^2 +\\ &\left\|U_Rf^-\right\|^2\left\|(N+1)^{1/2}\Psi_k\right\|^2 = \left\|f^+\right\|^2\left\|(N+1)^{1/2}\Psi_k\right\|^2 + \left\|f^-\right\|^2\left\|(N+1)^{1/2}\Psi_k\right\|^2,\end{aligned}$$

where in the last lines we used the Cauchy-Schwarz inequality, the bounds given in [GL07] and the unitarity of U_R.

Hence, due to the bounds of the deformed field we argue along the same lines in the proof for the coordinate operator (see Lemma 5.21), that the deformation using the special conformal operator is well-defined. □

6.6 Properties of the deformed quantum field

The Wightman properties of the deformed field are proved in this section. The Wightman axioms of covariance and locality are not satisfied, but are replaced by wedge covariance and wedge-locality. The relation between the fields defined on a deformed spacetime and fields defined on the wedge is given by the the constructed map in [BLS11, GL07]. To use this map we give the transformation property of the deformed quantum field ϕ_θ under Lorentz transformations and thus relate the skew-symmetric matrices to wedges. Furthermore, we prove that the field obtained by the construction is a wedge Lorentz-covariant and wedge-local quantum field.

6.6.1 Wightman properties of the deformed QF

Let us prove first prove that the deformed field $\phi_{\theta,K}$ satisfies the Wightman properties with the exception of covariance and locality.

Proposition 6.2. *Let θ be a real skew-symmetric matrix w.r.t. the Minkowski scalar-product on $\mathbb{R}^d$ and $f \in \mathscr{S}(\mathbb{R}^d)$.*

a) *The dense subspace $\mathcal{D}$ of vectors of finite particle number is contained in the domain $\mathcal{D}^{\theta,K} = \{\Psi \in \mathscr{H}_0^+ \,|\, \|\phi_{\theta,K}(f)\Psi\|^2 < \infty\}$ of any $\phi_{\theta,K}(f)$. Moreover, $\phi_{\theta,K}(f)\mathcal{D} \subset \mathcal{D}$ and $\phi_{\theta,K}(f)\Omega = \phi(f)\Omega$.*

b) *For scalar fields deformed via warped convolutions and $\Psi \in \mathcal{D}$,*

$$f \longmapsto \phi_\theta(f)\Psi$$

is a vector valued tempered distribution.

c) *For $\Psi \in \mathcal{D}$ and $\phi_{\theta,K}(f)$ the following holds*

$$\phi_{\theta,K}(f)^*\Psi = \phi_{\theta,K}(\overline{f})\Psi.$$

For real $f \in \mathscr{S}(\mathbb{R}^d)$, the deformed field $\phi_\theta(f)$ is essentially self-adjoint on $\mathcal{D}$.

d) *The Reeh-Schlieder property holds: Given an open set of spacetime $\mathcal{O} \subset \mathbb{R}^d$ then*

$$\mathcal{D}_\theta(\mathcal{O}) := span\{\phi_\theta(f_1)\ldots\phi_\theta(f_k)\Omega : k \in \mathbb{N}, f_1 \ldots f_k \in \mathscr{S}(\mathcal{O})\}$$

is dense in $\mathscr{H}_0^+$.

Proof. a) The fact that $\mathcal{D} \subset \mathcal{D}^\theta$, follows directly from Lemma 6.2 because the deformed scalar field satisfies the well known bounds of the free field. The fact that the deformed field acting on the vacuum is the same as the free field acting on Ω, can be easily shown due to the property of the unitary operators $U(b)\Omega = \Omega$.

b) By using Lemma 6.2 one can see that the right hand side depends continuously on the function f, hence the temperateness of $f \longmapsto \phi_{\theta,K}(f)\Psi$, $\Psi \in \mathcal{D}$ follows.

c) The hermiticity of the deformed field $\phi_{\theta,K}(f)^*$ is proven in the following

$$\begin{aligned}
\phi_{\theta,K}(f)^*\Psi &= (2\pi)^{-d} \left(\lim_{\epsilon \to 0} \iint dy\, dv\, e^{-iyv}\, \chi(\epsilon y, \epsilon v)\alpha_{\theta y}(\phi(f))U(v) \right)^* \Psi \\
&= (2\pi)^{-d} \lim_{\epsilon \to 0} \iint dy\, dv\, e^{-iyv}\, \overline{\chi(\epsilon y, -\epsilon v)}\, U(-v)\alpha_{\theta y}(\phi(f))^*\Psi \\
&= (2\pi)^{-d} \lim_{\epsilon \to 0} \iint dy\, dv\, e^{-iyv}\, \overline{\chi(\epsilon(y+\theta^{-1}v), -\epsilon v)}\, \alpha_{\theta y}(\phi(\overline{f}))U(-v)\Psi \\
&= \phi_{\theta,K}(\overline{f})\Psi.
\end{aligned}$$

In the last lines we performed a variable substitution$(v_\mu \to -v_\mu)$ and $\big(y_\mu \to y_\mu + (\theta^{-1}v)_\mu\big)$.

For real f essential self-adjointness of the deformed field $\phi_{\theta,K}(f)$ is proven by showing that the field has a dense set of analytic vectors $\mathcal{D}$. For $\Psi_k \in \mathscr{H}_{0,k}$ the estimates of the l-power of the deformed field $\phi_{\theta,K}(f)$, are given in the following

$$\left\| \phi_{\theta,K}(f)^l \Psi_k \right\| \leq 2^{l/2}(k+l)^{1/2}(k+l-1)^{1/2} \cdots (k+1)^{1/2} \left\| f \right\|^l \left\| \Psi_k \right\|,$$

where in the last lines we used Lemma 6.2 to use the same estimates for the deformed field as for the undeformed field. Therefore, we can write the sum

$$\sum_{l \geq 0} \frac{|t|^l}{l!} \left\| \phi(f)^l \Psi_k \right\| \leq \sum_{l \geq 0} \frac{(\sqrt{2}|t|)^l}{l!} \left(\frac{(k+l)!}{k!} \right)^{1/2} \left\| f \right\|^l \left\| \Psi_k \right\| < \infty$$

for all $t \in \mathbb{C}$. Thus each $\Psi \in \mathcal{D}$ is an analytic vector for the deformed field $\phi_{\theta,K}(f)$ and since $\mathcal{D}$ is dense in $\mathscr{H}_0^+$, it follows from Nelson's analytic vector theorem (see Theorem 8.4), that $\phi_{\theta,K}(f)$ is essentially self-adjoint on $\mathcal{D}$.

d) For the proof of the Reeh-Schlieder property we make use of the unitary equivalence given in (6.8). First note that the spectral properties of the representation of the special conformal transformations $U(y)$ are the same as for the representation of translations. This leads to the application of the standard Reeh-Schlieder argument [SW89] which states that that $\mathcal{D}_\theta(\mathcal{O})$ is dense in $\mathscr{H}_0^+$ if and only if $\mathcal{D}_\theta(\mathbb{R}^d)$ is dense in $\mathscr{H}_0^+$. We choose the functions $f_1, \dots, f_k \in \mathscr{S}(\mathbb{R}^d)$ such that the Fourier transforms of the functions do not intersect the past light cone and therefore the domain $\mathcal{D}_\theta(\mathbb{R}^d)$ consists of the following vectors

