Rayleigh-Bénard convection: bounds on the Nusselt number

Der Fakultät für Mathematik und Informatik
der Universität Leipzig
eingereichte

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

DOCTOR RERUM NATURALIUM
(Dr.rer.nat.)

im Fachgebiet

Mathematik

vorgelegt

von M.Sc.
Camilla Nobili
geboren am 03/10/1987 in Rom

Die Annahme der Dissertation wurde empfohlen von

- Professor Dr. Charles Doering (University of Michigan - Ann Arbor)
- Professor Dr. Felix Otto (MPI MIS - Leipzig)

Die Verleihung des akademischen Grades erfolgt mit Bestehen der Verteidigung am 09.11.2015
mit dem Gesamtprädikat magna cum laude .

Bibliographische Daten

Rayleigh-Bénard convection: bounds on the Nusselt number
Rayleigh-Bénard Konvektion: Schranken an die Nusselt-Zahl
Nobili, Camilla
Universität Leipzig, Dissertation, 2015
110 Seiten, 0 Abbildungen, 22 Referenzen

Contents

Acknowledgements

I will always be grateful to Prof. Felix Otto for this important scientific and life experience. I thank him for his supervision, for the trust he has given me from the very beginning, for the patience and inestimable support. I am glad to have had the possibility to work at Max-Planck Institute in such a nice environment and with such a stimulating group. I thank Antoine Choffrut for the very nice and instructive collaboration and for the close friendship. Among the many beautiful people I have met in Leipzig, I want to thank Philippe Logaritsch and Lisa Krüger for the true friendship and inestimable sustain. A special thank to Michael Goldman, Andrea Marchese and Daniela Marchese, for the many beautiful times spent together. I dedicate this thesis to Alessandro, Anna and Elena with the hope that you will grow up happy and curious. Always ask yourself "why?" and do not get disappointed if the answer does not come immediately. It might just be a matter of time! A big thank to my family for the immense love and support and to Steffen Pottel to be always at my side, enriching every single day with a special meaning. Last but not least I want to thank Dorothy, Cherie and Merida.

Chapter 1

Introduction

Thermal convection refers to a specific type of convection phenomena where temperature differences drive a fluid flow. More precisely temperature variations induce an unstable fluid-stratification which cause the transition of the fluid from a state of rest to a state of motion. It may occur that the fluid flow undergoes many successive instabilities, which reduce progressively the spatial coherence and the level of predictability of the details of motion. In this case the flow is called turbulent. Few example of (turbulent) thermal convection are air circulation, solar granulation, oceanic currents and convective flows in the earth's mantle and stars. Transport properties of turbulent convective flow are object of interest and investigation in many field ranging from physical sciences like geophysics, astrophysics, meteorology and oceanography to engineering and industrial applications. In this thesis we are interested in deriving mathematically rigorous bounds for the heat transport when the flow is turbulent. For this purpose, consider a fluid enclosed between two rigid parallel and infinitely extended plates separated by a vertical distance h and held at different temperatures $T = T_{\text{bottom}}$ and $T = T_{\text{top}}$ at height 0 and h respectively, with $T_{\text{bottom}} > T_{\text{top}}$. This model of thermal convection goes under the name of Rayleigh-Bénard convection. The differences in temperature inside the container, which cause the expansion of the fluid of a material-specific factor α, are associated with density variations on which differing gravitational forces[1] per unit volume act and induce buoyancy forces. As a result, the hot parcels of fluid rise and the cold (denser) parcels sink generating kinetic energy. While the motion is favored by temperature gradients, it is opposed by the kinematic viscosity of the fluid ν, which acts against the relative motion of fluid layers by inner friction, and by the thermometric conductivity χ, that tends to remove the temperature differences. We remark that the thermometric conductivity is defined as $\chi = \frac{\kappa}{\rho c_p}$ where κ is the thermal conductivity, ρ is the density and c_p is the specific heat at constant pressure.

Already in his early experiments in 1916, Lord Rayleigh realized that when the nondimensional number

$$\mathrm{Ra} = \frac{g\alpha(T_{\text{bottom}} - T_{\text{top}})h^3}{\nu\chi}. \tag{1.1}$$

overcome a certain value, convection is activated. The number defined above is the *Rayleigh number* and it expresses the relative strength of the driving mechanisms: temperature differences, thermal expansion and gravity . If the Rayleigh number is small, pure conduction is the main transport mechanism; the fluid is at rest in the bulk with constant temperature gradient. Above an explicitly known critical Rayleigh number ($\mathrm{Ra}_c \sim 1$), the conduction state is unstable and the global attractor consists of stationary *convection rolls.* As the Rayleigh number increases further, the stationary convection rolls become unstable and for sufficiently high Rayleigh number ($\mathrm{Ra} \gg 1$), the temperature field features boundary layer, from which *plumes* detach. This stage is classified as turbulent.

[1]Proportional to the acceleration of gravity g.

We introduce a second parameter, the Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\chi} \tag{1.2}$$

which depends only on the properties of the fluid (and its absolute temperature). Its values, which vary from very small number (e.g. 0.015 for mercury) to large and very large numbers (e.g. 13.4 for seawater and 10^{24} for Earth's mantle) naturally affects convection. We will now see how the Rayleigh and the Prandtl number appear in the equation of motion determining the intensity of the buoyancy and inertial force, respectively.
In a $d-$dimensional container we follow the evolution equations of the velocity vector field $u(x,t)$, the temperature scalar field $T(x,t)$ and the pressure scalar field $p(x,t)$ where we indicate with x the $d-$dimensional spatial variable and with t the time variable. We specify with x' the first $d-1$ horizontal components and with z the vertical component of the vector x. In the Boussinesq approximation, where the variation of the density is neglected except insofar as it gives rise to a gravitational force, u, T and p are governed by

$$\begin{aligned}
\partial_t T + u\cdot\nabla T - \Delta T &= 0 && \text{for} \quad 0<z<1\,, && (1.3a)\\
\frac{1}{\mathrm{Pr}}(\partial_t u + (u\cdot\nabla)u) - \Delta u + \nabla p &= \mathrm{Ra} T e_z && \text{for} \quad 0<z<1\,, && (1.3b)\\
\nabla\cdot u &= 0 && \text{for} \quad 0<z<1\,, && (1.3c)\\
u &= 0 && \text{for} \quad z\in\{0,1\}\,, && (1.3d)\\
u &= u_0 && \text{for} \quad t=0\,, && (1.3e)\\
T &= 1 && \text{for} \quad z=0\,, && (1.3f)\\
T &= 0 && \text{for} \quad z=1. && (1.3g)
\end{aligned}$$

where e_z is the upward unit normal vector. We refer to [1] for a detailed description of the Boussinesq approximation. The temperature, which is set to be higher at the bottom plate than at the top plate, diffuses $(-\Delta T)$ and is advected by the velocity $(u\cdot\nabla T)$. The velocity field, which satisfies the continuity equation in the constant-density form $(\nabla\cdot u = 0)$ evolves according to the Navier-Stokes equations with right-hand side given by the buoyancy force $\mathrm{Ra}Te_z$. This force generates acceleration, which, in turn, is balanced by diffusion $(-\Delta u)$. The pressure p appears as a Lagrangian multiplier to enforce the divergence-free condition. The velocity field u satisfies the *no-slip* boundary condition at both plates ($z=0$ and $z=1$) and periodicity in the horizontal variables $x'\in[0,L)^{d-1}$ is imposed for all the functions. We notice that the *pure conduction state* $u=0, T=1-z$ is a stationary solution of the system (1.3) in absence of convection and it is stable for $\mathrm{Ra} < \mathrm{Ra}_c \sim 1$. When $\mathrm{Ra} \gtrsim 1$, the conduction profile is unstable and, in this context, one of the challenges of engineers and physicists is to measure the (convective) heat transport. Besides the importance of the physical phenomena itself, understanding the effectiveness of the heat transport when Pr and Ra vary, is fundamental for the industry, for example in the construction of high-standard cooling devices on semiconductor or nuclear power plants. The quantity of interest is the averaged upward heat flux, which measures the effectiveness in the heat transport. An inspection of the advection-diffusion equation for the temperature shows that the heat flux is given by $uT - \nabla T$. The appropriately non-dimensionalized measure of the time-space average of the upward heat flux is given by the Nusselt number Nu defined through

$$\mathrm{Nu} = \limsup_{t_0\uparrow\infty} \frac{1}{t_0}\int_0^{t_0}\int_0^1 \frac{1}{L^{d-1}}\int_{[0,L)^{d-1}} (uT-\nabla T)\cdot e_z dx'dzdt,$$

where we consider the limit superior in order to avoid the case in which the limit does not exist. We notice that the pure conduction state ($u=0$, $T=1-z$) gives rise to $\mathrm{Nu}=1$. Since convective fluid flow increases vertical heat transport beyond the purely conductive flux, it is interesting to work in the regime of large Rayleigh number. Throughout the thesis we assume to work in the regime $\mathrm{Ra} \gg 1$, where the spatial coherence of the flow pattern is lost and the fluid flow becomes turbulent. When the flow is turbulent the Nusselt number Nu is thought to be a function of the dimensionless parameters Ra and Pr and the aspect ratio $\Gamma = \frac{L}{h}$ of the container; one example of a "similarity law" in fluid dynamics. The aspect ratio can be assimilated to the artificial lateral periodicity L. This paper does not address the dependency on the aspect ratio: We will focus on (upper and lower) bounds that are independent of the period length L, which amounts to neglecting the (limiting) effects of the lateral boundary condition. Likewise, this paper does not address the specifics of two-dimensional flows: Our analysis is in fact oblivious to the dimension d, which in particular amounts to allowing for turbulent boundary layers.

There are classical heuristic arguments in favor of two (different) scaling laws for Nu in terms of Ra and Pr: The scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$ was proposed in 1954 by Malkus [2] appealing to the *marginal stability argument* (see Chapter 2) while Spiegel [3] in 1962 predicted the scaling $\mathrm{Nu} \sim \mathrm{Pr}^{\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$ for small Pr according to the Newton's law. We now argue that the scaling laws predicted by Malkus and Spiegel, respectively, can be simply deduced by rescaling the equations in the limiting case when the viscosity term wins over the inertial term and vice versa. On the one hand, if we assume that the inertial term is negligible (setting $\mathrm{Pr} = \infty$) the equation (1.3b) reduces to

$$\left\{\begin{array}{rcl} \partial_t T + u \cdot \nabla T - \Delta T &=& 0\,, \\ -\Delta u + \nabla p &=& \mathrm{Ra} T e_z\,, \\ \nabla \cdot u &=& 0\,. \end{array}\right.$$

Rescaling this equation according to

$$x = \mathrm{Ra}^{-\frac{1}{3}}\hat{x},\ t = \mathrm{Ra}^{-\frac{2}{3}}\hat{t},\ u = \mathrm{Ra}^{\frac{1}{3}}\hat{u},\ p = \mathrm{Ra}^{\frac{2}{3}}\hat{p} \ \text{ and thus } \ \mathrm{Nu} = \mathrm{Ra}^{\frac{1}{3}}\widehat{\mathrm{Nu}} \tag{1.4}$$

we end up with the parameter-free system

$$\left\{\begin{array}{rcl} \partial_{\hat{t}} T + \hat{u} \cdot \hat{\nabla} T - \hat{\Delta} T &=& 0\,, \\ -\Delta \hat{u} + \nabla \hat{p} &=& T e_{\hat{z}}\,, \\ \hat{\nabla} \cdot \hat{u} &=& 0\,, \end{array}\right.$$

which naturally lives in the half space. Since for the latter system, it is natural to expect that the heat flux is universal, i.e. $\widehat{\mathrm{Nu}} \sim 1$, we obtain $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$.
On the other hand, if we rewrite the system (1.3b) neglecting the diffusivity and the viscosity term

$$\left\{\begin{array}{rcl} \partial_t T + u \cdot \nabla T &=& 0\,, \\ \frac{1}{\mathrm{Pr}}\left(\partial_t u + (u \cdot \nabla) u\right) + \nabla p &=& \mathrm{Ra} T e_z\,, \\ \nabla \cdot u &=& 0\,, \end{array}\right.$$

and we rescale according to

$$t = \frac{1}{(\mathrm{PrRa})^{\frac{1}{2}}}\hat{t},\ u = (\mathrm{PrRa})^{\frac{1}{2}}\hat{u},\ p = \mathrm{Ra}\,\hat{p} \ \text{ and thus } \ \mathrm{Nu} = (\mathrm{PrRa})^{\frac{1}{2}}\widehat{\mathrm{Nu}} \tag{1.5}$$

we end up with the system

$$\left\{\begin{array}{rcl} \partial_{\hat{t}} T + \hat{u} \cdot \nabla T &=& 0\,, \\ \partial_{\hat{t}} \hat{u} + (\hat{u} \cdot \nabla)\hat{u} + \nabla \hat{p} &=& T e_z\,, \\ \nabla \cdot \hat{u} &=& 0\,. \end{array}\right.$$

Imitating the previous argument we can conclude that $\mathrm{Nu} \sim \mathrm{Pr}^{\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$.

Many more scaling regimes for Nu in the Pr-Ra-plane have been proposed on experimental and theoretical grounds in the physics literature. By means of mixing length theory, Kraichnan in [4] not only reproduced the scalings of Malkus and Spiegel for big and very small Pr, respectively, but also suggested a third scaling $\mathrm{Nu} \sim \mathrm{Pr}^{-\frac{1}{4}}\mathrm{Ra}^{\frac{1}{2}}$ for big Ra and moderately low Pr. A fairly complete theory has been worked out in [5]. It is based on global balance laws (which we also use in our rigorous treatment (see Section 3.2.2) on distinguishing the cases of the dissipation dominantly taking place in the bulk or in the boundary layer) and on assumptions on the structure of both the thermal and the viscous boundary layer (which becomes relevant for $\mathrm{Pr} < \infty$). However, these statements are more speculative when the viscous boundary layer is turbulent rather than laminar.

Measurements on the Nusselt number are provided by the large-scale convection facility, *Barrel of Ilmenau* (BoI) at the technical University of Ilmenau. In order to reach the regime of big Rayleigh numbers, the close cylindrical container (7.15 m in diameter) has been constructed in such a way that the material parameters (thermal expansion coefficient, kinematic viscosity and thermometric conductivity) are compensated by the height of the container. Indeed the heating plate and the free-hanging cooling plate can reach a (maximal) distance of 6.3 meters. In the experiments at BoI the side-wall effects are not negligible and therefore the aspect ratio (which can vary from 1 to 40) influences the heat flux and affects the measurements [2].
While most of the experiments aim at understanding how the global flow structure organizes itself when varying the Rayleigh number , the variation of Prandtl number is more difficult. The experiments at BoI are done using air ($\mathrm{Pr} = 0.7$) at normal pressure which simulate the earth atmosphere. Changing fluid in the container would be a big issue regarding the construction of a (very large) cell that close tightly. In particular, low Prandtl number fluids such as mercury and gallium are less accessible: the first requires high security level in the laboratory while the latter has a very high cost. Nevertheless the variation of the Prandtl number can be performed by means of direct numerical simulations, see e.g. [6].

1.1 Results

Despite the complexity of the phenomenon of Rayleigh-Bénard convection in the turbulent regime, there are rigorous upper bounds of Nu in terms of Ra and Pr. In the case of $\mathrm{Pr} = \infty$, Constantin and Doering proved $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{2}{3}}\mathrm{Ra}^{\frac{1}{3}}$ in their seminal 1999 paper [7]. They obtained this bound by combining global balance laws with the maximum principle for the temperature and a (logarithmically failing) maximal regularity estimate for the (quasi)-stationary Stokes equations in L^∞_x. The heuristic argument of marginal stability of the boundary layer (c.f. Chapter 2) has inspired the application of the *background temperature field method* (c.f. Section 2.1). This method, based on the decomposition of the temperature profile into a background profile $\tau(z)$ and a fluctuation field $\theta(x,t)$, produces an upper bound on the Nusselt number each time a profile τ satisfies the stability condition

$$\left\langle \int_0^1 \left(2\frac{d\tau}{dz} u^z \theta + |\nabla\theta|^2 \right) dz \right\rangle \geq 0 \tag{1.6}$$

(where $\langle\cdot\rangle$ is defined in Notation). Since we are interested in deriving the lowest upper bound within the method , we study the following variational problem

$$\widetilde{\mathrm{Nu}} := \inf_{\substack{\tau:(0,1)\to\mathbb{R},\\ \tau(0)=1,\tau(1)=0}} \left\{ \int_0^1 \left(\frac{d\tau}{dz}\right)^2 dz |\ \tau \text{ satisfies (1.6)} \right\}.$$

the solution of which will give us the optimal upper bound

$$\mathrm{Nu} \lesssim \widetilde{\mathrm{Nu}}\,.$$

In 2006 Doering, Otto and Westdickenberg (née Reznikoff) [8] obtained the bound (with a slightly improved power of the logarithm w.r.t [7]) $\widetilde{\mathrm{Nu}} \lesssim (\ln \mathrm{Ra})^{\frac{1}{3}}\mathrm{Ra}^{\frac{1}{3}}$ by proving the stability

[2]The mentioned experiments are realized by the group of Prof. Ronald de Puits

of a logarithmic profile. With a refinement of the argument in [8], Otto and Seis obtained $\widetilde{\mathrm{Nu}} \lesssim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$ which improves further the logarithmic correction. In this thesis we show that the upper bound $\widetilde{\mathrm{Nu}} \lesssim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$ (Otto & Seis 2011 [9]) is optimal for the background field method. This is shown by proving that $(\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$ is also a lower bound for the Nusselt number associated to the background field method. In particular we establish the following result

Theorem (Camilla Nobili and Felix Otto, [10]).
Suppose that a profile $\tau : (0,1) \to \mathbb{R}$ *satisfies* $\int_0^1 \frac{d\tau}{dz}\, dz = -1$ *and*

$$\mathcal{Q}_\tau[\theta] := \left\langle \int_0^1 \left(2\frac{d\tau}{dz} u^z \theta + |\nabla \theta|^2 \right) dz \right\rangle \geq 0 \,, \tag{1.7}$$

for all (θ, u^z) *related by the fourth order boundary value problem*

$$\Delta^2 u^z = -\mathrm{Ra}\, \Delta_{x'} \theta \;\; \textit{with} \;\; \theta = u^z = \partial_z u^z = 0 \;\; \textit{at} \;\; z = 0, 1.$$

Then

$$\widetilde{\mathrm{Nu}} \gtrsim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}.$$

In particular $\widetilde{\mathrm{Nu}} \sim (\ln \mathrm{Ra})^{\frac{1}{15}} \mathrm{Ra}^{\frac{1}{3}}$.

