Onsager's Conjecture

(Die Vermutung von Onsager)

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Onsager's Conjecture

Abstract

In 1949, Lars Onsager in his famous note on statistical hydrodynamics conjectured that weak solutions to the 3-D incompressible Euler equations belonging to Hölder spaces with Hölder exponent greater than $1/3$ conserve kinetic energy; conversely, he conjectured the existence of solutions belonging to any Hölder space with exponent less than $1/3$ which do not conserve kinetic energy. The first part, relating to conservation of kinetic energy, has since been confirmed (cf. [Eyi94, CWT94]). The second part, relating to the existence of non-conservative solutions, remains an open conjecture and is the subject of this dissertation.

In groundbreaking work of De Lellis and Székelyhidi Jr. [DLSJ12a, DLSJ12b], the authors constructed the first examples of non-conservative Hölder continuous weak solutions to the Euler equations. The construction was subsequently improved by Isett [Ise12, Ise13a], introducing many novel ideas in order to construct $1/5 - \varepsilon$ Hölder continuous weak solutions with compact support in time.

Adhering more closely to the original scheme of De Lellis and Székelyhidi Jr., we present a comparatively simpler construction of $1/5 - \varepsilon$ Hölder continuous non-conservative weak solutions which may in addition be made to obey a prescribed kinetic energy profile.[1] Furthermore, we extend this scheme in order to construct weak non-conservative solutions to the Euler equations whose Hölder $1/3 - \varepsilon$ norm is Lebesgue integrable in time.

The dissertation will be primarily based on three papers: [BDLSJ13], [Buc13] and [BDLS14] – the first and third paper being in collaboration with De Lellis and Székelyhidi Jr.

[1]The ability to prescribe an energy profile was also present in the original schemes of De Lellis and Székelyhidi Jr.

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Die Vermutung von Onsager

Zusammenfassung

Im Jahr 1949 stellte Lars Onsager in seiner berühmten Arbeit zur statistischen Hydrodynamik die Vermutung auf, dass alle schwachen Lösungen der 3-D Euler Gleichungen, welche Hölder-stetig mit Exponent $\theta > 1/3$ sind, die kinetische Energie erhalten. Zudem vermutete Onsager, dass es in jedem Hölder-Raum mit Exponent $\theta < 1/3$ Lösungen gibt, die *nicht konservativ* sind, das heißt ihre kinetische Energie bleibt nicht erhalten. Der erste Teil der Vermutung wurde in [Eyi94, CWT94] bewiesen. Ein Beweis für den zweiten Teil der Vermutung steht noch aus und ist Gegenstand der vorliegenden Dissertation.

Erste Beispiele von nicht-konservativen Hölder-stetigen schwachen Lösungen der Euler Gleichungen wurden in der bahnbrechenden Arbeit [DLSJ12a, DLSJ12b] von De Lellis und Székelyhidi Jr. konstruiert. Die in dieser Arbeit verwendete Methode wurde im Folgenden durch Isett in [Ise12, Ise13a] verbessert, dem es gelang $1/5 - \varepsilon$ Hölder-stetige schwache Lösungen mit kompaktem Träger in der Zeit zu konstruieren.

In dieser Arbeit präsentieren wir eine alternative, vergleichsweise einfachere Konstruktion, die näher an der ursprünglichen Konstruktion von De Lellis and Székelyhidi Jr. ist, und dabei nicht nur solche $1/5 - \varepsilon$ Hölder-stetigen, nicht konservativen, schwachen Lösungen liefert, sondern uns auch erlaubt, das Energieprofil vorzuschreiben (vgl. die ursprüngliche Methode von De Lellis und Székelyhidi Jr.). Darüberhinaus erzielen wir eine Verbesserung dieser Methode, insofern dass wir die Existenz einer solchen Lösung nachweisen, deren $1/3 - \varepsilon$ Hölder-Norm Lebesgue-integrierbar bezüglich der Zeit ist.

Diese Dissertation basiert hauptsätchlich auf den Arbeiten [BDLSJ13], [Buc13] und [BDLS14], wobei der erste und dritte Artikel gemeinsame Arbeiten mit De Lellis und Székelyhidi Jr. sind.

Contents

Dedicated to the memory of László *(Laci)* György Kovács (1936 – 2013), a kind and gentle man, whose guidance and support played a formative role in my pursuit of an academic career in mathematics.

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1

Introduction

1.1 The Euler Equation

Since their inception in the mid 18th century [Eul55], the *Euler equations* remain subject of both intense study and debate. The equations have broad applications, from modeling tidal flows to air flow over an airfoil, capturing the essential features of an idealised flow where viscous effects are negligible.

In the *incompressible* case, where the fluid is assumed to have constant material density, the Euler equations may be formally written as

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases}, \tag{1.1.1}$$

where here v is a vector field representing the velocity of the fluid and p is the pressure.

In three dimensions, the question of whether the Cauchy problem is globally well-posed for smooth initial data remains famously unresolved. However, when one relaxes one's notion of solutions and considers *weak solutions* to the Euler equation, then the solutions are known to exhibit bad and in some cases paradoxical behavior.

In testament to the paradoxical behavior of weak solutions, in the remarkable work of Scheffer [Sch93] and in the subsequent work of Shnirelman [Shn97], the existence of nontrivial weak solutions with compact support in time was proved (see also [DLSJ09, Wie11]). Despite this, weak solutions remain the subject of study due to their perceived connection with the theory of turbulence.

Specifically, a pair (v, p) is said to be a *weak solution* on the 3-dimensional torus $\mathbb{T}^3 = [-\pi, \pi]^3$ if for all test functions $\phi \in C_c^\infty(\mathbb{T}^3 \times (0,T), \mathbb{R}^3)$ and $\psi \in C_c^\infty(\mathbb{T}^3 \times (0,T))$, the following identities holds

$$\int_0^T \int_{\mathbb{T}^3} (\partial_t \phi \cdot v + \nabla \phi : v \otimes v + p \operatorname{div} \phi) \; dxdt = 0 \tag{1.1.2}$$

$$\int_0^T \int_{\mathbb{T}^3} v \cdot \nabla \psi \; dxdt = 0 \, . \tag{1.1.3}$$

Alternatively, we may replace the identity (1.1.2) by the requirement that

$$\int_0^T \int_{\mathbb{T}^3} (\partial_t \phi \cdot v + \nabla \phi : v \otimes v) \; dxdt = 0 \, , \tag{1.1.4}$$

holds for all *divergence free* test functions $\phi \in C_c^\infty(\mathbb{T}^3 \times (0,T), \mathbb{R}^3)$. If v belongs to L^2, then the pressure up to an arbitrary function in time can be recovered by the formula

$$-\Delta p = \operatorname{div} \operatorname{div} (v \otimes v) \, , \tag{1.1.5}$$

where again (1.1.5) is assumed to hold in a distributional sense. With these observation in mind, we also call an L^2 vector field v a *weak solution* if it satisfies the identities (1.1.3) and (1.1.4).

1.2 The Onsager Conjecture

A fundamental feature of turbulent flow is that of dissipation of kinetic energy [Ons49, Kol41a, FK95], where given a solution to (1.1.1), its *kinetic energy* is defined to be

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \; dx \, .$$

A simple calculation however yields the *conservation* of energy for any smooth solution of (1.1.1). This formal calculation does not however hold for distributional solutions to Euler as is demonstrated by the paradoxical solution of Scheffer. In his famous note [Ons49] on statistical hydrodynamics, Lars Onsager conjectured the following dichotomy:

Conjecture 1 (Onsager's conjecture)**.**

(a) *Any weak solution v belonging to the Hölder space $C^\theta(\mathbb{T}^3 \times [0,T])$ for $\theta > \frac{1}{3}$ conserves its kinetic energy.*

(b) *For any $\theta < \frac{1}{3}$ there exist weak solutions $v \in C^\theta(\mathbb{T}^3 \times [0,T])$ which do not conserve its kinetic energy.*

Part (a) of this conjecture has since been resolved: it was first considered by Eyink in [Eyi94] following Onsager's original calculations, and later proven by Constantin, E and Titi in [CWT94] (see also [DR00, CCFS08]):

Theorem 1.2.1 (Constantin, E, Titi[1]). *Let $v \in L^3([0,T], C^\theta(\mathbb{T}^3)) \cap C(\mathbb{T}^3 \times [0,T])$ be a weak solution of the 3-D incompressible Euler equation. Then if $\theta > 1/3$, we have conservation of energy:*

$$E(t) = E(0)\ ,$$

for all $t \in [0,T]$.

The proof is beautiful in its simplicity, involving a mollification of the flow and a commutator estimate (see Section 4.4, Chapter 4). Indeed such arguments will play an important role in the present work (cf. Proposition 4.3.5, Chapter 4).

Part (b) remains an open conjecture and is the subject of this dissertation. The first constructions of non-conservative $1/10 - \varepsilon$ Hölder-continuous[2] weak solutions appeared in work of De Lellis and Székelyhidi Jr. [DLSJ12b], which itself was based on their earlier seminal work [DLSJ12a] where continuous weak solutions were constructed. In the recent doctoral work of Isett [Ise12, Ise13a], a number of new ideas were introduced in order improve the Hölder exponent to $1/5 - \varepsilon$, for weak solutions with compact support in time. In this work we will provide an alternative, simplified construction of non-conservative $1/5 - \varepsilon$ Hölder continuous weak solutions, which in addition may be made to obey any prescribed smooth energy profile. Specifically, we will prove the following theorem:

Theorem 1.2.2. *Assume $e : [0,T] \to \mathbb{R}$ is a* strictly positive *smooth function. Then there exists a continuous vector field $v \in C^{1/5-\varepsilon}(\mathbb{T}^3 \times [0,T], \mathbb{R}^3)$ and a continuous scalar field $p \in C^{2/5-2\varepsilon}(\mathbb{T}^3 \times [0,T])$ which solve (1.1.1) in the weak sense and such that $E(t) = e(t)$.*

[1] In fact the precise result proved in [CWT94], which is written in terms of Besov spaces, is slightly stronger than the result stated here (see the remarks at the end of Section 1.3.1).

[2] Here and below we will let ε denote an arbitrarily small positive number.

Going beyond the exponent $1/5$ seems to be a particularly challenging problem. Owing to the beauty of the Constantin-E-Titi result, it may seem natural to attempt to construct solutions v belonging to the spaces $L^p([0,T], C^{1/3-\varepsilon}(\mathbb{T}^3)) \cap C(\mathbb{T}^3 \times [0,T])$ for some $p \geq 1$.[3] In this direction, we prove the following theorem:

Theorem 1.2.3. *Assume* $e : [0,T] \to \mathbb{R}$ *is a* strictly positive *smooth function. Then for every* $\delta > 0$*, there exists a weak solution* $v \in L^1([0,T], C^{1/3-\varepsilon}(\mathbb{T}^3)) \cap C(\mathbb{T}^3 \times [0,T])$ *to* (1.1.1) *such that* $|E(t) = e(t)| < \delta$ *for all* $t \in [0,T]$.

Observe that unlike Theorem 1.2.2, the solutions in Theorem 1.2.3 are not guaranteed to obey the prescribed energy profiles *exactly*. In particular, given a monotonically decreasing energy profile, we cannot guarantee that the solutions constructed in Theorem 1.2.3 also have monotonically decreasing energy. Monotonically decreasing energy has been proposed as a possible admissibility criteria for Euler flows [DLSJ10].

1.3 References and Remarks

The proof of Theorems 1.2.2 and 1.2.3 will be based primarily on the joint papers [BDLSJ13] and [BDLS14] respectively, written in collaboration with Camillo De Lellis and László Székelyhidi Jr. The work [BDLS14] in part builds on ideas introduced in [Buc13] which describes the construction of non-trivial, non-conservative $1/5 - \varepsilon$ Hölder continuous solutions which for almost every time belong to the $1/3 - \varepsilon$ Hölder regularity class (see Section 8.4, Chapter 8 for a discussion of the result).

A minor difference between Theorem 1.2.3 and [BDLS14] is that instead of constructing weak solutions approximately obeying a prescribed energy profile, in [BDLS14] weak solutions are constructed having compact temporal support. This difference does not play an important role in the proof of the theorem.

1.3.1 A Weak Version of Onsager's Conjecture

A key postulate of Kolmogorov's K41 theory [Kol41a, Kol41c, Kol41b, FK95] is that for homogeneous, isotropic turbulence, the dissipation rate is non-vanishing in the inviscid limit. In particular, let us define the *structure functions* for homogeneous, isotropic turbulence by

$$S_p(\ell) := \langle [\delta v_L(\ell)]^p \rangle \, ,$$

[3] In line with [CWT94], it may also be interesting to study the problem with the Hölder norms replaced by the appropriate Besov norms (see the remarks in Section 1.3.1 below).

where $\langle\cdot\rangle$ denotes an ensemble average and $\delta v_L(\ell)$ is the longitudinal difference

$$\delta v_L(\ell) := (v(x+\hat{\ell}) - v(x)) \cdot \frac{\hat{\ell}}{\ell}\,,$$

for a spatial vector $\hat{\ell}$ of length ℓ. Then Kolmogorov's famous four-fifths law can be stated as

$$S_3(\ell) \sim -\frac{4}{5}\varepsilon_d \ell\,, \tag{1.3.1}$$

where here ε_d denotes the mean energy dissipation per unit mass. More generally, Kolmogorov's scaling laws can be stated as

$$S_p(\ell) = C_p \varepsilon_d^{\zeta_p} \ell^{\zeta_p}\,, \tag{1.3.2}$$

for any positive integer p, for $\zeta_p = p/3$.

A well known consequence of the above scaling laws is the Kolmogorov spectrum, which postulates a scaling relation on the 'energy spectrum' of a turbulent flow (cf. [FK95, ES06]). It was this observation that provided the original motivation for Onsager's conjecture.

For the particular case of $p = 3$, the scaling (1.3.2) is generally supported by experimental and numerical studies; however, evidence suggests that the exponents ζ_p seem to deviate significantly from the conjectured $p/3$ for $p > 3$ [Kol62, AGHA84, CDK$^+$05].

Since the current work in concerned with individual realisations and not statistical averages, it is interesting to note that in the work [Eyi03], Eyink provides analytical evidence that suggests at the inviscid limit, the $4/5$ law should hold with just local space-time averaging and angular averaging over the direction of the separation vector. This viewpoint has both numerical and experimental support [SVB$^+$96].

We now recall that in [CWT94], Constantin, E and Titi actually prove a stronger version of Theorem 1.2.1 with the spatial Besov norm $B_3^{\theta,\infty}$ replacing the Hölder norm C^θ, where here the Besov space $B_p^{\theta,\infty}$ is defined as

$$\|f\|_{B_p^{\theta,\infty}} = \sup_y |y|^{-\theta} \|f(\cdot) - f(\cdot - y))\|_{L^p}\ .$$

Observing the trivial bound

$$|\delta v_L(\ell)|^p \le \ell^{\theta p} \|v\|^p_{L^p([0,T];(B_p^{\theta,\infty}(\mathbb{T}^3))}\ , \tag{1.3.3}$$

we are naturally lead to the following weak version of Onsager's conjecture:

Conjecture 2 (Weak Version of Onsager's conjecture)**.** *For any $\theta < 1/3$, there exists weak solutions $v \in C([0,T]; L^2(\mathbb{T}^3))$ to (1.1.1) belonging to the Besov space $L^3([0,T], B_3^{\theta,\infty}(\mathbb{T}^3))$ which do not conserve its kinetic energy.*

Theorem 1.2.3 can then be seen as a first step in this direction, proving the case for the space $L^3([0,T], B_3^{\theta,\infty}(\mathbb{T}^3))$ replaced by $L^1([0,T], B_\infty^{\theta,\infty}(\mathbb{T}^3))$.

2

Outline of Convex Integration Scheme

2.1 Convex Integration and the Approach of De Lellis and Székelyhidi Jr. to Onsager's Conjecture

Convex integration was first proposed by Gromov in 1973 as a general method for solving *soft/flexible* partial differential equations of a geometric nature [Gro73]. The method itself was based on the earlier work of Nash [Nas54] and Kuiper [Kui55] on C^1-isometric embeddings of Riemannian manifolds into Euclidean space.

More recently, these techniques have been extended and adapted to various problems arising in mathematical physics. In particular, building on a framework of plane-wave analysis introduced by Tartar [Tar79, Tar83, DiP85], the method was adapted by De Lellis and Székelyhidi Jr. to the Euler equation in order provide an alternative construction of Scheffer's paradoxical flows [DLSJ09]. As is typical with such methods, the solutions constructed were shown to be wildly non-unique [DLSJ10, Wie11].

In a breakthrough paper of De Lellis and Székelyhidi Jr. [DLSJ12a], an alternate convex integration scheme was proposed in order to attack the problem of Onsager's conjecture, resembling more closely the arguments of Nash and Kuiper. Specifically, they proved the existence of continuous weak solutions to the Euler equations satisfying a prescribed kinetic energy profile. The scheme involved constructing a sequence of triples $(v_q, p_q, \mathring{R}_q)$ solving the *Euler-Reynolds system*:

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0 \ . \end{cases} , \tag{2.1.1}$$

such that the pairs (v_q, p_q) converge uniformly to the desired weak solution to the Euler equations (1.1.1).

The Euler-Reynolds system arises naturally upon considering spatial averages of highly oscillatory flows: Suppose (v, p) is a solution to (1.1.1) and let $(\overline{v}, \overline{p})$ be a spatial average of (v, p) over some given length scale[1], then $(\overline{v}, \overline{p}, R)$ is a solution to (2.1.1) for $R = \overline{v \otimes v} - \overline{v} \otimes \overline{v}$. In this context the 3×3 symmetric tensor R is referred to as the *Reynolds stress*.

The velocity field v_q turns out to provide a good approximation of the final flow v, averaged over a spatial length scale $\sim \lambda_q^{-1}$: the parameter λ_q being the approximate frequency of the perturbation

$$w_q := v_q - v_{q-1} \ .$$

Owing to this observation, the symmetric tensor $\mathring{R}_q$, which we note without loss of generality may assumed to be traceless, is also referred to as the Reynolds stress.

Since the relation (2.1.1) is linear in the Reynolds stress, the right hand side can be split into three key components:

$$\begin{aligned} \operatorname{div}(w_q \otimes w_q + \mathring{R}_{q-1}) - \nabla p_q \\ \partial_t w_q + v_{q-1} \cdot \nabla w_q \\ w_q \cdot \nabla v_{q-1} \ , \end{aligned}$$

which we call the *oscillation error*, *transport error* and *Nash error* respectively. The Reynolds stress $\mathring{R}_q$ can then be constructed by applying an -1 order differential operator $\mathcal{R}$ (see Chapter 3) to the sum of the errors. Letting $\|\cdot\|_0$ denote the uniform norm, then heuristically, given a function $f : \mathbb{T}^3 \to \mathbb{R}^3$ with spatial frequency λ, we have $\|\mathcal{R}f\|_0 \approx \lambda^{-1} \|f\|_0$: i.e. we achieve a gain of a factor of λ.

The perturbation w_q is constructed by superimposing highly oscillatory waves known as *Beltrami flows* at frequency λ_q in such a way to cancel the low frequency component of the oscillation error (see Chapter 3). Analogous to the use of *Nash twists* and *Kuiper cor-*

[1] For concreteness, one may consider $(\overline{v}, \overline{p})$ to be a mollification of (v, p).

rugations in order to minimise metric error for the C^1 embedding problem, the problem of cancelling the low frequency error is essentially algebraic in nature (cf. Proposition 3.1.1 and Lemma 3.1.2 of Chapter 3), with the amplitude of the waves being proportionate to the square root of the size of the previous Reynolds stress error $\mathring{R}_{q-1}$.

The perturbation w_q must be further corrected in order to control the transport error. Then, as long the frequency $\lambda_q \gg \lambda_{q-1}$ is chosen sufficiently large, one can ensure the remaining error is small in the uniform norm: for the case of the Nash error, we have heuristically

$$\left\|\mathcal{R}\left(w_q \cdot \nabla v_{q-1}\right)\right\|_0 \sim \frac{\left\|w_q\right\|_0 \left\|v_{q-1}\right\|_1}{\lambda_q}, \tag{2.1.2}$$

where here $\|\cdot\|_N$ denotes the norm associated with the space $C([0,T];C^N(\mathbb{T}^3))$ (cf. Appendix A.1). Such errors are characteristic of errors encountered in the C^1 embedding problem, which motivates the naming of the error.

Proceeding in this manner, with the frequency parameters λ_q growing at a super exponential rate, De Lellis and Székelyhidi Jr. showed that the Reynolds stresses can be made to converge uniformly to zero, and consequently the pairs (v_q, p_q) converge uniformly to a weak continuous solution (v,p) to Euler's equaton (1.1.1).

By keeping better track of first order estimates of the components of the construction and employing mollification in order to resolve an inherent *loss of derivative* issue (discussed in Section 3.2) associated with the scheme, the convex integration scheme was improved in [DLSJ12b] in order to construct $C^{1/10-\varepsilon}$ Hölder continuous weak solutions obeying a prescribed kinetic energy profile.

2.2 The Convex Integration Scheme of Isett

Building on the work of De Lellis and Székelyhidi Jr., Isett proved in his doctoral thesis the existence of $1/5-\varepsilon$ Hölder continuous weak solutions to Euler's equation with compact support in time [Ise12, Ise13a]. The proof employs a convex integration scheme similar to that of [DLSJ12a, DLSJ12b], although with a number of notable improvements.

Principal among these improvements is the replacement of the Beltrami flows of [DLSJ12a, DLSJ12b] with *microlocal* Beltrami flows that are better transported by the previous flow v_{q-1}. This change necessitates the introduction of sharp time cut-offs which limit the *life span* of the oscillatory of waves of the perturbation w_q in order to control the effects of the flow v_{q-1} on the perturbation. The use of such time cut-offs are comparable to the use of Courant–Friedrichs–Lewy (CFL) conditions [CFL28] employed in numerical analysis to study evolutionary equations.

Isett also recognised the importance of keeping track of the *material derivative* $\partial_t + v_q \cdot \nabla$ associated with the flow v_q of the Reynolds stress $\mathring{R}_q$. Analogous to the use of mollification in [DLSJ12b] in order to resolve the problem of *loss of derivative*, the technique of *mollification along the flow* was introduced in order to resolve a problem of *loss of material derivative* (cf. Section 4.2).

2.3 An Examination of Scales

As part of L.F Richardson's celebrated treatise on weather forecasting [Ric65], Richardson introduced the concept of an energy cascade in turbulent flows, whereby energy is transfered from larger scales to smaller scales through a hierarchy of eddies:

> *Big whorls have little whorls. That feed on their velocity,. And little whorls have lesser whorls. And so on to viscosity.*

Such eddies are typically charaterised by their size ℓ, charateristic velocity v_ℓ and *turnover time* $t_\ell \sim \frac{\ell}{v_\ell}$. The turnover time t_ℓ being the typical time scale at which eddies of length scale ℓ experience significant distortion, or alternatively the time scale at which energy is expected to be transfered to smaller scales [FK95].

Suppose we have a discrete family of decreasing eddy length scales ℓ_q, with associated frequencies $\lambda_q \sim \ell_q^{-1}$, we may assume (as is often done [FK95]) that the associated velocities v_ℓ scale according to some asymptotic law $v_\ell \sim \lambda_q^{-\beta}$ for some *regularity exponent* $\beta > 0$. Applying this framework, together with the heuristic $\|f\|_N \sim \lambda_q^N \|f\|_0$ for functions at characteristic frequency λ_q, to the scheme of De Lellis and Székelyhidi Jr. leads to the estimates

$$\|w_q\|_N \leq C\lambda_q^{N-\beta} \tag{2.3.1}$$
$$\|\mathring{R}_q\|_N \leq C\lambda_q^N \lambda_{q+1}^{-2\beta} . \tag{2.3.2}$$

Observe that we have invoked the requirement that size of w_q is proportional to the square of size of $\mathring{R}_{q-1}$, which we recall was related to the algebraic cancellation of low frequencies in the oscillation error.[2] The higher order estimates of $\mathring{R}_q$ then follows as a consequence of the frequency support of w_q and v_{q-1}. The turnover time $t_q \sim \lambda_q^{\beta-1}$,

[2]Note that as long as the frequencies λ_q are sufficiently spaced out, then we expect v_q to be approximately the spatial average $\bar{v}$ of v at length scale λ for $\lambda_q \gg \lambda \gg \lambda_{q+1}$. It is then instructive to compare $\mathring{R}_q$ to the Reynolds stress $R := \overline{v \otimes v} - \bar{v} \otimes \bar{v}$. Applying Proposition 4.3.5 from Chapter 4, we obtain $\|R\|_0 \leq \lambda^{-2\beta}$, i.e. we obtain the correct scaling.

should play an important role with regards to the *temporal resolution* at which we examine the perturbation.

The present work began as an effort to better understand the scheme of De Lellis and Székelyhidi Jr. under the above framework, as well reconcile the scheme with the conjectured solutions of Onsager. Suppose we can construct a sequence of triples $(v_q, p_q, \mathring{R}_q)$ satisfying the Euler-Reynolds system (2.1.1), and let us further assume the frequency parameters λ_q grow at *(at least)* an exponential rate, then upon application of interpolation (A.1.1) we obtain that the sequence v_q converges to a weak solution $v \in C^\theta$ of the Euler equations (1.1.1) for any $\theta < \beta$. In particular, in order to prove Onsager's conjecture we would need to show such a sequence of triples exist converging to a non-conservative weak solution for any given $\beta < 1/3$.

Let us assume the *super* exponential rate $\lambda_q \sim \lambda_{q-1}^b$ for some $b > 1$.[3] Now consider for the moment the estimates of the Nash error under the above framework. Observe that from (2.3.1) we have

$$\left\|v_{q-1}\right\|_1 \leq \sum_{q'=0}^{q-1} \left\|w_{q'}\right\|_1 \leq C \sum_{q'=0}^{q-1} \lambda_{q'}^{1-\beta} \leq C\lambda_{q-1}^{1-\beta},$$

and hence from (2.1.2) we obtain

$$\left\|\mathcal{R}\left(w_q \cdot \nabla v_{q-1}\right)\right\|_0 \sim \frac{\|w\|_0 \left\|v_{q-1}\right\|_1}{\lambda_q} \leq \frac{C\lambda_{q-1}}{\lambda_q^{1+\beta}\lambda_{q-1}^{\beta}} \leq C\lambda_{q-1}^{-(1+b)\beta+1-b}. \qquad (2.3.3)$$

Then from (2.3.2) and (2.3.3) we obtain the restriction $-(1+b)\beta + 1 - b < -2\beta b^2$, which itself leads to the requirement

$$\beta < \frac{1}{2b+1}.$$

Taking b arbitrarily close to 1 leads naturally to a constraint compatible with Onsager's conjecture. Unfortunately for us, while the Nash error appears to be relatively harmless and does not seem to impose an obstruction to Onsager's conjecture, the two other errors, namely the oscillation error (discussed in Chapter 3) and the transport error (discussed in Chapter 4) seem to be far from harmless.

As was observed by Isett, in order to obtain better estimates on the transport error, the Beltrami waves used in the scheme of Dellis and Székelyhidi Jr. need to be modified in order that they are better transported by the flow of the previous iteration. However

[3]The requirement of super exponential growth is a technical consideration (cf. Section 3.3, Chapter 3).

unlike the scheme of Isett, where microlocal Beltrami waves were used, we will instead employ the comparatively simpler solution of solving the transport equation directly. Analogous to Isett's scheme, this will necessitate the introduction of time cutoffs in order to partition time into intervals of length comparable in scale to the turnover time t_q.