$$\begin{aligned}
&\Gamma(U_R)\phi_{\theta,K}(f_1)\dots\phi_{\theta,K}(f_k)\Omega = \Gamma(U_R)a^*_{\theta,K}(f_1^+)\dots a^*_{\theta,K}(f_k^+)\Omega \\
&= \Gamma(U_R)\Gamma(U_R)(\Gamma(U_R)a^*(f_1^+)\Gamma(U_R))_{\theta,P}\dots(\Gamma(U_R)a^*(f_k^+)\Gamma(U_R))_{\theta,P}\Gamma(U_R)\Omega \\
&= a^*_{\theta,P}(U_R f_1^+)\dots a^*_{\theta,P}(U_R f_k^+)\Omega = \sqrt{k!}P_k(S_k(U_R f_1^+ \otimes \cdots \otimes U_R f_k^+)),
\end{aligned}$$

where P_k denotes the orthogonal projection from $\mathscr{H}_1^{\otimes k}$ onto its totally symmetric subspace $\mathscr{H}_{0,k}^+$, and $S_k \in \mathscr{B}(\mathscr{H}_1^{\otimes k})$ is the multiplication operator given as

$$S_k(p_1, \dots, p_k) = \prod_{1 \le l < r \le k} e^{ip_l \theta p_r}.$$

Since the operator $\Gamma(U_R)$ is a unitary operator the functions $U_R f_k^+$ for $f_k^+ \in \mathscr{S}(\mathbb{R}^d)$ will give rise to dense sets of functions in $\mathscr{H}_1$. Following the same arguments as in [GL07] the density of $\mathcal{D}_\theta(\mathbb{R}^d)$ in $\mathscr{H}_0^+$ follows. Note that we proved the density for vectors $\Gamma(U_R)\phi_{\theta,K}(f_1)\dots\phi_{\theta,K}(f_k)\Omega$ and not for the vectors without $\Gamma(U_R)$ as stated in the proposition. We use the unitarity of $\Gamma(U_R)$ to argue that vectors dense in $\mathscr{H}_0^+$ stay dense after the application of a unitary operator. □

6.6.2 Wedge-covariant fields

By using the map $Q: W \mapsto Q(W)$ from a set $\mathcal{W}_0 := \mathscr{L}_+^\uparrow W_1$ of wedges to a set $\mathcal{Q}_0 \subset \mathbb{R}^-_{d\times d}$ of skew-symmetric matrices we consider the corresponding fields $\phi_W(x) := \phi(Q(W), x)$. The meaning of the correspondence is that the field $\phi(Q(W), x)$ is a scalar field living on a NC spacetime which can be equivalently realized as a field defined on the wedge.

To show the covariance properties of the deformed quantum fields we use the homomorphism $Q(W)$ to map the deformed scalar fields to quantum fields defined on a wedge. Before we use the map from the set of skew-symmetric matrices to the wedges we state the following lemma about the transformation properties of the deformed field.

Lemma 6.3. *The transformation of the deformed particle annihilation and creation operator $a_{\theta,K}(\mathbf{p}), a^*_{\theta,K}(\mathbf{p})$, for $\mathbf{p} \in \partial V_+$ and θ being admissible, under the adjoint action $V(0,\Lambda)$ of the Lorentz group, $\Lambda \in \mathcal{L}$, is the following*

$$V(0,\Lambda)a_{\theta,K}(\mathbf{p})V(0,\Lambda)^{-1} = a_{\gamma_\Lambda(\theta),K}(\pm\Lambda\mathbf{p}),$$
$$V(0,\Lambda)a^*_{\theta,K}(\mathbf{p})V(0,\Lambda)^{-1} = a^*_{\gamma_\Lambda(\theta),K}(\pm\Lambda\mathbf{p}),$$

where the first sign is for $\Lambda \in \mathcal{L}^\uparrow$ and the second sign is for $\Lambda \in \mathcal{L}^\downarrow$. Hence the deformed field $\phi_{\theta,K}(x)$ transforms

$$V(0,\Lambda)\phi_{\theta,K}(x)V(0,\Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),K}(\Lambda x).$$

Proof. The proof is done along the same lines of [BLS11]. $V(0,\Lambda)$ is a unitary operator for $\Lambda \in \mathcal{L}^\uparrow$ and an antiunitary operator for $\Lambda \in \mathcal{L}^\downarrow$. Due to the commutator relation of the special conformal operator and the generator of the Lorentz transformations one obtains

$$V(0,\Lambda)U(x)V(0,\Lambda)^{-1} = U(\Lambda x), \qquad x \in \mathbb{R}^d.$$

Therefore, the deformed scalar field $\phi_{\theta,K}$ transforms under the adjoint action of the Lorentz group as

$$
\begin{aligned}
&(2\pi)^{-d}V(0,\Lambda)\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)\alpha_{\theta y}(\phi(x))U(v)V(0,\Lambda)^{-1}\\
&\quad=(2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon y,\epsilon v)\alpha_{\Lambda\theta y}(V(0,\Lambda)\phi(x)V(0,\Lambda)^{-1})U(\Lambda v)\\
&\quad=(2\pi)^{-d}\lim_{\epsilon\to 0}\iint dy\,dv\,e^{-iyv}\,\chi(\epsilon\sigma\Lambda^{T}y,\epsilon\Lambda^{-1}v)\,\alpha_{\gamma_\Lambda(\theta)y}(\phi(\Lambda x))U(v)=\phi_{\gamma_\Lambda(\theta),K}(\Lambda x),
\end{aligned}
$$

where σ is $+1$ if V is unitary and -1 if V is antiunitary. Moreover in the last lines we performed the integration variable substitutions $(y,v)\to(\sigma\Lambda^{T}y,\Lambda^{-1}v)$. □

In the next step we use the homomorphism, given in Equation (5.37), to map the deformed field to a field defined on a wedge. Furthermore we show that the field deformed with the special conformal operator is a wedge-covariant quantum field which transforms covariantly under the adjoint action of the Lorentz group $V(0,\Lambda)$. For this purpose let us first introduce the notion of a wedge-covariant quantum field, [GL07].

Definition 6.5. Let $\phi=\{\phi_W : W\in\mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions of the Wightman axioms. Then the field ϕ is defined to be a wedge Lorentz-covariant quantum field if the following condition is satisfied:

- For any $W\in\mathcal{W}_0$ and $f\in\mathscr{S}(\mathbb{R}^d)$ the following holds

$$
\begin{aligned}
V(\Lambda)\phi_W(f)V(\Lambda)^{-1}&=\phi_{\Lambda W}(f\circ(\Lambda)^{-1}), \qquad \Lambda\in\mathcal{L}_+^{\uparrow},\\
V(j)\phi_W(f)V(j)^{-1}&=\phi_{jW}(\overline{f}\circ j)^{-1}.
\end{aligned}
$$

We use the homomorphism $Q:W\mapsto Q(W)$ to define the deformed fields as quantum fields defined on the wedge, this is done in the following way

$$
\phi_W(f):=\phi(Q(W),f)=\phi_{\theta,K}(f). \tag{6.13}
$$

Proposition 6.3. *The family of fields $\phi=\{\phi_W : W\in\mathcal{W}_0\}$ defined by the deformation with the special conformal operators are wedge-covariant quantum fields on the Bosonic Fock space, w.r.t. the unitary representation V of the Lorentz group.*

Proof. Following lemma (6.3), the deformed field $\phi_{\theta,K}(x)$ transforms under the adjoint action V of the Lorentz group in the following way

$$
V(0,\Lambda)\phi_W(x)V(0,\Lambda)^{-1}=V(0,\Lambda)\phi_{\theta,K}(x)V(0,\Lambda)^{-1}=\phi_{\gamma_\Lambda(\theta),K}(\Lambda x)=\phi_{\Lambda W,K}(\Lambda x),
$$

where in the last lines we applied the map $Q(\Lambda W)=\gamma_\Lambda(Q(W))=\gamma_\Lambda(\theta)$. Therefore, one obtains the wedge-covariance property of the scalar field under the Lorentz group. □

A few comments are in order. The covariance property is given in the four dimensional case as well. As already explained, a unitary representation for the whole conformal group does not exist due to the absolute value of the scale factor. Nevertheless, we show in the next section that the scale factor is positive for a field localized in the wedge and therefore one has a unitary representation of the whole conformal group.