Nevertheless, the combination of the maximal regularity estimate for the Stokes equation in [7] with the background field method in [8] yields the doubly logarithmic bound $\mathrm{Nu} \lesssim (\ln \ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$ (c.f. [9]) which is, to our knowledge, optimal.
We finally observe that our lower bound on the background field method together with the (last) optimal upper bound yields

$$\mathrm{Nu} \ll \widetilde{\mathrm{Nu}} \,,$$

meaning that the background field method does not carry physical meaning.

In the case of $\mathrm{Pr} < \infty$, the lack of instantaneous *slaving* of the velocity field to the temperature field increases the difficulty in bounding the convection term $\langle \int u^z T\, dz \rangle$ (see Notations) in the definition of the Nusselt number and the background field method turns out to be no longer fruitful. In their 1996 paper [11], Doering & Constantin among other results gave an easy argument for $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{2}}$ for all values of Pr. Besides [11], there is only one other rigorous result for $\mathrm{Pr} < \infty$: Wang [12] proved by a perturbative argument that the Constantin & Doering 1999 bound $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{2}{3}} \mathrm{Ra}^{\frac{1}{3}}$ persists for $\mathrm{Pr} \gg \mathrm{Ra}$ (see Section 3.2.2 for an argument why this is the classical scaling regime). In this case we establish the following upper bound

Theorem (Antoine Choffrut, Camilla Nobili and Felix Otto, [13]).
Provided the initial data satisfy $T_0 \in [0,1]$, $\int |u_0| dx < \infty$ *and for* $\mathrm{Ra} \gg 1$

$$\mathrm{Nu} \leq C \begin{cases} (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}} & \text{for} \quad \mathrm{Pr} \geq (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}, \\ (\ln \mathrm{Ra})^{\frac{1}{2}} \mathrm{Pr}^{-\frac{1}{2}} \mathrm{Ra}^{\frac{1}{2}} & \text{for} \quad \mathrm{Pr} \leq (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}, \end{cases} \tag{1.8}$$

where C *depends only on the dimension* d.

This result on the one hand implies that the Doering & Constantin 1996 bound $\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{2}}$ is suboptimal for $\mathrm{Pr} \gg 1$ and, on the other hand, tells us that the Constantin & Doering 1999 bound $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{2}{3}} \mathrm{Ra}^{\frac{1}{3}}$ (in its slightly improved form of $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$) persists in the much larger regime $\mathrm{Pr} \gtrsim (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$ and then crosses over to $\mathrm{Nu} \lesssim (\ln \mathrm{Ra})^{\frac{1}{2}} \mathrm{Pr}^{-\frac{1}{2}} \mathrm{Ra}^{\frac{1}{2}}$, which can be seen as an interpolation between the marginal stability bound and the Constantin & Doering 1996 bound as Pr decreases from large $\mathrm{Pr} = (\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$ to moderate $\mathrm{Pr} = 1$. We want to remark that like in Wang's argument, ours treats the convective nonlinearity $(u \cdot \nabla) u$ in (1.3b) perturbatively. However, there is a difference: We perturb around the *non-stationary*

Stokes equations and gain access to Ra-Pr-regimes where the effective *Reynolds* number Re is allowed to be large. In fact we work with Leray's solution and thus only appeal to the global energy estimate on the level of the Navier-Stokes equations, whereas Wang's regime is limited by the use of the small-data regularity theory for the Navier-Stokes equations and thus $\mathrm{Re} \ll 1$, which in his analysis translates into $\mathrm{Pr} \gg \mathrm{Ra}$. Loosely speaking our analysis just requires small Re in the thermal boundary layer, not in the entire container, for the $(\ln \mathrm{Ra})^{\frac{1}{3}} \mathrm{Ra}^{\frac{1}{3}}$ scaling to persist. This upper bound is based on a maximal regularity estimate in the interpolation between the two norms of interest $L^1\left(dt\,dx'\,\frac{1}{z(1-z)}\,dz\right)$ and $L^\infty_z\left(L^1_{t,x'}\right)$. As we will explain in Subsection 3.2.2 this estimates holds only under bandedness assumption (i.e. restriction to a packet of wave numbers in Fourier space); this is the source of the logarithmic correction in the bounds for the Nusselt number.

1.2 Summary

Here is a summary of the content of this thesis.

In the rest of this introductory chapter we recall a maximum principle for the temperature equation and we derive some useful representations and bounds on the Nusselt number, directly coming from the equations of motion. In Chapter 2 we consider the Rayleigh-Bénard convection when the inertial force in the velocity equation is neglected, i.e. $\mathrm{Pr} = \infty$. In Section 2.1 we introduce the background field method for the temperature field. This method, capitalizing on the instantaneous linear slaving of the velocity to the temperature field, reduces the problem of finding upper bounds for the Nusselt number to solving a variational method: find background profiles τ that satisfy the stability condition (2.34). For preparation, we analyze some stable profiles and highlight the "good" features (in terms of the upper bounds on Nusselt number). In Section 2.2, assuming initially the container to be of infinite lateral size (amounting to real wavelengths in Fourier space), we derive necessary condition on $\xi = \frac{d\tau}{dz}$ coming from the stability condition (2.34). Specifically, in Subsection 2.2.1, we analyze a reduced version of the stability condition and we prove that a stable background profile must be increasing and grow logarithmically. In Subsection 2.2.2 we go back to investigate the profiles τ that satisfy the original stability condition and we show that ξ must be approximately positive (positive in average approximately in the bulk), satisfy a logarithmic growth at the level of the antiderivative τ and be approximative positive in the boundary layers . These results, stated in Lemma 1, Lemma 2 and Lemma 3 (Subsection 2.2.3), constitute the proof of the non optimality of the background field method, Theorem 1. *This is a joint work with Felix Otto, [10].*

In Subsection 2.2.6 we recover the physical setting of a container with finite lateral size, amounting to integer wavelengths. Here, although the proof of the logarithmic grow of τ requires a different argument, we can deduce the same conclusion as in Proposition 1.

In Chapter 3 we consider the Rayleigh-Bénard convection reintroducing the inertial term in the equation of the velocity allowing the Prandtl number to be finite. The velocity field, that evolves according to the Navier-Stokes equation, is not instantaneously slaved to the temperature field and therefore the background field method is no longer fruitful. For preparation, in Section 3.1 we show how to apply maximal regularity estimates to derive upper bounds on the Nusselt number. In Theorem 2, Section 3.2, we state our result on the upper bound on the Nusselt number at finite Prandtl number. The main ingredient for the proof of Theorem 2 is the maximal regularity estimate for the non-stationary Stokes equation in the strip, Theorem 3. The proof on a maximal regularity in the strip is, in turn, based on the maximal regularity estimate in the upper half space stated in Proposition 3. *This is a joint work with Antoine Choffrut and Felix Otto, [13].*

1.3 Prerequisites

We start by recalling that the equation of the temperature

$$\partial_t T + u \cdot \nabla T - \Delta T = 0$$

satisfies the maximum principle

$$\text{if } T_0 \in [0,1] \ \text{ then } \ ||T||_{L^\infty} \leq 1\,, \tag{1.9}$$

which furnishes us an a-priori bound on the temperature T.
Indeed one can show that for weak solutions of the Boussinesq system, i.e. $u \in L_t^\infty L_x^2$, $\nabla u \in L_t^2 L_x^2$, $T \in L_t^\infty L_x^2$ and $\nabla T \in L_t^2 L_x^2$, the maximum principle for the temperature holds. Indeed T satisfies a level set energy inequality (see eq. A.1 with $\alpha = 1$ in [14] observing that the solution there has the same regularity as our T. The proof only uses that the velocity-field u is (weakly) divergence-free). Applying the inequality to $(T-1)_+ = \max\{T-1, 0\}$ and to $T_- = \max\{-T, 0\}$ we deduce that $||(T-1)_+||_{L^2}$ and $||T_-||_{L^2}$ vanish for all time. Therefore we get that $0 \leq T \leq 1$.
Exploiting the incompressibility condition (1.3c) we can rewrite the temperature equation (1.3a) in the divergence form

$$\partial_t T + \nabla \cdot (uT - \nabla T) = 0\,. \tag{1.10}$$

The vector $uT - \nabla T$, called *heat flux* , sums up the two opposing contribution coming from convection (uT) and conduction ($-\nabla T$). The Nusselt number is defined as the time and space-average of the vertical heat flux

$$\mathrm{Nu} = \left\langle \int_0^1 (u^z T - \partial_z T)\; dz \right\rangle, \tag{1.11}$$

where

$$\langle \cdot \rangle = \limsup_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} \langle \cdot \rangle'\, dt \qquad \text{and} \qquad \langle \cdot \rangle' = \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} \cdot\, dx'.$$

The term $\left\langle \int u^z T\, dz \right\rangle$ quantify the heat transported in the bulk by convection. Near the boundary layer the term $\langle uT \rangle'$ becomes smaller and smaller due to the boundary condition that enforce this term to vanish at $z = 0$ and $z = 1$. Therefore close to the boundary the conduction term $\left\langle \int \nabla T\, dz \right\rangle$ becomes larger. An important observation is that the Nusselt number is the same in each horizontal layer at which we are measuring it, namely

$$\mathrm{Nu} = \langle u^z T - \partial_z T \rangle \qquad \forall\, z \in [0,1]. \tag{1.12}$$

Indeed, starting from the equation (1.10) and averaging in the horizontal direction and using the periodic boundary condition in x' we obtain

$$\partial_t \langle T \rangle' + \partial_z \langle u^z T - \partial_z T \rangle' = 0.$$

Since, by the maximum principle for the temperature

$$\mathrm{osc}_z \left\{ \frac{1}{t_0} \int_0^{t_0} \langle (u^z T - \partial_z T) \rangle'\, dt \right\} \lesssim \frac{1}{t_0}\,,$$

we conclude that $\langle u^z T - \partial_z T \rangle$ does not depend on z and therefore (1.12) holds.
Again from the temperature equation (1.3a) and the property (1.12) we obtain

$$\mathrm{Nu} = \left\langle \int_0^1 |\nabla T|^2\, dz \right\rangle. \tag{1.13}$$

Indeed, multiplying (1.3a) with T and averaging in x' we have

$$\begin{aligned} 0 &= \langle \partial_t T T\rangle' + \langle \nabla\cdot(uT)T\rangle' + \langle -\Delta T T\rangle' \\ &= \frac{1}{2}\partial_t\langle T^2\rangle' + \frac{1}{2}\left\langle \nabla\cdot(uT^2)\right\rangle' - \langle\nabla\cdot(T\nabla T)\rangle' + \left\langle |\nabla T|^2\right\rangle' \\ &= \frac{1}{2}\partial_t\left\langle T^2\right\rangle' + \frac{1}{2}\partial_z\left\langle u^z T^2\right\rangle' - \partial_z\left\langle T\partial_z T\right\rangle' + \left\langle |\nabla T|^2\right\rangle' . \end{aligned}$$

Taking the vertical-space average, the time average and using the non-slip boundary condition for u^z in the expression above we obtain

$$\frac{1}{2}\int_0^{t_0}\int_0^1 \partial_t\langle T^2\rangle'\,dz\,dt - \int_0^{t_0}\langle\partial_z T\rangle'_{|z=0}\,dt + \int_0^{t_0}\int_0^1\langle|\nabla T|^2\rangle'\,dz\,dt = 0.$$

Finally, we pass to the long-time limit of the expression above. Observing that, due to the boundedness of T (see (1.9)), the first term of the left-hand side vanish, i.e.

$$\limsup_{t_0\to\infty}\frac{1}{t_0}\int_0^{t_0}\int_0^1\partial_t\langle T^2\rangle'\,dz\,dt = \limsup_{t_0\to\infty}\frac{1}{t_0}\left(\frac{1}{2}\int_0^1\langle T^2\rangle'_{|t=t_0}\,dz - \frac{1}{2}\int_0^1\langle T^2\rangle'_{|t=0}\,dz\right) = 0\,, \tag{1.14}$$

we have

$$\left\langle\int_0^1|\nabla T|^2\,dz\right\rangle = -\left\langle\partial_z T|_{z=0}\right\rangle\,,$$

which yields (1.13) once we observe that by the property (1.12), the Nusselt number can be represented as $\mathrm{Nu} = \langle(u^z T - \partial_z T)\rangle'_{|z=0}$. From the Navier-Stokes equation (1.3b) we find the energy inequality

$$\left\langle\int_0^1|\nabla u|^2\,dz\right\rangle \le \mathrm{Ra}(\mathrm{Nu}-1) \tag{1.15}$$

for Leray solutions. Indeed testing the Navier-Stokes equation (1.3b) with u, averaging in space, integrating by part, using the no-slip boundary condition for u and finally averaging in time we have

$$\begin{aligned} &\frac{1}{2}\frac{1}{\mathrm{Pr}}\frac{1}{t_0}\int_0^1\langle|u|^2\rangle'_{|t=t_0}\,dz + \frac{1}{t_0}\int_0^{t_0}\int_0^1\langle|\nabla u|^2\rangle'\,dz\,dt \\ &= \frac{1}{2}\frac{1}{\mathrm{Pr}}\frac{1}{t_0}\int_0^1\langle|u|^2\rangle'_{|t=0}dz + \mathrm{Ra}\frac{1}{t_0}\int_0^{t_0}\int_0^1\langle u^z T\rangle'\,dz\,dt\,, \end{aligned}$$

which reduces to

$$\begin{aligned} &\frac{1}{t_0}\int_0^{t_0}\int_0^1\langle|\nabla u|^2\rangle'\,dz\,dt \\ &\le \frac{1}{2}\frac{1}{\mathrm{Pr}}\frac{1}{t_0}\int_0^1\langle|u|^2\rangle'_{|t=0}dz + \mathrm{Ra}\frac{1}{t_0}\int_0^{t_0}\int_0^1\langle u^z T\rangle'\,dz\,dt\,. \end{aligned}$$

Passing to the limit for big t_0 we obtain

$$\begin{aligned} \left\langle\int_0^1|\nabla u|^2\,dz\right\rangle &\le \mathrm{Ra}\left\langle\int_0^1 u^z T\,dz\right\rangle \\ &= \mathrm{Ra}\left(\left\langle\int_0^1(u^z T - \partial_z T)\,dz\right\rangle - 1\right) \\ &= \mathrm{Ra}(\mathrm{Nu}-1)\,. \end{aligned}$$

For the Stokes equation (2.1b), (1.15) holds with the equality sign.

Since the vertical heat flux does not depend on the variable z (see (1.12)) we can write

$$\mathrm{Nu} = \frac{1}{\delta} \left\langle \int_0^\delta u^z T - \partial_z T \, dz \right\rangle \leq \frac{1}{\delta} \left\langle \int_0^\delta u^z T \, dz \right\rangle + \frac{1}{\delta} \ , \tag{1.16}$$

where we have used the boundary condition for T. Furthermore, using the maximum principle for the temperature (1.9) in the last inequality we obtain

$$\mathrm{Nu} \leq \frac{1}{\delta} \left\langle \int_0^\delta |u^z| \, dz \right\rangle + \frac{1}{\delta} \, . \tag{1.17}$$

We conclude this subsection observing that two important properties of u^z can be deduce by the incompressibility condition (1.3c): from the combination of (1.3c) with the no-slip boundary condition for u^z (1.3d) we deduce that

$$\partial_z u^z = 0 \qquad \text{at} \qquad z = 0, 1. \tag{1.18}$$

Furthermore averaging the equation (1.3c) in the horizontal direction and using the periodicity of u' in the horizontal direction we have

$$\partial_z \langle u^z \rangle' = 0.$$

Using again the no-slip boundary condition for u we find that the horizontal average of u^z vanishes, i.e,

$$\langle u^z \rangle' = 0 \, . \tag{1.19}$$

Chapter 2

Infinite Prandtl number convection

When the Prandtl number is very big, such as for Glycerin at 20°C (Pr = 12000) and the Earth's mantle ($\mathrm{Pr} \approx 10^{24}$), it is reasonable to consider the infinite-Prandtl-number limit of the equations (1.3)

$$\partial_t T + u \cdot \nabla T - \Delta T = 0 \qquad \text{for} \quad 0 < z < 1\,, \tag{2.1a}$$

$$-\Delta u + \nabla p = \mathrm{Ra} T e_z \qquad \text{for} \quad 0 < z < 1 \tag{2.1b}$$

$$\nabla \cdot u = 0 \qquad \text{for} \quad 0 < z < 1\,, \tag{2.1c}$$

$$u = 0 \qquad \text{for} \quad z \in \{0, 1\}\,, \tag{2.1d}$$

$$T = 1 \qquad \text{for} \quad z = 0\,,$$

$$T = 0 \qquad \text{for} \quad z = 1\,,$$

where the inertial term does not appear since we set $\mathrm{Pr} = \infty$. In this case the problem of obtaining bounds for the Nusselt number is simplified by the instantaneous slaving of the velocity to the temperature field, which provides a tight control on on the indefinite term $\int \langle u^z T \rangle' \, dz$ in the definition of the Nusselt number (1.11). When the inertia of the fluid is neglected, Malkus [2] proposes the following heuristic argument in favor of the scaling $\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}$: with the observation that $\mathrm{Ra} \gg 1$, in a boundary layer (to be determined) thickness $\delta \ll h$, the temperature drops from 1 to its average $\frac{1}{2}$ and the flow is suppressed. By definition of the Nusselt number in the dimensionless variable (1.11), this yields $\mathrm{Nu} \sim \frac{1}{2\delta}$. So that the Nusselt number is linked to the relative size of the (thermal) boundary layer. Here comes the crucial argument of a *marginally stable boundary layer*: the actual size δ is expected to be proportional to the largest height h^* of the container in which the pure conduction solution $T = 1 - z, u = 0$ is stable. A critical Rayleigh number Ra^* (critical in both the sense of linear and nonlinear stability) is associated to h^* via

$$\mathrm{Ra}^* = \frac{g\alpha(T_{h^*} - T_0)(hh^*)^3}{\nu\kappa}$$

and it is explicitly known. Since Ra^* must be universal, then $\mathrm{Ra}^* \sim 1$ and we obtain

$$\frac{1}{2} \frac{g\alpha(T_{\text{bottom}} - T_{\text{top}})(h)^3}{\nu\kappa} (h^*)^3 \sim 1.$$

Recalling the definition of the Rayleigh number we have $\frac{1}{2}\mathrm{Ra}(h^*)^3 \sim 1$ which implies $\delta \sim \mathrm{Ra}^{-\frac{1}{3}}$. The combination with $\mathrm{Nu} \sim \frac{1}{2\delta}$ yields the desired

$$\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}}. \tag{2.2}$$

Numerical simulations [15] and experiments [16] are in perfect agreement with this scaling. In the next subsection we will introduce the background field method which shows striking similar characteristics with the marginal stability method: If the profile is stable then it gives an upper bound on the Nusselt number. In this sense the background field method can be viewed as a rigorous implementation of the marginal stability argument.

