Equivariant Differential Cohomology

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1 Introduction

The interplay between geometry and topology is a widely occurring theme in modern mathematics, whose most elementary appearance is the formula of *Hopf's Umlaufsatz*: Let $c\colon [0,a] \to \mathbb{R}^2$ be a closed smooth curve in the plane. Then the winding number of the curve is given by the integral over the curvature:

$$n_c = \frac{1}{2\pi} \int_0^a \kappa(t) \|c'(t)\| dt.$$

This result is surprising: The quantity on the left-hand side is an integer and purely topological – vividly speaking this means: it does not depend on small alterations of the curve. Whereas, on the right-hand side, one integrates a real-valued function, which does depend on the geometry – how long the curve is and how strongly it is curved.

A first generalization of the Umlaufsatz is known as the Gauss-Bonnet theorem, which states that for any compact surface M of genus g in $\mathbb{R}^3$:

$$2(g-1) = \frac{1}{2\pi} \int_M \kappa,$$

where now κ denotes the Gaussian curvature of the surface.

The generalizations of these statements by characteristic classes are based on *de Rham cohomology*: The differential forms on a smooth manifold form a chain complex, which depends on the geometry of the space, but the cohomology of this chain complex is isomorphic to any real cohomology theory, e.g., to singular cohomology with real coefficients. This means that any real cohomology class – a topological object – can be represented by a closed differential form, a geometric object.

In these terms, the left-hand side of the equations above will be generalized by the image of an *integral* cohomology class in real cohomology; the curvature on the right-hand side will be replaced by a closed differential form (depending on the curvature) and the integral will be expressed by taking the cohomology class of this form.

In general, characteristic classes associate cohomology classes to (isomorphism classes of) vector bundles. For smooth bundles, there are two well-known procedures to construct them, one which applies the geometric structure and one which uses topology only:

The Chern-Weil-Homomorphism starts with a connection on the bundle and evaluates an invariant symmetric polynomial on the associated curvature form, which leads to a closed differential form, the characteristic form. As the difference of the characteristic forms of two connections is an exact form – the exterior derivative of the transgression form – one gets a class in de Rham cohomology which is independent of the chosen connection and called the characteristic class of the bundle.

On the other hand one may also obtain these classes by pulling back universal characteristic classes via the classifying map of the bundle.

Both construction have their own strengths: The characteristic form contains geometric data, while the class is purely topological. The class itself actually is not an element in real, but in integral cohomology, where algebraic torsion may deliver finer information, which cannot be reflected by the characteristic form, as there is no algebraic torsion over the field of real or complex numbers.

To use both, the geometric information of the characteristic form and its transgression and the algebraic torsion information from integral cohomology in one object, one defines differential cohomology and differentially refined characteristic classes. This was done first by Jeff Cheeger and James Simons in [14]. The differential cohomology theory extends integral cohomology by closed differential forms. A notable result is that – while the classical first Chern class classifies complex line bundles up to isomorphism – the first differential Chern class classifies complex line bundles with connection up to isomorphism.

From this starting point there are various ideas of differential refinements of cohomology theories: Besides the differential characters of Cheeger and Simons, there is an isomorphic model by smooth Deligne cohomology (see [8, 9]). On the other hand there are various models for differential K-theory (see [11] for a survey, which includes a discussion of the literature). A general framework for these differential refinements is given in [12] and [10].

We want to go back to the starting point and generalize the idea of the differential refinement to an equivariant setting, i.e., we have a Lie group G acting on a smooth manifold M and ask for characteristic classes of equivariant bundles over M. Therefore we need a differential form model for equivariant cohomology, which is capable to receive a homomorphism from integral cohomology. Moreover, there should be two constructions of real/complex equivariant characteristic classes, one via equivariant characteristic forms and one via integral equivariant characteristic classes, which should coincide under the homomorphism between the cohomology theories.

The construction of the differential refinement, which we will give, is an equivariant version of smooth Deligne cohomology, but to stress that it fits into the picture of differential refinements, we will use the term equivariant differential cohomology, even if we will not discuss equivariant differential refinements in general.

Equivariant cohomology and simplicial manifolds

Defining equivariant cohomology $H^*_G(M)$ is a simple business using two expected properties of this functor: homotopy invariance and that, for free actions, the equivariant cohomology should coincide with the cohomology of the quotient. Namely, let EG be a contractible space with a free G-action, then the diagonal action of G on $EG \times M$ is free and the map $EG \times M \to \{*\} \times M$ is a homotopy equivalence. Hence, we have described the well-known Borel construction, which is in formulas

$$H_G^*(M) = H_G^*(\{*\} \times M) = H_G^*(EG \times M) = H^*(EG \times_G M),$$

for any cohomology theory and any coefficient group, e.g. singular cohomology with values in $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$. Here $EG \times_G M$ is the quotient of the diagonal G-action on $EG \times M$.

As short and easy this construction is, it creates a task for us: EG is even in simple cases not a finite-dimensional manifold, hence we have no de Rham cohomology. But EG is something similar to a manifold: Namely there is a simplicial manifold ([16, 19]), i.e., a simplicial set such that the set of p-simplices forms a smooth manifold for each p and all face and degeneracy maps are smooth, and the geometric realization of this simplicial manifold is $EG \times_G M$. This will be introduced in Section 3.1.1 and we will explain how one defines (simplicial) differential forms on a simplicial manifold. They lead to a complex, which is bi-graded: by the form degree and the simplicial degree. The cohomology of this double complex calculates equivariant complex cohomology. In fact, simplicial differential forms also form a (graded) simplicial sheaf $\Omega_{\mathbb{C}}^{\bullet;*}$.

Using the language of simplicial sheaf cohomology, the de Rham homomorphism is induced by the inclusion of the locally constant simplicial sheaf $\underline{\mathbb{Z}} \to \Omega_{\mathbb{C}}^{\bullet;*}$, as locally constant functions.

In Section 3.1.2, we will introduce the reader to a more famous model of equivariant cohomology using differential forms, known as the Cartan model. This is given by the so-called equivariant differential forms, i.e., equivariant polynomial maps $\mathfrak{g} \to \Omega^*(M)$, where the differential d_C on $(\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$ is given by

$$(d_C\omega)(X) = d(\omega(X)) + \iota_{X^\sharp}(\omega(X)),$$

i.e. the sum of the exterior differential and the contraction with the fundamental vector field of X, and hence increases the grading given through

$$\text{twice the polynomial degree } + \text{ the differential form degree}$$

by one.

The Cartan model has the advantage that its cochain complex is substantially slimmer than the double complex $\Omega^{\bullet,*}$ defined above, but it is not directly capable to receive a homomorphism from integral cohomology. Therefore we apply ideas of [23] to compare the different models of equivariant cohomology. This comparison will enable our construction of a differential refinement of equivariant integral cohomology.

Equivariant characteristic classes and forms

After this introduction to equivariant cohomology, we will discuss equivariant characteristic classes and forms in Section 3.2.

Let G act on the vector bundle $E \to M$, i.e., we have an action on the total space and the base space, such that the projection is equivariant. Via the Borel construction, one can define equivariant characteristic classes easily: Take the usual characteristic classes of $EG \times_G E \to EG \times_G M$!

There is also a characteristic form construction (see [3]) which does not only depend on the curvature, but also uses the moment map μ^∇ of the connection ∇. This is a map from the Lie algebra of the acting group to the endomorphisms of the vector bundle (see Definition 3.59 for details). In this way, one obtains an equivariant characteristic form, which is a closed equivariant differential form, i.e., an element in the Cartan complex.

Our first important result is to prove that both constructions lead to the same class in equivariant complex Borel cohomology (see Section 3.2). This was, on the one hand, only shown for special cases (compare [5]) and, on the other hand, stated in general in [23] as a citation, but the original work of Johan Dupont, which is cited, only proves the correctness of the construction in the non-equivariant setting. Nevertheless, the methods of Dupont can be generalized to solve the equivariant problem.

Equivariant differential cohomology

After we have achieved this understanding of equivariant characteristic forms, we can review previous definitions critically to obtain a more satisfactory one.

There is a definition of equivariant smooth Deligne cohomology $\hat{H}^*_G(M,\mathbb{Z})$ in [25], and Kiyonori Gomi shows there that $\hat{H}^2_G(M,\mathbb{Z})$ classifies G-equivariant line bundles with connection. We will show that his definition fits, for actions of compact groups, into a differential cohomology hexagon (Theorem 4.15) and thus can be interpreted as a model for equivariant differential cohomology. But this definition neglects the secondary information of the moment map and is, thus, only satisfactory in the case of finite groups, where there is no moment and in low degrees, where the moment map does not play a role. There are also other, less elaborated, definitions (see Remark 4.30), which are all unsatisfactory from our insight to characteristic forms.

Therefore, in Section 4.1.3, we define (full) equivariant differential cohomology $\widehat{\mathbb{H}}^*_G(M,\mathbb{Z})$ (using a mapping cone construction similar to the non-equivariant case in [9]) and show (see Theorem 4.23) that for any compact Lie group G, one has the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & \Omega^{n-1}_G(M)/(d+\iota)\Omega^{n-2}_G(M) \arrow[rr, "d+\iota"] \arrow[dr, "a"] & & \Omega^n_G(M)_{\mathrm{cl}} \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \widehat{\mathbb{H}}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n_G(M,\mathbb{C}) \\
 & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur, hook] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

where the line along the top, the one along the bottom and the diagonals are exact.

In the case of the trivial group one obtains the classical differential cohomology. In degree up to two, our definition coincides with the one of Gomi. In higher degrees one has additional geometric data, e.g., in the case of the conjugation action of $S^3 = SU(2)$ on itself, as discussed in Section 4.3.3, one has $\hat{H}^4_{S^3}(S^3,\mathbb{Z}) = H^3_{S^3}(S^3,\mathbb{C}/\mathbb{Z}) \oplus H^4_{S^3}(S^3,\mathbb{Z}) =$

$\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$, while we have a short exact sequence

$$0 \to {}^{\Omega^1(S^3)^{S^1}} \diagup_{dC^\infty(S^3)^{S^1}} \to \widehat{\mathbb{H}}^4_{S^3}(S^3, \mathbb{Z}) \to \hat{H}^4_{S^3}(S^3, \mathbb{Z}) \to 0,$$

hence we have additional transgression data.

From the hexagon, one concludes that equivariant differential cohomology is the right group to define equivariant differential characteristic classes in, since they can refine both, the equivariant integral characteristic class and the equivariant characteristic form. The details of this constructions are worked out in Section 4.2.

Towards equivariant differential K-theory?

The question of defining equivariant differential K-theory will not be a topic in this thesis. There is a definition in the case of finite groups by Michael Luis Ortiz [40], who remarks the following difficulty: The construction of equivariant characteristic classes as mentioned above uses the map

$$K^0_G(M) \to K^0(EG \times_G M), E \mapsto EG \times_G E,$$

and applies the normal characteristic class to the last object. Ortiz stresses that maybe this construction is not fine enough as it does not reflect the fixed point sets and thus he gives a definition of the equivariant Chern character, which is no longer an element of equivariant (Borel-)cohomology.

Remarks on notations

It might be helpful for the reader to have the following notations in mind:

- Throughout the thesis, M will denote a smooth paracompact manifold.
- Simplicial sets will be marked by $\bullet$, which can occur as sub- and superscript depending on the situation without a change of meaning.
- The sub- or superscript $*$ can denote push-forwards respectively pullbacks or gradings of non-simplicial objects (and simplicial objects, if we only care about the induced chain complex and not about the simplicial structure).
- In general the group acts from the left and will be denoted by G. On principal fiber bundles the fiber group will act from the right and will mostly be denoted by K.
- The general rule for boundary operators is: d denotes the exterior derivative for differential forms, ∂ denotes the boundary of a simplicial set, δ is the boundary of resolutions.

- As the standard example of characteristic classes we have in mind are Chern classes, we work with complex cohomology. Hence differential forms are always complex-valued forms and the dual space of a real vector space V is $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$.
- pr_k will denote the projection to the k-th factor in a Cartesian product.

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2 Foundations

In this chapter we will fix some basic notations and recall some well-known facts.

2.1 Group actions

Let the Lie group G act from the left on the manifold M and denote its Lie algebra by $\mathfrak{g} = T_eG$. There is a list of standard notations (which can be found in [44, Chapter 3] or [17, Chapter 1]) we want to state:

1. G acts on itself by *conjugation* $G \times G \ni (g, h) \mapsto ghg^{-1} \in G$.
2. There is the *exponential map* $\exp\colon \mathfrak{g} \to G$. For a definition see, e.g., [44, Prop. 3.30].
3. The *adjoint action*
$$G \times \mathfrak{g} \ni (g, X) \mapsto \mathrm{Ad}_g(X) \in \mathfrak{g}$$
is the derivative of the conjugation in the second argument at the identity.
4. The *coadjoint action* $\mathrm{Ad}^\vee$ is the dual of the adjoint action on the dual of the Lie algebra, i.e., let $f \in \mathfrak{g}^\vee, X \in \mathfrak{g}$, then $(Ad_g^\vee f)(X) = f(Ad_g X)$.
5. Let $X \in \mathfrak{g}$. The *fundamental vector field* $X^\sharp$ is defined as
$$X^\sharp(m) = \frac{d}{dt}\bigg|_{t=0} (e^{tX} \cdot m) \in T_mM.$$
6. Let N be a second G-space, i.e., a space with G-action, then there is a natural G-action on the space of maps $\{f\colon M \to N\}$ defined as $(gf)(m) = gf(g^{-1}m)$ for any $g \in G$ and $m \in M$.
7. A superscript G marks the subspace of fixed points, i.e., those points x of the G space, such that $gx = x$ for any $g \in G$. In particular, $C^\infty(M, N)^G$ is the space of smooth *equivariant* maps from M to N.

2.2 Geometry

2.2.1 (Equivariant) bundles

Before we are going to recall the definition of bundles in detail, we want to remark the following: An important property of bundles that one would like to have, is that a bundle over $X \times [0, 1]$ is the same as an isomorphism of bundles over the topological space X. This does not hold in general, but there are two ways to obtain this property: restrict your base spaces X to be paracompact or restrict the bundle you investigate to so called 'numerable bundles' (compare [27, Section 4.9.]). We will choose the first option and hence assume, that *all topological spaces, occurring in this thesis, are paracompact*.

First, we will define principal bundles and afterwards we will turn attention to vector bundles.

Definition 2.1 (see, e.g., [17, p.56]) Let G and K be Lie groups. A *G-equivariant principal K-bundle* is a continuous map $\pi\colon E \to B$ of topological spaces, such that the following conditions hold:

1. K acts freely from the right on E and π is K-invariant.
2. for every point $b \in B$ there is a neighborhood U and an K-equivariant homeomorphism $\varphi\colon \pi^{-1}(U) \to U \times K$ over U, i.e., $\pi = \mathrm{pr}_1 \circ \varphi$ and
$$\varphi(xk) = (\mathrm{pr}_1(x), \mathrm{pr}_2(\varphi(x))k)$$
for any $x \in \pi^{-1}(U)$ and $k \in K$.
3. G acts from the left on E and B and π is G-equivariant.
4. The actions of G and K commute.

E is called *total space* and B *base space* of the bundle. Furthermore, the maps φ are called local trivializations.

The bundle is called *smooth*, if E and B are smooth manifolds, π is a smooth map, both actions are smooth and the trivializations φ above can be chosen to be diffeomorphisms.

If G is the trivial group, one omits the attribute equivariant.

Remark 2.2 There is a more general definition of equivariant bundles (compare [30]), where one imposes a stronger condition on the trivializations. This stronger condition can be omitted if K is compact [30, Cor. 1.5.] or if K is a closed subgroup of Gl_n [30, Prop 1.11.]. This point will not be relevant in our discussions.

Definition 2.3 (see [17, p. 56]) A morphism of (smooth) G-equivariant principal K-bundle is a continuous (smooth) map of the total spaces, which is equivariant with respect to G and K.

Given a principal K-bundle, there is the following description by local data, from which the bundle can be reconstructed: Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of the base space of some principal-K-bundle such that there are trivializations $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times K$. If $U_\alpha \cap U_\beta \neq \emptyset$ consider
$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon\ (U_\alpha \cap U_\beta) \times K \to (U_\alpha \cap U_\beta) \times K.$$

As both trivializations are over the base, this map is of the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x,a) = (x, g_{\beta\alpha}(p)\cdot a), \quad a \in K, x \in U_\alpha \cap U_\beta,$$

for some smooth map $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to K$. The set $\{g_{\beta\alpha}\}$ is called the collection of *transition functions* of the bundle with respect to the cover and clearly satisfy the cocycle conditions

$$\begin{aligned} g_{\gamma\beta}\cdot g_{\beta\alpha} &= g_{\gamma\alpha}, \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= e \in K \text{ at any point in } U_\alpha. \end{aligned}$$

Definition 2.4 (see, e.g., [17, p. 56]) The *pullback* of a principal-K-bundle $E \xrightarrow{p} M$ along the continuous (smooth) map $f\colon N \to M$ is the triple $(f^*E, \pi, \bar{f})$, such that

- $\pi\colon f^*E \to N$ is a (smooth) principal-K-bundle
- $\bar{f}\colon f^*E \to E$ is a homomorphism of (smooth) principal-K-bundle
- $f \circ \pi = p \circ \bar{f}$

and this triple is universal in the sense, that for any other triple (Q, q_1, q_2) with these three properties, there is a unique map u, such that

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & f^*E \arrow[r, "\bar{f}"] \arrow[d, "\pi"] & E \arrow[d, "p"] \\ & N \arrow[r, "f"] & M \end{tikzcd}$$

commutes. The pullback of a G-equivariant bundle is defined analogously by imposing all maps to be G-equivariant.

Note that the pullback is only defined up to unique isomorphism. The existence of a pullback is given by the construction $f^*E = \{(v,x) \in E \times N | f(x) = \pi(v)\} \subset E \times N$, which yields an equivariant bundle (with the diagonal action), if E and f are equivariant.

Even if the definition of morphism of principal K-bundles is more general, they are actually of a very special type:

Lemma 2.5 (compare, e.g., Proposition 8.6 of [17, Ch. I]) *Any homomorphism of principal K-bundles is a pullback.*

Proof. Let

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

be homomorphism of principal-K-bundles and the triple (Q, q_1, q_2) as above.

First assume $E' = M' \times K$ is trivial. Then define $u = (u_1, u_2)\colon Q \to M \times K$ by $u_1(q) = q_1(q)$ and $u_2(q) \in K$ such that $(\bar{f}(q_1(q), e))u_2(q) = q_2(q)$. This is well defined as K acts free and transitive on the fibers. Moreover, one sees that this map is smooth by taking local sections (what is the same as local trivializations) of $E \to M$. As $\bar{f}$ is equivariant, u is a map, such that all triangles commute. Assume $u' = (u'_1, u'_2)$ is another such map, then $\pi' \circ u' = q_1$ implies $u'_1 = u_1$ and $\bar{f} \circ u = q_2 = \bar{f} \circ u'$ implies $\bar{f}(q_1(q), e))u_2(q) = \bar{f}(q_1(q), e))u'_2(q)$, hence $u_2 = u'_2$.

For general E': Cover $M' = \bigcup_\alpha U_\alpha$ by open sets, which trivialize E'. For any open set U_α there is a map $u_\alpha\colon q_1^{-1}(U_\alpha) \to E'|_{U_\alpha}$ and by uniqueness, these maps coincide on intersections $U_\alpha \cap U_\beta$, hence they define a global map $u\colon Q \to E'$ as claimed. □

There is a special principal K-bundle such that all principal K bundles are (up to unique isomorphism) pullbacks of this bundle.

Definition 2.6 (see, e.g., [27, pp. 53-54]) The principal K-bundle $EK \to BK$ is called universal, if for each principal K-bundle $E \to X$, there exists a continuous map $f\colon X \to BK$, such that f^*EK is isomorphic to E and the map f is unique up to homotopy.

Remark 2.7 Note the following:

1. The base space of a universal principal K-bundle, BK, is called the *classifying space* of K
2. The universal bundle can also be characterized by the requirement that EK is a contractible topological space with free K-action (compare, e.g., [38, Theorem 7.4].
3. Universal K-bundles exist, as there is an explicit construction of an universal K-bundles for any Lie group, which goes back to [36]. We will apply, in particular in Section 3.2, a slightly different concrete construction of $EK \to BK$.

Let us now turn attention to vector bundles shortly.

Definition 2.8 (Def. 1.1 of [27, Chapter 3]) A (smooth) real (complex) *vector bundle* is a continuous (smooth) map $\pi\colon E \to B$, such that each fiber $E_b := \pi^{-1}(b),\ b \in B$, has the structure of a real (complex) vector space and for any point $b \in B$ there is an open neighborhood U of b and a homeomorphism (diffeomorphism) $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ ($\mathbb{C}^n$ respectively) over U, which is fiberwise linear.

The most important example of vector bundles are the tangent bundle TM of a smooth manifold M. In this case the fiber above $x \in M$ is denoted by T_xM.

Definition 2.9 (Def. 2.1 of [27, Chapter 3]) A morphism of (smooth) vector bundles $(f, \bar{f})\colon (E, B, \pi) \to (E', B', \pi')$ is a pair of continuous (smooth) maps $f\colon B \to B'$ and $\bar{f}\colon E \to E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{f} & B' \end{array}$$

commutes and such that the map between the total spaces $\bar{f}\colon E \to E'$ is fiberwise linear.

Definition 2.10 (see [17, p. 67]) A G-*equivariant vector bundle* is a vector bundle $\pi\colon E \to B$, where E and B are G-spaces π is equivariant and G acts by morphism of vector bundles.

Definition 2.11 A morphism of G-equivariant vector bundles is a morphism of vector bundles $(\bar{f}, f)$, such that both maps are G-equivariant.

The definition of pullbacks of vector bundles is completely analogous to the case of principal bundles: just replace the term 'principal K-' by 'vector'. Note that in the category of vector bundles, it does not hold, that any morphism is a pullback. This is only true for morphism which are fiberwise isomorphisms (see, e.g., [37, Lemma 2.3]).

One can construct new bundles out of old ones (compare [29, III,§4]): Given a (continuous) functor on the category of vector spaces, then one performs functorial operations on vector bundles by performing them fiberwise:

- Let E be a vector bundle. The fibers of the dual bundle $E^\vee$ are the dual spaces $E_x^\vee = \mathrm{Hom}(E_x, \mathbb{C})$ of the fibers of E. In particular $T^\vee M$ is called the cotangent bundle and smooth sections of $T^\vee M$ are called 1-forms.
- Given two bundle E and F over the same base. There is the Whitney sum $E \oplus F$ and the tensor product $E \otimes F$ with fibers $(E \oplus F)_x = E_x \oplus F_x$ and $(E \otimes F)_x = E_x \otimes F_x$ respectively.
- Combining direct sums and tensor products, one obtains the tensor algebra $\bigoplus_{k\in\mathbb{N}} E^{\otimes k}$, whose quotient by the ideal of symmetric tensor (i.e., the ideal generated in the fiber at x by $v \otimes w + w \otimes v$ for $v, w \in E_x$) is the exterior algebra $\Lambda^* E$ (where the degree is the smallest tensor degree of a representative). The image of the product $\otimes$ in $\bigoplus_{k\in\mathbb{N}} E^{\otimes k}$ is called $\wedge$-product. Sections of $\Lambda^* T^\vee M$ are called (complex valued) *differential forms* and the algebra of all differential forms is denoted by $\Omega^*(M)$.
- Given again the vector bundle E on M, $\Omega^*(M, E)$ denotes the space of sections of $\Lambda^* T^\vee M \otimes E$, so called differential forms with values in E. In particular, given a vector space V, $\Omega^*(M, V) = \Omega^*(M, M \times V)$

2.2.2 Derivations and Connections

There are more operations on differential forms, which will be applied later on. This section is a collection of [29, V,§§4-5] and [22, 1.F-G].

Definition 2.12 Let $\omega \in \Omega^n(M)$ be an n-form on M and X a vector field. The *contraction* of ω by X is the $n-1$-form $\iota_X \omega \in \Omega^{n-1}(M)$ defined by

$$\iota_X \omega(x)[v_2, \dots, v_n] = \omega(x)[X(x), v_2, \dots, v_n],$$

for any $x \in M$ and $v_2, \dots, v_n \in T_x M$.

Definition 2.13 The *Lie derivative* of a differential form $\omega \in \Omega^n(M)$ along a vector field X on M is defined as

$$L_X\omega = \frac{d}{dt}\bigg|_{t=0} \alpha_t^*\omega,$$

where $\alpha\colon \mathbb{R} \times M \supset U \to M$ is the flow of the vector field X, i.e., $\alpha'(t,x) = X(\alpha(t,x))$, which is uniquely defined on an open neighborhood of $0 \in \mathbb{R}$ for any $x \in M$ (see [29, IV,§1]).

Definition + Proposition 2.14 *For any $n \in \mathbb{N}$, there is a unique operator, the* exterior derivative*, $d\colon \Omega^n(M) \to \Omega^{n+1}(M)$, such that*

1. *d is compatible with restrictions to open subsets,*
2. *for $p = 0$, $d\colon C^\infty(M) \to \Omega^1(M)$ is the differential on functions, i.e., $df(X) = L_X f$,*
3. *for $f \in C^\infty(M)$, we have $d(df) = 0$,*
4. *for $\alpha \in \Omega^n(M)$ and $\beta \in \Omega^*(M)$, we have*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^n \alpha \wedge d\beta.$$

The following proposition collects important properties of these operations.

Proposition 2.15 *On any smooth manifold M, the following holds:*

1. *$d \circ d = 0$,*
2. *d is functorial, i.e., for any smooth map $\varphi\colon M \to N$, $d \circ \varphi^*\omega = \varphi^* \circ d$,*
3. *for a vector field X on M, $L_X \circ d = d \circ L_X$ and $L_X = d \circ \iota_X + \iota_X \circ d$.*

By the first assertion of Proposition 2.15, the pair $(\Omega^*(M), d)$ is a cochain complex, thus there is an cohomology theory.

Definition 2.16 (see, e.g, [29, p.490]) The *de Rham cohomology* of M is defined as

$$H^n_{\mathrm{dR}}(M) = {\ker\left(d\colon \Omega^n(M) \to \Omega^{n+1}(M)\right)}\big/{d\Omega^{n-1}(M)}.$$

Differential forms in the kernel of d are called *closed forms* and differential forms in the image of d are called *exact forms*. Thus, in this terminology, de Rham cohomology is defined as the quotient of closed forms modulo exact forms.

Remark 2.17 Observe that the chain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

induces for any n an exact sequence

$$0 \to H^n_{\mathrm{dR}}(M) \to {\Omega^n(M)}\big/{d\Omega^{n-1}(M)} \xrightarrow{d} \Omega^{n+1}(M)_{\mathrm{cl}} \to H^{n+1}_{\mathrm{dR}}(M) \to 0.$$

This follows directly from the definition, as $H^n_{\mathrm{dR}}(M) = \ker(d\colon {\Omega^n(M)}\big/{d\Omega^{n-1}(M)} \to \Omega^{n+1}(M)_{\mathrm{cl}})$ and $H^{n+1}_{\mathrm{dR}}(M)$ is the cokernel of this map.

In our discussions of vector bundles and principal fiber bundles later, we will make use of the notation of a connection. Different authors prefer different notations, what arises naturally as the object one wants to define is a split of some exact sequence (compare [29, p. 57]). Namely, let $\pi\colon E \to M$ be a vector bundle. For any $x \in M$, there is the inclusion $E_x \to E$ as fiber at x, which induces a map $T(E_x) \to TE$. Moreover, as $T(E_x) = E_x \times E_x$, there is an isomorphism $\pi^*E \to \bigcup_{x\in M} T(E_x)$. This map yields, together with $T\pi$, a short exact sequence of vector bundles

$$0 \to \pi^*E \to TE \to \pi^*TM \to 0$$

over E. There are many ways to say what a split of this sequence is. We prefer the following:

Definition + Proposition 2.18 (see, e.g.,[9, Section 2.1]) *Let E be vector bundle over M.*

1. *A connection on E is a mapping*

$$\nabla\colon \Omega^0(M,E) \to \Omega^1(M,E)$$

which satisfies a Leibniz rule

$$\nabla(f\varphi) = df \wedge \varphi + f\nabla\varphi \text{ for } f \in \Omega^0(M,\mathbb{C}), \varphi \in \Omega^0(M,E).$$

2. *The connection ∇ extends uniquely to a linear map $\nabla\colon \Omega^*(M,E) \to \Omega^{*+1}(M,E)$ satisfying the Leibniz rule*

$$\nabla(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge \nabla\varphi \text{ for } \omega \in \Omega^k(M,\mathbb{C}), \varphi \in \Omega^*(M,E)$$

3. *The map $\nabla \circ \nabla\colon \Omega^0(M,E) \to \Omega^2(M,E)$ is $C^\infty(M,\mathbb{C})$-linear. Hence there is a section $R^\nabla \in \Omega^2(M, \operatorname{End} E)$ of the endomorphism bundle, such that*

$$\nabla \circ \nabla\varphi = R^\nabla\varphi \text{ for all } \varphi \in \Omega^0(M,E).$$

One calls R^∇ the curvature *of the connection.*

One can do an analogous construction for principal G-bundles (compare [19, pp.45-49]): Let $\pi\colon E \to M$ be a principal G-bundle. The action of the Lie algebra on E

$$E \times \mathfrak{g} \ni (x,X) \mapsto \left.\frac{d}{dt}\right|_{t=0} x\exp(tX) \in TE$$

actually is a map onto the kernel of $T\pi$. Thus we have an exact sequence

$$0 \to E \times \mathfrak{g} \to TE \to \pi^*TM \to 0,$$

for which one defines a G-equivariant split:

Definition 2.19 A connection on a principal G-bundle $\pi\colon E \to M$ is a 1-form $\vartheta \in \Omega^1(E, \mathfrak{g})$ satisfying:

1. $\vartheta_x \circ \nu_x = \mathrm{id}_{\mathfrak{g}}$, where $\nu_x\colon \mathfrak{g} \to T_x(E)$ is the differential of the map $g \mapsto xg$.
2. $R_g^* \vartheta = \mathrm{Ad}_{g^{-1}} \circ \vartheta$ for any $g \in G$, where R_g denotes the right translation of G on E.

It is an immediate corollary of the definition that convex combinations of connections are a connection again. On a trivial principal G-bundle $M \times G \to M$, there is a connection given by the derivative of the left multiplication $L_{g^{-1}}\colon T_{(m,g)}(M \times G) \to T_g G \to T_e G = \mathfrak{g}$.

Recall, that one has the Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which induces by past composition a map $\Omega^2(E, \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^2(E, \mathfrak{g})$. For a 1-form $\vartheta \in \Omega^1(E, \mathfrak{g})$, $[\vartheta, \vartheta]$ denotes the image of $\vartheta \wedge \vartheta$ under the Lie bracket.

Definition 2.20 The *curvature* of a connection $\vartheta \in \Omega^1(E, \mathfrak{g})$ is defined to be the 2-form

$$\Omega^\theta := d\vartheta + \frac{1}{2}[\vartheta, \vartheta] \in \Omega^2(E, \mathfrak{g}).$$

2.2.3 Sheaves

Clearly, differential forms can not only defined for any manifold, but also for every open subset of a manifold as this is itself a manifold. Moreover, given two differential forms, which are defined on open subsets and coincide on the intersection, there is a form on the union such that the two given forms are its restrictions to the corresponding open set. This behavior of functions is axiomatized in the Definition of sheaf, what is one of the notational foundations of the definition of cohomology, which we will apply.

Definition 2.21 (see, e.g., [26, p. 61]) Let $\mathcal{B}$ be an abelian category. A *presheaf* $\mathcal{F}$ of objects of $\mathcal{B}$ on a topological space X consists of an object of $\mathcal{B}$, denoted by $\mathcal{F}(U)$, for any open subset $U \subset X$ and for any inclusion $V \subset U$ of open subsets of X a $\mathcal{B}$-morphism $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$, such that the following conditions hold:

1. $\mathcal{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,
2. ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
3. if $W \subset V \subset U$ are three open subsets, then $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$.

The presheaf is called *sheaf*, if additionally for any open subset $U \subset X$ and $\{V_\alpha\}$ an open cover of U

4. if $s \in \mathcal{F}(U)$ an element, such that $s|_{V_\alpha} := \rho_{UV_\alpha}(s) = 0$ for all α, then $s = 0$, and
5. if for each α $s_\alpha \in \mathcal{F}(V_\alpha)$ is an element, such that for each pair α, β, $s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_\alpha} = s_\alpha$.

Example 2.22 Let M be a manifold. The assignment

$$M \supset U \mapsto C^\infty(U, \mathbb{C})$$

is, as one checks directly from the definition, a sheaf of rings.

Definition 2.23 (see [26, p. 62]) Let $\mathcal{F}$ and $\mathcal{G}$ be (pre-)sheaves on a topological space X. A *morphism* of (pre-)sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ consists of morphisms $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subset X$, such that whenever $V \subset U$ is an inclusion, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

is commutative.

Definition + Proposition 2.24 (Prop.+Defn. 1.2 in [26, Chapter II]) *Given a presheaf $\mathcal{F}$, then there is a sheaf $\mathcal{F}^+$ and a morphism $\vartheta\colon \mathcal{F} \to \mathcal{F}^+$, with the property, that for any sheaf $\mathcal{G}$ and any morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \vartheta$. Furthermore the pair $(\mathcal{F}^+, \vartheta)$ is unique up to unique isomorphism and $\mathcal{F}^+$ is called the* sheaf associated *to the presheaf $\mathcal{F}$*

Definition 2.25 (see [26, p. 65]) Let $f\colon X \to Y$ be a map of topological spaces. The *inverse image sheaf* $f^{-1}\mathcal{F}$ of a sheaf $\mathcal{F}$ on Y is the sheaf associated to the presheaf $U \mapsto \lim_{V \supset f(U)} \mathcal{F}(V)$. Here, the limit is ranges over all open sets $V \in X$ containing $f(U)$.

Many of the sheaves, which will occur in this thesis, have another property, which is of high relevance in the context of cohomology.

Definition 2.26 (see 5.10 of [44]) A sheaf $\mathcal{F}$ on X is called *fine*, if for each locally finite open cover $\{U_\alpha\}$ of X there exists for each α an endomorphism f_α of $\mathcal{F}$ such that

1. $\operatorname{supp} f_\alpha \subset U_\alpha$ and
2. $\sum_\alpha f_\alpha = \mathrm{id}$.

Here, $\operatorname{supp} f_\alpha$ is the support of f_α, i.e., the complement of the union of all open sets U such that $f_\alpha(U)\colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the zero morphism.

Example 2.27 (compare 5.10 of [44]) Let M be smooth manifold. The sheaf C^∞ is fine, as the existence of a partition of unity $\{\varphi_\alpha\}$ subordinated to a locally finite cover $\{U_\alpha\}_{\alpha \in A}$ of M (see, e.g., [44, Theorem 1.11]) induces endomorphisms

$$f_\alpha(U)\colon C^\infty(U) \ni s \mapsto s \cdot \varphi_\alpha \in C^\infty(U).$$

On the other hand the sheaf $U \mapsto \{h\colon U \to \mathbb{Z} | h \text{ continuous}\}$, called locally constant sheaf $\underline{\mathbb{Z}}$, is not fine, if the manifold has positive dimension.

Definition 2.28 (compare [26, p. 109]) Let M be a manifold. A sheaf of C^∞-modules is a sheaf of abelian groups $\mathcal{F}$ on M, such that for each open set $U \subset M$, $\mathcal{F}(U)$ is an $C^\infty(U)$-module and the module structure is compatible with restrictions, i.e., the diagram

$$\begin{tikzcd} C^\infty(U) \times \mathcal{F}(U) \arrow[r] \arrow[d, "\rho^{C^\infty}_{UV} \times \rho^{\mathcal{F}}_{UV}"] & \mathcal{F}(U) \arrow[d, "\rho^{\mathcal{F}}_{UV}"] \\ C^\infty(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes for any inclusion of open sets $V \subset U \subset M$.