6.6.3 Wedge-local fields

The wedge-covariant quantum field defined in the last section, fulfills the locality property of the wedges. Hence, the deformed field is a *wedge-local* field. We first define the notion of the wedge-local field.

Definition 6.6. The fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are said to be wedge-local if the following commutator relation is satisfied

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \qquad \Psi \in \mathcal{D},$$

for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$.

To show that the fields defined in the last section are wedge-local, we use the following Proposition, (see also Proposition 2.1).

Proposition 6.4. *Let the scalar fields $\phi(f)$, $\phi(g)$ be such that $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] = 0$ for all $v, u \in spU$ and for $f, g \in C_0^\infty(\mathbb{R}^d)$. Then*

$$[\phi_{\theta,K}(f), \phi_{-\theta,K}(g)]\Psi = 0, \qquad \Psi \in \mathcal{D}. \tag{6.14}$$

Lemma 6.4. *The special conformal transformations $U_{\theta v}$, with $v \in spU$ and θ being admissible, map the right wedge into the right wedge $U_{\theta v}(W_1) \subset W_1$. Furthermore, the special conformal transformations $U_{-\theta u}$, with $u \in spU$ and θ being admissible, map the left wedge into the left wedge $U_{-\theta u}(-W_1) \subset -W_1$.*

Proof. We first prove for $x^\mu \in W_1$, $v \in$ spU, θ being admissible and $\kappa > 0$, that the vector $x'^\mu := x^\mu + \kappa(\theta v)^\mu \in W_1$.

$$x'^1 > |x'^0|$$
$$x^1 + \kappa\lambda v^0 > |x^0 + \kappa\lambda v^1|.$$

The right hand side is obviously greater than zero and therefore we square both sides and obtain

$$\kappa^2\lambda^2(v_0^2 - v_1^2) - (x_0^2 - x_1^2) - 2\kappa\lambda(v^1x^0 - v^0x^1) > 0.$$

Due to the fact that the sum of the first two terms is greater than zero, we are only left with proving that the following inequality

$$\lambda v^1x^0 - \lambda v^0x^1 \leq 0$$

is satisfied. Equality only holds if $v_0 = 0$ or λ is zero. So if $v_0, \lambda \neq 0$ we have to show the following

$$x^1 > \frac{v^1}{v^0}x^0. \tag{6.15}$$

(6.15) is satisfied, because the stronger inequality

$$x_1 > \underbrace{|\frac{v_1}{v_0}|}_{0<\cdots<1} \; |x_0|$$

holds. By using the vector x'^{μ} we now can easily prove that $x^{\mu}_{\theta v} \in W_1$. To show that $x^{\mu}_{\theta v} \in W_1$ the following inequality must be satisfied.

$$x^1_{\theta v} > |x^0_{\theta v}|$$

$$(x^1 - (\theta v)^1 x^2)/(1 - 2(\theta v) \cdot x + (\theta v)^2 x^2) > |(x^0 - (\theta v)^0 x^2)/(1 - 2(\theta v) \cdot x + (\theta v)^2 x^2)|. \tag{6.16}$$

Positivity of the denominator can be seen by taking the vector x'^{μ} as defined above and setting $\kappa = -x^2 > 0$. From $x'^2 < 0$ we obtain

$$x'^2 = (x^{\mu} - x^2(\theta v)^{\mu})(x_{\mu} - x^2(\theta v)_{\mu}) = \underbrace{x^2}_{<0}\left(1 - 2x_{\mu}(\theta v)^{\mu} + x^2(\theta v)_{\mu}\right) < 0.$$

From the inequality it follows that the the denominator in (6.16) is positive and therefore one is left with proving

$$(x^1 - (\theta v)^1 x^2) > |(x^0 - (\theta v)^0 x^2)|.$$

By choosing $\kappa = -x^2$ this is exactly the inequality for the vector $x'^{\mu} \in W_1$. Therefore, the special conformal transformed coordinate is still in the right wedge. The proof that the special conformal transformations map the left wedge into the left wedge is analogous. □

Proposition 6.5. *For $n = 4l+1$, where $l \in \mathbb{N}_0$ the family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are wedge-local fields on the Bosonic Fockspace $\mathscr{H}^+$.*

Proof. We first prove that the expression $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))]$ vanishes for all $v, u \in$ spU and for $f \in C_0^{\infty}(W_1)$, $g \in C_0^{\infty}(-W_1)$. By using Proposition 6.4 it follows that the commutator $[\phi_{W_1}(f), \phi_{-W_1}(g)]$ vanishes.

$$\begin{aligned}
[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] &= (2\pi)^{-2(n+1)} \iint d^{n+1}x d^{n+1}y f(x) g(y) [\alpha_{\theta v}(\phi(x)), \alpha_{-\theta u}(\phi(y))] \\
&= (2\pi)^{-2(n+1)} \iint d^{n+1}x d^{n+1}y f(x) g(y) \sigma_{\theta v}(x)^{\frac{1-n}{2}} \sigma_{-\theta u}(y)^{\frac{1-n}{2}} \\
&\qquad \times [\phi(x_{\theta v}), \phi(y_{-\theta u})] \\
&= 0.
\end{aligned}$$

In the last line we applied Lemma 6.4 to prove that after the special conformal transformation, the support of the field ϕ_{W_1} stays in the right wedge and the support of the field ϕ_{-W_1} stays in the left wedge. Therefore, the supports of the fields are space-like separated, hence they commute. □

Lemma 6.5. *In four dimensions a unitary representation for the whole conformal group, which gives the correct transformation law (6.9), exists for the fields $\phi_{\theta,K}(f)$ with $f \in C_0^{\infty}(W_1)$. The same holds for the field $\phi_{-\theta,K}(g)$ with $g \in C_0^{\infty}(-W_1)$.*

Proof. The problem with the absence of a unitary representation for the whole conformal group that gives the correct transformation law (6.9) is due to the absolute value of the scale factor $\sigma_b(x)$. Nevertheless, we showed in Lemma 6.4 that the scale factor for a field localized in the right wedge is positive. The positivity of the scale factor in turn means that a unitary representation for the whole conformal group in four dimensions exists, [SV73].

$$\begin{aligned}\phi_{W_1}(f)\Psi &= (2\pi)^{-4}\int d^4x f(x)\lim_{\epsilon\to 0}\iint d^4u\, d^4v\, e^{-iuv}\,\chi(\epsilon u,\epsilon v)\alpha_{\theta v}(\phi(f))U(u)\Psi \\ &= (2\pi)^{-4}\int d^4x f(x)\lim_{\epsilon\to 0}\iint d^4u\, d^4v\, e^{-iuv}\,\chi(\epsilon u,\epsilon v)\sigma_{\theta v}(x)^{-1}\phi(x_{\theta v})U(u)\Psi.\end{aligned}$$

For a quantum field defined on the left wedge the proof is done analogously. □

Proposition 6.6. *For $n = 3$, the fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ are wedge-local fields on the Bosonic Fockspace $\mathscr{H}^+$.*

Proof. Due to the existence of a unitary representation shown in Lemma 6.5 the deformed field can be defined for $n = 3$. Furthermore, by applying Proposition 6.4 one shows that the expression $[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))]$ vanishes for all $v, u \in \mathrm{spU}$ and for $f \in C_0^\infty(W_1)$, $g \in C_0^\infty(-W_1)$.

$$\begin{aligned}[\alpha_{\theta v}(\phi(f)), \alpha_{-\theta u}(\phi(g))] &= (2\pi)^{-8}\iint d^4x d^4y f(x)g(y)[\alpha_{\theta v}(\phi(x)), \alpha_{-\theta u}(\phi(y))] \\ &= (2\pi)^{-8}\iint d^4x d^4y f(x)g(y)\sigma_{\theta v}(x)^{-1}\sigma_{-\theta u}(y)^{-1} \\ &\qquad\qquad \times [\phi(x_{\theta v}), \phi(y_{-\theta u})] \\ &= 0,\end{aligned}$$

Analogously to the proof of Proposition (6.5) we use Lemma 6.4 in the last line. □

This is an interesting result. The deformed case improves the representations such that one does not have to deal with representations of the covering of the conformal group.