Following the basic principles outlined above, we will show that is it possible to construct a convex integration scheme producing the weak solutions of Theorem 1.2.2. We note that in additional to having the ability to prescribe the kinetic energy profile and being comparatively simpler to the construction of Isett, the numerology of the scalings involved in the scheme of Theorem 1.2.2 will be considerably more opaque (cf. Chapter 7).

In contrast to the proof of Theorem 1.2.2, in order to prove Theorem 1.2.3, the parameter β will be allowed to depend on the time t and the iteration q, with the additional constraints

$$\beta(t, q) \geq \beta_0 \tag{2.3.4}$$

$$\{t : \beta(t, q) < r\} \leq C\lambda_q^{r-\beta_\infty+\varepsilon} \quad \text{for } \beta_0 < r < \beta_\infty\,, \tag{2.3.5}$$

for some constants $0 < \beta_0 < \beta_\infty < 1/3$ and $\varepsilon > 0$. Then the appropriate interpolation argument yields $v \in L^1([0, T]; C^\theta(\mathbb{T}^3)) \cap C^{\theta'}(\mathbb{T}^3 \times [0, T])$ for any $\theta < \beta_\infty - \varepsilon$ and $\theta' < \beta_0$. Under the phenomenology of turbulence introduced above, the eddies at a particular length scale will have characteristic velocities and turnover times depending on time. The variable turnover times will complicate the partitioning of time and will require us to keep an elaborate bookkeeping system (see Section 8.1, Chapter 8). We note in passing that such temporal irregularity is not entirely unnatural in the theory of turbulence [Sig77, OY89].

2.4 Convergence of the Energy

Observe that in the previous section, we made no mention of the estimates required in order to ensure the convergence of our convex integration schemes to a energy profile satisfying the requirements of Theorems 1.2.2 and 1.2.3. These estimates will be detailed below.

In order to simplify matters somewhat, we begin by considering a *normalised* energy profile $e : [0, T] \to \mathbb{R}$ satisfying the following properties:

$$\max_t e(t) = c_0\lambda_1^{-2\bar{\beta}}, \quad \inf_t e(t) \gg \lambda_2^{-2\bar{\beta}}, \quad \max_t e'(t) \leq 1\,, \tag{2.4.1}$$

for some small constant $c_0 > 0$ to be specified later, where here we write $\bar{\beta} = \beta$ for the proof of Theorem 1.2.2 and $\bar{\beta} = \beta_0$ for the case of Theorem 1.2.3.

In the case of Theorem 1.2.2, we want to show that the energy of the approximate solutions v_q converge to the given energy profile $e : [0, T] \to \mathbb{R}$ from *below*. To this aim, we impose the following estimate along the iteration

$$\left| e(t) - c_0 \lambda_{q+1}^{-2\beta} - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx \right| \leq c_0 \lambda_{q+1}^{-2\beta} . \tag{2.4.2}$$

For Theorem 1.2.3 we need only show that the energy of approximate solutions v_q converge to a function in a $C\lambda_2^{-2\beta_0} > 0$ neighbourhood of e in the uniform norm. In particular, this will be achieved given

$$\left| e(t) - \int_{\mathbb{T}^3} |v_1(x,t)|^2 \, dx \right| \leq C\lambda_2^{-2\beta_0} , \tag{2.4.3}$$

and

$$\sum_{q=2}^{\infty} \int_{\mathbb{T}^3} |w_q(x,t)|^2 \, dx \leq C\lambda_2^{-2\beta_0} . \tag{2.4.4}$$

Remark 2.4.1. The difficulty of obtaining convergence to the exact energy profile e arises from the fact that for the scheme used to prove Theorem 1.2.3, we do not necessarily have $\lambda_{q+1}^{-\beta(t,q+1)} \leq \lambda_q^{-\beta(t,q)}$ for a given time t.

For the case of general energy profiles which do not necessarily satisfy the inequalities (2.4.1), we apply a simply scaling argument in order reduce the problem to the case of a normalised profile. First note that the Euler equations are invariant under the transformation

$$(v, p) \mapsto (\tau v(x, \tau t), \tau^2 p(x, \tau t)) \tag{2.4.5}$$

for any $\tau > 0$. Now fix an energy profile $\bar{e}$ and define

$$e(t) := \frac{1}{c_0 \lambda_1^{2\bar{\beta}} \bar{e}_{\max}} e \left(x, \frac{t}{\lambda_1^{\bar{\beta}} \sqrt{c_0 \bar{e}_{\max}}} \right) ,$$

where here $\bar{e}_{\max} = \max_t \bar{e}(t)$. Hence assuming λ_0 to be sufficiently large (depending or $\bar{e}$, b and $\bar{\beta}$) we obtain (2.4.1).

Suppose then that Theorem 1.2.2 is satisfied for the normalised profile e, then it follows by (2.4.5) that Theorem 1.2.2 holds for $\bar{e}$. Similarly, if Theorem 1.2.3 is satisfied for the normalised profile e with $\delta = C\lambda_2^{-2\bar{\beta}}$, then Theorem 1.2.3 holds for $\bar{e}$ and $\delta = C\lambda_1^{2\bar{\beta}}\lambda_2^{-2\bar{\beta}}$. Assuming λ_0 to be sufficiently large we can make this rescaled δ as small as required. Thus, we can safely restrict ourselves to considering normalised profiles.

2.5 References and Remarks

Following the pioneering work of De Lellis and Székelyhidi Jr. [DLSJ09], the general framework of incorporating plane-wave analysis in the context of convex integration (cf. [MŠ03, KMŠ03, CFMM05]) has seen a number of implementations in the theory of evolutionary equations besides the incompressible Euler equations. In particular, the framework has been used in the context of the incompressible porous media equation [CFG11], a class of active scalar equations [Shv11] and the isentropic compressible Euler equations [CDLK13].

The refined convex integration of De Lellis and Székelyhidi Jr. introduced in [DLSJ12a, DLSJ12b] has also been adapted to the 2-D Euler equatons [CDLS12] and indeed it seems that the methods presented here are also adaptable to 2-D case. It should also be noted that as was the case with L^2 non-conservative weak solutions to the Euler equations, the convex integration schemes presented here and in [DLSJ12a, DLSJ12b, Ise12, Ise13a] construct solutions which are highly non-unique [Cho12, Dan14, Ise12, Ise13a].

3
Cancellation of low frequency error

In this chapter, we will study how by superimposing highly oscillatory Beltrami flows, we can cancel low frequency error. This will be used to construct an ansatz for the definition of the perturbation w_q. The oscillation error of the resulting ansatz will then be estimated.

3.1 Beltrami Flows

A stationary divergence free vector field v is called a *Beltrami flow* if it satisfies the *Beltrami condition*:

$$\lambda(x)v(x) = \operatorname{curl} v(x), \qquad \lambda(x) > 0\,, \tag{3.1.1}$$

for all x. The function λ is called the *Beltrami coefficient*.

Given a Beltrami flow v, from the divergence free condition we have the following identity

$$\operatorname{div}(v \otimes v) = v \cdot \nabla v = \nabla \frac{|v|^2}{2} - v \times (\operatorname{curl} v) = \nabla \frac{|v|^2}{2} - \lambda v \times v = \nabla \frac{|v|^2}{2}\,. \tag{3.1.2}$$

In particular setting $p := \frac{|v|^2}{2}$, then (v, p) is a stationary solution to the Euler equations.

In the mathematical physics literature, it has been postulated that that in regions of turbulence, flows organise themselves into hierarchies of weakly interacting superimposed approximate Beltrami flows [YOY+87, CM88]. With this thought in mind, and with the aim to mininise the oscillation error, we consider the ansatz

$$w_{q+1} = \sum W_k\,, \tag{3.1.3}$$

where W_k are approximate Beltrami flows oscillating at frequency λ_{q+1}, and the projection of $w_{q+1} \otimes w_{q+1}$ onto low frequencies ($\ll \lambda_{q+1}$) provides a good approximation of $\mathring{R}_q$ modulo the addition of a function depending solely on time.

The two propositions below will be used to describe the construction of the approximate Beltrami flows W_k.

Proposition 3.1.1. *Let $\lambda \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that*

$$A_k \cdot k = 0,\ |A_k| = \tfrac{1}{\sqrt{2}},\ A_{-k} = A_k\,,$$

for $k \in \mathbb{Z}^3$ with $|k| = \lambda$. Furthermore, let

$$B_k = A_k + i\frac{k}{|k|} \times A_k \in \mathbb{C}^3\,.$$

For any choice of $a_k \in \mathbb{C}$ with $\overline{a_k} = a_{-k}$ the vector field

$$W(\xi) = \sum_{|k|=\lambda} a_k B_k e^{ik\cdot\xi}\,, \tag{3.1.4}$$

is a real-valued Beltrami flow with constant Beltrami coefficient λ satisfying

$$\langle W \otimes W \rangle = \fint_{\mathbb{T}^3} W \otimes W \, d\xi = \frac{1}{2}\sum_{|k|=\lambda} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)\,. \tag{3.1.5}$$

Proof. By definition $a_k B_k = \overline{a_{-k} B_{-k}}$ and hence by symmetry it follows that W is real valued. By direct calculation we have

$$\operatorname{div} W = \sum_{|k|=\lambda} ik \cdot B_k a_k e^{ik\cdot\xi} \equiv 0,$$

since $k \cdot B_k = 0$ for each k. Moreover, we have

$$\operatorname{curl} W(\xi) = \sum_{|k|=\lambda} ik \cdot ik \times B_k a_k e^{ik\cdot\xi}$$

$$
\begin{aligned}
&= \lambda \sum_{|k|=\lambda} \left(i\frac{k}{|k|} \times A_k - \frac{k}{|k|} \times \left(\frac{k}{|k|} \times A_k \right) \right) \\
&= \lambda \sum_{|k|=\lambda} \left(i\frac{k}{|k|} \times A_k + A_k \right) \\
&= \lambda W(\xi) \,,
\end{aligned}
$$

and hence W is a real-valued Beltrami flow with Beltrami coefficient λ.

It remains to show (3.1.5). Averaging in space yields

$$
\begin{aligned}
\langle W \otimes W \rangle &= \sum_{|k|=\bar{\lambda}} W_k \otimes W_{-k} \\
&= \sum_{|k|=\bar{\lambda}} |a_k|^2 \, B_k \otimes \overline{B}_k \\
&= \sum_{|k|=\bar{\lambda}} |a_k|^2 \operatorname{Re} \left(B_k \otimes \overline{B}_k \right) \\
&= \sum_{|k|=\bar{\lambda}} |a_k|^2 \left(A_k \otimes A_k + \left(\frac{k}{|k|} \times A_k \right) \otimes \left(\frac{k}{|k|} \times A_k \right) \right) . \qquad (3.1.6)
\end{aligned}
$$

Finally, observe that $\sqrt{2}A_k$, $\frac{k}{|k|}$ and $\sqrt{2}\frac{k}{|k|} \times A_k$ form an orthonormal basis and hence we have the identity

$$
2A_k \otimes A_k + 2 \left(\frac{k}{|k|} \times A_k \right) \otimes \left(\frac{k}{|k|} \times A_k \right) = \mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \,.
$$

With this identity together with (3.1.6) we obtain (3.1.5). □

In order to choose the coefficients a_k in such a way to cancel the the low frequencies of the Reynolds stress we will require the following lemma.

Lemma 3.1.2. *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\lambda > 1$ with the following property. Let $B_{r_0}(\mathrm{Id})$ denote the ball of symmetric 3×3 matrices, centred at Id, of radius r_o. Then, there exist pairwise disjoint subsets*

$$
\Lambda_j \subset \{ k \in \mathbb{Z}^3 : \, |k| = \lambda \} \qquad j \in \{1, \ldots, N\} \,,
$$

and smooth positive functions

$$
\gamma_k^{(j)} \in C^\infty \left(B_{r_0}(\mathrm{Id}) \right) \qquad j \in \{1, \ldots, N\}, k \in \Lambda_j \,,
$$

such that

(a) $k \in \Lambda_j$ *implies* $-k \in \Lambda_j$ *and* $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;

(b) *For each* $R \in B_{r_0}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(R)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \qquad \forall R \in B_{r_0}(\mathrm{Id}) \,. \tag{3.1.7}$$

Proof. First consider the case for $N = 1$. We set e_1, e_2, e_3 to be the standard orthonormal basis for $\mathbb{R}^3$ and define

$$\Lambda = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq 3\} \subseteq \mathbb{Z}^3 \cap \{|k| = \sqrt{2}\} \,,$$

and

$$\Lambda^+ = \{(e_i \pm e_j) \mid 1 \leq i < j \leq 3\} \,.$$

With these choice we make the following observations:

1. The tensors

$$\mathcal{B} = \left\{\mathrm{Id} - \frac{k \otimes k}{|k|^2} \mid k \in \Lambda^+\right\} \tag{3.1.8}$$

are linearly independent, and thus form a basis for the space of symmetric matrices.

2. We have the identity

$$\frac{1}{2} \sum_{k \in \Lambda} \left(\mathrm{Id} - \frac{k \otimes k}{|k|^2}\right) = 4\mathrm{Id} \,. \tag{3.1.9}$$

Hence applying the inverse function theorem we obtain property (b).

Now consider the case for $N > 1$. Let B be the rotation by angle $\arccos \frac{3}{5}$ about the e_1 axis, i.e.

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix} .$$

Since $\pi^{-1} \arccos \frac{3}{5}$ is irrational ($\pi^{-1} \arccos \mathbb{Q} \cap \mathbb{Q} = \{0, 1/3, 1/2, 2/3, 1\}$) it follows that $\{B^j \Lambda\}_{j \in \{1,\ldots,N\}}$ form a disjoint family of sets of rational vectors satisfying properties 1 and 2. Thus there exists an integer M such that $\Lambda_j = MB^j\Lambda$ is a disjoint family of sets of vectors with integer coefficients satisfying property (a). Again, property (b) then follows upon applying the inverse function theorem. $\square$

With the help of the above propositions, we may now realise our ansatz (3.1.3). First define

$$R(x,t) := \rho(t)\mathrm{Id} - \mathring{R}_q(x,t)\,, \tag{3.1.10}$$

where $\rho : [0,T] \to \mathbb{R}$ is a scalar function depending on time, satisfying the constraints

$$\left\| \frac{\mathring{R}_q}{\rho} \right\|_0 \le r_0, \qquad \|\rho\|_0 \le \lambda_{q+1}^{-2\beta}\,. \tag{3.1.11}$$

Let Λ_1 be as in Lemma 3.1.2 (with $N=1$), and for each $k \in \Lambda_1$ define the coefficient functions a_k by

$$a_k(x,t) = \sqrt{\rho(t)}\gamma_1\left(\frac{R(x,t)}{\rho(t)}\right)\,. \tag{3.1.12}$$

Our *principle* perturbation w_o is then defined to be

$$w_o(x,t) := \sum_{k\in\Lambda_1} a_k B_k e^{i\lambda_{q+1}k\cdot x}\,. \tag{3.1.13}$$

The function ρ will allow us later better control of the energy of v_{q+1} which will be essential in ensuring our convex integration scheme converges to a flow satisfying our prescribed energy profile. Unfortunately, since the functions a_k depend on the spatial variable a_k, the vector field w_o does not necessarily satisfy the divergence free condition. Hence we will define a corrector w_c such that for

$$w_{q+1} = w_o + w_c\,, \tag{3.1.14}$$

we have $\operatorname{div} w \equiv 0$. First we note the identity

$$\frac{1}{\lambda_{q+1}}\operatorname{curl}\left(ia_k\frac{k\times B_k}{|k|^2}e^{i\lambda_{q+1}k\cdot x}\right) = \left(a_k B_k + \frac{i}{\lambda_{q+1}}\nabla a_k \times \left(\frac{k\times B_k}{|k|^2}\right)\right)e^{i\lambda_{q+1}k\cdot x}.$$

It follows by defining the *corrector* w_c to be

$$w_c := \frac{i}{\lambda_{q+1}}\nabla a_k \times \left(\frac{k\times B_k}{|k|^2}\right)e^{i\lambda_{q+1}k\cdot x}\,, \tag{3.1.15}$$

we have from the elementary identity $\operatorname{div}\operatorname{curl} \equiv 0$ that $\operatorname{div} w = \operatorname{div}(w_o + w_c) \equiv 0$.

A secondary consequence of having non-constant coefficients a_k is that the identity

(3.1.2) is no longer satisfied, rather we have the identity:

Lemma 3.1.3. *For w_o and R defined above we have the identity*

$$\begin{aligned} \operatorname{div}(w_o \otimes w_o + R) &= \nabla \frac{|w_o|^2}{2} \\ &+ \sum_{k+k' \neq 0} \left(B_k \otimes B_{k'} - \tfrac{1}{2}(B_k \cdot B_{k'})\mathrm{Id}\right) \nabla(a_k a_{k'}) e^{i\lambda_{q+1}(k+k')\cdot x} . \end{aligned} \tag{3.1.16}$$

Proof. Let us write

$$w_o(y, \xi, t) := \sum_{k \in \Lambda_1} a_k(y, t) B_k e^{ik\cdot\xi} ,$$

where here y is the *slow* variable and ξ is the *fast* variable. In particular we have $w_o(x,t) = w_o(x, \lambda_{q+1}x, t)$. With this notation and the identification $\xi = \lambda_{q+1}x$ and $y = x$, the left hand side of (3.1.16) becomes

$$\lambda_{q+1} \operatorname{div}_\xi (w_o \otimes w_o) + \operatorname{div}_y (w_o \otimes w_o + \mathring{R}) = I + II .$$

From (3.1.2) we have

$$\operatorname{div}_\xi (w_o \otimes w_o) = \nabla_\xi \frac{|w_o|^2}{2} ,$$

and from the choice of a_k we have

$$\operatorname{div}_y (w_o \otimes w_o + \mathring{R}) = \sum_{k+k' \neq 0} (B_k \otimes B_{k'})\mathrm{Id}) \nabla(a_k a_{k'}) e^{i\lambda_{q+1}(k+k')\cdot x} .$$

Finally we calculate

$$\begin{aligned} \nabla_y \frac{|w_o|^2}{2} &= \nabla \rho(t) + \frac{1}{2} \sum_{k+k' \neq 0} ((B_k \cdot B_{k'})\mathrm{Id}) \nabla(a_k a_{k'}) e^{i\lambda_{q+1}(k+k')\cdot x} \\ &= \frac{1}{2} \sum_{k+k' \neq 0} ((B_k \cdot B_{k'})\mathrm{Id}) \nabla(a_k a_{k'}) e^{i\lambda_{q+1}(k+k')\cdot x} . \end{aligned}$$

Combining the above identities we arrive at our claim. □

3.2 The Operator $\mathcal{R}$

A stated goal for this chapter was to construct a perturbation w_{q+1} which minimises the oscillation error, which we write as $\operatorname{div} \mathring{R}_o$ where $\mathring{R}_o$ is a solution to the equation

$$\operatorname{div} \mathring{R}_o = \operatorname{div}(w_{q+1} \otimes w_{q+1} + \mathring{R}_q) - \nabla p_{q+1} . \tag{3.2.1}$$

To solve the above equation for $\mathring{R}_o$ we define a singular operator $\mathcal{R}$ which acts as a partial inverse to the divergence operator.

Definition 3.2.1. Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ be a smooth vector field. We then define $\mathcal{R}v$ to be the matrix-valued periodic function

$$\mathcal{R}v := \frac{1}{4}\left(\nabla\mathcal{P}_{\mathbb{T}^3}u + (\nabla\mathcal{P}_{\mathbb{T}^3}u)^T\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^T\right) - \frac{1}{2}(\operatorname{div} u)\mathrm{Id}\,,$$

where $u = \Delta_{\mathbb{T}^3}^{-1} f \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is defined to be the solution of

$$\Delta u = v - \fint_{\mathbb{T}^3} v\,,$$

with $\fint_{\mathbb{T}^3} u = 0$ and $\mathcal{P}_{\mathbb{T}^3}$ is the Leray projection onto divergence-free fields with zero average.

Lemma 3.2.2 ($\mathcal{R} = \operatorname{div}^{-1}$). *For any $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ we have*

(a) $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;

(b) $\operatorname{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v$.

Proof. The matrix $\mathcal{R}v(x)$ is symmetric by definition. To see that it is also traceless, we note that since $\operatorname{div}\mathcal{P}v \equiv 0$, it follows that

$$\operatorname{tr}(\mathcal{R}v) = \frac{3}{4}(2\operatorname{div} u) - \frac{3}{2}\operatorname{div} u \equiv 0\,.$$

Moreover, from the identity $\Delta(\mathcal{P}u) = \Delta u - \nabla\operatorname{div} u$ we obtain

$$\operatorname{div}(\mathcal{R}v) = \frac{1}{4}(\Delta u - \nabla\operatorname{div} u) + \frac{3}{4}(\nabla\operatorname{div} u + \Delta u) - \frac{1}{2}\nabla\operatorname{div} u = \Delta u\,. \qquad (3.2.2)$$

Recall $\Delta u = v - \fint v$ and thus we obtain (b). □

Hence if we define $\mathring{R}_o$ by the formula

$$\mathring{R}_o = \mathcal{R}\left(\operatorname{div}(w_q \otimes w_q + \mathring{R}_{q-1}) - \nabla p_{q-1}\right)\,,$$

we obtain (3.2.1).

As mentioned in Chapter 2, since $\mathcal{R}$ is a -1 order differential operator, we have the rough heuristic that for a function f with frequency λ, $\|\mathcal{R}f\|_0 \approx \lambda^{-1}\|f\|_0$: i.e. we achieve a gain of a factor of λ. This heuristic is made precise in the proposition below.

Proposition 3.2.3. *Fix $\lambda \geq 1$ and let $k \in \mathbb{Z}^3$ be a vector satisfying $|k| = \lambda$. Then for a smooth vector field $a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$, if we set $F(x) := a(x)e^{ik\cdot x}$, we have*

$$\|\mathcal{R}(F)\|_0 \leq \frac{C}{\lambda^{1-\varepsilon}}\|a\|_0 + \frac{C}{\lambda^m}\|a\|_m \,, \tag{3.2.3}$$

where $C = C(\varepsilon, m)$ and $m \geq 1$.

In order to prove the Proposition 3.2.3 we will need the following standard singular integral estimate.

Lemma 3.2.4. *For any $\varepsilon \in (0,1)$ and any $m = 0, 1, \dots$ there exists constants $C(m)$ and $C(m,\varepsilon)$ such that we have the following estimate*

$$\|\mathcal{R}v\|_m \leq C(m)\|v\|_m \tag{3.2.4}$$

$$\|\mathcal{R}v\|_{m+1} \leq C(m,\varepsilon)\|v\|_{m+\varepsilon} \,. \tag{3.2.5}$$

Proof. We first consider a related operator $\mathcal{R}_{\mathbb{R}^3}$ defined by the formula

$$\mathcal{R}_{\mathbb{R}^3}(f) := \frac{1}{4}\left(\nabla\mathcal{P}_{\mathbb{R}^3}u + (\nabla\mathcal{P}_{\mathbb{R}^3}u)^T\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^T\right) - \frac{1}{2}(\operatorname{div} u)\mathrm{Id}\,,$$

for any $f \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with support contained in a ball of radius 8π, where here $u := \Delta^{-1}_{\mathbb{R}^3} f$ is the unique smooth rapidly decaying solution to the Laplace equation $\Delta u = f$ and $\mathcal{P}_{\mathbb{R}^3}$ is the Leray projection operator acting on $\mathbb{R}^3$ (see Appendix A.2).

By inspection, one sees that the composition $\mathcal{R}_{\mathbb{R}^3}\nabla$ can be written in terms of sums and compositions of Riesz operators (see (A.2.2) and (A.2.3)). In particular it follows from (A.2.4) that $\mathcal{R}_{\mathbb{R}^3}\nabla$ is a bounded operator on L^p spaces for $1 < p < \infty$. Hence applying Sobolev inequalities (Lemma A.2.1) we have

$$\|\mathcal{R}_{\mathbb{R}^3} f\|_{C^0(\mathbb{R}^3)} \leq C\,\|\mathcal{R}_{\mathbb{R}^3} f\|_{\dot{W}^{1,4}} \leq C\,\|f\|_{L^4(\mathbb{R}^3)} \,, \tag{3.2.6}$$

and

$$[\mathcal{R}_{\mathbb{R}^3} f]_{\dot{C}^{N+1}(\mathbb{R}^3)} \leq C\,\|\mathcal{R}_{\mathbb{R}^3} f\|_{W^{N+\varepsilon,p}(\mathbb{R}^3)} \,, \tag{3.2.7}$$

for any $p > 3/\varepsilon$.