2.1 Prerequisite: Background field method

The *background field method* consists of decomposing the temperature field T into a steady *background field profile* τ satisfying the driven boundary conditions

$$\tau = 1 \quad \text{at } z = 0 \qquad \text{and} \qquad \tau = 0 \quad \text{at } z = 1$$

and into temperature fluctuations θ, satisfying

$$\theta = 0 \qquad \text{at } z \in \{0, 1\}\,.$$

Therefore the temperature T can be written as

$$T(x', z, t) = \tau(z) + \theta(x', z, t)\,.$$

Imposing the decomposition into (2.1a), one finds that the fluctuations evolve according to

$$\partial_t \theta + u \cdot \nabla\theta = -\frac{d\tau}{dz} u^z + \Delta\theta + \frac{d^2\tau}{dz^2} \tag{2.3}$$

and the Nusselt number can be rewritten as

$$\mathrm{Nu} \overset{(1.13)}{=} \int_0^1 \left(\frac{d\tau}{dz}\right)^2 dz + \left\langle \int_0^1 |\nabla\theta|^2\, dz \right\rangle - 2\left\langle \int_0^1 \frac{d^2\tau}{dz^2}\theta\, dz \right\rangle. \tag{2.4}$$

Testing equation (2.3) with θ and averaging in space we obtain

$$\frac{1}{2}\int \langle \partial_t \theta^2\rangle'\, dz = -\int_0^1 \frac{d\tau}{dz}\langle u^z\theta\rangle'\, dz - \int_0^1 \langle |\nabla\theta|^2\rangle'\, dz + \int_0^1 \frac{d^2\tau}{dz^2}\langle\theta\rangle'\, dz\,, \tag{2.5}$$

where we observed that $\frac{1}{2}\int \langle \nabla(u\theta^2)\rangle'\, dz$ vanishes due to the no-slip boundary condition for u and the periodicity in the horizontal variables. Considering the long-time average of the expression above, the first term vanishes due to the boundedness of θ and we are left with

$$\left\langle \int_0^1 \frac{d^2\tau}{dz^2}\theta\, dz \right\rangle = \left\langle \int_0^1 \frac{d\tau}{dz} u^z\theta\, dz \right\rangle + \left\langle \int_0^1 |\nabla\theta|^2\, dz \right\rangle.$$

It is now easy to see that the Nusselt number representation (2.4) can be rewritten as

$$\mathrm{Nu} = \int_0^1 \left(\frac{d\tau}{dz}\right)^2 dz - \left\langle \int_0^1 \left(2\frac{d\tau}{dz}u^z\theta + |\nabla\theta|^2\right) dz \right\rangle\,, \tag{2.6}$$

which turns out to be revealing. Indeed (2.6) suggests the following idea: if one can construct a background field τ that satisfies

$$\mathcal{Q}_\tau[\theta] := \left\langle \int_0^1 \left(2\frac{d\tau}{dz}u^z\theta + |\nabla\theta|^2\right) dz \right\rangle \geq 0\,, \tag{2.7}$$

for every $\theta(x', z, t)$ satisfying homogeneous boundary conditions (and u^z defined through the Stokes equation (2.1b), the incompressibility condition (2.1c) and homogeneous boundary conditions (2.1d)), then the Dirichlet integral of $\tau(z)$ is an upper bound for the Nusselt number, i.e.

$$\mathrm{Nu} \leq \int_0^1 \left(\frac{d\tau}{dz}\right)^2 dz. \tag{2.8}$$

The constraint (2.7) is referred to as a *stability condition.*

We define the Nusselt number associated to the background field method

$$\widetilde{\mathrm{Nu}} := \inf_{\substack{\tau:(0,1)\to\mathbb{R},\\ \tau(0)=1,\tau(1)=0}} \left\{ \int_0^1 \left(\frac{d\tau}{dz}\right)^2 dz \middle|\ \tau \text{ satisfies } (2.7) \right\}, \tag{2.9}$$

which clearly bounds the Nusselt number from above,

$$\mathrm{Nu} \leq \widetilde{\mathrm{Nu}}\,.$$

Therefore via the background field method the problem of finding an optimal (within the method) upper bounds for the Nusselt number reduces to constructing a function τ that realizes the infimum of the variational problem defined by (2.8) constrained to (2.7). For the rest of the chapter we can think of functions which are independent on the time variable t. Eliminating the pressure from the Stokes equation (2.1b) via the incompressibility condition we deduce the direct relationship between $\theta(x',z)$ and $u^z(x',z)$:

$$\begin{cases} \Delta^2 u^z = -\mathrm{Ra}\Delta_{x'}\theta & \text{for } 0<z<1\,, \\ u^z = \partial_z u^z = 0 & \text{for } z\in\{0,1\}\,. \end{cases} \tag{2.10}$$

The stability condition (2.7) can be rewritten explicitly as follows

$$\mathcal{Q}_\tau[\theta] = 2\int_0^1 \frac{d\tau}{dz}\langle u^z\theta\rangle'\,dz + \int_0^1 \langle|\nabla_{x'}\theta|^2\rangle'\,dz + \int_0^1 \langle|\partial_z\theta|^2\rangle'\,dz \geq 0, \tag{2.11}$$

for all the functions $\theta(x',z)$ that vanish at $z\in\{0,1\}$, where the function $u^z(x',z)$ is determined by θ via the fourth-order boundary value problem (2.10). In terms of the horizontally Fourier-transformed variables (see Notation) $(\mathcal{F}'\theta)(k',z)$ and $(\mathcal{F}'u^z)(k',z)$, the relation (2.10) is

$$\begin{cases} \left(-\frac{d^2}{dz^2}+|k'|^2\right)^2 \mathcal{F}'u^z = \mathrm{Ra}|k'|^2\mathcal{F}'\theta & \text{for } 0<z<1\,, \\ \mathcal{F}'u^z = \frac{d}{dz}\mathcal{F}'u^z = 0 & \text{for } z\in\{0,1\} \end{cases} \tag{2.12}$$

and the constraint (2.7) is fulfilled if for every wavenumber $k'\in\frac{2\pi}{L}\mathbb{Z}^{d-1}\setminus\{0\}$

$$\mathcal{Q}_\tau[\mathcal{F}'\theta] = 2\int_0^1 \frac{d\tau}{dz}\mathrm{Re}[\mathcal{F}'u^z\overline{\mathcal{F}'\theta}]\,dz + \int_0^1 |k'|^2(\mathcal{F}'\theta)^2\,dz + \int_0^1 \left(\frac{d}{dz}\mathcal{F}'\theta\right)^2 dz \geq 0, \tag{2.13}$$

for all (complex valued) functions $\mathcal{F}'\theta(z)$, such that $\mathcal{F}'\theta(z)=0$ at $z\in\{0,1\}$.

Examples of stable profiles

The construction of stable profiles has been a challenging problem in the past years. In this subsection, in preparation for the next chapter we give some examples of stable profiles, highlighting their basic features. Each stable profile produces an upper bound for the Nusselt number through the variational problem (2.8). The idea that the major temperature drop occurs in the (thin) boundary layers, suggests the choice of a profile which is constant in the bulk, so that the support of $\frac{d\tau}{dz}$ is concentrated near the boundaries where u^z and θ vanish due to the boundary conditions. In particular one can consider

$$\tau(z) = \begin{cases} 1-\frac{1}{2\delta}z & 0\leq z\leq\delta\,, \\ \frac{1}{2} & \delta\leq z\leq 1-\delta\,, \\ \frac{1}{2\delta}(1-z) & 1-\delta\leq z\leq 1\,, \end{cases} \tag{2.14}$$

where δ is the boundary layer thickness. We first notice that the Dirichlet integral (2.8) is $\int_0^1(\frac{d\tau}{dz})^2\,dz = \frac{1}{2\delta}$ and δ will tell how thick the boundary layer should be in order to ensure the stability of the profile. Constantin and Doering in [17] showed the stability of this profile. In particular they showed that $\mathcal{Q}_\tau \geq 0$ once one chooses $\delta \sim \frac{1}{\mathrm{Ra}^{\frac{2}{5}}}$. This yields the (suboptimal) upper bound for Nusselt number

$$\mathrm{Nu} \leq \frac{1}{2\delta} \sim \mathrm{Ra}^{\frac{2}{5}},$$

(For more details the reader may consult [17]). Before passing to the next example we want to get some intuition on the term $\int \frac{d\tau}{dz} \langle u^z\theta\rangle' \, dz$. Let us assume for the moment the model (2.1) without the diffusion term in the temperature equation (2.1a) , i.e.

$$\begin{aligned}
\partial_t T + u\cdot\nabla T &= 0 && \text{for} \quad 0<z<1\,, && (2.15\text{a})\\
-\Delta u + \nabla p &= \mathrm{Ra} T e_z && \text{for} \quad 0<z<1\,, \\
\nabla\cdot u &= 0 && \text{for} \quad 0<z<1\,, && (2.15\text{b})\\
u &= 0 && \text{for} \quad z\in\{0,1\}\,, && (2.15\text{c})\\
T &= 1 && \text{for} \quad z=0\,,\\
T &= 0 && \text{for} \quad z=1\,.
\end{aligned}$$

In this case $T=\tau(z),\ u=0,\ p=p(z)=\mathrm{Ra}\int_0^z \tau(z')\,dz'$ is a stationary solution of the system (2.15). Monitoring the growth in time of the perturbation $\theta = T-\tau$ in the L^2- norm, we find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \langle\theta^2\rangle'\,dz &= \int_0^1 \langle\theta\partial_t\theta\rangle'\,dz\\
&= \int_0^1 \langle\theta\partial_t T\rangle'\,dz\\
&\overset{(2.15a)}{=} \int_0^1 \langle\theta(-\nabla\cdot(uT))\rangle'\,dz\\
&\overset{(2.15b)}{=} \int_0^1 \langle\theta(-\nabla\cdot(\theta u) - u^z\frac{d\tau}{dz})\rangle'\,dz\\
&\overset{(2.15c)}{=} -\int_0^1 \frac{d\tau}{dz}\langle u^z\theta\rangle'\,dz\,.
\end{aligned}$$

Therefore the term $\int \frac{d\tau}{dz}\langle u^z\theta\rangle'\,dz$ governs the growth of the finite (not just infinitesimal) perturbations of the stationary temperature field τ. For the full model (2.1) only the linear profile ($\tau = 1-z, u=0$) is a stationary solution. Again we compute the growth in time of the L^2- norm of the perturbation $\theta = T-\tau$ where $\tau = 1-z$. Using the homogeneous boundary condition for θ and the fact that

$$-\int_0^1 \langle|\nabla\theta|^2\rangle'\,dz = \int_0^1\langle\theta\Delta T\rangle'\,dz = -\int_0^1\langle\nabla\theta\nabla T\rangle'\,dz = -\int\langle|\nabla\theta|^2\rangle'\,dz + \int_0^1\langle\partial_z\theta\rangle'\,dz$$

we find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \left\langle\theta^2\right\rangle'\,dz &\overset{(2.5)}{=} -\int_0^1\frac{d\tau}{dz}\langle u^z\theta\rangle'\,dz - \int\langle|\nabla\theta|^2\rangle'\,dz + \int_0^1\frac{d^2\tau}{dz^2}\langle\theta\rangle'\,dz\\
&= \int_0^1\langle u^z\theta\rangle'\,dz - \int_0^1\langle|\nabla\theta|^2\rangle'\,dz - \int_0^1\langle\partial_z\theta\rangle'\,dz\\
&= \int_0^1\langle u^z\theta\rangle'\,dz - \int_0^1\langle|\nabla\theta|^2\rangle'\,dz\,.
\end{aligned}$$

Therefore this time we cannot infer that the term $\int \frac{d\tau}{dz}\langle u^z\theta\rangle'\,dz$ governs the stability of the stationary profile $\tau = 1-z$. Nevertheless the computation above suggests the following intuition:

$\tau(z)$ *is stable if the hot (lighter) fluid is on the top of the heavy (colder) fluid.* Indeed, in the next section, we will show that the condition

$$\frac{d\tau}{dz} \geq 0\,, \tag{2.16}$$

in the bulk, is necessary for the stability of a background profile τ. To validate this physical intuition we consider a linearly increasing profile $\tau = az + b$ with $a > 0$. We want to show that *in the bulk* this profile satisfies

$$\int_0^1 a\langle u^z\theta\rangle'\,dz \geq \frac{1}{\mathrm{Ra}}\int_0^1 a\langle|\nabla u^z|^2\rangle'\,dz\,. \tag{2.17}$$

Notice that this would immediately imply the stability for the profile τ since the other two terms in (2.11) are positive. For this purpose, after Fourier transforming the equations (2.12) we compute

$$\begin{aligned}
\mathrm{Re}\int_0^1 \mathcal{F}'u^z\overline{\mathcal{F}'\theta}\,dz &\overset{(2.12)}{=} \mathrm{Re}\frac{1}{\mathrm{Ra}}\frac{1}{|k'|^2}\int_0^1 \mathcal{F}'u^z\left(-\frac{d^2}{dz^2}+|k'|^2\right)^2\overline{\mathcal{F}'u^z}\,dz\\
&= \mathrm{Re}\frac{1}{\mathrm{Ra}}\frac{1}{|k'|^2}\int_0^1 \mathcal{F}'u^z\left(\frac{d^4}{dz^4}-2|k'|^2\frac{d^2}{dz^2}+|k'|^4\right)\overline{\mathcal{F}'u^z}\,dz\\
&= \frac{1}{\mathrm{Ra}}\left(\frac{1}{|k'|^2}\int_0^1\left|\frac{d^2}{dz^2}\mathcal{F}'u^z\right|^2+2\int_0^1\left|\frac{d}{dz}\mathcal{F}'u^z\right|^2dz+|k'|^2\int_0^1|\mathcal{F}'u^z|^2\,dz\right)\\
&\geq \frac{1}{\mathrm{Ra}}\left(\int_0^1\left|\frac{d}{dz}\mathcal{F}'u^z\right|^2dz+|k'|^2\int_0^1|\mathcal{F}'u^z|^2\,dz\right)\,,
\end{aligned}$$

which implies (2.17). Since the profile $\tau = az + b$ with $a > 0$ does not satisfy the boundary conditions, we need to modify it in such a way that the stable linear part occupy a big part of the bulk. Indeed, we can show that the profile

$$\tau(z) = \begin{cases} 1-\left(\frac{1-\delta}{\delta}\right)z & 0\leq z\leq\delta\,,\\ z & \delta\leq z\leq 1-\delta\,,\\ \left(\frac{1-\delta}{\delta}\right)(1-z) & 1-\delta\leq z\leq 1,\end{cases} \tag{2.18}$$

is stable, provided that the boundary layer thickness δ is small enough. Starting from the stability condition (2.11) , inserting the Ansatz (2.18) for τ and recalling the estimate (2.17) we have

$$\begin{aligned}
&Q_\tau[\theta] \geq\\
&\int_0^1\langle u^z\theta\rangle'\,dz-\frac{2}{\delta}\left(\int_0^\delta\langle u^z\theta\,\rangle'dz+\int_{1-\delta}^1\langle u^z\theta\rangle'\,dz\right)+\int_0^1\langle|\nabla\theta|^2\rangle'\,dz\\
&\overset{(2.17)}{\geq}\int_0^1\langle|\nabla u^z|^2\rangle'\,dz-\frac{2}{\delta}\left(\int_0^\delta\langle u^z\theta\rangle'\,dz+\int_{1-\delta}^1\langle u^z\theta\rangle'\,dz\right)+\int_0^1\langle|\nabla\theta|^2\rangle'\,dz\,.
\end{aligned}$$