6.7 NC Spacetime from special conformal operators

The main idea in this work is to use the special conformal operator to deform the free quantum field. We further proved that the deformed field satisfies some weakened covariance and locality properties. Now a natural question arises. What is the noncommutative spacetime that we obtain from the deformation with the special conformal operator? This question can be answered by calculating the deformed commutator of the coordinates.

$$[x_\mu \overset{\times_\theta}{,} x_\nu] = (2\pi)^{-d}\iint d^dv d^du e^{-ivu}\left(\alpha_{\theta v}(x_\mu)\alpha_u(x_\nu) - \alpha_{\theta v}(x_\nu)\alpha_u(x_\mu)\right). \tag{6.17}$$

To calculate the term $\alpha_{\theta v}(x_\mu)$ we insert the generator K_μ as a differential operator defined in ([DMS97]).

$$\alpha_{\theta v}(x_\mu) = \exp\left((\theta v)^\sigma\left(2x_\sigma x^\lambda\frac{\partial}{\partial x^\lambda} - x^2\frac{\partial}{\partial x^\sigma}\right)\right)x_\mu =: \exp\left((\theta v)^\sigma K_\sigma(x)\right)x_\mu$$

We could use the transformation of the coordinates under the special conformal generators, but in that case we would not be able to solve the integral. The ansatz we follow in this work is to solve the integral, order by order. This will be done by preforming a Taylor expansion of the exponentials.

Lemma 6.6. *Let the deformed product given in Definition (4.6) be defined by the generator of special conformal transformations K_μ. Then the deformed commutator (6.17), up to third order in θ is given as follows (see figure 6.1)*

$$[x_\mu \overset{\times_\theta}{,} x_\nu] = -2i\theta_{\mu\nu}x^4 - 4i\left((\theta x)_\mu x_\nu - (\theta x)_\nu x_\mu\right)x^2.$$

Proof. The deformed commutator gives the following

$$\begin{aligned}
[x_\mu \overset{\times_\theta}{,} x_\nu] &= (2\pi)^{-d}\iint d^dvd^due^{-ivu}\left(\alpha_{\theta v}(x_\mu)\alpha_u(x_\nu) - \mu \leftrightarrow \nu\right) \\
&= (2\pi)^{-d}\iint d^dvd^due^{-ivu}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{(\theta v)^\sigma K_\sigma(x)\cdots(\theta v)^\rho K_\rho(x)}_{k}x_\mu\right) \\
&\quad\times\left(\sum_{l=0}^{\infty}\frac{1}{l!}\underbrace{(u)^\lambda K_\lambda(x)\cdots(u)^\tau K_\tau(x)}_{l}x_\nu - \mu\leftrightarrow\nu\right).
\end{aligned}$$

There are two properties for the series that can be easily seen. First, the different orders between θv and u do not mix. The only terms which are not equal to zero are the terms of equal order. The vanishing of unequal orders between θv and u will be shown in the following calculation.

$$\begin{aligned}
&\iint d^dvd^due^{-ivu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}\underbrace{(\theta v)^\sigma\cdots(\theta v)^\rho}_{k}\underbrace{u^\lambda\cdots u^\tau}_{l}\left(\underbrace{K_\sigma(x)\cdots K_\rho(x)}_{k}x_\mu\right) \\
&\times\left(\underbrace{K_\lambda(x)\cdots K_\tau(x)}_{l}x_\nu\right) \\
&= \iint d^dvd^du\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-i)^k}{k!l!}\underbrace{\theta^\sigma_\kappa\cdots\theta^\rho_\gamma}_{k}\left(\underbrace{\frac{\partial}{\partial u_\kappa}\cdots\frac{\partial}{\partial u_\gamma}}_{k}e^{-ivu}\right)\underbrace{u^\lambda\cdots u^\tau}_{l} \\
&\times\left(\underbrace{K_\sigma(x)\cdots K_\rho(x)}_{k}x_\mu\right)\left(\underbrace{K_\lambda(x)\cdots K_\tau(x)}_{l}x_\nu\right) \\
&= \iint d^dvd^due^{-ivu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{i^k}{k!l!}\underbrace{\theta^\sigma_\kappa\cdots\theta^\rho_\gamma}_{k}\left(\left(\underbrace{\frac{\partial}{\partial u_\kappa}\cdots\frac{\partial}{\partial u_\gamma}}_{k}\right)\underbrace{u^\lambda\cdots u^\tau}_{l}\right) \\
&\times\left(\underbrace{K_\sigma(x)\cdots K_\rho(x)}_{k}x_\mu\right)\left(\underbrace{K_\lambda(x)\cdots K_\tau(x)}_{l}x_\nu\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \underbrace{\theta^{\sigma\lambda} \cdots \theta^{\rho\tau}}_{k} \left(\underbrace{K_\sigma(x) \cdots K_\rho(x)}_{k} x_\mu \right) \left(\underbrace{K_\lambda(x) \cdots K_\tau(x)}_{k} x_\nu \right)$$

In the third line we performed a partial integration. The expression vanishes in the case $k > l$, because the differentials annihilate the polynomial in u. It also vanishes if $k < l$ because non-vanishing polynomials in u stay and the integral sets the polynomials zero. Furthermore, by using the symmetry of the x-dependent differential operators K, one solves the integral. It is important to note that the result of the deformed product between the coordinates, is exactly the same result one would encounter by using twist-deformation with the special conformal operators, [ADK+09].

The second observation is that polynomials in u, v that are even vanish due to the antisymmetry of the commutator. This is shown in the following.

$$\iint d^dvd^due^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right)$$
$$- \iint d^dvd^due^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\nu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\mu \right)$$

Where m is a natural number. In the second integral we preform the integration variable substitution $(v, u) \to (\theta^{-1}u, \theta v)$ and obtain

$$\iint d^dvd^due^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right)$$
$$- \iint d^dvd^due^{ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right).$$

After preforming the integration variable substitution $u \to -u$ we obtain

$$\iint d^dvd^due^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right)$$
$$-(-1)^{2m} \iint d^dvd^due^{-ivu} \left(\underbrace{(\theta v)^\sigma K_\sigma(x) \cdots (\theta v)^\rho K_\rho(x)}_{2m} x_\mu \underbrace{(u)^\lambda K_\lambda(x) \cdots (u)^\tau K_\tau(x)}_{2m} x_\nu \right)$$
$$= 0.$$

Therefore, the only terms that do not vanish are those of equal odd order in v and u. In the following we calculate the noncommutativity of the coordinates up to the second order and obtain

$$[x_\mu \overset{\times}{,}_\theta x_\nu] = (2\pi)^{-d} \iint d^dvd^due^{-ivu} \left((\theta v)^\sigma (2x_\sigma x_\mu - x^2\eta_{\sigma\mu})(u)^\tau \left(2x_\tau x_\nu - x^2\eta_{\tau\nu} \right) - \mu \leftrightarrow \nu \right) \tag{6.18}$$

$$= -2i\theta_{\mu\nu}x^4 - 4i \left((\theta x)_\mu x_\nu - (\theta x)_\nu x_\mu \right) x^2 + \mathcal{O}(\theta^3). \tag{6.19}$$