To compare the original operator $\mathcal{R}$ with $\mathcal{R}_{\mathbb{R}^3}$, we fix a smooth 2π periodic vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ and let χ be a cut-off function identically 1 on the ball of radius 4π, with support contained in the ball of radius 8π. Now set $u : \mathbb{T}^3 \to \mathbb{R}^3$ to be the function $u := \Delta^{-1}_{\mathbb{T}^3} f$, and define $\tilde{u} := \Delta^{-1}_{\mathbb{R}^3}(\chi f)$. Obviously, by definition we have $\Delta(u - \tilde{u}) = \fint_{\mathbb{T}^3} f$

on the the torus. We also have $\Delta\left(\mathcal{P}_{\mathbb{T}^3}u - \mathcal{P}_{\mathbb{R}^3}\tilde{u}\right) \equiv 0$ on the torus: to see this we write

$$\mathcal{P}_{\mathbb{T}^3}u - \mathcal{P}_{\mathbb{R}^3}\tilde{u} = \underbrace{\mathcal{P}_{\mathbb{T}^3}u - \mathcal{P}_{\mathbb{T}^3}\tilde{u}}_{I} + \underbrace{\mathcal{P}_{\mathbb{T}^3}\tilde{u} - \mathcal{P}_{\mathbb{R}^3}\tilde{u}}_{II} .$$

Then by linearity, we have $\Delta I \equiv 0$ on the torus. Also by the definition of the projections $\mathcal{P}$ and $\mathcal{P}_{\mathbb{R}^3}$ we have $\operatorname{div} II = \nabla h$ for some scalar function h. Moreover, since $\operatorname{div} II \equiv 0$, it follows that h (and by implication II) is harmonic. We may apply a Sobolev inequality (Lemma A.2.1) and the harmonic function estimates (A.2.5) for $1/2 < (s-r)/3$ to obtain

$$\begin{aligned}
\|\mathcal{R}f - \mathcal{R}_{\mathbb{R}^3}\chi f\|_{C^r(\mathbb{T}^3)} &\leq \|\mathcal{R}f - \mathcal{R}_{\mathbb{R}^3}\chi f\|_{H^s(\mathbb{T}^3)} \\
&\leq C\,\|\mathcal{R}f - \mathcal{R}_{\mathbb{R}^3}\chi f\|_{L^2(\mathbb{T}^3)} \\
&\leq C\,\|f\|_{L^2(\mathbb{T}^3)} \\
&\leq C\,\|f\|_{L^\infty(\mathbb{T}^3)} .
\end{aligned} \tag{3.2.8}$$

The $L^2 \to L^2$ boundedness of $\mathcal{R}$ and $\mathcal{R}_{\mathbb{R}^3}$ follow as a consequence of Plancherel theorem. Then from (3.2.6), (3.2.7) and (3.2.8) we obtain our claim. □

Proof of Proposition 3.2.3. For $j = 0, 1, \ldots$ define

$$\begin{aligned}
A_j(x) &:= -i\left[\frac{k}{|k|}\left(i\frac{k}{|k|}\cdot\nabla\right)^j a(x)\right]e^{ik\cdot x} , \\
F_j(x) &:= \left[\left(i\frac{k}{|k|}\cdot\nabla\right)^j a(x)\right]e^{ik\cdot x} .
\end{aligned}$$

Direct calculation shows that

$$F_j = \frac{1}{|k|}\left(F_{j+1} + \operatorname{div} A_j\right) .$$

In particular, for any $m \in \mathbb{N}$, applying telescoping yields

$$a(x)e^{ik\cdot x} = F_0 = \frac{1}{|k|^m}F_m + \frac{1}{|k|}\sum_{j=0}^{m-1}\frac{1}{|k|^j}\operatorname{div}\left(A_j\right) .$$

Then from Lemma 3.2.4 we obtain that for any $\varepsilon > 0$ there exists a constant C such that

$$\left\|\mathcal{R}a(x)e^{ik\cdot x}\right\|_N = \frac{C}{|k|^m}\|a\|_{N+m} + \frac{C}{|k|}\sum_{j=0}^{m-1}\frac{1}{|k|^j}\|a\|_{N+j+\varepsilon} \,.$$

Finally, applying interpolation (A.1.1) we obtain the desired claim. □

Notice that Proposition 3.2.3 requires higher order derivatives of the Reynolds stress, although we will only keep track of first order estimates. This *loss of derivatives* problem, as it is known in Nash-Moser theory [And02], may be resolved by replacing $\mathring{R}_q$ in the definition of a_k with the its mollification $\mathring{R}_\ell$ at length scale $\ell = \lambda_{q+1}^{\varepsilon_0-1}$, for some small $\varepsilon_0 > 0$ such that $\lambda_q < \ell^{-1} < \lambda_{q+1}$. Precisely, let $\psi \in C^\infty(\mathbb{T}^3)$ be a standard mollifier: $\operatorname{supp}(\psi) \subset (-1,1)$, $\psi \geq 0$ and $\int_{\mathbb{T}^3}\psi = 1$; and define

$$\mathring{R}_\ell(x,t) = (\mathring{R}_q * \psi_\ell)(x,t) = \ell^{-3}(\mathring{R}_q * \psi(\cdot/\ell))(x,t) = \ell^{-3}\int_{\mathbb{T}^3}\mathring{R}_q(y)\psi\left(\frac{x-y}{\ell}\right)\,dy\,.$$

Now define the pressure p_{q+1} to be

$$p_{q+1} := p_q - \frac{|w_o|^2}{2} - \frac{|w_c|^2}{3} - \frac{2\langle w_o, w_c\rangle}{3}\,. \tag{3.2.9}$$

We can then replace $\mathring{R}_o$ with

$$\begin{aligned}
&\mathcal{R}\left(\operatorname{div}(w_o\otimes w_o + \mathring{R}_\ell) - \frac{\nabla|w_o|^2}{2}\right)\\
&+ w_c\otimes w_c + w_o\otimes w_c + w_c\otimes w_o - \left(\frac{|w_c|^2}{3} + \frac{2\langle w_o, w_c\rangle}{3}\right)\mathrm{Id}\\
&+ \mathring{R}_q - \mathring{R}_\ell\\
&\qquad = R_o' + R_o'' + R_o'''.
\end{aligned}$$

In particular with this definition we have

$$\operatorname{div}\left(R_o' + R_o'' + R_o'''\right) = \operatorname{div}\mathring{R}_o\,.$$

As a product of Lemma 3.1.3 and Proposition 3.2.3 (with $m > \frac{1}{\varepsilon}$) and the easily verifable bounds

$$\|a_k\|_N \leq C\lambda_{q+1}^{-\beta}(1 + \lambda_q\ell^{1-N})\,,$$

it is not difficult to show that for any small $\varepsilon > 0$

$$\left\|R_o'\right\|_0 \leq C \frac{\lambda_q}{\lambda_{q+1}^{1+\beta-\varepsilon} \lambda_q^{\beta}} \sim \lambda_q^{-(1+b)\beta+1-b} .$$

Similar estimates can be found to hold for R_o'' and R_o'''. Then, as was pointed out at the end of Section 2.3, Chapter 2, such estimates are compatible with Onsager's conjecture. Unfortunately, the definition of w_{q+1} given above does not lead to good estimates on the transport error and therefore this definition will need to be modified so that the perturbation is better transported by the flow v_q. This is the topic of the next chapter.

3.3 References and Remarks

The results of the chapter are almost entirely contained in the papers [DLSJ12a, DLSJ12b] of De Lellis and Székelyhidi Jr. The proof however of Lemma 3.1.2 follows more closely the style of an alternative proof given in [Ise12, Ise13a]. For the analogous results for the 2-D case Euler equation, we refer the reader to the papers [CDLS12, Cho12].

We note that the estimate (3.2.3) (and consequently (3.2.5)) can be further improved by observing that the Riesz operators are bounded on space of functions of bounded mean oscillation (BMO): one can then replace the use of Sobolev inequality with the logarithmic Sobolev inequality of Kozono and Taniuchi [KT00]. Such an improvement could potentially enable the convex integration schemes presented here to be modified in order to obtain better (slower) frequency growth rates. Recently, an entirely different approach to solving the equation $\operatorname{div} R = v$ was taken by Isett and Oh in [IO14] which allowed the authors to construct non-conservative $1/5 - \varepsilon$ Hölder continuous weak solutions to (1.1.1) in $\mathbb{R}^3$ with an exponential growth rate of characteristic frequencies.

4

Minimisation of Transport Error

4.1 The Principal Transport Error

As was pointed out in the previous chapter, we need to modify our definition of w_{q+1} in order that it is approximately transported by v_q. In particular, we need to minimise the transport error:

$$\partial_t w_{q+1} + v_q \cdot \nabla w_{q+1} \,. \tag{4.1.1}$$

The principle error arising from our previous definition of w_{q+1} in Section 3.1 arises when the material derivative $(\partial_t + v_q \cdot \nabla)$ falls on the oscillatory terms $e^{i\lambda_{q+1} k\cdot x}$. To fix this we introduce cut-off functions $\chi_\varsigma : [0,T] \to \mathbb{R}$ for indices $\varsigma \in \mathbb{N}$. We also introduce a family of large parameters $\mu_{q+1,\varsigma}$, and require that each cut-off χ_ς is identically 1 on a closed interval of length at least $\mu_{q+1,\varsigma}^{-1}$ and are supported on an interval of length at most $4\mu_{q+1,\varsigma}^{-1}$. The cut-offs will be constructed such that their squares provides a partition of unity of time, i.e. $\sum_\varsigma \chi_\varsigma(t)^2 \equiv 1$ for $t \in [0,T]$. Moreover, only cut-off functions with neighbouring indices will be allowed to have overlapping support. We then replace the terms $e^{i\lambda_{q+1} k\cdot x}$ in the definition of w_{q+1} with $\chi_\varsigma e^{i\lambda_{q+1} k\cdot \Phi_\varsigma}$, where Φ_ς are phase functions solving the transport equation

$$\begin{cases} \partial_t \Phi_\varsigma + v_\ell \cdot \nabla \Phi_\varsigma = 0 \\ \\ \Phi_\varsigma(x, t_\varsigma) = x \end{cases},$$

where t_ς is the centre of the interval $\operatorname{supp}(\chi_\varsigma)$ and v_ℓ is a mollification of v_q at length scale $\ell = \lambda_{q+1}^{\varepsilon_0 - 1}$. We will also replace the function $\rho : [0, T] \to \mathbb{R}$ by constants ρ_ς and in order to weaken the interaction between waves from neighbouring cut-off regions, we apply Lemma 3.1.2 with $n = 2$ to create to disjoint families of wave vectors Λ_0 and Λ_1. The principle perturbation w_o is then redefined to be

$$\begin{aligned} w_o(x, t) := & \sqrt{\rho_\varsigma} \sum_{k \in \Lambda_0, \varsigma \text{ odd}} \chi_\varsigma(t) \gamma_k \left(\frac{R_\varsigma(x, t)}{\rho_\varsigma} \right) B_k e^{i \lambda_{q+1} k \cdot \Phi_\varsigma(x,t)} + \\ & \sqrt{\rho_\varsigma} \sum_{k \in \Lambda_1, \varsigma \text{ even}} \chi_\varsigma(t) \gamma_k \left(\frac{R_\varsigma(x, t)}{\rho_\varsigma} \right) B_k e^{i \lambda_{q+1} k \cdot \Phi_\varsigma(x,t)} . \end{aligned} \tag{4.1.2}$$

We will defer the definition R_ς until the next section, however for now it suffices to say that R_ς will play the role of R in Chapter 3.1. Employing the notation

$$\varphi_{k\varsigma} := e^{i \lambda_{q+1} k \cdot (\Phi_\varsigma - x)} , \tag{4.1.3}$$

we define

$$a_{k,\varsigma}(x, t) := \sqrt{\rho_\varsigma} \chi_\varsigma(t) \gamma_k \left(\frac{R_\varsigma(x, t)}{\rho_\varsigma} \right) \varphi_{k\varsigma}(x, t) , \tag{4.1.4}$$

which will roughly replace a_k in the previous definition (3.1.12): in particular we may rewrite (3.1.13) as

$$w_o(x, t) := \sum_{k,\varsigma} a_{k,\varsigma}(x, t) B_k e^{i \lambda_{q+1} k \cdot x} , \tag{4.1.5}$$

where here and from now on we let $\sum_{k,\varsigma}$ denote the short hand for the sum over $k \in \Lambda_0 \cup \Lambda_1$ and indices ς.

Analogous to (3.1.15), the corrector w_c is then defined by the formula

$$w_c(x, t) := \sum_k \frac{i}{\lambda_{q+1}} \nabla a_{k,\varsigma}(x, t) \times \left(\frac{k \times B_k}{|k|^2} \right) e^{i \lambda_{q+1} k \cdot x} , \tag{4.1.6}$$

and as before we set $w_{q+1} := w_o + w_c$.

Clearly, assuming ℓ is sufficiently small and the parameters $\mu_{q+1,\varsigma}$ are sufficiently large, then the transport error arising when when the material derivative falls on the phase functions Φ_ς will be relatively small, and Φ_ς will provide a good approximation of the identity. As a trade off, a new error will be introduced when the time derivative falls on the cut-off functions in the regions of overlapping cut-offs.

Given an index ς, we denote the interval $\{s : \chi_\varsigma(s) = 1\}$ by K_ς^g and the overlapping region $\operatorname{supp}(\chi_\varsigma) \cap \operatorname{supp}(\chi_{\varsigma+1})$ by K_ς^b. Informally, we will refer to the union $\bigcup_\varsigma K_\varsigma^g$ as the set of *good* times, and conversely $\bigcup_\varsigma K_\varsigma^b$ will be referred to as the set of *bad* times. The rational for such a choice of terminology is that we will obtain better estimates on the *good* set than on the *bad* set.

In order to better parameterise the error obtained when time derivatives fall on the cut-offs, we introduce new small parameters $\eta_{q+1,\varsigma}$ and assume K_ς^b to be an open interval contained in a ball of radius at least $\max(\eta_{q+1,\varsigma}\mu_{q+1,\varsigma}^{-1}, \eta_{q+1,\varsigma+1}\mu_{q+1,\varsigma+1}^{-1})$. Furthermore, for $N = 0, 1, \ldots$ we assume the following estimate

$$\left\|\chi_\varsigma\right\|_N \leq C \min\left(\frac{\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}}, \frac{\mu_{q+1,\varsigma+1}}{\eta_{q+1,\varsigma+1}}\right)^N , \tag{4.1.7}$$

where the constant C depends only on N.

Remark 4.1.1. For the purpose of proving Theorem 1.2.2, one may assume the parameters $\mu_{q+1,\varsigma} := \mu_{q+1}$ to be chosen uniformly depending on the given iterate q and the parameters $\eta_{q+1,\varsigma}$ to be a uniform constant, say $\frac{1}{10}$. The cut-off functions χ_ς can then be defined in a uniform manner: set $\chi_\varsigma(t) := \chi\left(\mu_{q+1}t - \varsigma\right)$ for some smooth function χ, supported in $(-3/4, 3/4)$, bounded above by 1 and such that

$$\sum_{i \in \mathbb{Z}} \chi^2(t-i) \equiv 1 .$$

The choice of $\mu_{q+1,\varsigma}$, $\eta_{q+1,\varsigma}$ and χ_ς taken in the proof of Theorem 1.2.3 will be more delicate (see Chapter 8).

Finally, we replace the definition (3.2.9) of the pressure p_{q+1} with the following slightly modified definition

$$p_{q+1} = p_q - \frac{|w_o|^2}{2} - \frac{1}{3}|w_c|^2 - \frac{2}{3}\langle w_o, w_c\rangle - \frac{2}{3}\langle v_q - v_\ell, w_{q+1}\rangle . \tag{4.1.8}$$

The addition of the last term is a technical consideration that shifts the focus of estimating (4.1.1) to instead estimating

$$\partial_t w_{q+1} + v_\ell \cdot \nabla w_{q+1} \,.$$

In particular, we set

$$\mathring{R}_{q+1} = R^0 + R^1 + R^2 + R^3 + R^4 + R^5 \,,$$

where

$$R^0 = \mathcal{R}\left(\partial_t w_{q+1} + v_\ell \cdot \nabla w_{q+1}\right) \tag{4.1.9}$$
$$R^1 = \mathcal{R}\mathrm{div}\left(w_o \otimes w_o - \sum_\varsigma \chi_\varsigma^2 R_\varsigma - \tfrac{|w_o|^2}{2}\mathrm{Id}\right) \tag{4.1.10}$$
$$R^2 = \mathcal{R}\left(w_{q+1} \cdot \nabla v_\ell\right) \tag{4.1.11}$$
$$R^3 = w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c - \tfrac{|w_c|^2 + 2\langle w_o, w_c\rangle}{3}\mathrm{Id} \tag{4.1.12}$$
$$R^4 = w_{q+1} \otimes (v_q - v_\ell) + (v_q - v_\ell) \otimes w_{q+1} - \tfrac{2\langle (v_q - v_\ell), w_{q+1}\rangle}{3}\mathrm{Id} \tag{4.1.13}$$
$$R^5 = \mathring{R}_q + \sum_\varsigma \chi_\varsigma^2 \mathring{R}_\varsigma \,. \tag{4.1.14}$$

Note $\sum_\varsigma \chi_\varsigma^2 \mathrm{tr}\, R_\varsigma$ is a function of time only. Then by inspection one obtains

$$\begin{aligned}
&\mathrm{div}\, \mathring{R}_{q+1} - \nabla p_{q+1} \\
&\quad = \partial_t w_{q+1} + \mathrm{div}\left(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q + w_{q+1} \otimes w_{q+1}\right) \\
&\qquad + \mathrm{div}\, \mathring{R}_q - \nabla p_q \\
&\quad = \partial_t w_{q+1} + \mathrm{div}\left(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q + w_{q+1} \otimes w_{q+1}\right) \\
&\qquad + \partial_t v_q + \mathrm{div}\left(v_q \otimes v_q\right) \\
&\quad = \partial_t v_{q+1} + \mathrm{div}\, v_{q+1} \otimes v_{q+1} \,,
\end{aligned}$$

i.e. the triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ is a solution to the Euler-Reynolds system (2.1.1).

4.2 Transport Error of Previous Reynolds Stress

A secondary transport error arises when the material derivative $(\partial_t + v_q \cdot \nabla)$ falls on the functions R_ς, which will themselves be defined in terms of the Reynolds stress $\mathring{R}_q$. It then becomes necessary to keep track of the material derivatives of the Reynolds stress.

One potential pitfall is that the previous material derivative $(\partial_t + v_{q-1} \cdot \nabla)$ of the previous Reynolds stress $\mathring{R}_{q-1}$ appears in definition of the Reynolds stress $\mathring{R}_q$. It will then become convenient to approximate $\mathring{R}_q$ with a function $\mathring{R}_\varsigma$ that has good second order material derivative estimates.

In line with the definition of Φ_ς, a possible definition of R_ς (whose trace free part we denote as $\mathring{R}_\varsigma$) would be the solution to the free transport equation

$$\begin{cases} \partial_t R_\varsigma + v_\ell \cdot \nabla R_\varsigma = 0 \\ \\ R_\varsigma(x, t_\varsigma) = \rho_\varsigma \mathrm{Id} - \mathring{R}_\ell(x, t_\varsigma) . \end{cases} \tag{4.2.1}$$

Alternatively, another possibility is to *mollify along the flow*. Let ℓ_t be a small mollification parameter and $X_t(x, s)$ be the *flux* of v_ℓ with initial time t: $X_t(x, s)$ solves the ordinary differential equation

$$\begin{aligned} \frac{d}{ds} X_t(x, s) &= v_\ell(X_t(x, s), s) \\ X_t(x, t) &= x . \end{aligned}$$

The mollification of R_ℓ along the flow v_ℓ is then defined by the formula

$$R_\varsigma(x, t) = \rho_\varsigma \mathrm{Id} - \int_{\mathbb{R}} \mathring{R}_\ell(X_t(x, t+s), t+s)\, \tilde{\psi}_{\ell_t}(s)\, ds, \tag{4.2.2}$$

for standard mollifer $\tilde{\psi} \in C^\infty(\mathbb{R})$, where by abuse of notation $\mathring{R}_\ell$ denotes the vanishing temporal extension of $\mathring{R}_\ell$ to $\mathbb{R}$ — in application, such an extension will require us to be careful near the temporal boundary.

4.3 Transport Estimates

Before we state estimates for our Reynolds stress approximations $\mathring{R}_\varsigma$, we recall some elementary transport equation estimates. In what follows, we will assume $f : \mathbb{T}^3 \times [-T, T] \to \mathbb{R}$ to be a smooth solution to the *transport equation*

$$\begin{cases} \partial_t f + v \cdot \nabla f = g \\ f(x, 0) = f_0(x) , \end{cases} \tag{4.3.1}$$

for some smooth function g and smooth vector field v.

We will let $X(x, t)$ to be the *flux* of v from initial time 0, i.e. $X(x, t)$ is described by the ordinary differential equation

$$\frac{d}{dt}X(x,t) = v(X(x,t),t)$$
$$X(x,0) = x\,.$$

In particular note we have the identity

$$\frac{d}{dt}f(X(t,x),t) = g(X(t,x),t)\,. \tag{4.3.2}$$

The inverse flow to $X(x,t)$ will be denoted by $X^{-1}(t,\cdot)$, which by definition is a solution to the *free transport equation*, i.e with $g \equiv 0$. Furthermore, we will adopt the notation $D_t := \partial_t + v\cdot\nabla$ for the *material derivative* associated with v.

Proposition 4.3.1. *We have the following estimates on f*

$$\|f\|_0 \le \|f_0\|_0 + T\,\|g\|_0 \tag{4.3.3}$$
$$[f]_1 \le ([f_0]_1 + T[g]_1)\,e^{T[v]_1}\,, \tag{4.3.4}$$

and, more generally, for any $N = 2,3,\ldots$ there exists a constant C so that

$$[f]_N \le \Big([f_0]_N + CT[v]_N[f_0]_1\Big)e^{CT[v]_1} + CT\Big([g]_N + T[v]_N[g]_1\Big)e^{CT[v]_1}. \tag{4.3.5}$$

Let Φ be either the flux X or the inverse flux X^{-1}, then we have the following estimates:

$$\|D\Phi - \mathrm{Id}\|_0 \le e^{T[v]_1} - 1\,, \tag{4.3.6}$$
$$[\Phi]_N \le CT[v]_N e^{CT[v]_1} \qquad \forall N \ge 2\,. \tag{4.3.7}$$

Proof. Without loss of generality we may assume $t > 0$. To see this, simply replace v with $-v$.

We begin by considering the estimates on f. Integrating (4.3.2) in time we obtain

$$f(t,x) = f_0(X^{-1}(x,t)) + \int_0^t g(X(X^{-1}(t,x),s),s)\,ds\,, \tag{4.3.8}$$

from which (4.3.3) readily follows. Spatially differentiating (4.3.1) yields the identity

$$D_tDf = (\partial_t + v\cdot\nabla)Df = Dg - DfDv, \tag{4.3.9}$$

and thus

$$\|D_t Df(t)\|_0 \le [g(t)]_1 + [v(t)]_1 [f(t)]_1 \,, \tag{4.3.10}$$

where here we have employed the shorthand from Appendix A.1 where we write $[f(t)]_\alpha$ and $\|f(t)\|_\alpha$ to denote the seminorm/norm of f evaluated for the restriction of f to the t-time slice. Then from (4.3.8) and Gronwall's inequality we obtain (4.3.4). Further differentiating (4.3.9) and applying interpolation yields

$$\begin{aligned}\left\|D_t D^N f\right\|_0 &\le [g]_N + C\sum_{j=0}^{N-1}[v]_{j+1}[f(t)]_{N-j}\\ &\le [g]_N + C[v]_N[f(t)]_1 + C[v]_1[f(t)]_N \,.\end{aligned}$$

Hence applying (4.3.4), (4.3.8) and Gronwall's inequality we obtain (4.3.5).

We now consider the estimates on Φ. Again by replacing v by $-v$, we may without loss of generality assume that Φ is the inverse flux X^{-1}. Note that $\Phi - x$ is a solution to the transport equation with vanishing initial condition and nonlinearity $g(x) = -v$. Moreover, we have $D(\Phi - x) = D\Phi - \mathrm{Id}$. Applying (4.3.8) and (4.3.10) we obtain

$$\|D\Phi(t) - \mathrm{Id}\|_0 \le \int_0^t [v]_1 \left(\|D\Phi(s) - \mathrm{Id}\|_0 + 1\right)\, ds \,.$$

Then from Gronwall's inequality we have

$$\|D\Phi(t) - \mathrm{Id}\|_0 \le \int_0^t e^{(t-s)[v]_1}[v]_1\, ds \le e^{T[v]_1} - 1 \,.$$

Finally, since Φ solves (4.3.1) with $g = 0$ and $D^2\Phi(\cdot, 0) = 0$, the estimate (4.3.7) is a consequence of (4.3.5). $\square$

To state our estimates on $\mathring{R}_\varsigma$, we introduce amplitude parameters $\{\delta_{q,\varsigma}, \bar{\delta}_{q,\varsigma}\}$ satisfying $\delta_{q,\varsigma} \le \bar{\delta}_{q,\varsigma}$, and assume the following inductive estimates for times $t \in \operatorname{supp} \chi_\varsigma$:

$$\frac{1}{\lambda_q}\left\|v_q(t)\right\|_1 \le \delta_{q,\varsigma}^{1/2} \tag{4.3.11}$$

$$\frac{1}{\lambda_q}\left\|p_q(t)\right\|_1 + \frac{1}{\lambda_q^2}\left\|p_q(t)\right\|_2 \le \delta_{q,\varsigma} \tag{4.3.12}$$

$$\left\|\mathring{R}(t)q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q(t)\right\|_1 + \frac{1}{\lambda_q^2}\left\|\mathring{R}_q(t)\right\|_2 \le c_1\delta_{q+1,\varsigma} \tag{4.3.13}$$

$$\left\|(\partial_t + v_q\cdot\nabla)\mathring{R}_q(t)\right\|_0 \le c_1\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \,, \tag{4.3.14}$$

where $c_1 > 0$ is a small constant to be chosen later. The second order estimates of the pressure and Reynolds stress are important in controlling second order material derivative estimates of our approximation $\mathring{R}_{q,\varsigma}$, which in turn will be used in controlling the material derivative of our new Reynolds stress $\mathring{R}_{q+1}$, more specifically, the material derivative of the transport error (cf. Section 4.4).

Since our approximation of the Reynolds stress $\mathring{R}_\varsigma$ is constructed to be approximately transported by the *mollified* velocity v_ℓ, the notation D_t will from now be used solely to represent the operator $\partial_t + v_\ell \cdot \nabla$.

Remark 4.3.2. For the proof of Theorem 1.2.2, one may simply set $\bar{\delta}_{q,\varsigma} := \delta_{q,\varsigma} := \lambda_q^{-2\beta}$ uniformly for all ς; however for Theorem 1.2.3 the parameters $\delta_{q,\varsigma} \leq \bar{\delta}_{q,\varsigma}$ will depend on ς.

Lemma 4.3.3. *Assume the estimates* (4.3.11)-(4.3.14) *are satisfied for times* $t \in \operatorname{supp} \chi_\varsigma$. *Futhermore, assume* $\bar{\delta}_q^{1/2} = \delta_q^{1/2}$. *Then for* R_ς *defined by* (4.2.1) *and* $\mu_{q+1.\varsigma} \geq \delta_{q,\varsigma}^{1/2}\lambda_q$ *we obtain the following estimates*

$$\|\mathring{R}_\varsigma(t)\|_0 \leq c_1\delta_{q+1,\varsigma} \tag{4.3.15}$$

$$\|\mathring{R}_\varsigma(t)\|_N \leq C\delta_{q+1,\varsigma}\lambda_q\ell^{1-N} \qquad \text{for } N > 0 \tag{4.3.16}$$

$$\|(\mathring{R}_q - \mathring{R}_\varsigma)(t)\|_0 \leq C\delta_{q+1,\varsigma}\lambda_q\left(\delta_{q,\varsigma}^{1/2}\mu_{q+1,\varsigma}^{-1} + \ell\right)\,, \tag{4.3.17}$$

Moreover fixing N and assuming c_1 is sufficiently small, the constant C in the estimates above can be made arbitrarily small.

Proof. Restricting to times in the support of χ_ς and applying Proposition 4.3.1, we obtain (4.3.15) as direct consequence of (4.3.3) and (4.3.13). Similarly, from (4.3.4), (4.3.5) and (4.3.13), together with the observations

$$\sup_{t\in\operatorname{supp}\chi_\varsigma} \mu_{q+1.\varsigma}^{-1}[v(t)]_N \leq C\mu_{q+1.\varsigma}^{-1}\delta_{q,\varsigma}^{1/2}\lambda_q\ell^{1-N} \leq C\ell^{1-N},$$

for $N \in \mathbb{N}$, we conclude (4.3.16).

Again, applying Proposition 4.3.1, from (4.3.3) and (4.3.14), we obtain

$$\begin{aligned}
\|(\mathring{R}_q - \mathring{R}_\varsigma)(t)\|_0 &\leq \|\mathring{R}_q(t_\varsigma) - \mathring{R}_\ell(t_\varsigma)\|_0 + \mu_{q+1,\varsigma}^{-1}\left\|D_t\mathring{R}_q(t)\right\|_0 \\
&= \|\mathring{R}_q(t_\varsigma) - \mathring{R}_\ell(t_\varsigma)\|_0 + C\mu_{q+1,\varsigma}^{-1}\left\|(\partial_t + v_q\cdot\nabla)\mathring{R}_q(t)\right\|_0 \\
&\quad + C\mu_{q+1,\varsigma}^{-1}\left\|v_q(t) - v_\ell(t)\right\|_0\left\|\mathring{R}_q(t)\right\|_1 \\
&\leq C\delta_{q+1,\varsigma}\lambda_q\left(\delta_{q,\varsigma}^{1/2}\mu_{q+1,\varsigma}^{-1} + \ell + \delta_{q,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\ell\right)
\end{aligned}$$

$$\leq C\delta_{q+1,\varsigma}\lambda_q \left(\delta_{q,\varsigma}^{1/2}\mu_{q+1,\varsigma}^{-1} + \ell\right).$$

Here we used the decomposition $D_t = (\partial_t + v_q \cdot \nabla) + (v_\ell - v_q) \cdot \nabla$ and the inequality $\lambda_q \ell \leq 1$. □

Lemma 4.3.4. *Assume $\delta_{q,\varsigma}^{1/2}\lambda_q \leq \ell_t^{-1}$ and the estimates (4.3.11)-(4.3.14) are satisfied in a $4\ell_t$-neighbourhood of the support of χ_ς. Then if R_ς is defined by (4.2.2) and $t \in \operatorname{supp}(\chi_\varsigma)$, the following estimates are satisfied:*

$$\|\mathring{R}_\varsigma(t)\|_0 \leq c_1\delta_{q+1,\varsigma} \tag{4.3.18}$$

$$\|\mathring{R}_\varsigma(t)\|_N \leq C\delta_{q+1,\varsigma}\lambda_q\ell^{1-N} \qquad \text{for } N > 0 \tag{4.3.19}$$

$$\|D_t\mathring{R}_\varsigma(t)\|_N \leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\ell^{-N} \tag{4.3.20}$$

$$\|D_t^2\mathring{R}_\varsigma(t)\|_N \leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\ell_t^{-1}\ell^{-N} \tag{4.3.21}$$

$$\|(\mathring{R}_q - \mathring{R}_\varsigma)(t)\|_0 \leq C\delta_{q+1,\varsigma}\lambda_q(\bar{\delta}_{q,\varsigma}^{1/2}\ell_t + \ell), \tag{4.3.22}$$

Moreover, fixing N and assuming c_1 to be sufficiently small, the constant C in the estimates above can be made arbitrarily small.

