Concerning Triangulations of Products of Simplices

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Contents

List of Symbols

$[a,b]$	the set $\{a, a+1, \ldots, b-1, b\}$ for $a < b \in \mathbb{N}$ **(p. 28)**
$[d]^{[w]}$	set of maps from the set $[w]$ to the set $[d]$ **(p. 14)**
$[n]$	the set $\{1, 2, \ldots, n\}$ **(p. 10)**
$\mathbf{A}, \mathbf{A}', \mathbf{B}$	point configurations **(p. 74)**
$\mathbf{a}, \mathbf{v}, \mathbf{w}$	points in a point configuration **(p. 46)**
$\mathbf{A} \times \mathbf{B}$	cartesian product of the point configurations $\mathbf{A}, \mathbf{B}$ **(p. 75)**
$\mathbf{B}, \mathbf{B}'$	fine mixed cells in a Minkowski sum **(p. 86)**
$\widetilde{\mathbf{B}}$	candidate cell induced by a partial fine mixed subdivision $\mathcal{M}'$ **(p. 45)**
$\beta_{i,j}(\mathbf{M})$	Betti numbers of the graded $\mathbf{R}$-module $\mathbf{M}$ **(p. 66)**
$\binom{[n]}{k}$	set of k-element subsets of $[n]$ **(p. 44)**
$\boldsymbol{d}$	chain, or totally ordered set $\{1 < 2 < \ldots, d-1 < d\}$ **(p. 21)**
$\mathbf{C}$	circuit, i.e., minimally dependent point configuration **(p. 78)**
$\mathbf{c}$	affine depencence $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ on a point configuration **(p. 75)**
$\mathbf{C}^+, \mathbf{C}^-$	positive and negative parts of a circuit **(p. 78)**
$\mathfrak{C}_n^{(\ell)}$	$\{\varphi_\times^\ell(\mathbf{e}_i, \mathbf{e}_a) \in \Delta_{n-1} \times \Delta_{n-1} \colon i \leq a\}$ **(p. 50)**
$\mathcal{D}(\mathbf{A})$	vector space of affine dependences of the point configuration $\mathbf{A}$ **(p. 76)**
Δ_{n-1}	standard $(n-1)$-simplex $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \subset \mathbb{R}^n$ **(p. 75)**
$\mathfrak{D}_n^{(\ell)}$	ℓ-fold iterated image of the Dyck paths in $\mathcal{G}_{n\times d}$ under $\varphi_\times^\ell$ **(p. 50)**
$\dot\cup$	disjoint union **(p. 6)**
$\mathbf{F}, \mathbf{F}', \mathbf{G}$	face or a subconfiguration of a point configuration **(p. 75)**
$\mathcal{F}_*, \mathcal{G}_*$	free resolutions of an $\mathbf{R}$-module **(p. 64)**
Γ_n	incomparability graph of the Boolean lattice $\mathbf{2}^n$ **(p. 17)**
$\mathcal{G}_{n\times d}$	$n \times d$ rectangular grid **(p. 29)**
$\mathbf{I}$	ideal in a polynomial ring **(p. 63)**
$\mathbf{I}_G$	cut ideal of the graph G **(p. 12)**
$\mathcal{K}$	polyhedral complex **(p. 74)**
$\Lambda_{d,n-1}$	weak integer compositions of $n-1$ of length d **(p. 35)**

$\mathcal{M}', \mathcal{N}'$	(partial) fine mixed subdivision corresponding to a partial triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ **(p. 39)**
$\mathcal{M}, \mathcal{M}^\star$	mixed subdivision and its dual (when of $n\Delta_{d-1}$) **(p. 86)**
$\mathcal{M}_I$	fine mixed subdivision of $k\Delta_{n-1}$ with summands indexed by $I \subset [n]$ **(p. 45)**
$\partial(\mathbf{A})$	collection of facets of the point configuration **A** (i.e., boundary) **(p. 38)**
$\mathbf{p}$	discrete probability distribution **(p. 4)**
ϕ	module homomorphism **(p. 64)**
$\mathbf{2}^n$	power set, or Boolean lattice, of a set of n elements **(p. 21)**
$\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(k)$	projective $(n-1)$-dimensional space over the field k **(p. 66)**
$\mathbf{q}(\mathbf{B}), \mathbf{r}(\mathbf{B})$	"position" and "shape" of the cell **B** in a fine mixed subdivision of $n\Delta_{d-1}$ **(p. 33)**
$\mathbf{Q}_G$	polynomial ring with two variables for every edge of the graph G **(p. 12)**
$\mathbf{R}_G$	polynomial ring with variables indexed by cuts of the graph G **(p. 12)**
ρ	system of permutations of $[n]$ along the edges of $n\Delta_{d-1}$ **(p. 39)**
$\mathrm{skel}_l\mathbf{A}$	l-skeleton of the point configuration **A** **(p. 25)**
$\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\rho}$	simplices, either in a simplicial complex or in a triangulation **(p. 76)**
$\tilde{\boldsymbol{\sigma}}$	full-dimensional candidate simplex induced by a triangulation $\widetilde{\mathcal{T}}'$ of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ **(p. 46)**
$\mathcal{T}$	simplicial complex or triangulation **(p. 79)**
$\mathcal{T}'$	partial triangulation of a point configuration **(p. 82)**
$\widetilde{\mathcal{T}}'$	collection of full-dimensional candidate simplices induced by a triangulation $\widetilde{\mathcal{T}}'$ of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ **(p. 46)**
φ^ℓ (resp. $\varphi^\ell_\times$)	map $\mathcal{G}_{n\times n} \ni (i,\alpha) \mapsto (i+\ell \pmod n, \alpha+\ell \pmod n)$ (resp. $\Delta_{n-1} \times \Delta_{n-1} \ni (\mathbf{e}_i, \mathbf{e}_\alpha) \mapsto (\mathbf{e}_{i+\ell \pmod n}, \mathbf{e}_{\alpha+\ell \pmod n})$) **(p. 50)**
$\vartheta(\mathbf{B}), \mathrm{supp}(\vartheta(\mathbf{B}))$	stem of the fine mixed cell **B** and its set of positive entries (i.e., its support) **(p. 36)**
$\mathbf{V}$	affine or projective variety **(p. 13)**
$\mathbf{X}_G$	edge ideal of the graph G **(p. 10)**
$\mathbf{Y}_G$	polynomial ring with variables indexed by the vertices of the graph G **(p. 10)**
C_n	n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ **(p. 50)**
$Cuts(G)$	set of cuts of the graph G **(p. 15)**
$E(G)$	set of edges of the graph G **(p. 10)**
G	finite simple graph **(p. 11)**
$G_1 \# G_2$	0-sum of the graphs G_1, G_2 **(p. 13)**
$G_{ij}(\rho)$	directed graph associated to symbols i, j in a system of permutations ρ **(p. 39)**

$I \times_{Seg} J$	Segre product of the homogeneous ideals I, J **(p. 13)**
I, J	subset of indices of $[n]$ **(p. 26)**
k	field **(p. 61)**
$k[x_1, x_2, \ldots, x_n]$	ring of polynomials on n variables with coefficients in the field k **(p. 62)**
K_n	complete graph on n vertices **(p. 13)**
$K_{n,d}$	complete bipartite graph on $n+d$ vertices **(p. 26)**
s_e, t_e	indeterminates for cut and “uncut” edges in the polynomial ring $\mathbf{Q}_G$ **(p. 12)**
t	subtree of $K_{n,d}$ **(p. 46)**
T_n	tree on n vertices **(p. 15)**
v	vertex in a graph **(p. 13)**
$V(G)$	set of vertices of the graph G **(p. 10)**
X	discrete finite random variable **(p. 4)**

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“(...) si te quiero es porque sos
mi amor mi cómplice y mi todo
y en la calle codo a codo
somos mucho más que dos (...)”

Mario Benedetti “El amor, las mujeres y la vida”

Introduction

This thesis has its central motivation around the following general question:

Question 1: Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$ and $\mathbf{A}' = \{\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, \dots, \boldsymbol{\alpha}'_m\} \subset \mathbb{R}^{d'}$ be two point configurations. What is the relation between triangulations of their cartesian product:

$$\mathbf{A} \times \mathbf{A}' := \{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}'_j) : \boldsymbol{\alpha}_i \in \mathbf{A}, \boldsymbol{\alpha}'_j \in \mathbf{A}'\} \subset \mathbb{R}^{d+d'},$$

and triangulations of $\mathbf{A}$ and $\mathbf{A}'$?

A systematic treatment of triangulations of cartesian products of point configurations is currently nonexistent in the literature, and may well be out of reach in full generality. Indeed, even when both factors in the cartesian product are the standard simplices $\mathbf{A} = \Delta_{n-1}$ and $\mathbf{A}' = \Delta_{d-1}$, which only have the trivial triangulation consisting of one simplex, the collection of all triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ is quite intricate [GKZ08, DS04, San04, DRS10]. We briefly mention, in particular, that[1]:

- *(a)* a general description of the set of *regular triangulations* of $\Delta_{n-1} \times \Delta_{d-1}$ for $\min\{n, d\} > 2$ is unavailable [GKZ08, DRS10],
- *(b)* it is not known in general whether the set of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ is connected under *geometric bistellar flips* for $\min\{n, d\} > 3$ [San04, DRS10],
- *(c)* it took some time until *non-regular triangulations* of $\Delta_{n-1} \times \Delta_{d-1}$ were discovered, and their existence turned out to be closely related to realizability of matroids over the reals [GKZ08, De 96, Stu96, San02],
- *(d)* triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ have been used to model combinatorial structures coming from various fields of mathematics, such as *(i)* elimination theory [GKZ08, Stu96], *(ii)* tropical geometry [DS04, AD09], *(iii)* Schubert calculus [AB07].

Moreover, triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ have also been featured in many works in discrete geometry [Hai91, BB98, OS03, San04, AC13, San12]; in [San00], for instance, Santos used triangulations of products of simplices as building blocks to cleverly construct a triangulation of a point configuration that does not admit modifications via geometric bistellar flips.

In this thesis, we explore two problems concerning triangulations of cartesian products of simplices with a focus on their product structure and their relevance

[1] The reader can find the definitions of the italicized terms in chapter 1 and appendices A and B.

in combinatorial commutative algebra, as implied by Sturmfels' correspondence between *initial ideals of toric ideals* and triangulations of integer point configurations (Theorem A.3.5 or [Stu91, Theorem 3.1]).

In chapter 1, we present joint work with Samu Potka relating to *cut ideals of graphs* [PS12]. These are homogeneous polynomial ideals introduced by Sturmfels and Sullivant in [SS06] and further studied by Engström [Eng11] and Nagel and Petrovič [NP08]. Cut ideals of graphs can be thought of as defining a family of probability distributions on the cuts of a tree, where the probability of a cut only depends on the collection of edges it separates[2].

Concretely, we study *minimal free resolutions* for the family of cut ideals of tree graphs. Our main result consists of upper bounds for the *Betti numbers*[3] of cut ideals of trees. Our bounds take the form of exponential formulas on the number of vertices of the tree, which result from the enumeration of induced subgraphs of certain graph associated to the edges of the tree.

In order to achieve this combinatorial description, we start with the observation that the cut ideal of a tree in n vertices (henceforth denoted $\mathbf{I}_{T_n}$) equals the $(n-1)$-fold *Segre product* of the cut ideal of an edge (in particular, this implies that all cut ideals of trees in a fixed number of vertices are algebraically the same, regardless of the shape of the tree). In view of the correspondence between *toric ideals* and integer point configurations (cf. section A.3 or [Stu96, Chapter 4]), this translates into the statement that the point configuration associated to $\mathbf{I}_{T_n}$ is the cartesian product of $n-1$ segments: an $(n-1)$-dimensional cube. We exploit this product structure to define an *initial ideal* for $\mathbf{I}_{T_n}$ that equals the *edge ideal* of a certain graph Γ_{n-1}, denoted $X_{\Gamma_{n-1}}$. The vertices of Γ_{n-1} are subsets of the edge set of the tree T_n, and its edges are pairs of incomparable subsets of edges. By Sturmfels' correspondence (cf. theorem A.3.5 or [Stu91, Theorem 3.1]), choosing this initial ideal corresponds to choosing a *staircase triangulation* for the family of $(n-1)$-cubes [DRS10, Remark 6.2.17].

We then use an idea of Dochtermann and Engström from [DE09], and regard $X_{\Gamma_{n-1}}$ as the *Stanley-Reisner ideal* of the *independence complex* of Γ_{n-1}; the upper bounds for the Betti numbers follow from application of Hochster's formula (theorem 1.1.3 or [Hoc77, Theorem 5.1]) and enumeration of the induced subgraphs of Γ_{n-1}. We conclude chapter 1 discussing an extension of our methods for deriving upper bounds for the Betti numbers of the homogeneous ideal defining the *Segre embedding* of the cartesian product of several projective spaces of any dimension.

Chapter 2 undertakes a combinatorial study of triangulations of the cartesian product of two standard simplices, developed in collaboration with César

[2] A tree is a finite simple connected graph without cycles, and a cut is an unordered partition of its vertices into two subsets. Some edges of the tree may have their endpoints lying on different parts of a given cut, i.e., may be *separated* by the cut.

[3] Recall briefly that these invariants of homogeneous polynomial ideals give the numbers of minimal generators, in the different degrees, for the ideal and its syzygy modules (c.f. definition A.1.15), and that they are used to define further invariants of interest in commutative algebra [Eis05].

Ceballos and Arnau Padrol in [CPS13]. It departs from a general framework to study triangulations of products of point configurations through certain associated *partial triangulations* that are defined to reflect the product structure.

First, we recall some existing results about "full" and partial triangulations of $\Delta_{n-1} \times \Delta_{d-1}$. It turns out that the representation of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ as *fine mixed subdivisions of the dilated simplex* $n\Delta_{d-1}$ by the Cayley trick (cf. theorem B.4.3 or [HRS00, Theorem 3,1], [San04, Theorem 1.4]) provides a convenient representation for the partial triangulations we consider, as well as useful geometric intuition. Making use of this representation we are led to our main result, which is perhaps the strongest in the thesis; namely, a "finiteness theorem" that loosely reads:

Theorem (Theorem 2.4.25)**.** *If $d > n+1$,* **every** *triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is uniquely obtained by "gluing" compatible triangulations of the faces of $\Delta_{n-1} \times \Delta_{d-1}$ of the form $\Delta_{n-1} \times \Delta_n$.*

In other words, triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are not "much more complicated" than triangulations of $\Delta_{n-1} \times \Delta_n$, provided $d > n+1$.

With an explicit construction, that appears to be new, we prove moreover that the bound $d > n+1$ is tight:

Theorem (Theorem 2.4.32)**.** *For every $n \in \mathbb{N}$, $n \geq 2$, there is a collection of compatible triangulations of the faces of $\Delta_{n-1} \times \Delta_n$ of the form $\Delta_{n-1} \times \Delta_{n-1}$ that do not correspond to the restriction of any triangulation of $\Delta_{n-1} \times \Delta_n$.*

A key ingredient towards our construction is a family of triangulations of $\Delta_{n-1} \times \Delta_{n-1}$ that can be considered their second "simplest" triangulation (the simplest one being the staircase triangulations cf. theorem 2.2.8 or [DRS10, Theorem 6.2.13]). As it also seems to be new, and we call it the *Dyck path triangulation* of $\Delta_{n-1} \times \Delta_{n-1}$ (cf. theorem 2.4.27). Given its simplicity and its versatility to produce the family of non-extendable triangulations, we deem it deserves further study which, unfortunately grows out of the scope of this thesis. Based on the "finiteness" interpretation of theorem 2.4.25, we conclude chapter 2 formulating analogous finiteness conjectures for triangulations of the cartesian product of several simplices.

Since the concepts from commutative algebra we deal with tend to lie distant apart in a typical learning curve, and the theory of triangulations has only recently received a unified treatment in [DRS10], we have included two appendices that contain some definitions and facts pertinent to our presentation. However, we have not made an attempt towards being self-contained, and at several points refer the reader to the relevant literature.

Why triangulations of products?

Question 1 is a special case of the following question:

Question 2: Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$, $\mathbf{A}' = \{\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, \dots, \boldsymbol{\alpha}'_m\} \subset \mathbb{R}^{d'}$ and $\mathbf{A}'' = \{\boldsymbol{\alpha}''_1, \boldsymbol{\alpha}''_2, \dots, \boldsymbol{\alpha}''_l\} \subset \mathbb{R}^{d''}$ be point configurations such that the points of $\mathbf{A}$ and $\mathbf{A}'$ affinely project to the points of $\mathbf{A}''$; that is, such that there are affine maps $\pi_1 : \mathbf{A} \to \mathbf{A}''$ and $\pi_2 : \mathbf{A}' \to \mathbf{A}''$. What is the relation between triangulations of the point configuration:

$$\mathbf{A} \times_{\mathbf{A}''} \mathbf{A}' := \{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}'_j) : \boldsymbol{\alpha}_i \in \mathbf{A}, \boldsymbol{\alpha}'_j \in \mathbf{A}' \text{ with } \pi_1(\boldsymbol{\alpha}_i) = \pi_2(\boldsymbol{\alpha}'_j)\} \subset \mathbb{R}^{d+d'}, \qquad (1)$$

and triangulations of $\mathbf{A}$, $\mathbf{A}'$ and $\mathbf{A}''$?

Point configurations of the form (1) are called *fiber products*, and were introduced by Engström, Kahle and Sullivant in their study of composition of hierarchical log-linear statistical models in [EKS11]. Specifically, they showed that the *composite nature* of a hierarchical log-linear model induces the fiber product structure (1) on the point configuration associated to the toric ideal defining the hierarchical log-linear model.

Let us recall some definitions in order to make sense of the preceding paragraph; most are taken from [GMS06, DSS09, EKS11]. Consider a collection of random variables $X_1, X_2, \dots, X_N$, where X_α takes values in the set $[d_\alpha] = \{1, 2, \dots, d_\alpha\}$ (for $\alpha \in [n]$ and $d_\alpha \geq 2$), and let:

$$\mathbf{p} = \left\{p_{\lambda_1,\lambda_2,\dots,\lambda_N} := p(X_1 = \lambda_1, X_2 = \lambda_2, \dots, X_N = \lambda_N)\right\}_{(\lambda_1,\lambda_2,\dots,\lambda_N)\in\prod_{\gamma=1}^N [d_\gamma]},$$

be a joint probability distribution for the outcomes of the random variables. Given a subcollection of the random variables indexed by $\boldsymbol{\sigma} = \{\alpha_1, \alpha_2, \dots, \alpha_u\} \subset [N]$, we define the $\boldsymbol{\sigma}$*-marginal* of $\mathbf{p}$ as the joint probability distribution on the variables $\{X_\alpha : \alpha \in \boldsymbol{\sigma}\}$ gotten from $\mathbf{p}$ by summing over all indices not in $\boldsymbol{\sigma}$. In symbols:

$$p^{\boldsymbol{\sigma}}_{\lambda_{\alpha_1},\lambda_{\alpha_2},\dots,\lambda_{\alpha_u}} := \sum_{(\lambda_{\gamma_1},\lambda_{\gamma_2},\dots,\lambda_{\gamma_t})\in\prod_{\kappa\in[N]\setminus\boldsymbol{\sigma}}[d_\kappa]} p_{\lambda_1,\lambda_2,\dots,\lambda_N}, \qquad (2)$$

(i.e., $[N]\setminus\boldsymbol{\sigma} = \{\gamma_1, \gamma_2, \dots, \gamma_t\}$).

Let $\mathcal{T}$ be a simplicial complex on the base set $[N]$. The *hierarchical log-linear model* defined by $\mathcal{T}$ is a family of probability distributions for the joint outcomes of $X_1, X_2, \dots, X_N$ parameterized by the $\boldsymbol{\sigma}$-marginals of the outcomes, with $\boldsymbol{\sigma}$ ranging over the maximal faces of $\mathcal{T}$:

$$p_{\lambda_1,\lambda_2,\dots,\lambda_N} = \prod_{\substack{\boldsymbol{\sigma} \text{ maximal} \\ \text{face of } \mathcal{T}}} t^{\boldsymbol{\sigma}}_{(\lambda_1,\lambda_2,\dots,\lambda_N)|_{\boldsymbol{\sigma}}}; \qquad (3)$$

here, $(\lambda_1, \lambda_2, \dots, \lambda_N)|_{\boldsymbol{\sigma}} = (\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_u})$ for $\boldsymbol{\sigma} = \{\alpha_1, \alpha_2, \dots, \alpha_u\} \subset [N]$, and $t^{\boldsymbol{\sigma}}_{(\lambda_1,\lambda_2,\dots,\lambda_N)|_{\boldsymbol{\sigma}}} \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$[4].

[4]Although only real, nonnegative and normalized probability distributions are meaningful, the algebraic study of the parameterization (3) is commonly simplified by allowing domain and range of the parameterization to be complex [PS05, DSS09].

Example 1. *(a)* Let $N = 2$, $d_1 = r$ and $d_2 = c$ and $\mathcal{T} = \{\{1\},\{2\}\}$. Here $\mathcal{T}$ defines the hierarchical log-linear model given by the parametrization:

$$\begin{aligned} (\mathbb{C}^*)^{r+c} &\xrightarrow{\phi_{\mathcal{T}}} (\mathbb{C}^*)^{r\cdot c} \\ (t^{(1)}_{\lambda_1}, t^{(2)}_{\lambda_2}) &\longmapsto p_{\lambda_1,\lambda_2} = t^{(1)}_{\lambda_1} \cdot t^{(2)}_{\lambda_2}, \qquad 1 \le \lambda_1 \le r,\ 1 \le \lambda_2 \le c. \end{aligned}$$

The (projective closure of the) image of the parametrization $\phi_{\mathcal{T}}$ can be identified with the projectivized set of rank-1 $r \times c$ complex matrices (equivalently, the *Segre embedding* of $\mathbb{P}^{r-1}(\mathbb{C}) \times \mathbb{P}^{c-1}(\mathbb{C})$ into $\mathbb{P}^{rc-1}(\mathbb{C})$). In the statistics literature, such a model is called the *independence model* for the random variables X_1, X_2 [DSS09].

(b) Let $N = 3$, $d_1 = d_2 = d_3 = 2$ and $\mathcal{T} = \{\{1,2\},\{2,3\},\{1,3\}\}$. The hierarchical log-linear model is given by the parametrization:

$$\begin{aligned} (\mathbb{C}^*)^{4+4+4} &\xrightarrow{\phi_{\mathcal{T}}} (\mathbb{C}^*)^{2\cdot 2\cdot 2} \\ (t^{(12)}_{\lambda_1,\lambda_2}, t^{(23)}_{\lambda_2,\lambda_3}, t^{(13)}_{\lambda_1,\lambda_3}) &\longmapsto p_{\lambda_1,\lambda_2,\lambda_3} = t^{(12)}_{\lambda_1,\lambda_2} t^{(23)}_{\lambda_2,\lambda_3} t^{(13)}_{\lambda_1,\lambda_3}, \qquad 1 \le \lambda_1,\lambda_2,\lambda_3 \le 2, \end{aligned}$$

and is called the *binary no-threeway-interaction statistical model* [DSS09].

⬡

From the viewpoint of *algebraic statistics*, the study of hierarchical log-linear models proceeds by examining the projective variety $\mathbf{V}_{\mathcal{T}}$ obtained as the projective closure of the image of the parameterization (3) (which is taken for *the* hierarchical log-linear model) and the ideal $\mathbf{I}_{\mathcal{T}}$ of homogeneous polynomials vanishing on it. By virtue of the monomial form of the parametrization (3), a hierarchical log-linear model turns out to be a (irreducible) projective *toric variety* and its ideal of vanishing polynomials a (prime) homogeneous *toric ideal* [Stu96, DSS09].

A homogeneous toric ideal arising from a monomial parametrization as (3) has associated an integer point configuration $\mathbf{A}_{\mathcal{T}}$, consisting of the points $\mathbf{a}_{\lambda_1,\dots,\lambda_N} \subset \mathbb{R}^d$ with coordinates given by the exponents of the monomial in the right hand side of equation (3). Thus, ranging over the entries of $\mathbf{p}$ we obtain a collection of $d_1 \cdot d_2 \cdot \ldots \cdot d_N$ points in $\mathbb{R}^d$, where $d = \sum_{\boldsymbol{\sigma}\in\mathcal{T}} \prod_{\alpha\in\boldsymbol{\sigma}} d_\alpha$.

Example 2. *(a)* Suppose $N = 2$, $d_1 = d_2 = 2$ and $\mathcal{T} = \{\{1,2\}\}$: a 1-dimensional simplex. The monomial parametrization (3) then reads:

$$\begin{aligned} (\mathbb{C}^*)^{2\cdot 2} &\xrightarrow{\phi_{\mathcal{T}}} (\mathbb{C}^*)^{2\cdot 2} \\ t^{(12)}_{\lambda_1,\lambda_2} &\longmapsto p_{\lambda_1,\lambda_2} = t^{(12)}_{\lambda_1,\lambda_2}, \qquad 1 \le \lambda_1,\lambda_2 \le 2. \end{aligned}$$

The projective closure of the parametrization equals $\mathbb{P}^3(\mathbb{C})$, whose ideal of vanishing polynomials is $\{0\} \subset \mathbb{C}[p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}]$. The coordinates of the points in the point configuration associated to $\mathbf{I}_{\mathcal{T}}$ are the standard basis vectors of $\mathbb{R}^4$: $\mathbf{A}_{\mathcal{T}} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} \subset \mathbb{R}^4$.

(b) Let $N = 2$, $d_1 = d_2 = 4$ and $\mathcal{T} = \{\{1\},\{2\}\}$. We have that $\mathbf{V}_{\mathcal{T}}$ is the *Segre embedding* of $\mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C})$ into $\mathbb{P}^{15}(\mathbb{C})$. The point configuration associated

to $\mathbf{I}_{\mathcal{T}}$ is the cartesian product of two standard 3-simplices:

$$\mathbf{A}_{\mathcal{T}} = \{(\mathbf{e}_{\lambda_1}, \mathbf{e}_{\lambda_2}) : \mathbf{e}_{\lambda_1} \in \mathbb{R}^4, \mathbf{e}_{\lambda_2} \in \mathbb{R}^4\} = \Delta_3 \times \Delta_3 \subset \mathbb{R}^8.$$

(c) Let $N = 3$, $\mathcal{T} = \{\{1,2\},\{2,3\}\}$ and $d_1 = d_2 = d_3 = 2$. This defines the parameterization $p_{\lambda_1,\lambda_2,\lambda_3} = t^{(12)}_{\lambda_1,\lambda_2} t^{(23)}_{\lambda_2,\lambda_3}$, with $1 \le \lambda_1, \lambda_2, \lambda_3 \le 2$. We list the points in the associated point configuration as the columns of an 8×8 matrix:

$$\mathbf{A}_{\mathcal{T}} = \begin{array}{rc} {}_{\text{marginals}}\backslash^{x_1x_2x_3=} & \begin{array}{cccccccc} 111 & 112 & 121 & 122 & 211 & 212 & 221 & 222 \end{array} \\ \begin{array}{r} x_1x_2{=}11 \\ x_1x_2{=}12 \\ x_1x_2{=}21 \\ x_1x_2{=}22 \\ x_2x_3{=}11 \\ x_2x_3{=}12 \\ x_2x_3{=}21 \\ x_2x_3{=}22 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \subset \mathbb{R}^8$$

⬡

In the algebraic statistics literature, it had been observed that certain aspects of the *composite nature* of a simplicial complex $\mathcal{T}$ defining a hierarchical log-linear model were displayed by the associated point configuration $\mathbf{A}_{\mathcal{T}}$ (see for example [SS03, HS07]). A general “product” pattern underlying such point configurations was identified by Engström, Kahle and Sullivant in [EKS11], where they pursued an algebraic investigation of the hierarchical log-linear model specified by a simplicial complex $\mathcal{T}$, on account of $\mathcal{T}$'s being of composite nature. We briefly paraphrase their reasoning.

Let $\mathcal{T}$ be a simplicial complex on the base set $[N]$, and let $I, J \subset [N]$ be such that both $\mathcal{T}|_I$ and $\mathcal{T}|_J$ are subcomplexes of $\mathcal{T}$ and $\mathcal{T} = \mathcal{T}|_I \cup \mathcal{T}|_J$. Viewing $\mathcal{T}|_I$ and $\mathcal{T}|_J$ as individual simplicial complexes on base sets I and J, respectively, $\mathcal{T}$ can be considered as being composed of $\mathcal{T}|_I$ and $\mathcal{T}|_I$:

$$\mathcal{T} = \mathcal{T}|_I \cup \mathcal{T}|_J = {}^{\mathcal{T}|_I \dot{\cup} \mathcal{T}|_J}\Big/_{(\mathcal{T}|_I)|_{I\cap J} \sim (\mathcal{T}|_J)|_{I\cap J}}. \tag{4}$$

Broadly speaking, Engström, Kahle and Sullivant studied the question:

Question 3: How are the properties of the ideal $\mathbf{I}_{\mathcal{T}}$ related to the properties of the ideals $\mathbf{I}_{\mathcal{T}|_I}$, $\mathbf{I}_{\mathcal{T}|_J}$ and $\mathbf{I}_{\mathcal{T}|_{I\cap J}}$.

They observed that if $\boldsymbol{\tau} \in \mathcal{T}|_{I\cap J}$ and $\boldsymbol{\sigma} \in \mathcal{T}|_I$ are maximal faces with $\boldsymbol{\tau} \subset \boldsymbol{\sigma}$, the $\boldsymbol{\tau}$-marginal of a probability distribution on $X_1, X_2, \dots, X_N$ can be written in terms of the $\boldsymbol{\sigma}$-marginal, so that there is a natural affine projection π_1 mapping the points of $\mathbf{A}_{\mathcal{T}|_I}$ to the points of $\mathbf{A}_{\mathcal{T}|_{I\cap J}}$. Likewise, there is a natural affine projection π_2 mapping the points of $\mathbf{A}_{\mathcal{T}|_J}$ to the points of $\mathbf{A}_{\mathcal{T}|_{I\cap J}}$. With this, the point configuration $\mathbf{A}_{\mathcal{T}}$ proves to be related to the point configurations $\mathbf{A}_{\mathcal{T}|_I}$, $\mathbf{A}_{\mathcal{T}|_J}$ and $\mathbf{A}_{\mathcal{T}|_{I\cap J}}$ according to [EKS11, Proposition 5.1]:

$$\mathbf{A}_{\mathcal{T}} = \mathbf{A}_{\mathcal{T}|_I} \times_{\mathbf{A}_{\mathcal{T}|_{I\cap J}}} \mathbf{A}_{\mathcal{T}|_J}.$$

Example 3. Let $N = 3$, $\mathcal{T} = \{\{1,2\},\{2,3\}\}$ and $d_1 = d_2 = d_3 = 2$ as in example 2 *(c)*. Consider the decomposition $\mathcal{T} = \mathcal{T}|_I \cup \mathcal{T}|_J$ with $I = \{1,2\}$ and $J = \{2,3\}$. It follows that:

$$\mathbf{A}_{\mathcal{T}|_I} = \begin{array}{cc} {}_{marg.}\backslash x_1x_2 = & \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{array}{c} x_1x_2{=}11 \\ x_1x_2{=}12 \\ x_1x_2{=}21 \\ x_1x_2{=}22 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathbf{A}_{\mathcal{T}|_J} = \begin{array}{cc} {}_{marg.}\backslash x_2x_3 = & \begin{array}{cccc} 11 & 22 & 21 & 22 \end{array} \\ \begin{array}{c} x_2x_3{=}11 \\ x_2x_3{=}12 \\ x_2x_3{=}21 \\ x_2x_3{=}22 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

$$\mathbf{A}_{\mathcal{T}|_{I\cap J}=\{2\}} = \begin{array}{cc} {}_{marg.}\backslash x_2 = & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} x_2{=}1 \\ x_2{=}2 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

from where the reader readily verifies that $\mathbf{A}_{\mathcal{T}} = \mathbf{A}_{\mathcal{T}|_I} \times_{\mathbf{A}_{\mathcal{T}|_{I\cap J}}} \mathbf{A}_{\mathcal{T}|_J}$. ◇

The initial motivation for the subject of this thesis was that, together with Sturmfels' correspondence (theorem A.3.5), an understanding of triangulations of point configurations of the form (1) with respect to their product structure might, in principle, furnish insight into the combinatorics of Gröbner bases of toric ideals defining hierarchical log-linear models.

Chapter 1

Betti numbers of cut ideals of trees

The study of families of ideals associated to discrete objects, like graphs or simplicial complexes, is a subject of interest in combinatorial commutative algebra. Here, a typical problem asks to determine certain algebraic properties of a family of ideals that are associated to a family of discrete objects in terms of the combinatorial properties of the discrete object. In the framework of algebraic statistics, for instance, the discrete objects may be simplicial complexes encoding interaction structures in hierarchical log-linear models, and the algebraic properties of interest may be the degrees of minimal generating sets for the toric ideals defining them.

Such problems usually involve tracing the combinatorial features of the discrete object to the algebraic context, insofar as this is possible. In some situations, it is even possible to develop combinatorial interpretations or characterizations for algebraic properties of interest of the associated families of ideals.

In this chapter, we consider a family of homogeneous *toric* ideals that were introduced by Sturmfels and Sullivant in [SS06]: cut ideals of graphs. These can be thought of as defining a toy statistical model on the cuts of a graph. Concretely, we present joint work with Samu Potka concerning the estimation of the Betti numbers of cut ideals for the class of tree graphs [PS12] (cf. definition A.1.15). In this case, it turns out that the cut ideals are a "product" of the cut ideals of the individual edges of the tree, which makes their algebraic properties irrespective of the shape of the tree (cf. section 1.3).

The organization is as follows. In section 1.1, we introduce the objects from combinatorial commutative algebra relevant to our study, as well as the approach we have employed to obtain our results; this is an idea by Dochtermann and Engström from [DE09] to regard edge ideals of graphs as Stanley-Reisner ideals of certain associated simplicial complexes. Section 1.2 introduces cuts of graphs and their toric cut ideals. In section 1.3, we restrict to the class of trees and remark that their cut ideals equal the homogeneous ideal defining the Segre embedding of several projective lines [SS06]. Finally, in section 1.4 we develop our approach to minimal free resolutions of cut ideals of trees and derive our main result, consisting in exponential formulas giving upper bounds for the Betti numbers. We conclude in section 1.6 briefly framing our discussion within the context of triangulations of products of simplices, and listing possible further directions of research.

Although our results do not achieve a unified combinatorial characterization of the Betti numbers of cut ideals of trees, they do provide a succinct description of the computation of upper bounds, namely enumeration of induced graphs for a particular family of graphs and knowledge of their independence complexes. This leaves exposed various possible directions for further elaboration of our results.

Throughout we fix k to be an algebraically closed field of characteristic zero.

1.1 Edge ideals and combinatorial topology

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G) \subset \binom{V(G)}{2}$.

Definition 1.1.1. Consider the polynomial "vertex" ring $\mathbf{Y}_G = k[x_v : v \in V(G)]$. The *edge ideal* of G is the following monomial ideal in $\mathbf{Y}_G$:

$$\mathbf{X}_G := \langle x_v x_{v'} : \{v, v'\} \in E(G) \rangle.$$

Edge ideals are a classical object of study in combinatorial commutative algebra, and have motivated several investigations in the literature [Vil95, FVT07, MV12]. In [DE09], Dochtermann and Engström introduced a general method to systematically study edge ideals of graphs with tools from combinatorial topology. They were able to simplify proofs of existing results about the edge ideals associated to certain families of graphs, and to provide stronger versions in some cases.

The entry point of combinatorial topology in [DE09] is the theory of *square-free monomial ideals*, where ideals of this type are identified with simplicial complexes [MS05]. To every simplicial complex $\mathcal{T}$ on the base set $[n] := \{1, 2, \ldots, n\}$ we can associate a square-free monomial ideal $\mathbf{J}_{\mathcal{T}} \subset k[x_1, \ldots, x_n]$.

Definition 1.1.2. The *Stanley-Reisner ideal of $\mathcal{T}$* is the square-free monomial ideal in $k[x_1, \ldots, x_n]$ generated by the non-faces of $\mathcal{T}$:

$$\mathbf{J}_{\mathcal{T}} := \left\langle x_{i_1} x_{i_2} \ldots x_{i_q} : \tau = \{i_1, i_2, \ldots, i_q\} \subset [n], \tau \notin \mathcal{T} \right\rangle .$$

Conversely, every square-free monomial ideal $J \subset k[x_1, \ldots, x_n]$ can be associated the simplicial complex $\mathcal{T}_J := \{\{i_1, i_2, \ldots, i_r\} \subset [n] : x_{i_1} x_{i_2} \ldots x_{i_r} \notin J\}$ on the base set $[n]$. Among the many implications of this correspondence, its power can be demonstrated by Hochster's formula, which expresses the Betti numbers of square-free monomial ideals in terms of simplicial homology.

Theorem 1.1.3 (Theorem 5.1 in [Hoc77])**.** *Let $\mathcal{T}$ be a simplicial complex on the base set $[n]$ and $\mathbf{J}_{\mathcal{T}}$ be its Stanley-Reisner ideal. For $i \geq 0$, the Betti numbers of $\mathbf{J}_{\mathcal{T}}$ are given by:*

$$\beta_{i,j}(\mathbf{J}_{\mathcal{T}}) = \sum_{\substack{\mathbf{F}\subseteq[n] \\ |\mathbf{F}|=j}} \dim_k \tilde{H}_{j-i-2}(\mathcal{T}|_{\mathbf{F}}, k), \tag{1.1}$$

where $\mathcal{T}|_{\mathbf{F}}$ is the subcomplex of $\mathcal{T}$ induced by restricting $\mathcal{T}$ to the vertices in $\mathbf{F}$, and $\tilde{H}_{j-i-2}(\mathcal{T}|_{\mathbf{F}}, k)$ is its $(j-i-2)$-th reduced (simplicial) homology group.

The punch-line of the approach by Dochtermann and Engström to edge ideals was to regard these as Stanley-Reisner ideals of independence complexes of graphs. We recall the definition.

Definition 1.1.4. For a finite simple graph G, let $Ind(G)$ be the *independence complex of G*. This is the simplicial complex on the base set $V(G)$ where $\boldsymbol{\sigma} \subset V(G)$ is a simplex of $Ind(G)$ whenever $\{v, v'\} \notin E(G)$ for every $v, v' \in \sigma$. Equivalently, $Ind(G) = Cl(\bar{G})$, the *clique complex* of the (unlooped) complement of G.

We see therefore that $\mathbf{J}_{Ind(G)} = \mathbf{X}_G$, so knowledge of the complexes $Ind(G)$ for families of graphs can be entered in Hochster's formula and translated into algebraic information about the corresponding edge ideals.

As a sample of the results obtained from this observation in [DE09], Dochtermann and Engström used the fact that the independence complex of the complement of a chordal graph with d connected components is homotopy-equivalent to d disjoint points to obtain a simple formula for the Betti numbers of edge ideals of graphs in this class (a chordal graph is a graph where in every cycle of length four or larger there is an additional edge joining two non-consecutive vertices of it: a chord). A corollary of this is a celebrated result in combinatorial commutative algebra [DE09, Theorem 3.4].

1.2 Cut ideals

To define our main object of study, consider again a finite simple graph G.

Definition 1.2.1. A *cut of G* is an unordered partition of $V(G)$ into two subsets $A, B \subset V(G)$, which we denote $A|B$. If $|V(G)| = n$, then G has 2^{n-1} cuts.

Note that a cut $A|B$ of a graph G partitions the set of edges $E(G)$ into the subset $S_{A|B}$ of edges whose endpoints are *separated* by the cut (hence the S) and the subset $T_{A|B}$ of edges whose endpoints lie *together* within one part of the cut (hence the T).