Lemma 2.29 (compare 5.28 of [44]) *Any sheaf $\mathcal{F}$ of C^∞-modules is fine.*

Proof. Take a partition of unity and the endomorphisms as in Example 2.27. By the module structure on the sheaf $\mathcal{F}$ the endomorphisms of C^∞ turn into the desired endomorphisms of $\mathcal{F}$. □

Remark 2.30 Actually, another view on sheaves fits better to the functorial properties of cohomology theories. Objects like, e.g., differential forms are not only defined for a single manifold, but for any manifold in a functorial manner. Thus the object one has, is a functor from the category of smooth manifolds to, let's say, the category of abelian groups. How to formulate the properties with respect to covering of the sheaf? Therefore we need an additional structure on the category of manifolds, a so called Grothendieck topology, which turns the category into a site.

For the points, which we want to make, this language is not necessary, because we will not draw to much attention on the functorial properties of the cohomology theory we will define.

2.3 Topology

2.3.1 Simplices

The *standard n-simplex* Δ^n is the convex hull of the standard basis in $\mathbb{R}^{n+1}$. That is

$$\Delta^n = \left\{ t = (t_0, \dots, t_n) \in \mathbb{R}^{n+1} | t_i \geq 0 \text{ for all } i, \sum_i t_i = 1 \right\},$$

what is a closed subset of the hyperplane

$$V^n = \left\{ t = (t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \right\}.$$

The geometric structure of the simplex is given in the following way. The tangent space of the simplex equals the restriction of the tangent space of V^n to Δ^n, and a differential k-form on Δ^n is differential k-form on V^n, where forms which coincide on any open neighborhood of Δ^n are identified. (By a standard argument it is sufficient to define the form on any open neighborhood of the simplex). The face maps of the simplex are

$$\begin{aligned} \partial^i \colon \Delta^{n-1} &\to \Delta^n, i = 0, \ldots, n \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \end{aligned}$$

From the viewpoint of category theory on thinks differently about simplices (compare, e.g., [24, Section I.1]):

Definition 2.31 The *simplex category* Δ is the category, whose objects are non empty finite ordered sets

$$[p] = \{0 < 1 < \ldots < p\}, \text{ for } p \in \mathbb{N},$$

and the set of morphisms $\Delta(p, q), p, q \in \mathbb{N}$ is the set of order preserving maps $f \colon [p] \to [q]$.

The morphisms are generated by *coface maps* $\partial^i \colon [p-1] \to [p]$, which is the map that misses i and *codegeneracy maps* $\sigma^i \colon [p] \to [p-1]$, which is uniquely determined by hitting i twice. These maps satisfy the following *cosimplicial relations*:

$$\begin{aligned} \partial^j \partial^i &= \partial^i \partial^{j-1} && \text{if } i < j \\ \sigma^j \partial^i &= \partial^i \sigma^{j-1} && \text{if } i < j \\ \sigma^i \partial^i &= 1 = \sigma^i \partial^{i+1} \\ \sigma^j \partial^i &= \partial^{i-1} \sigma^j && \text{if } i > j+1 \\ \sigma^j \sigma^i &= \sigma^i \sigma^{j+1} && \text{if } i \le j \end{aligned}$$

2.3.2 Chain and double complexes

We will also need some homological algebra. The material can be found in Chapter 1 of [45].

Let $\mathcal{B}$ be an abelian category.

Definition 2.32 A *cochain complex* $C = (C, d)$ in $\mathcal{B}$ is a family $\{C^n\}_{\mathbb{Z}}$ of objects of $\mathcal{B}$, s.t. there exist some $k \in \mathbb{Z}$ with $C^n = 0$ for all $n < k$, together with morphisms $d = d^n \colon C^n \to C^{n+1}$ such that $d \circ d \colon C^n \to C^{n+2}$ is the zero. The kernel of d_n is the module Z^n of *cocycles* and the image of d^{n-1} is the module B^n of *coboundaries*. As $d^2 = 0$, we have

$$\{0\} \subset B^n \subset Z^n \subset C^n$$

and the quotient $H^n(C, d) := Z^n / B^n$ is called n-th *cohomology* of the cochain complex C.

Remark 2.33 In general one would prefer the term bounded below cochain complex for the object defined in the last definition. As all our cochain complexes will we bounded below, we skip the attribute.

Given two chain complex C, D. A morphism $f\colon (C, d_C) \to (D, d_D)$, also called *chain complex map*, is a family of morphisms $f^n\colon C^n \to D^n$ in $\mathcal{B}$ for any $n \in \mathbb{N}$, such that

$$f^{n+1} \circ d_C^n = d_D^n \circ f^n,$$

i.e., f commutes with the coboundary maps.

Thus, there is a category of Cochain complexes and in fact Cochain complexes in an abelian category form themselves an abelian category. Thus one may iterate the procedure and forms cochain complexes of cochain complexes. In general one prefers a slightly different, but equivalent object.

Definition 2.34 A *double complex* (C, d_v, d_h) in $\mathcal{B}$ is a family of $\{C^{p,q}\}_{p,q\in\mathbb{N}}$ of objects in $\mathcal{B}$ together with maps

$$d_v\colon C^{p,q} \to C^{p+1,q} \text{ and } d_h\colon C^{p,q} \to C^{p,q+1},$$

called vertical and horizontal coboundaries, such that $d_v^2 = d_h^2 = d_v d_h + d_h d_v = 0$.

One can easily turn a cochain complex of cochain complexes into a double complex: Just change the sign of the boundary operator in every other column. This turn the commuting boundaries into anti-commuting operators. To see, why the anti-commutativity is useful, take a look at the following procedure, which reduces a double complex to a cochain complex:

Definition 2.35 The *total complex* of a double complex (C, d_v, d_h) is the cochain complex

$$\operatorname{tot} C := \left(\left\{ \bigoplus_{p+q=n} C^{p,q} \right\}_{n\in\mathbb{N}}, d_v + d_h \right).$$

Clearly $d = d_v + d_h$ squares to zero. Moreover, the *cohomology of the double complex* is defined as

$$H^n(C, d_v, d_h) = H^n(\operatorname{tot} C, d_v + d_h).$$

Example 2.36 Let X be a topological space and A an abelian group. Denote by $S_n(X)$ the set of continuous maps $\Delta^n \to X$, so called singular n-simplices in X. The singular cochain groups $S^n(X, A)$ consists of all maps $S_n(X) \to A$ with the groups structure induced from A. The boundary of a cochain is defined as

$$df(\sigma) := \sum_{i=0}^{n+1} (-1)^i f(\sigma \circ \partial^i),$$

for any $\sigma\colon \Delta^{n+1} \to X$. It is a short exercise to see, that $d^2 = 0$. The groups

$$H^n(X, A) = H^n_{\text{sing}}(X; A) := H^n(S^n(X, A), d)$$

are called *singular cohomology* of X with values in A.

Remark 2.37 The short exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0$$

yields short exact sequences

$$0 \to S^n(X, \mathbb{Z}) \to S^n(X, \mathbb{C}) \to S^n(X, \mathbb{C}/\mathbb{Z}) \to 0$$

for any topological space X and any $n \in \mathbb{N}$, which commute with the coboundary maps. Thus by the Snake lemma (see/watch [15] for a proof), there is a long exact sequence

$$\ldots \to H^{n-1}(X, \mathbb{C}) \to H^{n-1}(X, \mathbb{C}/\mathbb{Z}) \xrightarrow{\beta} H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{C}) \to \ldots$$

known as the *Bockstein sequence*. The connecting morphism β is called Bockstein (morphism).

We will apply more operations on cochain complexes.

Definition 2.38 The *truncation* below n of a cochain complex is the cochain complex

$$\left(C^{\geq n}\right)^i = \begin{cases} 0 & \text{if } i < n \\ C^i & \text{if } i \geq n, \end{cases}$$

with the same boundary maps, if the C^i is not replaced by zero. The truncation above n is given by $C^{<n} = C/C^{\geq n}$.

Observe, in particular, that $H^n(C^{\geq n}) = Z^n$.

Definition 2.39 Let C be a cochain complex. The *shifted complex* $C[p]$ is defined as

$$C[p]^i := C^{i+p}, d^i_{C[p]} := (-1)^p d^{i+p}_C.$$

Definition 2.40 Let $f : C \to D$ be a map of cochain complexes in an abelian category. The *cone* of f is a chain complex defined by

$$\text{Cone}(f)^i := C^{i+1} \oplus D^i, d_{\text{Cone}}(x \oplus y) := (-d_C x \oplus (d_D y - f(x)))$$

Lemma 2.41 *The cone fits into a short exact sequence of chain complexes*

$$0 \to D \to \text{Cone}(f) \to C[+1] \to 0$$

and hence induces a long exact sequence in cohomology

$$\ldots \to H^{n-1}(\text{Cone}(f)) \to H^n(C) \xrightarrow{f} H^n(D) \to H^n(\text{Cone}(f)) \to \ldots .$$

2.3.3 Characteristic classes

Let A be a principal ideal domain, e.g., $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$.

Definition 2.42 (Def. 5.1 of [19]) A *characteristic class* c for principal K-bundles associates to every isomorphism class of topological principal K-bundles $\pi\colon E \to X$ a cohomology class of $c(E) \in H^*(X, A)$, such that $c((\bar{f}, f)^*E) = f^*c(E)$ for every pullback-diagram

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X. \end{array}$$

If $A = \mathbb{Z}$ the class is called integral.

Theorem 2.43 (Theorem 5.5 of [19]) *The map associating to a characteristic class c for principal K-bundles the element $c(EK) \in H^*(BK)$ is a one to one correspondence.*

Thus there is in particular a one to one correspondence between integral characteristic classes and $H^*(BK, \mathbb{Z})$.

For many important Lie groups, the cohomology of the classifying space can be described by polynomials.

Definition 2.44 (see, e.g., [19, p. 61]) Let V be a finite-dimensional vector space. For $p \geq 1$ let $S^p(V^\vee)$ denote the vector space of symmetric complex valued multilinear functions in p variables on V. In other words $P \in S^p(V^\vee)$ is a linear map $P\colon V \otimes \ldots \otimes V \to \mathbb{C}$ which is invariant under the action of the symmetric group on the entries. The map

$$\circ\colon S^p(V^\vee) \otimes S^q(V^\vee) \to S^{p+q}(V^\vee),$$

given by

$$P \circ Q(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_\sigma P(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) Q(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$

where the sum runs through all permutations of $1, \ldots, p+q$, defines a product on the graded sum $\bigoplus_{p \geq 0} S^p(V^\vee)$ (where $S^0(V^\vee) = \mathbb{C}$). The graded algebra thus obtained is denoted by $S^*(V^\vee)$.

Let K be a Lie group. The adjoint action on the Lie algebra $\mathfrak{k}$ induces an action on the $S^p(\mathfrak{k}^\vee)$ for every p:

$$(kP)(v_1, \ldots, v_p) = P(\operatorname{Ad}_{k^{-1}} v_1, \ldots, \operatorname{Ad}_{k^{-1}} v_p), \quad v_1, \ldots, v_p \in \mathfrak{k}, \quad k \in K.$$

Definition 2.45 (see, e.g., [19, p. 62]) Let K be a Lie group. The K-invariant part of the graded algebra $S^*(\mathfrak{k}^\vee)$ is called the algebra of *invariant symmetric polynomials*,

$$I^*(K) = \left(S^*(\mathfrak{k}^\vee)\right)^K.$$

Theorem 2.46 (Theorem 8.1 of [19]) *Let K be a compact Lie group, then there is a natural isomorphism*

$$I^*(K) \to H^*(BK, \mathbb{C}).$$

Moreover, note that any Lie group K with a finite number of connected components (see the Remark after [19, Theorem 8.1]) is contractible to a compact subgroup $\widetilde{K}$ and thus its cohomology is given by invariant polynomials on this subgroup $I^*(\widetilde{K}) \cong H^*(B\widetilde{K}, \mathbb{C}) \cong H^*(BK, \mathbb{C})$.

The polynomials which correspond to integral cohomology classes will be called *integral*.

There is a completely different way of constructing characteristic classes, known as Chern-Weil-Construction.

Definition 2.47 (compare Def. 2.39 of [9]) A *characteristic form* ω for principal K-bundles of degree n associates to each connection $\vartheta \in \Omega^1(E, \mathfrak{k})$ on a smooth principal K-bundles $\pi\colon E \to M$ a closed differential form $\omega(\vartheta) \in \Omega^n_{\mathrm{cl}}(M)$, such that $\omega((\bar{f}, f)^*\vartheta) = f^*\omega(\vartheta)$ for every pullback-diagram

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M. \end{array}$$

It can be shown (see, e.g., [9, Lemma 2.51]) that the difference between the characteristic forms of two connections is an exact form and hence the equivalence class of a characteristic form $\omega(\vartheta)$ in de Rham cohomology is a characteristic class, denoted by $\omega(E)$. Moreover, any invariant symmetric polynomial yields a characteristic form. We will discuss both procedures – from the form to the class and from the polynomial to the form – later in a slightly more complicated context, see Theorem 3.48.

Definition 2.48 (Def. 2.85 of [9]) . A characteristic form ω is called integral if $\omega(E) \in H^*(M, \mathbb{C})$ is integral, i.e., lies in the image of $H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{C})$, for every principal K-bundle $E \to M$.

We are now turning to principal Gl_n-bundles, what is the same as vector bundles. As the map $H^*(B\,\mathrm{Gl}_n, \mathbb{Z}) \to H^*(B\,\mathrm{Gl}_n, \mathbb{C})$ is injective, one can show the following statement:

Lemma 2.49 (Theorem 2.117 of [9]) *An integral characteristic form for principal* Gl_n*-bundles defines uniquely an integral characteristic class for principal* Gl_n*-bundles such that the image in complex cohomology coincides for every bundle.*

Example 2.50 An important example of integral characteristic forms and classes are Chern classes. The Lie algebra of Gl_n is the vector space of all $n \times n$ matrices. The Chern polynomials $C_k \in I^k(\mathrm{Gl}_n)$ are defined by

$$\det\left(\lambda \cdot \mathrm{id}_{n\times n} - \frac{1}{2\pi i}A\right) = \sum_k C_k(A, \dots, A)\lambda^{n-k} \tag{2.1}$$

and induce the integral Chern classes and Chern forms (compare, e.g., [19, p. 68],[9, Def 2.42]).

3 Equivariant cohomology

3.1 Models for equivariant cohomology

Let M be a smooth manifold acted on from the left by a Lie group G. To define equivariant cohomology one uses two properties which one expects from such a theory: it should be homotopy invariant and for free actions, the equivariant cohomology should be the cohomology of the quotient. Recall from Remark 2.7 that the total space of the classifying bundle EG is a contractible topological space with free G-action. Hence $EG \times M$ has the homotopy type of M and the diagonal action is free. Hence one defines

$$H_G^*(M) := H^*(EG \times_G M),$$

where $EG \times_G M$ is the quotient of $EG \times M$ by the diagonal action. We are interested in differential form models for equivariant cohomology, but in general EG is not a finite-dimensional manifold, hence we cannot use the usual de Rham cohomology. But there is a model for EG, which consist of finite dimensional manifold:

3.1.1 Simplicial manifolds and differential forms

The model of $EG \times_G M$ we are going to use is a given by a simplicial manifold.

Definition 3.1 (see, e.g., [19, p.89]) A *simplicial manifold* is contra-variant functor from the simplex category Δ to the category of smooth manifolds.

Explicitly this is an $\mathbb{N}$-indexed family of manifolds with smooth *face* and *degeneracy maps* satisfying the simplicial relations, i.e.

$$\begin{aligned}
\partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i, \text{ if } i < j \\
\sigma_i \circ \sigma_j &= \sigma_{j+1} \circ \sigma_i, \text{ if } i \leq j \\
\partial_i \circ \sigma_j &= \begin{cases} \sigma_{j-1} \circ \partial_i, & \text{if } i < j \\ \mathrm{id}, & \text{if } i = j, j+1 \\ \sigma_j \circ \partial_{i-1}, & \text{if } i > j+1 \end{cases}
\end{aligned}$$

Example 3.2 Our most important example of a simplicial manifold is the following (compare [25, p.316],[23, section 3.2]):

$$G^\bullet \times M = \{G^p \times M\}_{p \geq 0},$$

where G^p stands for the p-fold Cartesian product of G. The face maps $G^p \times M \to G^{p-1} \times M$ are given as

$$\begin{aligned}
\partial_0(g_1, \ldots, g_p, x) &= (g_2, \ldots, g_p, x) \\
\partial_i(g_1, \ldots, g_p, x) &= (g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_p, x) \text{ for } 1 \leq i \leq p-1 \\
\partial_p(g_1, \ldots, g_p, x) &= (g_1, \ldots, g_{p-1}, g_p x)
\end{aligned}$$

and the degeneracy maps for $i = 0, \ldots, p$ by

$$\begin{aligned}
\sigma_i : G^p \times M &\to G^{p+1} \times M \\
(g_1, \ldots, g_p, x) &\mapsto (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_p, x).
\end{aligned}$$

These maps satisfy the simplicial relations. In particular for $p = 1$ the map ∂_1 equals the group action, while ∂_0 is the projection onto the second factor, i.e. onto M.

Definition 3.3 (see, e.g., [19, p.75]) The (fat) *geometric realization* of a simplicial manifold $M_\bullet$, is the topological space

$$\|M_\bullet\| = \bigcup_{p \in \mathbb{N}} \Delta^p \times M_p / \sim$$

with the identifications

$$(\partial^i t, x) \sim (t, \partial_i x) \text{ for any } x \in M_p,\, t \in \Delta^{p-1}, i = 0, \ldots, n \text{ and } p = 1, 2, \ldots.$$

Example 3.4 The geometric realization of the simplicial manifold $G^\bullet \times M$ is a model of $EG \times_G M$ and in particular if M is single point the geometric realization of $G^\bullet \times pt$ is a model of the classifying space BG (compare [19, pp.75]).

Before giving a differential form model for equivariant cohomology, we will explain sheaves and sheaf cohomology for simplicial manifolds, as this is the technical basis for all further constructions and definitions.

Definition 3.5 (see [16, (5.1.6)]) A *simplicial sheaf* on the simplicial manifold $M_\bullet$ is a collection of sheaves $\mathcal{F}^\bullet = \{\mathcal{F}^p\}_{p \in \mathbb{N}}$, where, for each p, $\mathcal{F}^p$ is a sheaf on M_p and there are morphisms $\tilde{\partial}_i : \partial_i^{-1} \mathcal{F}^p \to \mathcal{F}^{p+1}$ and $\tilde{\sigma}_i : \sigma_i^{-1} \mathcal{F}^{p+1} \to \mathcal{F}^p$ satisfying the simplicial relations as stated above.

The *simplicial sheaf cohomology* is defined as the right derived functor of the global section functor [16, def. 5.2.2.], where global sections of a simplicial sheaf, are the equalizer

$$\ker\left(\tilde{\partial}_0 - \tilde{\partial}_1 \colon \mathcal{F}^0(M_0) \to \mathcal{F}^1(M_1)\right).$$

This definition opens the question: Are there enough injectives? As Pierre Deligne is quite short on this and there are mistakes in the literature (see Remark 3.8), we should give an answer.

Lemma 3.6 *The category of simplicial sheaves has enough injectives.*

Proof. Let $\mathcal{F}^\bullet$ be a simplicial sheaf. Let P_p be the functor from simplicial sheaves to sheaves, which sends a sheaf to its p-th level, i.e., $\mathcal{F}^\bullet$ is sent to the sheaf $\mathcal{F}^p$ on M_p. Pick for any $\mathcal{F}^p$ an injective sheaf I^p on $G^p \times M$, in which $\mathcal{F}^p$ embeds (for existence see e.g. [26, section III.2]).

Now we construct a right adjoint of P_p (analogous to [24, p.409]): Let B be a sheaf on $G^p \times M$. Define a simplicial sheaf on $G^\bullet \times M$ as

$$(S_pB)_q = \prod_{h \in \Delta(q,p)} h^{-1}B$$

By the adjointness of the functors, injectivity of B implies injectivity of S_pB. Moreover the equality

$$\operatorname{Hom}\left(\mathcal{F}^\bullet, \prod_p S_pI^p\right) = \prod_p \operatorname{Hom}(\mathcal{F}^\bullet, S_pI^p) = \prod_p \operatorname{Hom}(P_p\mathcal{F}^\bullet, I^p) = \prod_p \operatorname{Hom}(\mathcal{F}^p, I^p)$$

shows that the simplicial sheaf $\mathcal{F}^\bullet$ embeds into $\prod_p S_pI^p$ because for each $\mathcal{F}^p$ there is an injection into I^p. $\square$

Now let

$$0 \to \mathcal{F}^\bullet \to I^{\bullet,0} \xrightarrow{\delta} I^{\bullet,1} \xrightarrow{\delta} \dots$$

be an injective resolution. Omitting the first columns and taking global sections yields to a double complex

$$\left(I^{p,q}(M_p), \sum_{i=0}^{p}(-1)^i\tilde{\partial}_i + (-1)^p\delta\right),$$

whose cohomology is defined to be the cohomology

$$H^*(M_\bullet, \mathcal{F}^\bullet) = H^*\left(I^{p,q}(M_p), \sum_{i=0}^{p}(-1)^i\tilde{\partial}_i + (-1)^p\delta\right)$$

of the simplicial sheaf $\mathcal{F}^\bullet$ on the simplicial manifold $M_\bullet$.

The definition does not depend on the injective resolution chosen. In the non-simplicial case, this is a well-known fact: the identity on the space and the sheaf induces a morphism between two chosen injective resolutions, which is an isomorphism in cohomology. In the simplicial case, we need an additional argument: As before we obtain a morphism of the double complexes of global sections from the identity on the space. When taking cohomology in every horizontal line $(I^{p,*}(M_p), (-1)^p\delta)$, this morphism will induce an isomorphism between the bi-graded complexes. Hence we can apply the following lemma, to see, that we have an isomorphism in cohomology.

Lemma 3.7 (see e.g. [19, Lemma 1.19]) *Suppose* $f\colon (C_1^{*,*}, d_1' + d_1'') \to (C_2^{*,*}, d_2' + d_2'')$ *is homomorphism of double complexes and the induced homomorphism*

$$(H^q(C_1^{p,*}, d_1''), d_1') \to (H^q(C_2^{p,*}, d_2''), d_2')$$

is an isomorphism, then f *induces an isomorphism in the total cohomology of double complexes.*

Remark 3.8 One could have the idea (e.g. [7, p.3],[25, Section 3.2]) that an injective resolution on any simplicial level would be sufficient as the maps $\tilde{\partial}_i$ lift by the injectivity of the sheaf. But as this lift is not unique, no one ensures, that the simplicial relations hold and thus there is no general reason why $\partial = \sum_i (-1)^i \tilde{\partial}_i$ is a boundary operator. In fact one can construct the following counterexample: Take the trivial group, acting on a point, then all $\tilde{\partial}_i \colon \mathbb{Z} \to \mathbb{Z}$ are the identity. A injective resolution of the abelian group $\mathbb{Z}$ is given by $\mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Beside $\mathrm{id} \colon \mathbb{C} \to \mathbb{C}$, the complex conjugation is also a lift of $\mathrm{id}_{\mathbb{Z}}$. Making appropriate choices, for the lifts $\tilde{\partial}_i$ one finds an example where $\partial \circ \partial \neq 0$.

In practice, one usually uses acyclic resolutions, instead of injective ones, to calculate cohomology. This works in the simplicial case, too. Let

$$0 \to \mathcal{F}^\bullet \to \mathcal{A}^{\bullet,0} \xrightarrow{\delta} \mathcal{A}^{\bullet,1} \xrightarrow{\delta} \dots$$

be an acyclic resolution, i.e., each $\mathcal{A}^{\bullet,k}$ is a simplicial sheaf and all but the zeroth cohomology of each sheaf $\mathcal{A}^{p,q}$ vanish. Let $I^{\bullet,*}$ be a simplicial injective resolution. The identity map on the simplicial manifold and the sheaf $\mathcal{F}^\bullet$ induce a homomorphism of the double complex of global sections (by injectivity of I), which induces an isomorphism of the bi-complexes, $(H^q(\mathcal{A}^{p,*}, \delta), \partial) \to (H^q(I^{p,*}, \delta), \partial)$, as acyclic resolutions calculate cohomology. Thus the last lemma implies the isomorphism in the cohomology of the double complexes.

In the examples, which we study later, the simplicial sheaf will actually not just be a sheaf of abelian groups, but a cochain complexes of simplicial sheaves of abelian groups. A resolution for a chain complex goes by the name *Cartan-Eilenberg resolution* and exists for cochain complex in any abelian category with enough injectives (compare [45, Section 5.7]). In our context, the resolution of a cochain complex of simplicial sheaves is a triple instead of a double complex. Nevertheless, one can form a total complex of the global sections of the triple complex and the cohomology of the cochain complex of simplicial sheaves is defined as the cohomology of this total complex.

We will now discuss some explicit models for simplicial sheaf cohomology.

Simplicial de Rham cohomology

This exposition is based on [19, Section 6]. Let $M_\bullet = \{M_p\}$ be a simplicial manifold. For any p, differential forms on M_p form a the cochain complex of sheaves $(\Omega^*_{M_p}, d)$. The face and degeneracy maps of $M_\bullet$ induce, via pullback, face and degeneracy maps between the differential forms on M_p and $M_{p\pm1}$. Thus, one obtains the simplicial sheaf $\Omega^{\bullet,*}$ of differential forms on $M_\bullet$.

On the global sections of this sheaf

$$\Omega^{p,q}(M) = \Omega^q(M_p),$$

there is a horizontal differential $d : \Omega^{p,q}(M_\bullet) \to \Omega^{p,q+1}(M_\bullet)$, given by the exterior differential and vertical differential

$$\partial : \Omega^{p,q}(M_\bullet) \to \Omega^{p+1,q}(M_\bullet),$$

given by the alternating sum of pullbacks along the face maps

$$\partial(\omega) = \sum_{i=0}^{p+1} (-1)^i \partial_i^* \omega. \tag{3.1}$$

Proposition 3.9 $(\Omega^{p,q}(M_\bullet), d + (-1)^q \partial)_{p,q}$ *forms a double complex.*

Proof. $(d + (-1)^q \partial)^2 = 0$, as $d^2 = 0$ by Proposition 2.15, $\partial^2 = 0$ by the simplicial relations and $d\partial = \partial d$ as d is functorial. □

Moreover, since the differential forms form a sheaf of C^∞-module, they form a fine (Lemma 2.29) and hence acyclic sheaf.

In particular, for the simplicial manifold $G^\bullet \times M$, we have the double complex $\Omega^q(G^p \times M)$, what is a first de Rham type model for equivariant cohomology by the following Proposition.

Proposition 3.10 (Prop. 6.1 of [19]) *Let $M_\bullet$ be a simplicial manifold. There is a natural isomorphism*

$$H^*(\Omega^{\bullet,*}(M_\bullet), d + (-1)^* \partial) \cong H^*(\|M_\bullet\|, \mathbb{C}).$$

Simplicial Čech cohomology

Definition 3.11 (see [7, 25]) A *simplicial cover* for the simplicial manifold $M_\bullet$ is a family $\mathcal{U}^\bullet = \{\mathcal{U}^{(p)}\}$ of open covers such that

1. $\mathcal{U}^{(p)} = \{U_\alpha^{(p)} | \alpha \in A^{(p)}\}$ is an open cover of M_p, for each p, and
2. the family of index sets forms a simplicial set $A^\bullet = \{A^{(p)}\}$ satisfying
3. $\partial_i(U_\alpha^{(p)}) \subset U_{\partial_i \alpha}^{(p-1)}$ and $\sigma_i(U_\alpha^{(p)}) \subset U_{\sigma_i \alpha}^{(p+1)}$ for every $\alpha \in A^{(p)}$.

Definition 3.12 (see [7, 25]) Given a simplicial cover $\mathcal{U}^\bullet$, one forms the Čech chain groups $\check{C}^{\bullet,*}(\mathcal{U}^\bullet, \mathcal{F}^\bullet)$ by

$$\check{C}^{p,q}(\mathcal{U}^\bullet, \mathcal{F}^\bullet) = \prod_{\alpha_0^{(p)}, \dots, \alpha_q^{(p)} \in A^{(p)}} F^p \left(U_{\alpha_0^{(p)}}^{(p)} \cap \dots \cap U_{\alpha_q^{(p)}}^{(p)} \right),$$

with the usual Čech boundary operator $\delta \colon \check{C}^{p,q} \to \check{C}^{p,q+1}$ and the simplicial boundary map $\partial \colon \check{C}^{p,q} \to \check{C}^{p+1,q}$ defined as alternating sum as above.

Observe, that the third condition of the simplicial cover ensures that ∂ maps between the Čech groups. The *simplicial Čech cohomology*, denoted by

$$\check{H}^*(\mathcal{U}^\bullet, \mathcal{F}^\bullet),$$

is the cohomology of the double complex $(\check{C}^{p,q}, \partial, (-1)^p\delta)$. As in the non-simplicial case (see [26, section III.4]), any simplicial open cover induces a canonical homomorphism

$$\check{H}^*(\mathcal{U}^\bullet, \mathcal{F}^\bullet) \to H^*(M_\bullet, \mathcal{F}^\bullet).$$

Moreover, given a refinement $\mathcal{V}^\bullet$ of the simplicial open cover $\mathcal{U}^\bullet$, then the natural diagram

$$\begin{tikzcd} \check{H}^*(\mathcal{U}^\bullet, \mathcal{F}^\bullet) \arrow[r] \arrow[d] & H^*(M_\bullet, \mathcal{F}^\bullet) \\ \check{H}^*(\mathcal{V}^\bullet, \mathcal{F}^\bullet) \arrow[ur] & \end{tikzcd}$$

commutes. Thus one can form the limit over all refinements of simplicial open covers and obtains an isomorphism

$$\lim_{\mathcal{U}^\bullet} \check{H}^*(\mathcal{U}^\bullet, \mathcal{F}^\bullet) \to H^*(M_\bullet, \mathcal{F}^\bullet).$$

For more details see [7, 25].

Simplicial singular cohomology

Let A be an abelian group. Later, the most interesting cases for us will be $A \in \{\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{C}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}\}$. Then there is the locally constant sheaf $\underline{A}^\delta$, consisting of continuous maps to A furnished with the discrete topology, in any simplicial degree. The maps $\tilde{\partial}_i$ and $\tilde{\sigma}_i$ are given by pullback along ∂_i respectively σ_i. One can calculate $H^*(M_\bullet, \underline{A})$ via singular cohomology.

Definition 3.13 (see [19, p. 81]) The *simplicial singular cochain complex*

$$(C^{\bullet,\bullet}_{\text{sing}}(M_\bullet, A), \partial, \partial_{\text{sing}})$$

is the double complex consisting of groups

$$C^{p,q}_{\text{sing}} = C^q_{\text{sing}}(M_p) = \text{map}(C^\infty(\Delta^q, M_p), A)$$

of smooth singular cochains on each M_p with group structure induced from A, vertical boundary map induced from the simplicial manifold and horizontal boundary map given by the singular boundary operator.

To obtain a double complex one has to use the boundary map $\partial + (-1)^p\partial_{\text{sing}}$. A simplicial map $f_\bullet \colon M_\bullet \to M'_\bullet$ induces a map of double complexes $f^*_\bullet \colon C^{\bullet,\bullet}_{\text{sing}}(M'_\bullet, A) \to C^{\bullet,\bullet}_{\text{sing}}(M_\bullet, A)$.

Theorem 3.14 (Theorem 5.15. of [19]) *There are functorial isomorphisms*

$$H^*(\|M\|, A) = H^*_{\text{sing}}(M_\bullet, A) := H^*\left(C^{\bullet,\bullet}_{\text{sing}}(M_\bullet, A), \partial + (-1)^p\partial_{\text{sing}}\right)$$

To compare singular cohomology with general sheaf cohomology, one can use arguments of [44, pp. 191-200]. Sheafify the singular cochains $C^q_{\text{sing}}(M_p)$: Let $\mathcal{S}^q(M_p, A)$ be the sheaf associated to the presheaf

$$M \subset U \mapsto \text{map}(C^\infty(\Delta^q, U), A).$$

Then one has an acyclic resolution

$$0 \to \underline{A}_\bullet \to \mathcal{S}^0(M_\bullet, A) \to \mathcal{S}^1(M_\bullet, A) \to \dots$$

and hence

$$H^*(M_\bullet, A) = H^*(M_\bullet, \mathcal{S}^*(M_\bullet, A)).$$

On the other hand, the global sections of $\mathcal{S}^q(M_p, A)$ are exactly $C^q_{\text{sing}}(M_p)$.

Thus we have shown the following theorem.

Theorem 3.15

$$H^*(\|M\|, A) = H^*_{sing}(M_\bullet, A) = H^*(M_\bullet, \mathcal{S}^*) = H^*(M_\bullet, \underline{A}).$$

In particular, for $M_\bullet = G^\bullet \times M$, we obtain:

$$H^*_G(M, A) = H^*_{sing}(G^\bullet \times M, A) = H^*(G^\bullet \times M, \underline{A}).$$

Simplicial cellular cohomology

The most handy cohomology theory for calculation is cellular cohomology. Recall (compare [42, p. 12]) that a *CW complex* is a topological space X with a collection of subspaces, called cellular decomposition,

$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X,$$

such that X_0 is discrete, X_p is obtained from X_{p-1} by attaching p-cells, $X = \bigcup_i X_i$, and $U \subset X$ is closed, if and only if $U \cap X_p$ is closed in X_p for any $p \in \mathbb{N}$. A map $f\colon X \to Y$ between cellular complexes is called cellular, if $f(X_p) \subset Y_p$. The cellular chain complex (see [42, pp. 118-122]) is given by $C^n(X) = H^n_{\text{sing}}(X_n, X_{n-1}; A)$ and d^n_{cell} is the composition

$$H^n(X_n, X_{n-1}) \to H^n(X_n, \emptyset) \to H^{n+1}(X_{n+1}, X_n)$$

of the map induced from the inclusion $(X_n, \emptyset) \subset (X_n, X_{n-1})$ and the connecting morphism of $(X_n, \emptyset) \subset (X_{n+1}, \emptyset) \subset (X_{n+1}, X_n)$.

By a cellular decomposition of the simplicial manifold $G^\bullet \times M$, we understand a collection of topological spaces $(X_{p,q})_{p,q \in \mathbb{N}}$, such that $X_{p,*}$ is a cellular decomposition of $G^p \times M$ and all face and degeneracy maps are cellular. Thus we receive a double complex, the *simplicial cellular chain complex* $(C^q_{\text{cell}}(G^p \times M), d_{\text{cell}} + (-1)^q \partial)$. We define $H^*_{\text{cell}}(G^\bullet \times M, A)$ to be the cohomology of this double complex.

One has the following small proposition, for which I did not find a reference in the literature.