Before we can prove the above lemma, we will require the following commutator estimate which we will prove at the end of the section:

Proposition 4.3.5. *Let $f, g \in C^\infty(\mathbb{T}^3)$ and ψ the mollifier of Chapter 3. For any $r \geq 0$ we have the estimate*

$$\left\|(fg) * \chi_\ell - (f * \chi_\ell)(g * \chi_\ell)\right\|_N \leq C\ell^{2r-N}\|f\|_r\|g\|_r,$$

where the constant C depends only on $0 \leq r \leq 1$.

Proof of Lemma 4.3.4. Recall that in this case we have the formula

$$\mathring{R}_\varsigma(x,t) = \int \mathring{R}_\ell(X_t(x, t+s), t+s)\tilde{\psi}_{\ell_t}(s)\, ds. \tag{4.3.23}$$

From (4.3.13) we have for any $N \in \mathbb{N}$

$$\|\mathring{R}_\ell(t)\|_0 \leq c_1\delta_{q+1,\varsigma}$$

$$\|D^N\mathring{R}_\ell(t)\|_0 \leq C\delta_{q+1,\varsigma}\lambda_q\ell^{1-N}.$$

We immediately obtain (4.3.18) for $N = 0$. Then for $N \geq 1$ we apply (A.1.5) to obtain the estimate

$$\begin{aligned}\|D^N(\mathring{R}_\ell(X_t(t+s),t+s))\|_0 \leq & C\|D^N X_t(t+s)\|_0 \delta_{q+1,\varsigma}\lambda_q \\ & + C\|DX_t(t+s)\|_1^N \delta_{q+1,\varsigma}\lambda_q \ell^{1-N} \,. \end{aligned} \tag{4.3.24}$$

Taking $|s| \leq 4\ell_t$, by Proposition 4.3.1 we conclude

$$\|D^N X_t(t+s)\|_N \leq C\ell_t [v_\ell]_N e^{C\ell_t [v_\ell]_1} \leq C\ell^{1-N}, \tag{4.3.25}$$

where here we used the inequalities $[v_\ell(t)]_1 \leq \delta_{q,\varsigma}^{1/2}\lambda_q$ and $\ell_t \delta_{q,\varsigma}^{1/2}\lambda_q \leq 1$. Inserting in (4.3.24), we conclude

$$\|D^N(\mathring{R}_\ell(X_t(t+s),t+s))\|_0 \leq C\delta_{q+1,\varsigma}\lambda_q \ell^{1-N},$$

for all $N \geq 1$ Hence differentiating (4.3.23) we achieve (4.3.19) for any N.

We now observe the following identities:

$$D_t \mathring{R}_\varsigma(x,t) = \int (D_t \mathring{R}_\ell)(X_t(x,t+s),t+s)\, \tilde{\psi}_{\ell_t}(s)\, ds \tag{4.3.26}$$

$$\begin{aligned} D_t^2 \mathring{R}_\varsigma(x,t) &= \int (D_t^2 \mathring{R}_\ell)(X_t(x,t+s),t+s)\, \tilde{\psi}_{\ell_t}(s)\, ds \\ &\overset{(4.3.2)}{=} \int \frac{d}{ds}[(D_t \mathring{R}_\ell)(X_t(x,t+s),t+s)]\, \tilde{\psi}_{\ell_t}(s)\, ds \\ &= -\ell_t^{-1} \int (D_t \mathring{R}_\ell)(X_t(x,t+s),t+s)\, \tilde{\psi}'_{\ell_t}(s)\, ds \,. \end{aligned} \tag{4.3.27}$$

Hence we deduce from the following estimates

$$\|D_t \mathring{R}_\varsigma(t)\|_N \leq \sup_{|s|\leq 4\ell_t} C\|D_t \mathring{R}_\ell(X_t(t+s),t+s))\|_N \tag{4.3.28}$$

$$\|D_t^2 \mathring{R}_\varsigma(t)\|_N \leq \sup_{|s|\leq 4\ell_t} C\ell_t^{-1}\|D_t \mathring{R}_\ell(X_t(t+s),t+s))\|_N \,. \tag{4.3.29}$$

Observe the following decomposition

$$\begin{aligned} D_t \mathring{R}_\ell = & (D_t \mathring{R}_q) * \psi_\ell + \operatorname{div}\left(v_\ell \otimes \mathring{R}_\ell - (v_q \otimes \mathring{R}_q) * \psi_\ell\right) \\ & + [(v_q - v_\ell)\cdot \nabla \mathring{R}_q] * \psi_\ell \,. \end{aligned}$$

Therefore applying Proposition 4.3.5 on the second summand we conclude that taking

$|s| \leq 4\ell_t$ we have the estimate

$$\|D_t \mathring{R}_\ell(t+s)\|_N \leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \ell^{-N} + C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q^2 \ell^{1-N}$$
$$\leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \ell^{-N}\,. \tag{4.3.30}$$

Then from (4.3.28), (4.3.30),(4.3.25) and (A.1.5), we obtain

$$\begin{aligned}\|D^N D_t \mathring{R}_\varsigma(t)\|_0 &\leq C\|D^N X_t(t+s)\|_0 \bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \ell^{-1} \\ &\quad + C\|DX_t(t+s)\|_1^N \bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \ell^{-N} \\ &\leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q \ell^{-N}.\end{aligned}$$

Hence we conclude (4.3.20). The estimate (4.3.21) also follows analogously by utilising (4.3.29) in place of (4.3.28).

Finally, we note that

$$\left\|\mathring{R}_\varsigma(t) - \mathring{R}_\ell(t)\right\|_0 \leq \sup_{|s|\leq 4\ell_t} \left\|\mathring{R}_\ell(X_t(t+s), t+s) - \mathring{R}_\ell(t)\right\|_0 .$$

Since by definition $X_t(x,t) = x$, differentiating in s we conclude

$$\|\mathring{R}_\varsigma(t) - \mathring{R}_\ell(t)\|_0 \leq \ell_t \|D_t \mathring{R}_\ell\|_0 \,. \tag{4.3.31}$$

Hence (4.3.22) then follows from (4.3.31) and the mollification estimate

$$\|\mathring{R}_q(t) - \mathring{R}_\ell(t)\|_0 \leq C\delta_{q+1,\varsigma}\lambda_q \ell\,. \qquad \square$$

Proof of Proposition 4.3.5. Begin by noting that for a fixed x we have following identity

$$\begin{aligned}&(fg) * \psi_\ell - (f * \psi_\ell)(g * \psi_\ell) = \\ &\qquad [(f - f(x))(g - g(x))] * \psi_\ell - (f(x) - f) * \psi_\ell (g(x) - g) * \psi_\ell \,.\end{aligned}$$

Let α be a multi-index, then noting the identity $h(x) * D^\alpha \psi_\ell \equiv 0$, we obtain

$$\begin{aligned}D^\alpha\left[(fg) * \psi_\ell - (f * \psi_\ell)(g * \psi_\ell)\right] &= [(f - f(x))(g - g(x))] * D^\alpha \psi_\ell \\ &\quad - \sum_{\alpha'+\alpha''=\alpha} (f(x) - f) * D^{\alpha'}\psi_\ell (g(x) - g) * D^{\alpha''}\psi_\ell \,.\end{aligned}$$

Next, for $|y| \leq 2\ell$ note the trivial estimate

$$\begin{aligned}\|f(\cdot - y) - f(\cdot)\|_0 &\leq C\,|y|^r\,\|f\|_r \\ &\leq C\ell^r\,\|f\|_r\,,\end{aligned}$$

and similarly

$$\|g(\cdot - y) - g(\cdot)\|_0 \leq C\ell^{-r}\,\|g\|_r\,.$$

Then combining the above estimates with the above identities, we obtain our claim. □

4.4 References and Remarks

The basic construction of the perturbation w_{q+1} presented here was first introduced in [BDLSJ13] and later refined in [Buc13, BDLS14]. The construction itself being heavily influenced by the earlier papers of De Lellis and Székelyhidi Jr. [DLSJ12a, DLSJ12b].

The careful reader will note in contrast to inductive second order bounds of (4.3.12) and (4.3.13), in [BDLSJ13], only first order estimates of the pressure p_q and Reynolds stress $\mathring{R}_q$ were needed. The lack of second order estimates necessitated the careful choice of the mollification parameter ℓ; such a choice however seems incompatible with a scheme constructing solutions at Onsager-critical regularity such as Theorem 1.2.3. It seems however, at least for the purpose of proving Theorem 1.2.3, that only second order estimates on the pressure are required. Such an approach was taken in [Buc13]. For reasons of symmetry, and in the event that the resulting sharper estimates are required for future schemes, we decided to include the second order inductive estimate of $\mathring{R}_q$ — as we note was also done in [Ise12, Ise13a, BDLS14].

The parameter notation $(\bar{\delta}_{q,\varsigma}, \delta_{q,\varsigma}, \delta_{q+1,\varsigma}, \mu_{q+1,\varsigma}, \eta_{q+1,\varsigma})$ differs from that of [BDLSJ13, Buc13, BDLS14]: this is done in order to deal with Theorems 1.2.2 and 1.2.3 simultaneously in a coherent manner. The necessary translations between the differing notations will be dealt with in Chapters 7 and 8.

The convex integration scheme of Isett [Ise12, Ise13a] was the first to keep track of material derivatives of the Reynolds stress. Isett's scheme also introduced the concept of microlocal Beltrami waves in order to obtain better estimates on the *principal transport error* discussed in Section 4.1. The simpler solution of modifying the phase function of the Beltrami waves so that they solve the free transport equation was introduced in [BDLSJ13].

Isett's scheme was also the first to directly consider the transport error of the previous Reynolds stress (discussed in Section 4.2), where the technique of mollifying along the flow (4.2.2) was applied. The free transport solution (4.2.1) was introduced in [BDLSJ13] as an alternative solution to handling this error. We note that for the purpose of proving Theorem 1.2.2, either approximation may be used; however for the proof of Theorem 1.2.3 we will require both. Specifically, the mollification along flow provides a better approximation of the Reynolds stress in situations where the time mollification parameter ℓ_t is less than the size of the cut off $\mu_{q+1,\varsigma}^{-1}$. One minor issue however with the technique is that it requires estimates on the Reynolds stress in a *neighbourhood* of the support of the cut-off.[1]

The commutator estimate of Proposition 4.3.5 played an essential role in Constantin, E and Titi's elegant proof that Hölder continuous weak solutions to the Euler equations (1.1.1) with Hölder exponent greater than 1/3 conserve their kinetic energy [CWT94]. For the reader's convenience we present this proof below:

Proof of Theorem 1.2.1. [2] Let u_ℓ be the spatial mollification of u an length scale ℓ. By abuse of notation, let us extend u_ℓ smoothly in time to whole real line — the specific extension will play no role in later arguments. Define $u_{\ell,\tau}$ to be the time mollification of u_ℓ at length scale τ:

$$u_{\ell,\tau}(x,t) = u_\ell *_t \tilde{\psi}_\tau(x,t) = \int_{\mathbb{R}} u_\ell(x,s)\tilde{\psi}_\tau(t-s)\,ds\,.$$

Then for $t \in (2\tau, T-2\tau)$ we have that $u_{\ell,\tau}$ satisfies the differential equation

$$\partial_t u_{\ell,\tau} + \operatorname{div}(u\otimes u)_\ell *_t \tilde{\psi}_\tau + \nabla p_\ell *_t \tilde{\psi}_\tau\,.$$

Taking the inner product of the equation with $u_{\ell,\tau}$ and integrating on the range $(2\tau, t-2\tau)$, we obtain

$$\int_{\mathbb{T}^3} |u_\ell(x,t-2\tau)|^2\,dx - \int_{\mathbb{T}^3} |u_\ell(x,2\tau)|^2\,dx = 2\int_{2\tau}^{t-2\tau}\int_{\mathbb{T}^3} \mathrm{Tr}((u\otimes u)_\ell *_t \tilde{\psi}_\tau(\nabla u_{\ell,\tau}))dxds\,. \quad (4.4.1)$$

[1] This issue could potentially be resolved by replacing $\mathring{R}_q$ in the definition of $\mathring{R}_\varsigma$ with the free transport extension of the restriction of $\mathring{R}_q$ to the support of χ_ς, which in some sense would be an amalgamation of the two approximations.

[2] As was mentioned at the end of Section 1.3.1, the result of [CWT94], which is stated in terms of Besov spaces, is in fact stronger than Theorem 1.2.1, which is stated in terms of Hölder spaces. We note however the proof presented here easily transfers to the Besov case.

From the continuity of u, letting τ tend to zero, we obtain

$$\int_{\mathbb{T}^3} |u_\ell(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |u_\ell(x,0)|^2 \, dx = 2 \int_0^t \int_{\mathbb{T}^3} \mathrm{Tr}((u \otimes u)_\ell (\nabla u_\ell)) dx ds \, .$$

Since we have

$$\int_{\mathbb{T}^3} \mathrm{Tr}((u_\ell \otimes u_\ell)(\nabla u_\ell)) dx \equiv 0 \, ,$$

then we obtain the identity

$$\int_{\mathbb{T}^3} |u_\ell(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |u_\ell(x,0)|^2 \, dx = \\ 2 \int_0^t \int_{\mathbb{T}^3} \mathrm{Tr} \left(((u \otimes u)_\ell - (u_\ell \otimes u_\ell))(\nabla u_\ell) \right) dx ds \, .$$

Applying Proposition 4.3.5 we deduce

$$\left| \int_{\mathbb{T}^3} |u_\ell(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |u_\ell(x,0)|^2 \, dx \right| \leq C \ell^{3\theta - 1} \|u\|_\theta^3 \, .$$

Thus if $\theta > 1/3$ then the right hand side converges to zero as $\ell \to 0$. $\square$

5

Perturbation estimates

The goal of this chapter will be to collect a number of estimates involving the velocity perturbation w_{q+1} and the pressure perturbation $p_{q+1} - p_q$. These estimates will also be important in estimating the new Reynolds stress $\mathring{R}_{q+1}$ (see Chapter 6).

5.1 Additional Notation and Parameter Orderings

For future reference it is useful to introduce the notation

$$\tilde{a}_{k\varsigma} := \sqrt{\rho_\varsigma}\gamma_k\left(\frac{R_\varsigma}{\rho_\varsigma}\right), \tag{5.1.1}$$

and so in particular, we have the identity $a_{k\varsigma} = \chi_\varsigma \tilde{a}_{k\varsigma}\varphi_{k\varsigma}$. We also write

$$L^o_{k\varsigma} := \chi_\varsigma \tilde{a}_{k\varsigma} B_k \tag{5.1.2}$$

$$L^c_{k\varsigma} := \chi_\varsigma\left(\frac{i}{\lambda_{q+1}}\nabla\tilde{a}_{k\varsigma} - \tilde{a}_{k\varsigma}(D\Phi_\varsigma - \mathrm{Id})k\right) \times \frac{k \times B_k}{|k|^2} \tag{5.1.3}$$

$$L_{k\varsigma} := L^o_{k\varsigma} + L^c_{k\varsigma}, \tag{5.1.4}$$

which yields the additional formulas

$$w_o = \sum_{k,\varsigma} L^o_{k\varsigma} e^{i\lambda_{q+1} k\cdot\Phi_\varsigma} \tag{5.1.5}$$

$$w_c = \sum_{k,\varsigma} L^c_{k\varsigma} e^{i\lambda_{q+1} k\cdot\Phi_\varsigma} \tag{5.1.6}$$

$$w_{q+1} = \sum_{k,\varsigma} L_{k\varsigma} e^{i\lambda_{q+1} k\cdot\Phi_\varsigma} . \tag{5.1.7}$$

Before stating our estimates, let us list a number of parameter orderings that will assist in simplifying the statement of such estimates: for all indices ς we assume the following inequalities

$$\begin{gathered} \frac{\lambda_q}{\lambda_{q+1}} \le \lambda_q \ell \le \left(\frac{\delta_{q+1,\varsigma}}{\delta_{q,\varsigma}}\right)^{3/2} \ll 1 \\ \frac{\bar{\delta}_{q,\varsigma}^{1/2}\lambda_q}{\delta_{q+1,\varsigma}^{1/2}\lambda_{q+1}} \le \frac{\delta_{q,\varsigma}^{1/2}\lambda_q}{\mu_{q+1,\varsigma}} \le \frac{1}{\ell\lambda_{q+1}} = \frac{1}{\lambda_{q+1}^{\varepsilon_0}} \\ \bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1} \le \mu_{q+1,\varsigma}^2 . \end{gathered} \tag{5.1.8}$$

Observe that together, the identities yield $\mu_{q+1,\varsigma} \ge \delta_{q,\varsigma}^{1/2}\lambda_q$ which was a key constraint in Lemma 4.3.3. We also set

$$\ell_t = \delta_{q+1,\varsigma}^{1/2}\lambda_{q+1} , \tag{5.1.9}$$

and so from the above inequalities, we deduce $\delta_{q,\varsigma}^{1/2}\lambda_q \le \ell_t^{-1}$ which was a key constraint in Lemma 4.3.4.

Note that until now, we have not defined the value of ρ_ς. This will be left to Chapters 7 and 8, however in what follows we will require the following bounds on ρ_ς

$$4r^{-1}c_1\delta_{q+1,\varsigma} \le \rho_\varsigma \le 2c_0\delta_{q+1,\varsigma} , \tag{5.1.10}$$

where we recall c_0 was the constant appearing in Section 2.4 which is yet to be specified. The lower bound on ρ_ς together with Lemmas 4.3.3 and 4.3.4 ensures that for $t \in \operatorname{supp}\chi_\varsigma$ we have

$$\left\|\frac{R_\varsigma(t)}{\rho_\varsigma}\right\|_0 \le r_0 ;$$

in particular, $R_\varsigma\rho_\varsigma^{-1}$ is in the domain of the functions γ_k which is an essential requirement in order to ensure the perturbation w_{q+1} is well defined. The upper bound in

(5.1.10) is essential in order to control the size of the perturbation.

5.2 Estimates on Components of Perturbation

We begin by estimating the components in the definition of w_{q+1} and $p_{q+1} - p_q$.

Lemma 5.2.1. *Take $t \in \operatorname{supp}(\chi_\varsigma)$ and assume the estimates (4.3.11)-(4.3.14) hold. Then for $N > 0$ the following estimates are satisfied:*

$$\|D\Phi_\varsigma(t)\|_0 \leq C \tag{5.2.1}$$

$$\|D\Phi_\varsigma(t) - \mathrm{Id}\|_0 \leq C\delta_{q,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1} \tag{5.2.2}$$

$$\|D\Phi_\varsigma(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\ell^{-N} \tag{5.2.3}$$

$$\|\varphi_{k\varsigma}(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{1-N} \overset{(5.1.8)}{\leq} C\ell^{-N} \tag{5.2.4}$$

$$\|L^c_{k\varsigma}(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\ell^{-N} \overset{(5.1.8)}{\leq} C\delta_{q+1,\varsigma}^{1/2}\ell^{-N} \tag{5.2.5}$$

$$\|\tilde{a}_{k\varsigma}(t)\|_0 + \|a_{k\varsigma}(t)\|_0 + \|L^o_{k\varsigma}(t)\|_0 \leq C\delta_{q+1,\varsigma}^{1/2} \tag{5.2.6}$$

$$\|\tilde{a}_{k\varsigma}(t)\|_0 + \|L^o_{k\varsigma}(t)\|_N \leq C\delta_{q+1,\varsigma}^{1/2}\lambda_q\ell^{1-N} \tag{5.2.7}$$

$$\|a_{k\varsigma}(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{1-N} \overset{(5.1.8)}{\leq} C\delta_{q+1,\varsigma}^{1/2}\ell^{-N} \tag{5.2.8}$$

Moreover fixing N and assuming c_0 and c_1 to be sufficiently small, the constants C in the estimates (5.2.5)-(5.2.4) can be taken to be arbitrarily small.

Proof. First recall again as a result of (4.3.11) we have the mollification estimate

$$\|v_\ell(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\lambda_q\ell^{1-N},$$

for $N \geq 1$. Since $[v_\ell(t)]_1 \leq \mu_{q+1,\varsigma}$ and $\mu_{q+1,\varsigma}^{-1}$ bounds the length of $\operatorname{supp}(\chi_\varsigma)$, from Proposition 4.3.1 we deduce (5.2.1), (5.2.2) and (5.2.3).

To estimate $\varphi_{k\varsigma}$ we apply (A.1.5) and (5.2.2) to obtain for $N = 1, 2, \ldots$

$$\begin{aligned}
\|\varphi_{k\varsigma}(t)\|_N &\leq C\lambda_{q+1}\|D\Phi_\varsigma(t) - \mathrm{Id}\|_{N-1} + C\lambda_{q+1}^N\|D\Phi_\varsigma(t) - \mathrm{Id}\|_0^N \\
&\leq C\delta_{q,\varsigma}^{1/2}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{1-N} + C(\delta_{q,\varsigma}^{1/2}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1})^N \\
&\overset{(5.1.8)}{\leq} C\delta_{q,\varsigma}^{1/2}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{1-N},
\end{aligned}$$

which implies (5.2.4).

Let us now consider $\tilde{a}_{k\varsigma}$ and $L^o_{k\varsigma}$: estimating we have

$$\|\tilde{a}_{k\varsigma}(t)\|_0 + \left\|L^o_{k\varsigma}(t)\right\|_0 \leq C\rho_\varsigma^{1/2} \leq C\delta_{q+1,\varsigma}^{1/2}\,, \tag{5.2.9}$$

and by applying (A.1.5) we obtain for $N > 0$

$$[\tilde{a}_{k\varsigma}(t)]_N + [L^o_{k\varsigma}(t)]_N \leq C\rho_\varsigma^{-1/2}[\mathring{R}_\varsigma]_N + C\rho_\varsigma^{1/2-N}[\mathring{R}_\varsigma]_1^N \overset{\substack{(4.3.16)\\ \&(4.3.19)}}{\leq} C\delta_{q+1,\varsigma}^{1/2}\lambda_q\ell^{1-N}\,. \tag{5.2.10}$$

Next we estimate $L^c_{k\varsigma}$ making use of (A.1.3):

$$\begin{aligned}\|L^c_{k\varsigma}(t)\|_N \leq& \frac{1}{\lambda_{q+1}}\|\tilde{a}_{k\varsigma}(t)\|_{N+1} + \|\tilde{a}_{k\varsigma}(t)\|_N \|D\Phi_\varsigma(t) - \mathrm{Id}\|_0 \\ &+ \|\tilde{a}_{k\varsigma}(t)\|_0 \|D\Phi_\varsigma(t) - \mathrm{Id}\|_N\,,\end{aligned}$$

For $N = 0$ we have

$$\begin{aligned}\|L^c_{k\varsigma}(t)\|_0 &\overset{(5.2.2)\&(5.2.9)}{\leq} C\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1}^{-1} + C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1} \\ &\overset{(5.1.8)}{\leq} C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\,.\end{aligned}$$

Similarly, utilising in addition (5.2.3) and (5.2.10) for $N = 1, 2, \ldots$ we have

$$\begin{aligned}\|L^c_{k\varsigma}(t)\|_N &\leq C\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1}^{-1}\ell^{-N} + \delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q^2\mu_{q+1,\varsigma}^{-1}\ell^{1-N} + \delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\ell^{-N} \\ &\overset{(5.1.8)}{\leq} C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\ell^{-N}\,.\end{aligned}$$

Thus from the above estimates we obtain (5.2.5)-(5.2.8). □

We now present a number of material derivative estimates. Recall the notation $D_t = \partial_t + v_\ell \cdot \nabla$.

Lemma 5.2.2. *Assume $t \in \operatorname{supp}(\chi_\varsigma) \cap \operatorname{supp}(\chi_{\varsigma'})$ — importantly we do not exclude the possibility $\varsigma = \varsigma'$. Then the following estimates are satisfied:*

$$\|D_t v_\ell(t)\|_0 \leq C\delta_{q,\varsigma}\lambda_q \tag{5.2.11}$$

$$\|D_t v_\ell(t)\|_N \leq C\delta_{q,\varsigma}\lambda_q^2\ell^{1-N} \qquad \textit{for } N > 0 \tag{5.2.12}$$

$$\|D_t D\Phi_\varsigma(t)\|_N \leq C\delta_{q,\varsigma}\lambda_q^2\mu_{q+1,\varsigma}^{-1}\ell^{-N} \tag{5.2.13}$$

$$\|D_t^2 D\Phi_\varsigma(t)\|_N \leq C\delta_{q,\varsigma}^{3/2}\lambda_q^3\mu_{q+1,\varsigma}^{-1}\ell^{-N} \tag{5.2.14}$$

$$\|D_t \tilde{a}_{k\varsigma}(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q \ell^{-N} \tag{5.2.15}$$

$$\|D_t^2 \tilde{a}_{k\varsigma}(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1} \ell^{-N} \tag{5.2.16}$$

$$\|D_t L^o_{k\varsigma}(t)\|_N + \|D_t L^c_{k\varsigma}(t)\|_N \le C\delta_{q+1,\varsigma}^{1/2}\left(\bar{\delta}_{q,\varsigma}^{1/2}\lambda_q + \mu_{q+1,\varsigma'}\eta_{q+1,\varsigma'}^{-1}\right)\ell^{-N} \tag{5.2.17}$$

$$\|D_t^2 L^o_{k\varsigma}(t)\|_N + \|D_t^2 L^c_{k\varsigma}(t)\|_N \le C\delta_{q+1,\varsigma}^{1/2}\left(\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1} + \mu^2_{q+1,\varsigma'}\eta_{q+1,\varsigma'}^{-2}\right)\ell^{-N}\,. \tag{5.2.18}$$

$$\|D_t \nabla L^o_{k\varsigma}(t)\|_N \le C\delta_{q+1,\varsigma}^{1/2}\lambda_q\left(\bar{\delta}_{q,\varsigma}^{1/2}\ell^{-1} + \mu_{q+1,\varsigma'}\eta_{q+1,\varsigma'}^{-1}\right)\ell^{-N} \tag{5.2.19}$$

If in addition we have $t \in K^g_\varsigma$ then we have

$$\|D_t L^o_{k\varsigma}(t)\|_N + \|D_t L^c_{k\varsigma}(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\ell^{-N} \tag{5.2.20}$$

$$\|D_t^2 L^o_{k\varsigma}(t)\|_N + \|D_t^2 L^c_{k\varsigma}(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N}\,. \tag{5.2.21}$$

$$\|D_t \nabla L^o_{k\varsigma}(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\ell^{-N-1} \tag{5.2.22}$$

Moreover fixing N and assuming c_0 and c_1 to be sufficiently small, the constants C in the estimates (5.2.15)-(5.2.21) *can be taken to be arbitrarily small.*

Proof. We note the following decomposition

$$D_t v_\ell = \operatorname{div} \mathring{R}_q * \psi_\ell - \nabla p_q * \psi_\ell + \operatorname{div}(v_q * \psi_\ell \otimes v_q * \psi_\ell - (v_q \otimes v_q) * \psi_\ell)\,.$$

Applying Proposition 4.3.5 we deduce

$$\begin{aligned}\left\|\operatorname{div}\left[(v_q * \psi_\ell)(t) \otimes (v_q * \psi_\ell)(t) - ((v_q \otimes v_q) * \psi_\ell)(t)\right]\right\|_N &\le \left\|v_q(t)\right\|_1^2 \ell^{1-N} \\ &\le C\delta_{q,\varsigma}\lambda_q^2\ell^{1-N}\,.\end{aligned}$$

Together with the estimates (4.3.12) and (4.3.13) we can then conclude (5.2.11) and (5.2.12).