The motivation for the definition of the cut ideal of a graph is to regard two collections of cuts as indistinguishable if both involve the same subsets of edges being separated and kept together. To make this more precise, let $\mathbf{R}_G$ and $\mathbf{Q}_G$ be the polynomial rings in 2^{n-1} and $2|E(G)|$ variables, respectively:

$$\mathbf{R}_G := k[r_{A|B} : A|B \text{ cut of } G],$$
$$\mathbf{Q}_G := k[s_e, t_e : e \in E(G)].$$

Definition 1.2.2. Consider the monomial homomorphism:

$$\begin{aligned} \phi_G : \mathbf{R}_G &\longrightarrow \mathbf{Q}_G \\ r_{A|B} &\mapsto \prod_{e \in S_{A|B}} s_e \prod_{e \in T_{A|B}} t_e. \end{aligned}$$

The *cut ideal* of G is the homogeneous toric ideal in $\mathbf{R}_G$:

$$\mathbf{I}_G := \ker(\phi_G).$$

Example 1.2.3. Let G be the graph on 4 vertices displayed on Figure 1.1. The cut $A|B = \{2,3\}|\{1,4\}$ displayed in that figure has $S_{A|B} = \{(2,4),(3,4)\}$ and $T_{A|B} = \{(1,4)\}$, which corresponds to the assignment $r_{\{2,3\}|\{1,4\}} \overset{\phi_G}{\mapsto} t_{14}s_{24}s_{34}$. The remaining images of the variables in $\mathbf{R}_G$ under ϕ_G are:

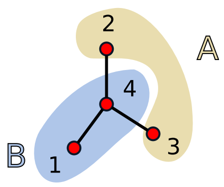


Figure 1.1: A cut on a graph.

$$\begin{array}{ll} r_{\emptyset|\{1234\}} \mapsto t_{14}t_{24}t_{34} & r_{\{13\}|\{24\}} \mapsto s_{14}t_{24}s_{34} \\ r_{\{1\}|\{234\}} \mapsto s_{14}t_{24}t_{34} & r_{\{12\}|\{34\}} \mapsto s_{14}s_{24}t_{34} \\ r_{\{2\}|\{134\}} \mapsto t_{14}s_{24}t_{34} & r_{\{4\}|\{123\}} \mapsto s_{14}s_{24}s_{34} \\ r_{\{3\}|\{124\}} \mapsto t_{14}t_{24}s_{34} & \end{array}$$

In this example, the cut ideal $\mathbf{I}_G = \ker(\phi_G)$ has a minimal generating set consisting of the 9 quadratic binomials:

$$\begin{array}{ccc} r_{2|134}r_{13|24} - r_{1|234}r_{14|23} & r_{3|124}r_{12|34} - r_{1|234}r_{14|23} & r_{3|124}r_{4|123} - r_{13|24}r_{14|23} \\ r_{2|134}r_{4|123} - r_{12|34}r_{14|23} & r_{1|234}r_{4|123} - r_{12|34}r_{13|24} & r_{\emptyset|1234}r_{4|123} - r_{1|234}r_{14|23} \\ r_{2|134}r_{3|124} - r_{\emptyset|1234}r_{14|23} & r_{1|234}r_{3|124} - r_{\emptyset|1234}r_{13|24} & r_{1|234}r_{2|134} - r_{\emptyset|1234}r_{12|34}; \end{array}$$

in particular, the multisets of edges cut by the cuts in the first and second terms of any binomial agree. Finally, a minimal free resolution of $\mathbf{I}_G$ takes the form:

$$0 \longrightarrow \mathbf{R}_G(-6) \longrightarrow \mathbf{R}_G(-4)^9 \longrightarrow \mathbf{R}_G(-3)^{16} \longrightarrow \mathbf{R}_G(-2)^9 \longrightarrow \mathbf{I}_G \longrightarrow 0$$

$\diamond$

In analogy with the work of Robertson, Seymour and others on graph minors, Sturmfels and Sullivant put forward several conjectures in [SS06] concerning the algebraic properties of cut ideals of families of minor-closed graphs. They conjectured, for instance, that cut-ideals of K_4-minor-free graphs are generated by quadrics. This was proved by Engström in [Eng11], where he first showed that the series-parallel composition characterizing K_4-minor-free graphs takes at the level of the cut ideal the form of a toric fiber product (cf. Introduction). With this, he was able to construct generating sets for the (cut ideal of the) series-parallel composition by gluing generating sets of the components.

1.3 Segre embeddings and cut ideals of trees

In [SS06], Sturmfels and Sullivant proved that the operation of 0-sum for two graphs translates in the algebraic setting to the *Segre product* of the cut ideals of the graphs. Let us recall the definitions involved.

Given two graphs G_1 and G_2 with one vertex v identified (i.e., $V(G_1) \cap V(G_2) = \{v\}$), recall that the 0-*sum* $G_1 \# G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

On the other hand, the *Segre embedding* provides a geometric realization of the cartesian product of two projective varieties as a projective variety [Har92]. Indeed, if $\mathbf{V}_1 \subset \mathbb{P}^{d_1-1}$ and $\mathbf{V}_2 \subset \mathbb{P}^{d_2-1}$ are projective varieties, the image of the cartesian product $\mathbf{V}_1 \times \mathbf{V}_2$ under the Segre embedding[1]:

$$\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1} \hookrightarrow \mathbb{P}^{d_1 d_2 - 1}$$

$$[x_1 : x_2 : \ldots : x_{d_1}], [y_1 : y_2 : \ldots : y_{d_2}] \mapsto \begin{bmatrix} z_{11} = x_1 y_1 : z_{12} = x_1 y_2 : \ldots : z_{1d_2} = x_1 y_{d_2} : \\ z_{21} = x_2 y_1 : z_{22} = x_2 y_2 : \ldots : z_{2d_2} = x_2 y_{d_2} : \\ \vdots \\ z_{d_1 1} = x_{d_1} y_1 : z_{d_1 2} = x_{d_1} y_2 : \ldots : z_{d_1 d_2} = x_{d_1} y_{d_2} \end{bmatrix},$$

is a projective variety in $\mathbb{P}^{d_1 d_2 - 1}$. Its defining ideal in $k[z_{ij} : ij \in [d_1] \times [d_2]]$ is the *Segre product* of the homogeneous ideals $\mathbf{I}(\mathbf{V}_1) \subset k[x_1, x_2, \ldots x_{d_1}]$ and $\mathbf{I}(\mathbf{V}_2) \subset k[y_1, y_2, \ldots, y_{d_2}]$ of polynomials vanishing on $\mathbf{V}_1$ and $\mathbf{V}_2$, denoted $\mathbf{I}(\mathbf{V}_1) \times_{Seg} \mathbf{I}(\mathbf{V}_2)$. A generating set for it consists of the following (infinitely many) polynomials [Sul07]:

$$\left\{ \begin{array}{c} 2 \times 2 \text{ minors} \\ \text{of } [z_{ij}]_{ij \in [d_1] \times [d_2]} \end{array} \right\} \cup \left\{ \begin{array}{c} \hat{f}\Big(\{z_{ij} : ij \in [d_1] \times [d_2]\}\Big) : \\ \hat{f}\Big(\{x_i y_j : ij \in [d_1] \times [d_2]\}\Big) \in \hat{\mathbf{I}}(\mathbf{V}_1) \end{array} \right\} \cup \left\{ \begin{array}{c} \hat{g}\Big(\{z_{ij} : ij \in [d_1] \times [d_2]\}\Big) : \\ \hat{g}\Big(\{x_i y_j : ij \in [d_1] \times [d_2]\}\Big) \in \hat{\mathbf{J}}(\mathbf{V}_2) \end{array} \right\}, \tag{1.2}$$

where $\hat{\mathbf{I}}(\mathbf{V}_1)$ and $\hat{\mathbf{J}}(\mathbf{V}_2)$ are the ideals in the larger polynomial ring $k[x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}]$ generated by the generators of $\mathbf{I}(\mathbf{V}_1)$ and of $\mathbf{I}(\mathbf{V}_2)$, respectively.

Remark 1.3.1. The set of polynomials in (1.2) can be reduced to a finite set of generators of $\mathbf{I}(\mathbf{V}_1) \times_{Seg} \mathbf{I}(\mathbf{V}_2)$, whose construction, presented by Sullivant in

[1] We slightly depart from the standard notation for the homogeneous coordinates of projective space. The reason is to parallel the notation for the standard simplex $\Delta_{n-1} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, whose associated toric ideal defines $\mathbb{P}^{n-1}$ (see appendix A).

[Sul07], is as follows. To every degree u generator f of $\mathbf{I}(\mathbf{V}_1) \in k[x_1, \ldots, x_{d_1}]$:

$$f = \sum_{\alpha=1}^{s} w_\alpha x_{i_1^\alpha} x_{i_2^\alpha} \ldots x_{i_u^\alpha},^2$$

associate a u-tuple $\mathbf{j} = (j_1, j_2, \ldots, j_u) \in [d_2]^{[u]}$ of indices, and define the polynomial:

$$f^{\mathbf{j}} := \sum_{\alpha=1}^{s} w_\alpha z_{i_1^\alpha j_1} z_{i_2^\alpha j_2} \ldots z_{i_u^\alpha j_u} \in k[z_{ij} : ij \in [d_1] \times [d_2]].$$

Ranging over all all generators of $\mathbf{I}(\mathbf{V}_1)$ and d-tuples in $[d_2]^{[u]}$, we obtaine a finite collection of polynomials which generate the ideal generated by the polynomials in the second term of (1.2); we schematically illustrate this lifting in figure 1.2. *Lifting* every generator of $\mathbf{I}(\mathbf{V}_2)$ of degree w likewise, according to some w-tuple in $[d_1]^{[w]}$, yields then the desired finite generating set.

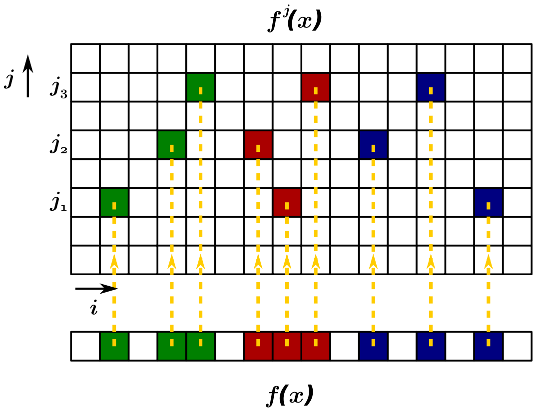


Figure 1.2: Each block in the bottom row represents a (ocurrence of a) variable, and blocks of the same color are variables in the same monomial. For a choice of j_1, j_2, j_3, $f^{\mathbf{j}}$ is gotten by concatenating j_1, j_2 and j_3 to the index in each variable.

⬡

With these definitions at hand, we turn to the cut ideals $\mathbf{I}_{G_1}, \mathbf{I}_{G_2}$ and $\mathbf{I}_{G_1\#G_2}$. The Segre product of $\mathbf{I}_{G_1}$ and $\mathbf{I}_{G_2}$ is the homogeneous ideal in the polynomial ring:

$$\mathbf{R} := k[z_{A_1|B_1, A_2|B_2} : A_1|B_1 \text{ cut of } G_1, A_2|B_2 \text{ cut of } G_2],$$

generated by the binomials coming from equation (1.2). To see that it equals $\mathbf{I}_{G_1\#G_2}$, first notice that the polynomial ring $\mathbf{R}$ equals precisely $\mathbf{R}_{G_1\#G_2}$, for any two cuts $A^{(1)}|B^{(1)}$ of G_1 and $A^{(2)}|B^{(2)}$ of G_2 define a unique cut $A|B$ of $G_1\#G_2$ given by $A^{(1)} \cup A^{(2)}|B^{(1)} \cup B^{(2)}$ (we implicitly set $v \in B^{(1)} \cap B^{(2)}$); conversely, every cut of $G_1\#G_2$ can be gotten this way. Next, recall that every edge in $E(G_1\#G_2)$ is either in $E(G_1)$ or in $E(G_2)$; consequently, the image of $r_{A|B} \in \mathbf{R}_{G_1\#G_2}$ under

[2] Here, each power of a variable is counted as a separate occurrence; e.g., x_1^2 would contribute $x_1 x_1$ to a term of f.

$\phi_{G_1\#G_2}$ can be written:

$$r_{A|B} \stackrel{\phi_{G_1\#G_2}}{\longmapsto} \prod_{e\in S_{A|B}} s_e \cdot \prod_{e\in T_{A|B}} t_e = \phi_{G_1}\left(r^{(1)}_{A\cap V(G_1)|B\cap V(G_1)}\right)\cdot \phi_{G_2}\left(r^{(2)}_{A\cap V(G_2)|B\cap V(G_2)}\right),$$

where $r^{(i)}_{A|B}$ denote the cut variables in $\mathbf{R}_{G_i}$ ($i = 1,2$), and the images of ϕ_{G_1} and ϕ_{G_2} are understood as lying inside the larger polynomial ring $\mathbf{Q}_{G_1\#G_2}$. In particular:

$$\mathbf{I}_{G_1\#G_2} = \ker\left(r_{A|B} \mapsto \phi_{G_1}\left(r^{(1)}_{A\cap V(G_1)|B\cap V(G_1)}\right)\cdot \phi_{G_2}\left(r^{(2)}_{A\cap V(G_2)|B\cap V(G_2)}\right)\right), \tag{1.3}$$

from which the containment $\mathbf{I}_{G_1} \times_{Seg} \mathbf{I}_{G_2} \subseteq \mathbf{I}_{G_1\#G_2}$ is easily seen to hold.

We verify the opposite containment for a quadratic binomial $b = r_{A_1|B_1}r_{A_2|B_2} - r_{C_1|D_1}r_{C_2|D_2} \in \mathbf{I}_{G_1\#G_2}$, the general case admitting the same argument. The assignment $z_{ij} \mapsto x_iy_j$ from equation (1.2) reads for cut variables:

$$r_{A|B} = r_{A^{(1)}\cup A^{(2)}|B^{(1)}\cup B^{(2)}} \mapsto r^{(1)}_{A^{(1)}|B^{(1)}}r^{(2)}_{A^{(2)}|B^{(2)}},$$

where $A^{(1)}, B^{(1)} \subset V(G_1)$ and $A^{(2)}, B^{(2)} \subset V(G_2)$. Applying this substitution to b, we obtain:

$$r^{(1)}_{A_1^{(1)}|B_1^{(1)}}r^{(2)}_{A_1^{(2)}|B_1^{(2)}}\ r^{(1)}_{A_2^{(1)}|B_2^{(1)}}r^{(2)}_{A_2^{(2)}|B_2^{(2)}} - r^{(1)}_{C_1^{(1)}|D_1^{(1)}}r^{(2)}_{C_1^{(2)}|D_1^{(2)}}\ r^{(1)}_{C_2^{(1)}|D_2^{(1)}}r^{(2)}_{C_2^{(2)}|D_2^{(2)}}.$$

From the expression (1.3), we know that the binomial $r^{(1)}_{A_1^{(1)}|B_1^{(1)}}r^{(1)}_{A_2^{(1)}|B_2^{(1)}} - r^{(1)}_{C_1^{(1)}|D_1^{(1)}}r^{(1)}_{C_2^{(1)}|D_2^{(1)}}$ lies in $\hat{\mathbf{I}}_{G_1}$, so we can divide b by it to obtain:

$$b = r^{(1)}_{C_1^{(1)}|D_1^{(1)}}r^{(1)}_{C_2^{(1)}|D_2^{(1)}}\left(r^{(2)}_{A_1^{(2)}|B_1^{(2)}}r^{(2)}_{A_2^{(2)}|B_2^{(2)}} - r^{(2)}_{C_1^{(2)}|D_1^{(2)}}r^{(2)}_{C_2^{(2)}|D_2^{(2)}}\right) \in \hat{\mathbf{I}}_{G_2},$$

confirming that $\mathbf{I}_{G_1\#G_2} \subset \mathbf{I}_{G_1\times_{Seg}\mathbf{I}_{G_2}}$.

Since any tree T_n on n vertices can be obtained as the 0-sum of $n-1$ edges K_2, it follows that $\mathbf{I}_{T_n} = \mathbf{I}_{K_2} \times_{Seg} \mathbf{I}_{K_2} \times_{Seg} \ldots \times_{Seg} \mathbf{I}_{K_2}$ ($n-1$ times). Thus, the cut ideals of all trees on n vertices are algebraically the same, irrespective of the shape of the tree. In fact, we see that $\mathbf{I}_{T_n}$ is the ideal defining the Segre embedding of $n-1$ copies of $\mathbb{P}^1$ in $\mathbb{P}^{2^{n-1}-1}$, for we have $\mathbf{I}_{K_2} = \{0\} \subset k[r_{\varnothing|12}, r_{1|2}]$, which defines the projective line $\mathbb{P}^1$.

The approach to cut ideals of trees in [PS12] tacitly arrived at the identification we just proved from the more specific remark below.

Proposition 1.3.2. *Let T_n be a tree on n vertices. The cuts of T_n are in bijection with the power set $\mathbf{2}^{E(T_n)}$ of the set $E(T_n)$.*

Proof. First observe that $|Cuts(T_n)| = |\mathbf{2}^{E(T_n)}|$. Let $\varepsilon : Cuts(T_n) \to \mathbf{2}^{E(T_n)}$ be the map defined by $\varepsilon(A|B) = S_{A|B}$. The assertion of the lemma follows once we verify that ε is injective. But this is true for any connected graph: if $A_1|B_1 \neq A_2|B_2$, both cuts differ at least in one vertex that lies in different parts; accordingly, there is at least one edge $e \in E(G)$ such that $e \in S_{A_1|B_1}$ but $e \notin S_{A_2|B_2}$. □

Proposition 1.3.2 says that we may consider the indeterminates in $\mathbf{R}_{T_n}$ to be as well labeled by subsets of $E(T_n)$. This allows us to define a normal form (modulo $\mathbf{I}_{T_n}$) for cut monomials in $\mathbf{R}_{T_n}$. For this purpose, we depict the edge subset associated to every $r_{A|B} \in \mathbf{R}_{T_n}$ as a string of the edges in T_n, with dashes on those edges that are cut by $A|B$; a cut monomial $m \in \mathbf{R}_{T_n}$ is represented by the strings of edges of each cut indeterminate stacked on top of each other. The normal form of m is then gotten by *sending all dashes representing cuts to the bottom*. This is drawn for the tree T_4 in Figure 1.3; here the normal form of $r_{14|23}r_{1|234}r_{2|134}$ is seen to be $r_{4|123}r_{2|134}r_{\varnothing|1234}$.

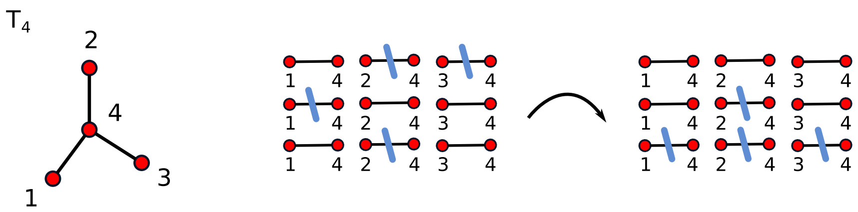


Figure 1.3: Normal form for the cut monomial of a tree on 4 vertices.

The precise definition for the normal form is the following one.

Lemma 1.3.3. *Let $\mathbf{I}_{T_n}$ be the cut ideal of a tree on n vertices. There is a term order $\prec$ in $\mathbf{R}_{T_n}$ with respect to which the set:*

$$\mathcal{G}_{\prec} := \{r_X r_Y - r_{X\cup Y} r_{X\cap Y} : X, Y \subseteq E(G) \text{ incomparable}\} \subset \mathbf{I}_{T_n},$$

is a Gröbner basis for $\mathbf{I}_{T_n}$.

We paraphrase the proof of this lemma from [PS12], which is based on the proof of [Stu96, Theorem 9.1].

Proof. Throughout we refer to cut monomials in $\mathbf{R}_{T_n}$ by their corresponding subset of edges cut. To every indeterminate r_X in $\mathbf{R}_{T_n}$ assign a weight equal to the number of indeterminates (labeled by subsets) incomparable with r_X, and let $\prec$ be any total order refining the partial order induced by the weight defined. We will prove that the initial term of any polynomial in $\mathbf{I}_{T_n}$ is divisible by some monomial in $\mathrm{in}_{\prec}(\mathcal{G}_{\prec}) := \{r_X r_Y : X, Y \text{ incomparable}\}$. To this end, suppose $b = m - m'$ is a binomial in $\mathbf{I}_{T_n}$ that gives a counterexample minimal with respect to the term order $\prec$, where m and m' have no common factors. By the minimality of b we mean that the initial term of any binomial $b' = n - n' \in \mathbf{I}_{T_n}$ with $nn' \prec mm'$ is divisible by some monomial in $\mathrm{in}_{\prec}(\mathcal{G}_{\prec})$. Since m is not divisible by any of the monomials in $\mathrm{in}_{\prec}(\mathcal{G}_{\prec})$, all the indeterminates appearing in m are labeled by comparable subsets of $E(T_n)$. We may actually assume that the same holds for m', for otherwise we could divide it by an element of $\mathcal{G}_{\prec}$, yielding a counterexample b' of smaller weight.

Note that the multisets of $E(T_n)$ of edges cut in m and m' coincide, because $b \in \mathbf{I}_{T_n}$. But the only way for this to hold while the indeterminates in m and in

m' are both indexed by collections of mutually comparable subsets of $E(T_n)$ is if $m = m'$, so we have a contradiction. □

1.4 Upper bounds for $\beta_{i,j}(I_{T_n})$

As stated in proposition A.2.5 (alternatively [MS05, Theorem 8.29]), the Betti numbers of $\text{in}_\prec(\mathbf{I}_{T_n})$ give upper bounds for the Betti numbers of $\mathbf{I}_{T_n}$. On the other hand, the Gröbner basis $\mathcal{G}_\prec$ gives us a simple combinatorial description for the monomials generating $\text{in}_\prec(\mathbf{I}_{T_n})$: they generate the edge ideal of the *incomparability graph* Γ_{n-1} of the power set $\mathbf{2}^{E(T_n)} = \mathbf{2}^{n-1}$, whose vertices are subsets of $E(T_n)$ and whose edges are pairs of incomparable subsets of $E(T_n)$.

In view of the idea from [DE09] to regard edge ideals of graphs as Stanley-Reisner ideals of independence complexes of graphs, application of Hochster's formula (1.1) will then reduce the estimation of the Betti numbers of $\mathbf{I}_{T_n}$ to the enumeration of induced subgraphs of Γ_{n-1}:

$$\beta_{i,j}(\text{in}_\prec(\mathbf{I}_{T_n})) = \beta_{i,j}(\mathbf{X}_{\Gamma_{n-1}}) = \beta_{i,j}(\mathbf{J}_{Ind(\Gamma_{n-1})}) = \sum_{\mathbf{F}\in\binom{[n]}{j}} \dim_k \left(\tilde{H}_{j-i-2}\left(Ind(\Gamma_{n-1}|_{\mathbf{F}}), k\right)\right) \quad (1.4)$$

(because $Ind(G)|_{\mathbf{F}} = Ind(G|_{\mathbf{F}})$ holds for every $\mathbf{F} \subset V(G)$). This is an easy combinatorial exercise that can be performed using the principle of inclusion-exclusion. We illustrate this with the enumeration of labeled occurrences in Γ_{n-1} of the graph $g =$  .

View the element $u \in \mathbf{2}^{E(T_n)}$ as an $(n-1)$-tuple in $\{0,1\}^{[n-1]}$. For two different $u, v \in \{0,1\}^{[n-1]}$, let a be the number of indices $i \in [n-1]$ such that $u_i = 0$ and $v_i = 1$ and b be the number of indices $j \in [n-1]$ such that $u_j = 1$ and $v_j = 0$; then $(u,v) \in E(\Gamma_{n-1}) \iff a > 0$ and $b > 0$. Likewise $(u,v) \notin E(\Gamma_{n-1}) \iff$ either $a = 0$ or $b = 0$.

Label the vertices of g as $V(g) = \{u,v,w\}$, so that $E(g) = \{(u,v),(v,w)\}$, and denote by $a,b,\ldots,h$ the number of indices $i \in [n-1]$ for which (u_i, v_i, w_i) equals $(0,0,0),(1,0,0),\ldots,(1,1,1)$, respectively, as shown in table 1.1.

	a	b	c	d	e	f	g	h
u	0	1	0	1	0	1	0	1
v	0	0	1	1	0	0	1	1
w	0	0	0	0	1	1	1	1

Table 1.1: Possible values of (u_i, v_i, w_i).

When comparing the three $(n-1)$-tuples u, v and w, we obtain a list of counts for $a,b,\ldots,h$ which sum up to $n-1$; there are $\binom{n-1}{a,b,\ldots,h}$ ways of obtaining such counts. Finally, $(u,v) \in E(\Gamma_{n-1}) \iff b+f > 0$ and $c+g > 0$, $(v,w) \in E(\Gamma_{n-1}) \iff c+d > 0$ and $e+f > 0$, and $(u,w) \notin E(\Gamma_{n-1}) \iff$ either $b+d = 0$ or $e+g = 0$. That is to say, the labeled occurrences of  in Γ_{n-1} are given by the following restricted sum:

$$
\begin{aligned}
\#g(\Gamma_{n-1}) = \sum_{\substack{a+b+\ldots+h=n-1,\\ b+f>0,\ c+g>0,\\ c+d>0,\ e+f>0,\\ b+d=0 \text{ xor } e+g=0}} \binom{n-1}{a,b,\ldots,h} &= \sum_{\substack{a+c+e+f+g+h=n-1,\\ f>0,\ c+g>0,\\ c>0,\ e+f>0,\\ b+d=0}} \binom{n-1}{a,c,e,f,g,h} \\
&+ \sum_{\substack{a+b+c+d+f+h=n-1,\\ b+f>0,\ c>0,\\ c+d>0,\ f>0,\\ e+g=0}} \binom{n-1}{a,b,c,d,f,h} \\
&- 2 \sum_{\substack{a+c+f+h=n-1,\\ f>0,\ c>0,\\ b+d=0, e+g=0}} \binom{n-1}{a,c,f,h},
\end{aligned}
$$

The terms on the right hand side are decomposed further according to inclusion-exclusion, for example:

$$
\sum_{\substack{a+c+f+h=n-1\\ f>0}} = \sum_{a+c+f+h=n-1} - \sum_{\substack{a+c+h=n-1\\ f=0}} .
$$

After arriving to a sum of unrestricted multinomial coefficients, we use:

$$
\sum_{\substack{a_1,\ldots,a_\ell \in \mathbb{N}\\ a_1+\ldots+a_\ell=N}} \binom{N}{a_1,\ldots,a_\ell} = \ell^N,
$$

and find:

$$
\#g(\Gamma_{n-1}) = 2 \cdot \left(6^{n-1} - 2\cdot 5^{n-1} + 2\cdot 3^{n-1} - 2^{n-1}\right).
$$

Observe now that $Ind(g)$ is homotopy equivalent to two isolated vertices, so $\dim_k \hat{H}_{3-1-2}(Ind(g)) = 0$; thus, the only contribution of the reduced homology groups of $Ind(g)$ to the Betti numbers of $\mathrm{in}_\prec(\mathbf{I}_{T_n})$ is to $\beta_{1,3}(\mathrm{in}_\prec(\mathbf{I}_{T_n}))$.

In [PS12], the information about $Ind(G)$ for graphs with up to 5 vertices displayed in table 1.2 allowed the authors to implement the strategy outlined above to obtain the following exponential formulas on n, giving upper bounds for some Betti numbers of $\mathbf{I}_{T_{n+1}}$.

Theorem 1.4.1. *Let $\prec$ be a term order in $\mathbf{R}_{T_{n+1}}$ with respect to which the binomials in $\mathcal{G}_\prec$ from lemma 1.3.3 form a Gröbner basis for $\mathbf{I}_{T_{n+1}}$. The following* *bounds hold for the Betti numbers of* $\mathbf{I}_{T_{n+1}}$*:*

$$\beta_{0,2}(\mathbf{I}_{T_{n+1}}) \leq \beta_{0,2}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{2}\left(4^n - 2{\cdot}3^n + 2^n\right)$$

$$\beta_{1,3}(\mathbf{I}_{T_{n+1}}) \leq \beta_{1,3}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{3}\left(8^n - 3{\cdot}6^n + 3{\cdot}4^n - 2^n\right)$$

$$\beta_{2,4}(\mathbf{I}_{T_{n+1}}) \leq \beta_{2,4}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{8}\left(16^n - 4{\cdot}12^n + 6{\cdot}8^n + 2{\cdot}7^n - 4{\cdot}6^n + 4{\cdot}5^n \right.$$
$$\left. - 9{\cdot}4^n + 2{\cdot}3^n + 2{\cdot}2^n\right)$$

$$\beta_{1,4}(\mathbf{I}_{T_{n+1}}) \leq \beta_{1,4}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{4}\left(7^n - 4{\cdot}6^n + 6{\cdot}5^n - 4{\cdot}4^n + 3^n\right)$$

$$\beta_{3,5}(\mathbf{I}_{T_{n+1}}) \leq \ \beta_{3,5}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{60}\left(2{\cdot}32^n - 10{\cdot}24^n + 30{\cdot}20^n - 120{\cdot}18^n + 30{\cdot}17^n \right.$$
$$- 40{\cdot}16^n + 180{\cdot}15^n + 375{\cdot}14^n - 420{\cdot}13^n - 180{\cdot}12^n$$
$$+ 200{\cdot}11^n - 280{\cdot}10^n - 220{\cdot}9^n + 985{\cdot}8^n - 720{\cdot}7^n$$
$$\left. + 655{\cdot}6^n - 710{\cdot}5^n + 35{\cdot}4^n + 340{\cdot}3^n - 132{\cdot}2^n\right)$$

$$\beta_{2,5}(\mathbf{I}_{T_{n+1}}) \leq \beta_{2,5}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}})) = \frac{1}{12}\left(3{\cdot}14^n - 12{\cdot}12^n - 2{\cdot}11^n + 22{\cdot}10^n - 2{\cdot}9^n \right.$$
$$\left. - 9{\cdot}8^n - 6{\cdot}7^n + 9{\cdot}6^n - 10{\cdot}5^n + 11{\cdot}4^n - 4{\cdot}3^n\right)$$

Graph	$\dim \tilde{H}_0$	$\dim \tilde{H}_1$
	1	0
	2	0

Graph	$\dim \tilde{H}_0$	$\dim \tilde{H}_1$
	3	0
	2	0
	1	0
	1	0
	1	0
	0	1

Graph	$\dim \tilde{H}_0$	$\dim \tilde{H}_1$	Graph	$\dim \tilde{H}_0$	$\dim \tilde{H}_1$
	4	0		3	0
	2	0		2	0
	1	0		0	1
	1	0		1	0
	2	0		1	0
	1	1		0	1
	1	0		0	1
	0	2		1	0
	1	0	 [1]	0	1

Table 1.2: Contributions to the reduced homology over $\mathbb{Z}$ of the independence complexes of graphs. Notice that there is no torsion in any of these homology groups. [1] does not contribute because incomparability graphs of posets are perfect, i.e., they have neither induced odd cycles of length ≥ 5 nor their complements [CB84].

Remark 1.4.2. The main algebraic ingredient we have used is Hochster's formula (1.1). Therefore, as long as there is no torsion in the (reduced) homology groups of independence complexes of the induced graphs of Γ_{n-1}, our bounds are independent of the characteristic of the underlying field k. This will be the case for the estimates $\beta_{i,j}(\mathrm{in}_{\prec}(\mathbf{I}_{T_{n+1}}))$ up to at least $j = 10$. This follows from the results in [Kat06] or [Ada12], which assert that the clique complexes of all graphs on less than 11 vertices are torsion-free. It is an interesting question whether the clique complexes of the class of comparability graphs (i.e., independence complexes of incomparability graphs) start having torsion at a larger number of vertices. ○

1.5 The triangulation viewpoint

The homogeneous ideal $\mathbf{I}_{\times_{Seg}(d_1,d_2,\dots,d_s)}$ defining the Segre embedding of s projective spaces $\mathbb{P}^{d_1-1}, \mathbb{P}^{d_2-1}, \dots, \mathbb{P}^{d_s-1}$ into $\mathbb{P}^{d_1 d_2 \dots d_s - 1}$ is the toric ideal resulting as the kernel of the monomial homomorphism:

$$k[\mathbf{z}] := k\left[z_{i_1 i_2 \dots i_s} : (i_1, i_2 \dots, i_s) \in [d_1] \times [d_2] \times \dots \times [d_s]\right] \longrightarrow k\begin{bmatrix} x_{1,1}, \dots, x_{1,d_1}, \\ x_{2,1}, \dots, x_{2,d_2}, \\ \vdots \\ x_{s,1}, \dots, x_{s,d_s} \end{bmatrix} \tag{1.5}$$

$$z_{i_1 i_2 \dots i_s} \longmapsto x_{1,i_1} x_{2,i_2} \dots x_{s,i_s}.$$

Hence, according to [Stu96, Chapter 4] or A.3, the point configuration associated to $\mathbf{I}_{\times_{Seg}(d_1,d_2,\dots,d_s)}$ is the cartesian product of s simplices: $\Delta_{d_1-1} \times \Delta_{d_2-1} \times \dots \times \Delta_{d_s-1}$. Together with the identification $\mathbf{I}_{T_n} = \mathbf{I}_{K_2} \times_{Seg} \dots \times_{Seg} \mathbf{I}_{K_2}$ ($n-1$ times) proved in section 1.3, we therefore see that the point configuration associated to $\mathbf{I}_{T_n}$ is a unit $(n-1)$-cube.

The term order we use in this chapter defines a square-free initial ideal of $\mathbf{I}_{T_n}$ generated by (products of indeterminates indexed by) pairs of incomparable elements of $\mathbf{2}^{n-1}$. By Sturmfels' correspondence in theorem A.3.5 and theorem A.3.7, this initial ideal corresponds to a unimodular triangulation of the $(n-1)$-cube $\Delta_1 \times \dots \times \Delta_1$ ($n-1$ times). This is the so-called *staircase triangulation* of

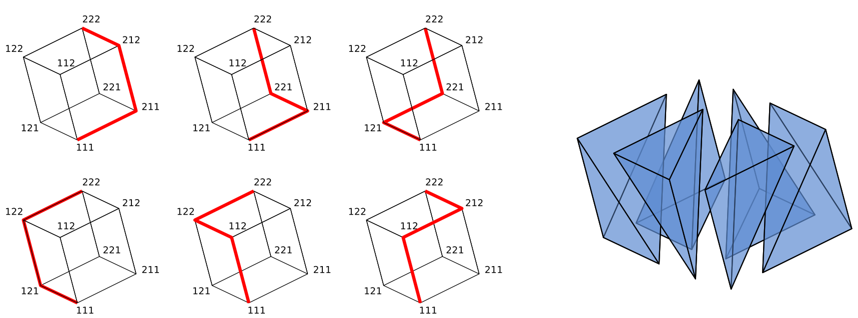


Figure 1.4: Staircase triangulation of the 3-cube, corresponding to $\mathrm{in}_{\prec}(\mathbf{I}_{T_4})$.

the product of several simplices (see figure 1.4), which generalizes the staircase triangulation of the product of two simplices (cf. section 2.2).

Its description is equally simple: we represent the points of $\Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_s-1}$ by positions in an s dimensional grid of dimensions $d_1 \times d_2 \times \ldots \times d_s$, and consider the collection of $(d_1+d_2+\ldots+d_s-s)$-simplices that correspond to connected paths in the grid from $(1,1,\ldots,1)$ to $(d_1,d_2,\ldots,d_s)$, where at each step exactly one coordinate increases. It is a well known fact that this triangulation is unimodular [DRS10, Theorem 6.2.18], and that its minimal non-faces are edges indexed by the pairs of incomparable elements of the distributive lattice $\boldsymbol{d_1} \times \boldsymbol{d_2} \times \ldots \times \boldsymbol{d_s}$[3], where $\boldsymbol{d_\alpha}$ stands for the *chain* (totally ordered set) $\{1 < 2 < \ldots < d_\alpha - 1 < d_\alpha\}$ (this is easily seen from [DRS10, Lemma 9.4.4]).

1.6 Concluding remarks

The *incomparability graph* method presented in this chapter can be used to effectively bound individual Betti numbers of $\mathbf{I}_{T_n}$, even when the algebraic computation from monomial initial ideals of $\mathbf{I}_{T_n}$ is not possible. The limiting factors are (i) knowledge of the independence complexes of graphs on 6 vertices and more, that for 5 or less vertices can be worked out "by eye", and (ii) eventually the computational cost of performing inclusion-exclusion which, although straightforward, required (in our implementation) exponentially many operations (on $\binom{j}{2}$), where j is the number of vertices of the induced subgraphs in equation (1.4)).

A major drawback of the enumerative computation of upper bounds for $\beta_{i,j}(\mathbf{I}_{T_n})$ presented is precisely that it focuses on individual Betti numbers, whereas to investigate general properties of algebraic interest of $\mathbf{I}_{T_n}$ we need collective information about the Betti numbers. Hence, we consider interesting a more general combinatorial-topological study of independence complexes of incomparability graphs of Boolean lattices $\mathbf{2}^n$, because it would have direct consequences in our work. To the best of our knowledge, such an investigation is currently unavailable in the literature.

The identification $\mathbf{I}_{T_n} \cong \{0\}(\subset k[x,y])^{\times_{Seg} n-1}$ remarked in [SS06], and rederived in section 1.3 by identifying cuts of T_n with elements of the Boolean lattice $\mathbf{2}^{n-1}$, points towards a direct extension of the incomparability graph method to obtain upper bounds for the Betti numbers of $\mathbf{I}_{\times_{Seg}(d_1,\ldots,d_s)}$. Indeed, viewing the indeterminates in $k[\mathbf{z}]$ as labelled by elements in $\boldsymbol{d_1} \times \boldsymbol{d_2} \times \ldots \times \boldsymbol{d_s}$, there is a term order $\prec'$ for which $\mathrm{in}_{\prec'}(\mathbf{I}_{\times_{Seg}(d_1,d_2,\ldots,d_s)}) = \langle z_{\mathbf{i}} z_{\mathbf{j}} : \mathbf{i},\mathbf{j} \text{ incomparable}\rangle$, and thus equals the edge ideal of the incomparability graph of $\boldsymbol{d_1} \times \boldsymbol{d_2} \times \ldots \times \boldsymbol{d_s}$. Needless to say, knowledge of topological properties of the independence complexes of these family of incomparability graphs would be of great significance for the study of this much more general family of ideals.

[3] Recall that the distributive lattice $\boldsymbol{d_1} \times \boldsymbol{d_2} \times \ldots \times \boldsymbol{d_s}$ has elements $\mathbf{i} = (i_1, i_2, \ldots, i_s)$ with $i_\alpha \in \boldsymbol{d_\alpha}$, and if $\mathbf{j} = (j_1, j_2, \ldots, j_s) \in \boldsymbol{d_1} \times \boldsymbol{d_2} \times \ldots \times \boldsymbol{d_s}$ we have $\mathbf{i} \leq \mathbf{j}$ whenever $i_\alpha \leq j_\alpha$ for every $\alpha \in [s]$

Chapter 2

Partial triangulations of cartesian products of simplices

In the introduction, we have argued that triangulations of products of point configurations are interesting from diverse perspectives. In this chapter, we propose an approach to triangulations of cartesian products of simplices that was developed in collaboration with César Ceballos and Arnau Padrol in [CPS13]. It concerns the study of the extendability of certain partial triangulations of cartesian products of simplices that are defined to reflect the product structure: triangulations of the polyhedral complex $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ (see definition B.2.19).

The organization of this chapter is as follows. In section 2.1, we state our general approach to triangulations of cartesian products of point configurations, and introduce the concepts and questions that we will consider in this chapter. Next, in section 2.2, we specialize to products of simplices, and recall some well-known properties and results that will be used in our proofs. In section 2.3, we present the connection between triangulations of products of simplices and mixed subdivisions of dilated simplices that was elaborated by Santos in [San04], together with some of their combinatorial aspects. In section 2.4, we first review some existing results about partial triangulations of products of simplices. Then, we present our main results. After observing that when $n \leq k < d$ a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ essentially defines a unique list of simplices of $\Delta_{n-1} \times \Delta_{d-1}$: (*i*) we obtain necessary and sufficient conditions for a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ to extend, (*ii*) we prove that the bound $n < k < d$ ensures the achievement of the conditions from (*i*), and (*iii*) we prove that this bound is tight by explicitly constructing an infinite family of non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$, which was unavailable in the literature and may be of independent interest. A key ingredient towards our construction is a triangulation of $\Delta_{n-1} \times \Delta_{n-1}$, which we call the *Dyck path* *triangulation*, that also seems to be new. Finally, in section 2.5, we briefly comment on the general scope of our constructions, and we state two conjectures generalizing our results to partial triangulations of the cartesian product of several simplices.

2.1 On triangulations of products of point configurations

In the Introduction, we mentioned that the composite nature of a simplicial complex defining a hierarchical log-linear statistical model is reflected at the level of its defining toric ideal as a product structure for the associated point configuration. In parallel to the algebraic questions about the composite nature of hierarchical log-linear models that were investigated by Engstr´om, Kahle and Sullivant in [EKS11] (cf. question 3 in the introduction), we raised the following general question in the Introduction:

Question 1: Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$ and $\mathbf{A}' = \{\boldsymbol{\alpha}'_1, \boldsymbol{\alpha}'_2, \ldots, \boldsymbol{\alpha}'_m\} \subset \mathbb{R}^{d'}$ be two point configurations. What is the relation between triangulations of their cartesian product:

$$\mathbf{A} \times \mathbf{A}' := \{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}'_j) : \boldsymbol{\alpha}_i \in \mathbf{A}, \boldsymbol{\alpha}'_j \in \mathbf{A}'\} \subset \mathbb{R}^{d+ed'},$$

and triangulations of $\mathbf{A}$ and $\mathbf{A}'$?

Part of the difficulty in question 1 can be understood with the following reasoning. Let $\mathcal{T}_{\mathbf{A}}$ be a triangulation of $\mathbf{A}$ and $\mathcal{T}'_{\mathbf{A}}$ be a triangulation of $\mathbf{A}'$. The cartesian product of $\mathcal{T}_{\mathbf{A}}$ and $\mathcal{T}'_{\mathbf{A}}$:

$$\mathcal{T}_{\mathbf{A}} \times \mathcal{T}'_{\mathbf{A}} := \{\boldsymbol{\sigma} \times \boldsymbol{\tau} : \boldsymbol{\sigma} \in \mathcal{T}_{\mathbf{A}}, \text{ and } \boldsymbol{\tau} \in \mathcal{T}'_{\mathbf{A}}\}$$

gives a polyhedral subdivision of the point configuration $\mathbf{A} \times \mathbf{A}'$ into polyhedral cells that are cartesian products of subsimplicies of $\mathbf{A}$ and of $\mathbf{A}'$. Any triangulation $\mathcal{T}$ of $\mathbf{A} \times \mathbf{A}'$ that is obtained by refining $\mathcal{T}_{\mathbf{A}} \times \mathcal{T}'_{\mathbf{A}}$ (i.e., further triangulating the cells in $\mathcal{T}_{\mathbf{A}} \times \mathcal{T}'_{\mathbf{A}}$) has two projections $\mathcal{T} \to \mathcal{T}_{\mathbf{A}}$ and $\mathcal{T} \to \mathcal{T}'_{\mathbf{A}}$ that are induced by the natural affine projections $\pi_1 : \mathbf{A} \times \mathbf{A}' \to \mathbf{A}$ and $\pi_2 : \mathbf{A} \times \mathbf{A}' \to \mathbf{A}'$. Therefore, $\mathcal{T}$ is specified after saying how the cells in $\mathcal{T}_{\mathbf{A}} \times \mathcal{T}'_{\mathbf{A}}$ are triangulated, and the combinatorics of $\mathcal{T}$ can be understood, to some extent, already from the combinatorics of triangulations of products of simplices.