Proposition 3.16 *There is an isomorphism*

$$H^*_{\mathrm{cell}}(G^\bullet \times M, A) = H^*_{\mathrm{sing}}(G^\bullet \times M, A).$$

Proof. Given a map between the singular and cellular chains, Lemma 3.7 would imply the result. Hence we are done, if we find such a map for normal, i.e., non-simplicial spaces, in a functorial manner. There is no map between singular and cellular chains in general, but one can construct a complex of so called simplicial singular chains (see [18, Section V.8]), and functorial quasi-isomorphisms to both, singular and cellular chains. □

3.1.2 The Cartan model

A well-known de Rham-like model for equivariant cohomology goes back to Henri Cartan ([13]). Our Exposition follows [31]. Let G be a compact Lie group acting smoothly on the smooth manifold M and denote the Lie algebra of G by $\mathfrak{g} = T_eG$. Let $S^*(\mathfrak{g}^\vee)$ be the symmetric tensor algebra of the (complex) dual of the Lie algebra $\mathfrak{g}^\vee$. The group G acts on this algebra by the coadjoint action and on $\Omega^*(M)$ by pulling back forms along the map $m \mapsto gm$. Hence we have a G-action on $S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)$. The invariant part of this algebra $(S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G$ is exactly what one calls the *Cartan complex* and is denoted by $\Omega^*_G(M)$. In other words: The Cartan complex consists of G-equivariant polynomial maps $\omega\colon \mathfrak{g} \to \Omega^*(M)$. Let $\omega_1, \omega_2 \in \Omega^*_G(M)$, then there is a wedge product

$$(\omega_1 \wedge \omega_2)(X) = \omega_1(X) \wedge \omega_2(X).$$

On this algebra one defines a differential as

$$d_C\omega(X) = d(\omega(X)) + \iota(X^\sharp)\omega(X),$$

for $\omega \in \Omega^*_G(M))$ and $X \in \mathfrak{g}$, i.e., one takes the differential on the manifold and adds the contraction with the fundamental vector field. To make this differential raise the degree by one, the grading on $\Omega^*_G(M)$ is given by

$$\text{twice the polynomial degree} + \text{the differential form degree.}$$

Lemma 3.17 (Ω^*_G, d_C) *is a cochain complex.*

Proof. First, observe that d_C increases the total degree by one, since d increases the differential form degree, and the contraction ι, while decreasing the form degree by one, increases the polynomial degree by one. Next, one has to check, that the differential really maps invariant forms to invariant forms and that it squares to zero.

Let $\omega \in \Omega^*_G(M))$ and $X \in \mathfrak{g}$.

$$\begin{aligned} d_C\omega(\mathrm{Ad}_g X) &= d(\omega(\mathrm{Ad}_g X)) + \iota((\mathrm{Ad}_g X)^\sharp)\omega(\mathrm{Ad}_g X) \\ &= d(g\omega(X)) + \iota(gX^\sharp g^{-1})g(\omega(X)) \\ &= gd(\omega(X)) + g\iota(X^\sharp)g^{-1}g(\omega(X)) \\ &= gd_C\omega \end{aligned}$$

Thus $d_C\omega$ is G-equivariant. Moreover, we have

$$d_C^2\omega(X) = d^2\omega(X) + d\iota(X)\omega(X) + \iota(X)d\omega(X) + \iota(X)^2\omega(X) = L_X\omega(X)$$

and

$$L_X\omega(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX)\omega(X) = \frac{d}{dt}\Big|_{t=0} \omega(\exp(-tX)X\exp(tX)) = \frac{d}{dt}\Big|_{t=0} \omega(X) = 0.$$

Thus d_C squares to zero, i.e., it is a boundary operator. □

In the special case of $M = pt$, i.e., of a single point, the Cartan algebra reduces to the algebra of *invariant symmetric polynomials*

$$I^k(G) = ((S^*(\mathfrak{g}^\vee) \otimes \Omega^*(pt))^G)^k = (S^k(\mathfrak{g}^\vee))^G.$$

3.1.3 Getzlers resolution

In order to investigate cohomology of actions of non-compact groups, Ezra Getzler [23, Section 2] defines a bar-type resolution of the Cartan complex. We will apply his ideas slightly different: The complex defined by Getzler will allow us to compare equivariant integral cohomology (defined via the simplicial manifold) with equivariant cohomology defined by the Cartan model.

Let, as before, a Lie group G act on a smooth manifold M from the left. Define $\mathbb{C}$-vector spaces $C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ consisting of smooth maps, from the p-fold Cartesian product

$$G^p \to S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M),$$

to the space of polynomial maps from $\mathfrak{g}$ to differential forms on M. We give these groups a bigrading: The horizontal grading is the one of $S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)$ defined above and the vertical grading is p. The Cartan boundary operator $d + \iota$ now induces a map $(-1)^p(d + \iota)$, which increases the horizontal grading by 1 in any row. As we are not restricted to the G-invariant part of $S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)$, this map will not square to zero, but

$$((-1)^p(d + \iota))^2 = d\iota + \iota d = L$$

is the Lie derivative (see Proposition 2.15). In vertical direction, there is a differential

$$\bar{d} \colon C^k(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \to C^{k+1}(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$$

defined by

$$(\bar{d}f)(g_0, \ldots, g_k|X) := f(g_1, \ldots, g_k|X) + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k|X) \\ + (-1)^{k+1} g_k f(g_0, \ldots, g_{k-1}|\operatorname{Ad}(g_k^{-1})X)$$

for $g_0, \ldots, g_k \in G$ and $X \in \mathfrak{g}$.

Note, in particular, that the kernel of

$$\bar{d}\colon C^0(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \to C^1(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$$

is exactly $\Omega^*_G(M)$. Moreover, in case of a discrete Group G, $\mathfrak{g} = 0$ and thus one checks, that

$$C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) = C^p(G, \Omega^*(M)) = \Omega^{p,*}(G^\bullet \times M)$$

and $\bar{d}$ is equal to ∂.

In the case of a compact Lie group, the map $\bar{d}$ admits a contraction (compare, e.g., [25, p. 322]):

Lemma 3.18 *Integration over the group, with respect to a right invariant probability measure, defines a map*

$$\int_G : C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \to C^{p-1}(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \tag{3.2}$$

$$\left(\int_G f\right)(g_1, \ldots, g_{p-1}, m) = (-1)^i \int_{g \in G} f(g, g_1, \ldots, g_{p-1}, m) dg$$

such that $\bar{d} \int_G f = f$ *if* $\bar{d} f = 0$.

Proof. This is proven by a direct calculation:

$$\begin{aligned}
&\left(\bar{d} \int_G \omega\right)(g_1, \ldots, g_p, m) \\
&= \left(\int_G f\right)(g_2, \ldots, g_p | X) + \sum_{i=2}^{p} (-1)^i \left(\int_G f\right)(g_1, \ldots, g_{i-1}g_i, \ldots, g_p | X) \\
&\quad + (-1)^{p+1} g_p \left(\int_G f\right)(g_1, \ldots, g_{p-1} | \operatorname{Ad}(g_p^{-1}) X) \\
&= \int_G f(g, g_2, \ldots, g_p | X) dg + \sum_{i=2}^{p} (-1)^i \int_G f(g, g_1, \ldots, g_{i-1}g_i, \ldots, g_p | X) dg \\
&\quad + \int_G g_p f(g, g_1, \ldots, g_{p-1} | \operatorname{Ad}(g_p^{-1}) X) dg \\
&= \int_G \Bigg(f(g, g_2, \ldots, g_p | X) + \sum_{i=2}^{p} (-1)^i f(g, g_1, \ldots, g_{i-1}g_i, \ldots, g_p | X) \\
&\quad + (-1)^{p+1} g_p f(g, g_1, \ldots, g_{p-1} | \operatorname{Ad}(g_p^{-1}) X) \Bigg) dg
\end{aligned}$$

Now we apply $\bar{d} f(g, g_1, \ldots, g_p | X) = 0$

$$\begin{aligned}
&= \int_G \left(f(g_1, \ldots, g_p | X) - f(gg_1, \ldots, g_p | X) + f(g, g_2, \ldots, g_p | X) \right) dg \\
&= f(g_1, \ldots, g_p | X) - \int_G f(gg_1, g_2, \ldots, g_p | X) dg + \int_G f(g, g_2, \ldots, g_p | X) dg \\
&= f(g_1, \ldots, g_p | X)
\end{aligned}$$

□

Thus, for compact groups, the vertical cohomology of this bi-graded collection of groups is the Cartan complex.

One can turn the bi-graded collection $C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ of groups into a double complex. Therefore Getzler defines another map,

$$\bar{\iota}\colon C^p(G, S^l(\mathfrak{g}^\vee) \otimes \Omega^m(M)) \to C^{p-1}(G, S^{l+1}(\mathfrak{g}^\vee) \otimes \Omega^m(M)),$$

given by the formula

$$(\bar{\iota} f)(g_1, \dots, g_{p-1}|X) := \sum_{i=0}^{p-1} (-1)^i \left.\frac{d}{dt}\right|_{t=0} f(g_1, \dots, g_i, \exp(tX_i), g_{i+1}, \dots, g_{p-1}|X),$$

where $X_i = \mathrm{Ad}(g_{i+1} \dots g_{p-1})X$.

Lemma 3.19 (Lemma 2.1.1. of [23]) *The map $\bar{\iota}$ has the following properties:*

$$\bar{\iota}^2 = 0 \text{ and } \bar{d}\bar{\iota} + \bar{\iota}\bar{d} = -L.$$

Proof. This is shown in [23] by recollection of the sums in the definition of $\bar{\iota}$ and $\bar{d}$. □

Moreover one obtains:

Lemma 3.20 (Corollary 2.1.2. of [23]) $d_G = \bar{d} + \bar{\iota} + (-1)^p(d + \iota)$ *is a boundary operator on the total complex* $\bigoplus_{p+2q+r=n} C^p(G, S^q(\mathfrak{g}^\vee) \otimes \Omega^r(M))$.

Proof. d_G increases the total index by one, as $\bar{d}$ increases the first index, d increases the third index, ι decreases the third, while it is increasing the second index and $\bar{\iota}$ decreases the first index, while it is increasing the second one.

As d and ι are equivariant under the G-action, they commute with $\bar{d}$. And as d and ι only act on the manifold M and not on the group part, the same is true for $\bar{\iota}$. Thus

$$\begin{aligned} d_G^2 &= (\bar{d} + \bar{\iota})^2 + (-1)^p(\bar{d} + \bar{\iota})(d + \iota) + (-1)^{p\pm 1}(d + \iota)(\bar{d} + \bar{\iota}) + (d + \iota)^2 \\ &= \bar{d}\bar{\iota} + \bar{\iota}\bar{d} + (d\iota + \iota d) \\ &= -L + L = 0. \end{aligned}$$

□

Remark 3.21 The reader, who compares this with the original paper of Getzler will note that we changed some signs. It just seems more natural to us in this way. Furthermore Getzler uses some reduced subcomplex, which is, by standard arguments on simplicial modules (compare Proposition 1.6.5 in [32]), quasi-isomorphic to the full complex, which we have taken.

One can check that

$$\left(\left(\bigoplus_{\substack{p+q=n \\ q+r=k}} C^p(G, S^q(\mathfrak{g}^\vee) \otimes \Omega^r(M)) \right)_{n,k}, \bar{d} + (-1)^p \iota, (-1)^p d + \bar{\iota} \right)$$

is a double complex. But this point of view will not fit to the construction, which we want to do with this bigraded module later: We want to turn the groups $C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ into simplicial sheaves on $G^\bullet \times M$.

Definition 3.22 A *simplicial homotopy cochain complex* of modules is a triple $(M^{\bullet,*}, f, s)$, where $M^{\bullet,*}$ is a $\mathbb{Z}$-graded simplicial module, f is a map of simplicial modules, which increases the degree by one and s is a simplicial zero homotopy of f^2 which commutes with f and squares to zero, i.e.,

$$s\partial + \partial s = -f^2, \quad sf = fs, \text{ and } s^2 = 0.$$

Example 3.23 Observe that

$$(C^\bullet(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)), d + \iota, \bar{\iota})$$

is a simplicial homotopy cochain complex.

Definition + Proposition 3.24 *The total complex of a simplicial homotopy cochain complex* $(M^{\bullet,*}, f, s)$ *is the chain complex*

$$\left(\left(\bigoplus_{p+q=n} M^{p,q} \right)_n, \partial + s + (-1)^p f \right).$$

Proof. We have to check that $\partial + s + (-1)^p f$ defines a boundary map. Therefore calculate

$$\begin{aligned}(\partial + s + (-1)^p f)^2 &= \partial^2 + s^2 + \partial s + s\partial + (-1)^p(\partial + s)f + (-1)^{p-1} f(\partial + s) + f^2 \\ &= s\partial + \partial s + f^2 \\ &= 0.\end{aligned}$$

□

Observe that the total complexes of the interpretation of $C^\bullet(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ as double complex and as simplicial homotopy cochain complex coincide. Moreover, note for our applications later, that in the first column, of the double complex interpretation and the degree zero part of the interpretation as simplicial homotopy cochain complex are equal. In formulas this means

$$\left(\bigoplus_{\substack{p+q=n \\ q+r=k}} C^p(G, S^q(\mathfrak{g}^\vee) \otimes \Omega^r(M)) \right)_{n,0} = C^n \left(G, S^0(\mathfrak{g}^\vee) \otimes \Omega^0(M) \right).$$

3.1.4 A quasi-isomorphism

In this section, we will discuss a map defined in [23, Section 2.2.]. It will relate the complex $C^*(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ from the last section to the double complex $\Omega^*(G^\bullet \times M)$, which consists in degree (p, q) of q-forms on $G^p \times M$ with horizontal boundary map $d = d_{G^p} + d_M$ and vertical boundary map ∂ from the simplicial manifold structure. Thus we have an explicit identifications of chains in the one complex with chains in the other complex. This will be of particular interest to us in the discussion of equivariant characteristic forms (Section 3.2.3) and will allow us to compare our definition of equivariant differential cohomology (Section 4.1.3) with definitions given before.

Definition 3.25 (Def. 2.2.1. of [23]) The map $\mathcal{J} \colon \Omega^*(G^p \times M) \to \bigoplus_{l=0}^p C^l(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$ is defined by the formula

$$\mathcal{J}(\omega)(g_1, \ldots, g_l|X) := \sum_{\pi \in S(l,p-l)} \operatorname{sgn}(\pi)\,(i_\pi)^* \left(\iota_{\pi(l+1)}(X_{l+1}^{(\pi)}) \ldots \iota_{\pi(p)}(X_p^{(\pi)})\omega\right).$$

Here $S(l, p-l)$ is the set of shuffles, i.e., permutations π of $\{1, \ldots, p\}$, satisfying

$$\pi(1) < \ldots < \pi(l) \text{ and } \pi(l+1) < \ldots < \pi(p),$$

$X_j^{(\pi)} = \operatorname{Ad}(g_m \ldots g_l)X$, where m is the least integers less than l, such that $\pi(j) < \pi(m)$, ι_j means, that the Lie algebra element should be a tangent vector at the j-th copy of G, and $i_\pi \colon G^l \times M \to G^p \times M$ is the inclusion $x \mapsto (h_1, \ldots, h_p, x)$ with

$$h_j = \begin{cases} g_m & \text{if } j = \pi(m), 1 \le m \le l \\ e \in G & \text{otherwise,} \end{cases}$$

which is covered by the bundle inclusion $TM \to T(G^p \times M)$.

Observe that the image of ω under $\mathcal{J}$ does only depend on the zero form part and, in direction of any copy of G, on the one form part at the identity $e \in G$.

The next Lemma – which is mainly a citation of [23, Lemma 2.2.2.], but with signs corrected – shows, that the map $\mathcal{J}$ can be interpreted as a map of double complex.

Lemma 3.26 *The map $\mathcal{J}$ respects the boundaries with the correct sign, i.e.,*

$$\mathcal{J} \circ \partial = \left(\bar{d} + (-1)^p \iota\right) \circ \mathcal{J}$$

and, after decomposing $d = d_G + d_M$ with respect to the Cartesian product $G^p \times M$

$$\mathcal{J} \circ ((-1)^p\, d_M) = (-1)^{p'} d \circ \mathcal{J} \text{ and } \mathcal{J} \circ (-1^p)\, d_G = \bar{\iota} \circ \mathcal{J},$$

where p is the simplicial degree before and p' the simplicial degree after application of the map $\mathcal{J}$,

Proof. The following four types of terms contribute to $\mathcal{J} \circ \partial$:

1. those terms where ∂ acts by the group multiplication $G \times G \to G$ and $\mathcal{J}$ take a one form component on one of these groups: these parts cancel by symmetry;

2. those parts where ∂ acts by the group multiplication or the action on M and $\mathcal{J}$ takes the zero form component on this part: these contribute to $\bar{d} \circ \mathcal{J}$;

3. those terms where ∂ corresponds to the action of G on M and $\mathcal{J}$ takes the one form on this corresponding G at e yield $\iota \circ \mathcal{J}$. The sign comes from the fact that ∂_p has this sign in ∂.

This proves the first equation. For the second decompose the exterior derivative $d = \sum_{k=1}^{p} d_{G^{(k)}} + d_M$ on $G^p \times M$ further into a part corresponding to each copy of G and one corresponding to M. One checks immediately that $\mathcal{J} \circ (-1)^p \, d_M = (-1)^{p'} d \circ \mathcal{J}$, where the sign difference comes from interchanging d_M with the contractions. For the last equation, which contains $\bar{\iota}$, note that one can restrict to $\omega \in \Omega^q(G^p \times M)$, whose degree on each copy of G is either zero or one. Let $\pi \in S(l, p-l)$ be a shuffle and $\omega \in \Omega^q(G^p \times M)$ be a form, such that the differential form degree on the $\pi(k)$-th copy of G is zero if $k \leq l$ and one if $k \geq l+1$. Then for any $X \in \mathfrak{g}$ we can calculate

$$
\begin{aligned}
&(\bar{\iota} \circ \mathcal{J}\omega)(X) \\
&\quad = \sum_{k=1}^{l} (-1)^k L^k_{X_k} (\mathcal{J}\omega)(X)
\end{aligned}
$$

where L^k should denote the Lie derivative on the k-th G,

$$
\begin{aligned}
&= \sum_{k=1}^{l} (-1)^k \operatorname{sgn}(\pi) \sigma^*_{k-1} \iota_k(X_k) d_{G^{(k)}} i^*_\pi \left(\iota_{\pi(l+1)}(X^{(\pi)}_{l+1}) \dots \iota_{\pi(p)}(X^{(\pi)}_p) \omega \right) \\
&= \sum_{k=1}^{l} (-1)^k \operatorname{sgn}(\pi) \sigma^*_{k-1} \iota_k(X_k) i^*_\pi \left(d_{G^{(\pi(k))}} \iota_{\pi(l+1)}(X^{(\pi)}_{l+1}) \dots \iota_{\pi(p)}(X^{(\pi)}_p) \omega \right) \\
&= \sum_{k=1}^{l} (-1)^k \operatorname{sgn}(\pi) \sigma^*_{k-1} \iota_k(X_k) i^*_\pi (-1)^{p-l} \left(\iota_{\pi(l+1)}(X^{(\pi)}_{l+1}) \dots \iota_{\pi(p)}(X^{(\pi)}_p) d_{G^{(\pi(k))}} \omega \right) \\
&\qquad + \text{terms which include a Lie derivative } L^k_{X_k}(X_{l+j})
\end{aligned}
$$

The sign in the first term comes from the fact that the d and ι anti-commute, since they act on different G's. Moreover the other terms vanish, as each of them contains a factor of the form $\left.\frac{d}{dt}\right|_{t=0} e^{tX} X e^{-tX} = [X, X] = 0$.

$$
= \sum_{k=1}^{l} (-1)^{p-l+k} \operatorname{sgn}(\pi) \sigma^*_{k-1} i^*_\pi \left(\iota_{\pi(k)}(X_k) \left(\iota_{\pi(l+1)}(X^{(\pi)}_{l+1}) \dots \iota_{\pi(p)}(X^{(\pi)}_p) d_{G^{(\pi(k))}} \omega \right) \right)
$$

Now define $\pi_k \in S(l-1, p-l+1)$ as the shuffle, which is obtained from π by transpose k and l and resorting the two groups.

$$= \sum_{k=1}^{l} (-1)^{p-l+k} \operatorname{sgn}(\pi) i^*_{\pi_k} \left(\iota_{\pi(k)}(X_k) \left(\iota_{\pi(l+1)}(X^{(\pi)}_{l+1}) \dots \iota_{\pi(p)}(X^{(\pi)}_p) d_{G^{(\pi(k))}} \omega \right) \right)$$

The sign of π and π_k differ by a $(-1)^{l-k}$ for transposing $\pi(k)$ into the second group and a sign change for every transposition which is necessary to reorder the second group. These reordering sign also occur a second time, when reordering the contractions. Thus they cancel out each other.

$$\begin{aligned}
&= \sum_{k=1}^{l} (-1)^{p} \operatorname{sgn}(\pi_k) i^*_{\pi_k} \left(\iota_{\pi_k(l)}(X^{(\pi_k)}_{l}) \dots \iota_{\pi_k(p)}(X^{(\pi_k)}_p) d_{G^{(\pi(k))}} \omega \right) \\
&= (-1)^p \sum_{k=1}^{l} \mathcal{J} \left(d_{G^{(\pi(k))}} \omega \right) (X) \\
&= (-1)^p \mathcal{J} \left(\sum_{k=1}^{l} d_{G^{(\pi(k))}} \omega \right) (X) \\
&= (-1)^p \mathcal{J} \left(d_{G^p} \omega \right) (X).
\end{aligned}$$

Note for the last step, that $\mathcal{J}$ vanishes on forms, whose degree on any copy of G is larger than one. ☐

Moreover, the map $\mathcal{J}$ induces an isomorphism in the cohomology of the associated total complexes.

Theorem 3.27 (Theorem 2.2.3. of [23]) *$\mathcal{J}$ is a quasi-isomorphism.*

3.2 Equivariant characteristic forms and classes

In this section G and K will denote Lie groups.

In analogy to the non-equivariant case (Def. 2.42), we define equivariant characteristic classes (with values in the ring $A \in \{\mathbb{Z}, \mathbb{R}, \mathbb{C}\}$):

Definition 3.28 An *G-equivariant characteristic class c* for G-equivariant principal K-bundles associates to every isomorphism class of topological G-equivariant principal K-bundles $\pi\colon E \to X$ a cohomology class $c(E) \in H^*_G(X, A)$, such that $c((\bar{f}, f)^* E) = f^* c(E)$ for every pullback diagram

$$\begin{tikzcd}
E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
X' \arrow[r, "f"] & X
\end{tikzcd}$$

of G-equivariant principal K-bundles.

If $A = \mathbb{Z}$ the class is called *integral.*

Any characteristic class for principal K bundles c yields to an equivariant characteristic class c^G by the following procedure (compare , e.g., [31, Section 5.4]): Let E be a G-equivariant principal K-bundles and let $EG \to BG$ denote the universal G bundle. One defines

$$c^G(E) := c(EG \times_G E) \in H^*(EG \times_G X) = H^*_G(X).$$

Note that the by

$$\pi_G = id \times \pi : EG \times_G E \to EG \times_G M =: M_G$$

one constructs a principal K-bundle from the G-equivariant principal K-bundle $E \to M$.

Lemma 3.29 *The association procedure $c \mapsto c^G$ is an one to one correspondence between characteristic classes and equivariant characteristic classes.*

Proof. Any principal K-bundle $\pi\colon E \to X$ can be understood as a G-equivariant principal K-bundle with trivial G-action. This holds for morphisms, too. Moreover, fix any point $p \in BG$, then the inclusion $i_p\colon X \to \{p\} \times X \to BG \times X$ induces a map $i_p^*\colon H^*_G(X) = H^*(EG \times_G X) = H^*(BG \times X) \to H^*(X)$. Thus an equivariant characteristic class naturally yields a characteristic class. We are now going to prove that this is an inverse to $c \mapsto c^G$.

Let c be a characteristic class and E be a principal K-bundle. Then $c^G(E) = c(EG \times_G E) = c(BG \times E)$. Let $\mathrm{pr}\colon BG \times X \to X$ denote the projection, then $BG \times E = \mathrm{pr}^* E$ and $i_p^* c(BG \times E) = i_p^* c(\mathrm{pr}^* E) = i_p^*(\mathrm{pr}^* c(E)) = (\mathrm{pr} \circ i_p)^* c(E) = c(E)$, since $\mathrm{pr} \circ i_p = \mathrm{id}_X$.

On the other hand, let c be G-equivariant characteristic class. We have to show that for any G-equivariant principal K-bundle $F \to Y$

$$c(F) = i^* c(EG \times_G F) \in H^*_G(Y) = H^*(EG \times_G Y),$$

where $i\colon EG \times_G Y \to BG \times (EG \times_G Y)$ is an inclusion as above. Both squares in the commutative diagram

$$\begin{tikzcd}
EG \times_G F \arrow[d] & EG \times F \arrow[l] \arrow[r] \arrow[d] & F \arrow[d] \\
EG \times_G Y & EG \times Y \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & Y
\end{tikzcd}$$

are pullbacks of G-equivariant bundles, hence $\mathrm{pr}_2^* c(F) = \mathrm{pr}_1^* c(EG \times_G F)$ in $H^*_G(EG \times Y)$. Since EG is contractible and Borel cohomology is invariant under (non-equivariant) contractions, pr_2^* is an isomorphism. Thus we only have to show that $(\mathrm{pr}_2^*)^{-1} \circ \mathrm{pr}_1^* = i^*$. This follows, because the diagram

$$\begin{tikzcd}
EG \times_Y \arrow[rr, "i"] & & BG \times (EG \times_G Y) \\
& (EG \times EG \times Y)/G \arrow[ul, "\mathrm{pr}_2"] \arrow[ur, "\mathrm{pr}_1"'] &
\end{tikzcd}$$

commutes up to homotopy, since EG is contractible. $\square$

Remark 3.30 This lemma is a reformulation of the basic statement of [34] and the reason why Jon Peter May thinks, that equivariant characteristic classes in Borel cohomology are too 'crude'.

For non-equivariant characteristic classes, one has the Chern-Weil-construction producing characteristic classes out of differential geometric data given by a connection and its curvature. One can do something similar in the equivariant case. The definition of equivariant characteristic forms goes back to [3] and is also discussed in [2, pp.204] and [31]. For compact connected Lie groups acting on the bundle, there is a proof that the equivariant characteristic form calculates the equivariant characteristic class as defined above via the Borel-construction give in [5]. This equality is generally assumed the hold, see, e.g., [28, p. 311] without giving or citing a proof.

We will motivate the definition of Nicole Berline and Michèle Vergne and proof that equivariant characteristic forms calculate the equivariant characteristic classes defined via the Borel construction for general compact Lie groups acting. The construction we apply therefore also shows how to extend the definition of characteristic form to non-compact groups. To do so, we will refine a construction of [19]: Johan Dupont uses simplicial manifolds to show that the Chern-Weil-construction is a way to construct characteristic classes. Hence it is natural to use bisimplicial manifolds to show the statement for simplicial manifolds. This section can be read as a proof to [23, Theorem 3.1.1], which Getzler claims to be proven in [20], where only a weaker statement is shown. Nevertheless, some arguments are influenced by [23].

3.2.1 Bisimplicial manifolds

In this section we will define bisimplicial manifolds, as this motivates a construction we will employ in the next section. In the end it will turn out that we don't have to care about bisimplicial manifold, as they are reducible to their diagonal, which as simplicial manifold.

Definition 3.31 (see [24, p. 196]) A *bisimplicial set* is a simplicial object in the category of simplicial sets or equivalently a functor $\Delta^{op} \times \Delta^{op} \to Sets$.

Remark 3.32 A bisimplicial set is a collection of Sets $\{X_{p,q}|p, q = 0, 1, \ldots\}$ together with vertical and horizontal face and degeneracy maps, such that vertical and horizontal maps commute.

Definition 3.33 A *bisimplicial manifold* is a collection of manifolds $\{X_{p,q}|p, q = 0, 1, \ldots\}$, which forms a bisimplicial set and all face and degeneracy maps are smooth.

Example 3.34 Let $M_\bullet$ be a simplicial manifold and $\mathcal{U}^\bullet$ a simplicial cover. Recall (from Definition 3.11) that this is a family of open covers $\mathcal{U}^\bullet = \{\mathcal{U}^{(p)}\}$, such that

1. $\mathcal{U}^{(p)} = \{U_\alpha^{(p)}|\alpha \in A^{(p)}\}$ is an open cover of M_p and
2. the index sets form a simplicial set $A^\bullet = \{A^{(p)}\}$ satisfying

3. $\partial_i(U_\alpha^{(p+1)}) \subset U_{\partial_i\alpha}^{(p)}$ and $\sigma_i(U_\alpha^{(p+1)}) \subset U_{\sigma_i\alpha}^{(p)}$.

Thus for any p we can construct the Čech simplicial manifold [19, Example 5, p.78]:

$$(N_2 M_{\mathcal{U}})_{p,q} := \coprod_{(\alpha_0,\ldots,\alpha_q)} U_{\alpha_0}^{(p)} \cap \ldots \cap U_{\alpha_q}^{(p)},$$

where the disjoint union is taken over all $(q+1)$-tuples $(\alpha_0,\ldots,\alpha_q) \in (A^{(p)})^{q+1}$ with $U_{\alpha_0}^{(p)} \cap \ldots \cap U_{\alpha_q}^{(p)} \neq \emptyset$. The face and degeneracy maps are given on the index sets by removing, respective doubling of the i-th index and on the open sets by the corresponding inclusions.

Lemma 3.35 *$N_2 M_{\mathcal{U}}$ is a bisimplicial manifold.*

Proof. Clearly, we have a bi-graded collection of manifolds. The third property of the simplicial cover ensure that the face and degeneracy maps of $M_\bullet$ restrict to the disjoint unions of intersections of covering sets and thus induce vertical face and degeneracy maps for the bisimplicial manifold. That these vertical maps compute with the horizontal 'Čech' maps follows, because it is the same, if one first restricts the neighborhood of a point, and then map the point, or doing it the other way around. □

There are evidently two ways to geometrically realize a bisimplicial space: 1) first realize vertically and afterwards realize the received simplicial space in horizontal direction or 2) do it the other way around. Moreover, there is also a third one: realize the diagonal!

Definition 3.36 (see [24, p. 197]) Let $(M_{\bullet,\bullet}, \partial_i, \sigma_i, \partial_i', \sigma_i')$ be a bisimplicial manifold. The *diagonal* is the simplicial manifold

$$(p \mapsto M_{p,p}, \partial_i \circ \partial_i', \sigma_i \circ \sigma_i').$$

Lemma 3.37 (see [41, p. 10/86/94]) *There are canonical homeomorphisms*

$$\|p \mapsto X_{p,p}\| = \|p \mapsto \|q \mapsto X_{p,q}\|\| = \|q \mapsto \|p \mapsto X_{p,q}\|\|.$$

This lemma motivates to reduce the bisimplicial manifold $N_2 M_{\mathcal{U}}$ to its diagonal.

Definition 3.38 Let $M_\bullet$ be a simplicial manifold with simplicial cover $\mathcal{U}^\bullet$. The simplicial manifold $NM_{\mathcal{U}}$ is defined to be the diagonal of $N_2 M_{\mathcal{U}}$.

Lemma 3.39 *The inclusions $\mathcal{U}^{(p)} \ni U_\alpha^{(p)} \subset M_p$ induce a map*

$$NM_{\mathcal{U}} \to M_\bullet, \tag{3.3}$$

which induces an functorial isomorphism in cohomology

$$H^*(\|M\|, A) \cong H^*(\|NM_{\mathcal{U}}\|, A),$$

with coefficients in any abelian group A.

Proof. We will combine Čech and simplicial singular cohomology to prove this statement. Recall from Section 3.1.1, that $H^*(\|M\|, A) = H^*(M_\bullet, \mathcal{S}^*)$.

Let $\mathcal{U}^\bullet$ be a (locally finite) simplicial open cover of $M_\bullet$, then we can calculate the simplicial sheaf cohomology $H^*(M_\bullet, \mathcal{S}^*(M_\bullet, A))$ explicitly by the Čech complex for the chain complex of simplicial sheaves $(\mathcal{S}^*(M_\bullet, \underline{A}))$

$$\check{C}^{p,q,r}(\mathcal{U}^\bullet) = \check{C}^{p,q}(\mathcal{U}^\bullet, \mathcal{S}^r(M_\bullet, \underline{A})$$

which is a triple complex with boundary map $\partial + (-1)^p\delta + (-1)^{p+q}\partial_{\mathrm{sing}}$. As the sheaves $(\mathcal{S}^*(M, \underline{A}))$ are fine, i.e., admit a partition of unity, one can contract the triple complex in the Cech direction and obtains the double complex $C^{p,q}_{\mathrm{sing},\mathcal{U}}(M_\bullet, A)$ of cochains with support in $\mathcal{U}$, i.e., one restricts to those simplices whose image is contained completely in one of the open sets of $\mathcal{U}$. Such simplices are known as $\mathcal{U}$-small simplices.

Fix some p. As any $\mathcal{U}$-small simplex is a simplex, there is a restriction map $C^{p,*}_{\mathrm{sing}}(M_p, A) \to C^{p,*}_{\mathrm{sing},\mathcal{U}^{(p)}}(M_p, A)$. The theorem of the small simplex (see, e.g., [44, pp.197]) states that this map induces an isomorphism in cohomology. By Lemma 3.7, this turns over to the simplicial manifold, hence we can conclude that

$$C^{\bullet,*}_{\mathrm{sing}}(M_\bullet, A) \to C^{\bullet,*}_{\mathrm{sing},\mathcal{U}^{(\bullet)}}(M_\bullet, A)$$

also induces an isomorphism in cohomology.

On the other hand, one has an equality of groups

$$C^{p,r}_{\mathrm{sing}}(NM_{\mathcal{U}}) = \check{C}^{p,p,r}(\mathcal{U}, \mathcal{S}^*(M_\bullet, \underline{A})).$$

In other words $C^{\bullet,r}_{\mathrm{sing}}(NM_{\mathcal{U}})$ coincides on the set-level with the diagonal of of the bisimplicial complex

$$\check{C}^{\bullet,\bullet,r}(\mathcal{U}, \mathcal{S}^*(M_\bullet, \underline{A})).$$

Moreover, one check from the definitions, that the face and degeneracy maps coincide, too. Thus the $C^{\bullet,r}_{\mathrm{sing}}(NM_{\mathcal{U}})$ and the diagonal of $\check{C}^{\bullet,\bullet,r}(\mathcal{U}, \mathcal{S}^*(M_\bullet, \underline{A}))$ are the same simplicial objects.

By the generalized Eilenberg-Zilber theorem ([24, Ch. IV, Theorem 2.4]), there is a chain homotopy equivalence between the diagonal and the total complex of a bisimplicial complex and hence from

$$\bigoplus_{p+r=n} C^{p,r}_{\mathrm{sing}}(NM_{\mathcal{U}}) \to \bigoplus_{p+q+r=n} \check{C}^{p,q,r}(\mathcal{U}, \mathcal{S}^*(M_\bullet, \underline{A})).$$

This induces the asserted isomorphism in cohomology. □

3.2.2 Classifying bundle and classifying map

Let K be a Lie group. A simplicial manifold, whose realization is $EK \to BK$ is given in the following way: $NK := K^\bullet \times pt$ is the simplicial manifold of the action of K on a point (see Example 3.2) and $\left(N\overline{K}\right)_\bullet$ is given as $N\overline{K}_p = K^{p+1}$ with face maps

$$\partial_i(k_0, \ldots, k_p) = (k_0, \ldots, \hat{k}_i, \ldots, k_p)$$

and degeneracy maps

$$\sigma_i(k_0, \ldots, k_p) = (k_0, \ldots, k_i, k_i, \ldots, k_p).$$

The K-action on $N\overline{K}$ given by

$$(k_0, \ldots, k_p)k = (k_0 k, \ldots, k_p k)$$

is compatible with the face and degeneracy maps and hence a simplicial action. Moreover, there is a simplicial map

$$\gamma : N\overline{K} \to NK, (k_0, \ldots, k_p) \mapsto (k_0 k_1^{-1}, \ldots, k_{p-1} k_p^{-1}).$$

Lemma 3.40 (Prop. 5.3 of [19]) $\|\gamma\| \colon \left\|N\overline{K}\right\| \to \|NK\|$ *is a principal* K*-bundle.*

The following lemma is is a special case of a standard argument in the theory of simplicial sets (see [32, Prop. 1.6.7]), which applies in the case that there is an additional degeneracy.