We recall the formula

$$D_t \nabla f = -Dv_\ell^T \nabla f + \nabla D_t f\,. \tag{5.2.23}$$

Taking a further material derivative of (5.2.23) and applying (A.1.3) yields

$$\begin{aligned}\|D_t^2 \nabla f(t)\|_N \le\, & C\left\|v_\ell(t)\right\|_{N+1}\left\|v_\ell(t)\right\|_1 [f(t)]_1 + C\left\|v_\ell(t)\right\|_1^2 [f(t)]_{N+1} \\ &+ C\left\|D_t v_\ell(t)\right\|_{N+1}[f(t)]_1 + C\left\|D_t v_\ell(t)\right\|_1 [f(t)]_{N+1} \\ &+ C\left\|v_\ell(t)\right\|_{N+1}[D_t f(t)]_1 + C\left\|v_\ell(t)\right\|_1 [D_t f(t)]_{N+1}\end{aligned}$$

$$
\begin{aligned}
&+ C\left\|D_t^2 f(t)\right\|_{N+1} \\
&\leq C\delta_{q,\varsigma}\lambda_q^2\left(\ell^{-N}[f(t)]_1 + [f(t)]_{N+1}\right) \\
&\quad + C\delta_{q,\varsigma}^{1/2}\lambda_q\left(\ell^{-N}[D_t f(t)]_1 + [D_t f(t)]_{N+1}\right) \\
&\quad + C\left\|D_t^2 f(t)\right\|_{N+1} \, .
\end{aligned} \tag{5.2.24}
$$

Now consider $D\Phi_\varsigma$ and observe

$$
D_t D\Phi_\varsigma = D_t(D\Phi_\varsigma - \mathrm{Id}) = -(D\Phi_\varsigma - \mathrm{Id})Dv_\ell \, , \tag{5.2.25}
$$

and thus, using Lemma 5.2.1 and (A.1.3) we obtain

$$
\|D_t D\Phi_\varsigma(t)\|_N \leq C\delta_{q,\varsigma}\lambda_q^2\ell^{-N}\mu_{q+1,\varsigma}^{-1} \, .
$$

Taking a further material derivative of (5.2.25), estimating in an analagous was to (5.2.24), and applying Lemma 5.2.1 yields

$$
\begin{aligned}
\|D_t^2 D\Phi_\varsigma(t)\|_N &\leq C\|v_\ell\|_{N+1}\|v_\ell\|_1\|D\Phi_\varsigma - \mathrm{Id}\|_0 + C\|v_\ell\|_1^2[D\Phi_\varsigma - \mathrm{Id}]_N \\
&\quad + C\|D_t v_\ell(t)\|_{N+1}\|D\Phi_\varsigma - \mathrm{Id}\|_0 + C\|D_t v_\ell(t)\|_1[D\Phi_\varsigma - \mathrm{Id}]_N \\
&\leq C\delta_{q,\varsigma}^{3/2}\lambda_q^3\mu_{q+1,\varsigma}^{-1}\ell^{-N} \, .
\end{aligned}
$$

Hence we conclude (5.2.13) and (5.2.14)

We next consider $\tilde{a}_{k\varsigma}$ and applying Lemmas 4.3.3 and 4.3.4 yields

$$
\|D_t\tilde{a}_{k\varsigma}(t)\|_0 \leq C\rho_\varsigma^{-1/2}\|D_t\mathring{R}_\varsigma(t)\|_0 \leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q \, ,
$$

and for $N > 0$, by the (A.1.3) and (A.1.5) we have

$$
\begin{aligned}
\|D_t\tilde{a}_{k\varsigma}(t)\|_N &\leq C\rho_\varsigma^{-1/2}\|D_t\mathring{R}_\varsigma(t)\|_N \\
&\quad + C\rho_\varsigma^{-\frac{3}{2}}\|D_t\mathring{R}_\varsigma(t)\|_0\left(\left\|\mathring{R}_\varsigma\right\|_N + \rho_\varsigma^{1-N}\left\|\mathring{R}_\varsigma\right\|_1^N\right) \\
&\leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\left(\ell^{-N} + \lambda_q\ell^{1-N} + \lambda_q^N\right) \\
&\overset{(5.1.8)}{\leq} C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\ell^{-N} \, ,
\end{aligned}
$$

from which (5.2.15) follows. Taking a further material derivative and applying Lemmas

4.3.3 and 4.3.4 yields

$$\begin{aligned}\|D_t^2\tilde a_{k\varsigma}(t)\|_0 &\le C\rho_\varsigma^{-1/2}\|D_t^2\mathring{R}_\varsigma(t)\|_0 + \rho_\varsigma^{-3/2}\|D_t\mathring{R}_\varsigma(t)\|_0^2\\ &\le C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1} + C\bar\delta_{q,\varsigma}\delta_{q+1,\varsigma}^{1/2}\lambda_q^2\\ &\overset{(5.1.8)}{\le} C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\end{aligned}$$

and by analogous arguments to those used to estimate $\|D_t\tilde a_{k\varsigma}(t)\|_N$ we obtain

$$\|D_t^2\tilde a_{k\varsigma}(t)\|_N \le C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N}\,.$$

We may apply (5.2.23), Lemma 5.2.1 and the estimates for $D_t\tilde a_{k\varsigma}$ to conclude

$$\|D_t\nabla\tilde a_{k\varsigma}(t)\|_N \le C\delta_{q+1,\varsigma}^{\frac12}\lambda_q\ell^{-N-1}\left(\delta_{q,\varsigma}^{1/2}+\bar\delta_{q,\varsigma}^{1/2}\right) \le C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{\frac12}\lambda_q^2\ell^{-N}\,.$$

Since

$$\|D_t\nabla L^o_{k\varsigma}(t)\|_N \le C\|D_t\nabla\tilde a_{k\varsigma}(t)\|_N + C\|\chi'_\varsigma\nabla\tilde a_{k\varsigma}(t)\|_N\,,$$

we obtain (5.2.19) and (5.2.22) with the help of (4.1.7), (5.2.6) and (5.2.7).

Using (5.2.24), we obtain with the help of Lemma 5.2.1

$$\begin{aligned}\|D_t^2\nabla\tilde a_{k\varsigma}(t)\|_N &\le C\delta_{q,\varsigma}\delta_{q+1,\varsigma}^{1/2}\lambda_q^3\ell^{-N}\\ &\quad + C\bar\delta_{q,\varsigma}^{1/2}\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q^2\ell^{-N-1}\\ &\quad + C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N-1}\\ &\overset{(5.1.8)}{\le} C\bar\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N-1}\,.\end{aligned}$$

Assume now $t\in K_\varsigma^g$. Applying (5.2.13)-(5.2.16) and (5.1.8) we obtain (5.2.20) and (5.2.21). With the addition of (4.1.7) we conclude (5.2.17) and (5.2.18). □

We now move on to estimating the perturbation w_{q+1} and consequently the new velocity and pressure. In the lemmas above we estimated components of the perturbation which correspond to a single index ς. However in order to estimate the perturbation we will require additional parameter orderings corresponding to neighbouring indices (and accordingly overlapping regions): if $\delta_{q+1,\varsigma}\ge\delta_{q+1,\varsigma'}$, for some $\varsigma'=\varsigma\pm1$ then we assume the following inequalities

$$\bar\delta_{q,\varsigma}\ge\bar\delta_{q,\varsigma'},\quad \delta_{q,\varsigma}\ge\delta_{q,\varsigma'},\qquad \frac{\delta_{q,\varsigma}^{1/2}}{\mu_{q+1,\varsigma}}\ge\frac{\delta_{q,\varsigma'}^{1/2}}{\mu_{q+1,\varsigma'}}\,. \tag{5.2.26}$$

Remark 5.2.3. In particular, the above inequality implies that for index ς the estimates in Lemma 5.2.1 and Lemma 5.2.2 are weaker than the corresponding estimates for ς'. This observation will be used repetitively in the lemmas below.

Lemma 5.2.4. *If t belongs to the non-overlapping zone K_ς^g and the constants c_0 and c_1 are chosen sufficiently small, we have*

$$\|w_c(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}^{-1}\lambda_{q+1}^N \tag{5.2.27}$$

$$\|w_o(t)\|_N \leq C\delta_{q+1,\varsigma}^{1/2}\lambda_{q+1}^N \tag{5.2.28}$$

$$\lambda_{q+1}^{-1}\|v_{q+1}(t)\|_1 + \|w_{q+1}(t)\|_0 \leq \delta_{q+1,\varsigma}^{1/2} \tag{5.2.29}$$

$$\lambda_{q+1}^{-2}\|p_{q+1}(t)\|_2 + \lambda_{q+1}^{-1}\|p_{q+1}(t)\|_1 + \|(p_{q+1} - p_q)(t)\|_0 \leq \delta_{q+1,\varsigma}\,, \tag{5.2.30}$$

Moreover, the same estimates hold if $t \in \operatorname{supp}(\chi_\varsigma) \cap \operatorname{supp}(\chi_{\varsigma'})$ for $\varsigma' = \varsigma \pm 1$ and $\delta_{q+1,\varsigma} \geq \delta_{q+1,\varsigma'}$.

Proof. Recall by definition, and (A.1.3), we have the following inequalities:

$$\begin{aligned}
\|w_o(t)\|_N &\leq \sum_{k,\sigma} \left\|a_{k\sigma}(t)e^{i\lambda_{q+1}k\cdot x}\right\|_N \leq \sum_{k,\sigma} C\,\|a_{k\sigma}(t)\|_N + C\lambda_{q+1}^N\,\|a_{k\sigma}(t)\|_0 \\
\|w_c(t)\|_N &\leq \sum_{k,\sigma} \left\|L_{k\sigma}^c(t)\varphi_{k\sigma}(t)e^{i\lambda_{q+1}k\cdot x}\right\|_N \\
&\leq \sum_{k,\sigma} C\,\|L_{k\sigma}^c(t)\|_N + C\,\|L_{k\sigma}^c(t)\|_0\left(\lambda_{q+1}^N + \left\|\varphi_{k\sigma}(t)\right\|_N\right)
\end{aligned}$$

$$\begin{aligned}
\left\|w_{q+1}(t)\right\|_N &\leq \|w_o(t)\|_N + \|w_c(t)\|_N \\
\left\|v_{q+1}(t)\right\|_N &\leq \left\|v_q(t)\right\|_N + \left\|w_{q+1}(t)\right\|_N \\
\left\|p_{q+1}(t)\right\|_N &\leq \left\|p_q(t)\right\|_N + \left\||w_o(t)|^2\right\|_N + \left\||w_c(t)|^2\right\|_N + \||\langle w_o(t), w_c(t)\rangle|\|_N \\
&\quad + \left\|\langle v_q - v_\ell, w_{q+1}\rangle\right\|_N \\
&\leq \left\|p_q(t)\right\|_N + C\,\|w_o(t)\|_N\left(\|w_o(t)\|_0 + \|w_c(t)\|_0\right) + \|w_c(t)\|_N\,\|w_c(t)\|_0 \\
&\quad + \left\|v_q - v_\ell\right\|_N\left\|w_{q+1}\right\|_0 + \left\|v_q - v_\ell\right\|_0\left\|w_{q+1}\right\|_N
\end{aligned}$$

Hence if t is in the good region K_ς^g (the only non-vanishing cut-off is χ_ς) then the claimed estimates follow directly from Lemma 5.2.1 and the inequalities (5.1.8). Now assume $t \in \operatorname{supp}(\chi_\varsigma) \cap \operatorname{supp}(\chi_{\varsigma'})$ for some $\varsigma' = \varsigma \pm 1$ and $\delta_{q+1,\varsigma} \geq \delta_{q+1,\varsigma'}$, then taking into account Remark 5.2.3, the claimed estimates again follow directly from Lemma 5.2.1 and the inequalities (5.1.8) and (5.2.26). $\square$

Finally, we list material derivative estimates of the principal perturbation w_o and the corrector w_c.

Lemma 5.2.5. *If t belongs to the non-overlapping region K_ς^g we then conclude*

$$\|D_t w_o(t)\|_N + \|D_t w_c(t)\|_N \le C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1}^N \tag{5.2.31}$$

$$\|\partial_t w_{q+1}(t)\|_0 \le C\left(1+\left\|v_q\right\|_0\right)\delta_{q+1,\varsigma}^{1/2}\lambda_{q+1} \tag{5.2.32}$$

$$\|\partial_t(p_{q+1}-p_q)(t)\|_0 \le C\left(1+\left\|v_q\right\|_0\right)\delta_{q+1,\varsigma}\lambda_{q+1}\,. \tag{5.2.33}$$

Moreover, if $t \in \operatorname{supp}(\chi_\varsigma)\cap\operatorname{supp}(\chi_{\varsigma'})$ for $\varsigma' = \varsigma \pm 1$ and $\delta_{q+1,\varsigma} \ge \delta_{q+1,\varsigma'}$, then the following estimates hold

$$\|D_t w_o(t)\|_N + \|D_t w_c(t)\|_N \le C\delta_{q+1,\varsigma}^{1/2}\eta_{q+1,\varsigma}^{-1}\mu_{q+1,\varsigma}\lambda_{q+1}^N \tag{5.2.34}$$

$$\|\partial_t w_{q+1}(t)\|_0 \le C\left(\lambda_{q+1}+\left\|v_q\right\|_0\lambda_{q+1}+\eta_{q+1,\varsigma}^{-1}\mu_{q+1,\varsigma}\right)\delta_{q+1,\varsigma}^{1/2} \tag{5.2.35}$$

$$\|\partial_t(p_{q+1}-p_q)(t)\|_0 \le C\left(\lambda_{q+1}+\left\|v_q\right\|_0\lambda_{q+1}+\eta_{q+1,\varsigma}^{-1}\mu_{q+1,\varsigma}\right)\delta_{q+1,\varsigma}\,. \tag{5.2.36}$$

Proof. Analogous to the proof of Lemma 5.2.4, the estimates (5.2.31) and (5.2.34) follow as a result of Remark 5.2.3, Lemma 5.2.1, (5.1.8), (5.2.26) and the additional estimates of Lemma 5.2.2.

Taking into account the identity $\partial_t = D_t - v_\ell\cdot\nabla$, the estimate (5.2.32) follows from (5.2.27), (5.2.28) and (5.2.31). Using (5.2.34), the estimate (5.2.35) on the overlapping region follows analogously.

To estimate $\partial_t(p_{q+1}-p_q)$, we observe by construction we have

$$\begin{aligned}\|\partial_t(p_{q+1}(t)-p_q(t))\|_0 \le&(\|w_c(t)\|_0+\|w_o(t)\|_0)(\|\partial_t w_c(t)\|_0+\|\partial_t w_o(t)\|_0)\\&+2\|w_{q+1}(t)\|_0\|\partial_t v_q(t)\|_0+\ell\|v_q(t)\|_1\|\partial_t w_{q+1}(t)\|_0\,,\end{aligned}$$

where here we used the fact $\|\partial_t v_\ell\|_0 \le \left\|\partial_t v_q\right\|_0$. By (5.1.8) we have $\ell\|v_q(t)\|_1 \le \delta_{q+1,\varsigma}^{1/2}$. From the $\partial_t = D_t - v_\ell\cdot\nabla$ and (5.2.11) we have

$$\left\|\partial_t v_q(t)\right\|_0 \le \left\|D_t v_q(t)\right\|_0+\left\|v_q(t)\right\|_0\left\|v_q(t)\right\|_1 \le C\delta_{q,\varsigma}\lambda_q+\left\|v_q\right\|_0\delta_{q,\varsigma}^{1/2}\lambda_q\,.$$

Applying (5.1.8) and (5.2.26), the estimates (5.2.33) and (5.2.36) then follow from (5.2.27), (5.2.28), (5.2.31) and (5.2.34). ◻

5.3 References and Remarks

The estimates presented here can essentially all be found in [BDLS14], which itself is based on [BDLSJ13] and [Buc13]. In particular, in [BDLSJ13], no distinction was made between the estimates on *good regions* and those on *bad regions*. This distinction was first introduced in [Buc13], being one of the key new ideas, enabling for the first time the ability to construct weak solutions to the Euler equations with Onsager critical regularity a.e. in time. In order to take advantage of these time localised estimates, we will require a very careful choice of the parameters $\mu_{q+1,\varsigma}$ and $\eta_{q+1,\varsigma}$, as well as the introduction of a sophisticated bookkeeping system (see Chapter 8).

We note that in comparing the scheme presented here to that of [BDLSJ13, Buc13, BDLS14], there are a number of notational differences. The amplitude functions $a_{k,\varsigma}$ are chosen here so that they more closely resemble the ansatz (3.1.13); in comparison the functions $a_{k,\varsigma}$ of [BDLSJ13, Buc13, BDLS14] correspond to the functions $\tilde{a}_{k,\varsigma}$ of the present work. The notation $L_{k,\varsigma}$ also differs slightly from that of [BDLSJ13, Buc13, BDLS14]. This is done in order to better organise some of the estimates of the new Reynolds stress $\mathring{R}_{q+1}$ in the next chapter.

6

Reynolds Stress Estimates

6.1 Reynolds Stress Estimates

To ensure convergence of our convex integration scheme, we will need to obtain good estimates on the new Reynolds stress $\mathring{R}_{q+1}$. We note that as a particular consequence of the parameter inequalities (5.1.8) we obtain the additional ordering:

$$\frac{\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q}{\lambda_{q+1}} \leq \frac{\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q}{\mu_{q+1,\varsigma}} \leq \frac{\delta_{q+1,\varsigma}^{1/2}\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}\lambda_{q+1}}\,. \tag{6.1.1}$$

Notice that with the identification $\bar{\delta}_{q,\varsigma} \sim \lambda_q^{-2\beta}$ and $\delta_{q+1,\varsigma} \sim \lambda_{q+1}^{-2\beta}$, then modulo an iteration index change $(q \mapsto q-1)$, the expression appearing on the left appeared previously in our preliminary estimate (2.3.3) of the contribution of Nash error to the Reynolds stress. The expression in the middle will make an appearance in the estimates of the oscillation error. Finally, the expression on the right will appear in estimates involving a time derivative falling on the cut-off functions χ_ς — since such errors will only appear in a subset of time, it seems natural to allow the expression to be considerably larger than the other two expressions.

Proposition 6.1.1. *Assume $t \in \operatorname{supp}(\chi_\varsigma)$. In the case $t \in \operatorname{supp}(\chi_{\varsigma'})$ for $\varsigma' = \pm\varsigma$ then we assume in additional that $\delta_{q+1,\varsigma} \geq \delta_{q+1,\varsigma'}$. We then have the following estimates:*

$$\|\mathring{R}_{q+1}(t)\|_0 + \frac{1}{\lambda_{q+1}}\|\mathring{R}_{q+1}(t)\|_1 + \frac{1}{\lambda_{q+1}^2}\|\mathring{R}_{q+1}(t)\|_2 \leq C\frac{\delta_{q+1,\varsigma}^{1/2}\mu_{q+1,\varsigma}\ell}{\eta_{q+1,\varsigma}} \tag{6.1.2}$$

$$\|\partial_t\mathring{R}_{q+1}(t) + v_{q+1}\cdot\nabla\mathring{R}_{q+1}(t)\|_0 \leq C\left(\delta_{q+1,\varsigma}^{1/2}\lambda_{q+1} + \frac{\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}}\right)\frac{\delta_{q+1,\varsigma}^{1/2}\mu_{q+1,\varsigma}\ell}{\eta_{q+1,\varsigma}}\,. \tag{6.1.3}$$

Furthermore if $t \in K_\varsigma^g$ then we have

$$\|\mathring{R}_{q+1}(t)\|_0 + \frac{1}{\lambda_{q+1}}\|\mathring{R}_{q+1}(t)\|_1 + \frac{1}{\lambda_{q+1}^2}\|\mathring{R}_{q+1}(t)\|_2 \leq C\frac{\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}^{\varepsilon_0}}{\mu_{q+1,\varsigma}} \tag{6.1.4}$$

$$\|\partial_t\mathring{R}_{q+1}(t) + v_{q+1}\cdot\nabla\mathring{R}_{q+1}(t)\|_0 \leq C\frac{\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{3/2}\lambda_q\lambda_{q+1}^{1+\varepsilon_0}}{\mu_{q+1,\varsigma}}\,. \tag{6.1.5}$$

In the proof of Proposition 6.1.1 we will make use of the following commutator estimate whose proof will be postponed until the end of the chapter.

Proposition 6.1.2. *Let $\lambda \geq 1$ and $0 < \alpha < 1$ be fixed. Then suppose we are given a vector $k \in \mathbb{Z}^3$ satisfying $|k| = \lambda$, a smooth vector field $a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ and a smooth function $b \in C^\infty(\mathbb{T}^3)$: if we set $F(x) := a(x)e^{ik\cdot x}$, we have*

$$\|[b,\mathcal{R}](F)\|_0 \leq C\lambda^{\alpha-2}\|a\|_0\|b\|_1 + C\lambda^{-m}\left(\|a\|_{m-1}\|b\|_1 + \|a\|_0\|b\|_m\right) \tag{6.1.6}$$

where $m \in \mathbb{N}$ and $C = C(\alpha, m)$.

Proof of Proposition 6.1.1. First note that from the decomposition $\partial_t + v_{q+1}\cdot\nabla = D_t + (w_{q+1} + v_q - v_\ell)\cdot\nabla$ we have

$$\begin{aligned}
\|\partial_t\mathring{R}_{q+1}(t) + v_{q+1}\cdot\nabla\mathring{R}_{q+1}(t)\|_0 \leq &\|D_t\mathring{R}_{q+1}(t)\|_0 \\
&+ \left(\|v_q(t) - v_\ell(t)\|_0 + \|w_{q+1}(t)\|_0\right)\|\mathring{R}_{q+1}(t)\|_1 \\
\leq &\|D_t\mathring{R}_{q+1}(t)\|_0 \\
&+ C(\delta_{q,\varsigma}^{1/2}\lambda_q\ell + \delta_{q+1,\varsigma}^{1/2})\|\mathring{R}_{q+1}(t)\|_1 \\
\overset{(5.1.8)}{\leq} &\|D_t\mathring{R}_{q+1}(t)\|_0 + C\delta_{q+1,\varsigma}^{1/2}\|\mathring{R}_{q+1}(t)\|_1\,.
\end{aligned}$$

Hence assuming (6.1.2) and (6.1.4), to prove (6.1.3) and (6.1.5), it suffices to prove

$$\|D_t \mathring{R}_{q+1}(t)\|_0 \le C \left(\delta_{q+1,\varsigma}^{1/2} \lambda_{q+1} + \frac{\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}} \right) \frac{\delta_{q+1,\varsigma}^{1/2} \mu_{q+1,\varsigma} \ell}{\eta_{q+1,\varsigma}} , \tag{6.1.7}$$

and for the case $t \in K_\varsigma^g$

$$\|D_t \mathring{R}_{q+1}(t)\|_0 \le C \frac{\delta_{q,\varsigma}^{1/2} \delta_{q+1,\varsigma}^{3/2} \lambda_q \lambda_{q+1}^{1+\varepsilon_0}}{\mu_{q+1,\varsigma}} . \tag{6.1.8}$$

Taking advantage of the arbitrary nature of the constant C, we prove (6.1.2), (6.1.4), (6.1.7) and (6.1.8) by showing that the estimates hold with $\mathring{R}_{q+1}$ replaced with R^σ for each $\sigma = 0, 1, 2, 3, 4, 5$, with the definition of R^σ given by (4.1.9)-(4.1.14).

Estimates on R^0. By direct calculation we have

$$D_t w_{q+1} = \sum_{k,\sigma} D_t L_{k\sigma} \varphi_{k\sigma} e^{i\lambda_{q+1} k\cdot x}.$$

Applying Lemmas 5.2.1 and 5.2.2, we obtain

$$\begin{aligned} \|D_t L_{k\sigma}(t) \varphi_{k\sigma}(t)\|_N &\le C \delta_{q+1,\varsigma}^{1/2} \left(\bar{\delta}_{q,\varsigma}^{1/2} \lambda_q + \mu_{q+1,\varsigma} \eta_{q+1,\varsigma}^{-1} \right) \\ &\overset{(5.1.8)}{\le} C \delta_{q+1,\varsigma}^{1/2} \mu_{q+1,\varsigma} \eta_{q+1,\varsigma}^{-1} \ell^{-N} , \end{aligned} \tag{6.1.9}$$

and for times $t \in K_\varsigma^g$

$$\|D_t L_{k\sigma}(t) \varphi_{k\sigma}(t)\|_N \le C \bar{\delta}_{q,\varsigma}^{1/2} \delta_{q+1,\varsigma}^{1/2} \lambda_q \ell^{-N}. \tag{6.1.10}$$

Hence the desired estimates for $\left\|R^0\right\|_0$ and $\left\|R^0\right\|_1$ follow as a consequence of Proposition 3.2.3 (with $m > \frac{1}{\varepsilon_0}$) together with (A.1.3) for the estimates of $\left\|R^0\right\|_1$ and $\left\|R^0\right\|_2$.

Estimates on $D_t R^0$. Again, by a direct calculation we have

$$D_t^2 w_{q+1} = \sum_{k,\sigma} D_t^2 L_{k\sigma} \varphi_{k\sigma} e^{i\lambda_{q+1} k\cdot x}.$$

Applying (A.1.3), Lemmas 5.2.1 and 5.2.2 we obtain

$$\begin{aligned}\|D_t^2 L_{k\sigma}(t)\varphi_{k\sigma}(t)\|_N &\leq C\delta_{q+1,\varsigma}^{1/2}\left(\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\lambda_{q+1} + \mu_{q+1,\varsigma}^2\eta_{q+1,\varsigma}^{-2}\right)\ell^{-N}\\ &\overset{(5.1.8)}{\leq} C\frac{\delta_{q+1,\varsigma}^{1/2}\mu_{q+1,\varsigma}^2\ell^{-N}}{\eta_{q+1,\varsigma}^2}\,,\end{aligned} \tag{6.1.11}$$

and for times $t \in K_\varsigma^g$

$$\|D_t^2 L_{k\sigma}(t)\varphi_{k\sigma}(t)\|_N \leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N}\,. \tag{6.1.12}$$

Next, observe that we can write

$$\begin{aligned}D_t R^0 &= \Big([D_t,\mathcal{R}] + \mathcal{R}D_t\Big)D_t w_{q+1}\\ &= \Big([v_\ell,\mathcal{R}]\nabla + \mathcal{R}D_t\Big)D_t w_{q+1}\\ &= \sum_{k,\sigma}\mathcal{R}(D_t^2 L_{k\sigma}\varphi_{k\sigma}e^{i\lambda_{q+1}k\cdot x}) + [v_\ell,\mathcal{R}](\nabla(D_t L_{k\sigma}\varphi_{k\sigma})e^{i\lambda_{q+1}k\cdot x})\\ &\quad + i\lambda_{q+1}[v_\ell\cdot k,\mathcal{R}](D_t L_{k\sigma}\varphi_{k\sigma}e^{i\lambda_{q+1}k\cdot x})\,.\end{aligned}$$

The desired estimates for $\left\|D_t R^0\right\|_0$ then follow from Proposition 6.1.2 and Proposition 3.2.3, together with the estimates (6.1.9), (6.1.10), (6.1.11) and (6.1.12).