However, not all triangulations of $\mathbf{A} \times \mathbf{A}'$ have this structure; see the triangulation of the 3-cube with 5 simplices in figure B.7 for an example.

In this case, we may still consider certain natural projections of $\mathcal{T}$ that highlight the product structure of $\mathbf{A} \times \mathbf{A}'$. Namely, recall from proposition B.2.18 that if $\mathcal{T}$ is a triangulation of a point configuration $\mathbf{A} \subset \mathbb{R}^d$ and $\mathbf{F} \preceq \mathbf{A}$ is a face of $\mathbf{A}$, then the restriction $\mathcal{T}|_{\mathbf{F}}$ is a well-defined triangulation of the point configuration $\mathbf{F}$. Then, it can be seen that a general triangulation $\mathcal{T}$ of the cartesian product

$\mathbf{A} \times \mathbf{A}'$ does not project to only one triangulation of each of $\mathbf{A}$ and $\mathbf{A}'$, but rather to a collection of them, gotten by restricting $\mathcal{T}$ to some faces of $\mathbf{A} \times \mathbf{A}'$:

$$\{\mathcal{T}|_{\mathbf{A}\times\{\mathbf{w}\}} : \mathbf{w} \text{ vertex of } \mathbf{A}'\} \sim \text{triangulations of } \mathbf{A}$$
$$\{\mathcal{T}|_{\{\mathbf{v}\}\times\mathbf{A}'} : \mathbf{v} \text{ vertex of } \mathbf{A}\} \sim \text{triangulations of } \mathbf{A}'.$$

More generally, associated to a triangulation $\mathcal{T}$ of $\mathbf{A} \times \mathbf{A}'$ we can consider the following collections of triangulations gotten by restricting $\mathcal{T}$:

$$\mathcal{T}^{\cdot\times\mathrm{skel}_l(\mathbf{A}')} := \{\mathcal{T}|_{\mathbf{A}\times\mathbf{F}} : \mathbf{F} \in \mathrm{skel}_l(\mathbf{A}')\}$$
$$\mathcal{T}^{\mathrm{skel}_k(\mathbf{A})\times\cdot} := \{\mathcal{T}|_{\mathbf{G}\times\mathbf{A}'} : \mathbf{G} \in \mathrm{skel}_k(\mathbf{A})\},$$

where $\mathrm{skel}_l(\mathbf{A}')$ is the set of faces of the point configuration $\mathbf{A}'$ of dimension smaller than or equal to l. With the terminology from appendix B, these are triangulations of the polyhedral complexes $\mathbf{A} \times \mathrm{skel}_l(\mathbf{A}')$ and $\mathrm{skel}_k(\mathbf{A}) \times \mathbf{A}'$, respectively (equivalently, partial triangulations of $\mathbf{A} \times \mathbf{A}'$ with respect to $\mathcal{K} := \mathbf{A} \times \mathrm{skel}_l(\mathbf{A}')$ and $\mathcal{K}' := \mathrm{skel}_k(\mathbf{A}) \times \mathbf{A}'$, respectively).

Our strategy to study triangulations of products of point configurations is summarized by the following particularization of question 1:

Question 4: What are necessary and sufficient conditions on $\mathbf{A}, \mathbf{A}'$ and l for a triangulation of the polyhedral complex $\mathbf{A} \times \mathrm{skel}_l(\mathbf{A}')$, where $0 \le l < \dim(\mathbf{A}')$, to equal the restriction of a triangulation $\mathcal{T}$ of $\mathbf{A} \times \mathbf{A}'$?

A solution to question 4 would give us a way to describe triangulations of $\mathbf{A} \times \mathbf{A}'$ as partial triangulations and, consequently, would say that the "complicatedness" of triangulations of $\mathbf{A} \times \mathbf{A}'$ is dominated by the complicatedness of triangulations of $\mathbf{A} \times \mathbf{F}$, for $\mathbf{F} \in \mathrm{skel}_l(\mathbf{A}')$.

Remark 2.1.1. We consider it worth pointing out the similarity of question 4 with analogous questions in fiber bundle theory. Indeed, we can intuitively think of a triangulation $\mathcal{T}'$ of $\mathbf{A} \times \mathrm{skel}_l(\mathbf{A}')$ as a "bundle of triangulations" over $\mathrm{skel}_l(\mathbf{A}')$. Then, the fact that $\mathcal{T}'$ equals the restriction of some triangulation $\mathcal{T}$ of $\mathbf{A} \times \mathbf{A}'$ can be interpreted as the bundle $\mathcal{T}'$ being "trivial". As in bundle theory, we can look for necessary conditons for a bundle $\mathcal{T}'$ to extend, along with corresponding "obstructions", that is, partial triangulations that do not extend to triangulations of $\mathbf{A} \times \mathbf{A}'$. This geometric picture provided the motivation for our investigation in this chapter. ○

2.2 Cartesian products of simplices

We now introduce our main objects of study in this chapter, and fix the notation and conventions to be used. All facts in this section are well-known, and can be found, for instance, in [GKZ08, Stu96, DRS10].

Simplices are the first family of point configurations for which we carry out the strategy outlined in section 2.1 for studying triangulations of cartesian products of point configurations. We have mentioned that triangulations of products of simplices are the building blocks of triangulations refining the product of two triangulations. Therefore, understanding triangulations of products of simplices is also of relevance for understanding triangulations of products of point configurations.

Recall from appendix B that the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$ consists of the independent points with coordinates $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{R}^n$. If $I \subset [n]$ stands for some index subset, we will refer to the face of Δ_{n-1} spanned by the points labeled by I with the symbol Δ_I.

In order to distinguish the two factors in the cartesian product $\Delta_{n-1} \times \Delta_{d-1}$, the vertices in the first factor will be labelled by the indices $1, 2, \ldots, i, \ldots, n$, and those in the second factor by $A, B, \ldots, \alpha, \ldots, d$. Moreover, to simplify the notation, we denote the points $(\mathbf{e}_i, \mathbf{e}_\alpha)$ of the point configuration $\Delta_{n-1} \times \Delta_{d-1} := \{(\mathbf{e}_i, \mathbf{e}_\alpha) \in \mathbb{R}^{d+n} : \mathbf{e}_i \in \Delta_{n-1},\ \mathbf{e}_\alpha \in \Delta_{d-1}\}$ just by the indices (i, α), or even by $i\alpha$.

In view of the simplicial complex structure of triangulations of point configurations, a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ will be determined by a collection of $(n+d-1)$-element subsets of the index set $\{i\alpha = (i, \alpha) : 1 \leq i \leq n,\ A \leq \alpha \leq d\}$, defining a collection of $(n+d-2)$-simplices that fulfill the properties in definition B.2.8.

For the study of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, there are various ways to represent subsets (i.e. subconfigurations) of $\Delta_{n-1} \times \Delta_{d-1}$, which aid considerably when examining some of their features (e.g. independence, dependence). We first present two common ones; the next section wil be devoted to the more recent geometric representation of mixed subdivisions that was developed by Santos in [San04].

Bipartite graph representation

Let $K_{n,d}$ be the complete bipartite graph on $n+d$ vertices with the bipartition $[n] \cup [d]$, where the vertices in the first part are labelled by $\{1, 2, \ldots, n\}$ and those in the second part by $\{A, B, \ldots, d\}$, as before. A point (i, α) in $\Delta_{n-1} \times \Delta_{d-1}$ is represented by the undirected edge (i, α) in $K_{n,d}$; correspondingly, a subset $\boldsymbol{\sigma}$ of $\Delta_{n-1} \times \Delta_{d-1}$ is represented by a subgraph of $K_{n,d}$. See figure 2.1.

One of the main advantages of the bipartite graph representation is the following characterization of dependent and independent subsets of $\Delta_{n-1} \times \Delta_{d-1}$.

Lemma 2.2.1 (Lemma 6.2.8 in [DRS10])**.** *Let $\boldsymbol{\sigma}$ be a subset of points of $\Delta_{n-1} \times \Delta_{d-1}$. Then:*

1. *$\boldsymbol{\sigma}$ is affinely independent if and only if the corresponding subgraph of $K_{n,d}$ is a forest (i.e. contains no cycles).*

2. *$\boldsymbol{\sigma}$ is affinely independent and of maximal dimension if and only if the corresponding subgraph of $K_{n,d}$ is a spanning tree.*

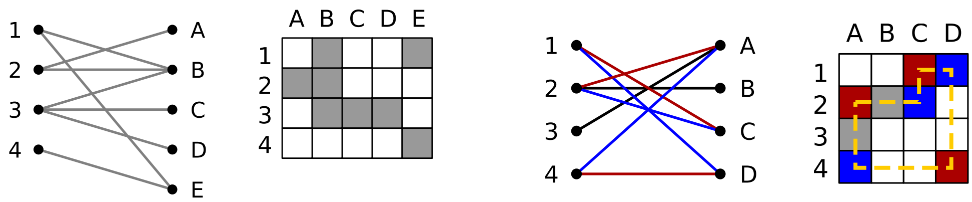
3. *$\boldsymbol{\sigma}$ is a minimal affinely dependent set, that is, a circuit, if and only if the corresponding subgraph of $K_{n,d}$ is a cycle. Moreover, each part in the partition of the circuit $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ consist of all the edges that are at an even distance from each other.*

Proof. We only prove item 3, from which items 1 and 2 directly follow. For the "if" direction, suppose $\boldsymbol{\sigma}$ is represented by a cycle C in $K_{n,d}$. To obtain the minimal affine dependence among the elements in $\boldsymbol{\sigma}$, assign alternatingly the coefficients $+1$ and -1 to the points of $\boldsymbol{\sigma}$ corresponding to the edges encountered when traversing the cycle C. Since each vertex in C has degree 2 and C has even length, every $\mathbf{e}_i$ appears exactly twice, with opposite signs, as points $(\mathbf{e}_i, \mathbf{e}_a)$ and $(\mathbf{e}_i, \mathbf{e}_b)$ of $\Delta_{n-1} \times \Delta_{d-1}$, so the assignment of coefficients yields a minimal affine dependence. For the "only-if" direction, we prove by induction on $m := n+d$ that if $\boldsymbol{\sigma}$ is represented by forest, then $\boldsymbol{\sigma}$ is an independent subset of $\Delta_{n-1} \times \Delta_{d-1}$. Indeed, for $m = 2$ the statement holds trivially. Suppose the statement holds for $m = n + d - 1$, and assume without loss of generality that $\boldsymbol{\sigma}$ corresponds to a spanning forest of $K_{n,d}$. Let $i \in [n]$ be a vertex in the forest with degree 1 (which exists since forest have no cycles). Its removal gives a spanning forest of $K_{n-1,d}$, which represents an independent set by the induction hypothesis. Since the edge of $K_{n,d}$ connecting to vertex i represents the only point $(\mathbf{e}_i, \mathbf{e}_a)$ of $\Delta_{n-1} \times \Delta_{d-1}$ having $\mathbf{e}_i$ in its first entry, we have that $\boldsymbol{\sigma}$ is also an independent set. □

Since all cycles in bipartite graphs are of even length, the next corollary is an easy consequence of lemma 2.2.1

Corollary 2.2.2. *The circuits of $\Delta_{n-1} \times \Delta_{d-1}$ are* balanced, *that is, they are all of the type (k, k) (cf. section B.2), where $2 \leq k \leq \min(n, d)$.*

Example 2.2.3. Let $n = 4$ and $d = 5$. In figure 2.1a, we see the representation of a 7-simplex as a spanning tree of $K_{4,5}$. In figure 2.1b, we see a dependent subset of $\Delta_3 \times \Delta_4$. Note that it is not a circuit, whereas the subset $\mathbf{C}$ of points spanned by the cycle of $K_{4,5}$ (shown with colored edges) is. The partition of this circuit is indicated by the coloring of the edges: $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-) = (\{(\mathbf{e}_4, \mathbf{e}_A), (\mathbf{e}_2, \mathbf{e}_C), (\mathbf{e}_1, \mathbf{e}_D)\}, \{(\mathbf{e}_2, \mathbf{e}_A), (\mathbf{e}_1, \mathbf{e}_C), (\mathbf{e}_4, \mathbf{e}_D)\})$. ○



(a) A maximal independent subset of $\Delta_3 \times \Delta_4$.

(b) A dependent subset of $\Delta_3 \times \Delta_4$.

Figure 2.1: Bipartite graph and grid representations of subsets of $\Delta_{n-1} \times \Delta_{d-1}$.

An interesting fact that can be conveniently established with the bipartite graph representation is the following one.

Proposition 2.2.4 (Proposition 6.2.11 in [DRS10])**.** *All the full-dimensional simplices in $\Delta_{n-1} \times \Delta_{d-1}$ are unimodular, that is, they have the same euclidean volume (in their affine span).*

Proof. Again we prove this by induction on $m = n + d - 2$. The case $m = 2$ is easily seen to be true: all the triangles spanned by vertices of the regular square $\operatorname{conv}(\{(\mathbf{e}_1, \mathbf{e}_a), (\mathbf{e}_1, \mathbf{e}_b), (\mathbf{e}_2, \mathbf{e}_a),\ (\mathbf{e}_2, \mathbf{e}_b)\})$ have area $\frac{1}{2}$. Suppose the statement is true for $(n-1, d)$ and $(n, d-1)$ with $m > 2$, and consider $\Delta_{n-1} \times \Delta_{d-1}$. By the induction hypothesis, all codimension-1 simplices on the facets of $\Delta_{n-1} \times \Delta_{d-1}$ have the same volume. The following two claims will yield the proof:

- *Let $\mathbf{F} = \Delta_{[n]\setminus\{i\}} \times \Delta_{d-1}$ be a facet of $\Delta_{n-1} \times \Delta_{d-1}$. The perpendicular distance between any point $\mathbf{q} \in (\Delta_{n-1} \times \Delta_{d-1}) \setminus \mathbf{F}$ and the $(n+d-3)$-dimensional plane of $\mathbb{R}^{n+d}$ spanned by $\mathbf{F}$ is the same*. Indeed, it equals $(\mathbf{q} - \mathbf{p}) \cdot \hat{\mathbf{s}}$, where $\mathbf{p}$ is any point on $\mathbf{F}$ and $\hat{\mathbf{s}}$ is the unit normal to (the plane spanned by) $\mathbf{F}$ pointing towards the interior of $\Delta_{n-1} \times \Delta_{d-1}$ in the affine space spanned by $\Delta_{n-1} \times \Delta_{d-1}$. We can always choose $\mathbf{p}$ so that $(\mathbf{q} - \mathbf{p})$ is of the form $(\mathbf{e}_i - \mathbf{e}_k, 0)$, with $k \neq i$. That the distance between any different $\mathbf{q}' \in (\Delta_{n-1} \times \Delta_{d-1}) \setminus \mathbf{F}$ and $\mathbf{F}$ is the same then holds because we have $(\mathbf{e}_i - \mathbf{e}_j, 0) \cdot \hat{\mathbf{s}} = \big((\mathbf{e}_i - \mathbf{e}_k, 0) + (\mathbf{e}_k - \mathbf{e}_j, 0)\big) \cdot \hat{\mathbf{s}} = (\mathbf{e}_i - \mathbf{e}_k, 0) \cdot \hat{\mathbf{s}}$, since $(\mathbf{e}_k - \mathbf{e}_j, 0) \cdot \hat{\mathbf{s}} = 0$ for every $j \in [n] \setminus \{i\}$.
- *For any two maximal simplices $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ of $\Delta_{n-1} \times \Delta_{d-1}$, and facets $\boldsymbol{\tau}_1 \preceq \boldsymbol{\sigma}_1$ and $\boldsymbol{\tau}_2 \preceq \boldsymbol{\sigma}_2$, there is a maximal simplex $\boldsymbol{\sigma}$ of $\Delta_{n-1} \times \Delta_{d-1}$ such that it has facets lying on the same facets of $\Delta_{n-1} \times \Delta_{d-1}$ as $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$.* In the bipartite graph representation, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are spanning trees, so each has at least one vertex of degree one. Consider the vertices of $K_{n,d}$ spanned by $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ after removing from each the vertex of degree one. Extend any tree spanning those vertices by adding edges to the two vertices of degree one we removed. We obtain a spanning tree of $K_{n,d}$ that represents a maximal simplex with the desired property.

The first claim establishes the equality of the euclidean volume of any two simplices with a facet lying on the same facet of $\Delta_{n-1} \times \Delta_{d-1}$. The second claim extends the equality of the volumes to any pair of simplices of $\Delta_{n-1} \times \Delta_{d-1}$. □

Remark 2.2.5. A further well-known property of maximal simplices of $\Delta_{n-1} \times \Delta_{d-1}$ is that they are in fact *totally unimodular*. This means that for every simplex $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n+d-1}\} \subset \Delta_{n-1} \times \Delta_{d-1}$ the vectors $\mathbf{w}_k := \mathbf{v}_k - \mathbf{v}_1$, where $k \in [2, n+d-1]$ satisfy:

$$\begin{aligned} \operatorname{cone}(\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_{n+d-1}) \cap \mathbb{Z}^{n+d-2} &= \mathbb{N}\{\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_{n+d-1}\} \\ &= \{z_2\mathbf{w}_2, z_3\mathbf{w}_3, \dots, z_{n+d-1}\mathbf{w}_{n+d-1} : z_2, \\ &\qquad z_3, \dots, z_{n+d-1} \in \mathbb{N}\}\,. \end{aligned}$$

A consequence of this is that all the (minimal) arithmetic relations that may hold among points of $\Delta_{n-1} \times \Delta_{d-1}$ precisely coincide with the (minimal) affine dependencies of the point configuration (the correct statement is [Stu96, Proposition 8.11]), which by (the proof of) lemma 2.2.1 involve only coefficients $+1$ or -1 (here by arithmetic relations we mean those encoded in the toric ideal associated to the points of $\Delta_{n-1} \times \Delta_{d-1}$; see discussion before proposition A.3.1.) ⬡

Grid representation

Consider an rectangular array of squares of height n and width d, that we refer to as the $n \times d$ *grid* $\mathcal{G}_{n\times d}$. A point (i, α) in $\Delta_{n-1} \times \Delta_{d-1}$ is represented by the square in $\mathcal{G}_{n\times d}$ in row $i \in [n]$ and column $\alpha \in [d]$, where the indices increase southwards and eastwards, as for an $n \times d$ matrix. A subset of $\Delta_n \times \Delta_d$ is represented as a subset of squares in $\mathcal{G}_{n\times d}$ as in figure 2.1, where these are shown grey.

Remark 2.2.6. Let $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-)$ be a circuit in $\Delta_{n-1} \times \Delta_{d-1}$. In $\mathcal{G}_{n\times d}$, $\mathbf{C}$ can be seen as a closed path whose corners alternatingly use squares (representing points) from $\mathbf{C}^+$ and $\mathbf{C}^-$. This merely expresses the fact that, in order to have a minimal affine dependence in $\Delta_n \times \Delta_d$, for every point in $\mathbf{C}^+$ with first coordinate $\mathbf{e}_i$ there must be exactly one further point in $\mathbf{C}^-$ with first coordinate $\mathbf{e}_i$, and similarly for the points with second coordinate $\mathbf{e}_\alpha$. ⬡

While the grid representation is slightly less apt to see that a subset of $\Delta_{n-1} \times \Delta_{d-1}$ is dependent or independent, it is very useful to explicitly describe some triangulations of $\Delta_{n-1} \times \Delta_{d-1}$. The triangulation below, for example, can be considered the “simplest” triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

Definition 2.2.7 (Definition 6.2.12 in [DRS10])**.** A *monotone staircase* in the $n \times d$ grid is a subset of $n + d - 1$ squares containing the squares $(1, A)$ and (n, d), such that if (i, α) is in the monotone staircase (for $i < n$ and $\alpha < d$), exactly one of $\{(i, \alpha + 1), (i + 1, \alpha)\}$ is also in the staircase. Said differently, a monotone staircase is a walk on the grid from $(1, A)$ to (n, d) where only steps southwards and eastwards are allowed.

Theorem 2.2.8 (Theorem 6.2.13 in [DRS10])**.** *The set of monotone staircases in a $n \times d$ grid gives the collection of $(n + d - 2)$-simplices of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, called the* staircase triangulation of $\Delta_{n-1} \times \Delta_{d-1}$*.*

To illustrate this construction, in figure 2.2 we see the simplices in a staircase triangulation of $\Delta_2 \times \Delta_4$.

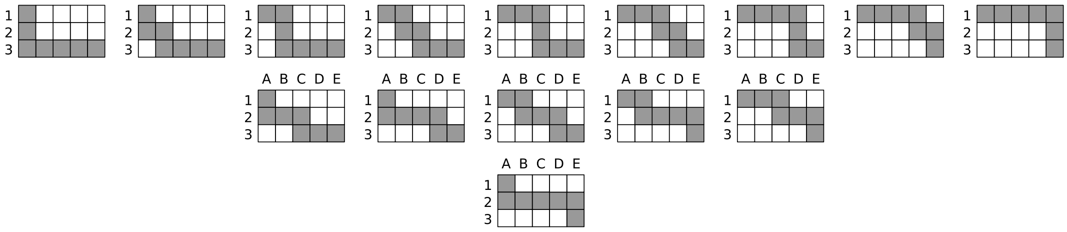


Figure 2.2: A staircase triangulation of $\Delta_2 \times \Delta_4$

Note that every linear ordering of the vertices of Δ_{n-1} and of Δ_{d-1} gives a different staircase triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, and therefore there are $\frac{n!d!}{2}$ different staircase triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

Example 2.2.9. Suppose $n = 2$. There are $d!$ staircase triangulations of $\Delta_1 \times \Delta_{d-1}$, which are in bijection with the $d!$ linear orderings of the labels of the vertices of Δ_{d-1}. Moreover, all triangulations of $\Delta_1 \times \Delta_{d-1}$ are of this form (see [DRS10, Proposition 6.2.3] or example 2.3.9 below for a proof of this fact). ⬡

Remark 2.2.10. The staircase triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ has been commonly used in algebraic topology to give a canonical structure of simplicial complex to the cartesian product of two simplicial complexes [GKZ08, p. 248]. Also, in topological combinatorics it can be interpreted as the order complex of the poset $\mathbf{n} \times \mathbf{d} := \{1 < 2 < \ldots < n\} \times \{1 < 2 < \ldots < d\}$, whose elements are $\{(i, a) : 1 \leq i \leq n,\ 1 \leq a \leq d\}$ and we have $(i, a) \leq (j, b)$ if and only if $i \leq j$ and $a \leq b$. Indeed, recall that the order complex of $\mathbf{n} \times \mathbf{d}$ is the simplicial complex whose base set are the elements of $\mathbf{n}\times\mathbf{d}$ and whose simplices are the chains (i.e. totally ordered subsets) of $\mathbf{n} \times \mathbf{d}$ (which coincide with the monotone staircases in a $n \times d$ grid). ⬡

Proposition 2.2.11. *The staircase triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ has $\binom{n+d-2}{n-1}$ full-dimensional simplices.*

(See [DRS10, Section 6.2] for a proof)

Remark 2.2.12. Propositions 2.2.4 and proposition 2.2.11 together say that all triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ have $\binom{n+d-2}{n-1}$ full-dimensional simplices. This property is called *equidecomposability*, and already follows from the weaker fact that all the circuits of $\Delta_{n-1} \times \Delta_{d-1}$ are balanced [Bay93]. A very helpful consequence of this property is that we can check whether a given collection of (distinct) $(n + d - 2)$-simplices of $\Delta_{n-1} \times \Delta_{d-1}$ cover the volume of $\Delta_{n-1} \times \Delta_{d-1}$ simply by checking that the number of simplices is $\binom{n+d-2}{n-1}$. ⬡

A third way to represent simplices in triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ is as *fine mixed cells in fine mixed subdivisions of the dilated simplex* $n\Delta_{d-1}$.

2.3 Fine mixed subdivisions of dilated simplices

Observe that if $\mathbf{P}_1 = \mathbf{P}_2 = \ldots = \mathbf{P}_n = \Delta_{d-1}$ are n copies of the standard $(d-1)$-simplex in $\mathbb{R}^d$, then the following holds (see example B.4.2):

$$\mathcal{C}\left(\underbrace{\Delta_{d-1}, \ldots, \Delta_{d-1}}_{n \text{ times}}\right) = \Delta_{n-1} \times \Delta_{d-1} \subset \mathbb{R}^{n+d},$$

where $\mathcal{C}(\Delta_{d-1}, \Delta_{d-1}, \ldots, \Delta_{d-1})$ denotes the n-fold Cayley embedding of Δ_{d-1}.

In [San04], Santos observed that, with this identification, the Cayley trick could be used to represent triangulations of the cartesian product $\Delta_{n-1} \times \Delta_{d-1}$ as fine mixed subdivisions of the dilated simplex:

$$n\Delta_{d-1} := \underbrace{\Delta_{d-1} + \ldots + \Delta_{d-1}}_{n \text{ times}}.$$

An immediate consequence of this correspondence is that we have a representation of triangulations of the $(n+d-2)$-dimensional object $\Delta_{n-1} \times \Delta_{d-1}$ as fine mixed subdivisions of the $(d-1)$-dimensional object $n\Delta_{d-1}$. Santos investigated further combinatorial and geometric aspects of fine mixed subdivisions of $n\Delta_{d-1}$, and was able to obtain several results about triangulations of products of simplices that had gone unnoticed before. It is the purpose of this section to expose some of Santos' results in preparation for our treatment of partial triangulations of the product of two simplices in section 2.4.

Before we start, let us present an example of the use of the Cayley trick for products of simplices.

Example 2.3.1. Let $n = d = 3$. The following collection of 4-simplices is a triangulation of $\Delta_2 \times \Delta_2$:

$$\boldsymbol{\sigma}_1 = \{1A, 1B, 1C, 2A, 3A\} \quad \boldsymbol{\sigma}_2 = \{1B, 2A, 2B, 2C, 3B\} \quad \boldsymbol{\sigma}_3 = \{1C, 2A, 3A, 3B, 3C\}$$
$$\boldsymbol{\sigma}_4 = \{1B, 1C, 2A, 3A, 3B\} \quad \boldsymbol{\sigma}_5 = \{1B, 1C, 2A, 2C, 3B\} \quad \boldsymbol{\sigma}_6 = \{1C, 2A, 2C, 3B, 3C\}$$

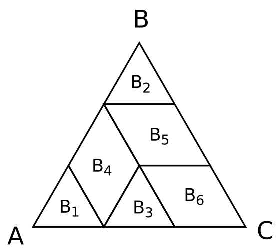
According to the Cayley trick (theorem B.4.3), the fine mixed cells corresponding to the simplices above are:

$$\mathbf{B}_1 = ABC + A + A \qquad \mathbf{B}_2 = B + ABC + B \qquad \mathbf{B}_3 = C + A + ABC$$
$$\mathbf{B}_4 = BC + A + AB \qquad \mathbf{B}_5 = BC + AC + B \qquad \mathbf{B}_6 = C + AC + BC,$$

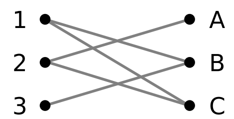
where the labels of the Minkowski summands are 1, 2 and 3. Note that we have used our convention for denoting subsimplices of Δ_2 just by the labels of the points; for instance, cell $\mathbf{B}_1$ above stands for the 2-dimensional point configuration:

$$\{\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_C\} + \{\mathbf{e}_A\} + \{\mathbf{e}_A\} \subset \mathbb{R}^3.$$

The cells $\{\mathbf{B}_1, \ldots, \mathbf{B}_6\}$ constitute a fine mixed subdivision of the dilated simplex $3\Delta_2$, which is illustrated in figure 2.3a. In relation with the bipartite graph rep-



(a) A fine mixed subdivision of $3\Delta_2$.



(b) The spanning tree in $K_{3,3}$ representing $\mathbf{B}_5$.

resentation, notice that the i-th summand of a fine mixed cell $\mathbf{B}$ in a fine mixed subdivision $\mathcal{M}$ of $n\Delta_{d-1}$ precisely says which vertices of the part $[d]$ of $K_{n,d}$ are adjacent to vertex $i \in [n]$. The cell $\mathbf{B}_5$, for instance, has the representation as spanning tree of $K_{3,3}$ shown in figure 2.3b. One can thus think of the fine mixed subdivision representation of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ as a "geometrization" of its bipartite graph representation. $\diamond$

Remark 2.3.2. Under the Cayley trick, triangulations of $\Delta_{n-1} \times \Delta_2$ are in bijection (up to symmetry and labeling) with *lozenge tilings* of the dilated regular triangle $n\Delta_2$. These are tilings using unit upward triangles △ and unit rhombi ▱, ▱, ◊, also called lozenges. Among other things, Santos used this correspondence to obtain asymptotic formulas for the number of triangulations of $\Delta_{n-1} \times \Delta_2$ and to establish the connectedness of the set of triangulations of $\Delta_{n-1} \times \Delta_2$ under *geometric bistellar flips* [San04] (see definition B.2.23). $\diamond$

Remark 2.3.3. Just as we identify $\Delta_{n-1} \times \Delta_{d-1}$ with the n-fold Cayley lifting of Δ_{d-1}, we may identify it with the d-fold Cayley lifting of Δ_{n-1}. In this case, the Cayley trick also establishes the equivalence of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ with fine mixed subdivisions of $d\Delta_{n-1}$. Hence, the triangulation in example 2.3.1 also corresponds to the fine mixed subdivision of $3\Delta_2$ with fine mixed cells:

$$\begin{array}{lll} \mathbf{B}_1^\star = 123 + 1 + 1 & \mathbf{B}_2^\star = 2 + 123 + 2 & \mathbf{B}_3^\star = 23 + 3 + 13 \\ \mathbf{B}_4^\star = 23 + 13 + 1 & \mathbf{B}_5^\star = 2 + 13 + 12 & \mathbf{B}_6^\star = 2 + 3 + 123, \end{array}$$

where the Minkowski summands of each cell are labelled by A, B, and C. This fine mixed subdivision is displayed in figure 2.3.

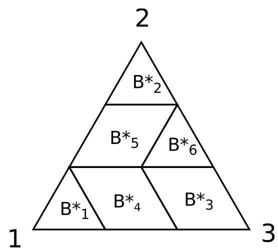


Figure 2.3: A fine mixed subdivision of $3\Delta_2$

We think of the fine mixed subdivisions of $n\Delta_{d-1}$ and of $d\Delta_{n-1}$ that correspond to a given triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ as being *dual* to each other [San04, AC13], and will denote the fine mixed subdivision of $d\Delta_{n-1}$ dual to a fine mixed subdivision $\mathcal{M}$ of $n\Delta_{d-1}$ by $\mathcal{M}^\star$. Clearly, every $(d-1)$-dimensional fine mixed cell $\mathbf{B}$ of $\mathcal{M}$ has a unique dual cell $\mathbf{B}^\star$ of $\mathcal{M}^\star$ which is $(n-1)$-dimensional and fine. $\diamond$

We argue that the representation of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ as a fine mixed subdivision of $n\Delta_{d-1}$ gives a "geometric" description that is rich enough to encode all the combinatorial information in the triangulation.

With a view towards justifying this assertion, let us begin with two observations about the geometry of mixed cells in a fine mixed subdivision of a dilated simplex, which the reader can inspect in example 2.3.1. Let $\mathbf{B}$ be such a cell of dimension $d-1$ in a fine mixed subdivision $\mathcal{M}$ of $n\Delta_{d-1}$.

First, $\mathbf{B}$ has a "position" inside $n\Delta_{d-1}$, which is specified by a *weak integer composition* of $n-1$ of length d. This is a tuple $\mathbf{q}(\mathbf{B}) = (q_1(\mathbf{B}), q_2(\mathbf{B}), \ldots, q_d(\mathbf{B})) \in \mathbb{N}^d$ such that $\sum_{a=1}^{d} q_a(\mathbf{B}) = n-1$, where $q_a(\mathbf{B})$ is the a-th barycentric coordinate of $\mathbf{B}$ $n\Delta_{d-1}$. To obtain it, we simply count the number of times $\mathbf{e}_a$ appears among all the Minkowski summands of $\mathbf{B}$ and subtract one. We call $\mathbf{q}(\mathbf{B})$ the *position vector* of $\mathbf{B}$.

Second, note that since the Minkowski summands of $\mathbf{B}$ all lie on independent affine spaces, $\mathbf{B}$ has the form of (is combinatorially equivalent to – see proposition B.3.1) a cartesian product of several simplices: there is one factor for every summand of $\mathbf{B}$ (with possibly some factors of dimension zero). Therefore, we can define the "shape" of $\mathbf{B}$ as a weak integer composition $\mathbf{r}(\mathbf{B}) = (r_1(\mathbf{B}), r_2(\mathbf{B}), \ldots, r_n(\mathbf{B})) \in \mathbb{N}^n$ of $d-1$ of length n, where $r_i(\mathbf{B})$ stands for the dimension of the i-th Minkowski summand of $\mathbf{B}$. For example, the possible shapes (modulo labeling) for 2-dimensional fine mixed cells in a fine mixed subdivision of $n\Delta_2$ are a triangle and a rhombus (or lozenge); in a fine mixed subdivision of $n\Delta_3$, the 3-dimensional fine mixed cells can be tetrahedra, triangular prisms or parallelepipeds. We call $\mathbf{r}(\mathbf{B})$ the *shape vector* of $\mathbf{B}$.

Remark 2.3.4. It follows from the definition of duality that, if $\mathbf{B}^\star$ is the cell in $\mathcal{M}^\star$ dual to $\mathbf{B}$ in $\mathcal{M}$, then the shape and position vectors of $\mathbf{B}^\star$ are related to those of $\mathbf{B}$ by:

$$\mathbf{r}(\mathbf{B}^\star) = \mathbf{q}(\mathbf{B})$$
$$\mathbf{q}(\mathbf{B}^\star) = \mathbf{r}(\mathbf{B})$$

⬡

In lemma 2.3.5 below, we prove that the notions of position and shape of a full-dimensional fine mixed cell are actually well defined.

Lemma 2.3.5. *Let $\mathcal{M}$ be a fine mixed subdivision of $n\Delta_{d-1}$, and let $\mathbf{B}, \mathbf{B}' \in \mathcal{M}$ be such that $\mathbf{q}(\mathbf{B}) = \mathbf{q}(\mathbf{B}')$ or $\mathbf{r}(\mathbf{B}) = \mathbf{r}(\mathbf{B}')$; i.e. $\mathbf{B}$ and $\mathbf{B}'$ have the same combinatorial shape or the same position inside $n\Delta_{d-1}$. Then $\mathbf{B} = \mathbf{B}'$.*

Proof. Suppose $\mathbf{B}$ and $\mathbf{B}'$ are two different full-dimensional cells in $\mathcal{M}$ such that $\mathbf{q}(\mathbf{B}) = \mathbf{q}(\mathbf{B}')$, and denote by $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ the full-dimensional simplices of $\Delta_{n-1} \times \Delta_{d-1}$ that $\mathbf{B}$ and $\mathbf{B}'$ represent under the Cayley trick. Using induction on the number ℓ of points by which $\mathbf{B}$ and $\mathbf{B}'$ differ, we will prove that for every $\mathbf{w} \in \boldsymbol{\sigma}' \backslash \boldsymbol{\sigma}$, there is a circuit $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-)$ of $\Delta_{n-1} \times \Delta_{d-1}$, with $\mathbf{w} \in \mathbf{C}^-$, on which $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ overlap. This is sufficient to contradict the assumption that $\mathcal{M}$ represents a legal triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

To this end, assume first that $\ell = 1$, that is, $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \backslash \{\mathbf{v}_1\} \cup \{\mathbf{w}_1\}$. Let $\mathbf{C}_1 = (\mathbf{C}_1^+, \mathbf{C}_1^-)$ be the unique circuit contained in the minimally dependent set $\boldsymbol{\sigma} \cup \boldsymbol{\sigma}' = \boldsymbol{\sigma} \cup \{\mathbf{w}_1\}$, and view $\mathbf{C}_1$ as an even cycle in $K_{n,d}$. Note that by the symmetry between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, $\mathbf{C}_1$ must use the edges representing $\mathbf{v}_1$ and $\mathbf{w}_1$.

Since $\mathbf{q}(\mathbf{B}) = \mathbf{q}(\mathbf{B}')$, $\mathbf{v}_1$ and $\mathbf{w}_1$ are represented in $n\Delta_{d-1}$ by different points in the same summand of $\mathbf{B}$ and of $\mathbf{B}'$ which, in accordance with the remark at the end of example 2.3.1, correspond to adjacent edges in $\mathbf{C}_1$; in particular, $\mathbf{v}_1$ and $\mathbf{w}_1$ are within an odd distance from each other in $\mathbf{C}_1$. Given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the only points by which $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ differ, the cycle $\mathbf{C}_1$ can be traversed using edges (representing points) from $\boldsymbol{\sigma}$ and from $\boldsymbol{\sigma}'$ alternatingly, that is, $\mathbf{C}_1^+ \subset \boldsymbol{\sigma}$ and $\mathbf{C}_1^- \subset \boldsymbol{\sigma}'$, where we set $\mathbf{v}_1 \in \mathbf{C}_1^+$ and $\mathbf{w}_1 \in \mathbf{C}^-$.

Now suppose $\ell = o$, that is, $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \backslash \{\mathbf{v}_1, \dots, \mathbf{v}_o\} \cup \{\mathbf{w}_1, \dots, \mathbf{w}_o\}$, and consider the full-dimensional cell $\mathbf{B}''$ represented by the simplex $\boldsymbol{\sigma}'' = \boldsymbol{\sigma} \backslash \{\mathbf{v}_1, \dots, \mathbf{v}_{o-1}\} \cup \{\mathbf{w}_1, \dots, \mathbf{w}_{o-1}\}$, so that $\boldsymbol{\sigma}' = \boldsymbol{\sigma}'' \backslash \{\mathbf{v}_o\} \cup \{\mathbf{w}_o\}$. The induction hypothesis asserts that for each $i \in [o-1]$ there is a circuit $\widetilde{\mathbf{C}}_i = (\widetilde{\mathbf{C}}_i^+, \widetilde{\mathbf{C}}_i^-)$, with $\mathbf{w}_i \in \widetilde{\mathbf{C}}_i^-$, on which $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}''$ overlap. Denote by $\widetilde{\mathbf{C}}_o$ the unique circuit on the minimally independent set $\boldsymbol{\sigma}'' \cup \boldsymbol{\sigma}' = \boldsymbol{\sigma}'' \cup \{\mathbf{w}_o\}$, for which we set $\mathbf{v}_o \in \widetilde{\mathbf{C}}_o^+ \subset \boldsymbol{\sigma}''$ and $\mathbf{w}_o \in \widetilde{\mathbf{C}}_o^- \subset \boldsymbol{\sigma}'$.

The following two reasons may prevent the collection of circuits $\{\widetilde{\mathbf{C}}_1, \dots, \widetilde{\mathbf{C}}_{o-1}, \widetilde{\mathbf{C}}_o\}$ from satisfying the induction hypothesis, namely that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ overlap on every such circuit:

- $\widetilde{\mathbf{C}}_o^+ \not\subset \boldsymbol{\sigma}$, i.e., there is some subset $\{\mathbf{w}_1, \dots, \mathbf{w}_p\} \subset \widetilde{\mathbf{C}}_o^+$,
- $\widetilde{\mathbf{C}}_i^- \not\subset \boldsymbol{\sigma}'$, i.e., $\mathbf{v}_o \in \widetilde{\mathbf{C}}_i^-$ for some $i \in [o-1]$.

To deal with these situations, we first get rid of the $\{\mathbf{w}_1, \dots, \mathbf{w}_p\} \subset \widetilde{\mathbf{C}}_o^+$ by adding the minimal affine dependences inducing the circuits $\widetilde{\mathbf{C}}_1, \dots, \widetilde{\mathbf{C}}_p, \widetilde{\mathbf{C}}_o^+$ together. This results in an affine dependence $\mathbf{c}^\star$ that may not be minimal, but by proposition B.2.6 can be decomposed into minimal affine dependencies. Let $\mathbf{C}_o$ be the circuit induced by the minimal dependence such that $\mathbf{w}_o \in \mathbf{C}_o^-$, which exists since $\mathbf{w}_o$ appears only once among all the minimal dependences that were added. Observe that the circuit $\mathbf{C}_o$ has the property that $\mathbf{C}_o^+ \subset \boldsymbol{\sigma}$ and $\mathbf{C}_o^- \subset \boldsymbol{\sigma}'$ (since all the $\mathbf{v}$'s and $\mathbf{w}$'s that may appear after the addition can only end up within $\mathbf{C}_o^+$ and $\mathbf{C}_o^-$, respectively).