Lemma 3.41 $\left\|N\overline{K}\right\|$ *is contractible.*

Proof. By definition $\left\|N\overline{K}\right\| = \bigcup \Delta^n \times K^{n+1}/\sim$. Define $h : [0,1] \times \left\|N\overline{K}\right\| \to \left\|N\overline{K}\right\|$ by

$$h_s(t_0, \ldots, t_n, k_0, \ldots, k_n) = (s, (1-s)t_0, \ldots, (1-s)t_n, e, k_0, \ldots, k_n)$$

$h_0((t_i),(k_i)) = (0,(t_i),e,(k_i)) = (\partial^0((t_i)),e,(k_i)) \sim ((t_i),\partial_0(e,(k_i))) = ((t_i),(k_i))$ and $h_1((t_i),(k_i)) = (1,(0),e,(k_i)) = ((\partial^1)^n(1),e,(k_i)) \sim (1,(\partial_1)^n(e,(k_i))) = (1,e) \in \Delta^0 \times K$. Hence the homotopy h is a contraction of $\left\|N\overline{K}\right\|$ to a point. $\square$

Thus we have a model of $EK = \left\|N\overline{K}\right\| \to BK = {}^{EK}\!/_{K} = \|NK\|$. The map $\gamma \colon N\overline{K} \to NK$ is a special case of the following object.

Definition 3.42 (compare [19, p. 93]) A simplicial K-bundle $\pi \colon E \to M$ is a sequence $\pi_p \colon E_p \to M_p$ of differential K-bundles, where $E = \{E_p\}$ and $M = \{M_p\}$ are simplicial manifolds, π is a simplicial map and the right action of K on E, $R_k : E \to E$, is simplicial, i.e., commutes with all face and degeneracy maps.

An G-equivariant principal K-bundle $\pi \colon E \to M$ leads to the simplicial K-bundle

$$\pi_\bullet = id_G \times \pi \colon G^\bullet \times E \to G^\bullet \times M,$$

where the action of K is given by trivial extension along G. We want to construct a classifying map of the bundle E, which will be the geometric realization of a map of simplicial manifolds. Therefore we are going to define an intermediate bundle, mapping to the classifying space and to $G^\bullet \times E \to G^\bullet \times M$.

Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M and let the induced open cover of E be denoted by $\pi^{-1}\mathcal{U} = \{V_\alpha\}$, $V_\alpha = \pi^{-1}(U_\alpha)$. These covers of M and E induce simplicial covers of $G^\bullet \times E$ and $G^\bullet \times M$ as follows (compare [25, p.319]):

Define a simplicial index set $A^{(p)} = A^{p+1}$ with face and degeneracy maps given by removing respective doubling of the i-th element. Then define the simplicial cover $\pi^{-1}\mathcal{U}^{(p)} = \{V_\alpha^{(p)}\}_{\alpha \in A^{(p)}}$ inductively by

$$V_\alpha^{(p)} = \bigcap_{i=0}^{p} \partial_i^{-1}\left(V_{\partial_i(\alpha)}^{(p-1)}\right),$$

where $V_\alpha^{(0)} = V_\alpha$ for any $\alpha \in A^{(0)} = A$. The following lemma gives an alternative description of this construction.

Lemma 3.43

$$V_\alpha^{(p)} = \{(g_1, \dots, g_p, m) \mid m \in V_{\alpha_p}, g_p m \in V_{\alpha_{p-1}}, \dots, g_1 \dots g_p m \in V_{\alpha_0}\}$$

Proof. We will prove this by induction. For $p = 0$ there is nothing to show. Let $p > 0$:

$$\begin{aligned} V_\alpha^{(p)} &= \bigcap_{i=0}^{p} \partial_i^{-1}(V_{\partial_i(\alpha)}^{(p-1)}) \\ &= \bigcap_{i=0}^{p} \left\{(g_1, \dots, g_p, m) \mid \partial_i(g_1, \dots, g_p, m) \in V_{\partial_i(\alpha)}^{(p-1)}\right\} \end{aligned}$$

For any $i = 0, \dots, p$ we can apply the induction hypothesis to $\partial_i(g_1, \dots, g_p, m) \in V_{\partial_i(\alpha)}^{(p-1)}$, what implies

$$\begin{aligned} m \in V_{\alpha_p}\ , \ \dots\ , \ g_{i+2} \dots g_p m \in V_{\alpha_{i+1}}, \\ g_i \dots g_p m \in V_{\alpha_{i-1}}\ , \ \dots\ , \ g_1 \dots g_p m \in V_{\alpha_0}. \end{aligned}$$

This is almost the right-hand side of the condition to be proven, just the i-th term is missing. As i runs from 0 to p, we get for the intersection exactly

$$V_\alpha^{(p)} = \{(g_1, \dots, g_p, m) \mid m \in V_{\alpha_p}, g_p m \in V_{\alpha_{p-1}}, \dots, g_1 \dots g_p m \in V_{\alpha_0}\}.$$

□

By the construction of Definition 3.38, we have a simplicial bundle

$$\pi \colon N(G^\bullet \times E)_{\pi^{-1}\mathcal{U}} \to N(G^\bullet \times M)_{\mathcal{U}}.$$

Suppose the cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ trivializes E with trivialization $\varphi_\alpha \colon V_\alpha = \pi^{-1}(U_\alpha) \to U_\alpha \times K$ and transition functions $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to K$. Then there is an induced map

$$\bar{\psi} \colon N(G^\bullet \times E)_{\pi^{-1}\mathcal{U}} \to N\overline{K},$$

which is given on the intersection of $p+1$ covering sets of $G^p \times E$

$$V = \bigcap_{j=0}^{p} V^{(p)}_{\alpha_0^j, \dots, \alpha_p^j}$$

by

$$(g_1, \dots, g_p, x) \mapsto (\varphi_{\alpha_0^0}(g_1 \dots g_p x), \varphi_{\alpha_1^1}(g_2 \dots g_p x), \dots, \varphi_{\alpha_p^p}(x)) \in K^{p+1},$$

where, on the right-hand side, the maps φ_α are understood to be composted with the projection to K.

Next, we want to define $\psi : N(G^\bullet \times M)_\mathcal{U} \to NK$, such that $\bar{\psi}$ covers ψ. Therefore we need some additional transition functions of the bundle. Define

$$h_{\alpha\beta} \colon G \times M \supset \partial_0^{-1} U_\alpha \cap \partial_1^{-1} U_\beta \to K, (g, x) \mapsto (\pi_2 \circ \varphi_\alpha(gX))(\pi_2 \circ \varphi_\beta(X))^{-1},$$

for any $X \in \pi^{-1}(x)$. This definition is independent of the chosen fiber element, as any other element in the fiber equals Xk for some $k \in K$ and

$$\begin{aligned}(\pi_2 \circ \varphi_\alpha(g(Xk))(\pi_2 \circ \varphi_\beta(Xk))^{-1} &= (\pi_2 \circ \varphi_\alpha(gX)k)(\pi_2 \circ \varphi_\beta(X)k)^{-1} \\ &= (\pi_2 \circ \varphi_\alpha(gX))kk^{-1}(\pi_2 \circ \varphi_\beta(X))^{-1}.\end{aligned}$$

As there exist local smooth section, G acts smoothly and the trivialization maps are smooth, $h_{\alpha\beta}$ is smooth too.

Define ψ on

$$U = \bigcap_{j=0}^{q} U^{(p)}_{\alpha_0^j, \dots, \alpha_p^j}$$

by

$$(g_1, \dots, g_p, m) \mapsto (h_{\alpha_0^0 \alpha_1^1}(g_1, g_2 \dots g_p m), h_{\alpha_1^1 \alpha_2^2}(g_2, g_3 \dots g_p m), \dots, h_{\alpha_{p-1}^{p-1} \alpha_p^p}(g_p, m), *).$$

One checks from the definitions that there is a commutative diagram

$$\begin{tikzcd} N(G^\bullet \times E)_{\pi^{-1}\mathcal{U}} \arrow[r, "\bar{\psi}"] \arrow[d, "\pi_\mathcal{U}"] & N\overline{K} \arrow[d, "\gamma"] \\ N(G^\bullet \times M)_\mathcal{U} \arrow[r, "\psi"] & NK \end{tikzcd} \tag{3.4}$$

of simplicial manifolds. Later on we will need the following statement.

Lemma 3.44 *The geometric realization of this diagram is a pullback.*

Proof. This follows, as the bundle map is K-equivariant, from Lemma 2.5. □

3.2.3 Dupont's simplicial forms, connections and transgression

Let $M_\bullet$ be a simplicial manifold. Dupont has given another definition for simplicial differential forms than the cochain complex of simplicial sheaves $\Omega^{\bullet,*}$

Definition 3.45 (Def. 6.2 of [19]) A *simplicial n-form* ω on M_* is a sequence $\omega = \{\omega^{(p)}\}_{p\in\mathbb{N}}$, where each $\omega^{(p)} \in \Omega^n(\Delta^p \times M_p)$, such that

$$(\partial^i \times \mathrm{id}_{M_p})^* \omega^{(p)} = (\mathrm{id}_{\Delta^{p-1}} \times \partial_i)^* \omega^{(p-1)}$$

on $\Delta^{p-1} \times M_p$ for $i = 0, \dots, p$ and $p \in \mathbb{N}$. The space of simplicial n-forms will be denoted by $\mathcal{A}^n(M_\bullet)$.

The differential $d\colon \mathcal{A}^n(M_\bullet) \to \mathcal{A}^{n+1}(M_\bullet)$ is in any simplicial level the sum of $d_{M_p} + d_{\Delta^p}$. We can compare this complex of differential forms to the simplicial de Rham complex.

Lemma 3.46 (Theorem 6.4 of [19]) *The map*

$$\int_\Delta : \mathcal{A}^n(M_p) \to \bigoplus_{p+q=n} \Omega^q(M_p)$$
$$\omega^{(p)} \mapsto \int_{\Delta^p} \omega^{(p)} \in \Omega^{n-p}(M_p),$$

given by integration over the simplices, induces a quasi-isomorphism of chain complexes.

Definition 3.47 A *connection* on a simplicial principal K-bundle $E \to M$ is a Dupont-1-form $\vartheta \in \mathcal{A}^1(E, \mathfrak{k})$ such that the restriction to any $\Delta^p \times E_p$ is a connection on the bundle

$$\Delta^p \times E_p \to \Delta^p \times M_p.$$

The curvature of the connection is defined as

$$\Omega = d\vartheta + \frac{1}{2}[\vartheta, \vartheta] \in \mathcal{A}^2(E, \mathfrak{k}).$$

Theorem 3.48 *Let $P \in I^q(K)$ be an invariant symmetric polynomial, $E_\bullet \to M_\bullet$ a simplicial principal K-bundle with connection ϑ and curvature Ω.*

1. *$P(\Omega^q) = P(\Omega, \ldots, \Omega) \in \mathcal{A}^{2q}(E)$ is a basic $2q$-form, i.e., it is an element of $\pi^*\mathcal{A}^{2q}(M)$ or in words a pullback from the base space M. The form $\omega_P(\vartheta) \in \mathcal{A}^{2q}(M)$, s.t., $\pi^*\omega_P(\vartheta) = P(\Omega^q)$, is called* characteristic form *of (E, ϑ).*

2. *$I^q(K) \ni P \mapsto \omega_P(\vartheta) \in \mathcal{A}^{2q}(M)$ is an algebra homomorphism.*

3. *$\omega_P(\vartheta) \in \mathcal{A}^{2q}(M)$ is closed.*

4. *Given two simplicial connections ϑ_0, ϑ_1 on $E_\bullet$, then there is a path of connections from the first to the second, i.e., a connection $\tilde{\vartheta}$ on $\mathbb{R} \times E_\bullet \to \mathbb{R} \times M_\bullet$ such that $\tilde{\vartheta}|_{\{i\}\times M} = \vartheta_i$ for $i = 0, 1$. The* transgression form

   $$\widetilde{\omega}_P(\vartheta_1, \vartheta_0) = \int_{[0,1]\times M/M} \omega_P(\tilde{\vartheta}) \in \mathcal{A}^{2q-1}(M)/d\mathcal{A}^{2q-2}(M),$$

   is independent of the path chosen and satisfies as well

   $$d\tilde{\omega}_P(\vartheta_1, \vartheta_0) = \omega_P(\vartheta_1) - \omega_P(\vartheta_0) \tag{3.5}$$

   as for any third connection ϑ_2 on E

   $$\tilde{\omega}_P(\vartheta_2, \vartheta_1) + \tilde{\omega}_P(\vartheta_1, \vartheta_0) = \tilde{\omega}_P(\vartheta_2, \vartheta_1). \tag{3.6}$$

5. *Let $f\colon N \to M$ be a smooth map, then $f^*\omega_P(\vartheta) = \omega_P((\bar{f}, f)^*\vartheta)$.*

6. *The class $c_P(E) = [\omega_P(\vartheta)] \in H^{2n}(M)$ is called* characteristic class *(defined via the Chern-Weil-Construction). It is independent of the connection and does only depend on the isomorphism class of the bundle.*

Proof. These statements are more or less standard, but we give proofs for convenience of the reader.

First, the simplicial form is basic, if this is true on any simplicial level, where it is a standard fact that the curvature is horizontal and equivariant (see e.g. [19, Prop 3.12 b)]) and as P is invariant, $P((\Omega^{(p)})^q)$ is basic. To the second assertion: the question about sums and scalars follows clearly from the definition. Let $Q \in I^l(K)$, then

$$(P \cdot Q)(\Omega^{q+l}) = \frac{1}{(q+l)!}\sum_{\sigma} P(\Omega^q) \wedge Q(\Omega^l) = P(\Omega^q) \wedge Q(\Omega^l).$$

Thus $P \mapsto \omega_P(\vartheta)$ is a homomorphism of algebras.

To show closedness, it is, since $\pi^* \colon \mathcal{A}^{2q}(M) \to \mathcal{A}^{2q}(E)$ is injective, sufficient to show that $dP(\Omega^q) \in \mathcal{A}^{2q+1}(E)$ vanishes. We compute

$$\begin{aligned}
dP(\Omega^q) &= qP(d\Omega \wedge \Omega^{q-1}) \quad \text{(as P is symmetric)}\\
&= qP((\tfrac{1}{2}d[\vartheta,\vartheta]) \wedge \Omega^{q-1})\\
&= qP(([d\vartheta,\vartheta]) \wedge \Omega^{q-1}) \quad (\text{as } d[\vartheta,\vartheta] = [d\vartheta,\vartheta] - [\vartheta, d\vartheta])\\
&= qP(([\Omega,\vartheta]) \wedge \Omega^{q-1}) \quad (\text{as } [[\vartheta,\vartheta],\vartheta] = 0 \text{ by the Jacobi identity}).
\end{aligned} \tag{3.7}$$

On the other hand, P is K-invariant. Let $X, Y_1, \dots, Y_q \in \mathfrak{k}$ and differentiating the equation

$$P(Y_1, \dots, Y_q) = P(\mathrm{Ad}(\exp(tX))Y_1, \dots, \mathrm{Ad}(\exp(tX))Y_q)$$

by t at $t = 0$ yields

$$0 = \sum_{i=1}^{q} P(Y_1, \dots, [X, Y_i], \dots, Y_q) = \sum_{i=1}^{q} P([X, Y_i], Y_1, \dots, \hat{Y_i}, \dots, Y_q),$$

what shows $P(([\Omega,\vartheta]) \wedge \Omega^{q-1}) = 0$. Thus $dP(\Omega^q) = 0$ by equation (3.7).

To prove statement four about the transgression, define the connection $\tilde{\vartheta}_t = (1 - t)\operatorname{pr}_E^* \vartheta_0 + t \operatorname{pr}_E^* \vartheta_1$ on $\mathbb{R} \times E$, which is obviously a path of connections from ϑ_0 to ϑ_1. Thus (3.5) follows from Stokes theorem (applied to any simplicial level). Let $A^2 = \{x_0 + x_1 + x_2 = 1\} \subset \mathbb{R}^3$ be the hyperplane, whose intersection with the positive octant is Δ^2. Define by $\hat{\vartheta} = \sum_i x_i \vartheta_i$ a the connection on $A^2 \times E_\bullet \to A^2 \times M_\bullet$. By Stokes one has

$$\tilde{\omega}_P(\vartheta_2, \vartheta_1) + \tilde{\omega}_P(\vartheta_1, \vartheta_0) - \tilde{\omega}_P(\vartheta_2, \vartheta_1) = d\int_{\Delta^2} \omega_P(\hat{\vartheta}), \tag{3.8}$$

from which (3.6) follows. To show the independence from the chosen path, take $\vartheta_2 = \vartheta_1$ and define another connection $\hat{\vartheta}$ on $A^2 \times E_\bullet \to A^2 \times M_\bullet$ in the following way: $\hat{\vartheta}$ restricts to ϑ_i on the i-th vertex of the simplex (which is the intersection of the hyperplane with the non-negative octant), it is constantly $\mathrm{pr}_E^* \vartheta_1$ on the edge $(1,2)$ (from vertex 1 to vertex 2), the convex combination on $(0,1)$, an arbitrary path on the last edge and an interpolation in the interior (say the convex combination on lines parallel to $(1,2)$). Then (3.8) implies the independence of the path, because $\tilde{\omega}_P(\vartheta_2, \vartheta_1)$ is zero as an integral over a pullback form.

The 5th statement, about pullbacks, follows from the facts, that, on the one hand, the curvature of the pullback connection $(\bar{f}, f)^*\vartheta$ is $(\bar{f}, f)^*\Omega$, i.e. the pullback of the curvature and, on the other hand, pullbacks are an algebra homomorphism on differential forms. For the last assertion: the difference of the characteristic forms for two connections is an exact form by 4., thus the class is independence of the connection. That the class only depends on the isomorphism class follows from 5. applied to an isomorphism of the bundles, which covers the identity map on the base space. □

Let G and K be Lie groups and E a smooth G-equivariant principal K-bundle over a smooth manifold M with G-invariant connection $\vartheta \in \Omega^1(E, \mathfrak{k})^G$.

Definition 3.49 (see [4, p.543]) The *moment map* of ϑ is defined as

$$\mu^\vartheta \colon \mathfrak{g} \ni X \mapsto \iota(X^\sharp)\vartheta \in C^\infty(E, \mathfrak{k})^K.$$

Lemma 3.50 *The moment map is G- and K-equivariant.*

Proof. The K-equivariance of μ follows from the K-equivariance of ϑ. Now let $g \in G$, then

$$\begin{aligned}
\mu(\mathrm{Ad}_g X)(gx) &= \vartheta(gx)\left[\left.\frac{d}{dt}\right|_{t=0}\Big(\exp(gXg^{-1})(gx)\Big)\right] \\
&= \vartheta(gx)\left[g\left.\frac{d}{dt}\right|_{t=0}(\exp(X)x)\right] \\
&= \vartheta(gx)[gX^\sharp] \\
&= \vartheta(x)[X^\sharp] \quad \text{by the } G\text{-invariance of } \vartheta \\
&= \mu(X)(x).
\end{aligned}$$

□

Given $\vartheta \in \Omega^1(E, \mathfrak{k})^G$, one defines a simplicial connection on $G^\bullet \times E$ as follows (compare [23, p.104]): Let ϑ_i be the pullback of ϑ to $\Delta^p \times G^p \times E$ along

$$\begin{aligned}
\Delta^p \times G^p \times E &\to E \\
(t_0, \ldots, t_p, g_1, \ldots, g_p, e) &\mapsto g_{i+1} \ldots g_p e.
\end{aligned}$$

Now define $\Theta \in \mathcal{A}^1(G^\bullet \times E, \mathfrak{k})$ on $G^\bullet \times E$ by

$$\Theta^{(p)} = t_0\vartheta_0 + t_1\vartheta_1 + \ldots + t_p\vartheta_p. \tag{3.9}$$

It can be seen directly, that Θ satisfies

$$(\partial^i \times \mathrm{id}_{G^p \times E})^*\Theta^{(p)} = (\mathrm{id}_{\Delta^{p-1}} \times \partial_i)^*\Theta^{(p-1)}$$

and hence it is a simplicial Dupont one form.

We are now going to calculate the characteristic form of the simplicial connection Θ. We will see that this actually leads to the equivariant characteristic form of ϑ as defined by Berline and Vergne, i.e., one replaces the curvature by the sum of the curvature and the moment map. This is a more detailed reformulation of [23, Section 3.3.].

Theorem 3.51 *Let $P \in I^*(K)$ be an invariant symmetric polynomial.*

$$\mathrm{pr}_0\left(\mathcal{J}\left(\int_\Delta \omega_P(\Theta)\right)\right) = P(\Omega^\vartheta + \mu^\vartheta) \in S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)$$

Here $\mathcal{J}$ is the map defined in 3.25 and pr_0 is the projection from the double complex to its zeroth vertical level. As ϑ is G-invariant, the equation actually holds in $\Omega_G(M) = (S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G$.

Proof. Let
$$\Omega = d\Theta + \frac{1}{2}[\Theta, \Theta]$$
denote the curvature of Θ. We should refine the grading of the simplicial Dupont forms (compare [19, p.91]): A form in $\mathcal{A}^*(G^\bullet \times M)$ restricts to a differential form on the product $\Delta^p \times (G^p \times M)$ for each p. Thus we can grade the form by the differential form degree part on the simplex Δ^p and on the form degree on the manifold part $G^p \times M$, where the normal degree is the sum of booth. By construction, the form degree of Θ in direction of the simplex is zero. Thus the form degree of Ω in simplex direction can be at most one. Therefore

$$\int_\Delta \Omega = \Omega_0 + \Omega_1 \in \Omega^2(E, \mathfrak{k}) \oplus \Omega^1(G \times E, \mathfrak{k}).$$

Here Ω_0 is exactly the curvature of ϑ. While

$$\begin{aligned}
\Omega_1 &= \int_{\Delta^1} d\Theta^{(1)} + \frac{1}{2}[\Theta^{(1)}, \Theta^{(1)}] \\
&= \int_{\Delta^1} dt_0\vartheta_0 + dt_1\vartheta_1 \\
&= \int_0^1 dt(\vartheta_0 - \vartheta_1) \\
&= \vartheta_0 - \vartheta_1 \\
&= \partial_1^*\vartheta - \partial_0^*\vartheta
\end{aligned}$$

Let $X = X_G + X_E$ be a vector field on $G \times E$, decomposed in the directions of G and E, then

$$\begin{aligned}
(\Omega_1)(g,p)[X] &= (\partial_1^*\vartheta)(g,p)[X] - (\partial_0^*\vartheta)(g,p)[X] \\
&= \vartheta(gp)[(T\partial_1)(X)] - \vartheta(p)[(T\partial_0)X] \\
&= \vartheta(gp)[X_G^\sharp] + \vartheta(gp)[gX_E] - \vartheta(p)[X_E] \\
&= \vartheta(p)[g^{-1}(X_G^\sharp(gp))],
\end{aligned}$$

where $X_G^\sharp$ denotes the fundamental vector field of X_G. Restricting this to $e \in G$, what is the same as applying the map $\mathrm{pr}_0 \mathcal{J}$, one obtains the definition of the moment map $\mu^\vartheta = \iota(X\sharp)\vartheta$.

The statement of the theorem now follows from the next lemma. □

Lemma 3.52 *The composition of maps*

$$\mathrm{pr}_0 \circ \mathcal{J} \circ \int_\Delta : \mathcal{A}^*(G^\bullet \times M) \to S^*[\mathfrak{g}^\vee] \otimes \Omega^*(M)$$

is a homomorphism of algebras.

Proof. The map is clearly a homomorphism of vector spaces. Hence we only have to show that

$$\mathrm{pr}_0 \mathcal{J} \int_\Delta \omega_1 \wedge \omega_2 = \left(\mathrm{pr}_0 \mathcal{J} \int_\Delta \omega_1\right) \wedge \left(\mathrm{pr}_0 \mathcal{J} \int_\Delta \omega_2\right)$$

for $\omega_i \in \mathcal{A}^*(G^\bullet \times M)$. Using the refined grading defined above, by additivity of the map, we can restrict ourselves to ω_i being a p_i-form in the direction of the simplex and a q_i-form in the direction of $G^p \times M$. Without loss of generality $\omega_i^{(p_i)} = dt_1 \wedge \ldots \wedge dt_{p_i} \wedge \bar{\omega_i}$

Let $X \in \mathfrak{g}$. We calculate:

$$\begin{aligned}
&\left(\mathrm{pr}_0 \mathcal{J} \int_\Delta \omega_1 \wedge \omega_2\right)(X)\\
&= i_M^* \iota_1(X) \ldots \iota_{p_1+p_2}(X) \int_\Delta \omega_1 \wedge \omega_2\\
&= i_M^* \int_{\Delta^{p_1+p_2}} \iota_1(X) \ldots \iota_{p_1+p_2}(X)(\omega_1 \wedge \omega_2)\\
&= i_M^* \int_{\Delta^{p_1+p_2}} \sum_{\pi \in S(p_1,p_2)} (-1)^{f(\pi)} \left(\iota_{\pi(1)}(X) \ldots \iota_{\pi(p_1)}(X) \omega_1^{(p_1+p_2)}\right) \wedge\\
&\qquad\qquad \left(\iota_{\pi(p_1+1)}(X) \ldots \iota_{\pi(p_1+p_2)}(X) \omega_2^{(p_1+p_2)}\right)
\end{aligned}$$

Here $S(p_1, p_2)$ is the shuffle group (see Definition 3.25). What is the sign $f(\pi)$? Take a shuffle π, let $c_\pi(k)$ be the number of indices in $\{\pi(1), \ldots, \pi(p_1)\}$, which are larger than $\pi(p_1 + p_2 - k)$. Then we get, when expanding the contraction, a $(-1)^{p_1+q_1-c_\pi(k)}$ for swapping $\iota_{\pi(p_1+p_2-k}(X)$ with the partially contracted ω_1. Summing this up for $k = 0, \ldots, p_2 - 1$ results in $f(\pi)$. On the other hand the sum of the $c_\pi(k)$ is exactly the number of inversions of π as the two groups of the shuffle are in order, thus $(-1)^{f(\pi)}(-1)^{(q_1-p_1)p_2} \operatorname{sgn}(\pi) = 1$.

$$\begin{aligned}
&= i_M^* \int_{\Delta^{p_1+p_2}} \sum_\pi (-1)^{f(\pi)} (\iota_{\pi(1)}(X) \ldots \iota_{\pi(p_1)}(X)(\mathrm{id}_\Delta \times (\partial_{\pi(p_1+1)} \ldots \partial_{\pi(p_1+p_2)}))^* \omega_1) \wedge\\
&\qquad\qquad (\iota_1(X) \ldots \iota_{p_2}(X)(\mathrm{id}_\Delta \times (\partial_{\pi(1)} \ldots \partial_{\pi(p_1)}))^* \omega_2)
\end{aligned}$$

We can add the ∂'s in front of the ω_i as $\sigma_i^*\partial_i^* = \mathrm{id}$, and $\sigma_i \circ i_M = i_M$. Now apply the property of simplicial Dupont forms with respect to the face maps.

$$
\begin{aligned}
&= i_M^* \int_{\Delta^{p_1+p_2}} \sum_\pi (-1)^{f(\pi)} (\iota_1(X)\dots\iota_{p_1}(X)((\partial^{\pi(p_1+1)}\dots\partial^{\pi(p_1+p_2)})\times \mathrm{id})^*\omega_1^{(p_1)})\wedge \\
&\qquad\qquad (\iota_1(X)\dots\iota_{p_2}(X)((\partial^{\pi(1)}\dots\partial^{\pi(p_1)})\times \mathrm{id})^*\omega_2^{(p_2)}) \\
&= i_M^* \int_{\Delta^{p_1+p_2}} \sum_\pi (-1)^{f(\pi)} (\iota_1(X)\dots\iota_{p_1}(X) dt_{\pi(1)}\wedge\dots\wedge dt_{\pi(p_1)}\wedge\bar{\omega_1})\wedge \\
&\qquad\qquad (\iota_1(X)\dots\iota_{p_2}(X) dt_{\pi(p_1+1)}\wedge\dots\wedge dt_{\pi(p_1+p_2)}\wedge\bar{\omega_2}) \\
&= i_M^* \int_{\Delta^{p_1+p_2}} \sum_\pi (-1)^{f(\pi)} (dt_{\pi(1)}\wedge\dots\wedge dt_{\pi(p_1)}\wedge dt_{\pi(p_1+1)}\wedge\dots\wedge dt_{\pi(p_1+p_2)}\wedge \\
&\qquad\qquad (-1)^{(q_1-p_1)p_2+p_1^2+p_2^2}(\iota_1(X)\dots\iota_{p_1}(X)\bar{\omega_1})\wedge(\iota_1(X)\dots\iota_{p_2}(X)\bar{\omega_2})
\end{aligned}
$$

The sign comes from rearranging the forms. Now recall that the volume of the p-simplex is $\frac{1}{p!}$ and the number of elements of the shuffle group is $\frac{(p_1+p_2)!}{p_1!p_2!}$. Thus we obtain:

$$
\begin{aligned}
&= \frac{1}{p_1!p_2!}(-1)^{p_1^2+p_2^2}\, i_M^*(\iota_1(X)\dots\iota_{p_1}(X)\bar{\omega_1})\wedge(\iota_1(X)\dots\iota_{p_2}(X)\bar{\omega_2}) \\
&= \left(i_M^*\int_\Delta \iota_1(X)\dots\iota_{p_1}(X)\omega_1\right)\wedge\left(i_M^*\int_\Delta \iota_1(X)\dots\iota_{p_2}(X)\omega_2\right) \\
&= \left(\mathrm{pr}_0\,\mathcal{J}\int_\Delta \omega_1\right)(X)\wedge\left(\mathrm{pr}_0\,\mathcal{J}\int_\Delta \omega_2\right)(X) \\
&= \left(\left(\mathrm{pr}_0\,\mathcal{J}\int_\Delta \omega_1\right)\wedge\left(\mathrm{pr}_0\,\mathcal{J}\int_\Delta \omega_2\right)\right)(X).
\end{aligned}
$$

This finishes the proof. □

This theorem motivates the following definition, which translates the definition of characteristic form and transgression to the Cartan model.

Definition 3.53 (compare [4, p. 543]) Let $\vartheta, \vartheta_0, \vartheta_1$ be G-invariant connections on a G-equivariant principal K-bundle $E \to M$. The *characteristic form* related to the invariant polynomial $P \in I^*(K)$ is defined as $P(\Omega^\vartheta + \mu^\vartheta) \in \Omega_G^*(M)$ and the *transgression form* is $\widetilde{\omega}_P(\vartheta_1,\vartheta_0) = \int_I P(\Omega^{\vartheta_t}+\mu^{\vartheta_t})$, where ϑ_t is the convex combination of ϑ_0 and ϑ_1 as in the proof of Theorem 3.48.

Let Θ_i be the connections associated to ϑ_i, $i = 1,2$ by (3.9). The following lemma shows that we did not run into notational difficulties.

Lemma 3.54 *We have*

$$
\widetilde{\omega}_P(\vartheta_1,\vartheta_0) = \mathrm{pr}_0\left(\mathcal{J}\left(\int_\Delta \widetilde{\omega}_P(\Theta_1,\Theta_0)\right)\right),
$$

where the right-hand side is the transgression form of Theorem 3.48.

Proof. As the composition of maps $\mathrm{pr}_0 \circ \mathcal{J} \circ \int_\Delta$ is linear and all integral are taken over compact sets, it commutes with $\int_I$ and thus the simplicial transgression form is mapped to the one in the Cartan model. □

There is a 'canonical' connection on the classifying bundle (compare [19, p.94]). As seen above $EK \to BK$ is the geometric realization of the simplicial bundle $\gamma\colon N\overline{K} \to NK$. Let $\vartheta_0 \in \Omega^1(K, \mathfrak{k})$ denote the Maurer-Cartan connection of the trivial bundle $K \to \mathrm{pt}$, i.e.,

$$\vartheta_0(k) = L_{k^{-1}}\colon T_kK \to T_eK = \mathfrak{k}.$$

Let

$$\pi_i\colon \Delta^p \times K^{p+1} \to K$$

denote the projection to the i-th coefficient, $i = 0, \dots, p$ and $\vartheta_i = \pi_i^*\vartheta_0$. Then we define $\bar{\vartheta}$ on $\Delta^p \times (N\overline{K})_p$ by

$$\bar{\vartheta} = \sum_i t_i \vartheta_i,$$

where $(t_0, \dots, t_p)$ are barycentric coordinates on the simplex. $\bar{\vartheta}|_{\Delta^p \times (N\overline{K})_p}$ is a connection on $\Delta^p \times (N\overline{K})_p$, as it is a convex combination of connections. It is seen easily from the definition, that $\bar{\vartheta}$ is a simplicial-1-form.

Now we are able to prove the central theorem of this section.

Theorem 3.55 *Let K be a Lie group.*

1. *There is a homomorphism*

$$\begin{aligned} c\colon I^*(K) &\to H^{2*}(BK) = {\mathcal{A}^{2*}(NK)_{\mathrm{cl}}}\big/{d\mathcal{A}^{2*-1}(NK)} \\ P &\mapsto c_P(N\overline{K}) = [\omega_P(\bar{\vartheta})], \end{aligned}$$

   which is, for compact groups, inverse to $\mathrm{pr}_0 \circ \mathcal{J} \circ \int_\Delta$.

2. *Let G be another Lie group, $P \in I^*(K)$ and let E be a smooth G-invariant principal-K-bundle over the smooth manifold M, then*

$$c_P(E) = c(P)(EG \times_G E),$$

   i.e., the definition via the Chern-Weil construction on the left-hand side equals the pullback definition via the universal class $c(P) = c_p(N\overline{K})$ on the right hand side.

Proof. The homomorphism property is just a special case of Theorem 3.48 and the statement about the inverse follows from Theorem 3.51, as the curvature of ϑ is equal to zero and the moment map equals the identity.