Applying the Cauchy-Schwartz estimate, the Poincare' estimate in z and the Young estimate we find

$$\begin{aligned}
\frac{2}{\delta}\int_0^\delta\langle u^z\theta\rangle'\,dz &\leq \frac{2}{\delta}\int_0^\delta\langle|\theta||u^z|\rangle'\,dz\\
&\leq \frac{2}{\delta}\left(\int_0^\delta\langle|\theta|^2\rangle'\,dz\right)^{\frac{1}{2}}\left(\int_0^\delta\langle|u^z|^2\rangle'\,dz\right)^{\frac{1}{2}}\\
&\leq 2\delta\left(\int_0^\delta\langle|\nabla\theta|^2\rangle'\,dz\right)^{\frac{1}{2}}\left(\int_0^\delta\langle|\nabla u^z|^2\rangle'\,dz\right)^{\frac{1}{2}}\\
&\leq \int_0^1\langle|\nabla\theta|^2\rangle'\,dz+4\delta^2\int_0^1\langle|\nabla u^z|^2\rangle'\,dz
\end{aligned}$$

and similarly

$$\frac{2}{\delta}\int_{1-\delta}^{1}\langle u^z\theta\rangle' \, dz \le \int_0^1 \langle|\nabla\theta|^2\rangle' \, dz + 4\delta^2\int_0^1 \langle|\nabla u^z|^2\rangle' \, dz \,. \tag{2.19}$$

Combining the last three estimates together we obtain

$$Q_\tau[\theta] \ge \left(\frac{1}{\mathrm{Ra}} - 4\delta^2\right)\int_0^1 \langle|\nabla u^z|^2\rangle' \, dz \,.$$

The biggest δ for which the quadratic form $Q_\tau[\theta]$ is positive is $\delta \sim \mathrm{Ra}^{-\frac{1}{2}}$ and this choice gives us the upper bound on the Nusselt number

$$\mathrm{Nu} \lesssim \frac{2}{\delta} \sim \mathrm{Ra}^{\frac{1}{2}} \,.$$

Therefore the "good" linearly increasing part in the bulk can compensate the "bad" decreasing part in the boundary layers only provided that the boundary layers are very thin ($\delta \sim \mathrm{Ra}^{-\frac{1}{2}}$). In conclusion, the argument above provides (again) a sub-optimal upper bound which does not reproduce the physical scaling. Doering, Otto & Reznikoff in [8] showed that, in order to reduce the effect of the "bad" boundary layers and (at the same time) keep the function increasing in the bulk without loosing stability, one can choose a logarithmic profile (being it steep close to the boundary layers and slowly grows away from them). It is easy to see that the profile $\tau(z) = a\ln(z+b)$ is stable in the bulk, namely one can prove that

$$\int_0^1 \frac{1}{z+b}\langle u^z\theta\rangle' \, dz \ge \frac{1}{2}\frac{1}{\mathrm{Ra}}\int_0^1 \frac{1}{z+b}\langle|\nabla u^z|^2\rangle' \, dz. \tag{2.20}$$

The argument is the following: After changing variable our goal is to prove

$$\int_I \frac{1}{z}\langle u^z\theta\rangle' \, dz \ge \frac{1}{2}\frac{1}{\mathrm{Ra}}\int_I \frac{1}{z}\langle|\nabla u^z|^2\rangle' \, dz \,, \tag{2.21}$$

where $I = (b, 1+b)$. In the (horizontally) Fourier transformed variables estimate (2.21) can be restated as

$$\mathrm{Re}\int_I \frac{1}{z}\mathcal{F}'u^z\overline{\mathcal{F}'\theta} \, dz \ge \frac{1}{2}\frac{1}{\mathrm{Ra}}\int_I \frac{1}{z}\left(|k'|^2|\mathcal{F}'u^z|^2 + \left|\frac{d}{dz}\mathcal{F}'u^z\right|^2\right) dz \,, \tag{2.22}$$

which by (2.12) turns into

$$\begin{aligned}\mathrm{Re}\frac{1}{\mathrm{Ra}}\left(|k'|^2\int_I \frac{1}{z}|\mathcal{F}'u^z|^2 \, dz - 2\int_I \frac{1}{z}\mathcal{F}'u^z\frac{d^2}{dz^2}\overline{\mathcal{F}'u^z} \, dz + \frac{1}{|k'|^2}\int_I \frac{1}{z}\mathcal{F}'u^z\frac{d^4}{dz^4}\overline{\mathcal{F}'u^z} \, dz\right)\\ \ge \frac{1}{2}\frac{1}{\mathrm{Ra}}\int_I \frac{1}{z}\left(|k'|^2|\mathcal{F}'u^z|^2 + \left|\frac{d}{dz}\mathcal{F}'u^z\right|^2\right) dz.\end{aligned} \tag{2.23}$$

Let us call $g := \frac{1}{z}\mathcal{F}'u^z$ and observe that $\frac{d^2}{dz^2}\mathcal{F}'u^z = 2\frac{d}{dz}g + z\frac{d^2}{dz^2}g$, $\frac{d^4}{dz^4}\mathcal{F}'u^z = 4\frac{d^3}{dz^3}g + z\frac{d^4}{dz^4}g$ and using the boundary conditions $g = \frac{d}{dz}g = 0$ at $z = \{b, 1-b\}$ we can rewrite the last two terms of the left-hand side of (2.23) as

$$\begin{aligned} -2\int_I \frac{1}{z}\mathcal{F}'u^z\frac{d^2}{dz^2}\overline{\mathcal{F}'u^z} \, dz &= 2\int_I z\left(\frac{d}{dz}g\right)^2 dz \quad \text{and}\\ \int_I \frac{1}{z}\mathcal{F}'u^z\frac{d^4}{dz^4}\overline{\mathcal{F}'u^z} \, dz &= \int_I z\left(\frac{d^2}{dz^2}g\right)^2 dz.\end{aligned}$$

Again, by the homogeneous boundary condition for g we can rewrite the second term of the right-hand side as

$$\int_I \frac{1}{z}\left|\frac{d}{dz}\mathcal{F}'u^z\right|^2 dz = \int_I \frac{1}{z}g^2 \, dz + \int_I z\left(\frac{d}{dz}g\right)^2 dz.$$

Therefore (2.23) can be rewritten as

$$\frac{1}{\mathrm{Ra}}\left(|k'|^2\int_I zg^2\,dz+2\int z\left(\frac{d}{dz}g\right)^2dz+\frac{1}{|k'|^2}\int_I z\left(\frac{d^2}{dz^2}g\right)^2dz\right)$$
$$\geq\frac{1}{2}\frac{1}{\mathrm{Ra}}\left(|k'|^2\int_I zg^2\,dz+\int_I\frac{1}{z}g^2\,dz+\int_I z\left(\frac{d}{dz}g\right)^2dz\right).$$

Absorbing the first and last terms of the right-hand side in the left-hand side and dropping out the term $\int z(\frac{dg}{dz})^2\,dz$ in the right-hand side we are left to prove

$$|k'|^2\int_I zg^2\,dz+\frac{1}{|k'|^2}\int_I z\left(\frac{d^2}{dz^2}g\right)^2dz\geq\frac{1}{2}\int_I\frac{1}{z}g^2\,dz\,. \tag{2.24}$$

Setting $h:=\frac{1}{z}g$, noticing that $\int_I z\left(\frac{d^2}{dz^2}g\right)^2dz=\int z^3\left(\frac{d^2}{dz^2}h\right)^2dz$ and applying the Young inequality (since k' is arbitrary) we are left to prove the following inequality

$$\left(\int_I z^3h^2\,dz\int_I z^3\left(\frac{d^2}{dz^2}h\right)^2dz\right)^{\frac{1}{2}}\geq\int_I zh^2\,dz. \tag{2.25}$$

By the Hardy's inequality applied to the second term of the left-hand side, i.e. $\int_I z^3\left(\frac{d^2}{dz^2}h\right)^2dz\geq\int_I z\left(\frac{d}{dz}h\right)^2dz$ [1], the estimate (2.25) follows immediately. Indeed

$$\int zh^2\,dz=\int_I\frac{1}{2}\left(\frac{d}{dz}z^2\right)h^2\,dz=-\int z^2h\frac{d}{dz}h\,dz\leq\left(\int z^3h^2\,dz\int z\left(\frac{d}{dz}h\right)^2dz\right)^{\frac{1}{2}}. \tag{2.26}$$

Estimate (2.20) is proved by setting $b=0$. The logarithmic profile is therefore stable in the bulk, and in order to fulfill the boundary conditions, τ can be chosen of the form

$$\tau(z)=\begin{cases}1-\frac{z}{\delta} & 0\leq z\leq\delta\,,\\ \frac{1}{2}+\lambda(\delta)\ln\left(\frac{z}{(1-z)}\right) & \delta\leq z\leq 1-\delta\,,\\ \frac{(1-z)}{\delta} & 1-\delta\leq z\leq 1\,,\end{cases} \tag{2.27}$$

where $\lambda(\delta)=\frac{1}{2\ln(\frac{1-\delta}{\delta})}$. In [8], the authors show that the stability of the profile is preserved if $\delta\sim\left(\frac{1}{\mathrm{Ra}\ln(\mathrm{Ra})}\right)^{\frac{1}{3}}$. Since $\int_0^1(\frac{d\tau}{dz})^2\,dz\sim\frac{2}{\delta}$, by (2.8) we immediately obtain the upper bound

$$\mathrm{Nu}\lesssim(\ln\mathrm{Ra})^{\frac{1}{3}}\mathrm{Ra}^{\frac{1}{3}},$$

which successfully reproduces the physical scaling suggested by Malkus (2.2). (For more details the reader may consult [8])

[1] The Hardy inequality states that for $g=0$ at ∂I we have

$$\int_I z^\alpha g^2\,dz\leq C(\alpha)\int_I z^{\alpha+2}\left(\frac{d}{dz}g\right)^2dz$$

for $\alpha\neq-1$.

2.2 Result: Lower bound for the background field method

Joint work with Felix Otto, [10].

With the rescaling

$$x = \mathrm{Ra}^{-\frac{1}{3}}\hat{x}, \quad t = \mathrm{Ra}^{-\frac{2}{3}}\hat{t}, \quad u = \mathrm{Ra}^{\frac{1}{3}}\hat{u}, \quad p = \mathrm{Ra}^{\frac{2}{3}}\hat{p}$$

and setting $H := \mathrm{Ra}^{\frac{1}{3}}$ the equations (2.1) turn into

$$\begin{aligned}
\partial_{\hat{t}}T + \hat{u}\cdot\hat{\nabla}T - \hat{\Delta}T &= 0 && \text{for} \quad 0 < \hat{z} < H\,,\\
-\hat{\Delta}\hat{u} + \hat{\nabla}\hat{p} &= Te_{\hat{z}} && \text{for} \quad 0 < \hat{z} < H\,,\\
\hat{\nabla}\cdot\hat{u} &= 0 && \text{for} \quad 0 < \hat{z} < H\,,\\
\hat{u} &= 0 && \text{for} \quad \hat{z} \in \{0, H\}\,,\\
T &= 1 && \text{for} \quad \hat{z} = 0\,,\\
T &= 0 && \text{for} \quad \hat{z} = H.
\end{aligned}$$

From now on we will omit the ˆ for the rescaled quantities and we will work with the following system

$$\begin{aligned}
\partial_t T + u\cdot\nabla T - \Delta T &= 0 && \text{for} \quad 0 < z < H\,, && (2.29a)\\
-\Delta u + \nabla p &= Te_z && \text{for} \quad 0 < z < H\,, && (2.29b)\\
\nabla\cdot u &= 0 && \text{for} \quad 0 < z < H\,, && (2.29c)\\
u &= 0 && \text{for} \quad z \in \{0, H\}\,, && (2.29d)\\
T &= 1 && \text{for} \quad z = 0\,, && (2.29e)\\
T &= 0 && \text{for} \quad z = H. && (2.29f)
\end{aligned}$$

In this rescaling the turbulent regime is when $H \gg 1$. This is a condition that we will assume in the rest of the section.

Only in this section, in order to simplify the notations, we relabel the vertical component of the velocity as

$$w = u^z \qquad \text{i.e.} \qquad w = u\cdot e_z\,.$$

The starting point of this section is the stability condition in the Fourier-transformed variable $\mathcal{F}'w(z)$, $\mathcal{F}'\theta(z)$: for every wavenumber $k' \in \frac{2\pi}{L}\mathbb{Z}^{d-1}\setminus\{0\}$

$$\mathcal{Q}_\tau[\mathcal{F}'\theta] = 2\int_0^H \frac{d\tau}{dz}\mathrm{Re}[\mathcal{F}'w\overline{\mathcal{F}'\theta}]\,dz + \int_0^H |k'|^2(\mathcal{F}'\theta)^2\,dz + \int_0^H \left(\frac{d}{dz}\mathcal{F}'\theta\right)^2 dz \geq 0, \tag{2.30}$$

for all $\mathcal{F}'\theta(z)$ related to $\mathcal{F}'w(z)$ through the fourth-order boundary value problem

$$\begin{cases} \Delta^2 u^z &= -\Delta_{x'}\theta \qquad \text{for} \quad 0 < z < H\,,\\ u^z &= \partial_z u^z = 0 \qquad \text{for} \quad z \in \{0, H\}\,, \end{cases} \tag{2.31}$$

such that $\mathcal{F}'\theta = 0$ at $z \in \{0, H\}$. It is convenient to introduce the slope $\xi := \frac{d\tau}{dz}$ of the background temperature profile. Using the equation (2.12) we can eliminate θ from the stability condition, obtaining

$$\int_0^H \xi\mathcal{F}'w\left(-\frac{d^2}{dz^2} + |k'|^2\right)^2\overline{\mathcal{F}'w}\,dz + \int_0^H |k'|^{-2}\left|\frac{d}{dz}\left(-\frac{d^2}{dz^2} + |k'|^2\right)^2\mathcal{F}'w\right|^2 dz$$
$$+ \int_0^H \left|\left(-\frac{d^2}{dz^2} + |k'|^2\right)^2\mathcal{F}'w\right|^2 dz \;\geq\; 0\,, \tag{2.32}$$

for all $k' \in \frac{2\pi}{L}\mathbb{Z}^{d-1} \setminus \{0\}$ and all (complex valued) functions $\mathcal{F}'w(z)$ satisfying the three boundary conditions

$$\mathcal{F}'w \;=\; \frac{d}{dz}\mathcal{F}'w \;=\; \left(-\frac{d^2}{dz^2}+|k'|^2\right)^2 \mathcal{F}'w \;=\; 0 \quad \text{for } z \in \{0, H\}. \tag{2.33}$$

We denote with $\mathcal{S}_{d,L}$ the class of all the background profiles that satisfy the stability condition (2.32) parametrized by the dimension of the space d in which the container lies and by the lateral size of the container L. We observe that in (2.32) only the modulus of the wavelength k' appears. This means that the class $\mathcal{S}_{L,d}$ is independent of the dimension parameter d, i.e. $\mathcal{S}_{L,d} = \mathcal{S}_L$. Therefore the analysis that follows is the same in every dimension. Furthermore, since the background profile τ is stable for arbitrary horizontal length L, we may *assume* that our background profile τ belongs to the intersection of all these classes, i.e $\tau \in \cap_{L<\infty}\mathcal{S}_L$. We will therefore say that ξ satisfies the stability condition if

$$\int_0^H \xi\mathcal{F}'w\left(-\frac{d^2}{dz^2}+k'^2\right)^2\overline{\mathcal{F}'w}\,dz + \int_0^H k'^{-2}\left|\frac{d}{dz}\left(-\frac{d^2}{dz^2}+k'^2\right)^2\mathcal{F}'w\right|^2 dz + \int_0^H \left|\left(-\frac{d^2}{dz^2}+k'^2\right)^2\mathcal{F}'w\right|^2 dz \;\geq\; 0\,, \tag{2.34}$$

holds true for all $k' \in \mathbb{R}$ and all (complex valued) functions $\mathcal{F}'w(z)$ satisfying the three boundary conditions (2.33). The condition $k' \in \mathbb{R}$, which amounts to consider a container of infinite length, turns out to simplify the analysis that follows. In Subsection 2.2.6 we recover the physical setting of a container with finite lateral size, by setting $k' \in \frac{2\pi}{L}\mathbb{Z} \setminus \{0\}$.
In the rest of the paper, in order to simplify the notations, we will omit the symbol $\mathcal{F}'$ for the horizontal Fourier transform.

2.2.1 Reduced stability condition

Let us first observe that the stability condition (2.34)&(2.33) is invariant under the following transformation

$$z = L\hat{z} \;\text{ and thus }\; k' = \frac{1}{L}\hat{k}',\; H = L\hat{H} \;\text{ and }\; \xi = L^{-4}\hat{\xi}\,. \tag{2.35}$$

Hence in the bulk ($z \gg 1$ and $H - z \geq 1$) we expect that the first term in (2.34) dominates. This motivates to consider the **reduced stability condition**

$$\int_0^H \xi w\left(-\frac{d^2}{dz^2}+k'^2\right)^2\overline{w}\,dz \geq 0\,, \tag{2.36}$$

for all $k' \in \mathbb{R}$ and all (complex valued) functions $w(z)$ with the three boundary conditions (2.33). In the following proposition we characterize the profiles that satisfy the reduced stability condition (2.36).

Proposition 1. *Let $\tau : (0,H) \to \mathbb{R}$ satisfy the reduced stability condition, i.e. for all $k' \in \mathbb{R}$ and for all $w(z)$ satisfying (2.33), the condition (2.36) holds . Then*

-
$$\xi \geq 0\,, \tag{2.37}$$

-
$$\int_{1/e}^1 \xi dz \lesssim \frac{1}{\ln H}\int_1^H \xi\,dz\,. \tag{2.38}$$

The proof of these two statements is based on the inspection of the limits $k' \uparrow \infty$ and $k' \downarrow 0$.