Estimates on R^1. We recall that from the decomposition (3.1.16) we have

$$\begin{aligned}&\operatorname{div}\left(w_o\otimes w_o - \sum_l \chi_\sigma^2\mathring{R}_\sigma - \frac{|w_o|^2}{2}\mathrm{Id}\right) =\\ &= \sum_{\substack{(k,\sigma),(k',\sigma')\\ k+k'\neq 0}}\left(B_k\otimes B_{k'} - \tfrac{1}{2}(B_k\cdot B_{k'})\mathrm{Id}\right)\nabla(a_{k\sigma}a_{k'\sigma'})e^{i\lambda_{q+1}(k+k')\cdot x}\,.\end{aligned}$$

From (A.1.3), Lemma 5.2.1 and the orderings (5.1.8) we have for $N \geq 1$

$$\|a_{k\sigma}(t)a_{k'\sigma'}(t)\|_N \leq C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{1-N}\,, \tag{6.1.13}$$

Hence the desired estimates follow from Proposition 3.2.3.

Estimates on $D_t R^1$. As we did for the estimate for $D_t R^0$, we make use of the identity

$D_t\mathcal{R} = [v_\ell, \mathcal{R}]\nabla + \mathcal{R}D_t$ in order to write

$$D_t R^1 = \sum_{\substack{(k,\sigma),(k',\sigma')\\ k+k'\neq 0}} \Big([v_\ell, \mathcal{R}] \left(\nabla \Omega_{k\sigma k'\sigma'} e^{i\lambda_{q+1}(k+k')\cdot x} \right)$$
$$+ i\lambda_{q+1}[v_\ell \cdot (k+k'), \mathcal{R}] \left(\Omega_{k\sigma k'\sigma'} e^{i\lambda_{q+1}(k+k')\cdot x} \right) + \mathcal{R} \left(\Omega'_{k\sigma k'\sigma'} e^{i\lambda_{q+1}(k+k')\cdot x} \right) \Big),$$

where we have set $\Omega_{k\sigma k'\sigma'} = \left(B_k \otimes B_{k'} - \frac{1}{2}(B_k \cdot B_{k'})\mathrm{Id} \right) \nabla (a_{k\sigma} a_{k'\sigma'})$ and

$$\begin{aligned}
&\Omega'_{k\sigma k'\sigma'} = \\
&\left[D_t \nabla \left(L^o_{k\sigma} \otimes L^o_{k'\sigma'} - \tfrac{1}{2}(L^o_{k\sigma} \cdot L^o_{k'\sigma'})\mathrm{Id} \right) \right] \varphi_{k\sigma}\varphi_{k'\sigma'} + \\
&\left[D_t \left(L^o_{k\sigma} \otimes L^o_{k'\sigma'} - \tfrac{1}{2}(L^o_{k\sigma} \cdot L^o_{k'\sigma'})\mathrm{Id} \right) \right] \nabla \left(\varphi_{k\sigma}\varphi_{k'\sigma'} \right) + \\
&i\lambda_{q+1} \left(L^o_{k\sigma} \otimes L^o_{k'\sigma'} - \tfrac{1}{2}(L^o_{k\sigma} \cdot L^o_{k'\sigma'})\mathrm{Id} \right) \left[D_t \left(D\Phi_\sigma k + D\Phi_{\sigma'} k' \right) \right] \varphi_{k\sigma}\varphi_{k'\sigma'} \\
&= I + II + III,
\end{aligned}$$

where here we used the identity

$$\nabla \varphi_{k\sigma} e^{i\lambda_{q+1} k\cdot x} = \nabla e^{i\lambda_{q+1} k\cdot(\Phi_\sigma - x)} e^{ik\cdot x} = i\lambda_{q+1}((D\Phi_\sigma - \mathrm{Id})k) e^{i\lambda_{q+1} k\cdot \Phi_\sigma}.$$

Hence from Lemmas 5.2.1 and 5.2.2, together with the estimates (5.1.8) we obtain the following inequalities:

$$\begin{aligned}
\|I(t)\|_N \leq & C \left\| \left[(D_t \nabla L^o_{k,\sigma}(t)) \otimes L^o_{k'\sigma'}(t) \right] \varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right\|_N \\
& + C \left\| \left[D_t L^o_{k\sigma}(t) \otimes \nabla L^o_{k'\sigma'}(t) \right] \varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right\|_N \\
& + C \left\| \left[L^o_{k,\sigma}(t) \otimes (D_t \nabla L^o_{k'\sigma'}(t)) \right] \varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right\|_N \\
& + C \left\| \left[\nabla L^o_{k\sigma}(t) \otimes D_t L^o_{k'\sigma'}(t) \right] \varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right\|_N \\
\leq & C\delta_{q+1,\varsigma}\lambda_q \left(\bar{\delta}^{1/2}_{q,\varsigma}\lambda_q + \mu_{q+1,\varsigma'}\eta^{-1}_{q+1,\varsigma'} \right) \ell^{-N} \\
\leq & C\delta_{q+1,\varsigma}\lambda_q \mu_{q+1,\varsigma}\eta^{-1}_{q+1,\varsigma} \ell^{-N};
\end{aligned}$$

$$\begin{aligned}
\|II(t)\|_N \leq & C \left\| D_t \left[L^o_k(t) \otimes L^o_{k'}(t) \right] \nabla \left(\varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right) \right\|_N \\
\leq & C\delta^{1/2}_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\mu^{-1}_{q+1,\varsigma} \left(\bar{\delta}^{1/2}_{q,\varsigma}\lambda_q + \mu_{q+1,\varsigma}\eta^{-1}_{q+1,\varsigma} \right) \ell^{-N} \\
\leq & C\delta^{1/2}_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\eta^{-1}_{q+1,\varsigma} \ell^{-N};
\end{aligned}$$

and

$$
\begin{aligned}
\|III(t)\|_N \leq & C\lambda_{q+1} \sum_{\hat{\sigma}=\sigma,\sigma'} \left\| \left(L^o_{k\sigma}(t) \otimes L^o_{k'\sigma'}(t)\right) \left[D_t D\Phi_{\hat{\sigma}}(t)\right] \varphi_{k\sigma}(t)\varphi_{k'\sigma'}(t) \right\|_N \\
\leq & C\delta_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q^2\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{-N}.
\end{aligned}
$$

Similarly for $t \in K^g_\varsigma$ we obtain

$$
\begin{aligned}
\|I(t)\|_N \leq & C\bar{\delta}^{1/2}_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q\ell^{-N-1} \\
\|II(t)\|_N \leq & C\bar{\delta}^{1/2}_{q,\varsigma}\delta^{1/2}_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q^2\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{-N} \\
\|III(t)\|_N \leq & C\delta_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q^2\lambda_{q+1}\mu_{q+1,\varsigma}^{-1}\ell^{-N}.
\end{aligned}
$$

Applying (5.1.8) again we obtain

$$
\left\|\Omega'_{k\sigma k'\sigma'}(t)\right\|_N \leq C\left(\delta^{1/2}_{q+1}\lambda_{q+1,\varsigma} + \frac{\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}}\right)\frac{\delta^{1/2}_{q+1,\varsigma}\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}\ell^N},
$$

and for times $t \in K^g_\varsigma$

$$
\left\|\Omega'_{k\varsigma k'\varsigma'}(t)\right\|_N \leq C\bar{\delta}^{1/2}_{q,\varsigma}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}\ell^{-N}.
$$

The estimate (6.1.13) can be used to estimate $\Omega_{k\sigma k'\sigma'}$.

The estimate on $\|D_t R^1(t)\|_0$ now follows exactly as above for $D_t R^0$ applying Proposition 6.1.2 to the commutator terms Proposition 3.2.3 for the remaining terms.

Estimates on R^2 and $D_t R^2$.

Computing we have

$$
\begin{aligned}
w_{q+1}\cdot\nabla v_\ell = & \sum_{k,\sigma} L_{k\sigma}\cdot\nabla v_\ell \varphi_{k\sigma} e^{i\lambda_{q+1}k\cdot x} \\
D_t(w_{q+1}\cdot\nabla v_\ell) = & \sum_{k,\sigma}\left(D_t L_{k\sigma}\cdot\nabla v_\ell + L_{k\sigma}\cdot\nabla D_t v_\ell - L_{k\sigma}\cdot\nabla v_\ell\cdot\nabla v_\ell\right)\varphi_{k\sigma} e^{i\lambda_{q+1}k\cdot x}
\end{aligned}
$$

Applying Lemmas 5.2.1 and 5.2.2 we obtain

$$
\left\|L_{k\sigma}(t)\cdot\nabla v_\ell(t)\varphi_{k\sigma}(t)\right\|_N \leq C\delta^{1/2}_{q,\varsigma}\delta^{1/2}_{q+1,\varsigma}\lambda_q\ell^{-N},
$$

and

$$\begin{aligned}\left\|\left(D_t L_{k\sigma}(t)\cdot\nabla v_\ell(t)+L_{k\sigma}(t)\cdot\nabla D_t v_\ell(t)-L_{k\sigma}(t)\cdot\nabla v_\ell(t)\cdot\nabla v_\ell(t)\right)\varphi_{k\sigma}(t)\right\|_N&\\ \leq C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q(\mu_{q+1,\varsigma}\eta_{q+1,\varsigma}^{-1}+\bar{\delta}_{q,\varsigma}^{1/2}\lambda_q)\ell^{-N}&\\ \overset{(5.1.8)}{\leq} C\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q\mu_{q+1,\varsigma}\eta_{q+1,\varsigma}^{-1}\ell^{-N}\,.&\end{aligned}$$

Similarly for $t\in K_\varsigma^g$ we have

$$\begin{aligned}\left\|\left(D_t L_{k\sigma}(t)\cdot\nabla v_\ell(t)+L_{k\sigma}(t)\cdot\nabla D_t v_\ell(t)-L_{k\sigma}(t)\cdot\nabla v_\ell(t)\cdot\nabla v_\ell(t)\right)\varphi_{k\sigma}(t)\right\|_N&\\ \leq C\bar{\delta}_{q,\varsigma}^{-1/2}\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}^{1/2}\lambda_q^2\ell^{-N}\,.&\end{aligned}$$

The estimates on R^2 then follow by Proposition 3.2.3 together with the orderings (6.1.1) and (5.1.8). Again, making use of the identity $D_t\mathcal{R}=[v_\ell,\mathcal{R}]\nabla+\mathcal{R}D_t$, the estimates on D_tR^2 follow by Propositions 3.2.3 and 6.1.2.

Estimates on R^3 and D_tR^3. Using Lemma 5.2.4 and (A.1.3) we have

$$\|R^3(t)\|_N\leq C(\|(w_c(t))^2\|_N+\|w_o(t)w_c(t)\|_N)\leq C\frac{\delta_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\lambda_{q+1}^N}{\mu_{q+1,\varsigma}}\,.$$

Similarly, with the Lemmas 5.2.4 and 5.2.5 we achieve

$$\begin{aligned}\left\|D_tR^3(t)\right\|_0\leq& C\left\|D_tw_c(t)\right\|_0\left(\|w_o(t)\|_0+\|w_c(t)\|_0\right)\\ &+C\|D_tw_o(t)\|_0\|w_c(t)\|_0\\ \leq& C\delta_{q+1,\varsigma}\left(\bar{\delta}_{q,\varsigma}^{1/2}\lambda_q+\mu_{q+1,\varsigma}\eta_{q+1,\varsigma}^{-1}\right)\\ \overset{(5.1.8)}{\leq}& C\delta_{q+1,\varsigma}\mu_{q+1,\varsigma}\eta_{q+1,\varsigma}^{-1}\,,\end{aligned}$$

and for $t\in K_\varsigma^g$

$$\left\|D_tR^3(t)\right\|_0\leq C\bar{\delta}_{q,\varsigma}^{1/2}\delta_{q+1,\varsigma}\lambda_q\,.$$

Estimates on R^4 and D_tR^4. The estimates on $\|R^4(t)\|_0$, $\|R^4(t)\|_1$ and $\|R^4(t)\|_2$ are a direct consequence of mollification estimates together with Lemma 5.2.4. For D_tR^4 we have

$$\|D_tR^4(t)\|_0\leq\|v_q(t)-v_\ell(t)\|_0\|D_tw_{q+1}(t)\|_0+(\|D_tv_q(t)\|_0+\|D_tv_\ell(t)\|)\|w_{q+1}(t)\|_0$$

Concerning D_tv_q, since v_q solves the Euler Reynolds system (2.1.1), from our inductive estimates (4.3.11)-(4.3.13), we have

$$\begin{aligned}\|D_t v_q(t)\|_0 &\le \|\partial_t v_q(t) + v_q\cdot\nabla v_q(t)\|_0 + \|v_q(t)-v_\ell(t)\|_0\|v_q(t)\|_1\\ &\le \|p_q(t)\|_1 + \|\mathring{R}_q(t)\|_1 + C\delta_{q,\varsigma}\lambda_q^2\ell\\ &\le C\delta_{q,\varsigma}\lambda_q\,.\end{aligned}$$

Thus the required estimate on D_tR^4 follows from Lemmas 5.2.2 and 5.2.5.

Estimates on R^5 and D_tR^5. The required estimates follow directly from (4.3.13), (4.3.14) and Lemmas 4.3.3 and 4.3.4.

□

Proof of Proposition 6.1.2. We begin by noting that by setting

$$\mathcal{S}(v) := \nabla v + (\nabla v)^t - \frac{2}{3}(\operatorname{div} v)\mathrm{Id}\,,$$

we may rewrite $\mathcal{R}(v)$ as

$$\mathcal{R}(v) = S\left(\frac{1}{4}\mathcal{P}(u) + \frac{3}{4}u\right)\,,$$

where u is the mean zero solution to the equation $\Delta u = v - \fint v$.

The operator $\mathcal{S}$ satisfies the following property:

$$\operatorname{div}\mathcal{S}(v) = 0 \Leftrightarrow v \equiv \text{const}. \tag{6.1.14}$$

The implication $\Leftarrow$ is obvious. In order to show the implication $\Rightarrow$ holds, we observe that the identity $\operatorname{div}\mathcal{S}(v) = 0$ is equivalent to

$$\Delta v_j + \frac{1}{3}\partial_j \operatorname{div} v = 0\,. \tag{6.1.15}$$

Differentiating and summing the above identity in j yields the identity

$$\frac{4}{3}\Delta \operatorname{div} v = 0.$$

Therefore $\operatorname{div} v$ is a constant and moreover from (6.1.15) it follows that v is a constant. Since $\operatorname{div}\mathcal{S}(v)$ has mean zero, we obtain from the definition of $\mathcal{R}$ that $\operatorname{div} S(v) = \operatorname{div}\mathcal{R}(\operatorname{div} S(v))$. Hence from (6.1.14) we obtain

$$\mathcal{S}(v) - \mathcal{R}(\operatorname{div}\mathcal{S}(v)) = 0\,, \tag{6.1.16}$$

here we used the fact the fact $S(v)$ is mean zero.

For a given vector field $a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ and $k \in \mathbb{Z}^3$ satisfying $|k| = \lambda$, let us write

$$\mathscr{S}(ae^{ik\cdot x}) := -\mathcal{S}\left(\frac{3}{4}\frac{a}{\lambda^2}e^{ik\cdot x} + \frac{1}{4\lambda^2}\left(a - \frac{(a\cdot k)k}{\lambda^2}\right)e^{ik\cdot x}\right) .$$

In particular, by the product rule we have

$$ae^{ik\cdot x} - \operatorname{div}\mathscr{S}(ae^{ik\cdot x}) = \frac{B_1(a)}{\lambda}e^{ik\cdot x} + \frac{B_2(a)}{\lambda^2}e^{ik\cdot x} ,$$

for some homogeneous differential operators B_1 and B_2 of order 1 and 2 respectively with constant coefficients (depending only on $\frac{k}{\lambda}$). Moreover, again by the product rule we obtain

$$\mathscr{S}(bae^{ik\cdot x}) - b\mathscr{S}(ae^{ik\cdot x}) = \frac{aA(b)}{\lambda^2}e^{ik\cdot x} , \tag{6.1.17}$$

Then applying (6.1.16) we obtain the following decomposition

$$\begin{aligned} \mathcal{R}(bF) - b\mathcal{R}(F) &= \mathscr{S}(bae^{ik\cdot x}) - b\mathscr{S}(ae^{ik\cdot x}) \\ &\quad + \mathcal{R}\left(bF - \operatorname{div}\mathscr{S}(bae^{ik\cdot x})\right) - b\mathcal{R}\left(F - \operatorname{div}\mathscr{S}(ae^{ik\cdot x})\right) \\ &= \frac{aA(b)}{\lambda^2}e^{ik\cdot x} + \mathcal{R}\left(\frac{B_1(ab)}{\lambda}e^{ik\cdot x} + \frac{B_2(ab)}{\lambda^2}e^{ik\cdot x}\right) \\ &\quad - b\mathcal{R}\left(\frac{B_1(a)}{\lambda}e^{ik\cdot x} + \frac{B_2(a)}{\lambda^2}e^{ik\cdot x}\right) . \end{aligned} \tag{6.1.18}$$

Using the product rule to write $B_1(ab) = B_1(a)b + aB_1(b)$ and $B_2(ab) = B_2(a)b + aB_2(b) + C_1(a)C_1(b)$, for some homogeneous operator C_1 of order 1, we may rewrite the above decomposition as

$$\begin{aligned} -[b,\mathcal{R}](F) &= \frac{aA(b)}{\lambda^2}e^{ik\cdot x} \\ &\quad + \mathcal{R}\left(\frac{aB_1(b)}{\lambda}e^{ik\cdot x}\right) + \mathcal{R}\left(\frac{aB_2(b) + C_1(a)C_1(b)}{\lambda^2}e^{ik\cdot x}\right) \\ &\quad - \frac{1}{\lambda}[b,\mathcal{R}]\left(B_1(a)e^{ik\cdot x}\right) - \frac{1}{\lambda^2}[b,\mathcal{R}]\left(B_2(a)e^{ik\cdot x}\right) . \end{aligned} \tag{6.1.19}$$

Observe that no zero order terms in b appear on the first two lines. The two terms on the second line can be estimated by applying Proposition 3.2.3, with $m = N-1$ and $m = N-2$ to the first summand and second summand respectively. Applying interpolation, we conclude

$$
\begin{aligned}
\|[b,\mathcal{R}](F)\|_0 \leq{}& C\frac{\|a\|_0\|b\|_1}{\lambda^{2-\alpha}} + C\frac{\|a\|_{N-1}\|b\|_1 + \|a\|_{N-2}\|b\|_2}{\lambda^N} \\
&+ C\frac{\|a\|_1\|b\|_{N-1} + \|a\|_0\|b\|_N}{\lambda^N} \\
&+ \frac{1}{\lambda}\underbrace{\left\|[b,\mathcal{R}]\left(B_1(a)e^{ik\cdot x}\right)\right\|_0}_{II} + \frac{1}{\lambda^2}\left\|[b,\mathcal{R}]\left(B_2(a)e^{ik\cdot x}\right)\right\|_0 . \qquad (6.1.20)
\end{aligned}
$$

We proceed by applying the same idea to the term II in (6.1.20), which is of the form $\|[b,\mathcal{R}](F')\|_0$, where $F'(x) = B_1(a)(x)e^{ik\cdot x}$ and $B_1(a)$ are linear combinations of first order derivatives of a. However, this time we apply it with $N-1$ in place of N:

$$
\begin{aligned}
\|[b,\mathcal{R}](F)\|_0 \leq{}& C\lambda^{\alpha-2}\|b\|_1\left(\|a\|_0 + \lambda^{-1}\|a\|_1\right) \\
&+ C\frac{\|a\|_{N-1}\|b\|_1 + \|a\|_{N-2}\|b\|_2}{\lambda^N} \\
&+ C\frac{\|a\|_2\|b\|_{N-2} + \|a\|_1\|b\|_{N-1} + \|a\|_0\|b\|_N}{\lambda^N} \\
&+ \frac{1}{\lambda^2}\left\|[b,\mathcal{R}]\left(B_2'(a)e^{ik\cdot x}\right)\right\|_0 + \frac{1}{\lambda^3}\left\|[b,\mathcal{R}]\left(B_3'(a)e^{ik\cdot x}\right)\right\|_0 , \qquad (6.1.21)
\end{aligned}
$$

where $B_2' = B_2 + B_1 \circ B_1$ is a second order operator and $B_3' = B_2 \circ B_1$ a third order operator. Proceeding inductively yields

$$
\begin{aligned}
\|[b,\mathcal{R}](F)\|_0 \leq{}& C\lambda^{\alpha-2}\|b\|_1\sum_{i=0}^{N-2}\lambda^{-i}\|a\|_i + C\lambda^{-N}\sum_{i=0}^{N-1}\|a\|_i\|b\|_{N-i} \\
&+ \frac{1}{\lambda^{N-1}}\left\|[b,\mathcal{R}]\left(B_{N-1}'(a)e^{ik\cdot x}\right)\right\|_0 + \frac{1}{\lambda^N}\left\|[b,\mathcal{R}]\left(B_N'(a)e^{ik\cdot x}\right)\right\|_0 ,
\end{aligned}
$$

where B_{N-1}' and B_N' are two linear differential operators of order $N-1$ and N respectively.

Finally, we apply Proposition 3.2.3 and Lemma 3.2.4 to the final two terms and interpolate to reach the desired estimate. □

6.2 References and Remarks

As with Chapter 5, the estimates presented here can essentially all be found in [BDLS14], which itself is based on [BDLSJ13] and [Buc13]. The idea of splitting the estimates into *good* and *bad* regions was introduced in [Buc13].

For related estimates on the Reynolds stress defined in terms of frequency cut-offs of arbitrary Hölder continuous weak solutions to the Euler equations (1.1.1), we refer the interested reader to the following paper of Isett [Ise13b].

7

Proof of Theorem 1.2.2

IN THIS CHAPTER, we conclude the proof of Theorem 1.2.2. We complete our definition of the perturbation by defining our cut-off functions χ_ς and amplitude parameters ρ_ς. We then proceed in providing estimates on the energy of the approximate solutions v_q. Then after carefully selecting ours parameters, we utlitise our convex integration scheme in order to construct a sequence of approximate solutions (v_q, p_q) converging to a solution (v, p) satisfying the requirements of Theorem 1.2.2.

As noted in Remark 4.1.1: for the purposes of proving Theorem 1.2.2, we may fix $\mu_{q+1,\varsigma} := \mu_{q+1}$ and $\eta_{q+1,\varsigma} := \frac{1}{10}$ uniformly for all ς, and define the cut-off functions in terms of the appropriate translation and scaling of a fixed function χ. In Remark 4.3.2, we noted that we may define $\bar{\delta}_{q,\varsigma} := \delta_{q,\varsigma} := \lambda_q^{-2\beta}$ uniformly for all ς, for some yet to be chosen $0 < \beta < 1/5$. For the proof of Theorem 1.2.2, either choice of the approximation R_ς described in Section 4.2 will suffice.[1]

[1] There is a very minor issue regarding the temporal boundary if one decides to use the approximation (4.2.2). This can be rectified in a number of ways: for example one could simply smoothly extend the prescribed energy profile and ignore estimates at the new temporal boundary.

7.1 Estimates on the Energy

Recall from Section 2.4, we wish to show that the energies of our approximate solutions converge monotonically from below to our target energy profile $e : [0, T] \to \mathbb{R}$. In order to achieve this goal we define our amplitude parameter ρ_ς as follows

$$\rho_\varsigma := \frac{1}{3(2\pi)^3}\left(e(t_\varsigma) - c_0\delta_{q+2} - \int_{\mathbb{T}^3} \left|v_q(x, t_\varsigma)\right|^2 \, dx\right) , \tag{7.1.1}$$

where we recall that t_ς is defined to be the midpoint of the support of χ_ς.

Lemma 7.1.1. *Let v_{q+1} be as described in Chapter 4 with amplitude parameters ρ_ς given by (7.1.1), then we have the following estimate on the energy of v_{q+1}*

$$\left|e(t) - c_0\delta_{q+2} - \int_{\mathbb{T}^3} \left|v_{q+1}\right|^2 \, dx\right| \leq C\mu_{q+1}^{-1} + C\lambda_q^{1-\beta}\lambda_{q+1}^{-2\beta}\mu_{q+1}^{-1}. \tag{7.1.2}$$

Proof. We begin by setting

$$\bar{e}(t) := 3(2\pi)^3 \sum_\varsigma \chi_\varsigma^2(t)\rho_\varsigma \, .$$

Then it follows as a consequence of (3.1.7) and the definition of w_o that we have the following decomposition

$$|w_o(x,t)|^2 = \sum_{\varsigma,\varsigma'} \chi_\varsigma^2(t) \operatorname{tr} R_\varsigma(x,t) + \tag{7.1.3}$$

$$\sum_{(k,\varsigma),(k',\varsigma'),k\neq -k'} a_{k\varsigma}(x,t)a_{k'\varsigma'}(x,t)B_k \cdot B_{k'} e^{i\lambda_{q+1}(k+k')\cdot x}$$

$$= (2\pi)^{-3}\bar{e}(t) + \sum_{(k,\varsigma),(k',\varsigma'),k\neq -k'} a_{k\varsigma}(x,t)a_{k'\varsigma'}(x,t)B_k \cdot B_{k'} e^{i\lambda_{q+1}(k+k')\cdot x} \, . \tag{7.1.4}$$

Therefore since $(k + k') \neq 0$ in the sum above, we apply Lemma 5.2.1 and integration by parts to conclude

$$\begin{aligned}\left|\int_{\mathbb{T}^3} |w_o(x,t)|^2 \, dx - \bar{e}(t)\right| &\leq C\frac{\sum_{k,k',\varsigma,\varsigma'} \left\|a_{k\varsigma}a_{k'\varsigma'}\right\|_1}{\lambda_{q+1}} \\ &\leq C\lambda_q^{1-\beta}\lambda_{q+1}^{-2\beta}\mu_{q+1}^{-1} \, .\end{aligned} \tag{7.1.5}$$

Now recall the identity

$$w_{q+1} = \sum_{k,\varsigma} \frac{1}{\lambda_{q+1}} \text{curl} \left(i a_{k\varsigma} \frac{k \times B_k}{|k|^2} e^{i\lambda_{q+1} k\cdot x} \right).$$

Then applying integration by parts yields

$$\left| \int_{\mathbb{T}^3} v_q(x,t) \cdot w_{q+1}(x,t)\, dx \right| \le C\lambda_q^{1-\beta} \lambda_{q+1}^{-1-\beta} \overset{(6.1.1)}{\le} C\lambda_q^{1-\beta} \lambda_{q+1}^{-2\beta} \mu_{q+1}^{-1}. \tag{7.1.6}$$

Moreover, as a consequence of Lemma 5.2.4, we obtain

$$\int_{\mathbb{T}^3} |w_c(x,t)|^2 + |w_c(x,t) w_o(x,t)|\, dx \le C\lambda_q^{1-\beta} \lambda_{q+1}^{-2\beta} \mu_{q+1}^{-1}. \tag{7.1.7}$$

Combining the above estimates, we obtain

$$\left| \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2\, dx - \left(\bar{e}(t) + \int_{\mathbb{T}^3} |v_q(x,t)|^2\, dx \right) \right| \le C\lambda_q^{1-\beta} \lambda_{q+1}^{-2\beta} \mu_{q+1}^{-1}. \tag{7.1.8}$$

Thus it remains to estimate the difference $|\bar{e} - e|$. Note by definition we have

$$\begin{aligned} \bar{e}(t) &= 3(2\pi)^3 \sum_{\varsigma} \chi_\varsigma^2(t) \rho_\varsigma \\ &= \sum_{\varsigma} \chi_\varsigma^2(t)(e(t_\varsigma) - c_0 \delta_{q+2}) - \sum_{\varsigma} \chi_\varsigma^2(t) \int_{\mathbb{T}^3} |v_q(x,t_\varsigma)|^2\, dx. \end{aligned}$$

Since $|t - t_\varsigma| < \mu_{q+1}^{-1}$ on the support of χ_ς and since $\sum_\varsigma \chi_\varsigma^2 \equiv 1$, we have

$$\left| e(t) - \sum_{l} \chi_\varsigma^2 e(t_\varsigma) \right| \le C\mu_{q+1}^{-1}.$$

Since the triple $(v_q, p_q, \mathring{R}_q)$ solves the the Euler-Reynolds system (2.1.1), we deduce

$$\begin{aligned} &\int_{\mathbb{T}^3} \left(|v_q(x,t)|^2 - |v_q(x,t_\varsigma)|^2 \right) dx \\ &= \int_{t_\varsigma}^{t} \int_{\mathbb{T}^3} \partial_t |v_q(x,t)|^2\, dx \\ &\quad - \int_{t_\varsigma}^{t} \int_{\mathbb{T}^3} \text{div} \left(v_q(x,t) \left(|v_q(x,t)|^2 + 2p_q(x,t) \right) \right) dx \\ &\quad + 2\int_{t_\varsigma}^{t} \int_{\mathbb{T}^3} v_q(x,t) \cdot \text{div}\, \mathring{R}_q(x,t)\, dx \end{aligned}$$

$$= -2\int_{t_\varsigma}^{t}\int_{\mathbb{T}^3} Dv_q : \mathring{R}_q(x,t)\,dx\,,$$

where we $A : B$ denotes tensor contraction.