□

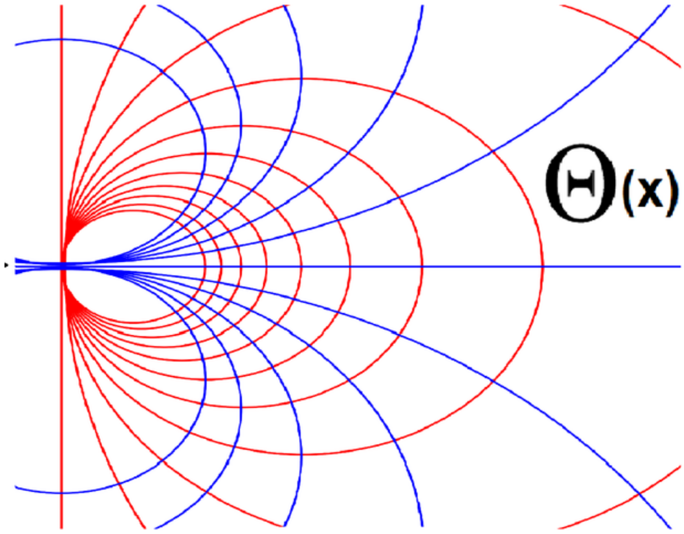

Figure 6.1: Example of a Conformal-Moyal-Weyl plane $\mathbb{R}^2_{\theta,K}$

The deformed commutator of the coordinates shows that the deformation induced by the special conformal operators spans a nonconstant noncommutative spacetime. This is very interesting because the spacetime that we obtain is a curved noncommutative spacetime and the curvature of the noncommutative spacetime is induced by the special conformal operators. In the case of using the momentum operator, i.e. the generator of translations in Minkowski, for the deformation one obtains a flat noncommutative spacetime. The special conformal operators induce a conformal flat spacetime on Minkowski and therefore, one obtains a conformally flat noncommutative spacetime when deforming with K_μ. Some examples of a nonconstant noncommutative spacetime exist in literature, where the highest order of noncommutativity known is the so called quantum space structure, [KS97, Wes90, Wes98]. The quantum space structure has an x-polynomial dependence up to second order.

6.8 Generalization of the deformation

The deformation of an operator by either using the momentum operator P_μ or the special conformal operator K_μ can be written in a general form. The generalization can be accomplished by using a linear combination of generators of the pseudo-orthogonal group $SO(2,d)$. First, we redefine the operators P^μ and K^ν in the following way

$$\tilde{P}^\mu := \begin{pmatrix} \lambda' P^0 \\ \lambda' P^1 \\ \eta' P^2 \\ \eta' P^3 \end{pmatrix}, \qquad \tilde{K}^\mu := \begin{pmatrix} \lambda K^0 \\ \lambda K^1 \\ \eta K^2 \\ \eta K^3 \end{pmatrix},$$

where $\lambda', \lambda \in \mathbb{R}^+$ and $\eta', \eta \in \mathbb{R}$. In the next step we redefine the Lorentz generators $J^{4,\mu}$, $J^{5,\mu}$, $J^{\pm,\mu}$ and the skew-symmetric matrix θ as follows

$$\tilde{J}^{4,\mu} := \frac{1}{2}\left(\tilde{P}^\mu - \tilde{K}^\mu\right), \qquad \tilde{J}^{5,\mu} := \frac{1}{2}\left(\tilde{P}^\mu + \tilde{K}^\mu\right)$$

$$\tilde{J}^{\pm,\mu} := \tilde{J}^{5,\mu} \pm \tilde{J}^{4,\mu},$$

$$\tilde{\theta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6.20}$$

Definition 6.7. Let $\tilde{\theta}$ be a real skew-symmetric matrix given in (6.20) and let $A \in C^\infty$. Then the generalized warped convolutions, i.e. the deformation of A denoted as $A_\theta^\pm$ is defined on $\Psi \in \mathcal{D}$, follows

$$A_\theta^\pm \Psi := (2\pi)^{-4} \lim_{\epsilon \to 0} \iint d^4y\, d^4v\, e^{-iyv}\, \chi(\epsilon y, \epsilon v) U^\pm(\theta y) A U^\pm(\theta y) U^\pm(v) \Psi,$$

where the unitary operator $U^\pm(v)$ is defined as $U^\pm(v) := \exp\left(iv^\mu \tilde{J}^\pm_\mu\right)$.

The generalization of the deformation is interesting because it is obtained as a linear combination of generators of $SO(2,4)$. By choosing the plus sign, one obtains the Moyal-Weyl case and by choosing the minus sign one gets the special conformal model introduced in this work.

Chapter 7

Conclusion and Outlook

7.1 Conclusion

The aim of the thesis was to obtain a physical quantum spacetime from deformation.

For the entire work we used the method of warped convolutions for deformation. The novel tool was constructed in [BLS11], for bounded operators that are smooth w.r.t. the unitary groups, used for the deformation. For this thesis we had to prove, on various stages, that the deformation formula is still mathematically well-defined if the deformed operators are unbounded. This was done by using the Hörmander definitions of a an oscillatory integral and of symbol classes.

Before we studied deformations in quantum field theory, we studied the quantum mechanical case. In the quantum mechanical chapter the investigation of deformation with the coordinate operator led to a variety of physical effects. In particular, deformations induced by the coordinate operator, correspond to effects emerging from the presence of a magnetic field. We reproduced the Landau-levels, the Zeeman effect and the Aharanov-Bohm effect. This in particular means that deformation with the coordinate operator, induces magnetic fields. Thus, a new interpretation of magnetic fields appearing in QM context can be given. These effects can be induced by a deformation procedure.

Furthermore, we were able to interpret the NC space that emerges in the case of the lowest Landau levels, as a deformation of the coordinate operator with the momentum operator. These two ideas are used as guiding principles for the realization of a NC space in the realm of QFT. The two paths leading to the same result are graphically displayed in the following diagram. The two paths leading to the same result are graphically displayed in Figure 7.1.

Following the path, given in the QM context, we deform physical objects in QFT with the coordinate operator. For the investigation of a coordinate operator in QFT we use the results of [SS09]. The authors constructed in view of the QFT-relativistic context a temporal coordinate operator as well. In this construction on the level of

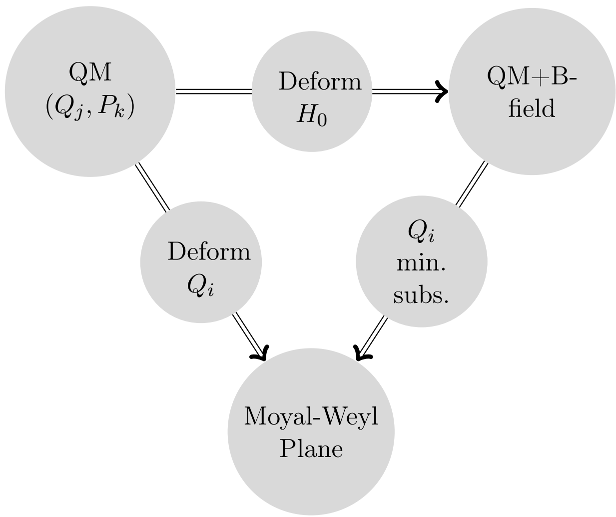

Figure 7.1: QM deformation

the coordinate operators one is already led to a noncommutative relations between the temporal and the spatial component. Thus, prior to deformation, we investigated the noncommutativity. This was done by calculating the expectation value of the noncommutativity given by the relations between the coordinate operators. By gauging the parameters of the expectation value we were able to reproduce the Moyal-Weyl plane. This plane displays an important example of a constant noncommutative spacetime.