Now we proceed to remove the $\mathbf{v}_o$ that may lie within some of the $\widetilde{\mathbf{C}}_i^-$ (for some $i \in [o-1]$). This we do by adding the affine dependencies inducing the circuits $\widetilde{\mathbf{C}}_i$ and $\mathbf{C}_o$. In the resulting, perhaps non-minimal, affine dependence, we let $\mathbf{C}_i$ be the circuit induced by the minimal dependence such that $\mathbf{w}_i \in \mathbf{C}_i^-$. Note that there is no risk of accidentally disposing of $\mathbf{w}_i$ in the addition, because $\mathbf{C}_o^+ \subset \boldsymbol{\sigma}$ and $\mathbf{C}_o^- \subset \boldsymbol{\sigma}'$ (in other words, there are no $\mathbf{w}$'s inside $\mathbf{C}_o^+$). The simplices $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ overlap on every circuit in the collection $\{\mathbf{C}_1, \dots, \mathbf{C}_{o-1}, \mathbf{C}_o\}$, thus the proof is concluded. $\square$

It should come as no surprise that no two full-dimensional cells in a fine mixed subdivision $\mathcal{M}$ of $n\Delta_{d-1}$ may "occupy" the same position in the dilated simplex. On the other hand, the statement that no two full dimensional cells may have the same combinatorial shape is not directly evident from the *primal* representation $\mathcal{M}$, which somehow conceals the symmetry between the factors in $\Delta_{n-1} \times \Delta_{d-1}$. Despite this "asymmetry", the geometry of fine mixed subdivisions of $n\Delta_{d-1}$ is so constrained that we can always recover the more symmetric representation as a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. Part of the constraint results from the interesting fact that actually all possible positions and combinatorial shapes appear in a fine mixed subdivision of $n\Delta_{d-1}$.

Proposition 2.3.6. *Let $\mathcal{M}$ be a fine mixed subdivision of $n\Delta_{d-1}$. The full-dimensional cells in $\mathcal{M}$ are in bijection with the sets of weak integer compositions:*

$$\Lambda_{d,n-1} := \left\{ (q_1, q_2, \dots, q_d) \in \mathbb{N}^d : \sum_{j=1}^{d} q_j = n-1 \right\} \text{ and}$$
$$\Lambda_{n,d-1} := \left\{ (r_1, r_2, \dots, r_n) \in \mathbb{N}^n : \sum_{i=1}^{n} r_i = d-1 \right\}.$$

Proof. The cardinalities of the sets of weak compositions $\Lambda_{n,d-1}$ and $\Lambda_{d,n-1}$ equal both $\binom{n+d-2}{n-1}$ (see [Sta11, p.25]), which coincides by proposition 2.2.11 with the number of full-dimensional simplices in a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. The statement follows from the Cayley trick in theorem B.4.3 and lemma 2.3.5, which says that the mappings $\mathbf{q}$ and $\mathbf{r}$ in:

$$\Lambda_{d,n-1} \xleftarrow{\mathbf{q}} \left\{ \begin{array}{c} \text{full-dimensional} \\ \text{cells in } \mathcal{M} \end{array} \right\} \xrightarrow{\mathbf{r}} \Lambda_{n,d-1}, \tag{2.1}$$

are injective. □

Corollary 2.3.7. *The map that sends a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ to the unique bijection $\Lambda_{d,n-1} \longleftrightarrow \Lambda_{n,d-1}$ induced according to proposition 2.3.6 is injective.*

Remark 2.3.8. It is an interesting open problem to combinatorially characterize those bijections between $\Lambda_{n,d-1}$ and $\Lambda_{d,n-1}$ that arise from fine mixed subdivisions of $n\Delta_{d-1}$, thus developing a fourth representation of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ ○

Example 2.3.9. Let $n = 2$; there are exactly $d!$ triangulations of $\Delta_1 \times \Delta_{d-1}$. Indeed, a triangulation of $\Delta_1 \times \Delta_{d-1}$ defines a bijection between the vertices of Δ_{d-1} (that is, $\Lambda_{d,1}$) and $\Lambda_{2,d-1} = \{(x,y) \in \mathbb{N}^2 : x+y = d-1\}$. Since there are $d!$ such bijections , we have an upper bound of $d!$ for the number of triangulations of $\Delta_1 \times \Delta_{d-1}$ (cf. 2.2.9). But we know that there are at least $d!$ such triangulations (corresponding to the staircase triangulations).

Alternatively, the triangulations of $\Delta_1 \times \Delta_{d-1}$ are in bijection with the subdivisions of the dilated edge $d\Delta_1$. Every maximal cell in such a subdivision is an unmixed segment, and the subdivision is specified by the positions of the d unmixed segments along $d\Delta_1$ (for which there are $d!$ possibilities). ○

Let $\mathbf{B}$ be a fine mixed cell in $n\Delta_{d-1}$. To record the indices that appear in the summands of $\mathbf{B}$ of positive dimension, which correspond to non-zero coordinates of the shape vector $\mathbf{q}(\mathbf{B})$, we introduce the following notation.

Definition 2.3.10. Let **B** be a full-dimensional cell in a subdivision $\mathcal{M}$ of $n\Delta_{d-1}$. By the *stem* of **B**, written $\vartheta(\mathbf{B})$, we mean the collection of Minkowski summands of **B** of positive dimension. The *support* of $\vartheta(\mathbf{B})$, written $\operatorname{supp}(\vartheta(\mathbf{B}))$, is the subset of $[n]$ consisting of the indices that label the Minkowski summands in $\vartheta(\mathbf{B})$, i.e., the indices of the Minkowski summands of **B** of positive dimension.

The stem of a full-dimensional fine mixed cell in a fine mixed subdivision of a dilated simplex can be thought of as its "combinatorial type". It follows directly from lemma 2.3.5 that no two full-dimensional fine mixed cells in a fine mixed subdivision may have the same combinatorial type.

Remark 2.3.11. Let $\mathcal{M}$ be a fine mixed subdivision of $n\Delta_{d-1}$. Proposition 2.3.6 says that n full-dimensional fine mixed cells in $\mathcal{M}$ have the simplest stem possible: a single $(d-1)$-simplex. In [AB07, AC13] these cells were called the *unmixed simplices of* $\mathcal{M}$; they have the importance that choosing a labeling for them completely specified the labeling of $\mathcal{M}$. This is made clear in theorem 2.3.12 below. ○

So far we have seen how some geometric features of the cells in a fine mixed subdivision of a dilated simplex are constrained by the fact that they represent subsimplices in the cartesian product of two simplices. One of the punch-lines in the theory of fine mixed subdivisions is the following result of Santos, which says that already the requirement that some fine mixed cells fit in and fill a dilated simplex guarantees that the geometric polyhedral subdivision resulting **always** correspond to a fine mixed subdivision that represents, up to labeling, a triangulation of the product of two simplices.

Theorem 2.3.12 (Theorem 2.6 in [San04])**.** *Let $\mathcal{S}$ be a polyhedral subdivision of $n\Delta_{d-1}$, and assume that every cell of it can be written as a Minkowski sum of faces of Δ_{d-1} lying on independent affine spaces. Then, up to reordering, there is a unique fine mixed subdivision $\mathcal{M}$ of $n\Delta_{d-1}$ consistent with $\mathcal{S}$.*

We explain how to obtain the labeled mixed subdivision $\mathcal{M}$ from $\mathcal{S}$ [San04]. First, we label all unmixed simplices in $\mathcal{S}$, of which there are n, with the indices from 1 to n; this corresponds to identifying them by the Cayley trick with copies of the simplex $\{\mathbf{e}_i\} \times \Delta_{d-1}$, $1 \le i \le n$, inside the product $\Delta_{n-1} \times \Delta_{d-1}$. We can recover the i-th summand of every cell by induction on its codimension as follows.

Suppose we already know the i-th summand for all cells for which that summand has dimension $d' + 1 < d - 1$, and consider the $(d' + 1)$ dimensional summand $\mathbf{B}_i$ for a cell $\mathbf{B} \in \mathcal{S}$, together with some d' dimensional face $\mathbf{B}'_i \prec \mathbf{B}_i$. We want to determine which cells of $\mathcal{S}$ have $\mathbf{B}'_i$ as their i-th summand.

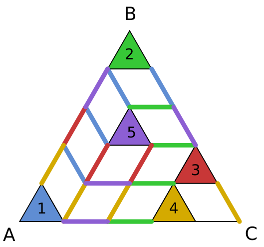
Since the Minkowski summands of every cell lie on independent affine spaces, **B** has a face **F** parallel to $\mathbf{B}'_i$. Every cell in $\mathcal{S}$ sharing the face **F** with **B** will have $\mathbf{B}'_i$ contained in its i-th summand, so its i-th summand is either of dimension $d'' > d' + 1$, or equals $\mathbf{B}'_i$. Moreover, if a full-dimensional cell $\mathbf{B}^{(1)}$ of $\mathcal{S}$ has $\mathbf{B}'_i$

as its i-th summand, then so does every full-dimensional cell $\mathbf{B}^{(2)}$ sharing a face parallel to $\mathbf{B}'_i$ with $\mathbf{B}^{(1)}$.

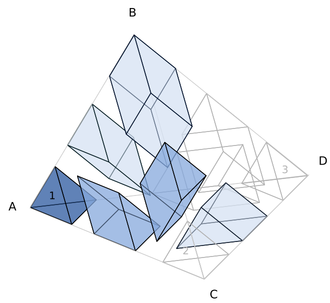
Rather than presenting a proof of theorem 2.3.12, which is available in [San04], we illustrate its geometric content in the following example.

Example 2.3.13. *(a)* Consider the polyhedral subdivision $\mathcal{S}_1$ of $5\Delta_2$ displayed in figure 2.4b. After fixing a labeling for each unmixed triangle, we determine which polyhedral cells in $\mathcal{S}_1$ have as Minkowski summand an edge of a given unmixed triangle. This is shown in figure 2.4b with the matching coloring of the edges in $\mathcal{S}_1$.

(b) Let $\mathcal{S}_2$ be the polyhedral subdivision of $3\Delta_3$ displayed in figure 2.4a. The cells having a face of the unmixed tetrahedron with label 1 as Minkowski summand are colored blue; cells where the summand has higher codimension have a lighter coloring.



(a) Edges of $\mathcal{S}_1$ with an edge of an unmixed triangle as Minkowski summand.



(b) Cells with a face of the unmixed tetrahedron 1 as a Minkowski summand (with the exception of 0-dimensional faces).

Figure 2.4: The inductive process in the proof of theorem 2.3.12

$\diamond$

We have found that the representation of triangulations of products of simplices as fine mixed subdivisions of dilated simplices provides a geometric intuition that is useful when considering partial triangulations. This will become more apparent in our development of partial triangulations in section 2.4 next. For now, note that with theorem 2.3.12 one can easily generate triangulations of $\Delta_{n-1} \times \Delta_3$ by "inspection", drawing the corresponding fine mixed subdivision of $n\Delta_3$.

2.4 Triangulations of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$

From this point on, since we always deal with triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, which correspond to fine mixed subdivisions of $n\Delta_{d-1}$, we will drop the adjective "fine mixed", and write only "subdivisions" when referring to fine mixed subdivisions of $n\Delta_{d-1}$. Likewise, when writing "cells", we mean fine mixed cells of a fine mixed subdivision.

Let n, d, k be natural numbers such that $n, d, k > 1$ and $k < d$. With the Cayley trick, we may regard a triangulation of the polyhedral complex $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$:

- as a subdivision of the $(k-1)$-*skeleton of the dilated simplex* $\mathrm{skel}_{k-1}(n\Delta_{d-1})$, that is, a collection of subdivisions of the $(k-1)$-faces of $n\Delta_{d-1}$, or
- as a collection of $\binom{d}{k}$ subdivisions of $k\Delta_{n-1}$, where the Minkowski summands in each subdivision are labeled by the indices in a k-element subset of $[d]$.

Since these are representations of a partial triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, in accordance with definition B.2.19 in the appendix, both satisfy some "consistency requirements", namely that the subdivisions agree on mutual faces of $\mathrm{skel}_{k-1}(n\Delta_{d-1})$ in the first representation, and that subdivisions of $k\Delta_{n-1}$ agree on the intersections of k-element subsets of $[d]$ in the second representation. Ocasionally we will refer to any of these representations as *partial subdivision of* $n\Delta_{d-1}$.

Remark 2.4.1. It is important to bear in mind that the notation $\mathrm{skel}_{k-1}(n\Delta_{d-1})$ refers throughout to the polyhedral complex $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$, and not to the $(k-1)$-skeleton of $\Delta_{n-1} \times \Delta_{d-1}$. When $k = d-1$, we will also write $\partial(n\Delta_{d-1})$ for $\Delta_{n-1} \times \partial(\Delta_{d-1})$: the *boundary* of the dilated simplex. ○

2.4.1 $k = 2$: systems of permutations

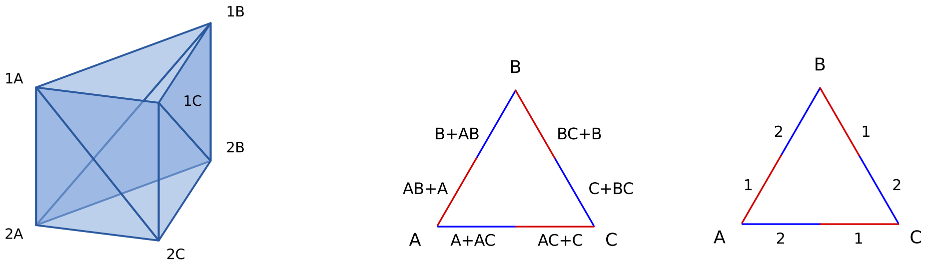
We start by reviewing the results around partial triangulations of products of simplices in [AC13, San12]. We introduce them with an example.

Example 2.4.2. Let $\mathcal{T}'$ be the triangulation of $\Delta_1 \times \partial(\Delta_2)$ shown in figure 2.5a. There is no triangulation of $\Delta_1 \times \Delta_2$ that induces the triangulation $\mathcal{T}'$ on the faces in $\Delta_1 \times \partial(\Delta_2)$. Indeed, if there was such a triangulation, then the 2-simplices $\boldsymbol{\tau}_1 = \{1B, 2A, 2B\}$ and $\boldsymbol{\tau}_2 = \{1B, 1C, 2B\}$ would be two facets of the 3-simplex $\boldsymbol{\tau}_1 \cup \boldsymbol{\tau}_2$. Otherwise, if $\boldsymbol{\sigma}_1 \supset \boldsymbol{\tau}_1$ and $\boldsymbol{\sigma}_2 \supset \boldsymbol{\tau}_2$ were different 3-simplices, then each of $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$ would contain an edge whose relative interior intersects the relative interior of an edge of $\boldsymbol{\tau}_2$ and of $\boldsymbol{\tau}_1$, respectively. However, the relative interior of the edge $\{1A, 2B\}$ of $\boldsymbol{\tau}_1 \cup \boldsymbol{\tau}_2$ intersects that of the edge $\{1B, 2A\}$ in $\mathcal{T}'$, contradicting property (**HP**) in B.2.8 (see discussion preceding lemma B.2.20). In other words, $\mathcal{T}'$ does not *extend* to a triangulation of $\Delta_1 \times \Delta_2$.

○

The situation in example 2.4.2 is probably the best known instance of a non-extendable partial triangulation, and has been called the "mother of all examples" in the triangulations literature.

We examine it from the perspective of subdivisions. In figure 2.5b we see the subdivision $\mathcal{M}'$ of $\partial(2\Delta_2)$ corresponding to the triangulation $\mathcal{T}'$ in figure 2.5a. Observe that the labels $\{1, 2\}$ of the unmixed simplices in the subdivisions of



(a) A non-extendable triangulation of $\Delta_1 \times \partial(\Delta_2)$

(b) Subdivision of $\partial(2\Delta_2)$ corresponding to the partial triangulation triangulation to the left.

Figure 2.5: Triangulation of $\Delta_1 \times \partial(\Delta_2)$

the dilated edges in $\partial(2\Delta_2)$ appear cyclically as we "go around" the dilated triangle.

In [AC13], Ardila and Ceballos set out to characterize triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ in terms of the triangulations of $\Delta_{n-1} \times \mathrm{skel}_1(\Delta_{d-1})$ they induce by restriction. Starting from the observation in the previous paragraph, they identified an appropriate generalization of the cyclic situation, and were able to formulate a necessary *acyclicity condition*, for arbitrary n and d, for a triangulation of $\Delta_{n-1} \times \mathrm{skel}_1(\Delta_{d-1})$ to extend.

Ardila and Ceballos proved that this acyclicity condition was also sufficient to guarantee extendability of a triangulation of $\Delta_{n-1} \times \mathrm{skel}_1(\Delta_{d-1})$ for $\min\{n, d\}$ up to 3, and conjectured this to be also the case for any n and d (conjecture 2.4.7 below). Shortly after, Santos presented a counterexample to the acyclicity conjecture in [San12], which motivated the investigation in this chapter.

Recall that, by the Cayley trick, triangulations of $\Delta_{n-1} \times \Delta_1$ are in bijection with subdivisions of the dilated segment $n\Delta_1$, which are specified by an ordering of $[n]$ along $n\Delta_1$ (see examples 2.2.9 and 2.3.9). Ardila and Ceballos observed that the restriction of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ to the prism faces $\{\Delta_{n-1} \times \mathbf{E} : \mathbf{E}$ edge in $\mathrm{skel}_1(\Delta_{d-1})\}$ could therefore be seen under the Cayley trick as a collection of permutations written along the edges of the dilated simplex $n\Delta_{d-1}$. They termed this a *system of permutations* on $\mathrm{skel}_1(n\Delta_{d-1})$. With this, the situation in figure 2.5b can be described as a *cyclic* system of permutations of $\{1, 2\}$ along the edges of $2\Delta_2$.

Definition 2.4.3 (Definition 5.5 in [AC13])**.** Let ρ be a system of permutations of $[n]$ along the edges of $n\Delta_{d-1}$. For a pair of indices $i, j \in [n]$ ($i \neq j$), define an oriented graph $G_{ij}(\rho)$ as follows: its vertices are the vertices of $n\Delta_{d-1}$, and there is an oriented edge $a \to b$ if the index i appears before the index j in the permutation of ρ along the *ordered edge* (a, b). The system of permutations ρ is said to be *acyclic* if, for every $i, j \in [n]$, the directed graph $G_{ij}(\rho)$ is acyclic.

Remark 2.4.4. In the definition of a system of permutations, we write ordered edge to emphasize that the permutations should be regarded as an ordered sequence of the numbers in $[n]$ written along an edge (a, b). That is, the permutation along the edge (b, a) is the reverse of the one along the edge (a, b). ○

Remark 2.4.5. Among other things, Ardila and Ceballos showed in [AC13] that an acyclic system of permutations on $\mathrm{skel}_1(n\Delta_{d-1})$ uniquely specifies the positions of the n unmixed simplices in $n\Delta_{d-1}$. However, this information does not suffice to uniquely determine the extended subdivision, provided one exists; see remark 2.4.13. The two cases are illustrated in figure 2.6: the two subdivisions

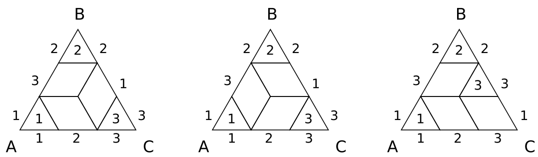


Figure 2.6: An acyclic system of permutations may or may not determine a unique subdivision.

in the left induce the same system of permutations, whereas the subdivision in the right is the unique extending the system of permutations. ○

Theorem 2.4.6 (Theorem 5.6 in [AC13]). *Let $\mathcal{M}$ be a fine mixed subdivision of $n\Delta_{d-1}$, and $\rho(\mathcal{M})$ be the associated system of permutations obtained by restriction of $\mathcal{M}$ to* $\mathrm{skel}_1(n\Delta_{d-1})$*. Then $\rho(\mathcal{M})$ is acyclic. If* $\min\{n, d\} \leq 3$*, then the converse is also true.*

Proof. Suppose indices $i, j \in [n]$ define a graph $G_{ij}(\rho)$ with a cycle around the vertices a, b, c of $n\Delta_{d-1}$. The restriction of ρ gives a triangulation of $\Delta_{\{i,j\}} \times \partial(\Delta_{\{a,b,c\}})$ that is not extendable. However, M represents a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ where, in particular, all its $\Delta_1 \times \Delta_2$ faces are properly triangulated, in contradiction to the assumption that the triangulation of $\Delta_{\{i,j\}} \times \partial(\Delta_{\{a,b,c\}})$ cannot be extended. The proof of the second statement is omitted (cf. [AC13, Theorem 4.2]). □

The conjecture Ardila and Ceballos proposed was the following.

Conjecture 2.4.7 (Acyclicity conjecture). *Let ρ be an acyclic system of permutations on* $\mathrm{skel}_1(n\Delta_{d-1})$*. Then there is a fine mixed subdivision $\mathcal{M}$ of $n\Delta_{n-1}$ such that ρ equals the restriction of $\mathcal{M}$ to* $\mathrm{skel}_1(n\Delta_{d-1})$*.*

Remark 2.4.8. In connection with the framework for triangulations of products of point configurations suggested in section 2.1, we can assert that conjecture 2.4.7 might have been fairly optimistic. Indeed, it only excludes the obstructions to extendability that arise at the level of $\Delta_{n-1} \times \mathrm{skel}_1(\Delta_{d-1})$ whereas, in general, higher-dimensional obstructions at the level of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ may also exist. One interpretation of the family of non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$ in theorem 2.4.32 ahead is precisely as an infinite family of obstructions for extendability of a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{n-1}(\Delta_{d-1})$. This sustains the bundle-theoretic intuition that a complete characterization of extendability of partial triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, for arbitrary n and d, cannot be achieved by a finite list of excluded obstructions. ○

Santos' counterexample to the acyclicity conjecture 2.4.7 in [San12] has as a main ingredient the construction of a subdivision $\mathcal{M}'$ of $\partial(3\Delta_3)$ (resp. triangulation of $\Delta_2 \times \partial(\Delta_3)$) that does not extend to a subdivision of $3\Delta_3$ (resp. triangulation of $\Delta_2 \times \Delta_3$). The partial subdivision $\mathcal{M}'$ is not uniquely determined by an acyclic system of permutations; however, this feature can be achieved by "embedding" $\mathcal{M}'$ in a non-extendable subdivision of $\partial(5\Delta_3)$ that is completely determined by a system of permutations on $5\Delta_3$. Thus, one obtains an acyclic system of permutations which does not correspond to any subdivision of $5\Delta_3$.

Instead of reproducing that counterexample here, we present a smaller counterexample –the smallest possible in view of theorem 2.4.6– which was constructed in collaboration with César Ceballos and Arnau Padrol in [CPS13]. Our counterexample also relies on a non-extendable triangulation of $\Delta_2 \times \partial(\Delta_3)$, which belongs to the family of non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$ that will be presented in generality later on in section 2.4.

The construction requires a result from [AC13] concerning the a notion of duality for subdivisions to systems of permutations.

Lemma 2.4.9. *Let ρ be an acyclic system of permutations on the edges of $n\Delta_{d-1}$. Then there is a unique acyclic system of permutations $\rho^\star$ on the edges of $d\Delta_{n-1}$* dual *to ρ. ρ and $\rho^\star$ are related by the rule: i lies before j along the (ordered) edge (a, b) of $n\Delta_{d-1}$ if and only if a lies before b along the (ordered) edge (i, j) of $d\Delta_{n-1}$.*

Proof. See [AC13, Section 6.2]. □

Lemma 2.4.10. *The subdivision $\mathcal{M}'$ of $\partial(3\Delta_3)$ in figure 2.7 does not extend to a subdivision of $3\Delta_3$.*

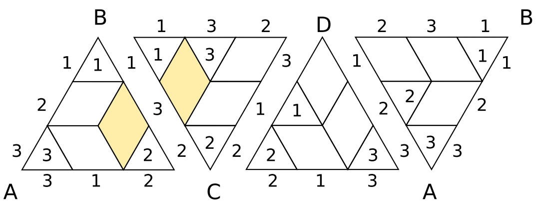
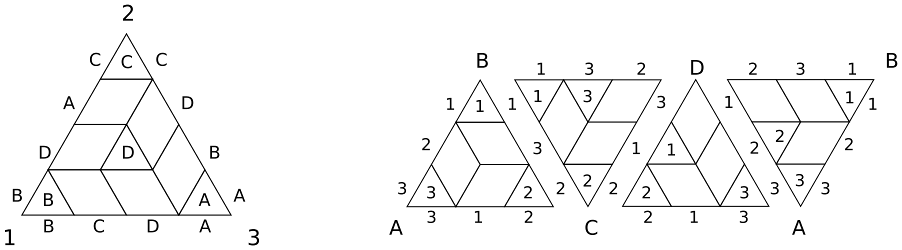


Figure 2.7: A non-extendable subdivision of $\partial(3\Delta_3)$

Proof. For the proof we will repeat Santos' argument in his construction in [San12, Proposition 3.1], postponing a more general idea to section 2.4. If $\mathcal{M}'$ extended, then the two rhombic cells colored yellow in figure 2.7 would be facets of a cubic cell with Minkowski sum decomposition $\mathbf{B} = CD + AB + BC$, whose dual cell has Minkowski sum decomposition $\mathbf{B}^\star = 2 + 23 + 13 + 1$ (a proof of this follows from lemma 2.3.5). Using lemma 2.4.9, we obtain the dual acyclic system of permutations $\rho(\mathcal{M}')^\star$ that uniquely determines the subdivision $\mathcal{M}^\star$ of $4\Delta_2$ displayed in figure 2.8a. However, there is not such a cell $\mathbf{B}^\star$ in $\mathcal{M}^\star$. We conclude



(a) A subdivision $\mathcal{M}^\star$ of $4\Delta_2$ determined by its system of permutations.

(b) Extendable subdivision of $\partial(3\Delta_3)$ induced by the subdivision dual to $\mathcal{M}^\star$.

Figure 2.8: Proof of lemma 2.4.10.

that $\mathcal{M}'$ cannot correspond to any subdivision of $3\Delta_3$, i.e., the corresponding triangulation of $\Delta_3 \times \partial(\Delta_3)$ does not extend. □

Remark 2.4.11*.* Note that the subdivision of $3\Delta_3$ dual to $\mathcal{M}^\star$ in figure 2.8a induces by restriction the subdivision of $\partial(3\Delta_3)$ depicted in figure 2.8b, which differs from the one in figure 2.7 only in the subdivision of the face spanned by the vertices $\{A, B, C\}$. We will again encounter this pattern in our family of non-extendable subdivisions of $\partial(n\Delta_n)$ in section 2.4. ○

Proposition 2.4.12. *The acyclic system of permutations* ρ *in figure 2.9 determines uniquely a non-extendable subdivision* $\mathcal{N}'$ *of* $\partial(4\Delta_3)$*. Thus, there is no subdivision of* $4\Delta_3$ *(that is, triangulation of* $\Delta_3 \times \Delta_3$*) that induces the acyclic system of permutations* ρ*.*

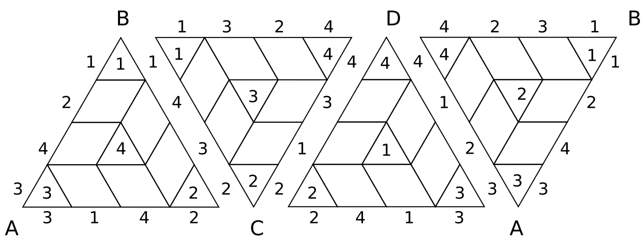


Figure 2.9: A smallest counterexample to conjecture 2.4.7.

Proof. If $\mathcal{N}'$ extended, so would the subdivisions of $\partial(3\Delta_3)$ gotten by eliminating the fourth Minkowski summand from the cells in $\mathcal{N}'$. This corresponds to restricting the triangulation of $\Delta_3 \times \partial(\Delta_3)$ to the faces in $\Delta_{\{1,2,3\}} \times \partial(\Delta_{\{A,B,C,D\}})$. However, this yields the non-extendable subdivision of $\partial(3\Delta_3)$ from lemma 2.4.10 (in fact, elimination of any summand gives a non-extendable subdivision of $3\Delta_3$ of the same sort as that of lemma 2.4.10), hence $\mathcal{N}'$ does not extend. □

Remark 2.4.13*.* That the systems of permutations in lemma 2.4.10 and proposition 2.4.12 completely specify a unique subdivision of a dilated simplex has been established by “eye”. It remains an open problem to characterize those systems of permutations on $n\Delta_{d-1}$ that uniquely determine a subdivision of $n\Delta_{d-1}$. ○

2.4.2 $n \le k < d$: necessary and sufficient conditions

After having presented the negative results for the cases $n = 2$, $d = 3$ in example 2.4.2 and for $n = 3$, $d = 4$ in lemma 2.4.10, we ask whether there are positive results at all:

Question 5: Are there conditions on n and d and k such that every triangulation of $\Delta_{n-1} \times \operatorname{skel}_{k-1}(\Delta_{d-1})$ extends?

This is a particularization of question 4 in section 2.1, and an affirmative answer would say that, under those conditions, every triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is gotten by appropriately gluing together triangulations of $\Delta_{n-1} \times \Delta_{k-1}$. In other words, an affirmative answer would assert that triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ cannot be “much more complicated” than triangulations of $\Delta_{n-1} \times \Delta_{k-1}$.

In this section, we present one of the most important results in the thesis, namely a precise affirmative answer to question 5. We prove that if $n < k < d$, **every** triangulation of $\Delta_{n-1} \times \operatorname{skel}_{k-1}(\Delta_{d-1})$ extends to a unique triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. This is derived as a corollary of the characterization of extendability of triangulations of $\Delta_{n-1} \times \partial(\Delta_{d-1})$ when $n < d-1$.

A second important result in this thesis is that the bound $n < k$ is tight, in the sense that if $k = n$, there are non-extendable triangulations of $\Delta_{n-1} \times \operatorname{skel}_{n-1}(\Delta_n)$. Although this general statement was to be expected in view of the situations for $n = 2$ in example 2.4.2 and for $n = 3$ in lemma 2.4.10, there was not an explicit construction of such non-extendable triangulations, for arbitrary n, in the literature.

Positive results

We start with two useful observations.

Proposition 2.4.14. *Let* $\mathbf{B}$ *be a cell in* $n\Delta_{d-1}$*, then* $1 \le |\operatorname{supp}(\vartheta(\mathbf{B}))| \le d-1$.

Proof. The weak integer composition $\mathbf{r}(\mathbf{B}) \in \Lambda_{n,d-1}$ of $d-1$ of length n has at least one non-zero entry (equal to $d-1$, when $\mathbf{B}$ is an unmixed simplex) and at most $d-1$ non-zero entries (all equal to one, when $\mathbf{B}$ is an $(d-1)$-cube). □

Lemma 2.4.15. *Suppose* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are different triangulations of* $\Delta_{n-1} \times \Delta_{d-1}$*, where* $n < d$*, and denote by* $\mathcal{T}_1'$ *and* $\mathcal{T}_2'$ *their restrictions to the faces in* $\Delta_{n-1} \times \operatorname{skel}_{n-1}(\Delta_{d-1})$*. Then* $\mathcal{T}_1' \neq \mathcal{T}_2'$.

Proof. Suppose on the contrary that $\mathcal{T}_1 \neq \mathcal{T}_2$ but $\mathcal{T}_1' = \mathcal{T}_2'$. There is at least one simplex $\boldsymbol{\sigma} \in \mathcal{T}_1$ which does not belong to $\mathcal{T}_2$. By lemma B.2.20, this means that there is one simplex $\boldsymbol{\sigma}' \in \mathcal{T}_2$ that overlaps with $\boldsymbol{\sigma}$ on a circuit $\mathbf{C}$; that is, such that $\mathbf{C}^+ \subseteq \boldsymbol{\sigma}$ and $\mathbf{C}^- \subseteq \boldsymbol{\sigma}'$. Since all the circuits of $\Delta_{n-1} \times \Delta_{d-1}$ are at most of type (n, n) (corollary 2.2.2), $\mathbf{C}$ is contained in a face $\mathbf{F}$ of $\Delta_{n-1} \times \mathrm{skel}_{n-1}(\Delta_{d-1})$. But then the restriction $\boldsymbol{\sigma}|_{\mathbf{F}}$ does not belong to $\mathbf{F}$, that is, $\boldsymbol{\sigma} \notin \mathcal{T}_2'$ and $\mathcal{T}_1' \neq \mathcal{T}_2'$: contradiction. □

Remark 2.4.16. Observe that lemma 2.4.15 fails if we instead consider the restriction of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ to $\Delta_{n-1} \times \mathrm{skel}_{n-2}(\Delta_{d-1})$. This can be checked by restricting the subdivisions of $3\Delta_2$ at the left and center in figure 2.6 to collections of subdivisions of $2\Delta_2$. By duality, for $n = 3$ this is merely the statement from remark 2.4.5 that a system of permutations on $3\Delta_2$ does not necessarily specify a subdivision of $3\Delta_2$. ⬡

Remark 2.4.17. Lemma 2.4.15 can also be easily proven in the language of mixed subdivisions using lemma 2.3.5. Indeed, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the subdivisions of $d\Delta_{n-1}$ corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ and $\mathcal{M}_1'$ and $\mathcal{M}_2'$ be collections of subdivisions of $n\Delta_{n-1}$ indexed by n-subsets of $[d]$ corresponding to $\mathcal{T}_1'$ and $\mathcal{T}_2'$. Assume that $\mathcal{M}_1' = \mathcal{M}_2'$ but $\mathcal{M}_1 \neq \mathcal{M}_2$; then, there will be a cell $\mathbf{B}_1 \in \mathcal{M}_1$ that is not present in $\mathcal{M}_2$. In accordance with proposition 2.3.6, there is a cell $\mathbf{B}_2 (\neq \mathbf{B}_1) \in \mathcal{M}_2$ with $\mathbf{q}(\mathbf{B}_2) = \mathbf{q}(\mathbf{B}_1)$, and by proposition 2.4.14, there is at least some $I \in \binom{[d]}{n}$ such that $\mathrm{supp}(\vartheta(\mathbf{B}_1)) \subset I$ and $\mathbf{B}_1|_I \neq \mathbf{B}_2|_I$ holds (cf. proposition B.3.8 for the definition of restriction of subdivisions). But we have $\mathbf{q}(\mathbf{B}_1|_I) = \mathbf{q}(\mathbf{B}_2|_I)$ (seen as compositions of $n-1$ of length n) whereas we assume $\mathcal{M}_1'|_I = \mathcal{M}_2'|_I$, in contradiction with lemma 2.3.5. ⬡

Corollary 2.4.18. *Let $\mathcal{T}'$ be a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$, where $k \geq n$. If $\mathcal{T}'$ extends to a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, then this triangulation is uniquely determined by $\mathcal{T}'$.*

Proof. This is almost a restatement of lemma 2.4.15, after realizing that $\mathcal{T}'$ induces by restriction a unique triangulation of $\Delta_{n-1} \times \mathrm{skel}_{n-1}(\Delta_{d-1})$. □

Remark 2.4.19. These results should be contrasted with the analogous fact that holds for triangulations of $\Delta_1 \times \Delta_{d-1}$. Here, there are $2^{\binom{d}{2}}$ ways to triangulate the faces in $\Delta_1 \times \mathrm{skel}_1(\Delta_{d-1})$. Of these, only the $d!$ triangulations that correspond to permutations of $[d]$ on $d\Delta_1$ (equivalently, to acyclic systems of permutations of $\{1, 2\}$ on the edges of $2\Delta_{d-1}$) extend to a triangulation of $\Delta_1 \times \Delta_{d-1}$, which they determine uniquely. Unfortunately, for $n > 2$ we lack such a succinct description of triangulations of $\Delta_{n-1} \times \mathrm{skel}_{n-1}(\Delta_{d-1})$ like permutations of $[d]$ for $n = 2$, which allow us to formulate the acyclicity condition. ⬡

Corollary 2.4.18 says that when $n < d$, there is exactly one collection of full-dimensional simplices of $\Delta_{n-1} \times \Delta_{d-1}$ compatible with a given triangulation of $\Delta_{n-1} \times \mathrm{skel}_{d-1}(\Delta_{n-1})$ which, in the extendable case, gives the extended triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. We show know how to actually build this unique collection of *candidate simplices* of $\Delta_{n-1} \times \Delta_{d-1}$. It turns out that It is possible to formulate a precise necessary and sufficient condition that establishes the extendability or not of $\mathcal{T}'$ in terms of these candidate simplices.

We consider that the description is most clear using the representation as fine mixed subdivisions. Recall from the beginning of the section that a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ can be identified with a collection of subdivisions of $k\Delta_{n-1}$:

$$\mathcal{M}' := \left\{ \mathcal{M}_I \text{ subdivision of } k\Delta_{n-1} : I \in \binom{[d]}{k} \right\}, \tag{2.2}$$

where the Minkowski summands of the cells in $\mathcal{M}_I$ are labeled by the indices in I, and any two subdivisions in $\mathcal{M}'$ agree on their restrictions to the intersection of their index sets (that may be empty). We will call such a collection a *consistent collection of subdivisions*.

Under the hypothesis $n \le k < d$, let us observe that proposition 2.4.14 implies that for every full-dimensional cell $\mathbf{B}$ in a subdivision $\mathcal{M}$ of $d\Delta_{n-1}$, there are at least $2 \le \binom{d-n+1}{k-n+1}$ subsets $I, J \in \binom{[d]}{k}$ such that $\mathbf{B}|_I$ and $\mathbf{B}|_J$ are full-dimensional cells in the restricted subdivisions $\mathcal{M}|_I$ and $\mathcal{M}|_J$.

On the same account, if $\mathcal{M}'$ is a consistent collection of subdivisions as (2.2) that extends, there are at least $2 \le \binom{d-n+1}{k-n+1}$ subsets $I, J \in \binom{[d]}{k}$ such that and $\mathbf{B} \in \mathcal{M}_I$ and $\mathbf{B}' \in \mathcal{M}_J$ are full-dimensional cells of $k\Delta_{n-1}$ and $\vartheta(\mathbf{B}) = \vartheta(\mathbf{B}')$. It follows by lemma 2.4.15 that $\mathbf{B}$ and $\mathbf{B}'$ must be restrictions of a unique cell $\widetilde{\mathbf{B}}$ in $d\Delta_{n-1}$.

Hence, we may define a *stem-equivalence relation* among the full-dimensional cells of $k\Delta_{n-1}$ in the subdivisions in $\mathcal{M}'$ by setting $\mathbf{B} \in \mathcal{M}_I \sim \mathbf{B}' \in \mathcal{M}_J$ whenever $\vartheta(\mathbf{B}) = \vartheta(\mathbf{B}')$. Furthermore, to every *stem-equivalence class* $\vartheta(\mathbf{B})$ corresponds the unique full-dimensional cell $\widetilde{\mathbf{B}}$ of $d\Delta_{n-1}$ such that $\vartheta(\widetilde{\mathbf{B}}) = \vartheta(\mathbf{B})$. The cell $\widetilde{\mathbf{B}}$ is obtained by "merging" the Minkowski sum decompositions of all full-dimensional cells $\mathbf{B}$ in $\mathcal{M}'$ with $\vartheta(\mathbf{B}) = \vartheta(\widetilde{\mathbf{B}})$ and is called the *candidate cell* associated to $\vartheta(\mathbf{B})$. By lemma 2.4.15 and the consistent nature of $\mathcal{M}'$, both the stem-equivalence relation and the corresponding candidate cell are well-defined. Moreover, they come in the correct amount.

Proposition 2.4.20. *Let $\mathcal{M}'$ be a collection of subdivisions as in* (2.2)*, where $n \le k < d$. There are $\binom{n+d-2}{n-1}$ stem-equivalence classes classifying the full-dimensional cells of $\mathcal{M}'$.*

Proof. Every stem-equivalence class (equivalently, every candidate cell) can be identified with a unique weak integer composition $\mathbf{q} \in \Lambda_{d,n-1}$ of $n-1$ of length d. By proposition 2.4.14, there are at least two $I, J \in \binom{[d]}{k}$ containing the support of $\mathbf{q}$ (that is, the labels of its positive entries). Therefore, ranging over the k-subsets of $[d]$ we obtain all the possible supports of the compositions in $\Lambda_{d,n-1}$. It follows that there are $\binom{n+d-2}{n-1}$ stem-equivalence classes. □

Remark 2.4.21. Observe that if we allow for $n-1 \le k < d$, we can still identify full-dimensional cells among the subdivisions in $\mathcal{M}'$ and get $\binom{n+d-2}{n-1}$ stem-equivalence classes. However, as already indicated in remark 2.4.16 it is not true that every stem-equivalence class defines a unique candidate cell; in detail, some 0-dimensional summands may not be recovered. ○

With all this, we specialize to the case $k = d-1$, that serves as an intermediate step for the case of general k, and for which the proofs of the following two results are notationally simpler.

Lemma 2.4.22. *Let $\mathcal{T}'$ be a triangulation of $\Delta_{n-1} \times \partial(\Delta_{d-1})$, where $d > n$, and denote by $\widetilde{\mathcal{T}}'$ the collection of candidate simplices obtained from $\mathcal{T}'$. Then $\mathcal{T}'$ extends to a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ (whose maximal simplices are the candidate simplices $\widetilde{\mathcal{T}}'$) if and only if the following condition is fulfilled by every candidate simplex $\widetilde{\boldsymbol{\sigma}} \in \widetilde{\mathcal{T}}'$:*

$$\text{for every } I \in \binom{[d]}{d-1},\ \widetilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_I} \text{ is a valid simplex in } \mathcal{T}'|_{\Delta_{n-1}\times\Delta_I} \qquad (\star)$$

Proof. We check that if the condition $(\star)$ is satisfied, then the collection of candidate simplices gives a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$; the converse is easily established, and the proof is omitted. To this end, we employ the characterization in theorem B.2.21 in the appendix.

That condition (**TV**) of theorem B.2.21 holds (the total volumes of the candidate simplices equals the volume of $\mathrm{conv}(\Delta_{n-1} \times \Delta_{d-1})$) was established in proposition 2.4.20: $\mathcal{T}'$ defines a total of $\binom{n+d-2}{n-1}$ candidate simplices.