Proof.
Argument for (2.37):
Letting $k' \uparrow \infty$, (2.36) reduces to

$$\int_0^H \xi |w|^2 \, dz \geq 0$$

for all compactly supported w, from which we infer (2.37).
Argument for (2.38):
Letting $k' \downarrow 0$, (2.36) reduces to

$$\int_0^H \xi w \frac{d^4}{dz^4} \bar{w} dz \geq 0 \,, \tag{2.39}$$

for all functions $w(z)$ satisfying the three boundary conditions

$$w = \frac{dw}{dz} = \frac{d^4 w}{dz^4} = 0 \qquad \text{for } z \in \{0, H\} \,. \tag{2.40}$$

In fact, besides Subsection 2.2.5.3, we will work with w compactly supported in $z \in (0, H)$, so that the boundary condition (2.40) are trivially satisfied. Focusing on the lower half of the container, i.e restricting to $z \in \left(0, \frac{H}{2}\right)$, we make the following Ansatz

$$w = z^2 \hat{w} \,,$$

where $\hat{w}(z)$ is a real function with compact support in $(0, H)$.
The merit of this Ansatz is that in the new variable $\hat{w}$, the multiplier in (2.39) can be written in the scale-invariant form

$$\phi = w \frac{d^4}{dz^4} \bar{w} = \hat{w} z^2 \frac{d^4}{dz^4} z^2 \hat{w} = \hat{w} \left(z \frac{d}{dz} + 2 \right) \left(z \frac{d}{dz} + 1 \right) z \frac{d}{dz} \left(z \frac{d}{dz} - 1 \right) \hat{w} \,. \tag{2.41}$$

Note that the fourth-order polynomial in $z \frac{d}{dz}$ appearing on the right hand side of (2.41) may be inferred, without lengthy calculations, from the fact that $z^2 \frac{d^4}{dz^4} z^2$ annihilates $\{\frac{1}{z^2}, \frac{1}{z}, 1, z\}$. This suggests to introduce the new variables

$$s = \ln z \;\text{ and }\; \xi = z^{-1} \hat{\xi}, \tag{2.42}$$

for which the stability condition turns into

$$\int_{-\infty}^{\ln H} \hat{\xi} \, \phi \, ds \geq 0 \quad \text{ where } \quad \phi = \hat{w} \left(\frac{d}{ds} + 2 \right) \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} \left(\frac{d}{ds} - 1 \right) \hat{w} \,, \tag{2.43}$$

for all functions $\hat{w}$ with compact support in $z \in (0, H)$. Here it comes the heuristic argument: For $H \gg 1$, we can think of test functions $\hat{w}$ that vary slowly in the logarithmic variable s. For these $\hat{w}$ we have

$$\phi = \hat{w} \left(\frac{d}{ds} + 2 \right) \left(\frac{d}{ds} + 1 \right) \frac{d}{ds} \left(\frac{d}{ds} - 1 \right) \hat{w} \approx -2 \hat{w} \frac{d}{ds} \hat{w} = -\frac{d}{ds} \hat{w}^2 \,, \tag{2.44}$$

which particular implies

$$0 \leq \int_{-\infty}^{\ln H} \hat{\xi} \, \phi \, ds \approx -\int_{-\infty}^{\ln H} \hat{\xi} \frac{d}{ds} \hat{w}^2 \, ds = \int_{-\infty}^{\ln H} \frac{d\hat{\xi}}{ds} \hat{w}^2 \, ds \,,$$

for all $\hat{w}(s)$ with compact support in $(-\infty, \ln H)$. Thus it follows that, approximately on large $s-$scales,

$$\frac{d\hat{\xi}}{ds} \geq 0 \,.$$

We expect that this implies that for any $1 \ll S_1 \leq \ln H$:

$$\int_{-1}^{0} \hat{\xi} ds \lesssim \frac{1}{S_1} \int_{0}^{S_1} \hat{\xi}\, ds\,, \tag{2.45}$$

which in the original variables (2.42), for $S_1 = \ln \frac{H}{2}$ turns into (2.38). We now prove that (2.39) and (2.37) imply (2.45).

Argument for (2.45):

We start by noticing that because of translation invariance in s, (2.43) can be reformulated as follows: For any function $\hat{w}(s)$ supported in $s \leq 0$, and any $s' \leq \ln H$ we have

$$\int_{-\infty}^{\infty} \hat{\xi}(s'')\phi(s''-s')\, ds'' \;=\; \int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi(s)\, ds \;\geq\; 0, \tag{2.46}$$

where the multiplier ϕ is defined as in (2.43):

$$\phi \;=\; w\frac{d^4}{dz^4}\overline{w} \;=\; \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{d}{ds}-1\right)\hat{w}.$$

We note that (2.45) follows from (2.46) once for given S_1 we construct

- a family $\mathfrak{F} = \{w_{s'}\}_{s'}$ of smooth functions $w_{s'}$ parameterized by $s' \in \mathbb{R}$ and compactly supported in $z \in (0,1)$ (i. e. $s \in (-\infty, 0]$) and
- a probability measure $\rho(ds') = \rho(s')\, ds'$ supported in $s' \in (-\infty, \ln H]$,

such that the corresponding convex combination of multipliers $\{\phi_{s'}\}_{s'}$ shifted by s', i. e.

$$\phi_1(s'') \;:=\; \int_{-\infty}^{\infty} \phi_{s'}(s''-s')\,\rho(s')ds'\,, \tag{2.47}$$

satisfies

$$\phi_1(s'') \leq \begin{cases} -1 & \text{for } \; -1 \leq s'' \leq 0, \\ \frac{C}{S_1} & \text{for } \quad 0 \leq s'' \leq S_1, \\ 0 & \text{else,} \end{cases} \tag{2.48}$$

for a (possibly large) universal constant C. Indeed, using (2.46), (2.48) in conjunction with the positivity (2.37) of the profile $\hat{\xi}$ we have

$$0 \leq \int_{-\infty}^{\infty} \hat{\xi}\phi_1 ds'' \leq -\int_{-1}^{0} \hat{\xi} ds'' + \frac{C}{S_1}\int_{0}^{S_1} \hat{\xi} ds'',$$

which implies (2.45) .

We first address the form of the family $\mathfrak{F}$. The heuristic observation (2.44) motivates the change of variables

$$s \;=\; \lambda \hat{s} \quad \text{with} \quad \lambda \geq 1, \tag{2.49}$$

our "(logarithmic) length scale", to be chosen sufficiently large. We fix a smooth, compactly supported "mask" $\hat{w}_0(\hat{s})$; it will be convenient to restrict its support to $\hat{s} \in (-1, 0]$, say

$$|\hat{w}_0|^2 > 0 \quad \text{in} \quad \left(-\frac{1}{2}, 0\right) \qquad \text{and } \hat{w}_0 = 0 \quad \text{else} \;\;, \tag{2.50}$$

and, in order to justify the language of "mollification by convolution" we think of the normalization $\int \hat{w}_0^2 d\hat{s} = 1$. By mask we mean that in (2.46) we choose

$$\hat{w}(\lambda \hat{s}) \;=\; \lambda^{-1/2}\hat{w}_0(\hat{s}). \tag{2.51}$$

With this change of variables, the multiplier can be rewritten as follows

$$\begin{aligned}\phi &= \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{d}{ds}-1\right)\hat{w}\\ &= \frac{1}{\lambda}\hat{w}_0\left(\frac{1}{\lambda}\frac{d}{d\hat{s}}+2\right)\left(\frac{1}{\lambda}\frac{d}{d\hat{s}}+1\right)\frac{1}{\lambda}\frac{d}{d\hat{s}}\left(\frac{1}{\lambda}\frac{d}{d\hat{s}}-1\right)\hat{w}_0\\ &= \hat{w}_0\left(\frac{1}{\lambda^5}\frac{d^4}{d\hat{s}^4}+\frac{2}{\lambda^4}\frac{d^3}{d\hat{s}^3}-\frac{1}{\lambda^3}\frac{d^2}{d\hat{s}^2}-\frac{2}{\lambda^2}\frac{d}{d\hat{s}}\right)\hat{w}_0\,,\end{aligned}$$

and reordering the terms we have

$$\phi = -\frac{2}{\lambda^2}\hat{w}_0\frac{d}{d\hat{s}}\hat{w}_0-\frac{1}{\lambda^3}\hat{w}_0\frac{d^2}{d\hat{s}^2}\hat{w}_0+\frac{2}{\lambda^4}\hat{w}_0\frac{d^3}{d\hat{s}^3}\hat{w}_0+\frac{1}{\lambda^5}\hat{w}_0\frac{d^4}{d\hat{s}^4}\hat{w}_0\,. \tag{2.52}$$

Heuristically, for $\lambda \gg 1$ the multiplier ϕ can be approximated by the first term on the right hand side

$$\phi(s) \approx -\frac{1}{\lambda^2}\frac{d}{d\hat{s}}\hat{w}_0^2\left(\frac{s}{\lambda}\right).$$

Inserting this approximation in the definition (2.47) of ϕ_1 we have

$$\begin{aligned}\phi_1(s'') &= \int_{-\infty}^{\infty}\phi(s''-s')\rho(s')ds' = \int_{-\infty}^{\infty}\phi(s)\rho(s''-s)ds\\ &\approx \int\left(-\frac{1}{\lambda^2}\frac{d}{d\hat{s}}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\right)\rho(s''-s)ds = -\int\frac{d}{ds}\left(\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\right)\rho(s''-s)ds\\ &= \int\left(\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\right)\frac{d}{ds}(\rho(s''-s))ds = -\int\left(\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right)\right)\left(\frac{d\rho}{ds'}\right)(s''-s)ds\,.\end{aligned}$$

For λ smaller than the characteristic scale on which ρ varies, we may think $\frac{1}{\lambda}\hat{w}_0^2\left(\frac{s}{\lambda}\right) \approx \delta_0(s)$, in view of our normalization. This yields

$$\phi_1 \approx -\frac{d\rho}{ds'} \tag{2.53}$$

which, in view of (2.48) suggests that ρ should have the form

$$\rho(s') = \begin{cases} s'+1 & \text{for } -1\le s'\le 0\\ 1-\frac{s'}{S_1} & \text{for } 0\le s'\le S_1\end{cases} \tag{2.54}$$

Now we will go through this heuristic argument assessing the error terms. Expanding ρ in a Taylor series around s''

$$\rho(s''-s) \approx \rho(s'')-\frac{d\rho}{ds'}(s'')s+\frac{1}{2}\frac{d^2\rho}{ds'^2}(s'')s^2\,,$$

we can write

$$\begin{aligned}\phi_1(s'') &= \int_{-\infty}^{\infty}\phi(s)\rho(s''-s)ds\\ &\approx \rho(s'')\int_{-\infty}^{\infty}\phi\,ds-\frac{d\rho}{ds'}(s'')\int_{-\infty}^{\infty}s\phi\,ds+\frac{1}{2}\frac{d^2\rho}{ds'^2}(s'')\int_{-\infty}^{\infty}s^2\phi\,ds\end{aligned}$$

We now note that the first term in (2.52), i.e $-\frac{2}{\lambda^2}\hat{w}_0\frac{d\hat{w}_0}{d\hat{s}} = -\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}$ gives the leading order contribution to the first and the second moment, thus

$$\int_{-\infty}^{\infty}s\phi\,ds \approx \int_{-\infty}^{\infty}s\left(-\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}\right)ds = \int_{-\infty}^{\infty}\hat{s}\left(-\frac{d\hat{w}_0^2}{d\hat{s}}\right)d\hat{s} = \int_{-\infty}^{\infty}\hat{w}_0^2\,d\hat{s} = 1 \tag{2.55}$$

and

$$\int_{-\infty}^{\infty} s^2\phi\, ds \approx \int_{-\infty}^{\infty} s^2\left(-\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}\right) ds = -\lambda\int_{-\infty}^{\infty}\hat{s}^2\left(\frac{d\hat{w}_0^2}{d\hat{s}}\right) d\hat{s} = \lambda\int_{-\infty}^{\infty} \hat{2}s\hat{w}_0^2\, d\hat{s}\,, \qquad (2.56)$$

while the second term in (2.52) gives the leading-order contribution to the zeroth moment of the multiplier ϕ:

$$\int_{-\infty}^{\infty}\phi\, ds \approx \int_{-\infty}^{\infty}\left(-\frac{1}{\lambda^3}\hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right) ds = \frac{1}{\lambda^2}\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s}\,. \qquad (2.57)$$

Hence we obtain the following specification of (2.53)

$$\begin{aligned}\phi_1(s'') &= \int_{-\infty}^{\infty}\rho(s''-s)\phi(s)ds\\ &\approx \frac{1}{\lambda^2}\rho(s'')\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s} - \frac{d\rho}{ds'}(s'') + \lambda\frac{d^2\rho}{ds'^2}(s'')\int_{-\infty}^{\infty}(-\hat{s})\hat{w}_0^2\, d\hat{s}\,.\end{aligned} \qquad (2.58)$$

Our goal is to specify the choice (2.54) of ρ such that (2.48) is satisfied. This show a dilemma: On the one hand, in the "plateau-region" $s''\sim S_1$, we would need $\lambda^2 \gg S_1$ so that the first term in (2.58) does not destroy the desired $\frac{1}{S_1}-$behavior. On the other hand in the "foot-region" $s''\in[0,1]$, we would need $\lambda\lesssim 1$ so that the last term does not destroy the effect of the middle term. This suggests that λ should be chosen to be small in the foot regions and large on the plateau region. Therefore it is natural to choose

$$\lambda = s'\,, \qquad (2.59)$$

and make the mask $\hat{w}$ depend on s' when we translate by s':

$$\int \phi_{s'}(s''-s')\rho(s')ds'.$$

Now (2.52) assumes the form

$$\phi_{s'} = -\frac{2}{(s')^2}\hat{w}_0\frac{d}{d\hat{s}}\hat{w}_0 - \frac{1}{(s')^3}\hat{w}_0\frac{d^2}{d\hat{s}^2}\hat{w}_0 + \frac{2}{(s')^4}\hat{w}_0\frac{d^3}{d\hat{s}^3}\hat{w}_0 + \frac{1}{(s')^5}\hat{w}_0\frac{d^4}{d\hat{s}^4}\hat{w}_0\,. \qquad (2.60)$$

Note that with the choice (2.59) and $s = s''-s'$, (2.49) turns into the nonlinear change of variables between s' and $\hat{s}$

$$\hat{s} = \frac{s''-s'}{s'} = \frac{s''}{s'} - 1 \Rightarrow s' = \frac{s''}{1+\hat{s}}\,. \qquad (2.61)$$

We consider this as a change of variables between s' and $\hat{s}$ (with s'' as a parameter); Thanks to the support restriction (2.50) on $\hat{w}_0$, it is invertible in the relevant range $\hat{s}\in[-\frac{1}{2},0]$: $\frac{d}{d\hat{s}} = -\frac{s''}{(1+\hat{s})^2}\frac{d}{ds'} = -\frac{(s')^2}{s''}\frac{d}{ds'}$ and $ds' = \frac{s''}{(1+\hat{s})^2}d\hat{s}$. From (2.47) and (2.60) we thus get the first representation

$$\begin{aligned}\phi_1(s'') &= -\frac{1}{s''}\int_{-\infty}^{\infty}\frac{d\hat{w}_0^2}{d\hat{s}}\,\rho\, d\hat{s} - \frac{1}{(s'')^2}\int_{-\infty}^{\infty}(1+\hat{s})\,\hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\rho\, d\hat{s}\\ &+ \frac{2}{(s'')^3}\int_{-\infty}^{\infty}(1+\hat{s})^2\,\hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\rho\, d\hat{s} + \frac{1}{(s'')^4}\int_{-\infty}^{\infty}(1+\hat{s})^3\,\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\rho\, d\hat{s}.\end{aligned}$$

An approximation argument in $\hat{w}_0$ below necessitates a second representation that involves $\hat{w}_0$ only up to second derivatives. For this purpose, we rewrite (2.60) in terms of the three quadratic quantities $\hat{w}_0^2$, $(\frac{d\hat{w}_0}{d\hat{s}})^2$, and $(\frac{d^2\hat{w}_0}{d\hat{s}^2})^2$:

$$
\begin{aligned}
\phi_{s'} &= -\frac{1}{(s')^2}\frac{d\hat{w}_0^2}{d\hat{s}} + \frac{1}{(s')^3}\left[\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 - \frac{1}{2}\frac{d^2\hat{w}_0^2}{d\hat{s}^2}\right] + \frac{1}{(s')^4}\left[-3\frac{d}{d\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{d^3\hat{w}_0^2}{d\hat{s}^3}\right] \\
&+ \frac{1}{(s')^5}\left[\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 - 2\frac{d^2}{d\hat{s}^2}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{2}\frac{d^4\hat{w}_0^2}{d\hat{s}^4}\right] \\
&= \left(-\frac{1}{(s')^2}\frac{d}{d\hat{s}} - \frac{1}{2}\frac{1}{(s')^3}\frac{d^2}{d\hat{s}^2} + \frac{1}{(s')^4}\frac{d^3}{d\hat{s}^3} + \frac{1}{2}\frac{1}{(s')^5}\frac{d^4}{d\hat{s}^4}\right)\hat{w}_0^2 \\
&+ \left(\frac{1}{(s')^3} - 3\frac{1}{(s')^4}\frac{d}{d\hat{s}} - 2\frac{1}{(s')^5}\frac{d^2}{d\hat{s}^2}\right)\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{(s')^5}\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 .
\end{aligned}
\tag{2.62}
$$