Thus, for $|t - t_\varsigma| \le \mu_{q+1}^{-1}$ we conclude

$$\left|\int_{\mathbb{T}^3} |v_q(x,t)|^2 - \left|v_q(x,t_\varsigma)\right|^2 \; dx\right| \le C\lambda_q^{1-\beta}\lambda_{q+1}^{-2\beta}\mu_{q+1}^{-1}\,.$$

Again using $\sum \chi_\varsigma^2 = 1$, we then conclude

$$\left|e(t) - c_0\delta_{q+2} - \left(\bar{e}(t) + \int_{\mathbb{T}^3} |v_q(x,t)|^2\,dx\right)\right| \le C\mu_{q+1}^{-1} + C\lambda_q^{1-\beta}\lambda_{q+1}^{-2\beta}\mu_{q+1}^{-1}\,. \quad (7.1.9)$$

Finally, the estimate (7.1.2) follows from (7.1.8) and (7.1.9). □

7.2 Main Proposition and Choice of Parameters

In this section we present our main proposition which will be used in order to construct our sequence of pairs (v_q, p_q) converging to a solution (v, q) to (1.1.1) satisfying the conditions stated in Theorem 1.2.2.

Proposition 7.2.1. *For every $0 < \beta < \frac{1}{5}$ there exists a $\bar{\lambda} > 1$ and $b > 1$ such that for any integer $\lambda_0 > \bar{\lambda}$ and normalised energy profile $e : [0,T] \to \mathbb{R}$ satisfying (2.4.1), the following holds: Suppose we have $\lambda_0^{b^i} \le \lambda_i \le 2\lambda_0^{b^i}$ for each $i \in \mathbb{N}$, and assume for some $q \in \mathbb{N}$, the triple $(v_q, p_q, \mathring{R}_q)$ is a solution to the Euler-Reynolds system satisfying (2.4.2) and (4.3.11)-(4.3.14). Then there exists a solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ to the Euler-Reynolds equation satisfying the aforementioned inequalities with q replaced by $q+1$. Furthermore, in addition we have the following estimates*

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_q}\|v_{q+1}\|_1 \le C\lambda_{q+1}^{-\beta} \quad (7.2.1)$$

$$\|p_{q+1} - p_q\|_0 + \frac{1}{\lambda_q}\|p_{q+1}\|_1 \le C\lambda_{q+1}^{-2\beta} \quad (7.2.2)$$

$$\|\partial_t(v_{q+1} - v_q)\|_0 \le C\left(1 + \left\|v_q\right\|_0\right)\lambda_{q+1}^{1-\beta} \quad (7.2.3)$$

$$\|\partial_t\left(p_{q+1} - p_q\right)\|_0 \le C\left(1 + \left\|v_q\right\|_0\right)\lambda_{q+1}^{1-2\beta}\,. \quad (7.2.4)$$

Proof. We begin by choosing $b > 1$ such that $b\beta < 1/5$ and then set

$$\mu_{q+1} := \lambda_0^{b^q(1+b)(1-\beta)/2} \tag{7.2.5}$$

$$\varepsilon_0 := \frac{(b-1)(1-5b\beta)}{10b} . \tag{7.2.6}$$

Observe that apart from the inequality

$$\frac{\delta_q^{1/2}\lambda_q}{\mu_{q+1}} \leq \frac{1}{\lambda_{q+1}^{\varepsilon_0}} , \tag{7.2.7}$$

the inequalities (5.1.8) follow by simply calculations. Taking logarithms and dividing by $b^q \ln \lambda_0$, the inequality (7.2.7) amounts to showing

$$\begin{aligned} 0 &\geq 1 - \beta - \frac{(1+b)(1-\beta)}{2} + b\varepsilon_0 + O\left(\frac{1}{b^q \ln \lambda_0}\right) \\ &\geq \frac{(1-b)(1-\beta)}{2} + 2b\varepsilon_0 \\ &\geq \frac{1-b}{5} , \end{aligned}$$

where in the first inequality we assumed λ_0 to be sufficiently large such that the last term on the right hand side is bounded by $b\varepsilon_0$.

Observe that (4.3.11), (4.3.12) with q replaced by $q+1$ follow as a consequence of Lemma 5.2.4. Likewise, we also obtain (7.2.1) and (7.2.2).

Note that as a consequence of the definition of μ_{q+1} above we have

$$\frac{\mu_{q+1}}{\eta_{q+1}} \leq \delta_{q+1}^{1/2}\lambda_{q+1},$$

and thus we from Lemma 5.2.5 we obtain (7.2.3) and (7.2.4).

Now consider (4.3.13) and (4.3.14) with q replaced by $q+1$. Applying Proposition 6.1.1, taking logarithms and dividing by $b^q \ln \lambda_0$, proving the mentioned inequalities amounts to showing

$$
\begin{aligned}
0 &\geq \ln\left(\frac{C\delta_{q+1}^{1/2}\mu_{q+1}\ell}{\delta_{q+2}}\right)(b^q \ln \lambda_0)^{-1} \\
&\geq 2b^2\beta - b\beta + \frac{(1+b)(1-\beta)}{2} - b + b\varepsilon_0 + O\left(\frac{1}{c_0 b^q \ln \lambda_0}\right) \\
&\geq \frac{(1-b)(1-(4b+1)\beta)}{2} + 2b\varepsilon_0 \\
&\geq \frac{3(1-b)(1-5b\beta)}{10} .
\end{aligned}
$$

Finally, since for large λ_0 we have $\delta_{q+1}\delta_q^{1/2}\lambda_q \geq \lambda_0^{b^q(1-3b\beta)} \geq 1$, then from Lemma 7.1.1, ordering (6.1.1) and the above calculation, we obtain (2.4.2) with q replaced by $q+1$. □

7.3 Conclusion of Proof of Theorem 1.2.2

We now apply Proposition 7.2.1 in order to conclude our proof of Theorem 1.2.2.

Observe by setting $(v_0, p_0, \mathring{R}_0) = ((0,0,0), 0, (0,0,0) \otimes (0,0,0))$ it follows that $(v_0, p_0, \mathring{R}_0)$ trivially satisfy the hypothesis of Proposition 7.2.1 with the exception of (2.4.2). Nevertheless, applying the same arguments as in the proof of Proposition 7.2.1 yields a new triple $(v_1, p_1, \mathring{R}_1)$ satisfying *all* requirements of Proposition 7.2.1 for $q = 1$. Applying Proposition 7.2.1 iteratively then leads to a sequence of approximate solutions (v_q, p_q) converging uniformly to a pair of continuous functions (v, p) solving (1.1.1) and satisfying (2.4.2).

From (7.2.1)-(7.2.4), by interpolation we conclude

$$
\begin{aligned}
\|v_{q+1} - v_q\|_{C^\theta(\mathbb{T}^3\times[0,T])} &\leq C\lambda_q^{\theta-\beta} \\
\|p_{q+1} - p_q\|_{C^{2\theta}(\mathbb{T}^3\times[0,T])} &\leq C\lambda_q^{2\theta-2\beta} .
\end{aligned}
$$

Thus, for every $\theta < \beta$, v_q converges in $C^\theta(\mathbb{T}^3 \times [0,T])$ to v and p_q converges in $C^{2\theta}(\mathbb{T}^3 \times [0,T])$ to p. Since β can be taken arbitrarily close to $1/5$, this concludes the proof of Theorem 1.2.2.

7.4 References and Remarks

The arguments of this chapter can be found in [BDLSJ13]. Slightly different numerological arguments are used here in a similar spirit to the papers [Buc13, BDLS14].

8

Proof of Theorem 1.2.3

8.1 Bookkeeping, Partitioning and Parameter Choice

To prove Theorem 1.2.3 we will need to construct the appropriate bookkeeping system in order to keep track of time localised estimates. Specifically, we will divide the time interval $[0, T]$ into a finite family of closed intervals $I_a^{(q)}$ for $a = 1, \ldots, N(q)$. The intervals will be ordered in ascending order with each pair $I_a^{(q)}, I_{a+1}^{(q)}$ intersecting at a single point. To each interval $I_a^{(q)}$ we will associate an amplitude exponent $\beta_j = \beta^{(q)}(a)$ for $j \in 0, 1, \ldots, q$, where $0 < \beta_j < \frac{1}{3}$ is defined by the inductive formula

$$\beta_{j+1} = \frac{\beta_j - \beta_\infty}{b} + \beta_\infty \,, \tag{8.1.1}$$

or alternatively

$$\beta_j = \frac{\beta_0}{b^j} + \left(1 - \frac{1}{b^j}\right)\beta_\infty \,, \tag{8.1.2}$$

where here $0 < \beta_0 < \beta_\infty$ are fixed exponents to be defined later. For notational convenience, we also introduce the additional exponent

$$\beta_{-1} = b\beta_0 + (1-b)\beta_\infty \,, \tag{8.1.3}$$

which we will also assume to be positive. Note that if we assume that for every $i \in \mathbb{N}$ we have $\lambda_0^{b^i} \leq \lambda_i \leq 2\lambda_0^{b^i}$, then we have the following useful inequality

$$\frac{1}{2}\lambda_i^{\beta_\infty}\lambda_{i+1}^{-\beta_\infty} \leq \lambda_i^{\beta_{j-1}}\lambda_{i+1}^{-\beta_j} \leq 2\lambda_i^{\beta_\infty}\lambda_{i+1}^{-\beta_\infty} . \tag{8.1.4}$$

We assume the following constraint on the length of the interval $I_a^{(q)}$

$$\left|I_a^{(q)}\right| \geq \frac{4}{\tilde{\mu}_{q+1,j}} , \tag{8.1.5}$$

where here j is chosen such that $\beta_j = \beta^{(q)}(a)$ and $\tilde{\mu}_{q+1,j}$ is a large parameter, related to the parameters $\mu_{q+1,\varsigma}$ of Chapter 4, defined by the following formula

$$\tilde{\mu}_{q+1,j} = \begin{cases} \lambda_{q+1}^{1-\beta_j} & \text{for } j > 1 \\ \lambda_q^{(1-\beta_0)/2}\lambda_{q+1}^{(1-\beta_0-\beta_\infty)/2}\lambda_{q+2}^{\beta_\infty/2} & \text{for } j \leq 1 . \end{cases} \tag{8.1.6}$$

We will later choose b, β_0 and β_∞ in such a way that the family of parameters $\tilde{\mu}_{q+1,j}$ is monotonically decreasing in j: if $j' > j$

$$\tilde{\mu}_{q+1,j'} \leq \tilde{\mu}_{q+1,j} . \tag{8.1.7}$$

Moreover, we will assume that for neighbouring intervals, the following constraint is satisfied

$$\beta^{(q)}(a) < \beta^{(q)}(a') \quad \Rightarrow \quad \beta^{(q)}(a) = 0 , \tag{8.1.8}$$

where $|a - a'| = 1$. For the endpoint intervals $a = 1, N(q)$ we further assume

$$\beta^{(q)}(a) = q . \tag{8.1.9}$$

For the special case $q = 0$ we assume $\mathring{R}_0 \equiv 0$. This requirement together with (8.1.9) are simply technical requirements in order to avoid potential issues involved with the *mollification along the flow* approximation of the Reynolds stress at the temporal boundaries (see end of Section 4.2).

We denote the union of all time intervals associated to a particular exponent β_j by $V_j^{(q)}$:

$$V_j^{(q)} = \bigcup_{\{a:\beta_j=\beta^{(q)}(a)\}} I_a^{(q)} . \tag{8.1.10}$$

For $t \in V_j^{(q)}$ we assume the following inductive estimates

$$\frac{1}{\lambda_q}\left\|v_q(t)\right\|_1 \leq \lambda_q^{-\beta_{(j-1)+}} \tag{8.1.11}$$

$$\frac{1}{\lambda_q}\left\|p_q(t)\right\|_1 + \frac{1}{\lambda_q^2}\left\|p_q(t)\right\|_2 \leq \lambda_q^{-2\beta_{(j-1)+}} \tag{8.1.12}$$

$$\left\|\mathring{R}_q(t)\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q(t)\right\|_1 + \frac{1}{\lambda_q^2}\left\|\mathring{R}_q(t)\right\|_2 \leq c_0\lambda_{q+1}^{-2\beta_j} \tag{8.1.13}$$

$$\left\|(\partial_t + v_q \cdot \nabla)\mathring{R}_q(t)\right\|_0 \leq c_0\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{-2\beta_j}\,, \tag{8.1.14}$$

where here we have adopted the notation $(a)_+ = max(a,0)$. The measure of the set $V_j^{(q)}$ will be assumed to satisfy the following constraint

$$\left|V_j^{(q)}\right| \leq \lambda_0\lambda_{q+1}^{\beta_j-\beta_\infty+\varepsilon_1}\,, \tag{8.1.15}$$

where $\varepsilon_1 > 0$ is a small constant.

Assuming that there exists such intervals $\{I_a^{(q)}\}_{a\in\{1,\ldots,N(q)\}}$ satisfying the properties above, we now describe the procedure for constructing the cut-off functions χ_ς as well as the amplitudes ρ_ς. In the process we will also define the inductive construction of new time intervals $\{I_a^{(q+1)}\}_{a\in\{1,\ldots,N(q+1)\}}$ satisfying the above conditions with $q+1$ replacing q.

For a given interval $I_a^{(q)} = [T_0, T_1]$ such that $\beta(a) = j$, we subdivide $I_a^{(q)}$ into closed subintervals $K_{a,1},\ldots,K_{a,n(a,q)}$ of uniform length, where $n(a,q)$ is the largest integer smaller than $\tilde{\mu}_{q+1,j}|I_a^{(q)}|/2$, with the intervals being indexed in ascending order, i.e.

- $T_0 \in K_{a,1}$
- $T_1 \in K_{a,n(a,q)}$
- For each $a' \in 1,\ldots,n(a,q)-1$ the intervals $K_{a,a'}$ and $K_{a,a'+1}$ intersect at a single point.

Observe that the estimate (8.1.5) ensures that such a subdivision is possible.

Now let us relabel the collection of interval $\{K_{a,a'}\}_{a\in 1,\ldots,N(q),a'\in 1,\ldots,n(a,q)}$, in ascending order as K_ς for $\varsigma = 1,\ldots,N'$. Then for a given interval $K_\varsigma \subset I_a^{(q)}$ such that $\beta^{(q)}(a) = \beta_j$, we set

$$\delta_{q,\varsigma} := \lambda_q^{-2\beta_{(j-1)+}}, \quad \bar{\delta}_{q,\varsigma} := \lambda_q^{-2\beta_{j-1}}, \quad \delta_{q+1,\varsigma} := \lambda_{q+1}^{-2\beta_j}$$
$$\mu_{q+1,\varsigma} := \tilde{\mu}_{q+1,j},$$

$$\eta_{q+1,\varsigma} := \tilde{\eta}_{q+1,j} := \lambda_{q+2}^{\beta_0-\beta_\infty} \lambda_{q+1}^{\beta_\infty-\beta_j} .$$

Observe then that if we choose R_ς according to the formula (4.2.2) for the case $\delta_{q+1,\varsigma} = \lambda_{q+1}^{-2\beta_0}$ and for all other cases according to the formula (4.2.1), then the hypotheses of Lemmas 4.3.4 and 4.3.3 will be satisfied for the respective cases.

We now define the *overlapping region* K_ς^b for $\varsigma = 1, \cdots, N' - 1$ in the following manner:

- If $\delta_{q,\varsigma} \leq \delta_{q,\varsigma+1}$ then let K_ς^b be the closed interval contained in K_ς of length $\mu_{q+1,\varsigma+1}^{-1}\eta_{q+1,\varsigma+1}$ with *maximal* endpoint coinciding with the common endpoint of K_ς and $K_{\varsigma+1}$.

- If $\delta_{q,\varsigma} > \delta_{q,\varsigma+1}$ then let K_ς^b be the closed interval contained in $K_{\varsigma+1}$ of length $\mu_{q+1,\varsigma}^{-1}\eta_{q+1,\varsigma}$ with *minimal* endpoint coinciding with the common endpoint of K_ς and $K_{\varsigma+1}$.

Taking into account (8.1.8), in order to ensure that the overlapping regions K_ς^b are each contained in a region $K_{\varsigma'}$, for some ς', we require the following parameter inequality to hold

$$\frac{\tilde{\eta}_{q+1,0}}{\tilde{\mu}_{q+1,0}} \leq \frac{1}{\tilde{\mu}_{q+1,j}} , \tag{8.1.16}$$

for $j \in \mathbb{N}$. Since $\tilde{\eta}_{q+1,j} < 1$, the above inequality is seen to be trivially weaker than (8.1.7).

Define the *non-overlapping region* K_ς^g as the *closure* of $K_\varsigma \setminus (K_{\varsigma-1}^b \cup K_\varsigma^b)$.

The cut-off functions $\chi_\varsigma : [0, T] \to [0, 1]$ are then defined such that

- $\sum \chi_\varsigma^2 \equiv 1$;

- χ_ς is identically 1 on K_ς^g and it is supported in $K_\varsigma^g \cup K_{\varsigma-1}^b \cup K_\varsigma^b$;

- On $K = K_\varsigma^b$ and $K = K_{\varsigma-1}^b$ we have the estimate

$$\|\partial_t^N \chi_\varsigma\|_0 \leq C|K|^{-N} .$$

In order that the above definition of the cut-off functions is compatible with (4.1.7) from Chapter 4.3.1 we require

$$\frac{\tilde{\eta}_{q+1,0}}{\tilde{\mu}_{q+1,0}} \geq \frac{\tilde{\eta}_{q+1,j}}{\tilde{\mu}_{q+1,j}} , \tag{8.1.17}$$

for $j \in \mathbb{N}$. The case $j = 1$ is trivial since $\tilde{\mu}_{q+1,0} = \tilde{\mu}_{q+1,1}$ and $\tilde{\eta}_{q+1,1} < \tilde{\mu}_{q+1,0}$. For $j \geq 2$, calculating we have

$$\frac{\tilde{\eta}_{q+1,j}}{\tilde{\mu}_{q+1,j}\tilde{\eta}_{q+1,0}} = \lambda_{q+1}^{\beta_0 - 1} \leq \frac{1}{\tilde{\mu}_{q+1,0}},$$

hence we obtain (8.1.17).

Note that with the above definition of the cut-off functions χ_ς, the inductive estimates (4.3.11)-(4.3.14) follow directly from the estimates (8.1.11)-(8.1.14).

In order to conclude the construction of the perturbation (v_{q+1}, p_{q+1}), we define the parameters ρ_ς:

$$\rho_\varsigma := c_0 \delta_{q+1,\varsigma}. \tag{8.1.18}$$

The new collection of intervals $\{I_a^{(q+1)}\}_{a \in \{1, N(q+1)\}}$ is then given by the collection of overlapping regions K_ς^b and non-overlapping regions K_ς^g indexed in ascending order.

We define the map $\beta^{(q+1)}(a)$ as follows

$$\beta^{(q+1)}(a) = \begin{cases} 0 & \text{if } I_a^{(q+1)} = K_\varsigma^b \text{ for some } \varsigma \\ \beta^{(q)}(a') + 1 & \text{otherwise} \end{cases}, \tag{8.1.19}$$

where here a' is chosen such that $I_a^{(q+1)} \subset I_{a'}^{(q)}$. Observe in particular that the above definition of $\beta^{q+1}(a)$ ensures that both (8.1.8) and (8.1.9) are satisfied for the new collection, i.e. with q replaced by $q + 1$. Note also that since

$$\{j : \beta^{(q)}(a), a = 1, \ldots, N(q)\} = \{0, \ldots, q\},$$

we deduce

$$\{j : \beta^{(q+1)}(a), a = 1, \ldots, N(q+1)\} = \{0, \ldots, q+1\}.$$

8.2 Main Proposition and Parameter Inequalities

We now state our main proposition which will be used to iteratively construct our solution (v, p) satisfying the conditions of Theorem 1.2.3.

Proposition 8.2.1. *Suppose $\beta_0 > 0$ and $\beta_\infty > 0$ satisfy the constraint $1/5 + \beta_0 < \beta_\infty < 1/3 - \beta_0$, then there exists $b > 1$ such that for sufficiently large integers λ_0 we have the following: Suppose we have $\lambda_0^{b^i} \leq \lambda_i \leq 2\lambda_0^{b^i}$ for each $i \in \mathbb{N}$. Furthermore, assume that for some $q \in \mathbb{N}$, the triple $(v_q, p_q, \mathring{R}_q)$ solves the Euler-Reynolds system (2.1.1) and* *let* $\{I_a^{(q)}\}_{a\in\{1,\cdots,N(q)\}}$ *be a subdivision of* $[0,T]$ *satisfying the requirements of Section 8.1. Then there exists a triple* $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ *solving the Euler-Reynolds system, together with a subdivision* $\{I_a^{(q+1)}\}_{a\in\{1,\cdots,N(q+1)\}}$ *satisfying the requirements of Section 8.1 with q replaced with* $q+1$*. Moreover, we have the following estimates*

$$\|v_{q+1}(t) - v_q(t)\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1}(t)\|_1 \le C\lambda_{q+1}^{-\beta_{(j-1)+}} \tag{8.2.1}$$

$$\|p_{q+1}(t) - p_q(t)\|_0 + \frac{1}{\lambda_{q+1}}\|p_{q+1}(t)\|_1 \le C\lambda_{q+1}^{-2\beta_{(j-1)+}} \tag{8.2.2}$$

$$\|\partial_t(v_{q+1} - v_q)(t)\|_0 \le C\left(1 + \|v_q\|_0\right)\lambda_{q+1}^{1-\beta_{(j-1)+}} \tag{8.2.3}$$

$$\|\partial_t\left(p_{q+1} - p_q\right)(t)\|_0 \le C\left(1 + \|v_q\|_0\right)\lambda_{q+1}^{1-2\beta_{(j-1)+}}\,. \tag{8.2.4}$$

for all $t \in V_j^{(q+1)}$.

Proof. We begin by choosing $b > 1$ sufficiently close to 1 such that the following conditions are satisfied

$$\beta_{-1} = b\beta_0 + (1-b)\beta_\infty > 0 \tag{8.2.5}$$

$$3b(\beta_0 + \beta_\infty) < 1 \tag{8.2.6}$$

$$b(1 + 3\beta_0) < 5\beta_\infty\,. \tag{8.2.7}$$

Now define $\varepsilon_0 > 0$ sufficiently small such that we have

$$\varepsilon_0 \le \frac{(b-1)(1 - 3b(\beta_0 + \beta_\infty))}{8b}\,. \tag{8.2.8}$$

With the above choices, let us check that the parameter orderings of Chapter 5 are satisfied, which will amount to proving the following lemma:

Lemma 8.2.2. *Assuming that* λ_0 *is appropriately large then we have the following parameter inequalities*

$$\frac{\lambda_q}{\lambda_{q+1}} \le \lambda_q \ell \le \lambda_q^{3\beta_{(j-1)+}}\lambda_{q+1}^{-3\beta_j} \ll 1 \tag{8.2.9}$$

$$\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{\beta_j - 1} \le \lambda_q^{1-\beta_{(j-1)+}}\tilde{\mu}_{q+1,j}^{-1} \le \lambda_{q+1}^{-\varepsilon_0} \tag{8.2.10}$$

$$\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{1-\beta_j} \le \tilde{\mu}_{q+1,j}^2 \tag{8.2.11}$$

$$\lambda_q^{-\beta_0}\tilde{\mu}_{q+1,0}^{-1} \ge \lambda_q^{-\beta_{(j-1)+}}\tilde{\mu}_{q+1,j}^{-1}\,. \tag{8.2.12}$$

In particular, the parameter orderings (5.1.8) *and* (5.2.26) *are satisfied.*

Proof. For $j = 0$, (8.2.9) follows from the restrictions (8.2.6) and (8.2.8). Similarly, for $j \geq 1$ the inequality (8.2.9) follows from (8.1.4), (8.2.6) and (8.2.8).

Now consider (8.2.10). For $j \geq 2$ we have $\tilde{\mu}_{q+1,j} = \lambda_{q+1}^{1-\beta_j}$ and thus we just need to check $\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{\beta_j-1} \leq \lambda_{q+1}^{-\varepsilon_0}$. Using (8.1.4), (8.2.6) and (8.2.8) we obtain the required inequality. For $j = 0, 1$ we must show

$$\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{\beta_j-1} \leq \lambda_q^{(1-\beta_0)/2}\lambda_{q+1}^{(\beta_0+\beta_\infty-1)/2}\lambda_{q+2}^{-\beta_\infty/2} \leq \lambda_{q+1}^{-\varepsilon_0}.$$

Applying (8.1.4) we have $\lambda_q^{1-\beta_{j-1}}\lambda_{q+1}^{\beta_j-1} \leq 2\lambda_q\lambda_{q+1}^{-1-\beta_\infty}\lambda_{q+2}^{\beta_\infty}$. Then from (8.2.6) we easily obtain the first inequality. Taking logarithms and dividing by $b^q \ln \lambda_0$ the second inequality is equivalent to showing

$$(1-b)(1-\beta_0+b\beta_\infty) \leq -2b\varepsilon_0 - O\left(\frac{1}{b^q \ln \lambda_0}\right), \tag{8.2.13}$$

which follows from (8.2.6) and (8.2.8)

Consider (8.2.11) for $j \geq 2$: the inequality follows as a simple consequence of the fact that $\lambda_q^{1-\beta_{j-1}} \overset{(8.2.9)}{\leq} \lambda_{q+1}^{1-\beta_j}$. The case for $j = 0$ is clearly stronger than the case for $j = 1$. For $j = 0$, by definition of $\tilde{\mu}_{q+1,0}$ we have $\tilde{\mu}_{q+1,0}^{-2} \leq \lambda_q^{\beta_0+\beta_\infty-1}\lambda_{q+1}^{\beta_0-\beta_\infty-1}$ and thus

$$\lambda_q^{1-\beta_{-1}}\lambda_{q+1}^{1-\beta_0}\tilde{\mu}_{q+1,0}^{-2} \leq C\left(\lambda_q^{-\beta_{-1}+\beta_\infty}\lambda_{q+1}^{\beta_0-\beta_\infty}\right)\lambda_q^{\beta_0}\lambda_{q+1}^{-\beta_0} \overset{(8.1.4)}{\leq} C\lambda_q^{\beta_0}\lambda_{q+1}^{-\beta_0} \ll 1.$$

Hence we obtain (8.2.11).