For the second assertion let $\mathcal{U}$ be a trivializing cover of $\pi\colon E \to M$. From (3.4) and (3.3) we obtain a commutative diagram

$$\begin{tikzcd} G^\bullet \times E \arrow[d] & N(G^\bullet \times E)_{\pi^{-1}\mathcal{U}} \arrow[l, "i"'] \arrow[r, "\bar{\phi}"] \arrow[d, "\pi_{\mathcal{U}}"] & N\overline{K} \arrow[d, "\gamma"] \\ G^\bullet \times M & N(G^\bullet \times M)_{\mathcal{U}} \arrow[l, "i"] \arrow[r, "\phi"] & NK. \end{tikzcd} \tag{3.10}$$

Now:

$$\begin{aligned} i^*c(P)(EG \times_G E) &= c(P)(\|N(G^\bullet \times E)_{\pi^{-1}\mathcal{U}}\|) \\ &= \|\phi\|^*c(P) \quad \text{(right square is a pullback by Lemma 3.44)} \\ &= \phi^*[\omega_P(\bar{\vartheta})] \quad \text{(isomorphism of Theorem 3.14)} \\ &= [\omega_P(\phi^*\bar{\vartheta})] \\ &= [\omega_P(i^*\vartheta)] \quad \text{(class is independent of connection)} \\ &= i^*c_P(E) \quad \text{(Chern-Weil definition)} \end{aligned}$$

The statement follows, since i^* is an isomorphism. □

Corollary *Let K be a compact Lie group. Any characteristic form ω for principal K-bundles of degree $2n$ corresponds to a polynomial $P \in I^n(K)$, which induces via the Chern-Weil construction the same characteristic class as ω.*

Proof. Let $\bar{\vartheta} = \left\{\bar{\vartheta}^{(p)}\right\}_{p\in\mathbb{N}}$ denote the simplicial connection on $N\overline{K} \to NK$ defined above. By the pullback property of the characteristic form, $\left\{\omega\left(\bar{\vartheta}^{(p)}\right)\right\}_{p\in\mathbb{N}} \in \mathcal{A}^n_{\mathrm{cl}}(NK)$ is a closed Dupont n-form. Thus $\mathcal{J} \circ \int_\Delta(\omega)$ is a cocycle in the Getzler model. Since K is compact, the Getzler model contracts to the Cartan model and thus $\mathcal{J} \circ \int_\Delta(\omega)$ is cohomologous to an element of $I^n(K)$, which we will call P. Since $N\overline{K} \to NK$ is a universal principal K-bundle, the assertion follows. □

Remark 3.56 There is a shorter way, to show the main theorem above, then the one we gave, but we will need the construction above later on, when defining differential refinements of characteristic classes. As this shorter construction is maybe interesting to the reader we will give a sketch (which generalizes arguments of [21]). Instead of the simplicial manifold of the covering one defines a special classifying space for each bundle: As before let $\pi : E \to M$ be an G-equivariant principal K-bundle and $G^\bullet \times E \to G^\bullet \times M$ the associated simplicial bundle. We define a simplicial manifold $N(G^\bullet \times E)^\bullet$, s.t., $N(G^p \times E)^p = G^p \times E^{p+1}$, face and degeneracy maps act on the G part as described above and on the E part be removing/doubling the ith entry. In particular $\partial_p(g_1, \ldots, g_p, b_0, \ldots, b_p) = (g_1, \ldots, g_{p-1}, g_pb_0, \ldots, g_pb_{p-1})$. K acts diagonal on the E's. This allows to define maps

$$\begin{tikzcd} G^\bullet \times E \arrow[r, "\bar{\psi}"] \arrow[d, "\pi"] & N(G^\bullet \times E)^\bullet \arrow[d] & N\overline{K} \arrow[l, "\bar{\phi}_x"'] \arrow[d, "\gamma"] \\ G^\bullet \times M \arrow[r, "\psi"] & N(G^\bullet \times E)^\bullet/K & NK. \arrow[l, "\phi_x"'] \end{tikzcd}$$

Here $\bar{\psi}$ is induced by the diagonal inclusion $E \to E^p$ and the map $\bar{\phi}$ is given by

$$\begin{aligned}\left(\bar{\phi}_x\right)_p &: K^{p+1} \to G^p \times E^{p+1} \\ (k_0, \dots, k_p) &\mapsto (e, \dots, e, xk_0, \dots, xk_p).\end{aligned}$$

As $\|N(G^\bullet \times E)\|$ is contractible, the right side of the diagram induces homotopy equivalences in the geometric realization.

3.2.4 Vector bundles alias principal $\mathrm{Gl}_n(\mathbb{C})$-bundles

As (complex) vector bundles are of specific interest, we want to translate the statements, about equivariant characteristic forms in the last sections, from principal $\mathrm{Gl}_n(\mathbb{C})$-bundles to their associated vector bundles. One can also replace $\mathrm{Gl}_n(\mathbb{C})$ by subgroups, e.g., $U(n)$ to obtain analogues statements.

Definition 3.57 Let E be a principal $\mathrm{Gl}_n(\mathbb{C})$-bundle. The associated vector bundle

$$\mathcal{E} = E \times_{\mathrm{Gl}_n} \mathbb{C}^n$$

is the quotient of $E \times \mathbb{C}^n$ by the diagonal action of Gl_n, where the action on $\mathbb{C}^n$ is given by matrix multiplication from the left.

Lemma 3.58 *There is an isomorphism $C^\infty(E, \mathfrak{gl}_\mathfrak{n})^{\mathrm{Gl}_n} \to C^\infty(M, \mathrm{End}\,\mathcal{E})$.*

Proof. This map is well known, but we give a proof for completeness.

$$\mathrm{End}\,\mathcal{E} = \mathrm{End}(E \times_{\mathrm{Gl}_n} \mathbb{C}^n) = E \times_{\mathrm{Gl}_n} \mathrm{End}(\mathbb{C}^n) = E \times_{\mathrm{Gl}_n} M_n(\mathbb{C}) = E \times_{\mathrm{Gl}_n} \mathfrak{gl}_n$$

Hence, it suffices to construct the isomorphism

$$C^\infty(E, \mathfrak{gl}_\mathfrak{n})^{\mathrm{Gl}_n} \to C^\infty(M, E \times_{\mathrm{Gl}_n} \mathfrak{gl}_n).$$

Therefore, let $f \in C^\infty(E, \mathfrak{gl}_\mathfrak{n})^{\mathrm{Gl}_n}$ and s be a local section of $E \to M$. Then,

$$\begin{aligned}M \supset U &\to E \times_{\mathrm{Gl}_n} \mathfrak{gl}_n \\ m &\mapsto (s(m), f(s(m)))\end{aligned}$$

defines a local section of $\mathrm{End}\,\mathcal{E} \to M$. Picking another local section s' around m, there exists $A \in \mathrm{Gl}_n$, s.t., $s'(m) = s(m)A$ and

$$\begin{aligned}(s'(m), f(s'(m))) &= (s(m)A, f(s(m)A)) \\ &= (s(m)A, A^{-1} f(s(m))A) \\ &= (s(m), f(s(m))) \in E_m \times_{\mathrm{Gl}_n} \mathfrak{gl}_n.\end{aligned}$$

Hence the image of f is independent of the local section and defines an element in $C^\infty(M, \mathrm{End}\,\mathcal{E})$.

On the other hand given $f \in C^\infty(M, E \times_{\mathrm{Gl}_n} \mathfrak{gl}_n)$, then for $x \in E$, $f(\pi(x)) = (xg, A_f(x)) = (x, gA_f(x)g^{-1})$ for some $A_f(x) \in \mathfrak{gl}_n$. The map $x \mapsto gA_f(x)g^{-1}$ is equivariant by definition and smooth as π, f, and the action are smooth. We only have to check that both map are inverse to each other.

Start with $f\colon E \to \mathfrak{gl}_n$. Let $x \in E$ and s be a local section of E around $\pi(x)$, then, for some $g \in \mathrm{Gl}_n$,

$$x \mapsto (s(\pi(x)), f(s(\pi(x)))) = (xg, f(xg)) = (x, gg^{-1}(f(x))gg^{-1}) = (x, f(x)) \mapsto f(x).$$

Now let $f \in C^\infty(M, E \times_{\mathrm{Gl}_n} \mathfrak{gl}_n)$ and s again be a local section of E. The composition is $m \mapsto (s(m), A_f(s(m))) = f(m)$, by the definition of A_f. □

A left G-action on E which commutes with the action of Gl_n clearly induces a left G-action on $\mathcal{E}$.

There is a one to one correspondence between connections on the principal-Gl_n-bundle and those one the associated vector bundle (see e.g. [1, Ex. 3.4]).

Definition 3.59 (Def. 2.23. of [9]) Let ∇ be a connection on the G-vector bundle $\mathcal{E}$. The *moment map* $\mu^\nabla \in \mathrm{Hom}(\mathfrak{g}, \Omega^0(M, \mathrm{End}(\mathcal{E})))^G$ is defined by

$$\mu^\nabla(X) \wedge \varphi := \nabla_{X_M^\sharp}\varphi + L_X^{\mathcal{E}}\varphi, \quad \varphi \in \Omega^0(M, \mathcal{E}).$$

Here $L_X^{\mathcal{E}}$ denotes the derivative

$$L_X^{\mathcal{E}}\varphi = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)^*\varphi.$$

Remark 3.60 Observe that for a function $f \in C^\infty(M, \mathbb{C})$,

$$L_X^{M\times\mathbb{C}} f(m) = \left.\frac{d}{dt}\right|_{t=0} (\exp(tX)^* f)(m) = \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(tX)^{-1}m\right) = -df_m(X_M^\sharp).$$

Therefore, we altered the sign in the definition of [9].

Theorem 3.61 *Let $\vartheta \in \Omega^1(E, \mathfrak{gl}_n)$ be a connection on the principal Gl_n-bundle E and ∇ be the associated connection on the associated vector bundle $\mathcal{E}$. Then, with respect to the isomorphism of Lemma 3.58, one has*

$$d\vartheta + \vartheta \wedge \vartheta = R^\nabla \quad \textit{and} \quad \mu^\vartheta = \mu^\nabla.$$

Proof. The first statement is standard, while the second is stated in [4, Lemma 3.2], where the proof is left to the reader. Here it is: The statement is local. Let $s = (s_1, \dots, s_n)$ be a local frame, i.e., a local section of E, $X \in \mathfrak{g}$ and $\varphi \in C^\infty(M, \mathcal{E})$.

Locally one can write $\varphi = \sum_i \varphi_i s_i$.

$$\begin{aligned}
s^*(\mu^\vartheta(X))\varphi &= s^*(\iota(X_E^\sharp)\vartheta)\sum_i \varphi_i s_i \\
&= \sum_i \varphi_i (s^*\vartheta[X_E^\sharp]) s_i \\
&= \sum_i \varphi_i (s^*(\vartheta[ds\circ d\pi(X_E^\sharp) + (1 - ds\circ d\pi)(X_E^\sharp)]) s_i \\
&= \sum_i \varphi_i \left(s^*(\vartheta[ds(X_M^\sharp)] + \vartheta[(1-ds\circ d\pi)(X_E^\sharp)])\right) s_i \\
&\overset{\text{see}}{\underset{\text{below}}{=}} \sum_i \varphi_i \left(s^*(\vartheta[ds(X_M^\sharp)]) s_i + \left.\frac{d}{dt}\right|_{t=0} \exp(tX)\cdot(s_i)\right) \\
&= \sum_i \varphi_i \left(\nabla_{X_M^\sharp} s_i + \left.\frac{d}{dt}\right|_{t=0} \exp(tX)^*(s_i)\right) \\
&= \sum_i (d\varphi_i(X_M^\sharp)s_i + \varphi_i \nabla_{X_M^\sharp} s_i + \left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)^*\varphi_i\right) s_i \\
&\qquad\qquad + \varphi_i \left.\frac{d}{dt}\right|_{t=0} \exp(tX)^*(s_i)) \\
&= \nabla_{X_M^\sharp}\sum_i \varphi_i s_i + \left.\frac{d}{dt}\right|_{t=0}\exp(tX)^*(\sum_i \varphi_i s_i) \\
&= \nabla_{X_M^\sharp}\varphi + L_X^M \varphi \\
&= \mu^\nabla(X)\varphi
\end{aligned}$$

For the step in the middle, we have to show that for any $m \in M$ the equation

$$\vartheta_{s(m)}\left[(1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m)\right)\right]\cdot s = \left.\frac{d}{dt}\right|_{t=0}\exp(tX)(s(\exp(-tX)m)) \tag{3.11}$$

holds, where the $\cdot$ on the left-hand side emphasizes that the action is from the right. This is shown as follows. The vector field $(1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m)\right)$ is horizontal, since

$$d\pi\left((1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m)\right)\right) = 0.$$

Hence there is a unique $Y \in \mathfrak{gl}_n$, such that

$$(1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m)\right) = s(m)Y$$

and thus $\vartheta_{s(m)}\left[(1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m)\right)\right] = Y$.

Now, calculate

$$\begin{aligned}
&(1-ds\circ d\pi)\left(\left.\frac{d}{dt}\right|_{t=0}\exp(tX)(s(m))\right) \\
&\qquad = \left.\frac{d}{dt}\right|_{t=0}\exp(tX)s(m) - \left.\frac{d}{dt}\right|_{t=0} s\circ\pi(\exp(tX)s(m)),
\end{aligned}$$

hence, by equivariance of π,

$$\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX)s(m) - s(\exp(tX)\pi \circ s(m))) \\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX)s(m) - s(\exp(tX)m)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp(tX)s(\exp(-tX)m).
\end{aligned}$$

Thus we have shown that shows that $Y = s(m)^{-1} \left.\frac{d}{dt}\right|_{t=0} \exp(tX)s(\exp(-tX)m)$ and hence equation 3.11 holds. $\square$

Let ϑ, ϑ' be connections on the principal Gl_n-bundle E and ∇, ∇' the associated connections on the associated vector bundle $\mathcal{E}$. Then one has the transgression forms

$$\widetilde{\omega}(\nabla, \nabla') = \int_I \omega(\nabla_t) = \widetilde{\omega}(\vartheta, \vartheta'),$$

which coincide in the sense of Lemma 3.58. Here ∇_t is, as above, the convex combination, which is a connection on $\mathbb{R} \times \mathcal{E} \to \mathbb{R} \times M$.

4 Equivariant differential cohomology

Differential cohomology is a refinement of integral cohomology for smooth manifold by differential forms. One motivation is to define refined characteristic classes, which reflect both, the integral characteristic class and the characteristic form. This is reflected by the following hexagon, see [9, Proposition 3.24], whose diagonals are exact in the middle.

$$\begin{tikzcd}
& \Omega^{n-1}(M,\mathbb{C})/\operatorname{im} d \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^n_{\mathrm{cl}}(M,\mathbb{C}) \arrow[dr] & \\
H^{n-1}(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \hat{H}^n(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, two heads, "I"] & & H^n(M,\mathbb{C}) \\
& H^{n-1}(M,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur, hook] & & H^n(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

The upper sequence, as discussed in Remark 2.17, is exact. The same is true for the bottom sequence, which is a part of the Bockstein sequence (see Remark 2.37) up to sign.

A particular model for differential cohomology is smooth Deligne cohomology:

Definition 4.1 (see, e.g., [9, Section 3.2]) Let M be a smooth manifold and $n \in \mathbb{N}$. The *smooth Deligne complex* in degree n is the complex of sheaves

$$\mathcal{D}(n) : \mathbb{Z} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1} \to 0 \to \dots$$

and the n-th *smooth Deligne cohomology* is the n-th hypercohomology of the Deligne complex:

$$\hat{H}^n(M,\mathbb{Z}) = H^n(M,\mathcal{D}(n)).$$

We will define a similar refinement in the case of the action of a Lie group on the manifold. The goal is a theory which sits in the middle of a hexagon as above, where the lower line is the Bockstein sequence in equivariant cohomology and the upper line is the tautological exact sequence of the Cartan complex

$$0 \to H^{n-1}_G(M,\mathbb{C}) \to \Omega^{n-1}_G(M)/_{d+\iota} \to \Omega^n_G(M)_{\mathrm{cl}} \to H^n_G(M,\mathbb{C}) \to 0.$$

The discussion of equivariant characteristic forms in the previous section motivates to enrich integral cohomology by all integral closed equivariant differential forms.

4.1 The definition

The are several attempts to a definition [33, 25, 40]. The most elaborated one is given by Kiyonori Gomi in [25], where he defines equivariant smooth Deligne cohomology of a smooth manifold M acted on by a Lie group G. His investigations (for G a compact group) can be summarized in the following diagram with exact diagonals

$$\begin{tikzcd}
\Omega^{n-1}(M,\mathbb{C})^G/\operatorname{im} d \arrow[dr, "a"] & & \Omega^n_{\mathrm{cl}}(M,\mathbb{C})^G \\
& \hat{H}^n(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[r] \arrow[dr, "I"'] & H^n_G(M,\mathbb{C}) \\
H^{n-1}(G^*\times M, \pi^{-1}\underline{\mathbb{C}/\mathbb{Z}}) \arrow[ur, hook] & & H^n_G(M,\mathbb{Z}).
\end{tikzcd} \tag{4.1}$$

The subscript G stands for equivariant cohomology and the superscript G for equivariant forms. Gomi defines the maps and shows that the diagonals are exact in the middle.

From our point of view, this diagram is not satisfactory: On the one hand, one does not have the Bockstein sequence. On the other hand, closed equivariant forms is not, what one expects in the upper right corner, as there indeed exists a map

$$\Omega^n_{\mathrm{cl}}(M,\mathbb{C})^G \to H^n_G(M,\mathbb{C}).$$

But this map is in general not surjective, as not every n-class in equivariant cohomology is represented by a closed equivariant n-form: There are classes represented by (non-zero degree) polynomials $\mathfrak{g} \to \Omega^*(M)$. As we have seen these are related to the moment map, which plays an important role when discussing equivariant characteristic classes and forms. Thus this information is neglected in Gomis curvature map.

We will start with the case of finite groups, as there is no moment map if the group is discrete and thus everything is much more similar to the non-equivariant case. In particular, the reader who is not used to differential cohomology hopefully will have an easier access in this way. Afterwards we will discuss the definition of Gomi and show how to define a better curvature map R, such that one obtains a hexagon with Gomi's definition of equivariant Deligne cohomology in the middle. By this we are motivated to give another definition, which incorporates additional geometric data. The difficulty is in general not to show *that* there is a hexagon, as this follows directly from the way of the definition by ideas of [10]. What the discussion is about, is *which* groups sit on the corners of the hexagon.

At the end of this section we will give some remarks on the definitions of [33, 40] for equivariant differential cohomology.

4.1.1 The case of finite groups

Let, in this subsection, G denote a finite group. Thus a de Rham-type complex for equivariant cohomology is given by

$$\Omega^0(M)^G \xrightarrow{d} \Omega^1(M)^G \xrightarrow{d} \Omega^2(M)^G \xrightarrow{d} \dots .$$

To prepare for the case of Lie groups, we will, nevertheless, work on the simplicial manifold $G^\bullet \times M$. For definition of the Deligne complex, we will use a cone construction similar to [9].

Definition 4.2 Let M be a G-manifold. The *equivariant Deligne complex* is defined as

$$\mathcal{D}(n)_{G^\bullet \times M} = \operatorname{Cone}\left(\underline{\mathbb{Z}} \oplus \sigma^{\geq n}\Omega^{\bullet,*} \to \Omega^{\bullet,*}, (z,\omega) \mapsto \omega - z\right)[-1].$$

Here $\underline{\mathbb{Z}}$ denotes the locally constant simplicial sheaf of $\mathbb{Z}$ and $\Omega^{\bullet,*}$ the cochain complex of simplicial sheaves of complex valued differential forms on the simplicial manifold $G^\bullet \times M$.

Definition 4.3 Let G be a finite group acting on a smooth manifold M. The *G-equivariant differential cohomology* of M is defined to be the hypercohomology

$$\hat{H}^n_G(M) := H^n(G^\bullet \times M, \mathcal{D}(n)_{G^p \times M}).$$

Note that $\mathcal{D}(n)_{G^\bullet \times M}$ is a cochain complex of simplicial sheaves, i.e., it induces a double complex of sheaves so that a resolution is given by triple complex. We give here the general idea of how to define the cohomology of such an object. For details see Section 3.1.1.

Denote the boundary map of $\mathcal{D}(n)$ by $\mathfrak{d}$ and pick an injective resolution $I^{\bullet,*,*}$, i.e., for any $p \in \mathbb{N}$ there is the following double complex of sheaves on $G^p \times M$

$$\begin{array}{ccccccccccc}
\underline{\mathbb{Z}} & \xrightarrow{\iota} & C^\infty & \xrightarrow{-d} & \Omega^1 & \xrightarrow{-d} & \dots & \xrightarrow{-d} & \Omega^{n-2} & \xrightarrow{-d} & \Omega^n \oplus \Omega^{n-1} \xrightarrow{(d,\iota\oplus -d)} \dots \\
\downarrow \delta & & \downarrow -\delta & & \downarrow \delta & & & & \downarrow (-1)^{n-1}\delta & & \downarrow (-1)^n\delta \\
I^{p,0,0} & \xrightarrow{\mathfrak{d}} & I^{p,1,0} & \xrightarrow{\mathfrak{d}} & I^{p,2,0} & \xrightarrow{\mathfrak{d}} & \dots & \xrightarrow{\mathfrak{d}} & I^{p,n-1,0} & \xrightarrow{\mathfrak{d}} & I^{p,n,0} \xrightarrow{\mathfrak{d}} \dots \\
\downarrow \delta & & \downarrow -\delta & & \downarrow \delta & & & & \downarrow (-1)^{n-1}\delta & & \downarrow (-1)^n\delta \\
I^{p,0,1} & \xrightarrow{\mathfrak{d}} & I^{p,1,1} & \xrightarrow{\mathfrak{d}} & I^{p,2,1} & \xrightarrow{\mathfrak{d}} & \dots & \xrightarrow{\mathfrak{d}} & I^{p,n-1,1} & \xrightarrow{\mathfrak{d}} & I^{p,n,1} \xrightarrow{\mathfrak{d}} \dots \\
\downarrow \delta & & \downarrow -\delta & & \downarrow \delta & & & & \downarrow (-1)^{n-1}\delta & & \downarrow (-1)^n\delta \\
I^{p,0,2} & \xrightarrow{\mathfrak{d}} & I^{p,1,2} & \xrightarrow{\mathfrak{d}} & I^{p,2,2} & \xrightarrow{\mathfrak{d}} & \dots & \xrightarrow{\mathfrak{d}} & I^{p,n-1,2} & \xrightarrow{\mathfrak{d}} & I^{p,n,2} \xrightarrow{\mathfrak{d}} \dots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The triple complex which calculates the cohomology is

$$K^{p,q,r} = \left(\Gamma(G^p \times M, I^{p,q,r}), \partial + (-1)^p\mathfrak{d} + (-1)^{p+q}\delta\right).$$

And in the same spirit as [9] and [25] one investigates differential cohomology by the following two short exact sequences

$$0 \to \operatorname{Cone}(\sigma^{\geq n}\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1] \xrightarrow{a} \mathcal{D}(n) \xrightarrow{I} \underline{\mathbb{Z}} \to 0 \tag{4.2}$$

$$0 \to \operatorname{Cone}(\underline{\mathbb{Z}} \xrightarrow{-\iota} \Omega^{\bullet,*})[-1] \to \mathcal{D}(n) \xrightarrow{R} \sigma^{\geq n}\Omega^{\bullet,*}_{\mathbb{C}} \to 0 \tag{4.3}$$

of complexes of simplicial sheaves and the exact triangle

$$\mathcal{D}(n)_{G^\bullet\times M} \to \underline{\mathbb{Z}} \oplus \sigma^{\geq n}\Omega^{\bullet,*}_{\mathbb{C}} \to \Omega^{\bullet,*}_{\mathbb{C}} \to \mathcal{D}(n)_{G^\bullet\times M}[1],$$

which has the following interesting part in its long exact cohomology sequence

$$\begin{aligned} &H^{n-1}(G^\bullet \times M, \underline{\mathbb{Z}}) \to H^{n-1}(G^\bullet \times M, \Omega^{\bullet,*}_{\mathbb{C}}) \to H^n(G^\bullet \times M, \mathcal{D}(n)) \\ &\qquad \overset{(I,R)}{\to} H^n(G^\bullet \times M, \underline{\mathbb{Z}}) \oplus H^n(G^\bullet \times M, \sigma^{\geq n}\Omega^{\bullet,*}) \overset{(-\iota,\iota)}{\to} H^n(G^\bullet \times M, \Omega^{\bullet,*}). \end{aligned} \tag{4.4}$$

Recall from Section 3.1.1 that

$$H^n(G^\bullet \times M, \underline{\mathbb{Z}}) = H^n(\|G^\bullet \times M\|, \mathbb{Z}) = H^n_G(M, \mathbb{Z})$$

and

$$H^n(G^\bullet \times M, \Omega^{\bullet,*}) = H^n_G(M, \mathbb{C}).$$

Moreover, we have:

Lemma 4.4

$$H^n(G^\bullet \times M, \sigma^{\geq n}\Omega^{\bullet,*}) = \Omega^n_{\mathrm{cl}}(M)^G.$$

Proof.

$$\begin{aligned} H^n(G^\bullet \times M, \sigma^{\geq n}\Omega^{\bullet,*}) &= \{\omega \in \Omega^n(M) | \partial\omega = 0, d\omega = 0\} \\ &= \{\omega \in \Omega^n(M) | \partial_0^*\omega = \partial_1^*\omega, d\omega = 0\} \\ &= \Omega^n_{\mathrm{cl}}(M)^G, \end{aligned}$$

as $\partial_0 \colon G \times M \to M$ is the projection to M and ∂_1 is the action of G on M. Thus we obtain closed equivariant differential n-forms. □

As $\Omega^{\bullet,*}$ is a resolution of $\underline{\mathbb{C}}^\delta$, the locally constant sheaf of continuous maps to the field $\mathbb{C}$ with discrete topology, we get a quasi isomorphism

$$\mathrm{Cone}(\underline{\mathbb{Z}} \overset{-\iota}{\to} \Omega^{\bullet,*}) \simeq \mathrm{Cone}(\underline{\mathbb{Z}} \overset{-\iota}{\to} \underline{\mathbb{C}}^\delta) \simeq \underline{\mathbb{C}^\delta / \mathbb{Z}}.$$

The long exact cohomology sequence of the exact triangle

$$\mathrm{Cone}(\underline{\mathbb{Z}} \overset{-\iota}{\to} \underline{\mathbb{C}}^\delta)[-1] \to \underline{\mathbb{Z}} \overset{-\iota}{\to} \underline{\mathbb{C}}^\delta \to \mathrm{Cone}(\underline{\mathbb{Z}} \overset{-\iota}{\to} \underline{\mathbb{C}}^\delta)$$

is the Bockstein sequence, up to a minus sign. Comparing this with (4.4) via the inclusion of (4.2), results in the follwing commutative diagram:

$$\begin{tikzcd} H^{n-1}_G(M,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{n-1}_G(M,\mathbb{C}) \arrow[r] \arrow[d, equal] & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[r, "-\beta"] \arrow[d] & H^n_G(M,\mathbb{Z}) \arrow[r, "-\iota"] \arrow[d, "\mathrm{id}\oplus 0"] & H^n_G(M,\mathbb{C}) \arrow[d, equal] \\ H^{n-1}_G(M,\mathbb{Z}) \arrow[r] & H^{n-1}_G(M,\mathbb{C}) \arrow[r] & \hat{H}^n_G(M,\mathbb{Z}) \arrow[r, "I\oplus R"] & H^n_G(M,\mathbb{Z}) \oplus \Omega^*_{\mathrm{cl}}(M)^G \arrow[r, "(-\iota,\iota)"] & H^n_G(M,\mathbb{C}). \end{tikzcd}$$

For the next steps, recall that one can define, similar to Lemma 3.18, a vertical contraction of the double complex $\Omega^{\bullet,*}(G^\bullet \times M)$.

Lemma 4.5 *'Integration' over the group, defines a map*

$$\int_G : \Omega^i_{\mathbb{C}}(G^k \times M) \to \Omega^i_{\mathbb{C}}(G^{k-1} \times M) \tag{4.5}$$
$$\left(\int_G \omega\right)(g_1, \ldots, g_{k-1}, m) = (-1)^i \frac{1}{|G|} \sum_G \omega(g, g_1, \ldots, g_{k-1}, m)$$

such that $\partial \int_G(\omega) = \omega$ *if* $\partial\omega = 0$.

Proof. The proof is given by the same calculation as for Lemma 3.18. □

One has $\mathrm{Cone}(\sigma^{\geq n}\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1] \simeq \sigma^{<n}\Omega^{\bullet,*}[-1]$. Thereby, the map in degree n, is given from right to left by $\eta \mapsto (d\eta, \eta)$. As differential forms form a fine sheaf (by Lemma 2.29), they are themselves their injective resolution. Thus we can calculate the hypercohomology $H^{n-1}(G^\bullet \times M; \Omega^{<n}_{\mathbb{C}})$, by the total complex of the double complex:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "-d"] \arrow[d, "\partial"] & \ldots \arrow[r, "-d"] & \Omega^{n-2}(M) \arrow[r, "-d"] \arrow[d, "\partial"] & \Omega^{n-1}(M) \arrow[r] \arrow[d, "\partial"] & 0 \\
\Omega^0(G \times M) \arrow[r, "d"] \arrow[d, "\partial"] & \ldots \arrow[r, "d"] & \Omega^{n-2}(G \times M) \arrow[r, "d"] \arrow[d, "\partial"] & \Omega^{n-1}(G \times M) \arrow[r] \arrow[d, "\partial"] & 0 \\
\Omega^0(G^2 \times M) \arrow[r, "-d"] \arrow[d, "\partial"] & \ldots \arrow[r, "-d"] & \Omega^{n-2}(G^2 \times M) \arrow[r, "-d"] \arrow[d, "\partial"] & \Omega^{n-1}(G^2 \times M) \arrow[r] \arrow[d, "\partial"] & 0 \\
\vdots & & \vdots & \vdots &
\end{tikzcd}$$

An $(n-1)$-cocycle is given by an n-tuple

$$\omega_i \in \Omega^{i-1}(G^{n-i} \times M), \quad i = 1, \ldots, n,$$

lying in the kernel, i.e., satisfying $\partial\omega_1 = 0$, $d\omega_i = \partial\omega_{i+1}$ for $i = 1, \ldots, n-1$ and $d\omega_n = 0$.

As $\partial\omega_1 = 0$, we have $\partial(\int_G(\omega_1)) = \omega_1$, hence by altering the cocycle (ω_i) by the coboundary $(\partial - d)(\int_G(\omega_1), 0, \ldots, 0)$ we get

$$(\omega_i) \sim (0, \omega_2 - d\int_G(\omega_1), \omega_3, \ldots, \omega_{n-1}).$$

We now can proceed inductively as $\partial(\omega_2 - d\int_G(\omega_1)) = \partial\omega_2 - d\partial\int_G(\omega_1)) = \partial\omega_2 - d\omega_1 = 0$, and hence obtain

$$(\omega_i) \sim (0, \ldots, 0, \tilde{\omega}_{n-1}).$$

This proved that there is a surjection $\Omega^{n-1}(M)^G \to H^{n-1}(G^\bullet \times M; \Omega^{<n})$. To complete the calculation of the cohomology we have to find the kernel of this map: We only have to care about forms on M, as forms on $G^k \times M$ for $k \neq 0$ are already discussed to be zero homologous. Let $\eta \in \Omega^{n-2}(M) \subset \bigoplus_{p=0}^{n-2} \Omega^{n-2-p}(G^p \times M)$. The boundary of η should conserve that the forms in positive simplicial degree are zero, therefore one

wants that $\partial\eta$ vanishes. $\partial\eta = 0$ is equivalent to $\partial_0^*\eta = \partial_1^*\eta$, i.e., η being G-invariant. Thus we obtain

$$H^{n-1}(G^\bullet \times M, \Omega^{<n}) = {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)},$$

and from (4.2) an induced map

$$a\colon {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)} \to H^n(G^\bullet \times M, \mathcal{D}(n))$$

Lemma 4.6 $R \circ a = d\colon {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)} \to \Omega^n_{\mathrm{cl}}(M)^G$

Proof. Let $\eta \in {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)}$. Recall that the quasi-isomorphism $\sigma^{<n}\Omega^{\bullet,*}[-1] \simeq \mathrm{Cone}(\sigma^{\geq n}\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1]$ is in degree n given by $\eta \mapsto (d\eta, \eta)$. Thus one calculates

$$R(a(\eta)) = R(d\eta, \eta) = d\eta,$$

since R comes from the projection to the first summand. □

Collecting these statements, we have proven the following theorem.

Theorem 4.7 *Let G be a finite group acting on the smooth manifold M, then there is a commutative diagram*

$$\begin{tikzcd}
 & \Omega^{n-1}_{\mathbb{C}}(M)^G\Big/d\left(\Omega^{n-2}_{\mathbb{C}}(M)^G\right) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^n_{\mathbb{C},\mathrm{cl}}(M)^G \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \hat{H}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n_G(M,\mathbb{C}) \\
 & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[ur, hook] \arrow[rr, "-\beta"] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

whose top line, bottom line and diagonals are exact.

Remark 4.8 If G is the trivial group this is exactly the hexagon of the non-equivariant case.

Proposition 4.9 *If G acts freely on M, then*

$$\hat{H}^n_G(M,\mathbb{Z}) = \hat{H}^n\left({M}\big/{G}, \mathbb{Z}\right)$$

Proof. Let Q denote the quotient manifold, $q\colon M \to Q$ the quotient map and $\{e\}$ the trivial group. The simplicial manifold $\{e\}^\bullet \times Q$ equals Q in any level and all face and degeneracy maps are the identity. This implies that

$$\hat{H}^n_{\{e\}}(Q,\mathbb{Z}) = H^n(\{e\}^\bullet \times Q, \mathcal{D}(n)) = \hat{H}^n(Q,\mathbb{Z}),$$

since ∂ alternately equals id or the zero map.

Moreover, q and $G \to \{e\}$ induce a smooth simplicial map $G^\bullet \times M \to \{e\}^\bullet \times Q$, whose geometric realization is a fattened, homotopy equivalent, version of $EG \times_G M \to Q$. This map induces homomorphism between the exact lines

$$\begin{tikzcd} H^{n-1}(Q,\mathbb{Z}) \arrow[r] \arrow[d] & H^{n-1}(Q,\mathbb{C}) \arrow[r] \arrow[d] & \hat{H}^n(Q,\mathbb{Z}) \arrow[r] \arrow[d] & H^n(Q,\mathbb{Z}) \oplus \Omega^n_{\mathrm{cl}}(Q) \arrow[r] \arrow[d] & H^n(Q,\mathbb{C}) \arrow[d] \\ H^{n-1}_G(M,\mathbb{Z}) \arrow[r] & H^{n-1}_G(M,\mathbb{C}) \arrow[r] & \hat{H}^n_G(M,\mathbb{Z}) \arrow[r] & H^n_G(M,\mathbb{Z}) \oplus \Omega^n_{\mathrm{cl}}(M)^G \arrow[r] & H^n_G(M,\mathbb{C}) \end{tikzcd}$$

As the two maps on the left and the right-hand side are isomorphisms, the same is true in the middle by the five lemma. □

How about homotopy invariance in equivariant differential cohomology? It is the same as in the non-equivariant case: It is not, but one can measure the deviation from being so.

Proposition 4.10 *Let $i_t \colon M \to [0,1] \times M$ be the inclusion at $t \in [0,1]$, let G act trivially on the interval and let $\hat{x} \in \hat{H}^n_G([0,1] \times M, \mathbb{Z})$, then*

$$i_1^*\hat{x} - i_0^*\hat{x} = a\left(\int_{[0,1]\times M/M} R(\hat{x})\right).$$

Proof. This is almost verbatim [9, Prop. 3.28]. As integral cohomology is homotopy invariant, there is $y \in H^n_G(M,\mathbb{Z})$, such that $I(x) = \mathrm{pr}_M^* y$. By surjectivity of I there is a lift $\hat{y} \in \hat{H}^n_G(M,\mathbb{Z})$, with $I(\hat{y}) = y$. Hence $I(\hat{x} - \mathrm{pr}_M^* \hat{y}) = 0$ and thus $\hat{x} = \mathrm{pr}_M^* \hat{y} + a(\omega)$ for some $\omega \in (\Omega^{n-1}([0,1] \times M))^G$. Note that $R(\hat{x}) = R(\mathrm{pr}_M^* \hat{y}) + d\omega$.

$$i_1^*\hat{x} - i_0^*\hat{x} = i_1^* a(\omega) - i_0^* a(\omega) = a\left(\int_{[0,1]\times M/M} d\omega\right) = a\left(\int_{[0,1]\times M/M} R(\hat{x})\right).$$

Where the second equality is Stokes theorem, and the third equality follows as the fiber integral of pullback forms is zero:

$$\int_{[0,1]\times M/M} R(\mathrm{pr}_M^* \hat{y}) = \int_{[0,1]\times M/M} \mathrm{pr}_M^*(R(\hat{y})) = 0.$$

□

4.1.2 A version for Lie groups by Gomi

From now on, let G denote a Lie group. In this section we restate the definition for equivariant smooth Deligne cohomology given in [25] and show how one can define a ‘curvature’ map, which does lead to a differential cohomology hexagon. Gomi notes the lack of his definition himself (see [25, Lemma 5.9]) and this lemma is the starting point for our alteration.