Now in this formula, using the change (2.61), we want to substitute the derivations $\frac{1}{(s')^m}\frac{d^n}{d\hat{s}^n}$ by linear combinations of derivations of the form $\frac{1}{(s'')^{m-k}}\frac{d^k}{ds'^k}(1+\hat{s})^{m-n-k}$ for $k = 0, \cdots, n$. The reason why this can be done is explained in the Appendix 2.2.7. The formulas (2.265), (2.266) & (2.267) allow to rewrite (2.62) as follows

$$
\begin{aligned}
\phi_{s'} &= \frac{1}{s''}\left(\frac{d}{ds'} - \frac{1}{2}\frac{d^2}{ds'^2}\frac{1}{(1+\hat{s})} - \frac{d^3}{ds'^3}\frac{1}{(1+\hat{s})^2} + \frac{1}{2}\frac{d^4}{ds'^4}\frac{1}{(1+\hat{s})^3}\right)\hat{w}_0^2 \\
&+ \left[\left(\frac{1}{(s'')^3} + \frac{6}{(s'')^4} - \frac{12}{(s'')^5}\right)(1+\hat{s})^3 + \left(\frac{3}{(s'')^3} - \frac{8}{(s'')^4}\right)\frac{d}{ds'}(1+\hat{s})^2\right. \\
&- \left.\frac{2}{(s'')^3}\frac{d^2}{ds'^2}(1+\hat{s})\right]\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{(s'')^5}(1+\hat{s})^5\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 .
\end{aligned}
$$

The advantage of this form is that integrations by part in s' become easy, so that we obtain

$$
\begin{aligned}
\phi_1 &= \frac{1}{s''}\int_{-\infty}^{\infty}\hat{w}_0^2\left(-\frac{d\rho}{ds'} - \frac{1}{2}\frac{1}{1+\hat{s}}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s})^2}\frac{d^3\rho}{ds'^3} + \frac{1}{2}\frac{1}{(1+\hat{s})^3}\frac{d^4\rho}{ds'^4}\right)ds' \\
&+ \left(\frac{1}{(s'')^3} + \frac{6}{(s'')^4} - \frac{12}{(s'')^5}\right)\int_{-\infty}^{\infty}(1+\hat{s})^3\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\rho\, ds' \\
&- \left(\frac{3}{(s'')^3} - \frac{8}{(s'')^4}\right)\int_{-\infty}^{\infty}(1+\hat{s})^2\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d\rho}{ds'}\, ds' \\
&- \frac{2}{(s'')^3}\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d^2\rho}{ds'^2}\, ds' \\
&+ \frac{1}{(s'')^5}\int_{-\infty}^{\infty}(1+\hat{s})^5\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2\rho\, ds' .
\end{aligned}
$$

Finally, using the substitution $\frac{ds'}{s''} = \frac{d\hat{s}}{(1+\hat{s})^2}$, the last formula turns into the desired second representation

$$\begin{aligned}\phi_1 &= \int_{-\infty}^{\infty} \hat{w}_0^2 \left(-\frac{1}{(1+\hat{s})^2}\frac{d\rho}{ds'} - \frac{1}{2}\frac{1}{(1+\hat{s})^3}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s})^4}\frac{d^3\rho}{ds'^3} + \frac{1}{2}\frac{1}{(1+\hat{s})^5}\frac{d^4\rho}{ds'^4}\right) d\hat{s} \\
&+ \left(\frac{1}{(s'')^2} + \frac{6}{(s'')^3} - \frac{12}{(s'')^4}\right)\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \rho \, d\hat{s} \\
&- \left(\frac{3}{(s'')^2} - \frac{8}{(s'')^3}\right)\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \frac{d\rho}{ds'} \, d\hat{s} \\
&- \frac{2}{(s'')^2}\int_{-\infty}^{\infty}\frac{1}{1+\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \frac{d^2\rho}{ds'^2} \, d\hat{s} \\
&+ \frac{1}{(s'')^4}\int_{-\infty}^{\infty}(1+\hat{s})^3\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 \rho \, d\hat{s} \, . \end{aligned} \tag{2.63}$$

From this representation we learn the following: If we assume that $\rho(s')$ varies on large length scales only, so that

- $\frac{d\rho}{ds'}, \frac{d^2\rho}{ds'^2}, \dots \ll \rho$
- $\frac{d^2\rho}{ds'^2}, \frac{d^3\rho}{ds'^3}, \dots \ll \frac{d\rho}{ds'}$

then for $s'' \gg 1$, we obtain to leading order from the above

$$\phi_1 \approx -\int_{-\infty}^{\infty}\frac{1}{(1+\hat{s})^2}\hat{w}_0^2 \frac{d\rho}{ds'} \, d\hat{s} + \frac{1}{(s'')^2}\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \rho \, d\hat{s} \, . \tag{2.64}$$

If $\rho(s')$ varies slowly even on a logarithmic scale (so that e. g. $s'\frac{d\rho}{ds'}$ is negligible with respect to ρ), the above further reduces to

$$\phi_1 \approx -\frac{d\rho(s'')}{ds'}\int_{-\infty}^{\infty}\frac{1}{(1+\hat{s})^2}\hat{w}_0^2 \, d\hat{s} + \frac{\rho(s'')}{(s'')^2}\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s} \, . \tag{2.65}$$

This is the specification of (2.58): We see that the first, negative, right-hand-side term of (2.65) dominates the second positive term provided

$$\frac{d\rho}{ds'} \gg \frac{1}{(s')^2}.$$

This is satisfied if ρ is of the form

$$\rho(s') = 1 - \frac{S_0}{s' - S_0}$$

for some $S_0 \gg 1$ to be chosen later; indeed

$$\frac{d\rho}{ds'} = \frac{S_0}{(s'-S_0)^2} \approx \frac{S_0}{(s')^2} \gg \frac{1}{(s')^2} \quad \text{for } s' \gg S_0 \gg 1 \, .$$

This motivates the following Ansatz for ρ in the range $1 \ll s' \ll S_1$: We fix a smooth mask $\rho_0(\hat{s}')$ such that

$$\rho_0 \;=\; 0 \;\text{ for } \hat{s}' \le 0, \quad \frac{d\rho_0}{d\hat{s}'} \;>\; 0 \;\text{ for } 0 < \hat{s}' \le 2, \quad \rho_0 \;=\; 1 - \frac{1}{\hat{s}'} \;\text{ for } 2 \le \hat{s}' \, . \tag{2.66}$$

For $S_0 \gg 1$, we consider the rescaled version

$$\rho(S_0(\hat{s}'+1)) \;=\; \rho_0(\hat{s}'), \quad \text{i. e. the change of variables } s' \;=\; S_0(\hat{s}'+1) \, . \tag{2.67}$$

It is convenient to rescale s'' accordingly:

$$s'' \;=\; S_0 \hat{s}''. \tag{2.68}$$

With this new rescaling, (2.63) turns into

$$\begin{aligned}
\phi_1 \;=\; & -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\frac{d\rho_0}{d\hat{s}'}\,d\hat{s} - \frac{1}{2S_0^2}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^3}\frac{d^2\rho_0}{d\hat{s}'^2}\,d\hat{s} \\
& +\frac{1}{S_0^3}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^4}\frac{d^3\rho_0}{d\hat{s}'^3}\,d\hat{s} + \frac{1}{2S_0^4}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^5}\frac{d^4\rho_0}{d\hat{s}'^4}\,d\hat{s} \\
& +\left(\frac{1}{S_0^2}\frac{1}{(\hat{s}'')^2} + \frac{1}{S_0^3}\frac{6}{(\hat{s}'')^3} - \frac{1}{S_0^4}\frac{12}{(\hat{s}'')^4}\right)\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\rho_0\,d\hat{s} \\
& -\left(\frac{1}{S_0^4}\frac{3}{(\hat{s}'')^3} - \frac{1}{S_0^5}\frac{8}{(\hat{s}'')^4}\right)\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d\rho_0}{d\hat{s}'}\,d\hat{s} \\
& -\frac{1}{S_0^5}\frac{2}{(\hat{s}'')^3}\int_{-\infty}^{\infty}\frac{1}{1+\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d^2\rho_0}{d\hat{s}'^2}\,d\hat{s} \\
& +\frac{1}{S_0^4}\frac{1}{(\hat{s}'')^4}\int_{-\infty}^{\infty}(1+\hat{s})^3\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2\rho_0\,d\hat{s}\,.
\end{aligned} \tag{2.69}$$

Since in the integrals in formula (2.63), the argument of ρ was given by $s = \frac{s''}{1+\hat{s}}$, c. f. (2.61), it follows from (2.67) and (2.68) that the argument of ρ_0 is given by

$$\hat{s}' \;=\; \frac{\hat{s}''}{1+\hat{s}} - 1\,. \tag{2.70}$$

Thus (2.69) just depends on $\hat{s}''$, not on S_0. Hence the above representation makes the dependence of ϕ_1 on S_0 explicit. Our reduced goal is now to show that the constructions of w (c. f. (2.51) and (2.59)) and ρ (c. f. (2.66) and (2.67)) yield the bound

$$\phi_1^*(s'') := \int \phi_{s'}(s''-s')\rho(s')\,ds' \begin{cases} = 0 & \text{for all } s'' \le \frac{1}{2}S_0\,, \\ < 0 & \text{for } s'' > \frac{1}{2}S_0\,, \end{cases} \tag{2.71}$$

for $S_0 \gg 1$. *Note that the measure ρ in the definition of ϕ_1^* is not (yet) a probability measure and therefore the multiplier ϕ_1^* is not admissible. At the end of the proof we show how to construct an admissible multiplier and how to pass from the reduced goal (2.71) to the desired bound (2.48).* In order to establish (2.71) it is convenient to distinguish three regions (note that if $s'' \in (\infty, \frac{S_0}{2}]$ all the integrals in (2.69) vanish because the supports of $\hat{w}_0$ and ρ_0 do not intersect, see below):

The range of large s'':

$$s'' \ge 3S_0 \quad \text{or equivalently} \quad \hat{s}'' \ge 3. \tag{2.72}$$

Note that because of our support condition (2.50) on $\hat{w}_0$, all integrals in (2.69) are supported in $\hat{s} \in [-\frac{1}{2}, 0]$. Together with our range (2.72), this yields for the argument $\hat{s}' \overset{(2.70)}{=} \frac{\hat{s}''}{1+\hat{s}} - 1$ of ρ_0 and its derivatives: $\hat{s}' \ge 2$. Because of $\frac{d\rho_0}{d\hat{s}'} = \frac{1}{(\hat{s}')^2}$ for $\hat{s}' \ge 2$, c. f. our Ansatz (2.66), the first integral in (2.69) reduces to

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\frac{d\rho_0}{d\hat{s}'}\,d\hat{s} &= \int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\frac{1}{(\frac{\hat{s}''}{1+\hat{s}}-1)^2}\,d\hat{s} \\
&= \int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(\hat{s}''-(1+\hat{s}))^2}\,d\hat{s} \approx \frac{1}{(\hat{s}'')^2}\int_{-\infty}^{\infty}\hat{w}_0^2\,d\hat{s}\,,
\end{aligned} \tag{2.73}$$

for $\hat{s}'' \gg 1$, whereas all the other integrals in (2.69) are $O(\frac{1}{(s'')^2})$ or smaller in $\hat{s}'' \gg 1$ or have prefactors $\frac{1}{(\hat{s}'')^2}$ or smaller. Since only the term in (2.69) coming from integral (2.73) has prefactor $\frac{1}{S_0}$ while all the other terms have prefactors $\frac{1}{S_0^2}$ or smaller (for $S_0 \gg 1$), the first term in (2.69) *uniformly* dominates all other terms:

$$\phi_1^* \;\approx\; -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(\hat{s}''-(1+\hat{s}))^2}\,d\hat{s} \quad \text{uniformly in } \hat{s}'' \geq 3 \quad \text{for } S_0 \gg 1\,. \tag{2.74}$$

In conclusion, in the range $\hat{s}'' \geq 3$ we have

$$\phi_1^* \sim -\frac{1}{S_0}\frac{1}{(\hat{s}'')^2} < 0 \;\text{ for }\; S_0 \gg 1\,. \tag{2.75}$$

<u>The range of intermediate s'':</u>

$$s'' \;\in\; \left[\frac{3}{4}S_0, 3S_0\right] \quad \text{or equivalently} \quad \hat{s}'' \;\in\; \left[\frac{3}{4}, 3\right]. \tag{2.76}$$

Again, we consider the first integral in (2.69). Now we use that $\frac{\hat{w}_0^2}{(1+\hat{s})^2} \geq 0$ is *strictly positive* in $\hat{s} \in \left(-\frac{1}{2}, 0\right)$, c. f. (2.50), and that $\frac{d\rho_0}{d\hat{s}'} \geq 0$ is strictly positive in $\hat{s}' > 0$, c. f. (2.66), that is, in $\hat{s} < \hat{s}'' - 1$, c. f. (2.70). We note that the two $\hat{s}-$intervals $(-\frac{1}{2}, 0)$ and $(-\infty, \hat{s}''-1)$ intersect for $\hat{s}'' > \frac{1}{2}$. Hence by continuity of the first integral in (2.69) in its parameter $\hat{s}''$, there exists a universal constant C such that

$$\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\,\frac{d\rho_0}{d\hat{s}'}\,d\hat{s} \;\geq\; \frac{1}{C} \quad \text{for } \hat{s}'' \in \left[\frac{3}{4}, 3\right].$$

Hence also in this range the first term in (2.69) dominates all other terms:

$$\phi_1^* \;\approx\; -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\,\frac{d\rho_0}{d\hat{s}'}\,d\hat{s} \quad \text{uniformly in } \hat{s}'' \in \left[\frac{3}{4}, 3\right] \quad \text{for } S_0 \gg 1\,, \tag{2.77}$$

and we can conclude that in the range $s'' \in \left[\frac{3}{4}S_0, 3S_0\right]$ we have

$$\phi_1^* \approx -\frac{1}{S_0}\frac{1}{C} < 0 \;\text{ for }\; S_0 \;\gg 1\,. \tag{2.78}$$

Note that the above discussion on supports also yields that ϕ_1 is supported in $\hat{s}'' \in \left[\frac{1}{2}, \infty\right)$.

<u>The range of small s'':</u>

$$s'' \;\in\; \left(\frac{1}{2}S_0, \frac{3}{4}S_0\right) \quad \text{or equivalently} \quad \hat{s}'' \;\in\; \left(\frac{1}{2}, \frac{3}{4}\right). \tag{2.79}$$

We'd like ϕ_1 to be strictly negative in this range for $S_0 \gg 1$. Here, we encounter the second difficulty: No matter how large $\lambda = s'$ in (2.52) is, the behavior of $\phi_{s'}$ near the left edge $-\frac{1}{2}$ of its support $\left[-\frac{1}{2}, 0\right]$ (and also at its right edge 0, but there we don't care) is dominated by the $\frac{1}{\lambda^5}\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}$-term and thus automatically is *strictly positive*. Taking the $\rho(s')\,ds'$-average of the shifted $\phi_{s'}(s''-s')$ does not alter this behavior as long as ρ is non-negative in $[S_0, \infty)$, c. f. (2.66): ϕ_1^* is strictly positive near the left edge $\frac{S_0}{2}$ of its support. The way out for this problem is to give *give up smoothness* of $\hat{w}_0$ near the left $-\frac{1}{2}$ of its support $\left[-\frac{1}{2}, 0\right]$. In fact, we shall first assume that $\hat{w}_0$ satisfies in addition

$$\hat{w}_0 \;=\; \frac{1}{2}\left(\hat{s}+\frac{1}{2}\right)^2 \quad \text{for } \hat{s} \in \left[-\frac{1}{2}, -\frac{1}{4}\right]. \tag{2.80}$$

This means that $\hat{w}_0$ has a bounded but discontinuous second derivative (it is in $H^{2,\infty}$). This is the main reason why we expressed ϕ_1^* only in terms of up to second derivatives of $\hat{w}_0$. We argue that the so defined ϕ_1 is, as desired, strictly negative on $s'' \in (\frac{1}{2}S_0, \frac{3}{4}S_0]$ for all S_0. Indeed, in view of (2.60) and (2.59), the assumption (2.80) implies

$$\phi_{s'} \;=\; -\frac{1}{(s')^2}\left(\frac{s}{s'}+\frac{1}{2}\right)^3 - \frac{1}{2}\frac{1}{(s')^3}\left(\frac{s}{s'}+\frac{1}{2}\right)^2 \;<\; 0 \quad \text{for } s \in \left(-\frac{s'}{2}, -\frac{s'}{4}\right]. \tag{2.81}$$

In view of (2.66) & (2.67), $\rho \geq 0$ is strictly positive for $s' \in (S_0, \infty)$. On the other hand, it follows from (2.81) that $s' \mapsto \phi_{s'}(s''-s')$ is strictly negative for $s''-s' \in (-\frac{s'}{2}, -\frac{s'}{4}]$, that is, for $s' \in [\frac{4}{3}s'', 2s'')$ (and supported in $s' \in [s'', 2s''])$. Hence, by (2.47), ϕ_1^* is strictly negative for $S_0 \in [\frac{4}{3}s'', 2s'')$, that is, for $s'' \in (\frac{1}{2}S_0, \frac{3}{4}S_0]$, for any value of $S_0 > 0$. Now we approximate $\hat{w}_0$ with a sequence of functions $\hat{w}_0^\nu$ in $H^{2,2}$ and we call $\phi_1^{*,\nu}$ the associated multiplier. Since ϕ_1^* involves $\hat{w}_0$ only up to second derivatives (c. f. (2.69)) then $\phi_1^{*,\nu}$ converge locally and uniformly in $\hat{s}''$ to ϕ_1^*. We conclude that for $s'' \in \left(\frac{1}{2}S_0, \frac{3}{4}S_0\right)$

$$\phi_1^* < 0 \;\text{ for }\; S_0 \gg 1\,. \tag{2.82}$$

Finally we fix a sufficiently large but universal S_0, so that (2.82) together with (2.75) and (2.78) imply (2.71).