To verify that condition $\circ_b$ of theorem B.2.21 is fulfilled, let $\widetilde{\boldsymbol{\tau}}$ be a facet of the candidate simplex $\widetilde{\boldsymbol{\sigma}} = \widetilde{\boldsymbol{\tau}} \cup \{\mathbf{v}\}$ that lies in the interior of $\Delta_{n-1} \times \Delta_{d-1}$. Denote by $\mathbf{B}(\widetilde{\boldsymbol{\sigma}})$ and $\mathbf{B}(\widetilde{\boldsymbol{\tau}})$ the cells of $d\Delta_{n-1}$ that correspond to $\widetilde{\boldsymbol{\sigma}}$ and $\widetilde{\boldsymbol{\tau}}$ by the Cayley trick, respectively. Observe first that $\mathbf{B}(\widetilde{\boldsymbol{\sigma}})$, and hence also $\mathbf{B}(\widetilde{\boldsymbol{\tau}})$, have at least two 0-dimensional Minkowski summands, in accordance with proposition 2.4.14 (since $|\mathrm{supp}(\vartheta(\mathbf{B}(\widetilde{\boldsymbol{\sigma}})))| < d-1$). Therefore, for any two maximal simplices $\widetilde{\boldsymbol{\sigma}}'$ and $\widetilde{\boldsymbol{\sigma}}''$ that share $\widetilde{\boldsymbol{\tau}}$ it holds that $\mathbf{B}(\widetilde{\boldsymbol{\sigma}}')$ and $\mathbf{B}(\widetilde{\boldsymbol{\sigma}}'')$ have at least one 0-dimensional summand in common.

In the bipartite graph representation, $\widetilde{\boldsymbol{\sigma}}$ is a spanning tree t of $K_{n,d}$ and $\widetilde{\boldsymbol{\tau}}$ is two disconnected trees of $K_{n,d}$. Let us denote these trees by t_1 and t_2 and the edge representing $\mathbf{v}$ by (i, a). Any full-dimensional simplex containing the facet $\widetilde{\boldsymbol{\tau}}$ must include (a vertex represented by) an edge that connects the trees t_1 and t_2.

By proposition 2.4.14, there are at least two degree-1 vertices in the part $[d]$ of $K_{n,d}$. Of these at most one can be part of an isolated edge in the forest formed by t_1 and t_2; denote by b a degree-1 vertex in $[d]$ that does not belong to an isolated edge.

Suppose then that there is no point $\mathbf{v}'$ such that $\widetilde{\boldsymbol{\tau}} \cup \{\mathbf{v}'\}$ is a full-dimensional simplex and $\mathbf{v}'$ lies in the same part as $\mathbf{v}$ of the unique circuit contained in $\widetilde{\boldsymbol{\sigma}} \cup \{\mathbf{v}'\}$. This translates into the assumption that there is no edge (i', a') in $K_{n,d}$ connecting t_1 and t_2 such that (i', a') is at an even distance from (i, a) in the unique cycle contained in the spanning graph formed by t and the edge (i', a'). In particular, the same statement holds when we delete the degree-1 vertex b from t. But then, this implies that the full-dimensional simplex $\widetilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_{[d]\setminus\{b\}}}$ is not a valid simplex in $\mathcal{T}'_{\Delta_{n-1}\times\Delta_{[d]\setminus\{b\}}}$, contrary to the hypothesis $(\star)$.

On the other hand, suppose that there are two points $\mathbf{v}'$ and $\mathbf{v}''$, such that $\tilde{\boldsymbol{\tau}} \cup \{\mathbf{v}'\}$ and $\tilde{\boldsymbol{\tau}} \cup \{\mathbf{v}''\}$ are full-dimensional simplices, $\mathbf{v}$ and $\mathbf{v}'$ lie on the same part of the unique circuit contained in $\tilde{\boldsymbol{\sigma}} \cup \{\mathbf{v}'\}$, and $\mathbf{v}$ and $\mathbf{v}''$ lie on the same part of the unique circuit contained in $\tilde{\boldsymbol{\sigma}} \cup \{\mathbf{v}''\}$. Represent $\mathbf{v}'$ and $\mathbf{v}''$ in $K_{n,d}$ by the edges (i', a') and (i'', a''). Our assumption implies that (i', a') (resp. (i'', a'')) lies at an even distance from (i, a) in the unique cycle contained in $t \cup (i', a')$ (resp. $t \cup (i'', a'')$). But then, in the unique cycle contained in the graph g formed by $t_1 \cup t_2 \cup (i', a') \cup (i'', a'')$, (i', a') is at an odd distance of (i'', a''), and this still holds when deleting from the graph g a degree-1 vertex b in the part $[d]$ of $K_{n,d}$ (since $d > n$, there is at least one such b). This implies that the full-dimensional simplices $(\tilde{\boldsymbol{\tau}} \cup \mathbf{v}')|_{\Delta_{n-1} \times \Delta_{[d] \setminus \{b\}}}$ and $(\tilde{\boldsymbol{\tau}} \cup \mathbf{v}'')|_{\Delta_{n-1} \times \Delta_{[d] \setminus \{b\}}}$ overlap on a circuit of $\Delta_{n-1} \times \Delta_{[d] \setminus \{b\}}$, contrary to the fact that $\mathcal{T}'|_{\Delta_{n-1} \times \Delta_{[d] \setminus \{b\}}}$ is a “honest triangulations". □

Lemma 2.4.22 is the closest we get to completely characterizing extendability of a triangulation of $\Delta_{n-1} \times \partial(\Delta_{d-1})$. With it we prove the following lemma, which says that the conditions in lemma 2.4.22 are always satisfied whenever $n < d - 1$.

Lemma 2.4.23. *Let $\mathcal{T}'$ be a triangulation of $\Delta_{n-1} \times \partial(\Delta_{d-1})$, where $d > 2$ and $n < d - 1$. Then $\mathcal{T}'$ extends to a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ (that by corollary 2.4.18 and the subsequent discussion is completely determined by $\mathcal{T}'$).*

The proof of lemma 2.4.23 uses the following auxiliary result.

Lemma 2.4.24.

1. *Let $\mathcal{T}$ be a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, where $n > 1$ and $d > 2$, and let $\mathbf{F}$ be the facet $\Delta_{n-1} \times \Delta_{[d-1]}$ of $\Delta_{n-1} \times \Delta_{d-1}$. Suppose $\boldsymbol{\tau}, \boldsymbol{\tau}' \in \mathcal{T}|_{\mathbf{F}}$ are two adjacent $(n+d-3)$-simplices (that is, maximal in $\Delta_{n-1} \times \Delta_{[d-1]}$ but codimension-1 in $\Delta_{n-1} \times \Delta_{d-1}$) such that $\boldsymbol{\sigma} := \boldsymbol{\tau} \cup \{\mathbf{v}\}$ and $\boldsymbol{\sigma}' := \boldsymbol{\tau}' \cup \{\mathbf{v}'\}$ are full-dimensional simplices of $\mathcal{T}$. Then either $\mathbf{v} = \mathbf{v}'$, or the simplex $\boldsymbol{\sigma}'' := \boldsymbol{\tau} \cap \boldsymbol{\tau}' \cup \{\mathbf{v}, \mathbf{v}'\}$ is a full-dimensional simplex of $\mathcal{T}$.*

2. *Let $\mathcal{T}'$ be a triangulation of $\Delta_{n-1} \times \partial(\Delta_{d-1})$, where $n < d - 1$, and let $\mathbf{F}$ be the facet $\Delta_{n-1} \times \Delta_{[d-1]}$ of $\Delta_{n-1} \times \partial(\Delta_{d-1})$. Suppose $\boldsymbol{\tau}, \boldsymbol{\tau}' \in \mathcal{T}'|_{\mathbf{F}}$ are adjacent $(n+d-3)$-simplices, where $\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} \cup \{\mathbf{v}\}$ and $\tilde{\boldsymbol{\tau}}' = \boldsymbol{\tau}' \cup \{\mathbf{v}'\}$ are the respective candidate simplices. Then either $\mathbf{v} = \mathbf{v}'$ or there is a candidate simplex $\tilde{\boldsymbol{\tau}}'' = \boldsymbol{\tau} \cap \boldsymbol{\tau}' \cup \{\mathbf{v}, \mathbf{v}'\}$.*

Proof. *1*: Assume on the contrary that $\mathbf{v} \neq \mathbf{v}'$ but $\boldsymbol{\sigma}'' = \boldsymbol{\tau} \cap \boldsymbol{\tau}' \cup \{\mathbf{v}, \mathbf{v}'\}$ is not a valid full-dimensional simplex of $\mathcal{T}$. Then there is circuit $\mathbf{C} = \{\mathbf{C}^+, \mathbf{C}^-\}$ of $\Delta_{n-1} \times \Delta_{d-1}$ such that $\mathbf{C}^+ \subset \boldsymbol{\sigma}''$ and $\mathbf{C}^- \subset \boldsymbol{\rho}$ for some $\boldsymbol{\rho} \in \mathcal{T}$.

Observe first that $\mathbf{C}^+$ must contain at least one of $\mathbf{v}$ or $\mathbf{v}'$, otherwise $\mathbf{C}^+ \subset \boldsymbol{\tau} \cap \boldsymbol{\tau}'$, contradicting the hypothesis that $\boldsymbol{\tau} \cap \boldsymbol{\tau}'$ is a legal simplex in $\mathcal{T}$.

On the other hand, since $\mathbf{v}, \mathbf{v}' \notin \mathbf{F}$ (that is, $\mathbf{v}, \mathbf{v}' \in \Delta_{n-1} \times \Delta_{\{d\}}$), in the bipartite graph representation $\mathbf{v}$ and $\mathbf{v}'$ correspond to the only two edges of $\boldsymbol{\sigma}''$ adjacent to the vertex d of $K_{n,d}$ (in terms of the subdivision of $d\Delta_{n-1}$, $\mathbf{v}$ and $\mathbf{v}'$ form the

d-th summand of the cell representing $\boldsymbol{\sigma}''$, which is then 1-dimensional). In particular, $\mathbf{v}$ and $\mathbf{v}'$ lie within an odd distance of each other (they are adjacent), so that either $\mathbf{v} \in \mathbf{C}^+$ or $\mathbf{v}' \in \mathbf{C}^+$ but not both. The former case implies that $\boldsymbol{\sigma} \notin \mathcal{T}$, whereas the latter that $\boldsymbol{\sigma}' \notin \mathcal{T}$, either way yielding a contradiction.

2: Assume again that $\mathbf{v} \neq \mathbf{v}'$ while $\tilde{\boldsymbol{\tau}}''$ is not a candidate simplex in $\tilde{\mathcal{T}}'$; we will prove that this implies that one of $\tilde{\boldsymbol{\tau}}$ or $\tilde{\boldsymbol{\tau}}'$ is not a candidate simplex either, leading to a contradiction.

We handle the notation using the language of subdivisions. Denote by $\mathcal{M}'$ the consistent collection of subdivisions:

$$\mathcal{M}' := \left\{ \mathcal{M}_I \text{ subdivision of } (d-1)\Delta_{n-1} : I \in \binom{[d]}{d-1} \right\}$$

corresponding to $\mathcal{T}'$ by the Cayley trick. Here, $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are represented by the full-dimensional cells $\mathbf{B}$ and $\mathbf{B}'$ in the subdivision $\mathcal{M}_{[d-1]} \in \mathcal{M}'$, respectively. Let $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}'$ and $\mathbf{B}''$ be the cells of $d\Delta_{n-1}$ corresponding to $\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\tau}}'$, and $\tilde{\boldsymbol{\tau}}''$, respectively.

By proposition 2.4.14, $\mathbf{B}$ and $\mathbf{B}'$ have at least $2 \le d-1-(n-1)$ Minkowski summands of dimension 0, of which at least 1 is common to both $\mathbf{B}$ and $\mathbf{B}'$. Denote the label of that summand by $j \in [d-1]$, and consider the subdivision $\mathcal{M}_{[d-1]}|_{[d-1]\setminus\{j\}}$ of $(d-2)\Delta_{n-1}$, which we regard as the restriction $\mathcal{M}_{[d]\setminus\{j\}}|_{[d]\setminus\{j,d\}}$. We have that $\tilde{\mathbf{B}}_{[d]\setminus\{j,d\}}$ and $\tilde{\mathbf{B}}'_{[d]\setminus\{j,d\}}$ represent adjacent codimension-2 simplices on the facet $\mathbf{F}' = \Delta_{n-1} \times \Delta_{[d]\setminus\{j,d\}}$, whereas $\mathbf{v} = \tilde{\mathbf{B}}|_{\{d\}} \neq \tilde{\mathbf{B}}'|_{\{d\}} = \mathbf{v}'$. This is precisely the situation of statement *1* of the lemma, so there is a cell $\mathbf{B}''' \in \mathcal{M}_{[d]\setminus\{j\}}$ equal to $\tilde{\mathbf{B}}''|_{[d]\setminus\{j\}}$, whose candidate cell we assume to satisfy $\tilde{\mathbf{B}}'''|_{\{j\}} \neq \tilde{\mathbf{B}}''|_{\{j\}}$. Thus, $\tilde{\mathbf{B}}'''$ and $\tilde{\mathbf{B}}''$ differ only in their 0-dimensional j-th summand.

Lemma 2.3.5 says that $\tilde{\mathbf{B}}''$ and $\tilde{\mathbf{B}}'''$ overlap on a circuit of $\Delta_{n-1} \times \Delta_{d-1}$. But this implies that one of $\tilde{\mathbf{B}}$ or $\tilde{\mathbf{B}}'$ also overlaps with $\tilde{\mathbf{B}}'''$ on the same circuit, giving the desired contradiction.

□

Proof of Lemma 2.4.23. Let $\tilde{\boldsymbol{\sigma}}$ be a candidate simplex in $\tilde{\mathcal{T}}$, and let $I = [d-1] \in \binom{[d]}{d-1}$. Using induction on $i := \dim\left(\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_{\{d\}}}\right)$, we prove that $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_I}$ is valid simplex in $\mathcal{T}'|_{\Delta_{n-1}\times\Delta_I}$.

The case $i = 0$ follows from lemma 2.4.15. Now assume that $\dim\left(\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_{\{d\}}}\right) = i > 0$, and let $J \in \binom{[d]}{d-1}$ be such that $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_J}$ is maximal (i.e., of codimension 1 in $\Delta_{n-1}\times\Delta_{d-1}$). There is at least one maximal simplex $\boldsymbol{\sigma}' \in \mathcal{T}'|_{\Delta_{n-1}\times\Delta_J}$ adjacent to $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_J}$ with $\dim(\boldsymbol{\sigma}'|_{\Delta_{n-1}\times\Delta_{\{d\}}}) < i$. Write $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_J} \cup \{\mathbf{v}\}$ and $\tilde{\boldsymbol{\sigma}}' = \boldsymbol{\sigma}' \cup \{\mathbf{v}'\}$, where $\tilde{\boldsymbol{\sigma}}'$ is the candidate simplex defined by $\boldsymbol{\sigma}'$

By lemma 2.4.24 *2*, either $\mathbf{v} = \mathbf{v}'$ so that $\tilde{\boldsymbol{\sigma}}$ is adjacent to $\tilde{\boldsymbol{\sigma}}'$ in $\Delta_{n-1}\times\Delta_{d-1}$, or there is some candidate simplex $\tilde{\boldsymbol{\sigma}}'' = \tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_J} \cap \boldsymbol{\sigma}' \cup \{\mathbf{v}, \mathbf{v}'\}$. We have $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_I} \subset \tilde{\boldsymbol{\sigma}}'|_{\Delta_{n-1}\times\Delta_I}$ (resp. $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_I} \subset \tilde{\boldsymbol{\sigma}}''|_{\Delta_{n-1}\times\Delta_I}$) together with $\dim(\tilde{\boldsymbol{\sigma}}'|_{\Delta_{n-1}\times\Delta_{\{d\}}}) < i$ (resp. $\dim(\tilde{\boldsymbol{\sigma}}''|_{\Delta_{n-1}\times\Delta_{\{d\}}}) < i$, because $\tilde{\boldsymbol{\sigma}}''|_{\Delta_{n-1}\times\Delta_{\{d\}}} = \tilde{\boldsymbol{\sigma}}'|_{\Delta_{n-1}\times\Delta_{\{d\}}} \subset \tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_{\{d\}}}$).

It then follows from the induction hypothesis that $\tilde{\boldsymbol{\sigma}}'|_{\Delta_{n-1}\times\Delta_I} \in \mathcal{T}'|_{\Delta_{n-1}\times\Delta_I}$ (resp. $\tilde{\boldsymbol{\sigma}}''|_{\Delta_{n-1}\times\Delta_I} \in \mathcal{T}'|_{\Delta_{n-1}\times\Delta_I}$), and therefore $\tilde{\boldsymbol{\sigma}}|_{\Delta_{n-1}\times\Delta_I}$ is a valid simplex in $\mathcal{T}'|_{\Delta_{n-1}\times\Delta_I}$.

$\square$

We now use lemma 2.4.23 in an inductive step to generalize the result to any k such that $n < k < d$.

Theorem 2.4.25. *Let $\mathcal{T}'$ be a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$, where $n, d, k > 2$ satisfy $n < k < d$. Then $\mathcal{T}'$ extends to a unique triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.*

Proof. We prove this by induction on $l = d - k$, where we consider d fixed; lemma 2.4.23 proves the assertion for $l = 1$.

Assume $\mathcal{M}'$ is a collection of subdivisions of $k\Delta_{n-1}$ corresponding to a triangulation $\mathcal{T}'$ of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$, where $d - k = l > 1$ (and $k > n$). Let $J \in \binom{[d]}{d-1}$. Since $\mathcal{M}'$ is a consistent collection of subdivisions, those subdivisions $\mathcal{M}'(J) := \{\mathcal{M}_I \in \mathcal{M}' : I \in \binom{[d]}{k},\ I \subset J\}$ represent a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_J)$, which corresponds to the case $(d-1) - k = l - 1$. With the induction hypothesis, we can extend the subdivisions in the collection $\mathcal{M}'(J)$ to a unique subdivision of $(d-1)\Delta_{n-1}$, where the summands are indexed by the elements of J; this corresponds to a triangulation of $\Delta_{n-1} \times \Delta_J$. Repeating the same procedure for every subset in $\binom{[d]}{d-1}$, we end up with a collection of subdivisions corresponding to a triangulation of $\Delta_{n-1} \times \partial(\Delta_{d-1})$, which extends to a unique subdivision of $d\Delta_{n-1}$ by lemma 2.4.23. Observe that lemmas 2.3.5 and 2.4.15 guarantee that this inductive procedure is consistent throughout, since the candidate simplices of the extension of $\mathcal{T}'$ are already determined by $\mathcal{T}'$. $\square$

Remark 2.4.26. From lemma 2.4.22 we immediately see that $k \geq 2d - 1$ is enough to prove (unique) the extendability of all triangulations of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$. In that case, in every face $\mathbf{F} = \Delta_{n-1} \times \Delta_{k-1}$ there is enough "room" to establish that condition $(\star)$ holds. Theorem 2.4.25 therefore constitutes a good improvement of this naive bound. $\circ$

Negative results

As remarked at the beginning of the section, the triangulations in example 2.4.2 and lemma 2.4.10 indicate that it is not to be expected that a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ always be extendable when $k = n$, so that the lower bound $k > n$ guaranteeing extendability seems to be tight. We now prove that this is indeed the case by explicitly constructing for every $n \geq 2$ a family of non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$ that generalize the known examples for $n = 2$ and $n = 3$.

The idea underlying our construction is that when a triangulation of a point configuration has a flip supported on a full-dimensional circuit, that is, when it admits a flip of codimension zero, its flipping only alters those simplices in the triangulation of that circuit, leaving the rest of the triangulation intact (see remark B.2.24). For example, the subdivisions of $\partial(3\Delta_3)$ in figures 2.7 and 2.8b differ by only one flip supported on a 4-dimensional circuit in one of the facets, which can be performed without affecting the subdivisions in the remaining facets, thus rendering the partial subdivision in figure 2.8b non-extendable.

Our first aim is then to produce a triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ that admits a codimension-zero flip. While this is in principle easy (for instance, by adequately perturbing a weight vector that induces a desired triangulation of a circuit of type (n,n) of $\Delta_{n-1} \times \Delta_{n-1}$), a deterministic construction was not previously available in the triangulations literature, to the best of our knowledge. Somewhat expectedly, our construction is described most naturally using the grid representation for cartesian products of simplices.

A *Dyck path* in an $n \times n$ grid $\mathcal{G}_{n\times n}$ [1] is a path from the square $(1,A)$ to the square $(n, d=n)$ consisting only of squares (i,α) such that $i \geq \alpha$ (since $n = d$, we identify the label sets of the columns and rows of $\mathcal{G}_{n\times n}$; in this sense is $i \geq \alpha$ to be understood). That is to say, a Dyck path is a staircase from $(1,A)$ to $(n, d=n)$ which does not pass above the $i = \alpha$ diagonal of $\mathcal{G}_{n\times n}$.

By theorem 2.2.8, each Dyck path represents a $(2n-2)$-simplex in $\Delta_{n-1} \times \Delta_{n-1}$, and every pair of such simplices intersect properly. Let $\mathfrak{D}_n^{(n)}$ stand for the collection of simplices obtained from all the Dyck paths, which gives a triangulation of the subconfiguration $\mathfrak{C}_n^{(n)} := \{(\mathbf{e}_i, \mathbf{e}_\alpha) \in \Delta_{n-1} \times \Delta_{n-1} : i \geq \alpha\}$. It is a well-known fact that the number of Dyck paths in an $n\times n$ grid equals the $(n-1)$-*th Catalan number* $C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ (see e.g. [Sta11, Chapter 1]), so that $\mathfrak{D}_n^{(n)}$ consists of C_{n-1} full-dimensional simplices. We can “replicate” this triangulation of $\mathfrak{C}_n^{(n)}$ in order to obtain a full triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ as follows.

Let φ be the map acting on the index sets $[n]$ and $[d=n]$ of the points in the two factors of $\Delta_{n-1} \times \Delta_{n-1}$ according to $i \mapsto i+1 \bmod n$ and $\alpha \mapsto \alpha + 1 \bmod n$. Define the collection of $(2n-2)$-simplices:

$$\mathfrak{D}_n^{(\ell)} := \left\{\varphi_\times^\ell(\boldsymbol{\sigma}) \subset \Delta_{n-1} \times \Delta_{n-1} : \boldsymbol{\sigma} \text{ is a Dyck path in } \mathcal{G}_{n\times n}\right\},$$

where $\varphi_\times^\ell$ is the ℓ-fold iteration of $\varphi_\times$: the (“direct sum”) extension of the action of φ to the points in $\Delta_{n-1} \times \Delta_{n-1}$ according to $(\mathbf{e}_i, \mathbf{e}_\alpha) \overset{\varphi_\times}{\mapsto} \mathbf{e}_{\varphi(i)}, \mathbf{e}_{\varphi(\alpha)})$. The collection $\mathfrak{D}_n^{(\ell)}$ gives a triangulation of the subconfiguration $\mathfrak{C}_n^{(\ell)} := \{(\varphi_\times^\ell(\mathbf{e}_i, \mathbf{e}_\alpha) : i \geq \alpha\}$. Considering all possible iterations of $\varphi_\times$, we end up with the following object.

Theorem 2.4.27. *The collections of* $(2n-2)$*-simplices:*

$$\boldsymbol{\mathfrak{D}}_n := \left\{\mathfrak{D}_n^{(\ell)} : 1 \leq \ell \leq n\right\},$$

constitute a triangulation that we call the Dyck path triangulation *of* $\Delta_{n-1} \times \Delta_{n-1}$.

Our proof uses the next simple but useful lemma.

[1] This slightly differs from the usual definition of a Dyck path, that uses the grid points in $\mathcal{G}_{n\times n}$ rather than the squares in $\mathcal{G}_{n\times n}$, and require $i \leq \alpha$. Hence the number of Dyck paths of size n in our convention is C_{n-1} rather than C_n.

Lemma 2.4.28. *Let $\mathcal{G}_{n\times n}$ be an $n \times n$ square grid, and let $s = (j, b)$ be a square in $\mathcal{G}_{n\times n}$. Then $(j, b) = (\varphi^{\ell}(i), \varphi^{\ell}(a))$ for $i \geq a$ if and only if:*

$$\underbrace{\begin{Bmatrix} j \geq b \text{ and} \\ b > \ell \end{Bmatrix}}_{I} \text{ or } \underbrace{\begin{Bmatrix} j \geq b \text{ and} \\ j \leq \ell \end{Bmatrix}}_{II} \text{ or } \underbrace{\begin{Bmatrix} j \leq \ell \text{ and} \\ b > \ell \end{Bmatrix}}_{III}. \tag{2.3}$$

(If $\ell = n$, we may still consider the inequalities using $\ell \bmod n$, that is, $\ell = 0$). In particular, $(\mathbf{e}_j, \mathbf{e}_b)$ is a point in $\mathfrak{C}_n^{(n)} \cap \mathfrak{C}_n^{(\ell)}$ if and only if $j \geq b$ and $\{j \leq \ell$ or $b > \ell\}$.

Proof. Follows by noting that the action of φ on each point in $\mathcal{G}_{n\times n}$ translates it parallel to the diagonal $i = a$ in direction of increasing i and a (modulo n). In figure 2.10, the grey area represents points in the image of $\mathfrak{C}_n^{(n)}$ under $\varphi_\times^{\ell}$. □

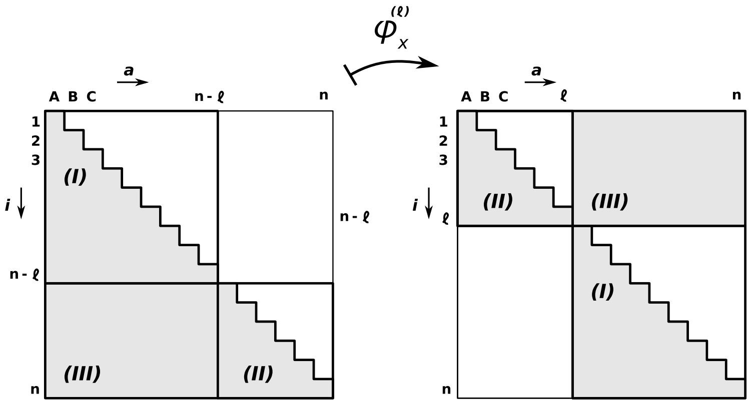


Figure 2.10: The squares in the grey region represent points in $\mathfrak{C}_n^{(\ell)}$.

Proof of theorem 2.4.27. For the proof we will use the characterization of triangulations in theorem B.2.21. The verification that the volume of all the simplices in the Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ equals the volume of $\mathrm{conv}(\Delta_{n-1} \times \Delta_{n-1})$ is by construction easy: there are $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$ full-dimensional simplices in each collection $\mathfrak{D}_n^{(\ell)}$, and there are n such collections, giving a total of $\binom{2(n-1)}{n-1}$ full-dimensional simplices. By proposition 2.2.4, their total volume equals the volume of $\Delta_{n-1} \times \Delta_{n-1}$.

To verify the property $\circ_b$, let $\boldsymbol{\tau}$ be a codimension-1 simplex $\mathfrak{D}_n^{(n)}$ (clearly, the restriction does not cause loss of generality). If $\boldsymbol{\tau}$ is gotten by removing a (point corresponding to a) square that is the only square in its row or in its column, then $\boldsymbol{\tau}$ lies on a facet of $\Delta_{n-1} \times \Delta_{n-1}$. If $\boldsymbol{\tau}$ is gotten by removing a square $\mathbf{s} = (j, b)$ lying on a corner of a Dyck path, then $\boldsymbol{\tau}$ contains squares in all the rows and columns of $\mathcal{G}_{n\times n}$. We distinguish two cases:

- Either $j - b > 1$ and $\boldsymbol{\tau}$ contains the squares $(j-1, b)$ and $(j, b+1)$, or $j - b \geq 0$ and $\boldsymbol{\tau}$ contains the squares $(j+1, b)$ and $(j, b-1)$.

- $j - b = 1$ and $\boldsymbol{\tau}$ contains the squares $(j-1, b)$ and $(j, b+1)$.

In the first case, either $\mathbf{s}' = (j-1, b+1)$ or $\mathbf{s}' = (j+1, b-1)$ is the only square in $\mathcal{G}_{n\times n}$ such that $\boldsymbol{\tau} \cup \mathbf{s}'$ is a simplex in $\mathfrak{D}_n$, which is, in fact, a Dyck path in $\mathfrak{C}_n^{(n)}$.

In the second case, all the squares $(i, a) \in \boldsymbol{\tau}$ satisfy $i \geq a$ and $\{i \leq b$ or $a > b\}$, but we saw in lemma 2.4.28 that these are precisely the conditions for $\boldsymbol{\tau}$ to lie inside $\mathfrak{C}_n^{(n)} \cap \mathfrak{C}_n^{(\ell)}$, with $\ell = b$. Now, the unique square $\mathbf{s}'$ that makes $\mathbf{s}' \cup \boldsymbol{\tau} = \varphi_\times^\ell(\boldsymbol{\sigma})$ for some (simplex represented by a) Dyck path $\boldsymbol{\sigma} \in \mathfrak{D}_n^{(n)}$ is $\mathbf{s}' = (1, n)$.

To conclude, we check for the second case that $\mathbf{s}$ and $\mathbf{s}'$ have the same sign in the unique circuit $\mathbf{C}$ contained in $\boldsymbol{\tau} \cup \mathbf{s} \cup \mathbf{s}'$ (the first case being equivalent to the situation for the staircase triangulation, where property $\circ_b$ of theorem B.2.21 holds). Indeed, suppose $\mathbf{s} \in \mathbf{C}^+$ and $\mathbf{s}' \in \mathbf{C}^-$; then, since the circuit has to be "closed" using $\mathbf{s}'$ (cf. remark 2.2.6), for all squares $\mathbf{z} = (i, a) \in \mathbf{C}^+$ with $a > b (= \ell)$, there is a square $\mathbf{z}' = (i', a) \in \mathbf{C}^-$ with $i' > i$, so that $\mathbf{z}$ can only be "reached from below". By the same token, every square $\mathbf{y} = (i, a)$ with $i \leq b (= \ell)$ can only be "reached from the left". Therefore, if $\mathbf{s} \in \mathbf{C}^+$, it is not possible to close the circuit $\mathbf{C}$. This reasoning is depicted in figure 2.11, where we draw squares in $\mathbf{C}^-$ red and squares in $\mathbf{C}^+$ blue. $\square$

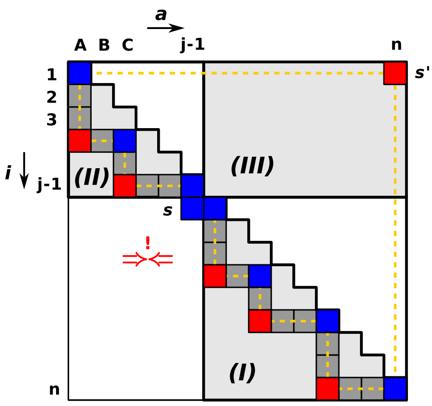


Figure 2.11: Illustration of the proof of theorem 2.4.27.

The Dyck path triangulation of $\Delta_3 \times \Delta_3$ is displayed in figure 2.12 below as a mixed subdivision of $4\Delta_3$. The reader may have noticed already that the subdivision of $3\Delta_2$ resulting after deleting the summand D from the subdivision of $4\Delta_2$ in figure 2.8a represents the Dyck path triangulation for $n = 3$.

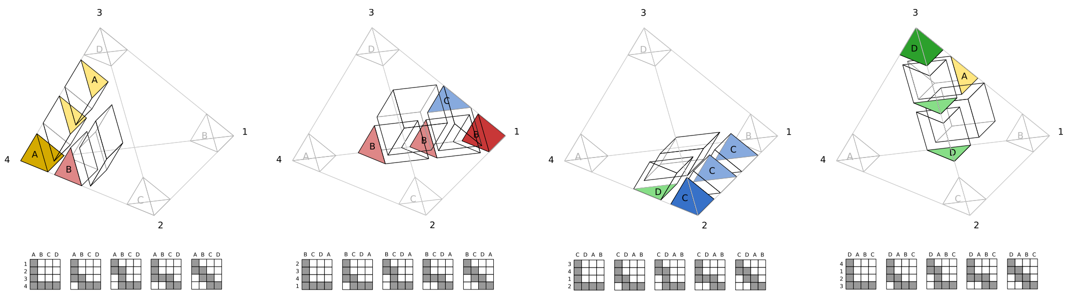


Figure 2.12: The Dyck path triangulation of $\Delta_3 \times \Delta_3$ drawn as a subdivision of $4\Delta_3$.

Remark 2.4.29. The Dyck path triangulation is a natural refinement of a coarse polyhedral subdivision that was presented (with slightly different conventions) by Gelfand, Kapranov and Zelevinsky in [GKZ08, Chapter 7, example 3.14]. Each polyhedral cell in their subdivision is gotten as the convex hull of the $\mathfrak{C}_n^{(\ell)}$. Although it was remarked in [GKZ08, Chapter 7, example 3.14] that the normalized volume of every such cell equals the $(n-1)$-th Catalan number C_{n-1}, we have not found explicit mention of the Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ in the literature. ○

The Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ has the property that it admits a geometric bistellar flip supported on the (n,n) circuit:

$$\mathbf{Z} = (\mathbf{Z}^+, \mathbf{Z}^-) := (\{1A, 2B, \ldots, nn\}, \{2A, 3B, \ldots, n(n-1), 1n\}).$$

The reader can verify that the "right-most" Dyck path:

$$\{1A, 2A, 2B, 3B, 3C, \ldots, n(n-1), nn\},$$

in $\mathfrak{D}_n^{(n)}$ together with its images under $\varphi_\times^\ell$ for $1 \le \ell \le n-1$ comprise the triangulation $\mathcal{T}^- = \{\mathbf{Z} \backslash \{\mathbf{v}\} : \mathbf{v} \in \mathbf{Z}^-\}$ of $\mathbf{Z}$. Due to the fact that $\mathbf{Z}$ is a full-dimensional circuit, the common link in $\mathfrak{D}_n$ of the simplices in $\mathcal{T}^-$ is empty, so $\mathcal{T}^-$ can be flipped to $\mathcal{T}^+$ without altering the remaining simplices in $\mathfrak{D}_n$.

We will now present a triangulation $\mathcal{T}'_{n\uparrow}$ of $\Delta_{n-1} \times \partial(\Delta_n)$ in which the face $\mathbf{F} = \Delta_{n-1} \times \Delta_{[n]}$ has the triangulation $\mathfrak{D}_n$. The freedom to flip the circuit $\mathbf{Z}$ in the triangulation of $\mathbf{F}$ will then be used to produce a non-extendable triangulation $\mathcal{T}'_{n\downarrow}$ of $\Delta_{n-1} \times \partial(\Delta_n)$. The triangulation $\mathcal{T}'_{n\uparrow}$ is gotten from the restriction of a natural extension of the Dyck path triangulation to a triangulation of $\Delta_{n-1} \times \Delta_n$ whose construction, that relies on the $\varphi_\times$-action, is explained next.

For a point $\mathbf{v} = (\mathbf{e}_i, \mathbf{e}_\alpha) \in \Delta_{n-1} \times \Delta_{n-1}$, let $\mathbf{v}^\varphi$ be the following collection of points of $\Delta_{n-1} \times \Delta_n$:

$$\mathbf{v}^\varphi := \left\{(\mathbf{e}_\ell, \mathbf{e}_{n+1}) \in \Delta_{n-1} \times \Delta_n : \mathbf{v} = \varphi_\times^\ell(\mathbf{e}_j, \mathbf{e}_b) \text{ with } j \ge b\right\}.$$

The indices of the points in $\mathbf{v}^\varphi$ register for which values of ℓ does $\mathbf{v} = (\mathbf{e}_i, \mathbf{e}_\alpha)$ lie in the image of a Dyck path in $\mathcal{G}_{n\times n}$ under $\varphi_\times^\ell$.

Now, associate to every simplex $\boldsymbol{\sigma} \in \mathfrak{D}_n$ the simplex of $\Delta_{n-1} \times \Delta_n$:

$$\boldsymbol{\sigma}^\varphi := \boldsymbol{\sigma} \cup \bigcap_{\mathbf{v} \in \boldsymbol{\sigma}} \mathbf{v}^\varphi.$$

It is a remarkable fact that the resulting collection of simplices gives a triangulation of $\Delta_{n-1} \times \Delta_n$.

Theorem 2.4.30. *The collection of simplices:*

$$\mathfrak{D}_n^\varphi := \{\boldsymbol{\sigma}^\varphi : \boldsymbol{\sigma} \in \mathfrak{D}_n\},$$

yields a triangulation of $\Delta_{n-1} \times \Delta_n$*, which we call the* φ-extension of $\mathfrak{D}_n$.

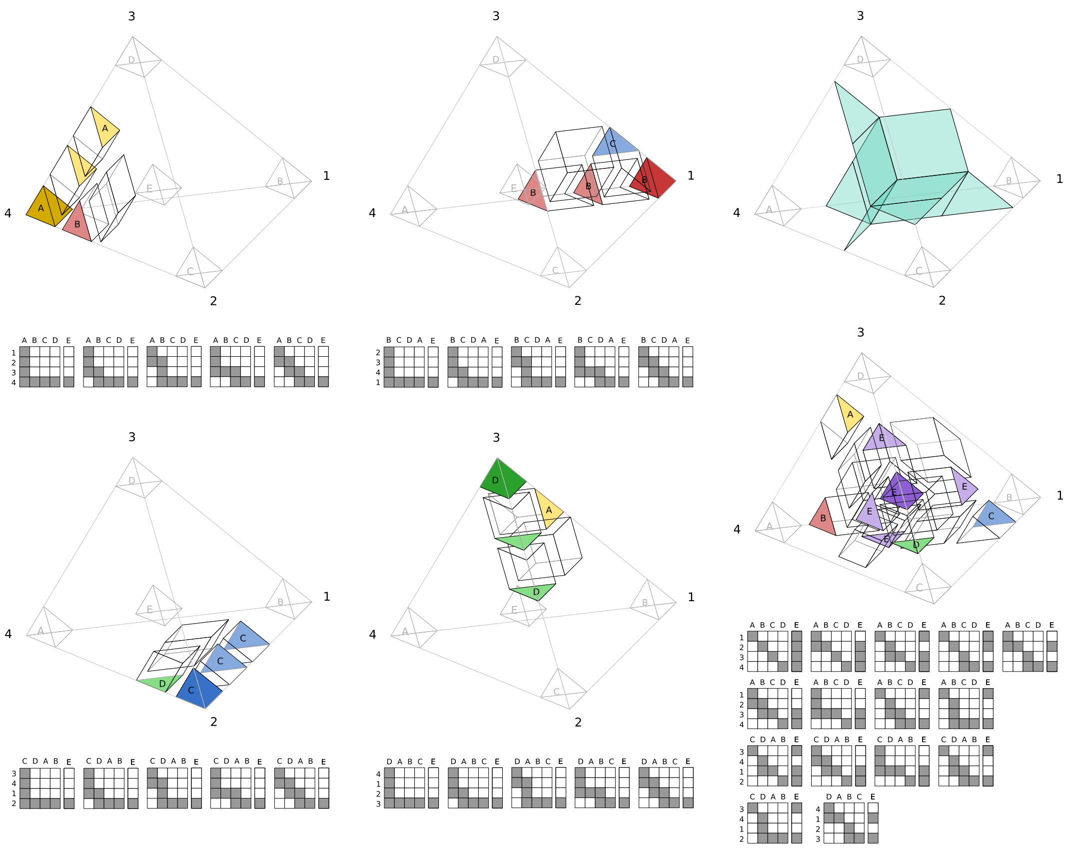


Figure 2.13: $\mathfrak{D}_4^{\varphi}$ as a mixed subdivision. In the third column at the top, we see the mixed cell representation of those low-dimensional simplices $\boldsymbol{\sigma} \subset \Delta_{n-1} \times \Delta_{n-1}$ whose extension $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$ is full-dimensional; at the bottom, we see their full-dimensional extensions.

The reader will find a mixed subdivision representation of the φ-extension of $\mathfrak{D}_3$ in figure 2.8a. The φ-extension of $\mathfrak{D}_4$ is displayed in figure 2.13.

We split the following fact from the proof of theorem 2.4.30.

Lemma 2.4.31. *There is a (non-canonical) bijection between the full-dimensional simplices in $\mathfrak{D}_n^{\varphi}$ and the weak integer compositions of n of lenght n. Therefore*[2]*, the number of full-dimensional simplices in $\mathfrak{D}_n^{\varphi}$ is $\binom{2n-1}{n-1}$.*

Proof. Before deriving the bijection in generality, let us remark that a point $(\mathbf{e}_i, \mathbf{e}_\alpha) \in \Delta_{n-1} \times \Delta_{d-1}$ lies in $\mathfrak{C}_n^{(\ell_1)} \cap \mathfrak{C}_n^{(\ell_2)} \cap \ldots \cap \mathfrak{C}_n^{(\ell_k)} \cap \mathfrak{C}_n^{(n)}$, where $\ell_1 < \ell_2 < \ldots < \ell_k < n$, if and only if:

$$\{i \geq \alpha\} \text{ and } \left\{ \begin{Bmatrix} i \leq \ell_1 \text{ and} \\ \alpha > 0 \end{Bmatrix} \text{ or } \begin{Bmatrix} i \leq \ell_2 \text{ and} \\ \alpha > \ell_1 \end{Bmatrix} \text{ or } \ldots \text{ or} \begin{Bmatrix} i \leq \ell_k \text{ and} \\ \alpha > \ell_{k-1} \end{Bmatrix} \text{ or} \begin{Bmatrix} i \leq n \text{ and} \\ \alpha > \ell_k \end{Bmatrix} \right\}. \quad (2.4)$$

As a consequence, a full-dimensional simplex $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$ such that $\boldsymbol{\sigma} \subset \Delta_{n-1} \times \Delta_{n-1}$ is contained in the region (2.4) looks like the simplex in figure 2.14.