In the QM case we obtained a physical NC space by deforming the coordinate operator with the momentum operator. The same is done in the QFT context and we obtain a noncommutativity that we call the QFT Moyal-Weyl. This gives a new kind of noncommutativity, because it has terms depending on the velocity. Thus the relations give a new example of a nonconstant noncommutative spacetime.

Now prior to deforming with the spatial part of the coordinate operator, we have to investigate the uniqueness of such an operator. This is done by using a definition, common in QM, of the spatial coordinate operator as unitarily equivalent to the momentum operator. We take this expression as a starting point and perform a second quantization on the operator level. The second quantized spatial coordinate operator obtained by such a procedure, turns out to be exactly the pre-coordinate operator found in [SS09].

Furthermore, in the context of coordinate operators the Newton-Wigner-Pryce operator is usually mentioned. It is the relativistic analog of the coordinate operator on a one particle level. A second quantization was also performed on the Newton-Wigner-Pryce operator and the outcome was once more the operator already found by [SS09]. Thus this operator is from an intuitive and mathematical standpoint the rightful n-particle spatial coordinate operator. This point of view is additionally supported by calculating the Heisenberg equation of the spatial coordinate operator and obtaining the relativistic velocity operator for an n-particle system. The following diagram summarizes the results. Figure 7.2 summarizes the results.

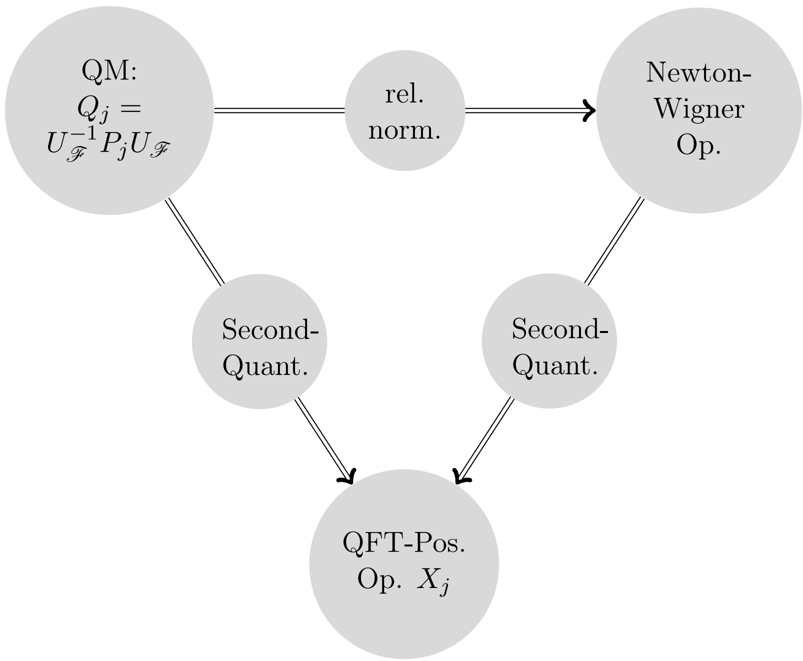

Figure 7.2: Second quantized coordinate operator

Next, we investigated the effect of deformation directly on the scalar quantum field. To proceed with the deformation, we were obliged to prove that the operators used are self-adjoint. In the case of the spatial part of the coordinate operator, we were able to show essential self-adjointness, by taking the QM coordinate operator and performing a second quantization. In particular, the coordinate operator in QM can be formulated as unitarily equivalent to the momentum operator. The unitary equivalence is given by the Fourier transformation. Thus the investigation shows that operator obtained by second quantization is an essentially self-adjoint. The temporal part seizes to be self-adjoint by the curse of Pauli's theorem. Thus by taking the Fourier transformation to be the unitary operator connecting the momentum and the coordinates also in the temporal part we are able to deform the QFT.

Now actually dealing with the deformed QF we found that indeed certain Wightman properties of the deformed QF are satisfied. Furthermore, by using the map from skew-symmetric matrices to a set of wedges, we were able to show that the deformed field has wedge-covariance and wedge-locality properties. Since locality is usually lost in the context of NCQFT, the locality found is highly nontrivial. To obtain the NC space we calculated the momentum plane, induced by deformation with the coordinate operators. This was done by using the deformed product. As in the QM case, we minimally substituted the coordinate operator using the deformation matrix of the NC momentum plane. The commutator of these new coordinate operators turns out to be equal to the commutation relations of the deformed coordinate operators. Thus, the newly defined coordinate operator generate the QFT-Moyal Weyl.

By comparing warped convolutions with unitary transformations of the QF under

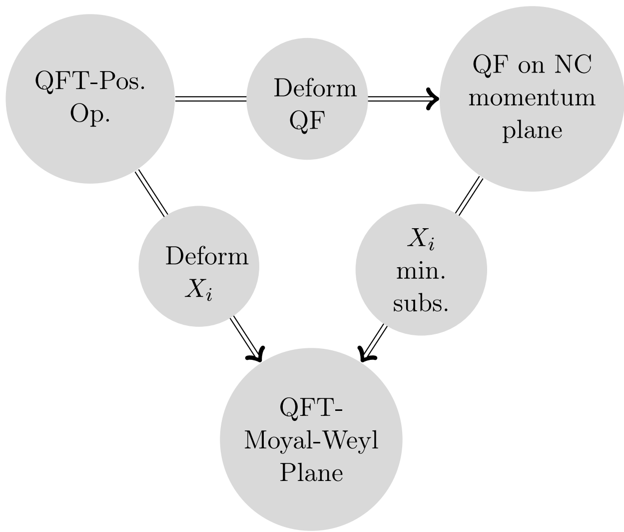

Figure 7.3: QFT deformation

the coordinate operator, we showed that the deformation parameter can be understood as the absolute value of the difference of two vectors that generate unitary transformations of the scalar field. The following diagram summarizes the results of the QFT-deformation. Figure 7.3 summarizes the results of the QFT-deformation.

In the last chapter we deformed a quantum field theory with the special conformal operator, using the warped convolutions. As before, we used the map from the deformed field to a field defined on the wedge. Furthermore, it was proven that the deformed field transforms as a wedge-covariant field under the adjoint action of the Lorentz group. Wedge-locality for the deformed field was shown in all even dimensions. In four dimensions one usually has a problem with the existence of a unitary representation for the whole conformal group. The absence of a unitary representation is due to the absolute value of the scale factor induced by the special conformal transformations. We circumvented the problem by proving positivity of the scale factor. Positivity was proven by using the properties of the wedge and the spectrum condition of the special conformal operator.

Moreover, we used the deformed product to calculate the NC spacetime induced by deformation with the special conformal operators. The investigation shows that we obtain a nonconstant noncommutative spacetime.

The main difference in this work to other works in NCQFT, is the deformation with objects defined by the investigated physical theory. Moreover, the investigation of the coordinate operators, leads to many physical effects in the QM context. In a QFT context the coordinate operators are already endowed with a rich structure, that allow to investigate the realization of a NC spacetime. This structure was thoroughly studied and the emergence of NC spacetimes is shown. For the case of the special conformal operator these ideas were applied and rewarded with a NC spacetime unkown up to the present investigation. Note that on NC spacetimes the notion of locality is usually lost, but on the

NC spacetimes that we investigated, we found that a modified version of the point-wise locality holds: the so called wedge-locality. Thus, by virtue of those locality properties it gives the deformation a possibility of being physically realized.