In order to conclude the proof we need to make the measure ρ, defined in (2.66) & (2.67), decay in the region $\frac{S_1}{2} \leq s' \leq S_1$. In this way the multiplier

$$\phi_1(s'') := \int_{-\infty}^{\infty} \phi_{s'}(s''-s')\rho(s')\rho\left(\frac{s'}{S_1}\right)\,ds'\,, \tag{2.83}$$

where η is a smooth cut-off equal to one for $s' \leq \frac{S_1}{2}$ and equal to zero for $s' \geq S_1$, is admissible. It is easy to see that $\phi_1(s'')$ satisfies

$$\phi_1(s'') \lesssim \frac{1}{S_1} \tag{2.84}$$

in the region $\frac{S_1}{4} \leq s'' \leq S_1$. Putting together (2.71) and (2.84) we obtain (2.48), after shifting. □

2.2.2 Original stability condition: main theorem

In this section we go back to the original stability condition (2.34) and we establish the following result

Theorem 1. *Assume that ξ satisfy $\int_0^H \xi\,dz = -1$ and*

$$\left\langle \int_0^H \left(2\xi\, w\,\theta \,+\, |\nabla\theta|^2\right) dz \right\rangle \geq 0\,, \tag{2.85}$$

holds for all (θ, w) that satisfy

$$\begin{cases} \Delta^2 w &= -\Delta_{x'}\theta & \text{for } 0<z<H\,, \\ w = \frac{dw}{dz} &= \theta = 0 & \text{for } z = 0, H\,. \end{cases} \tag{2.86}$$

Then $\int_0^H \xi^2\,dz \gtrsim (\ln H)^{\frac{1}{15}}$. In particular

$$\widetilde{\mathrm{Nu}} \gtrsim (\ln H)^{\frac{1}{15}}\,, \tag{2.87}$$

where $\widetilde{\mathrm{Nu}}$ is defined in (2.9).

This result has two direct consequences: On the one hand if we combine (2.87) with the upper bound $\widetilde{\mathrm{Nu}} \lesssim (\ln H)^{\frac{1}{15}}$ (c.f. Otto & Seis [9]) we obtain

$$\widetilde{\mathrm{Nu}} \sim (\ln H)^{\frac{1}{15}} .$$

On the other hand, (2.87) together with the upper bound $\mathrm{Nu} \lesssim (\ln\ln H)^{\frac{1}{3}}$ (Otto & Seis [9]), implies

$$\mathrm{Nu} \lesssim (\ln\ln H)^{\frac{1}{3}} \ll (\ln H)^{\frac{1}{15}} \lesssim \widetilde{\mathrm{Nu}} ,$$

which tells us that the Nusselt number associated to the background field method, $\widetilde{\mathrm{Nu}}$, does not carry a physical meaning.

2.2.3 Characterization of stable profiles

In this section we enunciate the lemmas that will be the main ingredients for the proof of Theorem 1. Recalling that by (2.42) $\hat{\xi} = z\xi$ and $s = \ln z$, we state the first lemma:

Lemma 1.
There exists a ϕ_0, which will play the role of a convolution kernel, with the properties

$$\phi_0(s) \geq 0, \qquad \int_{-\infty}^{\infty} \phi_0(s)\, ds = 1, \qquad \operatorname{supp} \phi_0(z) \subset \left(\frac{1}{4}, \frac{3}{4}\right) , \tag{2.88}$$

such that, for all $s' \leq \ln H$

$$\int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi_0(s)\, ds \geq -C \exp(-3s') , \tag{2.89}$$

and C denotes a universal constant.

This lemma tells us that under the stability condition (2.34) we can infer that $\hat{\xi} = z\xi$ is nearly positive on average in the logarithmic variable $s = \ln z$ for $1 \ll s \leq \ln H$. The right-hand side term of (2.89) estimates the deviation from the average positivity in the bulk. Much of the effort of this construction will consist in designing the kernel in such a way that it is both non-negative and compactly supported. Non-negativity of $\phi_0(s)$ and its fast decay for $s \downarrow -\infty$ and support in $s \leq 0$ will be crucial in Subsection 2.2.5.2, where we will work with the convolution (2.90). Let us define the convolution

$$\hat{\xi}_0(s') := \int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi_0(s)\, ds . \tag{2.90}$$

that, in virtue of Lemma 1, is approximately positive in the bulk. In Section 2.2.1 we proved that if a profile τ satisfies the reduced stability condition (2.36) then it grows logarithmically in the bulk. When ξ satisfies the weaker stability condition (2.34) we want to prove an analogous result for $\hat{\xi}_0$ defined in (2.90).

Lemma 2. *Let $\tau : (0, H) \to \mathbb{R}$ satisfy the original stability condition (2.34) and consider $\hat{\xi}_0$ defined in (2.90). Then for $S_1 \geq C$ we have*

$$\int_{-1}^{0} \hat{\xi}_0\, ds \leq C \left(\frac{1}{S_1} \int_{0}^{S_1} \hat{\xi}_0\, ds + 1 \right) , \tag{2.91}$$

where C denotes a (possibly large) generic universal constant.

The approximate non-negativity of $\hat{\xi}_0$ is lost in the boundary layer $s \ll -1$. However, we can show that $\hat{\xi}_0$ cannot be too negative in the boundary layer *provided $\hat{\xi}_0$ is sufficiently small in the transition region $|s| \lesssim 1$.*

Lemma 3. *Let $\tau : (0, H) \to \mathbb{R}$ satisfy the original stability condition (2.34) and consider $\hat{\xi}_0$ defined in (2.90). Then for all $S_2 \geq C$ and $\varepsilon \leq 1$ there exists a specific universal constant C_2 (with $C_2 < C$), such that we have*

$$\int_{-S_2}^{-1} \hat{\xi}_0 \, ds \geq -C_2 \left(\frac{1}{\varepsilon} \int_{-1}^{0} \hat{\xi}_0 \, ds \right) - C \left(\frac{1}{\varepsilon} + \int_{-S_2}^{-S_2+1} |\hat{\xi}_0| \, ds + \varepsilon \exp(5S_2) \right), \tag{2.92}$$

where C denotes a (possibly large) generic universal constant.

2.2.4 Proof of the main theorem

Let us recall that by assumption we have

$$\int_0^H \xi(z) dz = \tau(H) - \tau(0) = -1 \tag{2.93}$$

(see Subsection 3.2.6 for notations). Without loss of generality we can assume

$$\int_0^H \xi^2 dz \lesssim \widetilde{\mathrm{Nu}} \lesssim (\ln H)^{\frac{1}{15}}, \tag{2.94}$$

otherwise there is nothing to show. The proof of Theorem 1 consists of two steps:
Step1:
We claim that $\int_0^H \xi(z) dz = -1$ translates (by the up-down symmetry) into

$$\int_{-\infty}^{\ln H} \hat{\xi}_0 \, ds' \lesssim -\frac{1}{2}. \tag{2.95}$$

Argument for (2.95):
Let us introduce the change of variable

$$z = \frac{\hat{z}}{k}, \tag{2.96}$$

where $k > 0$, and the logarithmic variables

$$s \; = \; \ln \hat{z} \quad \text{and} \quad s' \; = \; -\ln k \,. \tag{2.97}$$

We note that by definition (2.90) of the convolution $\hat{\xi}_0$, by definitions (2.96) & (2.97) of the variables $\hat{z}$, s and s', by definition (2.42) of $\hat{\xi}$ we have

$$\begin{aligned}
\int_{-\infty}^{\ln H} \hat{\xi}_0 \, ds' \quad & \overset{(2.90)}{=} \quad \int_{-\infty}^{\ln H} \int_{-\infty}^{\infty} \hat{\xi}(s + s') \phi_0(s) \, ds \, ds' \\
& \overset{(2.96)\&(2.97)}{=} \quad \int_{\frac{1}{H}}^{\infty} \int_0^1 \hat{\xi} \left(\frac{\hat{z}}{k} \right) \phi_0(\hat{z}) \, \frac{d\hat{z}}{\hat{z}} \, \frac{dk}{k} \\
& \overset{(2.42)}{=} \quad \int_0^1 \int_{\frac{1}{H}}^{\infty} \xi \left(\frac{\hat{z}}{k} \right) \frac{dk}{k^2} \, \phi_0(\hat{z}) \, d\hat{z} \\
& = \quad \int_0^1 \int_0^{H\hat{z}} \xi(z) \, \frac{dz}{\hat{z}} \, \phi_0(\hat{z}) \, d\hat{z} \\
& = \quad \int_0^H \xi(z) \int_{\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} \, dz \,.
\end{aligned}$$

In view of this identity and the up-down symmetry (i. e. the symmetry of the problem under $z \rightsquigarrow H - z$), (2.95) will follow if we show that $\int_0^H \xi(z) \, dz = -1$ implies

$$\int_0^H \xi(z) \left(\int_{\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} + \int_{1-\frac{z}{H}}^1 \frac{1}{\hat{z}} \phi_0(\hat{z}) \, d\hat{z} \right) dz \lesssim -1, \tag{2.98}$$

using the normalization

$$\int_0^1 \frac{1}{\hat{z}}\phi_0(\hat{z})\,d\hat{z} = \int_{-\infty}^0 \phi_0(s)\,ds = 1\,. \tag{2.99}$$

As a further consequence of the up-down symmetry, we may assume that $\phi_0(\hat{z})$ is even w. r. t. $\hat{z} = \frac{1}{2}$, that is,

$$\phi_0(1-\hat{z}) \;=\; \phi_0(\hat{z})\,. \tag{2.100}$$

Indeed, in order to have this symmetry for ϕ_0, the distribution of w under the law $\rho(dw)$ from Subsection 2.2.5.1 has to be invariant under this symmetry transformation so that also the distribution ϕ_0 of $\phi = w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w$ under $\rho(dw)$ (c. f. (2.129)) is invariant under this symmetry transformation. In view of the Ansatz (2.131), this follows from the fact that $\hat{w}_0(\hat{\hat{z}})$ is even w. r. t. $\hat{\hat{z}} = 0$ and that $\rho_0(\hat{z}')$ is even w. r. t. $\hat{z}' = \frac{1}{2}$, c. f. (2.139).

<u>Argument for (2.98):</u>
<u>We argue that</u>

$$\int_0^H \xi(z)\left(1 - \int_{\frac{z}{H}}^1 \frac{1}{\hat{z}}\phi_0(\hat{z})\,d\hat{z} - \int_{1-\frac{z}{H}} \frac{1}{\hat{z}}\phi_0(\hat{z})\,d\hat{z}\right)dz \gtrsim -\frac{\widetilde{\mathrm{Nu}}^{\frac{1}{3}}}{H^{\frac{2}{3}}}\,. \tag{2.101}$$

Indeed, Claim (2.101) together with (2.93) and (2.94) imply (2.98) in the regime of $H \gg 1$.
<u>Argument for (2.101):</u>
<u>Let us reformulate (2.101) as</u>

$$\int_0^H \xi(z)\rho(z)dz \gtrsim -\frac{\widetilde{\mathrm{Nu}}^{\frac{1}{3}}}{H^{\frac{2}{3}}}\,, \tag{2.102}$$

where we introduced

$$\rho(z) := \rho_0\left(\frac{z}{H}\right) \quad\text{with}\quad \rho_0\left(\hat{z}\right) := 1 - \int_{\hat{z}}^1 \frac{1}{\hat{z}'}\phi_0(\hat{z}')\,d\hat{z}' - \int_{1-\hat{z}}^1 \frac{1}{\hat{z}'}\phi_0(\hat{z}')\,d\hat{z}'\,. \tag{2.103}$$

We notice that in view of the normalization (2.99), since $\phi_0(\hat{z})$ is compactly supported in $(0,1)$, so is $\rho_0(\hat{z})$. Moreover, the symmetry (2.100) of $\phi_0(\hat{z})$ implies that

$$\frac{d\rho_0}{d\hat{z}}(\hat{z}) \;=\; \left(\frac{1}{\hat{z}} - \frac{1}{1-\hat{z}}\right)\phi_0(\hat{z}) \begin{cases} \geq 0 & \text{for } \hat{z} \leq \frac{1}{2}\,, \\ \leq 0 & \text{for } \hat{z} \geq \frac{1}{2}\,, \end{cases}$$

so that

$$\rho_0(\hat{z}) \geq 0\,, \tag{2.104}$$

and

$$\rho_0(\hat{z}) \leq 1\,, \tag{2.105}$$

following from $\phi_0(\hat{z}) \geq 0$. Hence (2.102) is yet another way of expressing approximate non-negativity of ξ, this time far away from the boundary layers. The idea to establish (2.101) is now to construct an *even mollification kernel* $\phi(z)$ of length scale $\ell \ll H$ such that

-
$$(\xi * \phi)(z) \gtrsim -\frac{1}{\ell^4} \text{ for } z \in (\ell, H-\ell), \tag{2.106}$$

-
$$\int_0^H (\phi * \rho - \rho)^2 dz \lesssim \frac{\ell^4}{H^3} \text{ for } \ell \ll H. \tag{2.107}$$

We first argue how (2.106) and (2.107) imply (2.101). Indeed by the evenness of ϕ we have the representation

$$\int_{-\infty}^{\infty} \xi\rho dz = \int_{-\infty}^{\infty} \xi * \phi\rho dz - \int_{-\infty}^{\infty} \xi(\rho * \phi - \rho)\, dz\,,$$

from which, since $\rho \geq 0$ (from (2.104)), we get

$$\int_{-\infty}^{\infty} \xi\rho dz \geq \inf_{z\in\mathrm{supp}\rho}(\xi * \phi)(z) \int_{-\infty}^{\infty} \rho dz - \left(\int_{-\infty}^{\infty} \xi^2 dz \int_{-\infty}^{\infty} (\rho * \phi - \rho)^2 dz\right)^{\frac{1}{2}}.$$

Using (2.106) and (2.107) together with (2.94) and $\int_0^H \rho\, dz \lesssim H$ (from (2.105)) we obtain the estimate

$$\int_{-\infty}^{\infty} \xi\rho dz \gtrsim -\frac{H}{\ell^4} - \left(\widetilde{\mathrm{Nu}}\frac{\ell^4}{H^3}\right)^{\frac{1}{2}}.$$

The balancing choice of $\ell = \frac{H^{\frac{5}{12}}}{\widetilde{\mathrm{Nu}}^{\frac{1}{12}}}$ turns this estimate into (2.101). We now turn to the construction of the *mollification kernel* ϕ. We select a (nonvanishing) smooth and even $w_0(\hat{z})$ compactly supported in $\hat{z} \in [-1,1]$ and consider the corresponding multiplier

$$\phi_0 = w_0\left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w_0\,.$$

Notice that $\int_0^H \phi_0(z) d\hat{z} = \int_{-\infty}^{\infty} \left(\left(\frac{d^2 w_0}{d\hat{z}^2}\right)^2 + \left(\frac{dw_0}{d\hat{z}}\right)^2 + w_0^2\right) d\hat{z} > 0\,$. By multiplying w_0 with a positive constant, we thus may assume that

$$\int_{-\infty}^{\infty} \phi_0\, d\hat{z} = 1.$$

We rescale the mask ϕ_0 by ℓ so as to preserve its integral

$$\ell\phi(\ell\hat{z}) = \phi_0(\hat{z})\,, \tag{2.108}$$

and note that

$$\phi = w\left(-\frac{d^2}{d\hat{z}^2} + \frac{1}{\ell^2}\right)^2 w\,, \tag{2.109}$$

provided w is the following rescaling of w_0:

$$\frac{1}{\ell^{\frac{3}{2}}} w(\ell\hat{z}) = w_0(\hat{z})\,. \tag{2.110}$$

For any translation $z' \in (\ell, H-\ell)$, the translated test function $z \mapsto w(z-z')$ is compactly supported in $z \in (0,H)$ and thus we may apply the stability condition (2.34) with $k = \frac{1}{\ell}$. Because of (2.109), this yields (2.106):

$$\begin{aligned}
&\int_0^H \xi(z)\, \phi(z-z')\, dz \\
&\quad \geq \quad -\int_0^H \left[\ell^2\left(\frac{d}{dz}\left(-\frac{d^2}{dz^2} + \frac{1}{\ell^2}\right)^2 w\right)^2 + \left(\left(-\frac{d^2}{dz^2} + \frac{1}{\ell^2}\right)^2 w\right)^2\right](z-z')\, dz \\
&\quad \overset{(2.110)}{=} \quad -\frac{1}{\ell^4}\int_{-1}^1 \left[\left(\frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w_0\right)^2 + \left(\left(-\frac{d^2}{d\hat{z}^2} + 1\right)^2 w_0\right)^2\right] d\hat{z} \sim -\frac{1}{\ell^4}\,.
\end{aligned}$$

We now turn to (2.107). From the representation

$$
\begin{aligned}
(\rho * \phi - \rho)(z') &\overset{(2.108)}{=} \int_{-\infty}^{\infty} (\rho(z'-z) - \rho(z'))\,\phi(z)\,dz \\
&\overset{\phi \text{ is even}}{=} \frac{1}{2}\int_{-\infty}^{\infty} (\rho(z'+z) + \rho(z'-z) - 2\rho(z'))\,\phi(z)\,dz\,,
\end{aligned}
$$

we obtain the inequality

$$
\begin{aligned}
|(\rho * \phi - \rho)(z')| &\leq \frac{1}{2}\sup\left|\frac{d^2\rho}{dz^2}\right| \int_{-\infty}^{\infty} z^2|\phi(z)|\,dz \\
&\overset{(2.103),(2.108)}{=} \frac{1}{H^2}\sup\left|\frac{d^2\rho_0}{d\hat{z}^2}\right| \ell^2 \int_{-\infty}^{\infty} \hat{z}^2|\phi_0(\hat{z})|\,d\hat{z}, \,,
\end{aligned}
$$

that yields (2.107) after integration in $z' \in [0, H]$.
Step2:
We start by noticing that

$$
\widetilde{\mathrm{Nu}} \overset{(2.94)}{\gtrsim} \int_0^H \xi^2 dz \overset{(2.42)}{=} \int_{-\infty}^{\ln H} \exp(-s)\hat{\xi}^2\,ds \overset{(2.90)}{\gtrsim} \int_{-\infty}^{\ln H} \exp(-s)\hat{\xi}_0^2\,ds\,, \tag{2.111}
$$

where in the last inequality we used the (weighted) Young's inequality for convolution and the property (2.88) of the convolution kernel ϕ_0. We claim that

$$
\int_{-\infty}^{\ln H} \exp(-s)\hat{\xi}_0^2 ds \geq \ln^{\frac{1}{15}} H\,. \tag{2.112}
$$