Note that $\boldsymbol{\sigma}$, and therefore also $\boldsymbol{\sigma}^{\varphi}$, can be characterized by a unique monotonically increasing (not necessarily injective) map $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)} : [d = n] \to [n]$ such

[2]Recall that the number of weak integer compositions of n of length n equals $\binom{2n-1}{n-1}$ [Sta11, Section 1.2].

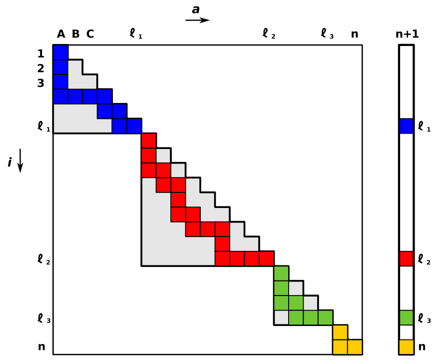


Figure 2.14: A "generic" full-dimensional simplex $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$. It can be represented by the weak composition (4, 0, 0, 1, 1, 0, 3, 1, 2, 1, 0, 2, 0, 0, 0, 3, 1, 0, 0, 2, 0) of 21 of length 21. The collection of partial sums is (3, 2, 1, 1, 1, 0, 2, 2, 3, 3, 2, 3, 2, 1, 0, 2, 2, 1, 0, 1, 0); observe that the positions of the zeros indicate the labels of the points in $\Delta_{20} \times \Delta_{\{21\}}$ of $\boldsymbol{\sigma}^{\varphi}$.

that $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a)$ gives the "height" of the "tallest"[3] point in $\boldsymbol{\sigma}$ with second coordinate $a \in [d = n]$ (as measured by the row-indices of $\mathcal{G}_{n\times n}$). In turn, the map $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}$ can be described by the weak integer composition $\mathbf{h}_{\boldsymbol{\sigma}} = (h_{\boldsymbol{\sigma}1}, h_{\boldsymbol{\sigma}2}, \dots, h_{\boldsymbol{\sigma}n}) \in \mathbb{N}^n$ of n given by:

$$h_{\boldsymbol{\sigma}a} = \begin{cases} \mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(1) & \text{if } a = 1 \\ \mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a) - \mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a-1) & \text{if } a > 1 \end{cases},$$

whose entries record the relative increase in height between $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a)$ and $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a-1)$ (the total height increase for $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}$ is n, so the entries of $\mathbf{h}_{\boldsymbol{\sigma}}$ do sum up to n). Since $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}$ is monotonically increasing, the partial sums giving the "vertical" distance between $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)}(a)$ and the $i = a$ diagonal in $\mathcal{G}_{n\times n}$ satisfy:

$$\sum_{b=1}^{a} h_{\boldsymbol{\sigma}b} - a \geq 0, \quad \text{for every } 1 \leq a \leq n.$$

We now prove that actually every simplex $\boldsymbol{\sigma} \subset \Delta_{n-1} \times \Delta_{n-1}$ such that $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$ is full-dimensional can be described by a unique weak composition of n of length n, and conversely that every such weak composition gives rise to a unique simplex $\boldsymbol{\sigma} \subset \Delta_{n-1} \times \Delta_{n-1}$ such that $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$ is full-dimensional (the proof is reminiscent of the so-called cycle, or Raney's, lemma; see [DZ90] or [GKP94, Section 7.5]).

Let $\boldsymbol{\sigma} \subset \Delta_{n-1} \times \Delta_{n-1}$ be a simplex in the conditions of the preceding paragraph. Although in general $\boldsymbol{\sigma}$ cannot be described by a monotonically increasing map $\mathfrak{m}_{\boldsymbol{\sigma}}^{(n)} : [n] \to [n]$, it always admits a description by a monotonically increasing map[4]:

$$\mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)} : \{\ell{+}1 \prec \ell{+}2 \prec \dots \prec n \prec 1 \prec 2 \prec \dots, \prec \ell\} \longrightarrow \{\ell{+}1 \prec \ell{+}2 \prec \dots \prec n \prec 1 \prec 2 \prec \dots, \prec \ell\},$$

for any ℓ indexing (the first component of) a point in $\boldsymbol{\sigma}^{\varphi} \backslash \boldsymbol{\sigma} \subset \Delta_{n-1} \times \{\mathbf{e}_{n+1}\}$. If there is more than one such point, we choose the largest index (with respect to

[3]Which in our figures would be the "depth" of the "deepest" point.
[4]As customary, sums of indices are understood modulo n.

the usual order $1 < 2 < \ldots < n$), thus fixing the monotonic map $\mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)}$ representing $\boldsymbol{\sigma}$. Once $\mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)}$ has been fixed, we set the corresponding weak composition $\mathbf{h}_{\boldsymbol{\sigma}} \in \mathbb{N}^n$ according to:

$$h_{\boldsymbol{\sigma}a} = \begin{cases} \mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)}(\ell+1) & \text{if } a = \ell+1 \\ \mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)}(\ell+a) - \mathfrak{m}_{\boldsymbol{\sigma}}(\ell+a-1) \pmod n & \text{if } a \succ \ell+1 \end{cases}.$$

Moreover, since $\mathfrak{m}_{\boldsymbol{\sigma}}^{(\ell)}$ is monotonically increasing, we have:

$$\sum_{b=1}^{a} h_{\boldsymbol{\sigma}\varphi^\ell(b)} - \varphi^\ell(a) \geq 0, \quad \text{for every } 1 \leq a \leq n.$$

For the inverse map, let $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{N}^n$ be a weak composition of n of length n. In general, some of the partial sums

$$g_a := \sum_{b=1}^{a} h_b - a,$$

may be negative. Set $\ell_{\mathbf{h}} := \max_< \{\arg\min\{g_a : 1 \leq a \leq n\}\}$, that is, the largest index (w.r.t. $<$) for which g_ℓ is most negative. By construction,

$$g_a^{\ell_{\mathbf{h}}} := \sum_{b=1}^{a} h_{\varphi^{\ell_{\mathbf{h}}}(b)} - \varphi^{\ell_{\mathbf{h}}}(a) \pmod n \geq 0, \quad \text{for every } 1 \leq a \leq n;$$

therefore, $\mathbf{h}$ defines a unique monotonically increasing map:

$$\begin{aligned} \mathfrak{m}_{\mathbf{h}} : \{\ell_{\mathbf{h}}+1 \prec \ell_{\mathbf{h}}+2 \prec \ldots \prec n \prec 1 \prec 2 \prec \ldots, \prec \ell_{\mathbf{h}}\} &\to \{\ell_{\mathbf{h}}+1 \prec \ell_{\mathbf{h}}+2 \prec \ldots \prec n \prec 1 \prec 2 \prec \ldots, \prec \ell_{\mathbf{h}}\} \\ a &\mapsto a + g_a^{\ell_{\mathbf{h}}} \bmod n, \end{aligned}$$

which characterizes a unique simplex $\boldsymbol{\sigma}_{\mathbf{h}} \subset \Delta_{n-1} \times \Delta_{n-1}$ such that $\boldsymbol{\sigma}_{\mathbf{h}}^{\varphi} \in \mathfrak{D}_n^{\varphi}$ is full-dimensional.

Clearly $\boldsymbol{\sigma}_{\mathbf{h}_{\boldsymbol{\sigma}}} = \boldsymbol{\sigma}$ and $\mathbf{h}_{\boldsymbol{\sigma}_{\mathbf{h}}} = \mathbf{h}$, and thus we obtain the desired bijection, concluding the proof. □

Proof of theorem 2.4.30. Again we do the proof by means of the characterization of triangulations in theorem B.2.21. We established in lemma 2.4.31 the fact that $\mathfrak{D}_n^{\varphi}$ has the required number of maximal simplices, $\binom{2n-1}{n-1}$, so condition (**TV**) of theorem B.2.21 holds.

To verify condition $\circ_b$ of theorem B.2.21, let $\boldsymbol{\tau}^{\varphi}$ be a codimension-1 simplex obtained by removing a square $\mathbf{s} = (i, a)$ from some maximal simplex $\boldsymbol{\sigma}^{\varphi} \in \mathfrak{D}_n^{\varphi}$, where $\boldsymbol{\sigma} \subset \mathfrak{C}_n^{(n)}$ (by symmetry, this choice does not cause loss of generality). Similar to the proof of theorem 2.4.27, if $\mathbf{s}$ is the only square in its column or row, $\boldsymbol{\tau}^{\varphi}$ lies on a facet of $\Delta_{n-1} \times \Delta_n$, and there are two further cases to consider:

- Either $i-a > 1$ and $\boldsymbol{\tau}^{\varphi}$ contains the squares $(i-1, a)$ and $(i, a+1)$, or $i-a \geq 0$ and $\boldsymbol{\tau}^{\varphi}$ contains the squares $(i+1, a)$ and $(i, a-1)$.

- $i - a = 1$ and $\boldsymbol{\tau}^{\varphi}$ contains the squares $(i-1, a)$ and $(i, a+1)$.

In the first case, the situation is the same as for the Dyck path triangulation $\mathfrak{D}_n$, namely the only square $\mathbf{s}'$ for which $\boldsymbol{\tau}^\varphi \cup \mathbf{s}'$ a simplex in $\mathfrak{D}_n^\varphi$ is either $(i-1, a+1)$ or $(i+1, a-1)$.

In the event that $i - a = 1$, note that after removing $\mathbf{s}$, the restriction of $\boldsymbol{\tau}^\varphi$ to the (points represented by vertices in the) first n columns of $\mathcal{G}_{n\times(n+1)}$ results in a codimension-2 (in $\Delta_{n-1} \times \Delta_n$) simplex $\boldsymbol{\tau}'$, with no points $(\mathbf{e}_j, \mathbf{e}_b)$ such that $j > i-1$ or $b \leq a = i-1$. From lemma 2.4.28, we deduce that $\boldsymbol{\tau}'$ is a simplex in $\mathfrak{C}_n^{(\ell)}$ with $\ell = i-1$. Therefore, $\boldsymbol{\tau}' = \varphi_\times^{i-1}(\boldsymbol{\rho})$ for some $\boldsymbol{\rho} \subset \mathfrak{C}_n^{(n)}$, so that $\mathbf{s}' = (i-1, n+1)$ is the only square $\mathbf{s}'$ that makes $\boldsymbol{\tau} \cup \mathbf{s}'$ a simplex in $\mathfrak{D}_n^\varphi$.

The proof that when $a - i = 1$, $\mathbf{s} = (i, a)$ and $\mathbf{s}' = (i-1, n+1)$ lie on the same part of the circuit contained in $\boldsymbol{\tau}^\varphi \cup \mathbf{s} \cup \mathbf{s}'$ proceeds by an argument similar to the one in the proof of theorem 2.4.27; hence, we only include the pictorial version shown in figure 2.15.

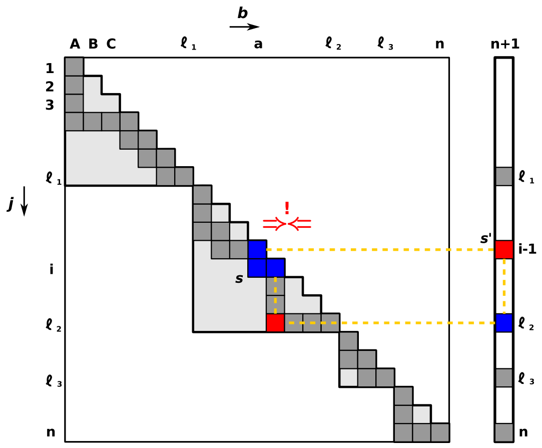


Figure 2.15: Illustration of the fact that no two adjacent full-dimensional simplices of $\mathfrak{D}_n^\varphi$ may overlap on a circuit.

$\square$

Let $\mathcal{T}'_{n\uparrow}$ be the triangulation of $\Delta_{n-1} \times \partial(\Delta_n)$ obtained by restricting $\mathfrak{D}_n^\varphi$. Let $\mathcal{T}'_{n\downarrow}$ be the triangulation of $\Delta_{n-1} \times \partial(\Delta_n)$ that results after flipping the triangulation of the (n,n) circuit $\mathbf{Z}$ in the face $\mathbf{F} = \Delta_{n-1} \times \Delta_{[n]}$ of $\Delta_{n-1} \times \partial(\Delta_n)$ from $\mathcal{T}^-$ to $\mathcal{T}^+$. $\mathcal{T}'_{n\downarrow}$ gives a family of non-extendable partial triangulations that generalize the known examples in example 2.4.2 and lemma 2.4.10 (so the $\mathcal{T}'_{n\downarrow}$ are infinitely many relatives of the mother of all examples).

Theorem 2.4.32. *The triangulation $\mathcal{T}'_{n\downarrow}$ of $\Delta_{n-1} \times \partial(\Delta_n)$ is non-extendable.*

Proof. This follows easily from lemma 2.4.22 in the previous section. View $\mathcal{T}'_{n\downarrow}$ as a consistent collection of subdivisions:

$$\mathcal{M}'_{n\downarrow} := \left\{ \mathcal{M}_I \text{ subdivision of } n\Delta_{n-1} : I \in \binom{[n+1]}{n} \right\}$$

The unmixed simplex with full-dimensional $(n+1)$-th summand defines a stem-equivalence class with the candidate simplex:

$$\mathbf{\Omega} = \{1A, 2B, 3C, \dots, nn, 1(n+1), 2(n+1), 3(n+1), \dots, n(n+1)\} \supset \mathbf{Z}^+,$$

(note that $\mathbf{\Omega} = \{1A, 2B, 3C, \dots, nn\}^{\varphi}$). Let $\mho$ be any simplex in the triangulation $\mathcal{T}^+$ of the (n, n) circuit $\mathbf{Z}$ in the restriction $\mathcal{T}'_{n\downarrow}|_{\mathbf{F}}$, where $\mathbf{F} = \Delta_{n-1} \times \Delta_{[n]}$. We observe that $\mathbf{Z}^- \subset \mho$, and therefore $\mathbf{\Omega}|_{\mathbf{F}}$ is not a valid simplex in the restricted triangulation $\mathcal{T}'_{n\downarrow}|_{\mathbf{F}}$, for $\mathbf{\Omega}|_{\mathbf{F}}$ overlaps with $\mho$ on the circuit $\mathbf{Z}$. □

Remark 2.4.33. The representation of triangulations of $\Delta_{n-1} \times \Delta_{n-1}$ as subdivision of $n\Delta_{n-1}$ provided the geometric intuition underlying the construction of the φ-extension of $\mathfrak{D}_n$. Observe that the circuit $\mathbf{Z}$ in $\mathfrak{D}_n$ is represented in the mixed subdivision picture as a (non-fine) mixed cell Ξ gotten as the Minkowski sum of n minimally dependent segments in $\mathbb{R}^n$ (in the sense that the removal of any segment results in an $(n-1)$-cube); for $n = 3$ this cell is a hexagon, for $n = 4$ it is a rhombic dodecahedron and in general it is called a *cubical flip* (see [DRS10, Section 9.1.2]). There are exactly two ways to tile such a cubical flip with n $(n-1)$-cubes, that correspond to the two ways of triangulating the circuit $\mathbf{Z}$, and one can switch freely between these two tilings without affecting the rest of the mixed subdivision. However, by inserting an unmixed $(n-1)$-simplex in the "middle" of Ξ, it is possible to "freeze" one of the two tilings (see figure 2.16 for an depiction of this). This fixes the tiling of Ξ that ought to appear in the facet of $\partial(n\Delta_n)$ representing the facet $\Delta_{n-1} \times \Delta_{[n]}$ of $\Delta_{n-1} \times \Delta_n$ (that bears the triangulation $\mathfrak{D}_n$), so by flipping to the alternative the tiling of Ξ we obtain a non-extendable partial subdivision. ○

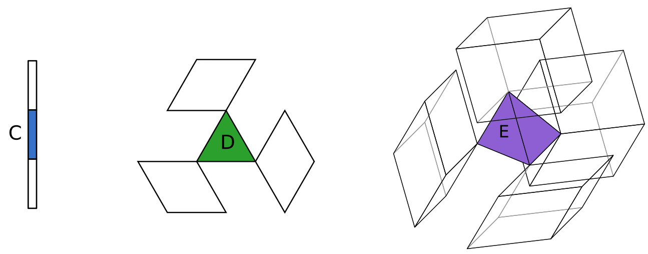


Figure 2.16: Inserting an unmixed simplex in the "middle" of a cubical flip "freezes" one of its tilings.

2.4.3 $k < d \leq n$: unsolved case

In the search for necessary conditions generalizing the acyclicity condition for systems of permutations of Ardila and Ceballos (cf. theorem 2.4.6), Ceballos, Padrol and the author had put forward the following conjecture

Conjecture 2.4.34. *Let $\mathcal{M}'$ be a subdivision of $\partial(n\Delta_{d-1})$, where $d \leq n$. Then $\mathcal{M}'$ extends to a subdivision of $n\Delta_{d-1}$ if and only if every restriction of $\mathcal{M}'$ to a subdivision of $\partial((d-1)\Delta_{d-1})$ extends to a subdivision of $(d-1)\Delta_{d-1}$.*

However, close to the submission of this thesis, a counterexample was found for $n = d = 4$ by Suho Oh and Hwanchul Yoo (personal communication).

In analogy with the study of triangulations of $\Delta_{n-1} \times \mathrm{skel}_1(\Delta_{d-1})$ by Ardila and Ceballos in [AC13] and the counterexamples to their conjecture 2.4.7, we expect that when $k < d \leq n$ it may only be possible to characterize extendability of triangulations of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ through necessary conditions, that is excluding obstructions, in line with remark 2.4.8.

The non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$ from theorem 2.4.32 can be thought of as an infinite family of obstructions to extendability of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ for $k < d \leq n$ arbitrary. However, there are further non-extendable triangulations of $\Delta_{n-1} \times \partial(\Delta_n)$ of different "types" (as, for example, Santos' construction in [San12]), so it seems unfeasible to obtain a complete list of (families of) obstructions.

On the other hand, it is not clear how to prove sufficiency of such necessary conditions for given fixed values of n and d; the proof in [AC13] that the acyclicity condition of theorem 2.4.6 is also sufficient when $\min\{n, d\} \leq 3$ is geometric, and therefore does not generalize for larger parameters. As observed in remark 2.4.13 for the case $k = 2$, one difficulty when studying triangulations of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ when $k < d \leq n$ is that it is not clear how to characterize those partial triangulations that uniquely specify a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

2.5 Concluding remarks

The results in this chapter can be interpreted as *finiteness results* for triangulations of $\Delta_{n-1} \times \Delta_{d-1}$. Indeed, theorem 2.4.25 says that if $k \geq n + 1$, every triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is obtained by "gluing together" triangulations of $\Delta_{n-1} \times \Delta_n$[5][6].

Generally speaking, we conclude that the framework proposed in section 2.1 for studying triangulations of cartesian products of point configurations seems convenient to state and derive finiteness results of the sort of theorem 2.4.25. Needless to say, it would be interesting to discover other families of products of point configurations admit an analysis in terms of partial triangulations, and therefore display similar finiteness phenomena.

We conjecture that this is true for triangulations of cartesian products of several simplices:

Conjecture 2.5.1.

1. *Let $d_1, d_2, \ldots, d_r$ be natural numbers larger than 2. There is a number $\widetilde{d} = \widetilde{d}(d_1, d_2, \ldots, d_r)$ such that if $k \geq \widetilde{d}$, every triangulation of $\Delta_{k-1} \times \Delta_{d_1-1} \times$*

[5] In big quotation marks, an alternative statement is: *"as $d \to \infty$, every triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is* generated *by gluing compatible triangulations of $\Delta_{n-1} \times \Delta_n$ together"*.

[6] Yet a more geometric interpretation is: *"If $k \geq n + 1$, extendability of a triangulation of $\Delta_{n-1} \times \mathrm{skel}_{k-1}(\Delta_{d-1})$ can be determined locally; by theorem 2.4.32, the same does not hold if $k = n$"*.

$\Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1}$ is obtained by gluing together compatible triangulations of the faces $\Delta_{\widetilde{d}-1} \times \Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1}$ of $\Delta_{k-1} \times \Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1}$.

2. *Let d_0 be a natural number larger than 2. There is a number $\widetilde{m} = \widetilde{m}(d_0, r)$ such that if $d_1 + d_2 + \ldots + d_r \geq \widetilde{m}$, then every triangulation of $\Delta_{d_0-1} \times \Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1}$ is obtained by gluing together compatible triangulations of the faces in $\Delta_{d_0-1} \times \mathrm{skel}_{\widetilde{m}-1}(\Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1})$ of $\Delta_{d_0-1} \times \Delta_{d_1-1} \times \Delta_{d_2-1} \times \ldots \times \Delta_{d_r-1}$.*

The pattern behind the construction of the triangulation $\mathfrak{D}_n^{\varphi}$ can be proposed as a general strategy to produce triangulations displaying a form of "rigidity", and deserves further study. We have:

- a family $\mathbf{P}_n$ of point configurations indexed by the number of vertices,
- a family of groups G_n acting on $\mathbf{P}_n$, whose elements can be labeled by the vertices of $\mathbf{P}_n$,
- a triangulation $\mathcal{T}_n^{(G_n)}$ of $\mathbf{P}_n \times \mathbf{P}_n$ whose maximal simplices are partitioned in orbits under the action of G_n.

Then, an extension of $\mathcal{T}_n^{(G_n)}$ to a triangulation of $\mathbf{P}_n \times \mathbf{P}_{n+1}$ is constructed by considering an additional coordinate that "graphs" for every simplex in $\mathcal{T}_n^{(G_n)}$ the elements g of G_n (which are indexed by points of $\mathbf{P}_n$) for which that simplex is a face of the g-th image of a reference maximal simplex in its orbit.

Appendix A

Very brief account of commutative algebra

This short appendix is meant to provide some basic definitions from commutative algebra that are worked with throughout the thesis, rather than to make the exposition self-contained. The presentation is therefore rather brief and narrow, but the reader can refer to one of excellent treatises on commutative algebra available for a deeper background. We paraphrase mostly content from the book [CLO07], along with some further material from [CLO05, Eis05, MS05, Stu96] (since the majority of the definitions and the results from commutative algebra are standard, we omit explicit references to the literature).

A.1 Graded minimal free resolutions

Modules over (commutative) rings are the central objects of study in commutative algebra. They can be thought of as a generalization of vector spaces in that the coefficients of "vectors" are taken over a commutative ring instead of over a field. A main difference with linear algebra, that partly gives rise to the much richer phenomena in commutative algebra, is that if we have a module (for example, coming from a system of polynomial equations), it may not be possible to obtain a generating set for it consisting of linearly independent elements.

The necessity to adequately describe modules by means of generating sets gives rise to the notion of a free resolution, from which many properties of modules can be characterized. The study of properties of free resolutions of modules is an active area of research in commutative algebra; we will restrict here to the case of graded modules, for which a unique minimal free resolution exists that admits the definition of invariant quantities of the module, such as the Betti numbers of the graded module (cf. definition A.1.15).

We start with the basic definitions. **Throughout we let k be an algebraically closed field**.

Definition A.1.1. A *commutative ring* consists of a set $\mathbf{R}$, and two binary operations '$\cdot$' and '$+$', defined on $\mathbf{R}$, for which the following conditions are satisfied:

1. '$\cdot$' and '$+$' are both associative,
2. '$\cdot$' and '$+$' are both commutative,
3. '$\cdot$' distributes over '$+$': $a \cdot (b+c) = a \cdot b + a \cdot c$, $\forall\ a, b, c \in \mathbf{R}$,
4. '$\cdot$' and '$+$' have identities, denoted by 1 and 0, respectively: $a \cdot 1 = a + 0 = a$, $\forall\ a \in \mathbf{R}$,
5. There is an additive inverse: $\forall\ a \in \mathbf{R}\ \exists\ (-a) \in \mathbf{R}$ such that $a + (-a) = 0$.

The only ring we consider in this thesis is the ring $\mathbf{R} = k[x_1, x_2, \ldots, x_n]$ of polynomials in n variables with coefficients in k; **we will therefore drop the adjective "commutative" everywhere**. Further examples of rings are the ring of integers $\mathbb{Z}$ or the ring of continuous complex-valued functions on a topological space.

The following is the generalization of vector spaces when taking coefficients in a ring.

Definition A.1.2. A *module* over a ring $\mathbf{R}$ (or $\mathbf{R}$-*module*) is a set $\mathbf{M}$, together with a binary operation, written $+$, and an operation of $\mathbf{R}$ on $\mathbf{M}$, called *scalar multiplication*, satisfying the following properties:

1. $\mathbf{M}$ is an abelian group under addition: addition is commutative, associative, there exists an additive identity in $\mathbf{M}$, and every element in $\mathbf{M}$ has an additive inverse,
2. $a(\mathbf{f} + \mathbf{g}) = a\mathbf{f} + a\mathbf{g}$, for all $a \in \mathbf{R}$, $\mathbf{f}, \mathbf{g} \in \mathbf{M}$,
3. $(a+b)\mathbf{f} = a\mathbf{f} + b\mathbf{f}$, for all $a, b \in \mathbf{R}$, $\mathbf{f} \in \mathbf{M}$,
4. $(a \cdot b)\mathbf{f} = a(b\mathbf{f})$, for all $a, b \in \mathbf{R}$, $\mathbf{f} \in \mathbf{M}$,
5. If 1 is the multiplicative identity in $\mathbf{R}$, then $1\mathbf{f} = \mathbf{f}$ for all $\mathbf{f} \in \mathbf{M}$.

Let $\mathbf{N} \subseteq \mathbf{M}$. If $\mathbf{N}$ is an $\mathbf{R}$-module with the addition and scalar multiplication inherited from $\mathbf{M}$, then $\mathbf{N}$ is called an $\mathbf{R}$-*submodule* of $\mathbf{M}$.

Let $\mathbf{M}$ be an $\mathbf{R}$-module and $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\} \subset \mathbf{M}$. The set:

$$\langle \mathbf{f}_1, \ldots, \mathbf{f}_s \rangle := \left\{ \sum_{i=1}^{s} a_i \mathbf{f}_i : a_1, \ldots, a_s \in \mathbf{R} \right\},$$

is an $\mathbf{R}$-submodule of $\mathbf{M}$ called the $\mathbf{R}$-*module generated by* $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$. If for a module $\mathbf{M}$ we have $\mathbf{M} = \langle \mathbf{f}_1, \ldots, \mathbf{f}_s \rangle$ for some $\mathbf{f}_1, \ldots, \mathbf{f}_s \subset \mathbf{M}$, we say that $\mathbf{M}$ is *finitely generated*.

A further example of a module is the ring $\mathbf{R}$ itself; its submodules are important objects in commutative algebra.

Definition A.1.3. Let $\mathbf{R}$ be a commutative ring. A subset $\mathbf{I} \subset \mathbf{R}$ is an *ideal* if it satisfies the following properties:

1. $0 \in \mathbf{I}$
2. $a, b \in \mathbf{I}$ implies $a + b \in \mathbf{I}$
3. $a \in \mathbf{I}$ and $b \in \mathbf{R}$ implies $a \cdot b \in \mathbf{I}$

Remark A.1.4. Let $\mathbf{R} = k[x_1, \ldots, x_n]$. If $n = 1$, then every ideal of $\mathbf{R}$, such as $\mathbf{R}$, is generated by a single polynomial, just as every one-dimensional vector space is generated by a single element. In contrast with this, when $n > 1$ we may need more than one element to generate an ideal of $\mathbf{R}$. ○

Let t be a natural number. Probably the simplest example of an $\mathbf{R}$-module is $\mathbf{R}^t$, consisting of t-tuples of elements of $\mathbf{R}$ with the addition defined componentwise and the scalar product distributing over the components of a t-tuple. In symbols:

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_t \end{pmatrix} = \begin{pmatrix} f_1 + g_1 \\ f_2 + g_2 \\ \vdots \\ f_t + g_t \end{pmatrix} \in \mathbf{R}^t, \qquad \text{where } f_1, \ldots, f_t, g_1, \ldots, g_t, h \in \mathbf{R}.$$

$$h \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{pmatrix} = \begin{pmatrix} h\, f_1 \\ h\, f_2 \\ \vdots \\ h\, f_t \end{pmatrix} \in \mathbf{R}^t,$$

The reason why $\mathbf{R}^t$ is so simple is that it is the closest we can get to vector spaces. That precise property R^t satisfies is identified next.

Definition A.1.5. Let $\mathbf{M}$ be a module over a ring $\mathbf{R}$. We say $\mathbf{M}$ is a *free module* if $\mathbf{M}$ has a module basis, that is, if there is a set $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\} \subset \mathbf{M}$ of $\mathbf{R}$-linearly independent elements such that every $\mathbf{f} \in \mathbf{M}$ can be written (uniquely) as $\mathbf{f} = a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \ldots + a_m\mathbf{f}_m$, for some $a_1, \ldots, a_m \in \mathbf{R}$.

When an $\mathbf{R}$-module $\mathbf{M}$ is not free, we examine the "lack of freeness" by means of the syzygy module.

Definition A.1.6 (Proposition 1.9, Chapter 6, [CLO05])**.** Let $(\mathbf{f_1}, \ldots, \mathbf{f_s})$ be an ordered tuple of elements of $\mathbf{M}$. The set of all $(a_1, \ldots, a_s)^T \in \mathbf{R}^s$ such that $a_1\mathbf{f_1} + \ldots a_s\mathbf{f_s} = 0$ is an $\mathbf{R}$-submodule of $\mathbf{R}^s$ called the *(first) syzygy module* of $(\mathbf{f_1}, \ldots, \mathbf{f_s})$, and denoted $\mathrm{Syz}(\mathbf{f_1}, \ldots, \mathbf{f_s})$. An element of $\mathrm{Syz}(\mathbf{f_1}, \ldots, \mathbf{f_s})$ is called a *syzygy* on $(\mathbf{f_1}, \ldots, \mathbf{f_s})$.

For polynomial rings in finitely many variables, *Hilbert's basis theorem* says that taking syzygy modules of finitely generated modules we always stay within the class of finitely generated modules.

Theorem A.1.7 (Hilbert's basis theorem). *Let* $\mathbf{R} = k[x_1, x_2, \ldots, x_n]$. *Then every submodule of* $\mathbf{R}^t$ *is finitely generated.*

Definition A.1.8 (Chapter 5, Definition 1.7, [CLO05]). An $\mathbf{R}$-*module homomorphism* between two $\mathbf{R}$-modules $\mathbf{M}$ and $\mathbf{N}$ is an $\mathbf{R}$-linear map $\phi : \mathbf{M} \to \mathbf{N}$, that is, a map ϕ such that:

$$\phi(a\mathbf{f} + \mathbf{g}) = a\phi(\mathbf{f}) + \phi(\mathbf{g})$$

for all $a \in \mathbf{R}$ and $\mathbf{f}, \mathbf{g} \in \mathbf{M}$

Choosing a generating set $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ for an $\mathbf{R}$-module $\mathbf{M}$ is equivalent to choosing a matrix representation ϕ_0 for a surjective $\mathbf{R}$-module homomorphism:

$$\mathbf{R}^s \xrightarrow{\phi_0} \mathbf{M} \longrightarrow 0$$
$$(a_1, a_2, \ldots, a_s) \mapsto a_1\mathbf{f}_1 + a_2\mathbf{f}_2 + \ldots + a_s\mathbf{f}_s$$

Being an $\mathbf{R}$-module itself, a generating set for $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s)$ may never consist of $\mathbf{R}$-linearly independent elements, that is, $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s)$ need not be a free module. Thus, when we choose a generating set for $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s)$, i.e., a matrix representing a surjective $\mathbf{R}$-module homomorphism $\mathbf{R}^t \xrightarrow{\phi_1} \mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s) \subset \mathbf{R}^s$ such that $\phi_0 \circ \phi_1 = 0$, we should also look at the syzygies among the elements of the generating set. This *second syzygy module* need not be free, so we consider the syzygy module for a generating set, etc... The procedure is represented by the following object.

Definition A.1.9 (Definition 1.9, Chapter 6, [CLO05]). Let $\mathbf{M}$ be an $\mathbf{R}$-module. A *free resolution* of $\mathbf{M}$ is an exact sequence $\mathcal{F}_*$ of the form:

$$\ldots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0,$$

where, for all i, $F_i \cong \mathbf{R}^{r_i}$, is a free $\mathbf{R}$-module. In other words, $\phi_0, \phi_1, \phi_2, \phi_3, \ldots$ are $\mathbf{R}$-module homomorphism which satisfy $\phi_{i-1} \circ \phi_i = 0$ for all $i \geq 1$. If there is an l such that $F_{l+1} = F_{l+2} = \ldots = 0$, but $F_l \neq 0$, we say that the resolution is finite of length l. In this case, we write the resolution as:

$$0 \longrightarrow F_l \xrightarrow{\phi_l} F_{l-1} \xrightarrow{\phi_{l-1}} \ldots\ldots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0.$$

For polynomial rings in a finite number of variables, *Hilbert's syzygy theorem* ensures that taking a free resolution stops at a finite number of steps.

Theorem A.1.10 (Hilbert's syzygy theorem, Theorem 2.1 Chapter 6, [CLO05]). *Let* $\mathbf{R} = k[x_1, \ldots, x_n]$. *Then every* $\mathbf{R}$-*module has a finite free resolution of length at most* n.

From this point on we fix $\mathbf{R} = k[x_1, \ldots x_n]$.

The type of ideals we deal with in this thesis are homogeneous, that is, generated by homogeneous polynomials, so their syzygy modules possess an additional structure, called a *grading*. Recall that for the ring of polynomials we have the following decomposition into finite dimensional vector spaces:

$$k[x_1,\dots,x_n] = \bigoplus_{i\geq 0} k[x_1,\dots,x_n]_i,$$

where $k[x_1,\dots,x_n]_i$, denotes the set of homogeneous polynomials of degree i.

The corresponding notion for modules reads.

Definition A.1.11 (Definition 3.2, Chapter 6, [CLO05])**.** A *graded module* over $\mathbf{R}$ is a module $\mathbf{M}$ with a direct-sum decomposition into submodules $\mathbf{M} = \bigoplus_{i\geq 0}\mathbf{M}_i$ that is compatible with the grading in the polynomial ring:

$$\mathbf{R}_i \cdot \mathbf{M}_j \subset \mathbf{M}_{i+j}.$$

The *shifted module* $\mathbf{M}(d)$ is the direct sum $\bigoplus_{i\geq 0}\mathbf{M}(d)_i$, where $\mathbf{M}(d)_i := \mathbf{M}_{i+d}$. It is isomorphic to $\mathbf{M}$ as a module, but has a different grading. (For example, in $\mathbf{R}(-d_1)\oplus\dots\oplus\mathbf{R}(-d_p)$ the standard basis vector $\mathbf{e}_i$ has degree d_i, for $1\leq i\leq p$.)

Definition A.1.12 (Definition 3.5, Chapter 6, [CLO05])**.** Let $\mathbf{M},\mathbf{N}$ be graded modules over $\mathbf{R}$. A homomorphism $\phi : \mathbf{M} \to \mathbf{N}$ is said to be a *graded homomorphism of degree d* if $\phi(\mathbf{M}_i) \subset \mathbf{N}_{i+d}$ for all $i \in \mathbf{N}$

Definition A.1.13 (Definition 3.7, Chapter 6, [CLO05])**.** Let $\mathbf{M}$ be a $\mathbf{R}$-module. A *graded free resolution of* $\mathbf{M}$ is a resolution $\mathcal{F}_*$ of the form:

$$\dots \longrightarrow \mathbf{F}_2 \xrightarrow{\phi_2} \mathbf{F}_1 \xrightarrow{\phi_1} \mathbf{F}_0 \xrightarrow{\phi_0} \mathbf{M} \longrightarrow 0,$$

where $\mathbf{F}_l$ is a shifted free graded module $\mathbf{R}(-d_1)\oplus\dots\oplus\mathbf{R}(-d_p)$ and each homomorphism ϕ_l is a graded homomorphism of degree zero.

To understand the last part of the definition, observe that we may achieve degree zero for a graded homomorphism between graded modules by shifting the direct summands in each free module. For instance, a $m\times p$ matrix A where the entry $a_{ij}\in\mathbf{R}$ is homogeneous of degree $d_j - c_i$ defines a graded homomorphism of degree zero [CLO05]:

$$\mathbf{R}(-d_1)\oplus\dots\oplus\mathbf{R}(-d_p) \xrightarrow{A} \mathbf{R}(-c_1)\oplus\dots\oplus\mathbf{R}(-c_m)$$

If every homomorphism ϕ_l in a graded free resolution is such that it sends the standard basis of $\mathbf{F}_l$ to a minimal generating set of $\mathrm{im}(\phi_l)$, we say it is a *graded minimal free resolution*.

Hilbert's graded syzygy theorem says that every graded module over $\mathbf{R}$ has a graded free resolution of length at most n. A further commodity of graded modules is that their graded minimal free resolutions are unique (up to isomorphism).

Theorem A.1.14 (Definition 3.11 and Theorem 3.13, Chapter 6, [CLO05])**.** *Let $\mathcal{F}_*$ and $\mathcal{G}_*$ be two graded minimal free resolutions:*

$$\mathcal{F}_* : \ \dots \xrightarrow{\phi_2} \mathbf{F}_1 \xrightarrow{\phi_1} \mathbf{F}_0 \xrightarrow{\phi_0} M \longrightarrow 0,$$

$$\mathcal{G}_* : \ \dots \xrightarrow{\psi_2} \mathbf{G}_1 \xrightarrow{\psi_1} \mathbf{G}_0 \xrightarrow{\psi_0} M \longrightarrow 0,$$

of the graded module **M**. *Then* $\mathcal{F}_*$ *is isomorphic to* $\mathcal{G}_*$; *that is, there are graded isomorphisms of degree zero* $\alpha_l : \mathbf{F}_l \to \mathbf{G}_l$ *such that* $\alpha_{l-1} \circ \phi_l = \psi_l \circ \alpha_l$ *for every* l.

Hence, invariant quantities can be defined for a graded module from the minimal free resolution. The invariant we consider in chapter 1 is the following one.

Definition A.1.15 (Section 1B, [Eis05], Definition 1.9 [MS05])**.** Let $\mathcal{F}_*$ be a graded minimal free resolution of a graded **R**-module **M**. If $\mathbf{F}_i = \bigoplus_{j\in\mathbf{N}} \mathbf{R}[-j]^{\beta_{i,j}}$, we call the invariant $\beta_{i,j} = \beta_{i,j}(\mathbf{M})$ the i-th *Betti number* of **M** in degree j.

Said differently, the Betti number $\beta_{i,j}(\mathbf{M})$ gives the number of minimal generators of degree j for the i-th syzygy module of **M** (e.g., for $i = 0$, $\beta_{0,j}(\mathbf{M})$ is the number of minimal generators of **M** of degree j).

Example A.1.16. Let $k = \mathbb{R}$ and consider the points $\{[1:1:0], [0:1:0], [1:0:1]\} \in \mathbb{P}^2(k)$. The homogeneous ideal of polynomials in $\mathbf{R} = k[x,y,z]$ vanishing on them is:

$$\mathbf{I} = \langle x-y, z\rangle \cap \langle x, z\rangle \cap \langle x-z, y\rangle = \langle x^2 - xy - z^2, xz - z^2, yz, \rangle$$

We compute with Macaulay2 [GS] the minimal free resolution of **I** (with respect to the *graded reverse lexicographic order* – see [CLO07, Definition 6, Section 2.2]); it is displayed below:

$$0 \longrightarrow \mathbf{R}(-3)^2 \xrightarrow{\phi_1} \mathbf{R}(-2)^3 \xrightarrow{\phi_0} \mathbf{I} \longrightarrow 0,$$

with the matrices representing **R**-module homomorphisms:

$$\phi_0 = \begin{pmatrix} x^2 - xy - z^2, xz - z^2, yz \end{pmatrix}$$
$$\phi_2 = \begin{pmatrix} -z & 0 \\ x-y+z & -y \\ -z & x-y \end{pmatrix}.$$

The nonzero Betti numbers of **I** are therefore $\beta_{0,2} = 3$ and $\beta_{1,3} = 2$. ○

A.2 Gröbner bases

Let **I** be an ideal in the commutative ring $\mathbf{R} = k[x_1, x_2, \ldots, x_n]$. Gröbner bases are special classes of generating sets of ideals in **R** that serve the purposes of providing consistent representations of equivalence classes of polynomials in **R**/**I**, also called *normal forms*. This makes them suitable to produce algorithms to solve problems such as elimination of variables from a system of polynomial equations.

The definition relies on an ordering that we introduce on the monomials of **R**.

Definition A.2.1 (Chapter 1, [Stu96])**.** A *monomial ordering* on $\mathbf{R} = k[x_1, x_2, \ldots, x_n]$ is a total ordering $\prec$ of the monomials in **R** that satisfies:

- 1 is the smallest monomial in $\mathbf{R}$ with respect to $\prec$,
- $m \prec n \Rightarrow m \cdot p \prec n \cdot p$ for every monomials $m, n, p \in \mathbf{R}$.

A monomial ordering $\prec$ on $\mathbf{R}$ allows us to identify the largest term in any polynomial $f \in \mathbf{R}$. It will be called the *initial term of f with respect to $\prec$*, and written $\mathrm{in}_\prec(f)$. Initial terms of polynomials come into play when we carry out the division algorithm for polynomials: they are the first terms we divide and divide by.

The output of the division usually depends on the initial term chosen; fortunately, it is possible to appropriately choose a set of "denominators" and leading terms so that we can definitely establish whether a polynomial can be written as linear combination with polynomial coefficients of the denominators.