Combining the ideas of Gomi with the cone construction, we reformulate the definition for the equivariant Deligne complex for Lie groups by Gomi:

Definition 4.11 Let M be a G-manifold for a Lie group G. The equivariant Deligne complex in degree n is defined as

$$\mathcal{D}_{\mathrm{Gomi}}(n)_{G^\bullet \times M} = \mathrm{Cone}(\underline{\mathbb{Z}} \oplus \mathcal{F}^1_n\Omega^{\bullet,*} \to \Omega^{\bullet,*}, (z,\omega) \mapsto \omega - z)[-1].$$

Here $\mathcal{F}^1_n\Omega^*_{\mathbb{C}}$ is the simplicial sub-sheaf achieved from the simplicial sheaf of differential forms on $G^\bullet \times M$ by imposing the following conditions: in simplicial level zero, i.e., on M, forms shall have at least degree n and on any other level the differential form degree on the G-part is at least 1, if the total form degree is less then n.

In particular, if G is discrete, this is the same complex as in the last section, because on discrete groups G there are no positive degree differential forms.

Definition 4.12 Let G be a Lie group acting on a smooth manifold M. The *G-equivariant differential cohomology* of M is defined to be the hypercohomology

$$\hat{H}^n_G(M) := H^n(G^\bullet \times M, \mathcal{D}_{\mathrm{Gomi}}(n)_{G^p \times M}).$$

We investigate equivariant differential cohomology for Lie groups with the same methods as for finite groups namely with the following two short exact sequences

$$0 \to \mathrm{Cone}(\mathcal{F}^1_n\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1] \xrightarrow{a} \mathcal{D}_{\mathrm{Gomi}}(n) \xrightarrow{I} \underline{\mathbb{Z}} \to 0 \tag{4.6}$$

$$0 \to \mathrm{Cone}(\underline{\mathbb{Z}} \xrightarrow{-\iota} \Omega^{\bullet,*})[-1] \to \mathcal{D}_{\mathrm{Gomi}}(n) \xrightarrow{R} \mathcal{F}^1_n\Omega^{\bullet,*} \to 0 \tag{4.7}$$

of complexes of simplicial sheaves and also with the exact triangle

$$\mathcal{D}_{\mathrm{Gomi}}(n)_{G^\bullet \times M} \to \underline{\mathbb{Z}} \oplus \mathcal{F}^1_n\Omega^{\bullet,*} \to \Omega^{\bullet,*} \to \mathcal{D}_{\mathrm{Gomi}}(n)_{G^\bullet \times M}[1],$$

which has the following interesting part in its long exact cohomology sequence

$$\begin{aligned} H^{n-1}(G^\bullet \times M, \underline{\mathbb{Z}}) \to H^{n-1}(G^\bullet \times M, \Omega^{\bullet,*}) \to H^n(G^\bullet \times M, \mathcal{D}_{\mathrm{Gomi}}(n)) \\ \xrightarrow{(I,R)} H^n(G^\bullet \times M, \underline{\mathbb{Z}}) \oplus H^n(G^\bullet \times M, \mathcal{F}^1_n\Omega^{\bullet,*}) \xrightarrow{(-\iota,\iota)} H^n(G^\bullet \times M, \Omega^{\bullet,*}). \end{aligned}$$

As before one has ([19, Prop. 5.15 and Prop. 6.1])

$$H^n(G^\bullet \times M, \underline{\mathbb{Z}}) = H^n_G(M, \mathbb{Z}), \qquad H^n(G^\bullet \times M, \Omega^{\bullet,*}) = H^n_G(M, \mathbb{C})$$

and

$$\mathrm{Cone}(\underline{\mathbb{Z}} \xrightarrow{-\iota} \Omega^{\bullet,*}) \simeq \underline{\mathbb{C}^\delta / \mathbb{Z}}.$$

Thus, the only things left to discuss, are the differential form sheaves on the left in (4.6) and on the right in (4.7).

Lemma 4.13 (Lemma 4.5 of [25]) *Let G be a compact Lie group, then*

$$H^n(G^\bullet \times M, \mathrm{Cone}(\mathcal{F}^1_n\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1]) = \Omega^{n-1}(M)^G / d(\Omega^{n-2}(M)^G)$$

and

$$H^{n+1}(G^\bullet \times M, \mathrm{Cone}(\mathcal{F}^1_n\Omega^{\bullet,*} \xrightarrow{\iota} \Omega^{\bullet,*})[-1]) = 0.$$

Proof. We proof the first equation first. Let (ω, η) be a cocycle in the total complex for the cohomology on the left-hand side, i.e., $\omega = (\omega_i)_{i=1\dots n}, \omega_i \in \Omega^i(G^{n-i} \times M)$ such that differential form degree on G of each ω_i is at least one for $i = 1, \dots, n-1$ and $\eta = (\eta_i)_{i=0\dots n-1}, \eta_i \in \Omega^i(G^{n-1-i} \times M)$ such that $(d+\partial)\eta = \omega$ (thus $(d+\partial)\omega = 0$).

Boundaries are $((d+\partial)\alpha, \alpha - (d+\partial)\beta)$ with $\alpha = (\alpha_i)_{i=1\dots n-2}, \alpha_i \in \Omega^i(G^{n-1-i} \times M)$ such that differential form degree on G of each α_i is at least one and $\beta = (\beta_i)_{i=0\dots n-2}, \beta_i \in \Omega^i(G^{n-2-i} \times M)$.

One may decompose the forms $\eta_i = \eta_i' + \eta_i''$, where the differential form degree of η_i' on the G part is at least one and of η_i'' is zero. Taking the boundary of $(\eta', 0)$, we see that (ω, η) is cohomologous to $(\tilde{\omega}, \tilde{\eta})$, where the differential form degree of $\tilde{\eta}$ is purely on M. Let dg denote a left invariant probability measure on G, then, in analogy to (4.5), the following integration formula for forms $\gamma \in \Omega^k(G^{p+1} \times M)$, such that the differential form degree on the first G is zero,

$$\left(\int_G \gamma\right)(g_1, \dots, g_p, m) = \int_G \gamma(g, g_1, \dots, g_p, m)dg,$$

results in a k-form on $G^p \times M$, which satisfies $\partial\left(\int_G \gamma\right) = \gamma$ if $\partial\gamma = 0$. This as shown by the same calculation as for (4.5).

We have $\partial\tilde{\eta}_0 = 0$, hence adding the boundary of $(0, \left(\left(\int_G \eta_0\right), 0, \dots 0\right))$, we can assume $\tilde{\eta}_0 = 0$ without altering the cohomology class. Of course, this also alters $\tilde{\eta}_1$, which, by another boundary argument as above, again may be assumed to have degree zero on the G-part. As $(d+\partial)\eta = \omega$, we have $\partial\eta_1 = \omega_1$. How can ∂ lead to a differential form degree on the G-part? Only by the action of G on M, hence the differential form degree on the first G of ω_1 is zero if $n \geq 3$ and hence $\left(\int_G \omega_1\right) \in \mathcal{F}^1\Omega^1_{\mathbb{C}}(G^{n-2} \times M)$. Altering by $\left(\left(\left(\int_G \omega_1\right), 0, \dots\right), 0\right)$, shows that we may assume $\omega_1 = 0$ in the cohomology class. Repeating the argument before, shows, that we may assume $\eta_1 = 0$. Repeating these steps (and forgetting about the $\sim$ for simplicity of notation) yield the situation, where the only non-vanishing forms are η_{n-1}, ω_{n-1} and ω_n, satisfying $d\eta_{n-1} = \omega_n$ and $\partial\eta_{n-1} = \omega_{n-1}$. Taking X to be a $(n-1)$-tuple of tangent fields on M, as ω_{n-1} has a positive differential form degree on G, we obtain

$$\begin{aligned}
0 &= \omega_{n-1}[X](s, m) \\
&= \partial\eta_{n-1}[X](s, m) \\
&= \eta_{n-1}[X](m) - \eta_{n-1}[sX](sm),
\end{aligned}$$

hence $\eta_{n-1} \in \Omega^{n-1}_{\mathbb{C}}(M)$ is G-invariant. The only boundaries we still can mod out are from $((\dots, 0, \alpha_{n-2}), (\dots, 0, \beta_{n-2}))$, satisfying $\alpha_{n-2} - \partial\beta_{n-2} = 0$ to avoid changing $\eta_{n-2} = 0$. $\alpha_{n-2} = \partial\beta_{n-2}$ implies β_{n-2} is G-invariant, as if not, α_{n-2} would have a summand with degree zero on G. Hence, the first claim is proved.

To proof the second statement, define the tuples $\omega, \eta, \beta, \alpha$ analogously to above. Observe first that the tuples ω, η, β get one additional element, whereas tuple α increases by two elements, as $\mathcal{F}^1_n\Omega^{n-1}_{\mathbb{C}}(M) = 0$, but $\mathcal{F}^1_n\Omega^n_{\mathbb{C}}(M) = \Omega^n_{\mathbb{C}}(M)$ may not. Now, we can repeat exactly the same arguments as above, until only the three forms η_n,ω_n and ω_{n+1} satisfying the relations $d\eta_n = \omega_{n+1}$ and $\partial\eta_n = \omega_n$ are left over. This element is a boundary, because it is exactly the image of $(\alpha, \beta) = ((0, \dots, \eta_n), 0)$ under $d + \partial$. Thus we have shown that any $(n+1)$-cocycle is zero homologous, hence there is no cohomology in degree $n+1$. □

From (4.6) we obtain an induced map

$$a\colon {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)} \to H^n(G^\bullet \times M, \mathcal{D}_{\mathrm{Gomi}}(n)).$$

Lemma 4.14 $R \circ a = d + \partial\colon {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)} \to \Omega^n_{\mathrm{cl}}(M)^G \oplus \mathfrak{g}^\vee \otimes \Omega^{n-2}(M)$

Proof. Let $\eta \in {(\Omega^{n-1}(M))^G}\Big/{d\left(\Omega^{n-2}(M)^G\right)}$. From the proof of Lemma 4.13 we get that $a(\eta) = (0, (\dots, 0, \partial\eta, d\eta), (\dots 0, \eta))$, and as R is the projection to the tuple of forms in the middle, we obtain the assertion. □

Collecting these statements we obtain the following theorem.

Theorem 4.15 *Let G be a compact Lie group acting from the left on the smooth manifold M. Then there is the following commutative diagram*

$$\begin{tikzcd}
 & {(\Omega^{n-1}(M))^G}\big/{d\left(\Omega^{n-2}(M)^G\right)} \arrow[rr, "d+\partial"] \arrow[dr, "a"] & & H^n(G^\bullet \times M, \mathcal{F}^1_n\Omega^*) \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \hat{H}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n_G(M,\mathbb{C}), \\
 & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd} \tag{4.8}$$

where the top line, the bottom line and the diagonals are exact.

Remark 4.16 Parts of this diagram are due to Gomi ([25]), but, as he – partially – defined maps to different groups in the corners, he did not achieve the entire hexagon.

If G is a discrete group, this reduces to the diagram of Theorem 4.7. If G is non-discrete and acting freely on M, such that the quotient space is a manifold, one would like to compare equivariant differential cohomology with differential cohomology of the quotient. In general, one can not expect, that $\hat{H}^n_G(M,\mathbb{Z}) = \hat{H}^n(M/G,\mathbb{Z})$ as $\left(\Omega^{n-1}(M)\right)^G$ is different from $\Omega^{n-1}(M/G)$. To see this in a very explicit example, take $M = G$, then, in degree $n = 2$, $\Omega^{n-1}(M)^G = \Omega^1(G)^G = \mathfrak{g}^\vee$, but $\Omega^{n-1}(M/G) = \Omega^1(pt) = 0$.

Moreover, one can not expect, that the map $H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \to \hat{H}^n_G(M,\mathbb{Z})$ is injective as in the discrete case, because $H^{n-1}(G^\bullet \times M, \mathcal{F}^1_n\Omega^*)$ will not vanish in general. To see this, take the following example for any positive dimensional Lie group G:

$$H^2(G^\bullet \times M, \mathcal{F}^1_3\Omega^*) = \ker\left(d + \partial\colon \mathcal{F}^1\Omega^1(G \times M) \to \Omega^2(G \times M) \oplus \Omega^1(G \times G \times M)\right)$$

If $\omega \in \Omega^1(G \times M)$ has form degree one on G, then $\partial\omega = 0$ means that for any $g_1, g_2 \in G$, $m \in M$ and any vector field $X = X_1 + X_2 + X_M$, decomposed into the tangent direction of the first copy of G, the second copy of G and M, one has

$$\begin{aligned} 0 &= (\partial\omega)(g_1, g_2, m)[X] \\ &= \omega(g_2, m)[X_2] - \omega(g_1 g_2, m)[X_1 g_2 + g_1 X_2] + \omega(g_1, g_2 m)[X_1] \end{aligned} \tag{4.9}$$

Taking $X_1 = 0$ this implies, that actually $\omega = f \in C^\infty(M, g^\vee \otimes \mathbb{C})$. Moreover, taking $X_2 = 0$ in (4.9), we obtain $Ad_g \circ f = L_g^* f$ for any $g \in G$. Finally, since $d\omega = 0$, one has $d_M f = 0$. Hence

$$H^2(G^\bullet \times M, \mathcal{F}_3^1\Omega^*) = \mathrm{map}(\pi_0(M), g^\vee) \neq \emptyset.$$

Example 4.17 When constructing characteristic classes, the cohomology of the classifying space is highly interesting. Let G be a group and $EG \to BG$ the universal bundle. Then for cohomology with any coefficient group one has

$$H^*(BG) = H^*(EG/G) = H_G^*(EG) = H_G^*(pt),$$

where pt is the manifold consisting of a single point. Hence our question is: What is $\hat{H}_G^*(pt, \mathbb{Z})$?

In this case the hexagon becomes

$$\begin{tikzcd} & \Omega^{n-1}(pt) \arrow[rr, "d+\partial"] \arrow[dr, "a"] & & H^n(G^\bullet \times pt, \mathcal{F}_n^1\Omega^*) \arrow[dr] & \\ H^{n-1}(BG, \mathbb{C}) \arrow[ur] \arrow[dr] & & \hat{H}_G^n(pt, \mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n(BG, \mathbb{C}). \\ & H^{n-1}(BG, \mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur] & & H^n(BG, \mathbb{Z}) \arrow[ur] & \end{tikzcd} \tag{4.10}$$

Hence $\hat{H}_G^n(pt, \mathbb{Z}) = H_G^n(pt, \mathbb{Z})$ if $n \neq 1$ and, as $H^1(G^\bullet \times pt, \mathcal{F}_1^1\Omega^*) = 0$, we get $\hat{H}_G^1(pt) = H_G^0(pt, \mathbb{C}/\mathbb{Z}) = \mathbb{C}/\mathbb{Z}$ if G is connected. Maybe one wonders whether this $\mathbb{C}/\mathbb{Z}$ yields some characteristic class like information. The answer is: Pulling back an element of $\mathbb{C}/\mathbb{Z}$ via the classifying map of some principal G bundle, just gives a constant function on the base space.

In Section 3.1.2 we defined the Cartan complex

$$(d + \iota)_n \colon \Omega_G^n(M) \to \Omega_G^{n+1}(M)$$

which calculates equivariant cohomology, where $\Omega_G^n(M) = \left((S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G \right)^n$. We want to compare the group $H^n(G^\bullet \times M, \mathcal{F}_n^1\Omega^*)$ in the upper right corner of (4.8) with the Cartan model.

Proposition 4.18 *There is a natural isomorphism*

$$H^n(G^\bullet \times M, \mathcal{F}_n^1\Omega^*) \to \ker(d+\iota)_n \Big/ (d+i)\left(\bigoplus_{k=1}^{n/2} \left(S^k(\mathfrak{g}^\vee) \otimes \Omega^{n-1-2k}(M) \right)^G \right).$$

Proof. In Section 3.1.4 we defined a quasi-isomorphism

$$\mathcal{J} \colon \Omega^*(G^p \times M) \to \bigoplus_{l=0}^{p} C^l(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)).$$

Let

$$X^{l,k,m} = \begin{cases} 0 & \text{if } k = 0 \text{ and } m < n, \\ C^l(G, S^k(\mathfrak{g}^\vee) \otimes \Omega^m(M) & \text{otherwise.} \end{cases}$$

The double complex $(X^{\bullet,(2*+*)}, d+\iota+\bar{d}+\bar{\iota})$ is a subcomplex of $C^\bullet(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$: One has to check, that the inclusion commutes with boundaries. By the way X is defined, the only reason for which it is maybe not a subcomplex, could arise from the maps which are turned into zero maps, as they map to the zero space. Thus, the problem can only come from maps lowering indices, namely ι and $\bar{\iota}$, but these two raise the second index, hence there image does not lie in one the spaces $X^{l,0,m}$, with $m < n$.

From the definition of $\mathcal{J}$ one checks that

$$\mathcal{J}(\mathcal{F}^1_n \Omega^*(G^\bullet \times M)) \subset X^{\bullet,*,*}.$$

Moreover, $\mathcal{J}$ is the identity on those forms, which have vanishing degree on the group part and

$$H^{n-1}\left(C^\bullet(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \big/ X^{\bullet,2*,*} \right) = \Omega^{n-1}(M)^G \big/ d\left(\Omega^{n-2}(M)^G\right)$$

by integration over the first copy of G (compare Lemma 4.5). Hence, $\mathcal{J}$ and the inclusion of the Cartan complex into Getzler's resolution induce the following commutative diagram with exact rows

$$\begin{tikzcd}
H^{n-1}_G(M,\mathbb{C}) \arrow[r] \arrow[d, "\mathcal{J}_*"] & (\Omega^{n-1}(M))^G \big/ d\left(\Omega^{n-2}(M)^G\right) \arrow[r, "d+\partial"] \arrow[d, "\mathrm{id}"] & H^n(G^\bullet \times M, \mathcal{F}^1_n\Omega^*) \arrow[r] \arrow[d, "\mathcal{J}_*"] & H^n_G(M,\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
H^{n-1}_G(M,\mathbb{C}) \arrow[r] & (\Omega^{n-1}(M))^G \big/ d\left(\Omega^{n-2}(M)^G\right) \arrow[r, "d+\iota"] & H^n(X^{*,(2*+*)}) \arrow[r] & H^n_G(M,\mathbb{C}) \arrow[r] & 0 \\
H^{n-1}_G(M,\mathbb{C}) \arrow[r] \arrow[u] & (\Omega^{n-1}(M))^G \big/ d\left(\Omega^{n-2}(M)^G\right) \arrow[r, "d+\iota"] \arrow[u, "\mathrm{id}"] & \ker(d+\iota)_n \big/ \sim \arrow[r] \arrow[u] & H^n_G(M,\mathbb{C}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $\ker(d+\iota)_n \big/ \sim$ should denote the right-hand side of the assertion. By the five lemma this diagram shows that there is the isomorphism as claimed. □

The discussion of this section thus manifests in the following alteration of Theorem 4.15.

Theorem 4.19 *For any compact Lie group acting on a smooth manifold M, there is the commutative diagram*

$$\begin{tikzcd}
 & {(\Omega^{n-1}(M))^G}\big/ d\left(\Omega^{n-2}(M)^G\right) \arrow[rr, "d+\iota"] \arrow[dr, "a"] & & \ker(d+\iota)_n \big/ (d+i)\left(\bigoplus_{k=1}^{n/2} S^k(\mathfrak{g}^\vee)\otimes\Omega^{n-1-2k}(M)\right)^G \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \hat{H}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, two heads, "I"] & & H^n_G(M,\mathbb{C}) \\
 & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd} \tag{4.11}$$

whose top line, bottom line and diagonals are exact.

4.1.3 The new version for Lie groups

In the last section, the geometric refinement was done only with respect to the manifold. In this section, we will give another solution, where one enriches the equivariant cohomology by all closed Cartan forms.

Therefore we want to use the model for equivariant cohomology defined by Getzler, which we introduced in Section 3.1.3. As noted there, this model is not a cochain complex of simplicial modules, but only a simplicial homotopy cochain complex. To proceed as in the previous Section and do a similar cone-construction, we first have to investigate the algebraic structure of simplicial homotopy cochain complexes in more detail.

Definition 4.20 A *simplicial sheaf homotopy cochain complex* of modules on a simplicial manifold $M_\bullet$ is a triple $(\mathcal{F}^{\bullet,*}, f, s)$, where $\mathcal{F}^{\bullet,*}$ is a $\mathbb{Z}$-graded simplicial sheaf of modules on $M_\bullet$, which is bounded from below[1], f is a map of simplicial sheaves, which increases the $\mathbb{Z}$-grading by one and s is a simplicial zero homotopy of f^2, i.e., in simplicial degree p, $s = (s_i)_{i=0,\dots,p-1}$, where

$$s_i \colon \sigma_i^{-1}\mathcal{F}^{p,q} \to \mathcal{F}^{p-1,q+1}, \quad i = 0,\dots,p-1$$

are maps of sheaves, such that the simplicial relations of degeneracy maps hold, s commutes with f and

$$s_p \circ \tilde{\partial}_{p+1} = -f^2 \colon \left(\sigma_p^{-1}\left(\partial_{p+1}^{-1}\mathcal{F}^{p,q}\right)\right) = \mathcal{F}^{p,q} \to \mathcal{F}^{p,q+1}$$

$$s_i \circ \tilde{\partial}_j = \begin{cases} \tilde{\partial}_j \circ s_{i-1} & \text{if } i < j \\ \tilde{\partial}_{j-1} \circ s_i & \text{if } i > j+1 \end{cases}$$

$$s_j \circ \tilde{\partial}_j = s_j \circ \tilde{\partial}_{j+1} s_0 \circ \tilde{\partial}_0 = 0.$$

A *morphism* of simplicial sheaf homotopy cochain complex is a map of the simplicial sheaves, which respects the grading and commutes with both the ‘boundary map’ f and the zero homotopy.

[1]This means, there is an Integer k such that each $\mathcal{F}^{p,q} = 0$ if $q < k$.

Definition + Proposition 4.21 *Let $w\colon (\mathcal{F}^{\bullet,*}, f, s) \to (\widetilde{\mathcal{F}}^{\bullet,*}, \tilde{f}, \tilde{s})$ be a morphism of simplicial homotopy cochain complex. The* cone *of w is the simplicial sheaf homotopy cochain complex*

$$\mathrm{Cone}(w) := \left(\left(\mathcal{F}^{\bullet,k+1} \oplus \widetilde{\mathcal{F}}^{\bullet,k} \right)_{k\in\mathbb{N}}, \begin{pmatrix} -f & -w \\ 0 & \tilde{f} \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & \tilde{s} \end{pmatrix} \right).$$

Proof. The only point, which is worth to check, is the relation between the 'boundary map' and the homotopy:

$$\begin{aligned} -\begin{pmatrix} -f & -w \\ 0 & \tilde{f} \end{pmatrix}^2 = -\begin{pmatrix} f^2 & fw - w\tilde{f} \\ 0 & \tilde{f}^2 \end{pmatrix} &= \begin{pmatrix} -f^2 & 0 \\ 0 & -\tilde{f}^2 \end{pmatrix} \\ &= \begin{pmatrix} s\partial + \partial s & 0 \\ 0 & \tilde{s}\partial + \partial\tilde{s} \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & \tilde{s} \end{pmatrix} \partial + \partial \begin{pmatrix} s & 0 \\ 0 & \tilde{s} \end{pmatrix}. \end{aligned}$$

□

We are now going to define the cohomology of a simplicial sheaf homotopy cochain complex $(\mathcal{F}^{\bullet,*}, f, s)$ using a Čech model. Let $\mathcal{U}^\bullet$ be a simplicial cover of the simplicial manifold $M_\bullet$. This defines for each q a resolution of the simplicial sheaf $\mathcal{F}^{\bullet,q}$ (compare Section 3.1.1)

$$\check{C}^{\bullet,q,*}(\mathcal{U}^\bullet, \mathcal{F}^{\bullet,k})$$

with Čech boundary map δ. The properties of the simplicial cover imply, that ∂ and s restrict to the Čech groups. Hence, on the total complex of this triple graded collection of modules, we have a boundary map

$$\left(\bigoplus_{p+q+r=n} \check{C}^{p,q,r}, \partial + s + (-1)^p f + (-1)^{p+q}\delta \right),$$

where ∂ and s are the alternating sums over the maps $\tilde{\partial}_i$ and s_i respectively.

Thus we can define $\check{H}(\mathcal{U}^\bullet, (\mathcal{F}^{\bullet,*}, f, s))$ to be the cohomology of this cochain complex. As for classical Čech cohomology, refinements of the simplicial cover induce homomorphisms of the associated cohomology theories. Thus we define

$$\check{H}(M_\bullet, (\mathcal{F}^{\bullet,*}, f, s)) = \lim_{\mathcal{U}^\bullet} \check{H}(\mathcal{U}^\bullet, (\mathcal{F}^{\bullet,*}, f, s))$$

to be the limit over all refinements of open covers.

If the simplicial sheaf homotopy cochain complex $(\mathcal{F}^{\bullet,*}, f, s) = (\mathcal{F}^{\bullet,*}, d, 0)$ actually is a cochain complex of simplicial sheaves, the total complex of the Čech resolution of both types (compare Section 3.1.1) coincides, and hence the cohomology defined here, coincides with the simplicial sheaf cohomology. Moreover, if the sheaves of $(\mathcal{F}^{\bullet,*}, f, s)$ are fine, then the Čech direction contracts by the standard argument and the cohomology of $(\mathcal{F}^{\bullet,*}, f, s)$ is the cohomology of the total complex $(\oplus_{p+q=n}\mathcal{F}^{p,q}(G^p \times M), (-1)^p f + s + \partial)$.

Now, turn to our specific case, i.e., we would like to find a simplicial sheaf homotopy cochain complex $\mathcal{C}^\bullet = \mathcal{C}^{\bullet,*}$ consisting of fine sheaves, such that its global sections are given by $C^\bullet(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$.

The map $\pi\colon G^\bullet \times M \to \{e\}^\bullet \times M, (g_1, \dots, g_p, m) \mapsto (g_1 \dots g_p m)$ is a morphism of simplicial manifolds. $S^*(\mathfrak{g}^\vee) \otimes \Lambda^* T^\vee M$ is a bundle over M, with left action of G on M, the induced action on the cotangent bundle and coadjoint action on the polynomial, whose global sections are $S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)$. We can interpret this bundle as simplicial bundle on the simplicial manifold $\{e\}^\bullet \times M$, with all face and degeneracy maps being the identity. The global sections of the pullback bundle $\pi^*(S^*(\mathfrak{g}^\vee) \otimes \Lambda^* T^\vee M)$ in simplicial level p are $C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))$, thus take for $U \subset G^p \times M$ open

$$\mathcal{C}^p(U) := \Gamma(U, (\pi^*(S^*(\mathfrak{g}^\vee) \otimes \Lambda^* T^\vee M)_p).$$

This is a sheaf of $C^\infty(G^p \times M)$-modules, hence fine (see Lemma 2.29). The morphism between the simplicial levels $\tilde{\partial}_i : \partial_i^{-1}\mathcal{C}^p \to \mathcal{C}^{p+1}$ and $\tilde{\sigma}_i : \sigma_i^{-1}\mathcal{C}^p \to \mathcal{C}^{p-1}$ are given by pullback along the simplicial bundle maps.

The map $d + \iota : \mathcal{C}^{\bullet,l} \to \mathcal{C}^{\bullet,l+1}$ increases the second grading and is clearly a map of sheaves, as booth operations are local. The maps $\bar{d}$ and $\bar{\iota}$ operate between different simplicial levels: On global sections $\bar{d}$ is the alternating sum of the maps $\tilde{\partial}_i$, while $\bar{\iota}$

$$\bar{\iota}\colon C^k(G, S^l(\mathfrak{g}^\vee) \otimes \Omega^m(M)) \to C^{k-1}(G, S^{l+1}(\mathfrak{g}^\vee) \otimes \Omega^m(M))$$

is given by the formula $\bar{\iota} = \sum_{i=0}^{k-1} (-1)^i \bar{\iota}_i$, where each $\bar{\iota}_i$ is the map of sheaves

$$\begin{aligned} \bar{\iota}_i\colon \sigma_i^{-1}\mathcal{C}^k &\to \mathcal{C}^{k-1} \\ (\bar{\iota}_i f)(g_1, \dots, g_{k-1}|X) &= \left.\frac{d}{dt}\right|_{t=0} f(g_1, \dots, g_i, \exp(tX_i), g_{i+1}, \dots, g_{k-1}|X), \end{aligned}$$

with $X_i = \mathrm{Ad}(g_{i+1} \dots g_{k-1})X$.

From the discussion of the maps $d + \iota$, $\bar{\iota}$ and $\bar{d}$ in Section 3.1.3 one obtains that

$$(\mathcal{C}^{\bullet,*}, d + \iota, \bar{\iota})$$

is a simplicial sheaf homotopy cochain complex.

$\mathcal{C}^{\bullet,0}$ is the simplicial sheaf of smooth functions, in which the simplicial sheaf $\underline{\mathbb{Z}}$ injects. This induces a map of simplicial sheaf homotopy cochain complexes

$$(\underline{\mathbb{Z}}, 0, 0) \to (\mathcal{C}^{\bullet,*}, d + \iota, \bar{\iota})\,,$$

where $\underline{\mathbb{Z}}$ is located in degree zero. With respect to this injection, we define

$$\mathcal{D}_C(n)_{G^\bullet \times M} = \mathrm{Cone}(\underline{\mathbb{Z}} \oplus C^{\bullet, \geq n} \to \mathcal{C}^{\bullet,*}, (z, \omega) \mapsto \omega - z)[-1].$$

Definition 4.22 Let G be a Lie group acting on a smooth manifold M. The *full G-equivariant differential cohomology* of M is defined to be the cohomology of simplicial sheaf homotopy cochain complexes $\mathcal{D}_C(n)$:

$$\widehat{\mathbb{H}}^n_G(M) := H^n(G^\bullet \times M, \mathcal{D}_C(n)_{G^\bullet \times M}).$$

Theorem 4.23 *If G is a compact group, one has the following hexagon*

$$
\begin{tikzcd}
& \Omega^{n-1}_G(M)\big/(d+\iota) \arrow[rr, "d+\iota"] \arrow[dr, "a"] & & \Omega^n_G(M)_{\mathrm{cl}} \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & \widehat{\mathbb{H}}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n_G(M,\mathbb{C}) \\
& H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur, hook] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
\tag{4.12}
$$

where the line along top, the one along the bottom and the diagonals are exact.

Proof. This follows by the same arguments as in the last section, and the fact, that for compact Lie groups, the Getzler resolution contracts to the Cartan complex (see Section 3.1.3). □

Example 4.24 Let $M = pt$ be a point, then the hexagon (4.12) reduces in even degrees to

$$
\begin{tikzcd}
& 0 \arrow[rr, "d+\iota"] \arrow[dr, "a"] & & (S^n(\mathfrak{g}^\vee))^G \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \widehat{\mathbb{H}}^{2n}_G(pt,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^{2n}(BG,\mathbb{C}) \\
& H^{n-1}(BG,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur, hook] & & H^{2n}(BG,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

and in odd degrees to

$$
\begin{tikzcd}
& (S^n(\mathfrak{g}^\vee))^G \arrow[rr, "d+\iota"] \arrow[dr, "a"] & & 0 \arrow[dr] & \\
H^{2n}(BG,\mathbb{C}) \arrow[ur] \arrow[dr] & & \widehat{\mathbb{H}}^{2n+1}_G(pt,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & 0 \\
& H^{2n}(BG,\mathbb{C}/\mathbb{Z}) \arrow[rr, "-\beta"] \arrow[ur, hook] & & H^{2n+1}(BG,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

Hence

$$
\widehat{\mathbb{H}}^n_G(pt,\mathbb{Z}) = \begin{cases} H^n(BG,\mathbb{Z}) & \text{if } n \text{ is even} \\ H^{n-1}(BG,\mathbb{C}/\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}
$$

The contravariant functor $\widehat{\mathbb{H}}_G$ assigning an abelian group to the G-manifold M is not homotopy invariant, but its deviation from homotopy invariance is measured by the homotopy formula.

Lemma 4.25 *Let $i_t\colon M \to [0,1] \times M$ be the inclusion determined by $t \in [0,1]$ and let G act trivially on the interval. Let $\omega \in (S^*(\mathfrak{g}^\vee) \otimes \Omega^*([0,1] \times M))^n)^G$*

$$
(d_M + \iota)\left(\int_{[0,1]\times M/M} \omega\right) = i_1^*\omega - i_0^*\omega + \int_{[0,1]\times M/M} (d_M + \iota)\omega
$$

Proof. Going to local coordinates (using a partition of unity), this is the derivative of the integral by the lower bound, the upper bound and the interior derivative. □

Proposition 4.26 *If $\hat{x} \in \widehat{\mathbb{H}}^n_G([0,1] \times M, \mathbb{Z})$, then*

$$i_1^*\hat{x} - i_0^*\hat{x} = a\left(\int_{[0,1]\times M/M} R(\hat{x})\right),$$

where we have kept the notions of the previous lemma.

Proof. As equivariant integral cohomology is homotopy invariant, there is a class $y \in H^n(M, \mathbb{Z})$, such that $p_M^* y = I(\hat{x})$. As I is surjective, choose a lift $\hat{y} \in \widehat{\mathbb{H}}^n_G(\times M; \mathbb{Z})$ with $I(\hat{y}) = y$. Thus $I(p_M^*\hat{y} - \hat{x}) = 0$ and hence $\hat{x} = p_M^*\hat{y} + a(\omega)$ for some $\omega \in (S^*(\mathfrak{g}^\vee) \otimes \Omega^*([0,1] \times M))^{n-1})^G$. Therefore $(d + \iota)\omega = R(a(\omega)) = R(\hat{x}) - R(p_M^*\hat{y})$. We can write $\omega = dt \wedge \alpha + \beta$, where dt corresponds to the interval and α, β are forms on p_M^*TM On the one hand

$$i_1^*\hat{x} - i_0^*\hat{x} = a\left(i_1^*\omega - i_0^*\omega\right) = a\left(i_1^*\beta - i_0^*\beta\right).$$

On the other hand

$$a\left(\int_{[0,1]\times M/M} R(\hat{x})\right) = a\left(\int_{[0,1]\times M/M} R(\hat{x}) - p_M^* R(\hat{y})\right),$$

and, as fiber integrals over basic forms vanish,

$$\begin{aligned}
&= a\left(\int_{[0,1]\times M/M} (d+\iota)\omega\right)\\
&= a\left(\int_{[0,1]\times M/M} (d_M+\iota)\omega\right) + a\left(\int_{[0,1]\times M/M} d_{[0,1]}\omega\right)\\
&= a\left(\int_{[0,1]\times M/M} (d_M+\iota)dt\wedge\alpha\right) + a\left(\int_{[0,1]\times M/M} d_{[0,1]}\beta\right)\\
&= a\left((i_0^* - i_1^*)dt\wedge\alpha + (d_M+\iota)\left(\int_{[0,1]\times M/M} dt\wedge\alpha\right)\right)\\
&\qquad + a\left((i_1^* - i_0^*)\beta\right)\\
&= a\left(i_1^*\beta - i_0^*\beta\right).
\end{aligned}$$

In the last step we use that a vanishes on exact forms. □

To compare our definition with the construction in the last section, we define a subsheaf $\mathcal{F}_n^1\mathcal{C}^{\bullet,*} \subset \mathcal{C}^{\bullet,*}$. In the bundle $S^*(\mathfrak{g}^\vee) \otimes \Lambda^*(T^\vee M)$, we have the subbundle

$$S^{\geq 1}(\mathfrak{g}^\vee) \otimes \Lambda^{<n}(T^\vee M) + \left(S^*(\mathfrak{g}^\vee) \otimes \Lambda^*(T^\vee M)\right)^{\geq n}.$$

$\mathcal{F}^1_n\mathcal{C}^{\bullet,*}$ is defined to be the sheaf of sections of (the pullback to the simplicial manifold of) this bundle. As one checks immediately

$$\mathcal{F}^1_n\mathcal{C}^{0,n-1}(M) = \left(\bigoplus_{k=1}^{n/2} S^k(\mathfrak{g}^\vee)\otimes\Omega^{n-1-2k}(M)\right),$$

i.e., the space, whose G-invariant part is known from Proposition 4.18.