7.2 Outlook

In the QM chapter, effects of a magnetic field were reproduced by deformation. This was done for the explicit deformation by using the spatial coordinate operator. It is nevertheless interesting to point out that the number of operators used for the deformation does not depend on the number of dimensions. Thus, the construction of a temporal component or a so called time operator would be interesting in this context.

Of course, this operator would have to commute with the spatial operator to build a unitary group for deformation and the time operator would have to be self-adjoint. After the deformation the question of the physical interpretation of this operator would have to be posed.

Moreover, supposing such an operator has been found, a second quantization would have to be performed. Thus, by following the path taken in this thesis, a deformation using the second quantized operator would be performed on the quantum field and the locality properties would have to be further investigated.

For the QFT-coordinate operator we defined a zero component, but were unable to show that the definition is in full agreement with Lorentz-covariance. If covariance could be achieved, we would have solved a problem as old as QFT itself. Namely, finding for each QFT the appropriate operator describing coordinates, in a fully Lorentz-covariant manner.

Moreover, if the defined coordinate operator is found to be truly Lorentz-covariant the deformed field would not only be wedge-covariant w.r.t. a subgroup of the Lorentz group but under the full group. Furthermore, a deformation with the zero component of the coordinate operator on the momentum plane, will supply further nontrivial relations.

In the last chapter we deformed with the special conformal operator and obtained a wedge-local field that can be interpreted as a QF on a nonconstant noncommutative spacetime. An interesting but still not fully exploited path, would be to take some other abelian Fockspace operators and deform on the level of the scalar QF.

Last but not least, the question of nonabelian deformations is open and not far developed. Is the emergence of a wedge-local field by such deformations possible? What are the resulting NC spacetimes in those cases?

These and many other questions are to this date open and provide a broad, interesting and exciting area of research.

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Chapter 8

Appendix

8.1 Banach Space

On of the most basic spaces in functional analysis is the metric space. We give the definition of the space in the following,([RS75a], Ch. I, page 4),

Definition 8.1. A **metric space** is a set M and a real-valued function $d(.,.)$ on $M \times M$ which satisfies,

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, z) + d(y, z)$,

where the last inequality is called the **triangle inequality**. The function d is called a **metric** on M.

Another important notion on metric spaces is known under Cauchy sequences.

Definition 8.2. A sequence of elementes $\{x_n\}$ of a metric space (M, d) is called a **Cauchy sequence** if $(\forall \epsilon > 0)(\exists N)$ n, $m \geq N$ implies $d(x_n, x_m) < \epsilon$.

In the following we define two terms that we consistently make use of

Definition 8.3. A metric space in which all Cauchy sequences converge is called **complete**.

Definition 8.4. A set B in a metric space M is called **dense** if every $m \in M$ is a limit of elements in B.

Moreover, the normed linear space is important in the discussion of bounded linear transformations and is defined as

Definition 8.5. A **normed linear space** is a vector space, V, over $\mathbb{R}$ (or $\mathbb{C}$) and a function, $||.||$ from V to $\mathbb{R}$ which satisfies,

1. $||v|| \geq 0$, for all $v \in V$

2. $||v|| = 0$ if and only if $v = 0$,
3. $||\alpha v|| = |\alpha|||v||$, for all $v \in V$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$)
4. $||v + w|| \leq ||v|| + ||w||$, for all $v, w \in V$

Definition 8.6. We say $< V, ||.|| >$ is **complete** if it is complete as a metric space in the induced metric.

By taking all former definition into account we can give the definition of a Banach space.
Definition 8.7. A complete normed linear space is called a **Banach space**.

8.2 Hilbert Space

In this Section we give the basic definitions and theorems for Hilbert spaces, [RS75a]. We start by defining an inner product space.

Definition 8.8. A complex vector space $\mathcal{V}$ is called an **inner product space** if there is a complex-valued function $\langle \cdot, \cdot \rangle$ on $\mathcal{V} \times \mathcal{V}$ that satisfies the following conditions

1. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
2. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,
3. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$,
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

where $x, y, z \in \mathcal{V}$ and $\alpha \in \mathbb{C}$. The function $\langle \cdot, \cdot \rangle$ is called an inner product.

Now a Hilbert space $\mathscr{H}$ can be defined as follows.

Definition 8.9. A complete inner product space is called a **Hilbert space** $\mathscr{H}$.

Definition 8.10. Two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are said to be **isomorphic** if there is a linear operator U from $\mathscr{H}_1$ *onto* $\mathscr{H}_2$ such that $\langle Ux, Uy \rangle_{\mathscr{H}_2} = \langle x, y \rangle_{\mathscr{H}_1}$ for all $x, y \in \mathscr{H}_1$. Such an operator is called **unitary**.

An important theorem follows from the former definition of inner product spaces and normed linear spaces.

Theorem 8.1. *Every inner product space $\mathcal{V}$ is a normed linear space with the norm $||x|| = \langle x, y \rangle^{1/2}$.*

8.3 Unbounded operators

An unbounded operator A is defined on dense linear subset of the Hilbert space $\mathscr{H}$ as linear map from its domain, a dense subspace of $\mathscr{H}$ to $\mathscr{H}$, i.e. $D(A) \to \mathscr{H}$.

Definition 8.11. The **graph** of the linear transformation T, denoted by $\Gamma(T)$ is the set of pairs

$$\{< \varphi, T\varphi > | \varphi \in D(T)\}. \tag{8.1}$$

Definition 8.12. Let T_1 and T be operators on $\mathscr{H}$. If $\Gamma(T) \subset \Gamma(T_1)$, then T_1 is said to be an **extension** of T and one writes $T \subset T_1$. Equivalently, $T \subset T_1$ iff $D(T) \subset D(T_1)$ and $T\varphi \subset T_1\varphi$ for all $\varphi \in D(T)$.

Definition 8.13. An operator T is **closable** if it has a closed extension. Every closable operator has a smallest closed extension, called its **closure**, which is denoted by $\overline{T}$.

Definition 8.14. Let T be a densely defined linear operator on a Hilbert space $\mathscr{H}$. Let $D(T^*)$ be the set of $\varphi \in \mathscr{H}$ for which there is an $\eta \in \mathscr{H}$ with

$$\langle T\psi, \varphi \rangle = \langle \psi, \eta \rangle, \qquad \forall \psi \in D(T). \tag{8.2}$$

For each such $\varphi \in D(T^*)$, we define $T^*\varphi = \eta$. T^* is called the **adjoint** of T. By the Riesz lemma, $\varphi \in D(T^*)$ iff $|\langle T\psi, \varphi\rangle| \leq C||\psi||$ for all $\psi \in D(T)$.

Definition 8.15. A densely defined operator T on a Hilbert space is called **Hermitian** (or **symmetric**) if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$. Equivalently T is Hermitian iff

$$\langle T\varphi, \psi \rangle = \langle \varphi, T\psi \rangle, \qquad \forall \varphi, \psi \in D(T). \tag{8.3}$$

Definition 8.16. T is called a self-adjoint operator, iff T is symmetric and $D(T) = D(T^*)$.

Definition 8.17. A symmetric operator T is called **essentially self-adjoint** if its closure $\overline{T}$ is self-adjoint.

In the following we give an important proposition which will be made use of during the entire thesis. We start by giving the definition of unitarily equivalent operators.

Definition 8.18. Let T, S be densely defined unbounded operators. Then, T, S are said to be **unitarily equivalent** if there is a unitary operator $U : D(S) \to D(T)$ such that $T = USU^{-1}$. This condition in particular means, that $D(T) = UD(S)$.

The following proposition deals with the hermiticity and self-adjointness of unitarily equivalent operators, [BEH08].