The set of denominators to appropriately perform polynomial division is given by a Gröbner basis. That is, a Gröbner basis consists of a generating set for $\mathbf{I}$ that interacts adequately with the monomial ordering. To formalize this idea, first associate to an ideal $\mathbf{I}$ in $\mathbf{R}$ the *initial ideal of $\mathbf{I}$ with respect to $\prec$*, which is the following monomial ideal:

$$\mathrm{in}_\prec(\mathbf{I}) := \langle \mathrm{in}_\prec(f) : f \in \mathbf{I} \rangle \subset \mathbf{R}.$$

Definition A.2.2 (Chapter 1, [Stu96])**.** Let $\prec$ be a monomial ordering on $\mathbf{R}$ and $\mathbf{I}$ be an ideal in $\mathbf{R}$. A generating set $\mathcal{G}_\prec := \{g_1, g_2, \ldots, g_s\}$ for $\mathbf{I}$ is called a *Gröbner basis for $\mathbf{I}$ with respect to $\prec$* if the following holds:

$$\mathrm{in}_\prec(\mathbf{I}) = \langle \mathrm{in}_\prec(g_1), \mathrm{in}_\prec(g_2), \ldots, \mathrm{in}_\prec(g_s) \rangle.$$

Example A.2.3. Let $f = x^2 + 3y^2 - 3x$, $g = xy^2 - 3y^2$ be polynomials in $\mathbb{C}[x, y]$, where we choose the graded lexicographic (see [CLO07, Definition 5, Section 2.2]) ordering for the monomials. Notice that:

$$y^2 \cdot (x^2 + 3y^2 - 3x) - x \cdot (xy^2 - 3y^2) = 3y^4 \in \langle f, g \rangle.$$

However, $3y^4$ is neither divisible by $\mathrm{in}_\prec(f) = x^2$ nor by $\mathrm{in}_\prec(g) = xy^2$. By naively running the division algorithm with the given ordering we would conclude that $f \nmid 3y^4$ and $g \nmid 3y^4$, so we would suspect $3y^4 \notin \langle f, g \rangle$, whereas we have showed the opposite. In this case, a Gröbner basis with respect to the glex ordering would be $\mathcal{G} = \{f, g, y^4\}$. ○

Proposition A.2.4 (Proposition 1 and Corollary 2 in [CLO07] Chapter 2 §6)**.** *Let $\mathcal{G}_\prec := \{g_1, g_2, \ldots, g_s\}$ be a Gröbner basis for $\mathbf{I} \subset \mathbf{R}$ with respect to $\prec$, and let $f \in \mathbf{R}$. There is a unique $r \in \mathbf{R}$, called the* normal form *of f with respect to $\prec$, such that:*

- *r is not divisible by any of $\{\mathrm{in}_\prec(g_1), \mathrm{in}_\prec(g_2), \ldots, \mathrm{in}_\prec(g_s)$*
- *There exists some $g \in \mathbf{I}$ with $f = g + r$.*

Moreover, $f \in \mathbf{I} \iff r = 0$.

In chapter 1, our interest in initial ideals and Gröbner bases will be that they constitute a systematic way to *approximate* an arbitrary ideal $\mathbf{I}$ by means of the simpler monomial ideal $\text{in}_{\prec}(\mathbf{I})$. The precise sense of the approximation we will be concerned with is the following result that we quote from [MS05].

Proposition A.2.5 (Upper semicontinuity, Theorem 8.29 in [MS05])**.** *Let* $\mathbf{R}$ *be the polynomial ring* $\mathbf{R} = k[x_1, x_2, \dots, x_n]$*,* $\prec$ *be a monomial ordering on* $\mathbf{R}$ *and* $\mathbf{I}$ *be a homogeneous ideal in* $\mathbf{R}$*. The following inequality holds:*

$$\beta_{i,j}(\mathbf{I}) \leq \beta_{i,j}(\text{in}_{\prec}(\mathbf{I})) \text{ for all } i, j \in \mathbb{N}.$$

An important fact about monomial orderings in a polynomial ring $k[x_1, x_2, \dots, x_n]$ is that they can always be induced by a sufficiently generic *weight vector* $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{R}^n$ [Stu96]. Let $f = \sum_{i=0}^{m} c_i x_1^{a_{i1}} x_2^{a_{i2}} \dots, x_n^{a_{in}} \in \mathbf{R}$; we define the initial term of f with respect to ω to be the sum of the monomials of f on which the scalar $\omega_1 a_{i1} + \omega_2 a_{i2} + \dots + \omega_n a_{in}$ achieves its maximum value. This polynomial is written $\text{in}_\omega(f)$ and is a monomial when ω is sufficiently generic.

Hence, as ω ranges over $\mathbb{R}^n$ all initial ideals of $\mathbf{I}$ can be obtained, of which there are only finitely many [Stu96, Theorem 1.2]. It is a remarkable result [Stu96, Propositions 2.3 and 2.4] that the possible initial ideals of $\mathbf{I}$ define a stratification of $\mathbb{R}^n$ into equivalence classes: set $\omega \sim \omega'$ whenever $\text{in}_\omega(\mathbf{I}) = \text{in}_{\omega'}(\mathbf{I})$. Moreover, the resulting equivalence classes are relatively open polyhedral cones that fit together to form a polyhedral fan covering $\mathbb{R}^n$ (see appendix B for definition). This fan is called the *Gröbner fan of* $\mathbf{I}$, *GrF*($\mathbf{I}$). In the next section, we will mention that this polyhedral structure associated with an ideal is closely related to the combinatorics of regular triangulations of polytopes in the case when the ideal is toric.

A.3 Toric ideals

The polynomial ideals we study in this thesis belong to the class of toric ideals. These are the ideals defining toric varieties, and enjoy a marked combinatorial nature. Here we present one particular construction of toric ideals, and refer the reader to [CLS11] for a thorough exposition. The material in this section is mostly taken from [Stu96].

Let $\mathbf{A}$ be a matrix in $\mathbb{N}^{d\times n}$ with the property that for some vector $\omega \in \mathbb{R}^d$ we have $\omega^t \mathbf{A} = (1, 1, \dots, 1)$, so that we may regard the columns $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ of $\mathbf{A}$ as giving the coordinates of the elements of a $(d-1)$-dimensional integer point configuration in $\mathbb{R}^d$.

The matrix $\mathbf{A}$ can be used to define a homogeneous polynomial ideal in the polynomial ring $\mathbf{R} = k[x_1, x_2, \dots, x_n]$. Concretely, consider the polynomial ring

$S = k[t_1, t_2, \ldots, t_d]$, together with the monomial homomorphism:

$$\begin{aligned} \phi_{\mathbf{A}} : k[x_1, x_2, \ldots, x_n] &\longrightarrow k[t_1, t_2, \ldots, t_d] \\ x_i &\longmapsto \mathbf{t}^{\boldsymbol{\alpha}_i} := t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_d^{a_{id}}, \end{aligned} \tag{A.1}$$

where $\boldsymbol{\alpha}_i = (a_{i1}, a_{i2}, \ldots, a_{id})^t \in \mathbb{N}^d$. The *toric ideal associated to* $\mathbf{A}$ is the homogeneous prime ideal:

$$\mathbf{I_A} := \ker(\phi_{\mathbf{A}}). \tag{A.2}$$

Equivalently, $\mathbf{I_A}$ is seen to be the ideal defining the projective *toric variety* $\mathbf{V}(\mathbf{I_A})$ resulting as the Zariski closure of the image of the monomial parametrization:

$$\begin{aligned} \mathbb{P}^{d-1} &\longrightarrow \mathbb{P}^{n-1} \\ [t_1 : t_2 : \ldots : t_d]^t &\longmapsto [\mathbf{t}^{\boldsymbol{\alpha}_1} : \mathbf{t}^{\boldsymbol{\alpha}_2} : \ldots : \mathbf{t}^{\boldsymbol{\alpha}_n}]^t, \end{aligned} \tag{A.3}$$

where $\mathbb{P} = \mathbb{P}(k)$.

By identifying monomials $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n}$ with integer points $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$, we see that the toric ideal $\mathbf{I_A}$ consists of all the arithmetic relations that hold between the columns of $\mathbf{A}$. Indeed, the following fact asserts that every element in $\mathbf{I_A}$ can be written in terms of binomials.

Proposition A.3.1 (Lemma 4.1 and corollary 4.3 in [Stu96])**.** *The toric ideal* $\mathbf{I_A}$ *is generated as a vector space over k by the binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ and } \mathbf{Au} = \mathbf{Av}\}.$$

If we decompose $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ *as* $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, *where* $u_i^+ = \max\{0, u_i\}$ *and* $u_i^- = \max\{0, -u_i\}$, *we have:*

$$\mathbf{I_A} = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \mathbb{Z}^n \text{ and } \mathbf{Au} = 0 \rangle$$

Remark A.3.2*.* In the context of algebraic statistics [DS98, DSS09], $\mathbf{A}$ is a matrix that acts linearly on a joint probability distribution to compute its marginals according to some simplicial complex (cf. example 2 in the introduction). Given an observed *contingency table* u with fixed marginals (computed with $\mathbf{A}$), one can test against the hypothesis that u was produced by a probability distribution in the hierarchical log-linear model defined by $I_{\mathbf{A}}$ by sampling the collection of all contingency tables with the same marginals. The fundamental theorem of algebraic statistics says that this task can be performed with a random walk on these contingency tables using “steps” that are encoded by binomials in a generating set of $\mathbf{I_A}$. ⬡

Remark A.3.3*.* The problem of finding the normal form modulo $\mathbf{I_A}$ of a monomial $\mathbf{x}^{\mathbf{u}} \in k[x_1, x_2, \ldots, x_n]$ with a Gröbner basis $\mathcal{G}_\omega(\mathbf{I})$ of $\mathbf{I_A}$ with respect to the term order induced by some $\omega \in \mathbb{R}^n$ is equivalent to the integer program:

$$\text{minimize } \omega \cdot \mathbf{v} \text{ subject to } \mathbf{Av} = \mathbf{Au} \text{ and } \mathbf{v} \in \mathbb{N}^n. \tag{A.4}$$

Indeed, if $\mathbf{x^u}$ is not in its normal form, by proposition A.2.4 we can divide it by some $g \in \mathcal{G}_\omega(\mathbf{I})$ to obtain a monomial $\mathbf{x^{u'}}$ with $\mathbf{Au'} = \mathbf{Au}$ and smaller ω-weight. Once $\mathbf{x^u}$ has achieved its normal form $\mathbf{x^w}$, no further (initial term of a) binomial of $\mathcal{G}_\omega(\mathbf{I})$ divides it, that is, $\mathbf{w}$ is a solution to (A.4). Polynomial division by the elements of a Gröbner basis of a toric ideal can therefore be seen as analogous to the solution of linear programs by the simplex method [Tho95, ST97]. $\hexagon$

Example A.3.4.

1. Let $\mathbf{A} \subset \mathbb{R}^n$ be the matrix whose columns are the points in the standard $(n-1)$-simplex Δ_{n-1}; we write this as $\mathbf{A} = \Delta_{n-1}$. Here we have $\phi_{\mathbf{A}} : k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_n]$ and $\ker(x_i \xmapsto{\phi_{\mathbf{A}}} t_i) = \langle 0 \rangle \subset k[x_1, \ldots, x_n]$: the defining ideal of projective space $\mathbb{P}^{n-1}$.

2. Let $\mathbf{A} = \Delta_{d_1-1} \times \Delta_{d_2-1}$. Then:

$$\begin{aligned} \phi_{\mathbf{A}} : k[z_{11}, z_{12}, \ldots, z_{d_1 d_2}] &\to k[x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}] \\ z_{ij} &\mapsto x_i y_j, \end{aligned}$$

and $\mathbf{I_A} = \ker(\phi_{\mathbf{A}})$ is the defining ideal of the *Segre embedding* (see chapter 1) $\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1} \hookrightarrow \mathbb{P}^{d_1 d_2 - 1}$. In general, if $\mathbf{A} = \Delta_{d_1-1} \times \ldots \times \Delta_{d_s-1}$, $\mathbf{I_A}$ defines the Segre embedding:

$$\mathbb{P}^{d_1-1} \times \ldots \times \mathbb{P}^{d_s-1} \hookrightarrow \mathbb{P}^{d_1 \ldots d_s - 1}.$$

$\hexagon$

The underlying motivation of this thesis concerns a correspondence between initial ideals and triangulations that is of a similar nature as the relation between integer programming and linear programming in remark A.3.3. Roughly speaking, it gives the interpretation of regular triangulations of a point configuration as "linear relaxations" of initial ideals. The accurate statement is the following theorem of Sturmfels.

Theorem A.3.5 (Sturmfels' correspondence, Theorem 3.1 in [Stu91])**.** *Let* $\mathbf{A} \in \mathbb{N}^{d \times n}$ *with the property that* $(1, 1, \ldots, 1)$ *is contained in its row span, and let* $\omega \in \mathbb{R}^n$ *be a generic weight vector. The Stanley-Reisner ideal (see definition 1.1.2) of the regular triangulation* $\mathcal{T}_\omega$ *of* $\mathrm{conv}(\mathbf{A})$ *induced by* ω *(cf. example B.2.15) equals the radical of the initial ideal* $\mathrm{in}_\omega(\mathbf{I_A})$*.*

Recall that the radical of a monomial ideal $\mathbf{I} = \langle m_1, m_2, \ldots, m_s \rangle \subset \mathbf{R}$ is the ideal generated by $\{m'_1, m'_2, \ldots, m'_s\}$, where m'_i contains the same variables as m_i but to the power 1. Theorem A.3.5 thus says that, modulo non-square-free-ness, the combinatorial information of the monomials defining the initial ideal $\mathrm{in}_\omega(\mathbf{I_A})$ is contained in the regular triangulation $\mathcal{T}_\omega$. By comparing the stratifications of $\mathbb{R}^n$ induced by initial ideals of $\mathbf{I_A}$ and regular triangulations of $\mathrm{conv}(\mathbf{A})$, this interpretation is made more accurate by the result next.

Corollary A.3.6 (Proposition 8.15 in [Stu96])**.** *The Gröbner fan of* $\mathbf{I_A}$ *is a refinement of the secondary polytope of* $\mathbf{A}$*. That is, every (relatively open) polyhedral cone in* $GrF(\mathbf{I_A})$ *is contained in one (relatively open) polyhedral cone in* $\Sigma(\mathbf{A})$*.*

Among the many theorems that relate combinatorial or geometric information in $\mathcal{T}_\omega$ to algebraic information of $\mathbf{I_A}$, we cite the following one that is touched upon in section 1.5.

Theorem A.3.7 (Corollary 8.9 in [Stu96])**.** *The initial ideal* $\mathrm{in}_\omega(\mathbf{I_A})$ *is squarefree if and only if the regular triangulation* $\mathcal{T}_\omega$ *of* $\mathrm{conv}(\mathbf{A})$ *is unimodular, that is, iff all maximal simplices in* $\mathcal{T}_\omega$ *have (normalized) volume equal to one.*

Appendix B

Basics of polyhedral geometry and triangulations

In this appendix, we gather together some basic definitions and results concerning polyhedral geometry and triangulations that are used throughout the thesis. These are taken from the literature, and are not always the most general possible. Our main references are the books [DRS10, Stu96, Zie95], to which we often provide citations where the reader can find more complete statements and developments.

B.1 Preliminary definitions

A *convex polyhedron* in $\mathbb{R}^d$ is a subset obtained as the intersection of finitely many closed half-spaces in $\mathbb{R}^d$. Here, by a closed half-space we mean a subset of points:

$$\mathcal{H}_{\leq b} := \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : r_1 x_1 + r_2 x_2 + \ldots + r_d x_d \leq b\} \subset \mathbb{R}^d,$$

where $r_1, r_2, \ldots, r_d, b \in \mathbb{R}$.

Example B.1.1. The *cone* generated by the (non-zero) vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$:

$$\mathrm{cone}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) := \{r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \ldots + r_n\mathbf{x}_n : r_i \geq 0 \in \mathbb{R},\ i \in [n]\} \subset \mathbb{R}^d$$

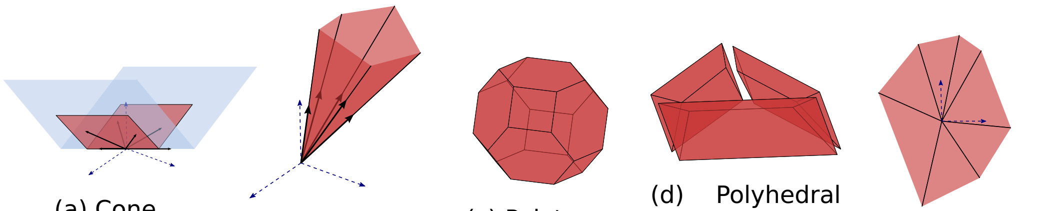
is an unbounded convex polyhedron. If no pair of vectors generating a cone are antiparallel, the cone is *pointed*. ⬡

A *convex polytope* **P** is a convex polyhedron that is bounded. Equivalently, a convex polytope **P** is the convex hull of finitely many points in $\mathbb{R}^d$. We will exclusively deal with convex polyhedra, so we will frequently omit the adjective “convex".

The *relative interior* of a polyhedron $\mathbf{P} = \bigcap_{i=1}^{m} \mathcal{H}^{(i)}_{\leq b_i}$ is the set of points that satisfy the inequalities defining $\mathbf{P}$ strictly; we denote it by relint($\mathbf{P}$).

A *polyhedral complex* $\mathcal{K}$ in $\mathbb{R}^d$ is a collection of finitely many polyhedra in $\mathbb{R}^d$ such that: (i) if $\mathbf{P} \in \mathcal{K}$ and $\mathbf{Q}$ is a face of $\mathbf{P}$, then $\mathbf{Q} \in \mathcal{K}$, and ($ii$) relint($\mathbf{P}$) $\cap$ relint($\mathbf{P}'$) $= \emptyset$ for every $\mathbf{P} \neq \mathbf{P}' \in \mathcal{K}$. Here, a face of a polyhedron $\mathbf{P}$ consists of those points in it that maximize some linear functional.

A polyhedral complex that consists of cones and entirely covers $\mathbb{R}^d$ is called a *polyhedral fan*. In figure B.1 we have drawn examples of each object defined.



(a) Cone. (b) Pointed cone. (c) Polytope. (d) Polyhedral complex. (e) Fan.

Figure B.1: Examples of convex polyhedra and polyhedral complexes.

An abstract *simplicial complex* on a base set $[n]$ is a collection $\mathcal{T}$ of subsets of $[n]$ with the property that if $\boldsymbol{\sigma} \in \mathcal{T}$ and $\boldsymbol{\tau} \subset \boldsymbol{\sigma}$, then $\boldsymbol{\tau} \in \mathcal{T}$.

B.2 Point configurations and their triangulations

In this thesis, we deal with triangulations of polytopes: roughly speaking, a dissection of a polytope into simplices. A triangulation is a natural step when studying arbitrary polytopes, for it amounts to decomposing the polytope into pieces that we know more of. To compute the volume of a polytope, for instance, we can compute the individual volumes of the simplices in a triangulation of it, for which there exists a formula, and then add up the contributions.

It turns out that the kind of triangulations we consider has a manifest combinatorial nature, with the structure of a simplicial complex, that lends itself conveniently to computation and to abstraction to higher dimensions. To make use of it, we need a precise combinatorial object, slightly more general than a convex polytope: a point configuration.

Definition B.2.1 (Definition 2.1.10 in [DRS10])**.** A *point configuration* $\mathbf{A}$ in $\mathbb{R}^d$ is a finite collection $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n\}$ of labelled points in affine space $\mathbb{R}^d$. In other words, a point configuration is a map $\mathbf{A} : [n] \to \mathbb{R}^d$ defined by $i \in [n] \mapsto \boldsymbol{\alpha}_i \in \mathbf{A} \subset \mathbb{R}^d$. We do not require $\mathbf{A}$ to be injective, therefore there may be two or more different points in $\mathbf{A}$ that have the same spatial coordinates.

With this in mind, we will think of a polytope $\mathbf{P} \subset \mathbb{R}^d$ in terms of the combinatorial information it contains, which is collected in the point configuration of the points in $\mathbb{R}^d$ of which $\mathbf{P}$ is the convex hull. Accordingly, the concepts we introduce below can be regarded as statements about the convex hull of the point configuration. This will be reflected in our figures, where we often represent a point configuration by its convex hull. However, when a point configuration has repeated points, or points that are not vertices of the convex hull of the point configuration, then we will explicity draw all the points for more clarity.

Let $\mathbf{A}$ be a point configuration in $\mathbb{R}^d$. The *dimension* of $\mathbf{A}$, written $\dim(\mathbf{A})$, is the dimension of its affine span. A *face* of $\mathbf{A}$ is a subset $\mathbf{F} \subseteq \mathbf{A}$ of the form:

$$\mathrm{face}_\omega(\mathbf{A}) := \{x \in \mathbf{A} : \omega \cdot x \geq \omega \cdot y \text{ for all } y \in \mathbf{A}\} \subseteq \mathbf{A} \subset \mathbb{R}^d,$$

where ω is some vector in $\mathbb{R}^d$ (after the identification of $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$ via the usual inner product). We will write $\mathbf{F} \preceq \mathbf{A}$ whenever $\mathbf{F}$ is a face of $\mathbf{A}$.

Example B.2.2. The standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$ consists of the points with coordinates given by the standard basis vectors of $\mathbb{R}^n$: $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. We denote it by Δ_{n-1}. ○

Evidently, a face of a point configuration is a point configuration itself; in particular, its dimension is well defined. Faces of dimension $0, 1, \ldots, d-2, d-1$ are respectively called *vertices*, *edges*, ..., *ridges*, *facets*. For notational convenience, we agree that the *empty face* is a face of every point configuration. More generally, we define a *subconfiguration* $\mathbf{A}'$ of a point configuration $\mathbf{A}$ in $\mathbb{R}^d$ as a subset of the points in $\mathbf{A}$.

Besides restriction to a subconfiguration, there are several ways to build new point configurations from existing ones. One of them is the cartesian product, with which we will concern ourselves mostly in this thesis.

Definition B.2.3. Let $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a point configuration in $\mathbb{R}^{d_1}$ and $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be a point configuration in $\mathbb{R}^{d_2}$. We define the *cartesian product* of $\mathbf{A}$ and $\mathbf{B}$ as the point configuration in $\mathbb{R}^{d_1+d_2}$:

$$\mathbf{A} \times \mathbf{B} := \{(\mathbf{a}_i, \mathbf{b}_j) : \ \mathbf{a}_i \in \mathbf{A}, \ \mathbf{b}_j \in \mathbf{B}\}.$$

Note that $\dim(\mathbf{A} \times \mathbf{B}) = \dim(\mathbf{A}) + \dim(\mathbf{B})$.

The faces of $\mathbf{A} \times \mathbf{B}$ are the point configurations of the form $\mathbf{F} \times \mathbf{G}$, where $\mathbf{F}$ and $\mathbf{G}$ are non-empty faces of $\mathbf{A}$ and of $\mathbf{B}$, respectively. See figure B.2 for an illustration.

We say a point configuration $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ in $\mathbb{R}^d$ is *affinely dependent*, or just *dependent*, if there is a nonzero vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^t \in \mathbb{R}^n$, with $\sum_{i=1}^n c_i = 0$, for which the equality:

$$\sum_{i=1}^{n} c_i \mathbf{a}_i = 0$$

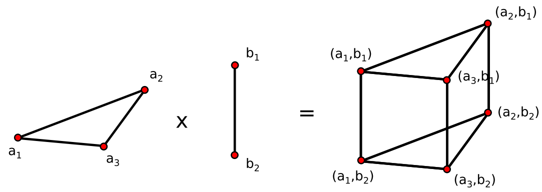


Figure B.2: The cartesian product of (the vertices of) a triangle with (the vertices of) a segment, that results in (the vertices of) a prism.

holds. In that case, we call $\mathbf{c}$ an *affine dependence* on $\mathbf{A}$. If no such $\mathbf{c} \in \mathbb{R}^n$ exists, we say $\mathbf{A}$ is *affinely independent*. If a subconfiguration $\boldsymbol{\sigma}$ of $\mathbf{A}$ is affinely independent, we say it is a *subsimplex* of $\mathbf{A}$, which is maximal if $|\mathbf{A}| = d+1$.

Clearly, the set of affine dependencies on a point configuration $\mathbf{A}$ in $\mathbb{R}^d$ constitutes a vector space of dimension $\min\{n - \dim(\mathbf{A}) - 1, 0\}$, that we write as $\mathcal{D}(\mathbf{A})$. Inside $\mathcal{D}(\mathbf{A})$ there is a collection of special elements that encode all the combinatorial information in the point configuration $\mathbf{A}$: they are called the *circuits* of $\mathbf{A}$. These elements span $\mathcal{D}(\mathbf{A})$ redundantly, and are crucial objects in the theory of point configurations. To "prove" the existence of circuits in a point configuration and introduce some of their properties, we start with the following well-known fact.

Theorem B.2.4 (Radon's theorem)**.** *Let* $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{d+2}\}$ *be a point configuration in* $\mathbb{R}^d$*. There are two disjoint subsimplices* $\boldsymbol{\sigma}_1$ *and* $\boldsymbol{\sigma}_2$ *of* $\mathbf{A}$ *such that the following holds:*

$$\operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2) \neq \{\emptyset\},$$

where for a point configuration $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\} \subset \mathbb{R}^e$*, the relative interior is given by:*

$$\operatorname{relint}(\mathbf{B}) := \left\{ \sum_{j=1}^{m} r_j \mathbf{b}_j : \sum_{j=1}^{m} r_j = 1 \text{ and } r_j > 0 \ \forall\ 1 \leq j \leq m \right\}.$$

(The relative interior of a point is the point itself.)

For the proof of Radon's theorem we use a basic lemma.

Lemma B.2.5 (Carathéodory's theorem)**.** *Let* $\mathbf{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\}$ *be an affinely dependent point configuration in* $\mathbb{R}^d$*, and suppose* $\mathbf{q} \in \operatorname{conv}(\mathbf{A})$*. There is a minimal independent subconfiguration* $\boldsymbol{\sigma} \subset \mathbf{A}$ *with* $|\boldsymbol{\sigma}| \leq d+1$*, such that* $\mathbf{q} \in \operatorname{relint}(\boldsymbol{\sigma})$*.*

Proof. Suppose $\mathbf{q} = \sum_{i=1}^n r_i \boldsymbol{\alpha}_i$ with $\sum_{i=1}^n r_i = 1$ and $r_i \geq 0$ for $1 \leq i \leq n$. Define $J := \{i : 1 \leq i \leq n \text{ and } r_i > 0\}$, the *support* of r. If $\mathbf{A}|_J := \{\boldsymbol{\alpha}_i : i \in J\}$ is independent, then we are done. Otherwise, let $\mathbf{c} \in \mathbb{R}^n$ be an affine dependence on $\mathbf{A}|_J$. Consider the following identity:

$$\mathbf{q} = \sum_{i \in J} r_i \boldsymbol{\alpha}_i - \mu \sum_{i \in J} c_i \boldsymbol{\alpha}_i,$$

where $\mu \in \mathbb{R}$ is arbitrary. We make the choice $\min_{i\in J}\left\{\frac{r_i}{c_i} : c_i > 0\right\} =: \frac{r_l}{c_l}$.This way we have:

$$\mathbf{q} = \sum_{i\in J, i\neq l} (r_i - \mu c_i)\boldsymbol{\alpha}_i =: \sum_{i\in J'} r_i' \boldsymbol{\alpha}_i,$$

with $J' = J\backslash\{l\}$ and $r_i' := r_i - \mu c_i$. Now, $\sum_{i\in J'} r_i' = 1$, and $r_i' \geq 0$ for every $i \in J'$ (since $r_i - \mu c_i \geq r_i - \frac{r_i}{c_i} c_i = 0$) so we have a convex combination giving $\mathbf{q}$ with smaller support. If $\mathbf{A}_{J'}$ is not independent, then we can repeat the same argument to further reduce the support of the convex combination giving $\mathbf{q}$. This procedure terminates as soon as $\mathbf{q}$ is written as a convex combination of some independent elements $\boldsymbol{\sigma} \subset \mathbf{A}$. The subsimplex $\boldsymbol{\sigma}$ is minimal since $\mathbf{q} \in \mathrm{relint}(\boldsymbol{\sigma})$. Finally, if $\boldsymbol{\sigma}$ is independent, then $|\boldsymbol{\sigma}| \leq d+1$, for otherwise the system of equations:

$$\sum_{\boldsymbol{\alpha}_i\in\boldsymbol{\sigma}} c_i\boldsymbol{\alpha}_i = 0, \qquad \sum_{\boldsymbol{\alpha}_i\in\boldsymbol{\sigma}} c_i = 0,$$

would have at least a nonzero solution, i.e., there would an affine dependence on $\boldsymbol{\sigma}$. □

Proof of Radon's theorem. The homogeneous system of $(d+1)$ linear equations in $(c_1, \ldots, c_{d+2})^t$:

$$\sum_{i=1}^{d+2} c_i\boldsymbol{\alpha}_i = 0$$
$$\sum_{i=1}^{d+2} c_i = 0,$$

has at least one non-trivial solution, which we can write:

$$\sum_{i:c_i>0} c_i\boldsymbol{\alpha}_i = -\sum_{i:c_i<0} c_i\boldsymbol{\alpha}_i.$$

If we let $\kappa = \sum_{i:c_i>0} c_i = -\sum_{i:c_i<0} c_i$, this expression says that the point $\mathbf{q} := \frac{1}{\kappa}(\sum_{i:c_i>0} c_i\boldsymbol{\alpha}_i)$ lies in the convex hull of the points $\mathbf{D}^+ := \{\boldsymbol{\alpha}_i : c_i > 0\}$ (which is disjoint from $\mathbf{D}^- := \{\boldsymbol{\alpha}_i : c_i < 0\}$). Using Carathéodory's theorem, we find the minimal independent $\boldsymbol{\sigma}_1 \subseteq \mathbf{D}^+$ containing $\mathbf{q}$ in its relative interior. The same argument gives $\boldsymbol{\sigma}_2$. □

Thus, in the case when $\dim(\mathbf{A}) = d$ and $n = d+2$, there is exactly one affine dependence on $\mathbf{A}$, which is unique up to a non-zero scalar multiple. Radon's theorem says that there are disjoint subsimplices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of $\mathbf{A}$ such that $\mathrm{relint}(\boldsymbol{\sigma}_1) \cap \mathrm{relint}(\boldsymbol{\sigma}_2)$ is equal to a single point, and that the subconfiguration $\mathbf{C} = \boldsymbol{\sigma}_1 \cup \boldsymbol{\sigma}_2 \subseteq \mathbf{A}$ is therefore minimally dependent, in the sense that the removal of any point from it yields an independent collection of points. We shall refer to such minimally dependent configurations, as well as to the unique (up to scaling) affine dependence holding on them, as *circuits*.

The smallest circuits up to dimension two are illustrated in figure B.3. Note that the smallest circuit, shown at the left (the one of type $(1,1)$ – see ahead), consists of two "different" points with the same coordinates

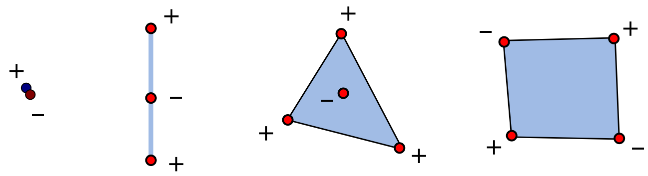


Figure B.3: Circuits up to dimension two.

When $\dim(\mathbf{A}) = d$ and $n > d+2$, it follows that $\mathbf{A}$ has at most one circuit for every d-dimensional $(d+2)$-element subconfiguration of $\mathbf{A}$, which can be obtained by application of Radon's theorem.

Let $\mathbf{C}$ be a circuit in a point configuration $\mathbf{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$ and $\sum_{i=1}^n c_i\boldsymbol{\alpha}_i = 0$, where $(c_1, \ldots, c_n)^t \in \mathbb{R}^n$, be the unique affine dependence on $\mathbf{C}$, so that $\mathbf{C} = \{\boldsymbol{\alpha}_i \in \mathbf{A} : c_i \neq 0\} \subseteq \mathbf{A}$. Observe that $\mathbf{C}$ comes with a partition of its elements into two nonempty subsets: $\mathbf{C}^+ := \{\boldsymbol{\alpha}_i \in \mathbf{A} : c_i > 0\}$ and $\mathbf{C}^- := \{\boldsymbol{\alpha}_i \in \mathbf{A} : c_i < 0\}$. If $|\mathbf{C}^+| = s$ and $|\mathbf{C}^-| = t$, we say $\mathbf{C}$ is a circuit of type (s,t). (The choice of which subconfiguration is $\mathbf{C}^+$ and which is $\mathbf{C}^-$ is irrelevant, as long as it remains fixed.)

From Radon's theorem, we thus see that circuits of a point configuration are the affine dependences with *minimal support*. That is, if $\mathbf{r} = (r_1, \ldots, r_n)$ defines the affine dependence $\sum_{i=1}^n r_i\boldsymbol{\alpha}_i = 0$, then there exists a circuit $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-)$ such that $\mathbf{C}^+ \subseteq \{\boldsymbol{\alpha}_i : \; r_i > 0\}$ and $\mathbf{C}^- \subseteq \{\boldsymbol{\alpha}_i : \; r_i < 0\}$. In fact, further development of Radon's theorem also says that circuits can be used to decompose affine dependences of a point configuration in a simple way: without cancellations. This is formalized in the next proposition that we will use in chapter 1.

Proposition B.2.6 (Lemma 6.7 in [Zie95] or corollary 4.1.13 in [DRS10])**.** *Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$ be a point configuration and suppose $\mathbf{r} = (r_1, \ldots, r_n)^t \in \mathbb{R}^n$ gives the affine dependence $\sum_{i=1}^n r_i\boldsymbol{\alpha}_i = 0$, for which we denote $\mathbf{R}^+ := \{\boldsymbol{\alpha}_i : r_i > 0\}$ and $\mathbf{R}^- := \{\boldsymbol{\alpha}_i : r_i < 0\}$. Then $\mathbf{r}$ can be written as a finite sum of minimal affine dependencies:*

$$\mathbf{r} = \sum_{i=1}^{s} \lambda_i \mathbf{c}_i,$$

where $\lambda_i > 0$, $\mathbf{C}_i^+ \subset \mathbf{R}^+$ and $\mathbf{C}_i^- \subset \mathbf{R}^-$ for every $i \in [s]$, where $\mathbf{C}_i$ is the circuit associated to $\mathbf{c}_i$.

Remark B.2.7*.* We mention that the combinatorial information present in the dependence relation between the points in a point configuration is abstracted in the notion of an *oriented matroid*. The axioms defining it can be stated in terms of the circuits of the point configuration, which justifies our assertion above that circuits contain all the combinatorial information of a point configuration. Below we will encounter more examples of the fundamental role played by circuits. For instance, they allow us to formulate useful properties and characterizations of triangulations of point configurations, and also to characterize the minimal modifications that a triangulation of a point configuration can undergo. ◯

We are now in the position to introduce the main definition in the thesis.

Definition B.2.8 (Theorem 4.5.4 in [DRS10]). A *triangulation* of a point configuration $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_n\}$ in $\mathbb{R}^d$ is a collection $\mathcal{T}$ of subsimplices of $\mathbf{A}$ that satisfy the following properties:

1. If $\boldsymbol{\sigma} \in \mathcal{T}$, and $\boldsymbol{\sigma}' \prec \boldsymbol{\sigma}$, then $\boldsymbol{\sigma}' \in \mathcal{T}$ as well
2. $\bigcup_{\boldsymbol{\sigma} \in \mathcal{T}} \operatorname{conv}(\boldsymbol{\sigma}) = \operatorname{conv}(\mathbf{A})$
3. For every pair $\boldsymbol{\sigma}, \boldsymbol{\sigma}' \in \mathcal{T}$ we have $\operatorname{conv}(\boldsymbol{\sigma}) \cap \operatorname{conv}(\boldsymbol{\sigma}') = \operatorname{conv}(\boldsymbol{\sigma} \cap \boldsymbol{\sigma}')$ (which may be empty)

Remark B.2.9. Property 1 is a *closure property* (**CP**) that states that the simplices forming $\mathcal{T}$ have the structure of a simplicial complex. The content of property 2 is clear, and we refer to it as the *union property* (**UP**) . Property 3 is called *hull property* (**HP**) , and says that any two simplices of $\mathcal{T}$ intersect always along a mutual face of both, which can be the empty face (note that we have adopted the notation from [DRS10]). ○

The fact that a triangulation $\mathcal{T}$ of a d-dimensional point configuration $\mathbf{A} = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n\}$ is a simplicial complex implies that it is completely specified by the list of subsets of points of $\mathbf{A}$ in every maximal simplex of $\mathcal{T}$. We may thus describe $\mathcal{T}$ by giving the $(d+1)$-subsets of $[n]$ that label the vertices of every d-dimensional simplex in $\mathcal{T}$.

Remark B.2.10. Triangulations of point configurations are a special case of the more general *polyhedral subdivisions*. These are dissections of point configurations into convex polytopes that define a polyhedral complex covering the point configuration, and morally satisfy similar properties as those in definition B.2.8. We remark that the analogous condition to property 3 above is more subtle, because not all subsets of the vertices of a polyhedral cell are faces of the cell. In this appendix, we will limit our exposition to the theory of triangulations, refering the reader to [Zie95, DRS10] for the general theory of polyhedral subdivisions of point configurations. ○

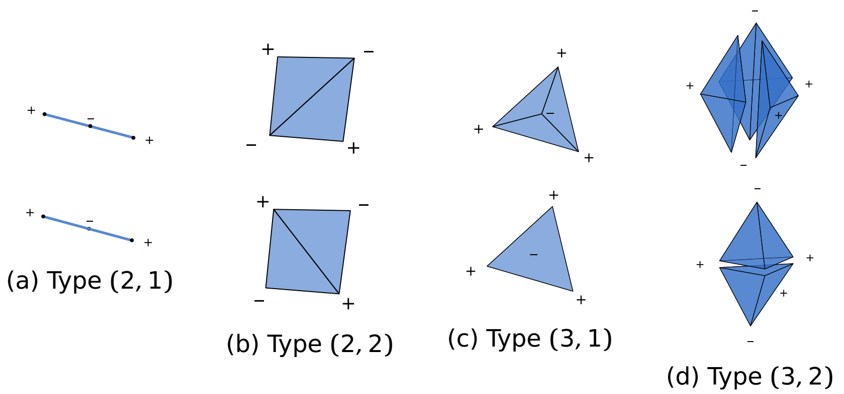
Remark B.2.11. In contrast to the definition of triangulation in other areas of mathematics, such as in geometry and topology or in discretizations of partial differential equations, in this one the points spanning the simplices of a triangulation are fixed from the beginning, and we are not allowed to add new points. ○

The simplest (non-trivial) triangulations are those of the minimally dependent point configurations, which have only two triangulations.

Proposition B.2.12 (Lemma 2.4.2 in [DRS10]). *Let $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-)$ be a circuit in $\mathbb{R}^d$. $\mathbf{C}$ has only two triangulations:*

$$\mathcal{T}^+ := \{\mathbf{C} \backslash \{\boldsymbol{\alpha}_i\} : \boldsymbol{\alpha}_i \in \mathbf{C}^+\}$$
$$\mathcal{T}^- := \{\mathbf{C} \backslash \{\boldsymbol{\alpha}_i\} : \boldsymbol{\alpha}_i \in \mathbf{C}^-\}$$

(See figure B.4 below for a depiction of this triangulations for the smallest circuits.)



(a) Type $(2, 1)$ (b) Type $(2, 2)$ (c) Type $(3, 1)$ (d) Type $(3, 2)$

Figure B.4: Unique triangulations of the smallest circuits. Top: $\mathcal{T}^+$, bottom: $\mathcal{T}^-$.

Remark B.2.13. We see in drawings B.4a and B.4c in figure B.4 that a triangulation of a point configuration $\mathbf{A}$ need not use all the points in $\mathbf{A}$. This may happen when the points in $\mathbf{A}$ are not in convex position. Also note that different triangulations of a point configuration may have different number of maximal simplices. ⬡

Example B.2.14. The five triangulations of a convex pentagon are shown in figure B.5 below.

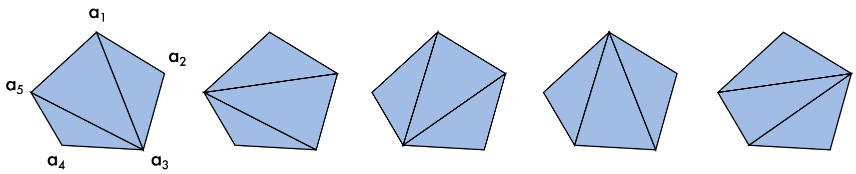


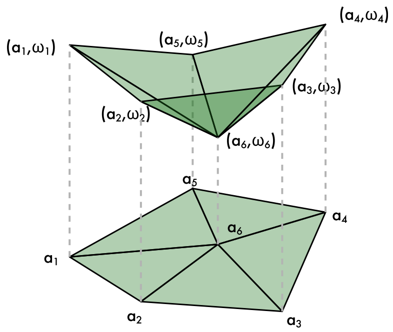
Figure B.5: The five triangulations of a convex pentagon.