Lemma 4.27 *The image of $\mathcal{F}^1_n\Omega^{\bullet,*}$ under the Getzler map $\mathcal{J}\colon\Omega^{\bullet,*}\to\mathcal{C}^{\bullet,*}$, defined in section 3.1.4, lies in $\mathcal{F}^1_n\mathcal{C}^{\bullet,*}$.*

Proof. Let $U\subset G^p\times M$ an open set and $\omega\in\mathcal{F}^1_n\Omega^{p,k}(U)$. If $k\geq n$ there is nothing to show. Let $k<n$. The projection of the image of $\mathcal{J}(\omega)$ to $\mathcal{C}^{\bullet,*}(U)/\mathcal{F}^1_n\mathcal{C}^{\bullet,*}(U)$ is the part of $\mathcal{J}(\omega)$ whose polynomial degree is zero. This is zero, since the form degree of ω on the G part is positive (by the condition $k<n$) and hence ω is mapped to zero in the quotient and hence to a positive degree polynomial. □

Let $\mathcal{D}_C(1,n)=\mathrm{Cone}(\underline{\mathbb{Z}}\oplus\mathcal{F}^1_n\mathcal{C}^{\bullet,*}\to\mathcal{C}^{\bullet,*},(z,\omega,\eta)\mapsto\omega+\eta-z)[-1]$

Lemma 4.28 *The map of chain complexes of simplicial sheaves*

$$\mathcal{J}_*\colon\mathcal{D}(n)_{G^\bullet\times M}\to\mathcal{D}_C(1,n)_{G^\bullet\times M}$$

induces an isomorphism $\hat{H}^*_G(M,\mathbb{Z})\to H^*(G^\bullet\times M,\mathcal{D}_C(1,n))$

Proof. The same arguments as given above show, that $H^*(G^\bullet\times M,\mathcal{D}_C(1,n))$ sits in the same hexagon (4.11), as $\hat{H}^*_G(M,\mathbb{Z})$ and the induced maps on all corners is the identity. □

We have an inclusion $\mathcal{D}_C(n)\to\mathcal{D}_C(1,n)$, which, composted with the isomorphism of Lemma 4.28, induces a map

$$f\colon\widehat{\mathbb{H}}^*_G(M,\mathbb{Z})\to\hat{H}^*_G(M,\mathbb{Z}).$$

Theorem 4.29 *f is an isomorphism in degree 0,1 and 2 and surjective in higher degrees.*

Proof. This again follows from the hexagons, which coincide in degree $0,1,2$. In higher degrees, the sequence along the bottom is the same and along the top one has surjections. □

Remark 4.30 Michael Luis Ortiz discusses an idea of a definition of equivariant differential cohomology in [40, p.7-9]. He gives a recipe what to do for general Lie groups, but does not make things precise. In particular he talks about differential forms on $M\times_G EG$. As you will have noted, giving them a precise meaning, in which one can compare them with integral cohomology and the Cartan model is one of the major lines in this thesis and found its final answer in this section.

On the other hand, there is a definition of Deligne cohomology for orbifolds by Ernesto Lupercio and Bernardo Uribe in [33]. This includes the 'action orbifold' of G on M with objects M and morphisms $G \times M$, whose nerve is our simplicial manifold $G^\bullet \times M$. Translating his definition to our language, one gets the complex

$$\operatorname{Cone}\left(\underline{\mathbb{Z}} \oplus \Gamma\left(\cdot, (\partial_1^*)^\bullet \Lambda^{\geq n} T^\vee M\right)^* \to \Gamma\left(\cdot, (\partial_1^*)^\bullet \Lambda^* T^\vee M\right), (z, \omega) \mapsto \omega - z\right)[-1],$$

of cochain complexes of simplicial sheaves on $G^\bullet \times M$, where $\Gamma(\cdot, E)$ denotes the sheaf of local sections of the bundle E. This yields (for G compact) the hexagon:

$$\begin{tikzcd}
& {(\Omega^{n-1}(M))^G \big/ d\left(\Omega^{n-2}(M)^G\right)} \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega^n_{\mathrm{cl}}(M,\mathbb{C})^G \arrow[dr] & \\
{(\Omega^{n-1}_{\mathrm{cl}}(M))^G \big/ d\left(\Omega^{n-2}(M)^G\right)} \arrow[ur] \arrow[dr] & & \hat{H}^n_G(M,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & {(\Omega^n_{\mathrm{cl}}(M))^G \big/ d\left(\Omega^{n-1}(M)^G\right)} \\
& H^{n-1}_G(M, \operatorname{Cone}(\underline{\mathbb{Z}} \to \Omega^*(\cdot)^G)) \arrow[ur, hook] \arrow[rr, "-\beta"] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

In the case of finite groups, one has $H^*_G(M,\mathbb{C}) = \Omega^n_{\mathrm{cl}}(M)^G / d\Omega^{n-1}(M)^G$, thus this is the same as we had before. In the case of positive dimensional Lie groups it is even less satisfactory then the definition of Gomi, as there is not even equivariant complex cohomology at the left and the right end.

4.2 Equivariant differential characteristic classes

Let us restrict to compact groups G acting on the manifold and on vector bundles. As rank n vector bundles admit a hermitian metric, they are in one to one correspondence with principal $U(n)$ bundles. Thus any characteristic form for vector bundles corresponds to an invariant polynomial $P \in I^*(U(n))$ by Corollary 3.2.3. The from from Section 3.2 that invariant polynomials give an important class of equivariant differential forms by $\omega(\nabla) = P(R^\nabla + \mu^\nabla)$. Moreover, if ω is integral, then there is an integral equivariant characteristic class c^ω coinciding with the class of ω in complex cohomology.

Definition 4.31 A differential refinement of ω associates to every G-equivariant vector bundle with connection (E, ∇) on M a class $\hat{\omega}(\nabla) \in \widehat{\mathbb{H}}_G(M;\mathbb{Z})$ such that

$$R(\hat{\omega}(\nabla)) = \omega(\nabla), \quad I(\omega(\nabla)) = c^\omega(E)$$

and for every map $f\colon M \to M'$, we have $f^*\hat{\omega}(\nabla) = \hat{\omega}(f^*\nabla)$.

As the intersection of the kernels

$$\ker(R) \cap \ker(I) = {H^{n-1}_G(M,\mathbb{C})} \big/ {H^{n-1}_G(M,\mathbb{Z})}$$

is in general non-trivial, the differentially refined class $\hat{\omega}(\nabla)$ can contain finer information than the pair $(\omega(\nabla), c^\omega(E))$. Thus it is a priori not clear that for a given equivariant characteristic form, there is only one equivariant differential characteristic class.

Theorem 4.32 *An integral equivariant characteristic form admits a unique equivariant differential extension.*

The line of arguments of this section is (almost) the following: As $\widehat{\mathbb{H}}^{2n}_{U(n)}(pt, \mathbb{Z}) = H^n(BU(n), \mathbb{Z})$, we would like to define a map of simplicial manifolds $G^\bullet \times M \to NU(n)$ classifying our bundle and pull back the universal class together with a corresponding connection. Now we can compare this connection with the one defined on our bundle and change the differential characteristic class according to this.

Lemma 4.33 *Let* ∇ *and* ∇' *are two connections on the same bundle, then*

$$\hat{\omega}(\nabla) - \hat{\omega}(\nabla') = a(\tilde{\omega}(\nabla, \nabla'))$$

Proof. Let ∇_t denote the convex combination of ∇ and ∇'. Then by Proposition 4.26

$$\begin{aligned}
\hat{\omega}(\nabla) - \hat{\omega}(\nabla') &= i_1^*\hat{\omega}(\nabla_t) - i_0^*\hat{\omega}(\nabla_t) \\
&= a\left(\int_{[0,1]\times M/M} R(\hat{\omega}(\nabla_t))\right) \\
&= a\left(\int_{[0,1]\times M/M} \omega(\nabla_t)\right) \\
&= a(\tilde{\omega}(\nabla, \nabla')).
\end{aligned}$$

□

This Lemma implies, in particular, that we are done, if we have defined the refined for hermitian bundles with hermitian connection, since any connection can by symmetrized (compare [9, Section 2.5]).

Let $\pi\colon E \to M$ be a G-equivariant hermitian vector bundle with hermitian connection ∇ and B be the associated principal $U(n)$-bundle (with respect to some metric) furnished with the associated principal connection ϑ. Given a trivializing cover $\mathcal{U} = \{U_\alpha\}$, trivializations $\varphi_\alpha\colon \pi^{-1}U_\alpha \to U_\alpha \times G$ and transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, we obtain a diagram of simplicial manifolds similar to (3.4):

$$\begin{array}{ccccc}
G^\bullet \times B & \xleftarrow[\bar{i}]{} & (N_G B)_{\pi^{-1}\mathcal{U}} & \xrightarrow{\bar{\phi}} & N\bar{U}(n) \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow & & \big\downarrow \\
G^\bullet \times M & \xleftarrow[i]{} & (N_G M)_{\mathcal{U}} & \xrightarrow{\phi} & NU(n)
\end{array}$$

Proposition 4.34 *The map* i *induces an isomorphism*

$$i^*\colon \widehat{\mathbb{H}}^n_G(M, \mathbb{Z}) \to H^n((N_G M)_{\mathcal{U}}, i^*\mathcal{D}_C(n))$$

and isomorphisms between all corners of the hexagons (4.12) *with the corresponding corners of*

$$\begin{tikzcd}
& H^n((N_GM)_{\mathcal{U}}, i^*\operatorname{Cone}(\mathcal{C}^{*,\geq n} \to \mathcal{C}^{*,*})[-1]) \arrow[rr, "d+\iota"] \arrow[dr, "a"'] & & H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{C}^{*,\geq n}) \arrow[dr] & \\
H^{n-1}_G(M,\mathbb{C}) \arrow[ur] \arrow[dr] & & H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{D}_C(n)) \arrow[ur, "R"] \arrow[dr, "I"] & & H^n_G(M,\mathbb{C}). \\
& H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[ur, hook] \arrow[rr, "-\beta"] & & H^n_G(M,\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

Proof. Recall that $\|i\|\colon \|(N_GM)_{\mathcal{U}}\| \to \|G^\bullet \times M\|$ is a homotopy equivalence. The short exact sequence of simplicial sheaves

$$0 \to \operatorname{Cone}(\underline{\mathbb{Z}} \to \mathcal{C}^{\bullet,*}) \to \mathcal{D}_C(n)_{G^\bullet \times M} \to C^{\bullet,\geq n} \to 0$$

and the map i induce the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="] & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[r] \arrow[d, "="] & \widehat{\mathbb{H}}^n_G(M,\mathbb{Z}) \arrow[r] \arrow[d, "i^*"] & \Omega^n_G(M)_{\mathrm{cl}} \arrow[r] \arrow[d, "i^*"] & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[d, "="] \\
0 \arrow[r] & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}) \arrow[r] & H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{D}_C(n)) \arrow[r] & H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{C}^{*,\geq n}) \arrow[r] & H^{n-1}_G(M,\mathbb{C}/\mathbb{Z}).
\end{tikzcd}$$

Thus, by the five lemma, it is sufficient to show, that

$$i^*\colon \Omega^n_G(M)_{\mathrm{cl}} \to H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{C}^{*,\geq n})$$

is an isomorphism. Observe that

$$\begin{aligned} H^n((N_GM)_{\mathcal{U}}, i^*\mathcal{C}^{*,\geq n}) = \ker\left(d+\iota\colon \mathcal{C}^{0,n}\left(\coprod U_\alpha\right) \to \mathcal{C}^{0,n+1}\left(\coprod U_\alpha\right)\right) \cap \\ \ker\left(\partial\colon \mathcal{C}^{0,n}\left(\coprod U_\alpha\right) \to \mathcal{C}^{1,n}\left(\coprod_{\alpha_1,\alpha_2,\beta_1,\beta_2} U^{(1)}_{\alpha_1\alpha_2} \cap U^{(1)}_{\beta_1\beta_2}\right)\right). \end{aligned}$$

Let $(\omega_\alpha) \in \mathcal{C}^{0,n}\left(\coprod U_\alpha\right)$. The definition of the map ∂

$$\begin{tikzcd}[row sep=small]
& m \in U_{\beta_2} \\
U^{(1)}_{\alpha_1\alpha_2} \cap U^{(1)}_{\beta_1\beta_2} \ni (g,m) \arrow[ur, mapsto, "\partial_0"] \arrow[dr, mapsto, "\partial_1"'] & \\
& gm \in U_{\alpha_1}
\end{tikzcd}$$

implies that $\partial(\omega_\alpha) = 0$ is equivalent to

$$\partial_0^*\omega_\beta|_{U^{(1)}_{\alpha\beta}} = \partial_1^*\omega_\alpha|_{U^{(1)}_{\alpha\beta}}.$$

Moreover, since $e \times (U_\alpha \cap U_\beta) \subset \partial_1^* U_\alpha \cap \partial_0^* U_\beta = U^{(1)}_{\alpha\beta}$, this equation implies that (ω_α) is the restriction of a global section $\omega \in C^{0,n}(M)$, which is by the same equation G-invariant. Hence $\omega \in \ker(d+\iota) = \Omega^n_G(M)_{\mathrm{cl}}$. This proves the first claim.

The claim about the hexagon follows by the same argument, because the 'de Rham' sequence along the top is exact. □

We have defined a connection $\bar\vartheta$ on $N\overline{U(n)} \to NU(n)$ in Section 3.2.3. Moreover, the isomorphism $I^*(U(n)) \to H^*(U(n), \mathbb{C})$ implies, that there is a polynomial P and a universal class $c_P \in H^n(BU(n), \mathbb{Z}) = \widehat{\mathbb{H}}^{2n}_{U(n)}(pt, \mathbb{Z})$ corresponding to ω_P.

Definition + Proposition 4.35 *The differential refinement is given by the formula*

$$\hat\omega(\nabla) = (i^*)^{-1}(\phi^* c_P + a(\widetilde{\omega_P}(i^*\vartheta, \bar\phi^*\vartheta_0))).$$

This definition is independent of the chosen cover and trivializations and defines the differential refinement of the integral characteristic form ω_P.

Proof. We will prove the independence of the cover in three steps:

Step 1: Let $\mathcal{U}' = \{U'_\beta\}$ be a refinement of the cover $\mathcal{U}$, i.e., for any β, there is some $\alpha(\beta)$, such that $U'_\beta \subset U_{\alpha(\beta)}$; let $\varphi'_\beta = \varphi_{\alpha(\beta)}|_{U'_\beta}$. The inclusion of the refinement yields a commutative diagram

$$\begin{tikzcd}
G^\bullet \times B \arrow[d] \arrow[dr, equal] & & (N_G B)_{\pi^{-1}\mathcal{U}'} \arrow[ll, "\bar i'"'] \arrow[rr, "\bar\phi'"] \arrow[d] \arrow[dr] & & N\bar U(n) \arrow[d] \arrow[dr, equal] & \\
& G^\bullet \times B \arrow[d] & & (N_G B)_{\pi^{-1}\mathcal{U}} \arrow[ll, "\bar i"'] \arrow[rr, "\bar\phi"] \arrow[d] & & N\bar U(n) \arrow[d] \\
G^\bullet \times M \arrow[dr, equal] & & (N_G M)_{\mathcal{U}'} \arrow[ll, "i'"'] \arrow[rr, "\phi'"] \arrow[dr] & & NU(n) \arrow[dr, equal] & \\
& G^\bullet \times M & & (N_G M)_{\mathcal{U}} \arrow[ll, "i"] \arrow[rr, "\phi"'] & & NU(n)
\end{tikzcd}$$

from which the independence of the cover follows, because the direct pullback is same as the one factorized over the coarser cover.

Step 2: Take one cover $\mathcal{U} = \{U_\alpha\}$, with two different families of trivialization maps $\varphi_\alpha, \varphi'_\alpha \colon \pi^{-1}U_\alpha \to U_\alpha \times G$.

Then there is a family of maps $\psi_\alpha \colon U_\alpha \to G$, such that $\psi_\alpha(\pi(b)) \cdot \varphi_\alpha(b) = \varphi'_\alpha(b)$ for any $b \in \pi^{-1}U_\alpha$ and any α.

The difference of the two definitions is

$$\begin{aligned}
\phi^* c_P + a(\widetilde{\omega_P}(i^*\vartheta, \bar\phi^*\vartheta_0)) - \phi'^* c_P - a(\widetilde{\omega_P}(i^*\vartheta, \bar\phi'^*\vartheta_0)) \\
= \phi^* c_P - \phi'^* c_P - a(\widetilde{\omega_P}(\bar\phi^*\vartheta_0, \bar\phi'^*\vartheta_0))
\end{aligned}$$

First assume each U_α is contractible, then there is a homotopy $\widetilde{\psi}_\alpha \colon [0,1] \times U_\alpha \to G$ such that $i_1^*\widetilde{\psi}_\alpha = \psi_\alpha$ and $i_0^*\widetilde{\psi}_\alpha$ maps any point to $e \in G$. These homotopies induce a homotopy

$$\widetilde{\phi} \colon [0,1] \times (N_G B)_{\pi^{-1}\mathcal{U}'} \to NU(n)$$

between $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}_1 = \phi'$ and one can calculate

$$\begin{aligned}
\phi^* c_P - \phi'^* c_P &= i_0^* \tilde{\phi}^* c_P - i_1^* \tilde{\phi}^* c_P \\
&= a\left(\int_{[0,1]} R(\tilde{\phi}^* c_P)\right) \\
&= a\left(\int_{[0,1]} \tilde{\phi}^* R(c_P)\right) \\
&= a\left(\int_{[0,1]} \tilde{\phi}^* \int_\Delta P(\vartheta_0)\right) \\
&= a\left(\int_{[0,1]} \int_\Delta P(\tilde{\phi}^* \vartheta_0)\right) \\
&= a(\tilde{\omega_P}(\bar{\phi}^* \vartheta_0, \bar{\phi'}^* \vartheta_0)).
\end{aligned}$$

In the last step, we use that $\widetilde{\omega}_P$ is independent of the path between the connections.

The case of non contractible U_α follows by Step 1.

Step 3: Let $(\mathcal{U}, (\varphi_\alpha)), (\mathcal{U}', (\varphi'_\beta))$ be two different covers with trivializations. Let $\tilde{\mathcal{U}} = \{U_\alpha \cap U'_\beta | \alpha, \beta\}$ be the common refinement on which there are two different families of trivializations are introduced by φ and φ'. Now the statement follows from the previous steps.

Next, we check the properties of the differential refinement:

$$I(\hat{\omega}(\nabla)) = I((i^*)^{-1}(\|\phi\|^* c_P)) = c^\omega(B)$$

and

$$\begin{aligned}
R(\hat{\omega}(\nabla)) &= R((i^*)^{-1}(\|\phi\|^* c_P) + a(\tilde{\omega}(i^*\vartheta, \bar{\phi}^* \vartheta_0)) \\
&= R((i^*)^{-1}(\|\phi\|^* c_P)) + (d + \iota)\tilde{\omega}(i^*\vartheta, \bar{\phi}^* \vartheta_0) \\
&= (i^*)^{-1}(\omega(\bar{\phi}^* \vartheta_0) + \omega(i^*\vartheta) - \omega(\bar{\phi}^* \vartheta_0)) \\
&= \omega(\nabla).
\end{aligned}$$

Let $(F, f)\colon (B, M) \to (B', M')$ be a pullback. As a trivialization of (B', M') induces a trivialization of (B, M), one has a commutative diagram

$$\begin{tikzcd}
G^\bullet \times B \arrow[dr] \arrow[dd] & & (N_G B)_{\pi^{-1} f^{-1} \mathcal{U}} \arrow[ll] \arrow[rr] \arrow[dr] \arrow[dd] & & N\bar{U}(n) \arrow[dr, equal] \arrow[dd] & \\
& G^\bullet \times B' \arrow[dd] & & (N_G B')_{\pi^{-1} \mathcal{U}} \arrow[ll] \arrow[rr] \arrow[dd] & & N\bar{U}(n) \arrow[dd] \\
G^\bullet \times M \arrow[dr] & & (N_G M)_{f^{-1} \mathcal{U}} \arrow[ll] \arrow[rr] \arrow[dr] & & NU(n) \arrow[dr, equal] & \\
& G^\bullet \times M' & & (N_G M')_{\mathcal{U}} \arrow[ll] \arrow[rr] & & NU(n)
\end{tikzcd}$$

which clearly implies the pullback property.

The refinement is unique, since we used for our definition only properties the differential refinement necessarily has, namely the pullback-property and Lemma 4.33. □

4.3 Examples for equivariant differential cohomology

4.3.1 The Hopf bundle

As a first example, we want to discuss the equivariant differential cohomology of the Hopf action of S^1 on S^3. This is defined in the following way: The complex numbers act on $\mathbb{C}^2$ by scalar multiplication. Restricting to elements of unit length, this is a free action of S^1 on S^3, with quotient S^2.

To calculate the equivariant differential cohomology groups, we will need some knowledge about S^1-invariant differential forms on S^3. On zero-forms, there is no difficulty

$$\Omega^0(S^3)^{S^1} = C^\infty(S^3)^{S^1} \cong C^\infty(S^2).$$

Lemma 4.36 *The projection map $q : S^3 \to S^2$ and contraction with the fundamental vector field of $1 \in \mathbb{R} = \mathfrak{s}^1$ induce a split short exact sequence*

$$0 \to \Omega^1(S^2) \xrightarrow{q^*} \Omega^1(S^3)^{S^1} \xrightarrow{\iota(1^\sharp)} \Omega^0(S^3)^{S^1} \to 0,$$

as $1^\sharp$ is S^1-invariant.

Proof. Let ω_0 denote the form dual to $1^\sharp$ (with respect to the standard scalar product in $\mathbb{C}^2$). Then $C^\infty(S^3)^{S^1} \ni f \mapsto f\omega_0$ defines a split. Moreover, this proves exactness at the right end. Injectivity on the left end is clear. Now if we think of $S^3 \to S^2$ as S^1-bundle, exactness in the middle is the fact that basic forms are exactly invariant forms which vanish on fundamental vector fields. □

Let $\omega \in \Omega^2(S^3)^{S^1}$. The 1-form $\iota(1^\sharp)\omega$ is again horizontal and invariant and the same is true for the 2-form $\omega - \omega_0 \wedge \iota(1^\sharp)\omega$, hence

$$\Omega^2(S^3)^{S^1} \cong \left(\omega_0 \wedge q^*\Omega^1(S^2)\right) \oplus q^*\Omega^2(S^2) \cong \left(\omega_0 \wedge q^*\Omega^1(S^2)\right) \oplus \left(q^*C^\infty(S^2,\mathbb{C})d\mathrm{vol}_{S^2}\right).$$

Repeating the same argument leads to

$$\Omega^3(S^3)^{S^1} \cong \omega_0 \wedge q^*\Omega^2(S^2) \cong \omega_0 \wedge q^*\left(C^\infty(S^2,\mathbb{C})d\mathrm{vol}_{S^2}\right)$$

Proposition 4.37

$$\hat{H}^k_{S^1}(S^3,\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ C^\infty(S^2,\mathbb{C})/\mathbb{Z} & k=1 \\ \mathbb{Z}\cdot d\mathrm{vol}_{S^2} \oplus {}^{\Omega^2(S^2)}\!\big/_{\mathbb{C}\cdot d\mathrm{vol}_{S^2}} \oplus C^\infty(S^2) & k=2 \\ \Omega^2(S^2)/\mathbb{Z} \cong \mathbb{C}/\mathbb{Z} \oplus \omega_0 \wedge q^*\left({}^{\Omega^1(S^2)}\!\big/_{dC^\infty(S^2,\mathbb{C})}\right) & k=3 \\ \mathbb{C} & k=4 \\ 0 & k\geq 5 \end{cases}$$

Moreover,

$$\widehat{\mathbb{H}}^3_{S^1}(S^3,\mathbb{Z}) = \mathbb{C}/\mathbb{Z}\cdot q^*\left(d\mathrm{vol}_{S^2}\right) \oplus \left(\omega_0 \wedge q^*\Omega^1(S^2)\right) \cong \mathbb{C}/\mathbb{Z} \oplus \Omega^1(S^2),$$

and for any $k \in \mathbb{N}$

$$\begin{aligned}\widehat{\mathbb{H}}^{4+2k}_{S^1}(S^3,\mathbb{Z}) &= \mathbb{C}d\mathrm{vol}_{S^3} \oplus q^*\Omega^1(S^2) \oplus \omega_0 \wedge C^\infty(S^2)\\ &\xrightarrow[d+\iota(1)]{\sim} \mathbb{C}d\mathrm{vol}_{S^2} \oplus q^* d\Omega^1(S^2) \oplus q^*C^\infty(S^2),\end{aligned}$$

and

$$\widehat{\mathbb{H}}^{5+2k}_{S^1}(S^3,\mathbb{Z}) = \omega_0 \wedge q^*\Omega^1(S^2) \cong \Omega^1(S^2),$$

Proof. Most of the proof is an application of (4.8) and (4.11).

$k=0$**:** ${(\Omega^{-1}(M))^S}\big/{d\left(\Omega^{-2}(M)^S\right)} = 0$, thus we have an isomorphism

$$\hat{H}^0_{S^1}(S^3,\mathbb{Z}) \overset{I}{=} H^0_{S^1}(S^3,\mathbb{Z}) = H^0(S^2,\mathbb{Z}) = \mathbb{Z}.$$

$k=1$**:** The map $a\colon C^\infty(S^3,\mathbb{C})^{S^1} \to H^1_{S^1}(S^3,\mathbb{Z})$ is a surjection, whose kernel is given by $H^0(S^2,\mathbb{Z})$, which injects as constant functions.

$k=2$**:** In this case (4.11) looks like

$$\begin{tikzcd}
& {(\Omega^1(S^3))^{S^1}\big/d\left(\Omega^0(S^3)^{S^1}\right)} \arrow[rr, "d+\iota"] \arrow[dr, hook, "a"] & & \Omega^2_{S^1}(S^3)_{\mathrm{cl}} \arrow[dr] & \\
H^1(S^3,\mathbb{C}) = 0 \arrow[ur] \arrow[dr] & & \hat{H}^2_{S^1}(S^3,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & H^2(S^3,\mathbb{C}) = \mathbb{C}. \\
& 0 = H^1(S^2,\mathbb{C}/\mathbb{Z}) \arrow[ur] \arrow[rr, "-\beta"] & & H^2(S^2,\mathbb{Z}) = \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

The map a is injective, as $H^1(S^2,\mathbb{Z}) = 0$. From the discussion above, one has

$$\begin{aligned}{(\Omega^1(S^3))^{S^1}}\big/{d\left(\Omega^0(S^3)^{S^1}\right)} &= {\Omega^0(S^3)^{S^1}\omega_0 \oplus q^*\Omega^1(S^2)}\big/{dq^*\Omega^0(S^2)}\\ &= q^*\Omega^0(S^3)^{S^1}\omega_0 \oplus q^*\left({\Omega^1(S^2)}\big/{d\Omega^0(S^2)}\right).\end{aligned}$$

Moreover ,

$$\Omega^2_{S^1}(S^3)_{\mathrm{cl}} = (\Omega^2(S^3)^{S^1} \oplus \Omega^0(S^3)^{S^1})_{\mathrm{cl}} = (q^*\Omega^2(S^2) \oplus \omega_0 \wedge q^*\Omega^1(S^2) \oplus q^*C^\infty(S^2))_{\mathrm{cl}}.$$

Let $(q^*\omega, \omega_0 \wedge q^*\eta, q^*f)$ be a triple in the right-hand side of the last equation, then, to be closed, is equivalent to $d\omega = 0, \eta = df$ and $d\eta = 0$, hence

$$\Omega^2_{S^1}(S^3)_{\mathrm{cl}} = q^*\Omega^2(S^2) \oplus q^*C^\infty(S^2).$$

The map $d+\iota$ becomes

$$d \oplus \iota \colon q^* \left({\Omega^1(S^2)}\big/{d\Omega^0(S^2)} \right) \oplus q^*\Omega^0(S^3)^{S^1}\omega_0 \to q^*\Omega^2(S^2) \oplus q^*C^\infty(S^2),$$

where the quotient space $\mathbb{C}$ corresponds to subspace generated by the harmonic volume form $d\mathrm{vol}_{S^2} \in \Omega^2(S^2)$.

$k=3$: $H^3_{S^1}(S^3,\mathbb{Z}) = H^3(S^2,\mathbb{Z}) = 0$. Hence $a\colon \Omega^2(S^3))^{S^1}/d\Omega^1(S^3)^{S^1} \to \hat{H}^3_{S^1}(S^3,\mathbb{Z})$ is surjective.

$$\begin{aligned}\left(\Omega^2(S^3)\right)^{S^1}\big/{d\Omega^1(S^3)^{S^1}} &\cong {\Omega^2(S^2)}\big/{d\Omega^1(S^2)} \oplus \omega_0 \wedge q^* \left({\Omega^1(S^2)}\big/{d\Omega^0(S^2)}\right) \\ &\cong \mathbb{C} \oplus \omega_0 \wedge q^*(d\Omega^1(S^2)) \cong \Omega^2(S^2)\end{aligned}$$

$H^2_{S^1}(S^3,\mathbb{Z}) \cong \mathbb{Z}$ where $1 \in \mathbb{Z}$ corresponds the volume form on S^2.

$k=4$: (4.8) reduces to

$$\begin{tikzcd}
{(\Omega^3(S^3))^{S^1}\big/d\left(\Omega^2(S^3)^{S^1}\right)} \arrow[rr, "d+\partial"] \arrow[dr, "a"'] & & H^4((S^1)^* \times S^3, \mathcal{F}^1_4\Omega^*) \\
& \hat{H}^4_{S^1}(S^3,\mathbb{Z}) \arrow[ur, "R"'] &
\end{tikzcd}$$

where a is surjective as $H^4_{S^1}(S^3,\mathbb{Z}) = 0$, R is injective as $H^3_{S^1}(S^3,\mathbb{C}/\mathbb{Z}) = 0$ and $d+\partial$ is an isomorphism as $H^3_{S^1}(S^3,\mathbb{C}) = H^4_{S^1}(S^3,\mathbb{C}) = 0$. Moreover as $R \circ a = d+\partial$, a must be injective and R must be surjective, thus

$$\hat{H}^4_S(M,\mathbb{Z}) = {(\Omega^3(M))^{S^1}}\big/{d\left(\Omega^2(M)^{S^1}\right)}.$$

Integrating over the sphere yields an isomorphism

$$\int_{S^3} : {(\Omega^3(M))^{S^1}}\big/{d\left(\Omega^2(M)^{S^1}\right)} \to \mathbb{C}.$$

$k \geq 5$: As the diagonal from the left top to the right bottom in (4.8) is exact and both groups at the ends are zero, there is a zero in the middle.

The statements about full differential cohomology follow from the calculation of the Cartan algebra. As the Lie algebra of S^1 is one dimensional, any monomial carries the same information as its evaluation at 1. For the hexagon of full differential cohomology in odd degree ≥ 3 we have

$$\begin{aligned}\Omega^2_{S^1}(S^3)\big/(d+\iota)\Omega^1_{S^1}(S^3) &= q^*C^\infty(S^2)\oplus\Omega^2(S^3)^{S^1}\big/(d+\iota)\Omega^1(S^3)^{S^1}\\ &= \frac{q^*C^\infty(S^2)}{\iota\omega_0\wedge q^*C^\infty(S^2)}\oplus\omega_0\wedge q^*\Omega^1(S^2)\oplus\frac{q^*\Omega^2(S^3)^{S^1}}{q^*d\Omega^1(S^2)}\\ &= \omega_0\wedge q^*\Omega^1(S^2)\oplus\mathbb{C}q^*d\mathrm{vol}_{S^2}\end{aligned}$$

and

$$\begin{aligned}&\Omega^{4+2k}_{S^1}(S^3)\big/(d+\iota)\Omega^{3+2k}_{S^1}(S^3)\\ &\quad= q^*C^\infty(S^2)\oplus\Omega^2(S^3)^{S^1}\big/(d+\iota)(\Omega^1(S^3)^{S^1}\oplus\Omega^3(S^3)^{S^1}\\ &\quad= \frac{q^*C^\infty(S^2)}{\iota\omega_0\wedge q^*C^\infty(S^2)}\oplus\omega_0\wedge q^*\Omega^1(S^2)\oplus\frac{q^*\Omega^2(S^3)^{S^1}}{q^*d\Omega^1(S^2)+\iota\Omega^3(S^3)^{S^1}}\\ &\quad= \omega_0\wedge q^*\Omega^1(S^2)\\ &\quad\cong q^*\Omega^1(S^2),\end{aligned}$$

respectively for the upper left corner and

$$\begin{aligned}\Omega^{3+2k}_{S^1}(S^3)_{\mathrm{cl}} &= \Big(\omega_0\wedge q^*C^\infty(S^2)\oplus q^*\Omega^1(S^2)\oplus\omega_0\wedge q^*\Omega^2(S^2)\Big)_{\mathrm{cl}}\\ &= (\omega_0\wedge q^*C^\infty(S^2))_{\iota=0}\oplus(q^*\Omega^1(S^2)\oplus\omega_0\wedge q^*\Omega^2(S^2))_{\mathrm{cl}}\\ &= q^*\Omega^1(S^2)\end{aligned}$$

in the closed forms.

Finally, in even degree ≥ 4 one calculates for the closed forms

$$\begin{aligned}\Omega^{4+2k}_{S^1}(S^3)_{\mathrm{cl}} &= (q^*C^\infty(S^2)\oplus\omega_0\wedge q^*\Omega^1(S^2))_{\mathrm{cl}}\oplus q^*\Omega^2(S^2)_{\mathrm{cl}}\\ &= q^*C^\infty(S^2)\oplus q^*\Omega^2(S^2)\\ &= q^*C^\infty(S^2)\oplus q^*d\Omega^1(S^2)\oplus\mathbb{C}q^*d\mathrm{vol}_{S^2}\end{aligned}$$

and for groups in the upper left corner

$$\begin{aligned}\Omega^{3+2k}_{S^1}(S^3)\big/(d+\iota)\Omega^{2+2k}_{S^1}(S^3) &= \Omega^1(S^3)^{S^1}\oplus\Omega^3(S^3)^{S^1}\big/(d+\iota)(\Omega^0(S^3)^{S^1}\oplus\Omega^2(S^3)^{S^1})\\ &= \frac{\omega_0\wedge q^*C^\infty(S^2)\oplus q^*\Omega^1(S^2)\oplus\omega_0\wedge\Omega^2(S^2)}{q^*dC^\infty(S^2)\oplus(d+\iota)\omega_0\wedge q^*\Omega^1(S^2)}\\ &= \omega_0\wedge q^*C^\infty(S^2)\oplus\frac{q^*\Omega^1(S^2)}{q^*dC^\infty(S^2)}\oplus\frac{\oplus\omega_0\wedge\Omega^2(S^2)}{\omega_0\wedge dq^*\Omega^1(S^2)}\\ &\cong q^*C^\infty(S^2)\oplus q^*d\Omega^1(S^2)\oplus\mathbb{C}q^*d\mathrm{vol}_{S^2}\,.\end{aligned}$$

□

It is maybe interesting to compare this to

$$\hat{H}^k(S^2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ C^\infty(S^2,\mathbb{C})/C^\infty(S^2,\mathbb{Z}) & k=1 \\ \mathbb{Z}dvol \oplus \Omega^2(S^2)/(\mathbb{C}dvol) & k=2 \\ \mathbb{C}/\mathbb{Z} & k=3 \\ 0 & k\geq 4 \end{cases}$$

Hence, we see that the additional terms are related to ω_0 in equivariant differential cohomology. Do these allow one to find a difference between the S^1 spaces S^3 and $S^1 \times S^2$?