Proposition 8.1. *Let T, S be densely defined unbounded operators that are unitarily equivalent, i.e. $T = USU^{-1}$, where U is a unitary operator. Then if S is densely defined, the same holds true for T and $T^* = US^*U^{-1}$. In particular, if S is symmetric or self-adjoint, then T is also symmetric or self-adjoint, respectively.*

The next theorem concerns the self-adjointness of an unbounded operator on a tensor product of Hilbert spaces, ([RS75a], Theorem VIII.33).

Theorem 8.2. *Let A_k be a self-adjoint operator on $\mathscr{H}_k$. Let $P(x_1, \cdots, x_N)$ be a polynomial with real coefficients of degree n_k in the k-th variable and suppose that D_k^e is a domain of essential self-adjointness for $A_k^{n_k}$. Then, $P(A_1, \cdots, A_N)$ is essentially self-adjoint on*

$$D^e = \bigotimes_{k=1}^{N} D_k^e.$$

Theorem 8.3. *Let A_k be a self-adjoint operator on $\mathscr{H}_k$. Let $P(x_1, \cdots, x_N)$ be a polynomial with real coefficients of degree n_k in the k-th variable and suppose that D_k^e is a domain of essential self-adjointness for $A_k^{n_k}$. Then, $P(A_1, \cdots, A_N)$ is essentially self-adjoint on*

$$D^e = \bigotimes_{k=1}^{N} D_k^e.$$

To prove the essential self-adjointness of a symmetric operator A on a Hilbert space $\mathscr{H}$, the Nelson's analytic vector theorem can be used. Before stating the theorem we first give the definition of an analytic vector for A, ([RS75b], Chapter X.6).

Definition 8.19. Let A be an operator on a Hilbert space $\mathscr{H}$. The set $C^\infty := \bigcap_{n=1}^\infty D(A^n)$ is called the **C^∞-vectors for A**. A vector $\phi \in C^\infty(A)$ is called an **analytic vector for A** if

$$\sum_{n=0}^{\infty} \frac{|A^n \phi|}{n!} t^n < \infty, \tag{8.4}$$

for some $t > 0$.

By using the former definition we can now give the Nelson's analytic vector theorem, ([RS75b], Chapter X.6, Theorem X.39).

Theorem 8.4. *Let A be a symmetric operator on a Hilbert space $\mathscr{H}$. If $D(A)$ contains a total set of analytic vectors, then A is essentially self-adjoint.*

8.4 Schwartz space

The Schwartz space is a dense subspace of the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$. The definition is given as follows.

Definition 8.20. Let $C^\infty(\mathbb{R}^n)$ be the set of all complex-valued functions which have partial derivatives of arbitrary order. For $f \in C^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ we set

$$\partial_\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha 1} \cdots \partial x_n^{\alpha n}}, \quad x^\alpha = x_1^{\alpha 1} \cdots x_n^{\alpha n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Then the **Schwartz space** denoted by $\mathscr{S}(\mathbb{R}^n)$ is given as

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \sup_x |x^\alpha (\partial_\beta f)(x)| < \infty, \alpha, \beta \in \mathbb{N}_0^n\},$$

and is dense in $L^2(\mathbb{R}^n)$.

8.5 Fourier Transformation

In this section we give a brief summary of the most basic facts of the Fourier transformation, [RS75b, Tes01].

Definition 8.21. For $f \in \mathscr{S}(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ is defined in 8.20, we define the **Fourier transformation** as follows

$$U_{\mathscr{F}}(f)(p) := \hat{f}(p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ipx} f(x) d^n x. \tag{8.5}$$

Lemma 8.1. *For any multi-index $\alpha \in \mathbb{N}_0^n$ and any $f \in \mathscr{S}(\mathbb{R}^n)$ we have*

$$U_{\mathscr{F}}(\partial_\alpha f)(p) = (ip)^\alpha (U_{\mathscr{F}} f)(p), \tag{8.6}$$

$$(\widehat{x^\alpha f(x)})(p) = i^{|\alpha|} \partial_\alpha \hat{f}(p), \tag{8.7}$$

where $\partial = (\partial_1, \cdots, \partial_n)$ is the gradient.

In particular $U_{\mathscr{F}}$ is an operator mapping $\mathscr{S}(\mathbb{R}^n)$ into itself.

Lemma 8.2. *The Fourier transform of the* ***convolution***

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) d^n y = \int_{\mathbb{R}^n} f(x-y) g(y) d^n y \tag{8.8}$$

of two functions $f, g \in \mathscr{S}(\mathbb{R}^n)$ is given by

$$\widehat{(f * g)}(p) = (2\pi)^{n/2} \hat{f}(p) \hat{g}(p). \tag{8.9}$$

Theorem 8.5. *The Fourier transform $U_{\mathscr{F}} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a bijection. Its inverse is given by*

$$U_{\mathscr{F}}^{-1}(f)(x) := \check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ipx} f(p) d^n p. \tag{8.10}$$

Furthermore, $U_{\mathscr{F}}^2(f)(x) = f(-x)$ and thus $U_{\mathscr{F}}^4 = \mathbb{I}$.

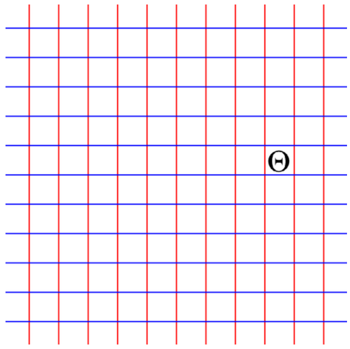

Figure 8.1: Example of a Moyal-Weyl plane $\mathbb{R}^2_\theta$

Theorem 8.6. *The Fourier transform $U_{\mathscr{F}}$ extend to a unitary operator $U_{\mathscr{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Its spectrum satisfies*

$$\sigma(U_{\mathscr{F}}) = \{z \in \mathbb{C} | z^4 = 1\} = \{1, -1, i, -i\}. \tag{8.11}$$

Lemma 8.1 allows to extend differentiation to a larger class.

Definition 8.22. The **Sobolev space** denoted as $H^r(\mathbb{R}^n)$ is defined as the following function space

$$H^r(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) | |\mathbf{P}|^r f \in L^2(\mathbb{R}^n)\}. \tag{8.12}$$

8.6 Moyal-Weyl Plane

We define the Moyal-Weyl plane as done in [DFR95].

Let us denote the Moyal-Weyl plane by $\mathbb{R}^d_\theta$, see figure 8.1. Where θ is a constant that reflects the strength of the noncommutativity. The *-algebra which is generated by the selfadjoint operators $\hat{x}_\mu$ has a center given by the commutator relation

$$[\hat{x}_\mu, \hat{x}_\nu] = -i\theta_{\mu\nu}, \tag{8.13}$$

where $\theta_{\mu\nu}$ is a constant non-degenerate tensor. An example for such an algebra is the quantum mechanical commutation relations between the momentum and the coordinate. Where in the QM case the deformation parameter θ is given by $\hbar$.

Notation and Conventions

We use in this thesis n to denote the spatial dimensions and d to denote the spacetime dimensions. While d-dimensional vectors have Greek letters, spatial vectors are denoted with Latin indexes. To denote a spatial vector, we often use bold letters.

The Einstein convention is used when an index variable appears twice, implying the summation over all possible values of the index, i.e.

$$\sum_{i=1}^{n} a_i b^i := a_i b^i. \tag{8.14}$$

Throughout this thesis we use the following convention for the Minkowski metric η

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8.15}$$

Not that we use this convention for the quantum mechanical case as well. This means for example that the free Hamiltonian is defined with a minus, i.e. $H_0 = -P^i P_i$.

Most of the time, except for cases were it becomes important we set the constants of speed of light c and the Planck constant $\hbar$ equal to

$$c = \hbar = 1. \tag{8.16}$$

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