From left to right, the following are the lists of maximal simplices in each of the triangulations:

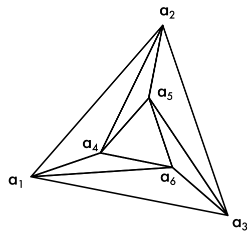
$$\begin{aligned}
\mathcal{T}_1 &= \{\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}, \{\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\}, \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_5\}\}, \\
\mathcal{T}_2 &= \{\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_5\}, \{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_5\}, \{\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\}\}, \\
\mathcal{T}_3 &= \{\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4\}, \{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}, \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\}\}, \\
\mathcal{T}_4 &= \{\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}, \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}, \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\}\}, \\
\mathcal{T}_5 &= \{\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_5\}, \{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}, \{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\}\}.
\end{aligned}$$

⬡

Example B.2.15. Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n\}$ be a point configuration in $\mathbb{R}^d$. A very important class of triangulations of $\mathbf{A}$ are the *regular triangulations* [Stu96, DRS10, Zie95], which are triangulations that can be induced by *lifting* the points in $\mathbf{A}$. Let $\omega \in \mathbb{R}^n$ be some weight vector, and consider the point configuration $\mathbf{A}^\omega := \{(\boldsymbol{\alpha}_1, \omega_1), (\boldsymbol{\alpha}_2, \omega_2), \ \ldots, (\boldsymbol{\alpha}_n, \omega_n)\} \subset \mathbb{R}^{d+1}$. Denote by $\mathbf{A}^\omega_\downarrow$ the collection of faces of $\mathbf{A}^\omega$ of the form $\mathrm{face}_\nu(\mathbf{A}^\omega)$, where $\nu = (\nu_1, \ldots, \nu_{d+1}) \in \mathbb{R}^{d+1}$ satisfies $\nu_{d+1} < 0$; if ω is chosen sufficiently generic, $\mathbf{A}^\omega_\downarrow$ consists of subsimplices of $\mathbf{A}^\omega$. The projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ that erases the last coordinate maps therefore maps $\mathbf{A}^\omega_\downarrow$ to a collection of subsimplices of $\mathbf{A}$ that constitutes a triangulation of $\mathbf{A}$, which we denote $\mathcal{T}_\omega$. See figure B.6a for an illustration of a regular triangulation. The triangulation in figure B.6b is the smallest example of a triangulation that cannot be obtained by lifting according to some weight vector: a *non-regular triangulation*. ○



(a) Lifting a point configuration (generically) induces a triangulation of it.



(b) No weight vector may induce this triangulation.

Figure B.6: Some triangulations are regular, others are not.

Remark B.2.16. The collection of regular triangulations of a point configuration $\mathbf{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n\} \in \mathbb{R}^d$ defines a stratification of $\mathbb{R}^d$ into equivalence classes: define $\omega_1 \sim \omega_2$ whenever $\mathcal{T}_{\omega_1} = \mathcal{T}_{\omega_2}$ [Zie95, GKZ08, Stu96, DRS10]. These equivalence clases are relatively open polyhedral cones that, together with the lower dimensional polyhedral cones of (equivalence classes of) weight functions inducing *coarser regular polyhedral subdivisions* of $\mathbf{A}$, form a fan that partitions $\mathbb{R}^n$. This fan is called the *secondary fan* of $\mathbf{A}$, and denoted $\Sigma(\mathbf{A})$. We briefly revisit $\Sigma(\mathbf{A})$ in appendix A, where it is compared with an analogous polyhedral stratification by a polyhedral fan that is induced by the initial ideals of a toric ideal. ○

Example B.2.17. Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_8\} \subset \mathbb{R}^4$ be the vertices of the 3-cube embedded in $\mathbb{R}^4$:

$$\begin{array}{lll} \boldsymbol{\alpha}_1 = (1,0,0,0)^t, & \boldsymbol{\alpha}_2 = (1,0,0,1)^t, & \boldsymbol{\alpha}_3 = (1,0,1,0)^t, \\ \boldsymbol{\alpha}_4 = (1,0,1,1)^t, & \boldsymbol{\alpha}_5 = (1,1,0,0)^t, & \boldsymbol{\alpha}_6 = (1,1,0,1)^t, \\ \boldsymbol{\alpha}_7 = (1,1,1,0)^t, & \boldsymbol{\alpha}_8 = (1,1,1,1)^t. & \end{array}$$

In figure B.7 we see two triangulation of the cube consisting of five and six simplices. ○

Looking at figure B.7, the reader may have realized the following fact.

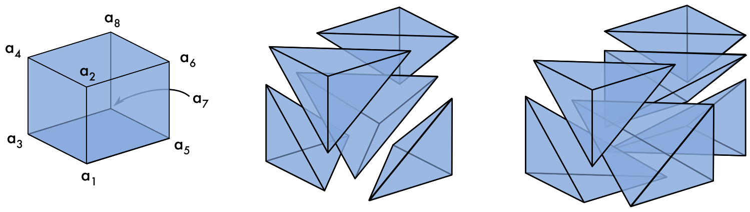


Figure B.7: Triangulation of the unit 3-cube into 5 and 6 simplices.

Proposition B.2.18. *Let $\mathcal{T}$ be a triangulation of a d-dimensional point configuration* $\mathbf{A}$*, and let* $\mathbf{F} \preceq \mathbf{A}$ *be a face of* $\mathbf{A}$*. The* restriction of $\mathcal{T}$ to $\mathbf{F}$*:*

$$\mathcal{T}|_{\mathbf{F}} := \{\boldsymbol{\sigma} \cap \mathbf{F} : \boldsymbol{\sigma} \in \mathcal{T}\},$$

is a triangulation of the point configuration $\mathbf{F}$*.*

Converse to starting with a triangulation of a point configuration and then restricting it to have a triangulation of a face of it, we may also consider starting with a collection of triangulations of some faces of a point configuration that agree on their intersections, which we may think of as restrictions of a triangulation of the whole point configuration. This is the notion of partial triangulations, that will come up in chapter 2 when we study products of simplices. There, we will partially solve the question of when does a partial triangulation of the product of two simplices completely determine a total triangulation of them.

Definition B.2.19. Let $\mathbf{A}$ be a point configuration in $\mathbb{R}^d$, and let $\mathcal{K} := \{\mathbf{A}_1, \ldots, \mathbf{A}_r\}$ be a collection of faces of $\mathbf{A}$. A *partial triangulation* of $\mathbf{A}$ with respect to $\mathcal{K}$, or equivalently a *triangulation of the polyhedral complex* $\mathcal{K} \subset \mathbf{A}$, is a collection $\mathcal{T}' := \{\mathcal{T}_{\mathbf{A}_1}, \ldots, \mathcal{T}_{\mathbf{A}_r}\}$ of triangulations of the faces in $\mathcal{K}$ such that $\mathcal{T}_{\mathbf{A}_i}|_{\mathbf{F}} = \mathcal{T}_{\mathbf{A}_j}|_{\mathbf{F}}$ whenever $\mathbf{F} \in \mathbf{A}_i \cap \mathbf{A}_j$ is a nonempty subconfiguration, and $i, j \in [r]$. Here we abuse notation and call a collection of point configurations $\mathcal{K} := \{\mathbf{A}_1, \ldots, \mathbf{A}_r\}$ a *polyhedral complex* whenever $\mathbf{A}_i \cap \mathbf{A}_j$ is a face of both and $\operatorname{relint}(\mathbf{A}_i) \cap \operatorname{relint}(\mathbf{A}_j) = \{\emptyset\}$ for every $i \neq j \in [r]$.

Characterization of triangulations

We have chosen the particular characterization of a triangulation in definition B.2.8 because it is one of the most familiar. The reader may refer to the book [DRS10], which contains a formidable collection of characterizations of a triangulation, and also of the more general *polyhedral subdivisions*, for a thorough treatment. Besides that, in [DRS10] one finds many different results to check a given collection of simplices fulfills the properties of a triangulation. They fit diverse situations, and the ones we found more adequate for our work are the presented next, that we paraphrase from [DRS10].

To verify property (**HP**) , first notice that it is equivalent to saying that any two simplices in a collection of simplices $\mathcal{T}$ satisfying (**CP**) intersect properly, in the sense that $\operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2) = \{\emptyset\}$ for any two different $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{T}$.

This property is called the *intersection property* (**IP**) , and when it holds for a collection of simplices, we say that they *intersect properly*.

Indeed, if $\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2 = \{\emptyset\}$, then from (**HP**) we have that $\operatorname{conv}(\boldsymbol{\sigma}_1) \cap \operatorname{conv}(\boldsymbol{\sigma}_2) = \{\emptyset\}$, which in turn implies that $\operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2) = \emptyset$. If $\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2 \neq \{\emptyset\}$, suppose $\mathbf{q} \in \operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2)$; in particular, it holds that $\mathbf{q} \in \operatorname{conv}(\boldsymbol{\sigma}_1) \cap \operatorname{conv}(\boldsymbol{\sigma}_2) = \operatorname{conv}(\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2)$, but this is only possible if $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2$, since both $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are independent and $\mathbf{q}$ belongs to the relative interiors of both.

Conversely, assume there exists some $\mathbf{r} \in \operatorname{conv}(\boldsymbol{\sigma}_1) \cap \operatorname{conv}(\boldsymbol{\sigma}_2)$ such that $\mathbf{r} \notin \operatorname{conv}(\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2)$ for two different $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{T}$. Using Caratheodory's theorem, we find the minimal $\boldsymbol{\tau}_1 \subseteq \boldsymbol{\sigma}_1$ and $\boldsymbol{\tau}_2 \subseteq \boldsymbol{\sigma}_2$ such that $\mathbf{r} \in \operatorname{relint}(\boldsymbol{\tau}_1) \cap \operatorname{relint}(\boldsymbol{\tau}_2)$. Since $\mathbf{r} \notin \operatorname{conv}(\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2)$, we have that $\boldsymbol{\tau}_1 \nsubseteq \boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2$ and $\boldsymbol{\tau}_2 \nsubseteq \boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2$, i.e., $\operatorname{relint}(\boldsymbol{\tau}_1) \cap \operatorname{relint}(\boldsymbol{\tau}_2) \neq \{\emptyset\}$ for some $\boldsymbol{\tau}_1 \neq \boldsymbol{\tau}_2 \in \mathcal{T}$, and the equivalence follows. See figure for an illustration of the situation $\operatorname{conv}(\boldsymbol{\sigma}_1) \cap \operatorname{conv}(\boldsymbol{\sigma}_2) \neq \operatorname{conv}(\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2)$.

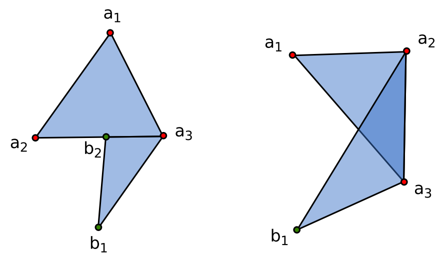


Figure B.8: Illustration of $\operatorname{conv}(\boldsymbol{\sigma}_1) \cap \operatorname{conv}(\boldsymbol{\sigma}_2) \neq \operatorname{conv}(\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2)$. In the left $\boldsymbol{\sigma}_1 := \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ and $\boldsymbol{\sigma}_2 := \{\mathbf{b}_1, \mathbf{b}_2, \boldsymbol{\alpha}_3\}$; in the right $\boldsymbol{\sigma}_1 := \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ and $\boldsymbol{\sigma}_2 := \{\mathbf{b}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$. In both cases $\operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2) \neq \{\emptyset\}$.

With the equivalence (for triangulations) of (**HP**) and (**IP**) at hand, we present the following criterion to verify that (**HP**) holds for a collection of simplices.

Lemma B.2.20 (Theorems 4.1.14 and 4.1.15 in [DRS10])**.** *Let $\mathcal{S}$ be a collection of subsimplices of a point configuration* **A** *satisfying the closure property (***CP***) . The following are equivalent:*

1. *For any two different simplices $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{S}$, $\operatorname{relint}(\boldsymbol{\sigma}_1) \cap \operatorname{relint}(\boldsymbol{\sigma}_2) = \{\emptyset\}$.*

2. *There is no circuit $\mathbf{C} = (\mathbf{C}^+, \mathbf{C}^-)$ of* **A** *such that $\mathbf{C}^+ \subseteq \boldsymbol{\tau}_1$ and $\mathbf{C}^- \subseteq \boldsymbol{\tau}_2$, where $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are two different simplices in $\mathcal{S}$.*

If $\mathbf{C}^+ \subseteq \boldsymbol{\tau}_1$ and $\mathbf{C}^- \subseteq \boldsymbol{\tau}_2$ for two different $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathcal{S}$, we say that $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ overlap on the circuit **C***.*

Proof. The proof follows directly from Radon's theorem B.2.4. □

Lemma B.2.20 is particularly useful when we know the circuits of a point configuration, because it gives us a way to combinatorially check whether a collection of simplices of a point configuration **A** intersects properly, namely by proving that no pair of simplices overlap on a circuit of **A**. This will be one of our tools of the trade in chapter 2, when we consider products of simplices.

The characterization next proved very convenient for verifying that the triangulations considered in this thesis are, in fact, triangulations.

Theorem B.2.21 (Corollary 4.5.20 in [DRS10])**.** *Let $\mathcal{T}$ be a collection of subsimplices of a point configuration $\mathbf{A}$ in $\mathbb{R}^d$. If:*

(**TV**) *The total volume of the simplices spanned by the elements of $\mathcal{T}$ equals the volume of* $\mathrm{conv}(\mathbf{A})$*,*

and

$\circ_a$ *For every interior simplex $\boldsymbol{\tau} \in \mathcal{T}$ of codimension 1, there are exactly two vertices $\mathbf{v}, \mathbf{v}' \in \mathbf{A}$ such that $\boldsymbol{\tau} \cup \{\mathbf{v}\}$ and $\boldsymbol{\tau} \cup \{\mathbf{v}'\}$ are full-dimensional simplices in $\mathcal{T}$ and $\mathbf{v}$ and $\mathbf{v}'$ lie on opposite sides of the plane spanned by the points in $\boldsymbol{\tau}$*

or, equivalently,

$\circ_b$ *For every interior simplex $\boldsymbol{\tau} \in \mathcal{T}$ of codimension 1, there are exactly two vertices $\mathbf{v}, \mathbf{v}' \in \mathbf{A}$ such that $\boldsymbol{\tau} \cup \{\mathbf{v}\}$ and $\boldsymbol{\tau} \cup \{\mathbf{v}'\}$ are full-dimensional simplices in $\mathcal{T}$ and $\mathbf{v}$ and $\mathbf{v}'$ have the same sign (i.e. lie in the same part) in the unique circuit contained in $\boldsymbol{\tau} \cup \{\mathbf{v}\} \cup \{\mathbf{v}'\}$,*

then $\mathcal{T}$ is a triangulation of $\mathbf{A}$.

Geometric bistellar flips

Let $\mathbf{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\}$ be a d-dimensional point configuration in $\mathbb{R}^d$ and $\mathcal{T}$ be a triangulation of $\mathbf{A}$. We are interested in the minimal modifications that can be performed on $\mathcal{T}$; intuition tells that these are obtained by removing some subcomplex of the simplicial complex $\mathcal{T}$ and properly gluing back a different subcomplex to obtain some triangulation $\mathcal{T}'$. The concept we need to define the proper removal or gluing back of a subcomplex $\boldsymbol{\sigma}$ in a simplicial complex $\mathcal{T}$ is the following.

Definition B.2.22. Let $\boldsymbol{\tau} \in \mathcal{T}$. The *link of $\boldsymbol{\tau}$ in $\mathcal{T}$* is the following subcomplex of $\mathcal{T}$:

$$\mathrm{link}_{\mathcal{T}}(\boldsymbol{\tau}) := \{\boldsymbol{\sigma} \in \mathcal{T} : \boldsymbol{\tau} \not\subset \boldsymbol{\sigma},\ \boldsymbol{\sigma} \cup \boldsymbol{\tau} \in \mathcal{T}\}\,.$$

Definition B.2.23 (Theorem 4.4.1 in [DRS10])**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of the point configuration $\mathbf{A}$. We say $\mathcal{T}$ and $\mathcal{T}'$ differ by a *geometric bistellar flip supported on the circuit* $\mathbf{C} \subset \mathbf{A}$, or just *flip*, if the following conditions are satisfied:

- $\mathcal{T}^-(\mathbf{C}) \subseteq \mathcal{T}$ and $\mathcal{T}^+(\mathbf{C}) \subseteq \mathcal{T}'$, where $\mathcal{T}^-(\mathbf{C})$ and $\mathcal{T}^+(\mathbf{C})$ denote the two unique triangulations of $\mathbf{C}$ (and the inclusion is defined simplex-wise).
- For any two maximal simplices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of $\mathcal{T}^-(\mathbf{C})$, $\mathrm{link}_{\mathcal{T}}(\boldsymbol{\sigma}_1) = \mathrm{link}_{\mathcal{T}}(\boldsymbol{\sigma}_2)$.

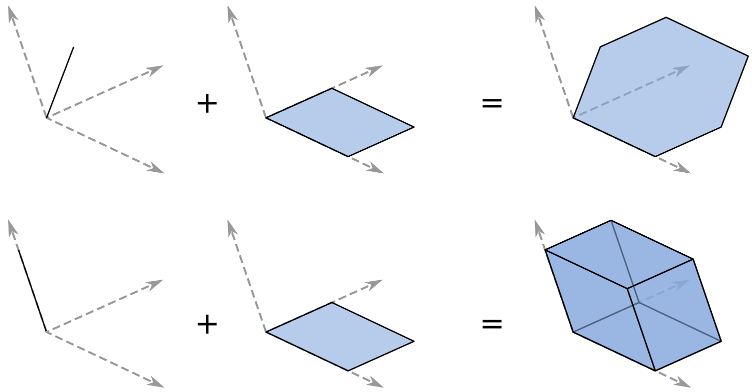


Figure B.9: Minkowski sum of (the vertices of) a segment and (the vertices of) a quadrilateral. Notice that the result depends on the relative positions of the summands.

- $\mathcal{T}' = \mathcal{T} \setminus \left(\mathcal{T}^-(\mathbf{C}) \cup \mathrm{link}_{\mathcal{T}}(\boldsymbol{\sigma})\right) \cup \left(\mathcal{T}^+(\mathbf{C}) \cup \mathrm{link}_{\mathcal{T}}(\boldsymbol{\sigma})\right)$, where $\boldsymbol{\sigma}$ is a maximal simplex of $\mathcal{T}^-(\mathbf{C})$.

Remark B.2.24. Let $\mathbf{A}$ be a point configuration in $\mathbb{R}^d$, and $\mathcal{T}$ be a triangulation of $\mathbf{A}$ that admits a flip supported on a circuit $\mathbf{C}$ of $\mathbf{A}$, so that $\mathcal{T}$ contains the triangulation $\mathcal{T}^+$ of $\mathbf{C}$. We can define the *codimension* of the corresponding flip of $\mathcal{T}$ as $\dim(\mathbf{A}) - \dim(\mathbf{C})$. If the codimension of a flip is 0, in particular, the link of every maximal simplex in $\mathcal{T}^+$ is empty, and the triangulation obtained after flipping $\mathcal{T}^+$ to $\mathcal{T}^-$ differs from $\mathcal{T}$ only in the simplices in $\mathcal{T}^+$. Said differently, the new triangulation is gotten after *carving* out $\mathbf{C}$ from $\mathcal{T}$ and gluing it back with the triangulation $\mathcal{T}^-$, without altering the rest of $\mathcal{T}$ in the process. ○

B.3 Fine mixed subdivisions

A second way to obtain new point configurations from existing ones is via Minkowski summation. We will use this in section 2.4 when working with partial triangulations of products of simplices. Our presentation here follows closely the articles [HRS00, San04] and the book [DRS10].

Let $\mathbf{P}$ and $\mathbf{Q}$ be point configurations in $\mathbb{R}^d$. We define their *Minkowski sum* to be the point configuration:

$$\mathbf{P} + \mathbf{Q} := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathbf{P},\ \mathbf{y} \in \mathbf{Q}\} \subset \mathbb{R}^d.$$

As opposed to the cartesian product, the result of the Minkowski sum $\mathbf{P}+\mathbf{Q}$, e.g. its dimension, depends on how the point configurations $\mathbf{P}$ and $\mathbf{Q}$ are embedded in $\mathbb{R}^d$, as shown in figure B.9.

In particular, not every pair of faces of $\mathbf{P}$ and $\mathbf{Q}$ makes a face of $\mathbf{P}+\mathbf{Q}$. Instead, we have:

$$\mathrm{face}_\omega(\mathbf{P} + \mathbf{Q}) = \mathrm{face}_\omega(\mathbf{P}) + \mathrm{face}_\omega(\mathbf{Q}),$$

where ω is some vector in $\mathbb{R}^d$ (representing a linear functional in $(\mathbb{R}^d)^*$). However, we have the following special case.

Proposition B.3.1 (See section 2.3 in [HRS00])**.** *Suppose* $\mathbf{P}$ *and* $\mathbf{Q}$ *are point configurations lying on independent affine subspaces. Then their Minkowski sum* $\mathbf{P}+\mathbf{Q}$ *is combinatorially equivalent to their cartesian product* $\mathbf{P}\times\mathbf{Q}$*. By this we mean that* $\mathbf{P}+\mathbf{Q}$ *and* $\mathbf{P}+\mathbf{Q}$ *have the same set of circuits. This can be seen in figure B.9.*

In this thesis, our interest in Minkowski sums derives from the Cayley trick: a correspondence between triangulations of a certain point configuration with fine mixed subdivisions of the Minkowski sum of certain point configurations.

Fine mixed subdivisions emerge as natural dissections of point configurations that are obtained as Minkowski sums. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be point configurations in $\mathbb{R}^d$ such that $\dim(\mathbf{P}_1 + \ldots + \mathbf{P}_n) = d$. A *fine mixed cell* in the Minkowski sum $\mathbf{P} = \mathbf{P}_1 + \ldots + \mathbf{P}_n$ is a point configuration $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_n$ such that each $\mathbf{B}_i$ is a (non-empty) subsimplex of $\mathbf{P}_i$ and $\dim(\mathbf{B}) = \sum_{i=1}^n \dim(\mathbf{B}_i)$, that is, the $\mathbf{B}_i$ lie on independent affine spaces.

Let $\mathbf{B} = \sum_{i=1}^n \mathbf{B}_i$ and $\mathbf{B'} = \sum_{i=1}^n$ be fine mixed cells in $\mathbf{P} = \mathbf{P}_1 + \ldots + \mathbf{P}_n$. We say $\mathbf{B'}$ is a face of $\mathbf{B}$, and write $\mathbf{B'} \preceq \mathbf{B}$, if $\mathbf{B}'_i \preceq \mathbf{B}_i$ for every $1 \le i \le n$.

Definition B.3.2 (See section 2.3 in [HRS00] or definition 1.1 in [San04])**.** With the notation from above, a *fine mixed subdivision* of $\mathbf{P}_1 + \ldots + \mathbf{P}_n$ is a collection $\mathcal{M}$ of fine mixed cells such that:

1. If $\mathbf{B} \in \mathcal{M}$ and $\mathbf{B'} \preceq \mathbf{B}$, then also $\mathbf{B'} \in \mathcal{M}$.
2. $\cup_{\mathbf{B}\in\mathcal{M}} \mathrm{conv}(\mathbf{B}) = \mathrm{conv}(\mathbf{P}_1 + \ldots + \mathbf{P}_n)$, that is, the mixed cells cover the polytope $\mathrm{conv}(\mathbf{P}_1 + \ldots + \mathbf{P}_n)$.
3. If $\mathbf{B} = \sum_i \mathbf{B}_i$ and $\mathbf{B'} = \sum_i \mathbf{B}'_i$ are mixed cells in $\mathcal{M}$, then $\mathbf{B} \cap \mathbf{B'} = \sum_i \mathbf{B}_i \cap \mathbf{B}'_i$ is a face of both $\mathbf{B}$ and $\mathbf{B'}$ (here, if one of the $\mathbf{B}_i \cap \mathbf{B}'_i$ is empty, then $\mathbf{B} \cap \mathbf{B'}$ is empty).

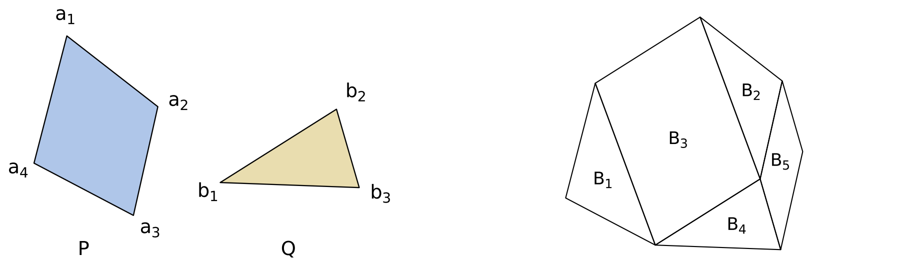
Remark B.3.3. Just as triangulations of point configurations are the "finest" instance of polyhedral subdivisions, fine mixed subdivisions are the finest instance of more general mixed subdivisions. ○

Remark B.3.4. Note that the point configurations corresponding to fine mixed cells are combinatorially isomorphic to cartesian products of simplices, as follows from proposition B.3.1. ○

Example B.3.5. Suppose $\mathbf{P}$ and $\mathbf{Q}$ are the (vertices of the) quadrilateral and the triangle depicted in figure B.10a. In figure B.10b we see a fine mixed subdivision of the Minkowski sum $\mathbf{P}+\mathbf{Q}$, where the full-dimensional fine mixed cells are:

$$\mathbf{B}_1 = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\} + \{\mathbf{b}_1\} \quad \mathbf{B}_2 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} + \{\mathbf{b}_2\} \quad \mathbf{B}_3 = \{\mathbf{a}_1, \mathbf{a}_3\} + \{\mathbf{b}_1, \mathbf{b}_2\}$$
$$\mathbf{B}_4 = \{\mathbf{a}_3\} + \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \quad \mathbf{B}_5 = \{\mathbf{a}_2, \mathbf{a}_3\} + \{\mathbf{b}_2, \mathbf{b}_3\}$$

○



(a) The polytopes **P** and **Q**.

(b) A fine mixed subdivision of $\mathbf{P}+\mathbf{Q}$.

Figure B.10: The mixed subdivision in example B.3.5.

The properties in the definition of a fine mixed subdivision should be regarded in analogy to those in the definition of a triangulation. Still, we consider it crucial to stress once more the fact that the dissection of $\operatorname{conv}(\mathbf{P}_1+\ldots+\mathbf{P}_n)$ in a fine mixed subdivision is in the combinatorial sense of the properties in definition B.3.2, rather than only in the geometric sense. The reason is that frequently (more than for triangulations) there will be several mixed cells in the Minkowski sum $\mathbf{P}_1+\ldots+\mathbf{P}_n$ that have the same “geometric shape”, while only one (or none) fits in properly (in the sense of fine mixed subdivisions).

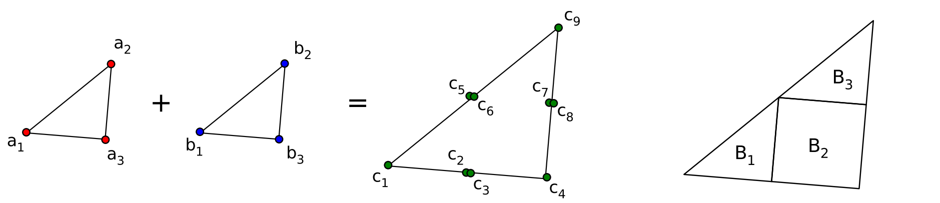
Example B.3.6. Let $\mathbf{P}_1=\mathbf{P}_2$ be the vertices of a triangle, and consider $\mathbf{P}_1+\mathbf{P}_2$, whose vertices are:

$$\begin{aligned}
&\mathbf{c}_1=\mathbf{a}_1+\mathbf{b}_1, && \mathbf{c}_2=\mathbf{a}_1+\mathbf{b}_3, && \mathbf{c}_3=\mathbf{a}_3+\mathbf{b}_1,\\
&\mathbf{c}_4=\mathbf{a}_3+\mathbf{b}_3, && \mathbf{c}_5=\mathbf{a}_1+\mathbf{b}_2, && \mathbf{c}_6=\mathbf{a}_2+\mathbf{b}_1,\\
&\mathbf{c}_7=\mathbf{a}_3+\mathbf{b}_2, && \mathbf{c}_8=\mathbf{a}_2+\mathbf{b}_3, && \mathbf{c}_9=\mathbf{a}_2+\mathbf{c}_2.
\end{aligned}$$

This is illustrated in figure B.11a. The fine mixed cells:

$$\begin{aligned}
&\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}+\{\mathbf{b}_1\}, && \{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}+\{\mathbf{b}_2\}, && \{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}+\{\mathbf{b}_3\},\\
&\{\mathbf{a}_1\}+\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}, && \{\mathbf{a}_2\}+\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}, && \{\mathbf{a}_3\}+\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\},
\end{aligned}$$

all correspond to the same polytope: a triangle; however they are regarded as different fine mixed cells.



(a) Notice that several points have the same geometric coordinates.

(b) The mixed subdivision $\mathcal{M}$ from example B.3.6.

Figure B.11: Minkowski sum of two point configurations.

Now, consider the mixed cells:

$$\mathbf{B}_1 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} + \{\mathbf{b}_1\}, \qquad \mathbf{B}_2 = \{\mathbf{a}_1, \mathbf{a}_3\} + \{\mathbf{b}_1, \mathbf{b}_3\},$$
$$\mathbf{B}_3 = \{\mathbf{a}_2\} + \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}, \qquad \mathbf{B}'_3 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} + \{\mathbf{b}_2\}.$$

The collection $\mathcal{M} = \{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ (and all faces thereof) is a fine mixed subdivision of $\mathbf{P}_1 + \mathbf{P}_2$ (see figure B.11b), but $\mathcal{M}' = \{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}'_3\}$ is not: even though $\mathbf{B}_3$ and $\mathbf{B}'_3$ define the same polytope, the fine mixed cells $\mathbf{B}_2$ and $\mathbf{B}'_3$ do not intersect properly. ⬡

Remark B.3.7. In view of example B.3.6 and the paragraph preceeding it, let us remark that fine mixed cells in a Minkowski sum $\mathbf{P}_1 + \ldots + \mathbf{P}_n$ are perhaps better thought of as ordered n-tuples of subconfigurations of the $\mathbf{P}_1, \ldots, \mathbf{P}_n$. This way, we fix the "identity" of the summands of a fine mixed cell, and regard the proper intersection property as "entry-wise" proper intersection of the n-tuples of subconfigurations giving the fine mixed cells. ⬡

In analogy to proposition B.2.18, the following result establishes a natural notion of restriction for mixed subdivisions.

Proposition B.3.8 (Lemma 2.1 in [San04])**.** *Let $\mathcal{M}$ be a fine mixed subdivision of $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$, and $I \subset [n]$ be the indices of some summands of $\mathbf{P}$. The restriction:*

$$\mathcal{M}|_I := \left\{ \mathbf{B}|_I := \sum_{i \in I} \mathbf{B}_i : \mathbf{B} = \sum_{i=1}^{n} \mathbf{B}_i \in \mathcal{M} \right\},$$

is a fine mixed subdivision of the Minkowski sum $\mathbf{P}|_I := \sum_{i \in I} \mathbf{P}_i$. In particular, if $j \in [n]$, then $\mathcal{M}|_{\{j\}}$ is a triangulation of $\mathbf{P}_j$.

The truth of this proposition can be intuitively understood by picturing that all the summands of $\mathbf{P}$ not labelled by I "shrink" to have zero size, so that the "limit object" coincides with $\mathcal{M}|_I$. A more complete argument is postponed until the next section, where we use the Cayley trick to prove it easily.

B.4 The Cayley trick

Generally, the Cayley trick gives a bijection between the mixed subdivisions of the Minkowski sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$ in $\mathbb{R}^d$ and the polyhedral subdivisions of a certain point configuration $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$ in $\mathbb{R}^{n+d}$, called the *Cayley lifting* or *Cayley embedding* of $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$. This bijection restricts to a correspondence between fine mixed subdivisions of $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$ and triangulations of $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$, and we will restrict our presentation to this special case. The reader can find the general theory, its implications in elimination theory and discrete geometry, as well as many references, in [Stu94, HRS00, San04, DRS10].

We superficially comment that the Cayley trick had its original motivation in elimination theory, and was (re-)discovered by Sturmfels in [Stu94]. In that formulation, it identified the resultant of a system of polynomial equations with the discriminant of a certain polynomial in more variables [GKZ08, HRS00]. The polyhedral version of the Cayley trick is (the general version of) theorem B.4.3 below, and it served Sturmfels, among others, to reinterpret and generalize several existing results about the resultant in discrete-geometric terms.

Definition B.4.1 (Section 2.4 in [HRS00] or section 1.3 in [San04])**.** Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be point configurations in $\mathbb{R}^d$. Denote by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ the standard basis vectors in $\mathbb{R}^n$, and consider the inclusions $\mu_i : \mathbb{R}^d \to \mathbb{R}^n \times \mathbb{R}^d = \mathbb{R}^{n+d}$ given by $\mu_i(\mathbf{v}) = \{\mathbf{e}_i\} \times \mathbf{v} = (\mathbf{e}_i, \mathbf{v})$, for $1 \leq i \leq n$. The *Cayley lifting*, or *Cayley embedding*, of the point configurations $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ is the point configuration:

$$\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n) := \bigcup_{i=1}^{n} \mu_i(\mathbf{P}_i) \subset \mathbb{R}^{n+d} \tag{B.1}$$

Example B.4.2. The simplest examples are shown below.

- The Cayley embedding of n copies of a point is an $(n-1)$-simplex.
- The Cayley embedding of two copies of a segment is a square.
- More generally, the Cayley embedding of n copies of a point configuration $\mathbf{P} \subset \mathbb{R}^d$ equals the cartesian product $\Delta_{n-1} \times \mathbf{P} \subset \mathbb{R}^{n+d}$.

See figure B.12 for illustrations. ⬡

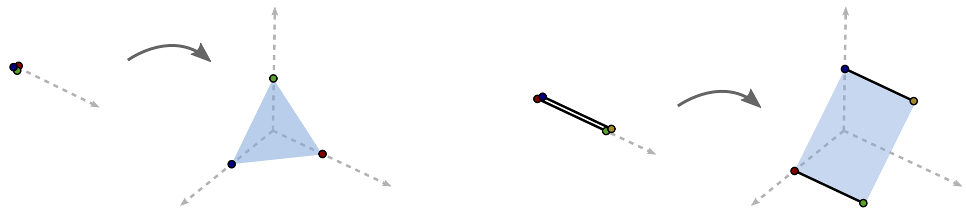


Figure B.12: Cayley embeddings of three copies of a point and of two copies of a segment

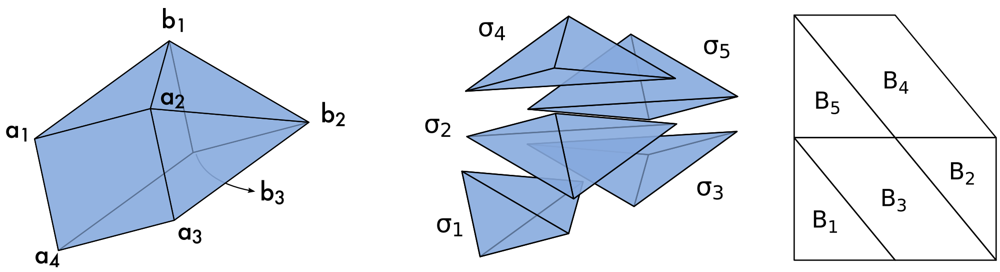
Theorem B.4.3 (Cayley trick [HRS00, San04, San12])**.** *In the notation of definition B.4.1, there is a bijection between triangulations of the point configuration $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$ and fine mixed subdivisions of the Minkowski sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$. The correspondence is as follows: let $\mathcal{T}$ be a triangulation of $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$. To each full-dimensional simplex $\boldsymbol{\sigma}$ in $\mathcal{T}$, assign the fine mixed cell $\mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_n$, where $\mathbf{B}_i$ is the simplex of $\mathbf{P}_i$ such that:*

$$\{\mathbf{e}_i\} \times \mathbf{B}_i = \boldsymbol{\sigma} \cap (\{\mathbf{e}_i\} \times \mathbf{P}_i)$$

In the other direction, let $\mathcal{M}$ be a fine mixed subdivision of $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$. To the full-dimensional fine mixed cell $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_n \in \mathcal{M}$, assign the simplex $\boldsymbol{\sigma}$ of $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$ gotten as:

$$\boldsymbol{\sigma} = \bigcup_{i=1}^{n} \{\mathbf{e}_i\} \times \mathbf{B}_i$$

Example B.4.4. We illustrate the content of the Cayley trick with an example. Let $\mathbf{P}_1$ be the vertices of a quadrangle and $\mathbf{P}_2$ be the vertices of a triangle. The Cayley embedding $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2)$ is displayed in the figure B.13a. In figure B.13b we see a triangulation $\mathcal{T}$ of $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2)$ having the maximal simplices $\boldsymbol{\sigma}_1 = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b}_3\}$, $\boldsymbol{\sigma}_2 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_2\}$, $\boldsymbol{\sigma}_3 = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{b}_2, \mathbf{b}_3\}$, $\boldsymbol{\sigma}_4 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2\}$ and $\boldsymbol{\sigma}_5 = \{\mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. The mixed subdivision of $\mathbf{P}_1 + \mathbf{P}_2$ corresponding to $\mathcal{T}$, in accordance with the Cayley trick, is shown at the right. The fine mixed cells are $\mathbf{B}_1 = \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} + \{\mathbf{b}_3\}$, $\mathbf{B}_2 = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} + \{\mathbf{b}_2\}$, $\mathbf{B}_3 = \{\mathbf{a}_1, \mathbf{a}_3\} + \{\mathbf{b}_2, \mathbf{b}_3\}$, $\mathbf{B}_4 = \{\mathbf{a}_1, \mathbf{a}_2\} + \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{B}_5 = \{\mathbf{a}_1\} + \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. ⬡



(a) The Cayley lifting of a square and a triangle

(b) A triangulation and the corresponding fine mixed subdivision

Figure B.13: Illustration of the Cayley trick

It is easy to check that the correspondences in theorem B.4.3 are inverse to each other. Rather than include a complete proof of the Cayley trick here, which can be found in [HRS00], we illustrate the underlying geometric idea with a concrete example.

Let $\mathbf{P}$ be the vertices of the unit square embedded in $\mathbb{R}^3$: $\mathbf{P} = \left\{(1,0,0)^t, (1,0,1)^t, (1,1,0)^t, (1,1,1)^t\right\}$. The Cayley embedding of two copies of $\mathbf{P}$ consists of the columns of the matrix:

$$\mathcal{C}(\mathbf{P}, \mathbf{P}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

This point configuration coincides with the vertices of a unit 3-cube. Let $\mathcal{T}$ be a triangulation of $\mathcal{C}(\mathbf{P}, \mathbf{P})$. The bijection in the Cayley trick is achieved by intersecting $\operatorname{conv}(\mathcal{C}(\mathbf{P}, \mathbf{P}))$ with the affine plane $\mathcal{H} = \{\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\} \times \mathbb{R}^3$. Such a plane intersects every maximal simplex of any triangulation of $\mathcal{C}(\mathbf{P}, \mathbf{P})$; in particular, $\mathcal{T} \cap \mathcal{H}$ can be seen as (a scaled copy of) a fine mixed subdivision of $\mathbf{P} + \mathbf{P}$, which is precisely the one associated to $\mathcal{T}$ by the bijection in the Cayley trick.

We have drawn $\mathcal{C}(\mathbf{P}, \mathbf{P})$ with a triangulation $\mathcal{T}$ in figure B.14. The intersection of $\mathcal{H}$ with the codimension 1 simplices of $\mathcal{T}$ is shown as thick red lines in the left. In the right, we see the fine mixed subdivision resulting. The reader can check that the simplex $\boldsymbol{\sigma}_i$ corresponds to the fine mixed cell $\mathbf{B}_i$ (for $1 \leq i \leq 6$), and that the collection of fine mixed cells so obtained is a fine mixed subdivision of $\mathbf{P}+\mathbf{P}$.

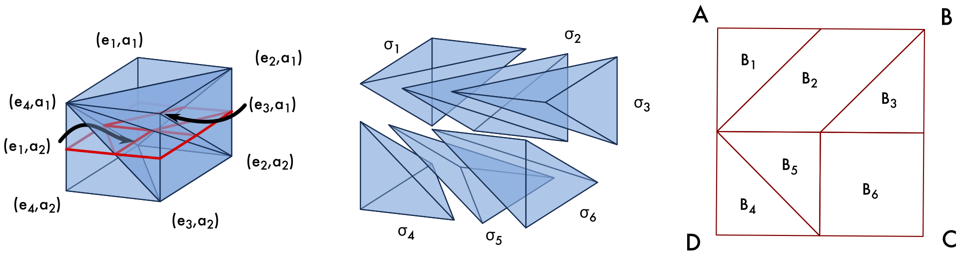


Figure B.14: Geometric idea behind the Cayley trick.

With the next simple property of Cayley embeddings, we formulate a short proof of proposition B.3.8.

Proposition B.4.5. *With the notation in definition B.4.1, we have that:*

$$\{\mathbf{e}_i\} \times \mathbf{P}_i = \mathrm{face}_{\omega_i}(\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)),$$

with $\omega_i = (\mathbf{e}_i, 0) \in \mathbb{R}^{n+d}$.

Proof of proposition B.3.8. With the Cayley trick, every mixed subdivision of $\sum_{i \in I} \mathbf{P}_i$ is in bijection with a triangulation of $\mathcal{C}(\{\mathbf{P}_i : i \in I\})$. The latter point configuration is of the form $\mathrm{face}_\omega(\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n))$ with $\omega = \left(\sum_{i \in I} \mathbf{e}_i, 0\right)$, to which we may restrict any triangulation of $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n)$ to get a fine mixed subdivision of $\sum_{i \in I} \mathbf{P}_i$. $\square$

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