The choice of a metric gives a canonical dual to the fundamental vector field of $1 \in \mathfrak{s}^1$, which further gives a canonical splitting, of the short exact sequence

$$0 \to \Omega^1(S^2) \xrightarrow{q^*} \Omega^1(S^3)^{S^1} \xrightarrow{\iota(1^\sharp)} \Omega^0(S^3)^{S^1} \to 0.$$

Hence one can't find a difference between $S^1 \times S^2$ with the left action of S^1 on the first factor and S^3 by calculating equivariant differential cohomology. We can generalize these arguments:

4.3.2 Free actions

Let the Lie group G act freely on the manifold M from the left. Does equivariant differential cohomology groups make a difference between the G manifolds M and $G \times M/G$? As equivariant cohomology does not make one, the question reduces to differential forms.

To discuss this, we collect the following statements:

Definition 4.38 (Def. 13.5. of [43]) The action is *proper*, if the action map

$$G \times M \to M \times M, (g,m) \mapsto (gm,m)$$

is proper, i.e., the pre-image of any compact set is compact.

Theorem 4.39 (Th. 13.8. of [43]) *Suppose G acts properly on M. Then each orbit $G \cdot m$ is an embedded closed submanifold of M, with*

$$T_m(G\cdot m) = \{X^\sharp_M(m) | X \in \mathfrak{g}\} = \mathfrak{g}^\sharp_m.$$

Theorem 4.40 (Th. 13.10. of [43]) *Suppose that G acts properly and freely on M, then the orbit space M/G is a manifold and the quotient map $\pi\colon M \to M/G$ is a submersion.*

Suppose the action is free and proper, thus M/G is a manifold. The quotient map always induce injections

$$q^* \colon \Omega^n(M/G) \to \Omega^n(M)^G$$

and

$$\mathrm{pr}^* \colon \Omega^n(M/G) \to \Omega^n(G \times M/G)^G.$$

These lead to two resolutions of $\Omega^*(M/G)$: The first one is given as the double complex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\Omega^n(M/G) \arrow[r, "q^*"] \arrow[d, "d"] & \Omega^n(M)^G \arrow[r, "\iota"] \arrow[d, "d"] & (\mathfrak{g}^\vee \otimes \Omega^{n-1}(M))^G \arrow[r, "\iota"] \arrow[d, "d"] & (S^2(\mathfrak{g}^\vee) \otimes \Omega^{n-2}(M))^G \arrow[r] \arrow[d, "d"] & \dots \\
\Omega^{n+1}(M/G) \arrow[r, "q^*"] \arrow[d, "d"] & \Omega^{n+1}(M)^G \arrow[r, "\iota"] \arrow[d, "d"] & (\mathfrak{g}^\vee \otimes \Omega^{n}(M))^G \arrow[r, "\iota"] \arrow[d, "d"] & (S^2(\mathfrak{g}^\vee) \otimes \Omega^{n-1}(M))^G \arrow[r] \arrow[d, "d"] & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

whose total complex is the Cartan complex $\Omega^*_G(M)$, while the total complex of the second resolution is $\Omega^*_G(G \times M/G)$. The question now is: Are these two complexes equivalent on the level of cycles? This is clearly true for zero forms as the two maps

$$C^\infty(M)^G \xleftarrow{q^*} C^\infty(M/G) \xrightarrow{\mathrm{pr}^*} C^\infty(G \times M/G)^G$$

are isomorphisms. For higher degrees let h be a G-invariant Riemannian metric on M. Then the tangent bundle

$$TM = \mathfrak{g}^\sharp \oplus \left(\mathfrak{g}^\sharp\right)^\perp$$

splits with respect to h. Moreover $dq_m \colon \left(\mathfrak{g}^\sharp_m\right)^\perp \to T_{q(m)}(M/G)$ is an isomorphism for any $m \in M$. Thus we have the following lemma, what shows the equivalence in degree one

Lemma 4.41 *Let G act properly and freely on M, then*

$$0 \to \Omega^1(M/G) \xrightarrow{q^*} \Omega^1(M)^G \xrightarrow{\iota} \left(\mathfrak{g}^\vee \otimes \Omega^0(M)\right)^G \to 0$$

splits.

Proof. Restriction to $\left(\mathfrak{g}^\sharp\right)^\perp \subset TM$ defines a map $\Omega^1(M)^G \to \Omega^1(M/G)$ which is left inverse of q^*. Thus it is a split. $\square$

For the higher degrees, recall the following relation between exterior algebras.

Proposition 4.42 (Prop. 10 of [6, Ch.III, §7.7]) *Let V, W be vector spaces. Then there is a natural isomorphism of algebras*

$$\Lambda^*(V) \otimes \Lambda^*(W) \to \Lambda^*(V \oplus W),$$

from the graded tensor product of the exterior algebras to the exterior algebra of the direct sum.

We will now restrict to the case, where the adjoint action of G on $\mathfrak{g}$ is trivial. This includes, in particular, the case of abelian Lie groups.

An element of $\Omega^*_G(G \times (M/G))$ is an invariant section of

$$S^*(\mathfrak{g}^\vee) \otimes \Lambda^* \left(T^\vee \left(G \times {}^{M}\!/_{G}\right)\right) \to G \times {}^{M}\!/_{G},$$

what by the splitting of the cotangent space and Proposition 4.42 is a G-invariant section of

$$S^*(\mathfrak{g}^\vee) \otimes \Lambda^* \left(\mathrm{pr}_1^* T^\vee G\right) \otimes \Lambda^* \left(\mathrm{pr}_2^* T^\vee {}^{M}\!/_{G}\right) \to G \times {}^{M}\!/_{G}.$$

This is the same as a section of

$$S^*(\mathfrak{g}^\vee) \otimes \Lambda^* \left(\mathfrak{g}^\vee\right) \otimes \Lambda^* \left(\mathrm{pr}_2^* T^\vee {}^{M}\!/_{G}\right) \to {}^{M}\!/_{G},$$

since the action of G on $S^*(\mathfrak{g}^\vee)$ is trivial. Pulling this section back to M along the quotient map yields a G-invariant section of

$$S^*(\mathfrak{g}^\vee) \otimes \Lambda^* \left(\mathfrak{g}^\vee\right) \otimes q^*\Lambda^* \left(T^\vee {}^{M}\!/_{G}\right) \to M.$$

Composition with $\mathrm{id} \otimes \sharp \otimes \left(dq\Big|_{(\mathfrak{g}^\sharp)^\perp}\right)^{-1}$ turns this section to an G-invariant section of

$$S^*(\mathfrak{g}^\vee) \otimes \Lambda^* T^\vee M \to M$$

and thus an element of $\Omega^*_G(M)$, because $X^\sharp_{gm} = g(g^{-1}Xg)^\sharp_m = g \cdot X^\sharp_m$. As any of these steps may be gone in the opposite direction, we have an isomorphism between $\Omega^*_G(M)$ and $\Omega^*_G(G \times (M/G))$.

Thus for free proper actions of abelian groups, there is no difference between M and $G \times (M/G)$ in equivariant differential cohomology. The most easy example for a free proper action of a non-abelian Lie group on a manifold is the left multiplication of $S^3 \subset \mathbb{H}$ on $S^7 \subset \mathbb{H}$. We will leave this discussion to future research.

Let $E \to M$ be a G-equivariant vector bundle with free and proper G-action on the base and the total space. Given a connection on ∇ on E, there is the question, whether this connection is a pullback from the quotient bundle

$$\begin{array}{ccc} E & \xrightarrow{\bar{q}} & \overline{E} := {}^{E}\!/_{G} \\ \downarrow & & \downarrow \\ M & \xrightarrow{q} & \overline{M} := {}^{M}\!/_{G}. \end{array}$$

Clearly, if the connection is a pullback, then every equivariant differential characteristic class $\hat{c}(\nabla)$ must lie in the image of

$$\tilde{q}^* \colon \hat{H}(\overline{M}, \mathbb{Z}) \to \widehat{\mathbb{H}}_G(M, \mathbb{Z}),$$

where $\tilde{q}$ is the projection of the simplicial manifolds $G^\bullet \times M \to \{e\}^\bullet \times \bar{M}$. In particular, the connection must be G-invariant and the moment map must vanish (compare also [9, Section 2.2]).

Now turn the question the other way around: Assume, the there is some collection of equivariant differential characteristic classes for a connection on $E \to M$, which all lie in the image of $\tilde{q}^*$. Does this imply that connection descends to the quotient bundle?

We want to remark to following observations according to an answer of this question: Let ∇ be a connection on the equivariant complex vector bundle $E \to M$ of rank n. Then total equivariant Chern form is given by

$$R(\hat{c}(\nabla) = \det\left(1 + \frac{1}{2\pi i} R^\nabla + \mu^\nabla\right).$$

For any $X \in \mathfrak{g}$, this form induces a polynomial

$$\begin{aligned} P_X(t) &= \det\left(1 + \frac{1}{2\pi i} R^\nabla + \mu^\nabla(tX)\right) \\ &= \det\left(1 + \frac{1}{2\pi i} R^\nabla + t\mu^\nabla(X)\right) \end{aligned}$$

in t. If the total equivariant Chern form lies in the image of the quotient map, then the degree of polynomial in t is zero.

In the case of $R^\nabla = 0$, $t^n P_X(\frac{1}{t})$ is exactly the characteristic polynomial of $\mu^\nabla(X)$ and hence all eigenvalues of $\mu^\nabla(X)$ are zero, if total equivariant Chern form lies in the image of the quotient map. In general, this does not imply that $\mu^\nabla(X)$ is zero, but if there is a metric on E, we can say more.

Let h be a hermitian metric on E and ∇ be compatible with h. Then E is in correspondence to a principal $U(n)$-bundle and, as the Lie algebra $\mathfrak{u}(n)$ consists of anti-hermitian matrices, the image of $\mu^\nabla(X)$ at any point of M is anti-hermitian. The Jordan normal form of an anti-hermitian matrix is diagonal, because the conjugate of an anti-hermitian matrix by an unitary one is anti-hermitian,

$$(U^*AU)^* = U^*A^*U = -U^*AU$$

and hence all 1's' in the first the upper diagonal must vanish. Since an invariant connection descends, if and only if the moment map vanishes (compare [9, Problem 2.24]), we have proven the following Proposition.

Proposition 4.43 *Let $(E, h) \to M$ be a G-equivariant hermitian vector bundle, such that the G-action is free and proper, and let ∇ be a G-invariant hermitian connection on E, such that the curvature R^∇ vanishes, then ∇ descends to a connection on*

$$E/G \to M/G,$$

if and only if the total Chern form vanishes.

4.3.3 Conjugation action on S^3

The manifold $S^3 \subset \mathbb{R}^4$ has a group structure. Recall that one defines on the vector space $\mathbb{R}^4$ a real (non-commutative) division algebra, the quaternions, with three imaginary units i, j, k squaring to -1 and satisfying $ij = -ji = k$. Now the space of unit quaternions is S^3 and has an induced multiplication. On the other hand, there is another description of the 3-sphere by the special unitary group of complex 2×2-matrices

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

The map

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto a + jb \in S^3 \subset \mathbb{H}$$

defines a group isomorphism between the two descriptions.

We want to investigate the conjugation action of S^3 on itself. Therefore note the following well-known fact:

Lemma 4.44 *Half the trace or the real part of the quaternion is an invariant surjective mapping*

$$\frac{1}{2}\operatorname{tr} \colon S^3 \to [-1, 1],$$

which induces an isomorphism of the quotient ${S^3}/{SU(2)} \to [-1,1]$. *The isotropy group of any point besides* 1 *and* -1 *is isomorphic to* S^1.

Proof. The map is surjective, since it is continuous, $1, -1 \in S^3 \subset \mathbb{H}$, and continuous maps map connected sets to connected sets. As

$$\operatorname{tr}(A^{-1}BA) = \operatorname{tr}(BAA^{-1}) = \operatorname{tr}(B),$$

the map is invariant. Applying this invariance, it suffices to show, that any orbit has a representative of the form $x+iy$, Therefore let $a, b, c, d \in \mathbb{R}$, with $\sqrt{a^2 + b^2 + c^2} = d > 0$. We have to show that there exists $g \in \mathbb{H}$, $\bar{g}g = 1$, such that $ai + bj + ck = d\bar{g}ig$. We write $g = z + jw$, with $z, w \in \mathbb{C} \subset \mathbb{H}$.

$$\bar{g}ig = (\bar{z} - jw)i(z + jw) = i\bar{z}z + \bar{z}kw - jiwz - jwkw = i\bar{z}z + 2kwz - i\bar{w}w$$

hence

$$a = d(|z|^2 - |w|^2), \quad b = 2d\Im(zw), \quad c = 2d\Re(zw), \quad |z|^2 + |w|^2 = 1$$

from which follows

$$|z| = \frac{\sqrt{d+a}}{\sqrt{2d}}, \quad |w| = \frac{\sqrt{d-a}}{\sqrt{2d}}, \quad e^{i\arg(zw)} = \frac{c + ib}{\sqrt{d^2 - a^2}}.$$

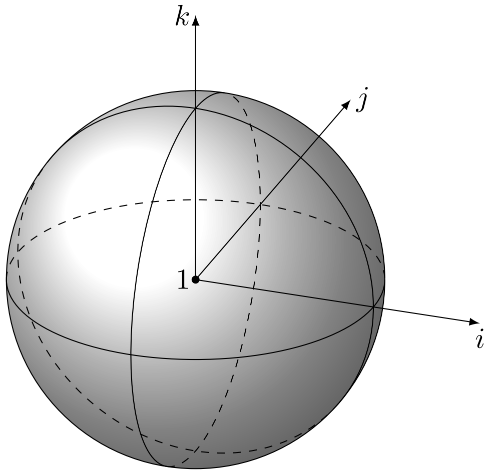
This defines the pair (z, w) up to an angle. This angle represents the S^1-isotopy. $\square$

Another helpful picture of S^3 is obtained from stereographic projection with projection point -1. In formulas this is expressed as

$$\mathbb{H} \supset S^3 \ni x = x_0 + ix_1 + jx_2 + kx_3 \mapsto \frac{1}{1+x_0}(x_1, x_2, x_3) \in \mathbb{R}^3 \cup \{\infty\},$$

where $1 \in \mathbb{H}$ is mapped to the $0 \in \mathbb{R}^3$ and -1 to ∞. Taking sets of fixed real value in $\mathbb{H}$, then these are mapped to 2-sphere of Radius $\sqrt{\frac{1-x_0}{1+x_0}}$.

The conjugation action acts transitive on each of these 2-spheres and leaves the midpoint and ∞ fixed.



This figure shows the stereographic projection of the 3-sphere $S^3 \setminus \{-1\}$ to $\mathbb{R}^3$, filled with 2-spheres. i, j and k are the imaginary units of the quaternions, which span the tangent space at $0 \in \mathbb{R}^3$.

The vector field in real direction, discussed in the text, points outward like the spines of a hedgehog, perpendicular to the corresponding 2-sphere and its length is the radius of this 2-sphere.

Taking a map $f \in C^\infty(S^3)^{S^3}$. It is clear that the map does only depend on the real value or, in the other picture, not on the point itself, but only on the 2-sphere, on which the point is located. To be smooth, the function must depend smoothly on the real value and the different direction must fit at 1 and -1. As the function has the same value in any direction of 1, fitting smoothly means that all odd derivatives must vanish. Thus

$$\begin{aligned} C^\infty(S^3)^{S^3} &\cong \left\{ f \in C^\infty([-1,1]) \middle| \frac{d^k f}{dt^k}(-1) = \frac{d^k f}{dt^k}(1) = 0, \text{ for all odd } k > 0 \right\} \\ &\subset C^\infty([-1,1], \mathbb{C}) \end{aligned}$$

Let $\omega \in \Omega^1(S^3)^{S^3}$. Let v be a tangent vector on one of the two fixed points. Then there exists $g \in S^3$, s.t. $g^{-1}vg = -v$, hence an invariant one form must be zero on the fixed points. As the real part of the quaternion is invariant under conjugation, the vector field pointing in this direction, projects to a invariant tangent field on S^3, which vanishes only at 1 and -1. In the $\mathbb{R}^3$ picture, this is the radial vector field pointing outward everywhere. Let X now denote the normalization of this vector field on $S^3 \setminus \{1, -1\}$, and ω_0 the one form dual to X. Let $\omega_1 = \omega - (\iota(X)\omega)\omega_0$, where ι is the contraction of the form by the field. A priori this forms are only defined on $S^3 \setminus \{1, -1\}$, but as ω is zero at 1 and -1, we can extend $(\iota(X)\omega)\omega_0$ and ω_1 by zero to obtain a smooth form on all of S^3. Taking any slice of S^3 with fixed real part in $(-1,1)$, this is isomorphic to S^2 and ω_1 actually is a one form on each of these 2-spheres. The S^1-isotropy found above, acts non trivially on tangent vectors. Hence with the same argument as above (Rotating the tangent vector to minus itself) one sees, that ω_1 actually is zero. Thus $\omega = (\iota(X)\omega)\omega_0$. Let f be the integral of $\iota(X)\omega \in C^\infty(S^3)^{S^3} \subset C^\infty([-1,1])$ over the interval, then $\omega = df$ and $f'(1) = f'(-1) = 0$ as ω vanishes at the fixed points. Thus we have shown

$$\Omega^1(S^3)^{S^3} \big/ dC^\infty(S^3)^{S^3} = 0.$$

Let $\omega \in \Omega^2(S^3)^{S^3}$. Contracting with the radial field X as defined in the last paragraph yields $\iota(X)\omega = f\omega_0$, for some function f. As $\iota^2 = 0$, $f = 0$. Thus, restricting ω to each of the level of fixed real part in the open interval, one obtains a multiple of the volume form on S^2. At the fixed points one gets a $SO(3)$-invariant 2-form on $\mathbb{R}^3$, since the adjoint action on the Lie algebra of $SU(2)$ is how one defines the double cover of $SU(2) \to SO(3)$. But there is no non-zero skew-symmetric matrix commuting with the whole $SO(3)$. Thus ω must vanish on the fixed points. Moreover, as any invariant 1-form is exact,

$$\begin{aligned}\Omega^2(S^3)^{S^3} \big/ d\Omega^1(S^3)^{S^3} &= \Omega^2(S^3)^{S^3} \\ &\cong \left\{ f \in C^\infty([-1,1]) \,\middle|\, f(-1) = f(1) = 0, \frac{d^k f}{dt^k}(\pm 1) = 0, k \text{ odd} \right\}.\end{aligned}$$

A volume form on the manifold induces an isomorphism $\Omega^3(S^3) \cong C^\infty(S^3)$. Since the standard volume is invariant, we get an isomorphism for invariant forms and functions. Let $X \in \mathfrak{s}^3 \subset \mathbb{H}$. Then

$$X^\sharp(m) = \frac{d}{dt}\bigg|_{t=0} (1+tX)m(1-tX) = Xm - mX.$$

Thus for $\omega \in \Omega^3(S^3)^{S^3}$

$$\begin{aligned}\iota(X^\sharp)\omega(m) = \iota(Xm - mX)\omega(m) &\overset{\omega = \mathrm{Ad}^*\omega}{=} \iota(Xm)\omega(m) - \iota(mX)Ad_m^*\omega(m) \\ &= \iota(Xm)\omega(m) - \iota(m^{-1}mXm)\omega(m) = 0. \quad (4.13)\end{aligned}$$

Moreover, d vanishes on top forms, hence the Cartan differential on $\Omega^3(S^3)^{S^3}$ is zero. As S^3 has empty boundary

$$\int_{S^3} : d\Omega^2(S^3)^{S^3} \to \mathbb{C}$$

is the zero map by Stokes theorem. Thus

$$\Omega^3(S^3)^{S^3} \big/ d\Omega^2(S^3)^{S^3} \to \mathbb{C}, \omega \mapsto \int_{S^3} \omega.$$

is a well defined injective homomorphism. From the calculation of the cohomology below, we see, that it is surjective.

What is the classical equivariant cohomology of the conjugation action of S^3 with values in $R \in \{\mathbb{Z}, \mathbb{C}, \mathbb{C}/\mathbb{Z}\}$? Taking the simplicial manifold model for $ES^3 \times_{S^3} S^3$ and a cellular resolution with cell structure on S^3 given by one zero cell, corresponding to the neutral element of S^3, and one three cell, then all simplicial maps are cellular and we obtain the following double complex with the cellular resolution horizontally to the right and the simplicial complex in vertical direction downwards (compare page 33).

$$
\begin{tikzcd}
S^3 & R \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\partial^{(0)}"] & 0 \arrow[r] \arrow[d] & \dots \\
S^3 \times S^3 & R \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^2 \arrow[r] \arrow[d, "\partial^{(1)}"] & 0 \arrow[r] \arrow[d] & \dots \\
(S^3)^2 \times S^3 & R \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^3 \arrow[r] \arrow[d, "\partial^{(2)}"] & 0 \arrow[r] \arrow[d] & \dots \\
(S^3)^3 \times S^3 & R \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^4 \arrow[r] \arrow[d, "\partial^{(3)}"] & 0 \arrow[r] \arrow[d] & \dots \\
(S^3)^4 \times S^3 & R \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R^5 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

The R in the 0-column corresponds to the zero cell and the R^k in the 3-column corresponds to the k 3-cells in $(S^3)^{\times k}$. The 3-cells in $S^3 \times S^3$ are $S^3 \times \{e\}$ and $\{e\} \times S^3$ and in $S^3 \times S^3 \times S^3$ are $S^3 \times \{e\} \times \{e\}$,$\{e\} \times S^3 \times \{e\}$ and $\{e\} \times \{e\} \times S^3$. One calculates directly for the conjugation action, that $\partial^{(0)} = 0$ and $\partial^{(1)}(a, b) = (0, 0, b)$, where the i-th entry corresponds to the i-th cell. Hence we obtain

$$H^k_{S^3}(S^3, R) = \begin{cases} R & k = 0, 3, 4 \\ 0 & k = 1, 2 \end{cases}$$

and can interpret this geometrically: The third cohomology is generated by the 3-cell in S^3 and the fourth cohomology is generating by the ‘acting’ 3-cell $S^3 \times \{e\} \subset S^3 \times S^3$.

Now the next proposition follows, in the main, by applying the hexagons (4.11) and (4.12).

Proposition 4.45 *For the conjugation action of the 3-sphere $S^3 = SU(2)$ on itself, we have*

$$\hat{H}^n_{S^3}(S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ C^\infty(S^3)^{S^3}/\mathbb{Z} & n = 1 \\ 0 & n = 2 \\ \Omega^2(S^3)^{S^3} \oplus \mathbb{Z}dvol_{S^3} \subset \Omega^3(S^3)^{S^3} & n = 3 \\ \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} & n = 4 \\ H^n_{S^3}(S^3, \mathbb{Z}) & n \geq 5 \end{cases}$$

and

$$\widehat{\mathbb{H}}^n_{S^3}(S^3,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ C^\infty(S^3)^{S^3}/\mathbb{Z} & n=1 \\ 0 & n=2 \\ \Omega^2(S^3)^{S^3} \oplus \mathbb{Z}dvol_{S^3} \subset \Omega^3(S^3)^{S^3} & n=3 \\ \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \oplus \Omega^1(S^3)^{S^1}/C^\infty(S^3)^{S^1} & n=4. \end{cases}$$

Proof. For $\hat{H}^n_{S^3}(S^3,\mathbb{Z})$, the only open question is the case $n=4$. There one obtains a short exact sequence $0 \to \mathbb{C}/\mathbb{Z} \to \hat{H}^4_{S^3}(S^3,\mathbb{Z}) \to \mathbb{Z} \to 0$ from the hexagon. This sequence splits, because $\mathbb{C}/\mathbb{Z}$ is an injective abelian group.

In the case of $\widehat{\mathbb{H}}^4_{S^3}(S^3,\mathbb{Z})$ one has the following hexagon from (4.12):

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C} \oplus {}^{(\mathfrak{s}^3)^\vee \otimes \Omega^1(S^3)))^{S^3}}\!\big/_{d((\mathfrak{s}^3)^\vee \otimes \Omega^0(S^3)))^{S^3})} \arrow[rr, "0\oplus(d+\iota)"] \arrow[dr, "a"] & & ((\mathfrak{s}^3)^\vee \otimes \Omega^2(S^3)))^{S^3} \oplus S^2((\mathfrak{s}^3)^\vee) \otimes \Omega^0(S^3)))^{S^3})_{\mathrm{cl}} \arrow[dr] & \\
\mathbb{C} \arrow[ur] \arrow[dr] & & \widehat{\mathbb{H}}^4_{S^3}(S^3,\mathbb{Z}) \arrow[ur, "R"] \arrow[dr, two heads, "I"] & & \mathbb{C} \\
 & \mathbb{C}/\mathbb{Z} \arrow[ur, hook] \arrow[rr, "0"] & & \mathbb{Z} \arrow[ur] &
\end{tikzcd} \tag{4.14}$$

As discussed above $\mathfrak{s}^3 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \subset \mathbb{H}$ and S^3 acts transitive on the unit sphere of this space. Moreover, the subgroup of S^3, which leaves $i \in \mathfrak{s}^3$ invariant, is exactly $S^1 \subset \mathbb{C} \subset \mathbb{H}$. Hence

$$\begin{gathered} \left(\left(\mathfrak{s}^3\right)^\vee \otimes \Omega^k\left(S^3\right)\right)^{S^3} \cong \Omega^k\left(S^3\right)^{S^1} \\ \left(\omega\colon \mathfrak{s}^3 \to \Omega^k\left(S^3\right)\right) \mapsto \omega(i) \end{gathered}$$

and, since the first and second de Rham cohomology of S^3 vanish, averaging over the S^1 implies that $d\colon \Omega^1(S^3)^{S^1}/dC^\infty(S^3)^{S^1} \to \Omega^2_{\mathrm{cl}}(S^3)^{S^1}$ is an isomorphism.

Further, let

$$(\omega, f) \in \left(\left((\mathfrak{s}^3)^\vee \otimes \Omega^2(S^3)\right)^{S^3} \oplus \left(S^2((\mathfrak{s}^3)^\vee) \otimes \Omega^0(S^3)\right)^{S^3}\right)_{\mathrm{cl}},$$

i.e., $d\omega = 0$ and $df = -\iota\omega$. Then $\omega = d\eta$ for one and only one

$$\eta \in {}^{(\mathfrak{s}^3)^\vee \otimes \Omega^1(S^3)))^{S^3}}\!\Big/_{d\left(\left(\left(\mathfrak{s}^3\right)^\vee \otimes \Omega^0\left(S^3\right)\right)^{S^3}\right)}$$

and $df = -\iota d\eta$. On the other f is given by a symmetric 3×3 matrix of smooth functions on S^3

$$\begin{pmatrix} f_{ii} & f_{ij} & f_{ik} \\ f_{ji} & f_{jj} & f_{jk} \\ f_{ki} & f_{kj} & f_{kk} \end{pmatrix}$$

and this matrix is determined, up to a constant matrix denoted by A, by the form η. By the transitive action of S^3 on the Lie algebra, it is clear, that the information of the matrix is contained in f_{ii} and f_{ij}. The conjugation by the element $\frac{1+k}{\sqrt{2}} \in S^3$ translates the pair (i,j) to $-(j,i)$. Hence $f_{ij} = -Ad^*_{\frac{1+k}{\sqrt{2}}} f_{ij}$. Thus the off-diagonal terms of the symmetric matrix A must vanish and hence A must be a multiple of identity matrix.

Thus, we have described an isomorphism

$$\mathbb{C} \oplus {}^{\Omega^1(S^3)^{S^1}} \diagup_{C^\infty(S^3)^{S^1}} \to \left(\left((\mathfrak{s}^3)^\vee \otimes \Omega^2(S^3) \right)^{S^3} \oplus \left(S^2((\mathfrak{s}^3)^\vee) \otimes \Omega^0(S^3) \right)^{S^3} \right)_{\mathrm{cl}}$$
$$(A, \eta) \mapsto (f, \omega).$$

Applying this isomorphism, the hexagon (4.14) changes to

$$\begin{tikzcd}
\Omega^1(S^3)^{S^1}/C^\infty(S^3)^{S^1} \arrow[rr, hook, "\mathrm{id}"] \arrow[dr, hook, "a"] & & \Omega^1(S^3)^{S^1}/C^\infty(S^3)^{S^1} \oplus \mathbb{C} \arrow[dr] & \\
& \widehat{\mathbb{H}}^4_{S^3}(S^3, \mathbb{Z}) \arrow[ur, "R"] \arrow[dr, "I"] & & \mathbb{C}, \\
\mathbb{C}/\mathbb{Z} \arrow[ur, hook] \arrow[rr, "0"] & & \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

where again the top line, the bottom line and the diagonals are exact. The map a is injective because the inclusion in the top line factors as $R \circ a$. $\square$

4.3.4 Actions of finite cyclic groups on the circle

Let $C_p = \mathbb{Z}/p\mathbb{Z}$ denote the cyclic group with p elements. There is an action of C_p on any odd sphere $S^{2n-1} \subset \mathbb{C}^n$, where a fixed generator acts by multiplication with $e^{\frac{1}{p}2\pi i}$. This diagonal action is also unitary on the infinite-dimensional separable Hilbert space $l^2(\mathbb{N}, \mathbb{C})$ and hence induces an action on the unit sphere S^∞. The inclusions of $\mathbb{C}^n$s as first coefficients induce equivariant inclusions

$$S^1 \to S^3 \to \ldots \to S^\infty$$

The sum of the tangent bundle and the normal bundle of $S^1 \subset \mathbb{C}$ is a complex line bundle, $TS^1 \oplus N \cong S^1 \times \mathbb{C}$, which we equip with the connection ∇, whose associated parallel transport respects the decomposition in tangent and normal space. Hence, the holonomy once around the circle equals 2π, thus is trivial. The sphere bundle (with respect to the standard metric) of $TS^1 \oplus N$ is the trivial S^1 bundle on S^1 with the S^1-invariant connection. Now we have a pullback diagram of bundles with connection with equivariant maps

$$\begin{tikzcd}
(S^1 \times S^1, \nabla) \arrow[r] \arrow[d] & H = S^3 \times_{S^1} S^1 \arrow[r] \arrow[d] & S^\infty \arrow[d] \\
S^1 \arrow[r, "f"] & S^3 \arrow[r] & S\infty/S^1.
\end{tikzcd}$$

Moreover the first Chern class $c_1(S\infty \to S^\infty/S^1) \in H^2(S^\infty/S^1) = H^2(BS^1)$ is a generator. Now for $\hat{H}^2_{C_p}(S^3,\mathbb{Z})$ we have the diagram

$$\begin{tikzcd} & \hat{H}^2_{C_p}(S^3,\mathbb{Z}) \arrow[dr, "I"] & \\ H^1_{C_p}(S^3,\mathbb{C}/\mathbb{Z}) \arrow[ur, hook] \arrow[rr, "-\beta"] & & H^2_{C_p}(S^3,\mathbb{Z}). \end{tikzcd}$$

As first and second cohomology are torsion, the Bockstein is an isomorphism, given by multiplication with p. As the connection on H is flat, $\hat{c}_1(H)$ actually is a class in $H^1_{C_p}(S^3,\mathbb{C}/\mathbb{Z})$. Let the cycle $\tau = \left[0,\frac{1}{p}\right] \subset \mathbb{R}/\mathbb{Z} \cong S^1$ be a fundamental domain of the C_p action on S^1. Evaluation at $f(\tau)$ induces the isomorphism $H^1_{C_p}(S^3,\mathbb{C}/\mathbb{Z}) \to \left(\frac{1}{p}\mathbb{Z}\right)/\mathbb{Z}$ under which $c_1(H)$ is mapped to $\frac{1}{p}$. Pulling back the class along f shows

$$\hat{c}_1(TS^1 \oplus N) = \frac{1}{p} \in \mathbb{C}/\mathbb{Z}.$$

A finer analysis shows that the bundle $S^1 \times S^1 \to S^1$, where C_p acts by multiplication with $e^{\frac{q}{p}2\pi i}$ on the fiber and $e^{\frac{1}{p}2\pi i}$ on the base space, has first equivariant differential Chern class $\frac{q}{p} \in \mathbb{C}/\mathbb{Z}$. One may interpret this as a measurement of holonomy along the fundamental domain.

4.3.5 G-Representations

In this section, we want to investigate actions of Lie groups on $\mathbb{R}^n$. This will lead to some implication to equivariant immersions. Equivariant immersions will be subject of further investigation. To generalize the well-known methods of characteristic classes applied to immersion, one has, in particular, to define multiplicative structures in equivariant differential cohomology and generalize the Whitney-Sum-Formula.

Let the Lie group G act smoothly on $\mathbb{R}^n$ with the standard metric. This induces an action on the tangent bundle $(T\mathbb{R}^n,\nabla) = (\mathbb{R}^n \times \mathbb{R}^n, d)$ with the trivial connection. As d^2 is zero and the Lie derivative coincides with d, the curvature and the moment map are zero

$$R^\nabla = 0, \quad \mu^\nabla = 0.$$

Hence for any equivariant differential characteristic class $\hat{c}$ with corresponding invariant polynomial $P \in I^*(O(n))$, one has

$$R(\hat{c}(G \curvearrowright \mathbb{R}^n)) = P(\mu^\nabla + R^\nabla) = P(0) \in \Omega^0_{\mathrm{cl}}(\mathbb{R}^n) = \mathbb{C}.$$

Hence, if the degree of the characteristic class is non zero, $I(c(G \curvearrowright \mathbb{R}^n)) \in H^n_G(\mathbb{R}^n,\mathbb{Z}) = H^n_G(pt,\mathbb{Z})$ is a torsion element. Thus it is in the image of the Bockstein homomorphism

$$\beta\colon H^n_G(pt,\mathbb{C}/\mathbb{Z}) \to H^n_G(pt,\mathbb{Z}) = H^n_G(\mathbb{R}^n,\mathbb{Z}).$$

Now let M be an Riemannian manifold and $f\colon M \to \mathbb{R}^n$ an isometric immersion. Then there is a normal bundle $NM \to M$, such that

$$TM \oplus NM = f^*T\mathbb{R}^n.$$

Moreover, the Levi-Civita connection on M is compatible with the pullback connection ∇_f of the trivial connection on $\mathbb{R}^n$ to $TM \oplus NM$.

Now one calculates

$$\begin{aligned}\hat{c}(TM \oplus NM, \nabla_f) &= f^*(\hat{c}(G \curvearrowright \mathbb{R}^n)) \\ &\in f^*(-\beta^{-1}(I(\hat{c}(G \curvearrowright \mathbb{R}^n)))) + h^*H_G^{n-1}(pt, \mathbb{C}) \subset \widehat{\mathbb{H}}_G^n(M, \mathbb{Z}),\end{aligned}$$

where $h\colon M \to pt$ is the unique map.

A discussion of multiplicative structures on equivariant differential cohomology, would – if a Whitney sum formula holds – enable one to turn the last equation into conditions on $\hat{c}(TM, \nabla)$. In particular if $H_G^{n-1}(pt, \mathbb{C})$ vanishes, there is a unique class, depending on the representation of G, to which $\hat{c}(TM, \nabla)$ must be in relation to. If, furthermore, the cohomology

$$H_G^*(pt, \mathbb{Z}) = H^*(BG, \mathbb{Z})$$

is torsion free, then the existence of an immersion would imply that $\hat{c}(TM \oplus NM, \nabla_f)$ vanishes.

In the non equivariant case, there is, along the way described in the last paragraph, a beautiful application of characteristic classes to give lower bounds to the minimal codimension of an immersion. In the world of classical characteristic classes this can be found, e.g., in [37, Theorem 4.8]). Differential characteristic classes apply for a result, that conformal immersions have a stronger bound for the minimal codimension, than smooth immersions (compare [35] and [14, §6] for the original work and [39] for a partly strengthened version).

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