Finally consider (8.2.12). For $j = 0, 1$ the inequalities are trivial: assume then $j \geq 2$ and apply (8.1.4) to deduce

$$\frac{\lambda_q^{-\beta_{j-1}}}{\tilde{\mu}_{q+1,j}} \leq C\lambda_q^{-\beta_\infty}\lambda_{q+1}^{\beta_\infty-1} \leq C\lambda_q^{-\beta_0}\tilde{\mu}_{q+1,0}^{-1}\underbrace{\lambda_q^{(1+\beta_0)/2}\lambda_{q+1}^{-(1+\beta_0)/2}\lambda_{q+1}^{-3\beta_\infty/2}\lambda_{q+2}^{3\beta_\infty/2}}_{I}.$$

Then from (8.2.6) applied to I we obtain our claim. □

Recall that in Section 8.1 we imposed the addition requirement that the family of parameters $\tilde{\mu}_{q+1,j}$ are monotonically decreasing in j: inequality (8.1.7). By inspection it suffices to check the inequality holds for $j = 0, 1$ and $j' = 2$. Taking logarithms and dividing by $b^q \ln \lambda_q$, we need to show

$$
\begin{aligned}
0 \geq & b - b\beta_2 - \frac{(1-\beta_0)(b+1) + b(b-1)\beta_\infty}{2} + O\left(\frac{1}{b^q \ln \lambda_0}\right) \\
\overset{(8.1.2)}{=} & \frac{(b-1)(b+\beta_0(b+2) - (b^2+2b+2)\beta_\infty)}{2b} + O\left(\frac{1}{b^q \ln \lambda_0}\right) .
\end{aligned}
$$

Applying (8.2.7) then yields (8.1.7).

It remains to check that that the new triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$, together with the family of intervals $\{I_a^{(q+1)}\}_{a \in \{1,\dots,N(q+1)\}}$, described in Chapter 4 and Section 8.1, satisfy the claimed properties.

Observe that (8.1.11),(8.1.12), (8.2.1) and (8.2.2) with q replaced by $q+1$ all follow as a consequence of Lemma 5.2.4.

To show (8.2.3) and (8.2.4) we will use the following inequality

$$
\lambda_{q+1}^{1-\beta_{-1}} \geq \tilde{\mu}_{q+1,j} \tilde{\eta}_{q+1,j}^{-1} . \tag{8.2.14}
$$

The estimates (8.2.3) and (8.2.4) will then follow as a consequence of Lemmas 5.2.4 and 5.2.5. To prove (8.2.14) we note that for $j \geq 2$ we have equality and consequently the case $j = 0$ follows from (8.1.17). Hence it suffices to consider the case $j = 1$, which is equivalent to showing

$$
\begin{aligned}
0 \geq & \frac{(1+b)(1-\beta_0) + 3\beta_\infty(b^2-b)}{2} + b\beta_1 - b^2\beta_0 + b(\beta_{-1} - 1) + O\left(\frac{1}{b^q \ln \lambda_0}\right) \\
\overset{(8.1.2)\&(8.1.3)}{=} & \frac{(b-1)(-1-\beta_0+(b+2)\beta_\infty)}{2} + O\left(\frac{1}{b^q \ln \lambda_0}\right) .
\end{aligned}
$$

Applying (8.2.6) and assuming λ_0 to be sufficiently large we obtain (8.2.14).

Now consider the estimates (8.1.13) and (8.1.14) with q replaced by $q+1$. Recall from Proposition 6.1.1 that if $t \in K_\varsigma^g$ or alternatively $t \in K_\varsigma \cap K_{\varsigma'}$ for $\varsigma' = \varsigma \pm 1$ and $\beta^{(q+1)}(\varsigma) = \beta_j \leq \beta^{(q+1)}(\varsigma')$ then the following estimates hold:

$$
\begin{aligned}
& \|\mathring{R}_{q+1}(t)\|_0 + \frac{1}{\lambda_{q+1}} \|\mathring{R}_{q+1}(t)\|_1 + \frac{1}{\lambda_{q+1}^2} \|\mathring{R}_{q+1}(t)\|_2 \leq C \frac{\tilde{\mu}_{q+1,j} \lambda_{q+1}^{\varepsilon_0}}{\tilde{\eta}_{q+1,j} \lambda_{q+1}^{1+\beta_j}} \\
& \|\partial_t \mathring{R}_{q+1}(t) + v_{q+1} \cdot \nabla \mathring{R}_{q+1}(t)\|_0 \leq C \frac{\tilde{\mu}_{q+1,j} \lambda_{q+1}^{\varepsilon_0}}{\tilde{\eta}_{q+1,j} \lambda_{q+1}^{\beta_j + \beta_{-1}}} ,
\end{aligned}
$$

where for the last inequality we applied (8.2.14) to eliminate the prefactor $\delta_{q+1,\varsigma}^{1/2} \lambda_{q+1} +$

$\frac{\mu_{q+1,\varsigma}}{\eta_{q+1,\varsigma}}$. Moreover, if $t \in K_\varsigma^g$ then we have

$$\|\mathring{R}_{q+1}(t)\|_0 + \frac{1}{\lambda_{q+1}}\|\mathring{R}_{q+1}(t)\|_1 + \frac{1}{\lambda_{q+1}^2}\|\mathring{R}_{q+1}(t)\|_2 \leq C\frac{\lambda_q\lambda_{q+1}^{\varepsilon_0}}{\tilde{\mu}_{q+1,j}\lambda_q^{\beta_{(j-1)+}}\lambda_{q+1}^{2\beta_j}}$$

$$\|\partial_t\mathring{R}_{q+1}(t) + v_{q+1}\cdot\nabla\mathring{R}_{q+1}(t)\|_0 \leq C\frac{\lambda_q\lambda_{q+1}^{1+\varepsilon_0}}{\tilde{\mu}_{q+1,j}\lambda_q^{\beta_{(j-1)+}}\lambda_{q+1}^{3\beta_j}}\,.$$

Therefore by inspection, the estimates (8.1.13) and (8.1.14) for q replaced by $q+1$ follow by the parameter orderings proved in the following lemma:

Lemma 8.2.3. *According to our choice of the parameters we have*

$$\lambda_q^{1-\beta_{(j-1)+}}\lambda_{q+1}^{-2\beta_j}\tilde{\mu}_{q+1,j}^{-1} \leq \lambda_{q+2}^{-2\beta_{j+1}}\lambda_{q+1}^{-2\varepsilon_0} \tag{8.2.15}$$

$$\lambda_{q+1}^{-\beta_j-1}\tilde{\mu}_{q+1,j}\tilde{\eta}_{q+1,j}^{-1} \leq \lambda_{q+2}^{-2\beta_0}\lambda_{q+1}^{-2\varepsilon_0}\,. \tag{8.2.16}$$

Proof. Consider the inequality (8.2.15) for $j = 0, 1$. Taking logarithms and dividing by $b^q \ln\lambda_0$, the inequality is equivalent to showing

$$\begin{aligned}0 \geq &-\beta_0 + 1 - 2\beta_j b - (1-\beta_0)\frac{b+1}{2} - b(b-1)\frac{\beta_\infty}{2} + 2\beta_{j+1}b^2 + 2b\varepsilon_0 \\ &+ O\left(\frac{1}{b^q\ln\lambda_0}\right) \\ =&\frac{b-1}{2}(-1+\beta_0+3b\beta_\infty) + 2b\varepsilon_0 + O\left(\frac{1}{b^q\ln\lambda_0}\right)\,,\end{aligned}$$

from which applying (8.2.6) and (8.2.8) the inequality (8.2.15) readily follows.

Next, consider the case $j \geq 2$, taking logarithms and dividing by $b^q\ln\lambda_0$ the inequality is equivalent to showing

$$\begin{aligned}0 &\geq 1 - \beta_{j-1} - b(1+\beta_j) + 2\beta_{j+1}b^2 + 2b\varepsilon_0 + O\left(\frac{1}{b^q\ln\lambda_0}\right) \\ &= (b-1)(-1+(2b+1)\beta_\infty) + 2b\varepsilon_0 + O\left(\frac{1}{b^q\ln\lambda_0}\right)\,,\end{aligned}$$

and thus the desired estimate is implied by (8.2.6) and (8.2.8)

To prove (8.2.16), we first note that since $\tilde{\mu}_{q+1,j} \leq \tilde{\mu}_{q+1,0} = \tilde{\mu}_{q+1,1}$ and $\delta_{q+1,j}^{1/2}\tilde{\eta}_{q+1,j}^{-1}$

is constant in j, it suffices to consider the case for $j = 0, 1$. In particular we need to show

$$\begin{aligned} 0 &\geq \frac{(1+b)(1-\beta_0) + 3\beta_\infty\left(b^2 - b\right)}{2} - b + b^2\beta_0 + 2b\varepsilon_0 + O\left(\frac{1}{b^q \ln \lambda_0}\right) \\ &= \frac{(b-1)(-1 + (2b+1)\beta_0) + 3b\beta_\infty)}{2} + 2b\varepsilon_0 + O\left(\frac{1}{b^q \ln \lambda_0}\right) , \end{aligned}$$

for which again we apply (8.2.6) and (8.2.8) to conclude (8.2.16). □

In order to conclude the proof of Proposition 8.2.1, we need to show that the family of intervals $I_\alpha^{(q+1)}$ and family of sets $V_j^{(q+1)}$ satisfy the constraints (8.1.5) and (8.1.15) respectively with $q+1$ replacing q.

Consider first (8.1.5) for intervals $I_\alpha^{(q+1)} = K_\varsigma^b$ for some ς, observe that

$$\left|K_\varsigma^b\right| \overset{(8.2.14)}{\geq} \lambda_q^{-1+\beta_{-1}} \geq \lambda_q^{(-1+\beta_0)/2}\lambda_{q+1}^{(-1+\beta_0)/2} \overset{(8.2.11)}{\geq} \tilde{\mu}_{q+1,0}^{-1} ,$$

from which — assuming that λ_0 is taken large enough — we obtain (8.1.5) on bad sets. For good sets, i.e. $I_\alpha^{(q+1)} = K_\varsigma^g$ for some ς, we have by construction

$$K_\varsigma^g - K_\varsigma^b - K_{\varsigma+1}^b \overset{(8.1.17)}{\geq} \frac{2}{\tilde{\mu}_{q+1}} - \frac{2\tilde{\eta}_{q+1,0}}{\tilde{\mu}_{q+1,0}} \overset{(8.1.7)}{\geq} \frac{1}{\tilde{\mu}_{q+1,j}} .$$

For $j = 0$, since $\tilde{\mu}_{q+2,0} = \tilde{\mu}_{q+2,1}$ the inequality follows by assuming λ_0 to be sufficiently large such that $\mu_{q+1,0}^{-1}\mu_{q+2,0} \geq 4$. For $j \geq 1$, we apply (8.2.10) twice to obtain

$$\frac{1}{\tilde{\mu}_{q+1,j}} \geq \lambda_{q+1}^{\beta_j - 1} \geq \frac{\lambda_{q+2}^{\varepsilon_0}}{\tilde{\mu}_{q+2,j+1}} .$$

Hence assuming λ_0 sufficiently large we obtain (8.1.5).

Observe now that $V_{j+1}^{(q+1)} \subset V_j^{(q)}$. The inductive estimate (8.1.15) will then be preserved, for $j \geq 1$, provided

$$\lambda_{q+2}^{\beta_{j+1} - \beta_\infty + \varepsilon_1} \geq \lambda_{q+1}^{\beta_j - \beta_\infty + \varepsilon_1} ,$$

which holds as long as λ_0 is sufficiently large depending on b and ε_1. Finally we have

$$|V_0^{(q+1)}| \;\leq\; \sum_{j=0}^{q} \tilde{\eta}_{q+1,j}|V_j^{(q)}| \leq 2\sum_{i=0}^{q} \lambda_{q+2}^{\beta_0 - \beta_\infty}\lambda_{q+1}^{\beta_\infty - \beta_j}\lambda_{q+1}^{\beta_j - \beta_\infty + \varepsilon_1} \leq 2q\lambda_{q+2}^{\beta_0 - \beta_\infty}\lambda_{q+1}^{\varepsilon_1} .$$

Thus $|V_0^{(q+1)}|$ satisfies (8.1.15) provided λ_0 is chosen sufficiently large enough such that

$\lambda_{q+2}^{\varepsilon_1}\lambda_{q+1}^{-\varepsilon_1} \geq 2q.$ □

8.3 Conclusion of the proof of Theorem 1.2.3

In this section we apply Proposition 8.2.1 in order to conclude our proof of Theorem 1.2.3.

We begin by fixing postive parameters θ, β_∞ and β_0 such that

$$\frac{1}{5} + \beta_0 < \theta < \beta_\infty < \frac{1}{3} - \beta_0 \,.$$

We set our initial triple as $(v_0, p_0, \mathring{R}_0) = ((0,0,0), 0, (0,0,0) \otimes (0,0,0))$, our initial family of intervals will consist of one element: $I_0^{(0)} = [0, T]$ with corresponding exponent $\beta^{(0)}(0) = \beta_0$. The triple $(v_0, p_0, \mathring{R}_0)$ together with the singleton set $\{I_0^{(0)}\}$ trivially satisfy the constraints of Section 8.1. To construct $(v_1, p_1, \mathring{R}_1)$ and the family $\{I_a^{(1)}\}$ we apply the same method presented in Chapter 4 and Section 8.1 with the exception that we define the amplitude parameters ρ_ς as follows:

$$\rho_\varsigma := \frac{1}{3(2\pi)^3} e(t_\varsigma) \,.$$

Taking into account this minor modification we may apply Proposition 8.2.1 in order to obtain our new triple $(v_2, p_2, \mathring{R}_2)$ satisfying *all* the requirements of Section 8.1. We now apply Proposition 8.2.1 inductively to obtain a sequence of triples $(v_q, p_q, \mathring{R}_q)$. From (8.2.1)-(8.2.4) and interpolation we see the sequence converges solution (v, p) to the Euler equations (1.1.1). Furthermore we have $v \in C^{\theta'}(\mathbb{T}^3 \times [0, T])$, $p \in C^{2\theta'}(\mathbb{T}^3 \times [0, T])$ for any $\theta' < \beta_0$.

Utilising (8.1.15) and calculating we have:

$$\begin{aligned}
\int_0^1 [v(t)]_\theta \, dt &\leq \sum_{q=1}^{\infty} \int_0^T [w_q(t)]_\theta \, dt \\
&\leq \sum_{q=0}^{\infty} \int_0^T \left\|w_q(t)\right\|_0^{1-\theta} \left\|w_q(t)\right\|_1^{\theta} \, dt \\
&\leq C \sum_{q=0}^{\infty} \sum_{j=0}^{q} \left|V_j^{(q)}\right| \lambda_q^{\theta-\beta_{j-1}} \\
&\leq C\lambda_0 \sum_{q=0}^{\infty} \sum_{j=0}^{q} \lambda_{q+1}^{\beta_j-\beta_\infty+\varepsilon_1} \lambda_q^{\theta-\beta_j}
\end{aligned}$$

$$\overset{(8.1.4)}{\leq} C\lambda_0 \sum_{q=0}^{\infty} (q+1)\lambda_q^{\theta-\beta_\infty+b\varepsilon_1}$$
$$\leq C\lambda_0 \,,$$

where in the last inequality we assume ε_1 is chosen small enough such that we have $b\varepsilon_1 < \beta_\infty - \theta$. An analogous calculation yields $p \in L^1([0,T]; C^{2\theta}(\mathbb{T}^3))$.

It remains to check the energy inequalities (2.4.3) and (2.4.4) are satisfied. From the definition of the cut-off functions χ_ς we have

$$\left| e(t) - \int_{\mathbb{T}^3} |v_1(x,t)|^2 \, dx \right| \leq C\tilde{\mu}_{1,0}^{-1} \leq C\lambda_2^{-2\beta_0} \,.$$

Assuming λ_0 is sufficiently large, then the right hand side can be made as small as desired in order to obtain (2.4.3). Moreover, since

$$\sum_{q=2}^{\infty} \int_{\mathbb{T}^3} |w_q(x,t)|^2 \, dx \leq C\lambda_q^{-2\beta_0} \leq C\lambda_2^{-2\beta_0} \,,$$

we obtain (2.4.4) by assuming λ_0 to be sufficiently large.

8.4 References and Remarks

The arguments of this chapter are based on the the work [BDLS14]. The parameters $(\mu_{q+1,j}, \eta_{q+1,j})$ of [BDLS14] correspond directly with the parameters $(\tilde{\mu}_{q+1,j}, \tilde{\eta}_{q+1,j})$ employed here.

The bookkeeping system of [BDLS14], which is presented here, is significantly more complex than the one originally presented in [Buc13]. We recall that [Buc13] describes the construction of non-trivial non-conservative $1/5 - \varepsilon$ Hölder continuous solutions which for almost every time belong to the $1/3 - \varepsilon$ Hölder regularity class. We now provide brief sketch of the arguments of [Buc13], written in the language of this dissertation:

As was done in Chapter 7, in [Buc13] the parameters $\mu_{q+1,\varsigma} := \mu_{q+1}$ and $\eta_{q+1,\varsigma} := \eta_{q+1}$ are chosen uniformly in ς. Indeed μ_{q+1} is chosen in the same manner described in Proposition 7.2.1; although in contrast to the approach taken in Chapter 7, the parameter η_{q+1} is chosen to be $\lambda_{q+1}^{-\varepsilon_2}$, for small suitably small parameter $\varepsilon_2 > 0$. In the notation of the bookkeeping system presented in Section 8.1 (and not that of [Buc13]) the regularity exponents β_j for $j \geq 1$ are chosen as follows:

$$\beta_j = \min\left(\beta_{j-1} + \frac{(b-1)^2}{100}, \beta_\infty\right).$$

The parameter β_0 is chosen in a similar manner as was the parameter β was chosen in Proposition 7.2.1, i.e. satisfying $\beta_0 < \frac{1}{5b}$. In addition, β_∞ is chosen such that $\beta_\infty < \frac{1}{3b}$ and $\beta_{-1} := \beta_0$. Then assuming ε_2 is chosen suitably small, by applying similar arguments to those given in Chapter 7, one can ensure the scheme converges *uniformly* in $C^{1/3-\varepsilon}(\mathbb{T}^3 \times [0,T])$.

Observe that by definition, there exists a finite integer N depending on b, β_0 and β_∞ such that for $j \geq N$ we have $\beta_j = \beta_\infty$. Hence defining

$$V_\infty^{(q)} = \bigcup_{\{a:\beta_j=\beta^{(q)}(a), j\geq N\}} I_a^{(q)}, \tag{8.4.1}$$

it is not difficulty to see that by the choice to η_{q+1} we have

$$\left|V_\infty^{(q)}\right| \geq T - C\sum_{q'=q-N}^{q} \lambda_{q'}^{-\varepsilon_2} \geq T - C\lambda_{q-N}^{-\varepsilon_2},$$

from which we infer

$$\lim_{q\to\infty}\left|\bigcap_{q'=q}^{\infty} V_\infty^{(q')}\right| \geq T - \lim_{q\to\infty} C\sum_{q'=q}^{\infty}\lambda_{q'-N}^{-\varepsilon_2} \geq T - C\lim_{q\to\infty}\lambda_{q-N}^{-\varepsilon_2} = T.$$

Hence, applying interpolation, for a.e. time $t \in [0,T]$, our constructed weak solution v is Hölder $1/3 - \varepsilon$ continuous. Indeed, as was pointed out in [Buc13], the set of times where v is not Hölder $1/3 - \varepsilon$ continuous is of Hausdorff dimension strictly less than 1.

A
Appendix

A.1 Hölder spaces

In this section we will introduce the standard *(spatial)* Hölder norms and seminorms. In what follows we let $m = 0, 1, 2, \ldots$, $\alpha \in (0, 1)$, and β be a multi-index. The standard supremum norm will be denoted by $\|f\|_0 := \sup_{(x,t)\in(\mathbb{T}^3\times[0,T])} |f(x,t)|$ and then we define the Hölder seminorms as

$$[f]_m = \max_{|\beta|=m} \|D^\beta f\|_0 \,,$$
$$[f]_{m+\alpha} = \max_{|\beta|=m} \sup_{x\neq y,t} \frac{|D^\beta f(x,t) - D^\beta f(y,t)|}{|x-y|^\alpha} \,,$$

where D^β are *space derivatives* only. The Hölder norms are then given by

$$\begin{aligned} \|f\|_m &= \sum_{j=0}^m [f]_j \\ \|f\|_{m+\alpha} &= \|f\|_m + [f]_{m+\alpha} \,. \end{aligned}$$

We also employ the above notation for functions in space only. For the analogous norms and seminorms defined on the Euclidean space $\mathbb{R}^3$ and the scaled torus $\lambda\mathbb{T}^3$, for

$\lambda > 0$ we will employ the notation $\|\cdot\|_{C^r(\mathbb{R})}$, $[\cdot]_{\dot{C}^r(\mathbb{R})}$, $\|\cdot\|_{C^r(\lambda\mathbb{T}^3)}$ and $[\cdot]_{\dot{C}^r(\lambda\mathbb{T}^3)}$ respectively.

For brevity, given a fixed time t, we write $[f(t)]_r$ and $\|f(t)\|_r$ to denote the semi-norm/norm of f evaluated for the restriction of f to the t-time slice.

Observe that since

$$[f]_{\dot{C}^s(\lambda\mathbb{T}^3)} \leq C(\|f\|_{C(\lambda\mathbb{T}^3)} + \|f\|_{\dot{C}^r(\lambda\mathbb{T}^3)}) ,$$

for any $0 < s < r$, and through homogeneity we have

$$[f]_{\dot{C}^s(\mathbb{T}^3)} = \varepsilon^{-s}[f(\varepsilon\cdot)]_{\dot{C}^s(\varepsilon^{-1}\mathbb{T}^3)} ,$$

then we obtain

$$[f]_s \leq C\big(\varepsilon^{r-s}[f]_r + \varepsilon^{-s}\|f\|_0\big) , \tag{A.1.1}$$

for $r \geq s \geq 0$, $\varepsilon > 0$. Setting $\varepsilon = \|f\|_0^{\frac{1}{r}}[f]_r^{-\frac{1}{r}}$ we obtain the standard interpolation inequalities

$$[f]_s \leq C\|f\|_0^{1-\frac{s}{r}}[f]_r^{\frac{s}{r}} . \tag{A.1.2}$$

Applying Young's inequality yields the following product estimate

$$[fg]_r \leq C\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) , \tag{A.1.3}$$

for any $r \geq 0$.

Finally, we state a classical estimate related to the Hölder norms of compositions.

Proposition A.1.1. *Let $\Psi : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^n \to \Omega$ be two smooth functions, with $\Omega \subset \mathbb{R}^N$. Then, for every $m \in \mathbb{N}$ there is a constant C (depending only on m, N and n) such that*

$$[\Psi \circ u]_m \leq C([\Psi]_1\|Du\|_{m-1} + \|D\Psi\|_{m-1}\|u\|_0^{m-1}\|u\|_m) \tag{A.1.4}$$

$$[\Psi \circ u]_m \leq C([\Psi]_1\|Du\|_{m-1} + \|D\Psi\|_{m-1}[u]_1^m) . \tag{A.1.5}$$

The proof of Proposition A.1.1 follows by a simple expansion of the the derivatives using the chain and product rule; then applying (A.1.2) to the resulting terms.

A.2 Linear Partial Differential Equation Theory

In this section we will recall a number of elementary results from linear partial differential equation theory on $\mathbb{R}^n$. For proofs of stated results, we refer the reader to [GT01].

We let $\|\cdot\|_{W^{m,p}}$ denote the usual Sobolev norm $\|\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta|=m} \|D^\beta\|_{L^p(\mathbb{R}^n)}$ for $1 \leq p \leq \infty$ and integers m. We also denote $\|\cdot\|_{W^{s,p}} = \left\|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}(\cdot)\right\|_{L^p(\mathbb{R}^n)}$ the canonical extension to non-integers s, where here $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the usual Fourier transform and Fourier inversion respectively.

We first recall a standard Sobolev inequality:

Lemma A.2.1. *Assume $1 < p \leq \infty$ and $s > 0$ is such that $1/p < s/n$. Then for any smooth function u on $\mathbb{R}^n$, there exists a constant C depending only on n, p and s such that*

$$\|u\|_{C(\mathbb{R}^n)} \leq C\,\|u\|_{W^{s,p}(\mathbb{R}^n)} \ . \tag{A.2.1}$$

We we now restate some properties of Riesz operators and the Leray projection operator. First observe that if f is a smooth function with compact support defined on $\mathbb{R}^n$, $n \geq 3$, defining

$$u := \Delta_{\mathbb{R}^n}^{-1} f := -\frac{1}{n(n-2)\omega(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy,$$

where $\omega(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Then u is a smooth function on $\mathbb{R}^n$ satisfying the Poisson equation

$$\Delta u = f.$$

The Riesz transform $R_j f$ for $j = 1, \ldots, n$ is then defined by the formula

$$R_j f(x) = \text{p.v.}\frac{1}{\pi\omega(n-1)} \int_{\mathbb{R}^n} \frac{(y_j - x_j) f(y)}{|x-y|^{n+1}} dy.$$

For $n \geq 3$ the Riesz transform and $\Delta_{\mathbb{R}^n}^{-1}$ can be related by the following formula

$$\partial_{x_i}\partial_{x_j}\Delta_{\mathbb{R}^n}^{-1} f = -R_i R_j f\,. \tag{A.2.2}$$

Moreover, we can write the standard Leray projection operator $\mathcal{P}_{\mathbb{R}^n}$, which projects vector fields onto its zero-divergence part as

$$(\mathcal{P}_{\mathbb{R}^n} f)_j := f_j - \sum_{k=1,\ldots,n} R_j R_k f_k\,. \tag{A.2.3}$$

A standard result from Harmonic Analysis is that the Riesz operators are bounded on L^p for $1 < p < \infty$:

Lemma A.2.2. *For* $1 < p < \infty$ *and* f *a smooth function on* $\mathbb{R}^n$ *for* $n \geq 2$ *we have*

$$\left\|R_j f\right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} , \tag{A.2.4}$$

where the constant C *depends on* p *and* n.

Finally, we state a well-known estimate on harmonic functions:

Lemma A.2.3. *Suppose* f *is a harmonic function* $(\Delta f \equiv 0)$ *on a bounded Lipschitz domain* $U \subset \mathbb{R}^n$ *then we have the following estimates on the derivatives of* f

$$\left\|D^\beta f\right\|_{L^\infty(U')} \leq C \|f\|_{L^1(U)} \tag{A.2.5}$$

for U' *compactly contained in* U, *where the constant* C *depends only on* n, *the* $|\beta|$, U *and* U'.

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