<u>Argument for (2.112)</u>: By (2.42) and the rescaling $z = L\hat{z}$, $z\xi = L^{-3}(\hat{z}\hat{\xi})$ we can write

$$
s = S_0 + \hat{s} \qquad\qquad \text{and} \qquad\qquad \hat{\xi} = \exp(-3S_0)\hat{\hat{\xi}}\,, \tag{2.113}
$$

where $\hat{s} = \ln\hat{z}$ and, $S_0 = \ln L$ and $\hat{\hat{\xi}} = \hat{z}\hat{\xi}$. In particular

$$
\hat{\xi}_0 = \exp(-3S_0)\hat{\hat{\xi}}_0\,.
$$

Let us rewrite the approximate logarithmic growth (2.91) in the new variables $\hat{s}$ and $\hat{\hat{\xi}}$: For $S_1 \geq C$ we have

$$
\int_{-1}^{0} \hat{\hat{\xi}}_0\,d\hat{s} \leq C_1\left(\frac{1}{S_1}\int_0^{S_1} \hat{\hat{\xi}}_0 d\hat{s} + 1\right)\,. \tag{2.114}
$$

By the definition (2.113) we can generalize (2.114) for all S_0

$$
\int_{S_0-1}^{S_0} \exp(3S_0)\hat{\xi}_0\,ds \;\leq\; C_1\left(\frac{1}{S_1}\int_{S_0}^{S_0+S_1} \exp(3S_0)\hat{\xi}_0\,ds + 1\right)\,. \tag{2.115}
$$

Dividing the above inequality by $\exp(3S_0)$ we obtain

$$
\int_{S_0-1}^{S_0} \hat{\xi}_0\,ds \;\leq\; C_1\left(\frac{1}{S_1}\int_{S_0}^{S_0+S_1} \hat{\xi}_0\,ds + \exp(-3S_0)\right)\,, \tag{2.116}
$$

and turning it around we have, for $S_1 \geq C$ and for all S_0

$$
\int_{S_0}^{S_0+S_1} \hat{\xi}_0\,ds \;\geq\; \frac{S_1}{C_1}\int_{S_0-1}^{S_0} \hat{\xi}_0\,ds - S_1\exp(-3S_0)\,. \tag{2.117}
$$

We now apply a similar argument to Lemma 3. Like above we obtain from (2.92): For $S_2 \geq C$, $\varepsilon \leq 1$ and all S_0 we have

$$\begin{aligned}\int_{S_0-S_2}^{S_0-1} \hat{\xi}_0\, ds \;&\geq\; -C_2 \left(\frac{1}{\varepsilon}\int_{S_0-1}^{S_0} \hat{\xi}_0\, ds\right) \\ &- \; C\left(\frac{1}{\varepsilon}\exp(-3S_0) + \int_{S_0-S_2}^{S_0-S_2+1} |\hat{\xi}_0|\, ds + \varepsilon \exp(5S_2 - 3S_0)\right) .\end{aligned} \tag{2.118}$$

Above we choose $\varepsilon = \frac{C_1 C_2}{S_1}$, where C_1 and C_2 are the universal constants appearing in (2.117) and (2.118). Therefore, for $S_2 \geq C$ and for all S_0 we have

$$\begin{aligned}\int_{S_0-S_2}^{S_0-1} \hat{\xi}_0\, ds \;&\geq\; -\frac{S_1}{C_1}\int_{S_0-1}^{S_0} \hat{\xi}_0\, ds \\ &- \; C\left(\frac{S_1}{C_1C_2}\exp(-3S_0) + \int_{S_0-S_2}^{S_0-S_2+1} |\hat{\xi}_0|\, ds + \frac{C_1C_2}{S_1}\exp(5S_2 - 3S_0)\right) .\end{aligned}$$

We now integrate $\hat{\xi}_0$ between $S_0 - S_2$ and $S_0 + S_1$

$$\int_{S_0-S_2}^{S_0+S_1} \hat{\xi}_0\, ds = \int_{S_0-S_2}^{S_0} \hat{\xi}_0\, ds + \int_{S_0-1}^{S_0} \hat{\xi}_0\, ds + \int_{S_0}^{S_0+S_1} \hat{\xi}_0\, ds$$

and apply (2.118), (2.89) and (2.117), respectively to the three terms of the right hand side, obtaining:

$$\begin{aligned}\int_{S_0-S_2}^{S_0+S_1} \hat{\xi}_0\, ds \;&\geq\; -\frac{S_1}{C_1}\int_{S_0-1}^{S_0} \hat{\xi}_0\, ds - \frac{CS_1}{C_1C_2}\exp(-3S_0) \\ &- \; C\int_{S_0-S_2}^{S_0-S_2+1} |\hat{\xi}_0|\, ds - \frac{CC_1C_2}{S_1}\exp(5S_2 - 3S_0) \\ &- \; C\int_{S_0-1}^{S_0} \exp(-3s)\, ds + \frac{S_1}{C_1}\int_{S_0-1}^{S_0} \hat{\xi}_0\, ds - S_1\exp(-3S_0) \\ &\geq\; -C\left(S_1\exp(-3S_0) + \int_{S_0-S_2}^{S_0-S_2+1} |\hat{\xi}_0|\, ds + \frac{1}{S_1}\exp(5S_2 - 3S_0)\right) \\ &- \; C\exp(-3S_0) - S_1\exp(-3S_0) \\ &\geq\; -C\left(S_1\exp(-3S_0) + \int_{S_0-S_2}^{S_0-S_2+1} |\hat{\xi}_0|\, ds + \frac{1}{S_1}\exp(5S_2 - 3S_0)\right) ,\end{aligned}$$

which we rewrite with $S_- := S_0 - S_2$ and $S_+ := S_0 + S_1$ as

$$\int_{S_-}^{S_+} \hat{\xi}_0\, ds \;\geq\; -C\left((S_+ - S_0)\exp(-3S_0) + \int_{S_-}^{S_-+1} |\hat{\xi}_0|\, ds + \frac{1}{(S_+ - S_0)}\exp(2S_0 - 5S_-)\right) . \tag{2.119}$$

In the regime $S_- \ll S_+$ we can choose S_0 in order to balance the first and the last right hand side term in (2.119), which is achieved for $S_+ - S_0 = \exp(\frac{5}{2}(S_0 - S_-))$. Thus in particular $S_+ - S_0 \approx S_+ - S_- \gg S_0 - S_-$ so that the balance is achieved for $S_+ - S_- \approx \exp(\frac{5}{2}(S_0 - S_-))$. Hence (2.119) turns into

$$\int_{S_-}^{S_+} \hat{\xi}_0\, ds \geq -C\left((S_+ - S_-)^{-\frac{1}{5}}\exp(-3S_-) + \int_{S_-}^{S_-+1} |\hat{\xi}_0|\, ds\right) , \tag{2.120}$$

and in case of $|S_-| \ll S_+$ it simplifies to

$$\int_{S_-}^{S_+} \hat{\xi}_0\, ds \geq -C\left((S_+)^{-\frac{1}{5}}\exp(-3S_-) + \int_{S_-}^{S_-+1} |\hat{\xi}_0|\, ds\right) . \tag{2.121}$$

In (2.121) we choose $S_+ = \ln H$

$$\int_{S_-}^{\ln H} \hat{\xi}_0 \, ds \geq -C \left((\ln H)^{-\frac{1}{5}} \exp(-3S_-) + \int_{S_-}^{S_-+1} |\hat{\xi}_0| \, ds \right) , \tag{2.122}$$

and we combine this last inequality with (2.95)

$$\begin{aligned}
1 & \overset{(2.95)}{\lesssim} & - \int_{-\infty}^{\ln H} \hat{\xi}_0(s) \, ds \\
& = & - \int_{-\infty}^{S_-} \hat{\xi}_0(s) \, ds - \int_{S_-}^{\ln H} \hat{\xi}_0(s) \, ds \\
& \overset{(2.122)}{\leq} & \left(\int_{-\infty}^{S_-} \exp(-s) \hat{\xi}_0^2(s) ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{S_-} \exp(s) \, ds \right)^{\frac{1}{2}} \\
& +C & \left(2(\ln H)^{-\frac{1}{5}} \exp(-3S_-) \right) + C \left(\int_{S_-}^{S_-+1} \exp(-s) \hat{\xi}_0^2 ds \right)^{\frac{1}{2}} \left(\int_{S_-}^{S_-+1} \exp(s) \, ds \right)^{\frac{1}{2}} ,
\end{aligned}$$

where we applied twice the Cauchy-Schwarz' inequality in the form

$$\int \hat{\xi}_0 ds = \int \exp\left(-\frac{s}{2}\right) \exp\left(\frac{s}{2}\right) \hat{\xi}_0 \, ds \leq \left(\int \left(\exp\left(-\frac{s}{2}\right) \hat{\xi}_0 \right)^2 ds \right)^{\frac{1}{2}} \left(\int \exp\left(\frac{s}{2}\right)^2 ds \right)^{\frac{1}{2}} . \tag{2.123}$$

Using (2.112) in the previous estimate we obtain

$$\begin{aligned}
1 & \lesssim & \left(\widetilde{\mathrm{Nu}} \exp(S_-) \right)^{\frac{1}{2}} + C \left((\ln H)^{-\frac{1}{5}} \exp(-3S_-) \right) + C \left(\widetilde{\mathrm{Nu}} \exp(S_-) \right)^{\frac{1}{2}} \\
& \leq & C \left((\widetilde{\mathrm{Nu}} \exp(S_-))^{\frac{1}{2}} + (\ln H)^{-\frac{1}{5}} \exp(-3S_-) \right) .
\end{aligned} \tag{2.124}$$

Balancing the two terms on the right-hand side of (2.124), we get the (optimal) condition on S_-:

$$(\widetilde{\mathrm{Nu}})^{\frac{3}{7}} (\ln H)^{\frac{6}{35}} = \exp(-3S_-).$$

This allows us to conclude that

$$1 \lesssim \left(\widetilde{\mathrm{Nu}} (\ln H)^{-\frac{1}{15}} \right)^{\frac{3}{7}} ,$$

$$1 \lesssim \left(\widetilde{\mathrm{Nu}} (\ln H)^{-\frac{1}{15}} \right)^{\frac{3}{7}} ,$$

which yields

$$\widetilde{\mathrm{Nu}} \gtrsim (\ln H)^{\frac{1}{15}}$$

where, with the symbol $\gtrsim$ we denote the inequality $\geq$ up to universal constants.

2.2.5 Proofs of technical lemmas

In this section we will give the detailed proofs of the lemmas stated in Subsection 2.2.3.

2.2.5.1 Approximate positivity in the bulk: Proof of Lemma 1

We are going to show that even under the weaker condition (2.34), $\xi = \frac{d\tau}{dz}$ is "positive on average approximately in the bulk" (i. e. for $1 \ll z$ and $1 \ll H - z$). As in Section 2.2.1, we will express this in terms of $\hat{\xi} = z\xi$ and argue that $\hat{\xi}$ is not too negative on average in the logarithmic variable $s = \ln z$ for $1 \ll s \leq \ln H$. The average is expressed in terms of a smooth "convolution kernel"

$\phi_0(s)$ with the properties (2.88), c. f. Subsection 2.2.3. As stated in Lemma 1 we will construct such a ϕ_0 so that for all $s' \leq \ln H$

$$\int_{-\infty}^{\infty} \hat{\xi}(s+s')\phi_0(s)ds \;\geq\; -C\exp(-3s')\,, \tag{2.125}$$

where C denotes a universal constant and the r. h. s. term estimates in which sense we have approximate positivity in the bulk. In order to infer non-negativity, we can no longer let $k \uparrow \infty$ in (2.34) (as in the proof of Proposition 1), since the two last terms would blow up. To quantify this qualitative observation, we restrict to $k > 0$ and recall the change of variable (2.96) so that (2.34) turns into

$$\int_0^{kH} \hat{\xi}\left(\frac{\hat{z}}{k}\right) w \left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \bar{w}\,\frac{d\hat{z}}{\hat{z}} \\ + k^3 \int_0^{kH} \left|\frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right|^2 d\hat{z} + k^3 \int_0^{kH} \left|\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right|^2 d\hat{z} \;\geq\; 0\,. \tag{2.126}$$

We shall restrict ourselves to k with $kH \geq 1$ and real functions $w(\hat{z})$ compactly supported in $\hat{z} \in (0,1]$ so that the boundary conditions (2.33) are automatically satisfied. In particular, an integration by parts (based on $(\frac{d^2}{dz^2}+1)^4 = \frac{d^8}{dz^8} - 4\frac{d^6}{dz^6} + 6\frac{d^4}{dz^4} - 4\frac{d^2}{dz^2} + 1$) in the two last terms of (2.126), yielding

$$\int_0^{kH} \left|\frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right|^2 d\hat{z} + \int_0^{kH} \left|\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right|^2 d\hat{z} \\ = \int_0^{\infty} \left[\left(\frac{d^5 w}{d\hat{z}^5}\right)^2 + 5\left(\frac{d^4 w}{d\hat{z}^4}\right)^2 + 10\left(\frac{d^3 w}{d\hat{z}^3}\right)^2 + 10\left(\frac{d^2 w}{d\hat{z}^2}\right)^2 + 5\left(\frac{dw}{d\hat{z}}\right)^2 + w^2\right] d\hat{z}\,, \tag{2.127}$$

shows that there are no fortuitous cancellations: Provided the multiplier $\phi := w(-\frac{d^2}{d\hat{z}^2}+1)^2 w$ of $\hat{\xi}$ is non-negligible in the sense of $\int_0^\infty \phi\,\frac{d\hat{z}}{\hat{z}} = O(1)$, the two last terms of (2.126) are at least of $O(k^3)$. Hence we are forced to work with $k \ll 1$ and thus, as expected, with $z = \frac{\hat{z}}{k} \gg 1$. With the logarithmic variables (2.97), the first term in (2.126) can be rewritten as follows

$$\int_0^{kH} \hat{\xi}\left(\frac{\hat{z}}{k}\right) w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \bar{w}\,\frac{d\hat{z}}{\hat{z}} = \int_{-\infty}^{\infty} \hat{\xi}(s+s') w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\,ds\,.$$

In view of this and (2.127), the stability condition (2.126) turns into: For all $s' \leq \ln H$ we have

$$\int_{-\infty}^{\infty} \hat{\xi}(s+s') w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\,ds \geq -\exp(-3s') \int_0^1 \left[\left(\frac{d^5 w}{d\hat{z}^5}\right)^2 + \cdots + w^2\right] d\hat{z}\,. \tag{2.128}$$

Let us consider the left hand side in (2.128) in more detail. In order to derive a result of the type of (2.125), it would be convenient to have a smooth *compactly supported* w such that the multiplier $\phi = w(-\frac{d^2}{d\hat{z}^2}+1)^2 w$ is *non-negative*. Although we don't have an argument, we believe that such a w does not exist.
Instead, we will construct

- a family $\mathfrak{F}$ of smooth functions w supported in $\hat{z} \in (0,1]$
- and a probability measure $\rho(dw)$ on $\mathfrak{F}$,

such that the convex combination

$$\phi_0(\hat{z}) \;:=\; \int_{\mathfrak{F}} \phi(\hat{z})\,\rho(dw) \quad \text{where} \quad \phi \;:=\; w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\,, \tag{2.129}$$

is non-negative (and non-trivial) — and thus satisfies (2.88) after normalization and is supported in $[\frac{1}{4},\frac{3}{4}]$. Roughly speaking, the reason why this can be achieved is the following: For any (non-trivial) smooth, compactly supported w we have

– $\phi = w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w$ is positive on average:

$$\int_0^1 \phi\, d\hat{z} \;=\; \int_0^1 \left[\left(\frac{d^2w}{d\hat{z}^2}\right)^2 + 2\left(\frac{dw}{d\hat{z}}\right)^2 + w^2\right] d\hat{z}\,.$$

– $\phi = w\frac{d^4w}{dz^4} + \cdots + w^2$ is positive near the edge of the support of w (incidentally this would *not* be true for the positive *second* order operator $-\frac{d^2}{d\hat{z}^2}+1$).

Before becoming much more specific let us address the error term stemming from the right hand side of (2.128) for our construction, that is

$$\int_{\mathfrak{F}}\int_0^1 \left[\left(\frac{d^5w}{d\hat{z}^5}\right)^2 + \cdots + w^2\right] d\hat{z}\,\rho(dw)\,. \tag{2.130}$$

The functions in our family $\mathfrak{F}$ will be of the form

$$w_{\ell,\hat{z}'}(\hat{z}) \;:=\; \left(\sqrt{\ell}\right)^3 w_0\left(\frac{\hat{z}-\hat{z}'}{\ell}\right)\,, \tag{2.131}$$

that is, translations and rescalings of a "mask" w_0. The mask w_0 is some compactly supported smooth function that we fix now, say

$$w_0(\hat{\hat{z}}) \;:=\; \left\{\begin{array}{cc} \frac{1}{\sqrt{C_0}}\exp\left(-\frac{1}{(1-\hat{\hat{z}}^2)^2}\right) & \text{for} \quad \hat{\hat{z}}\in(-1,1) \\ 0 & \text{else} \end{array}\right\}\,, \tag{2.132}$$

and the normalization constant C_0 chosen such that

$$\int \left(\frac{dw_0}{d\hat{\hat{z}}^2}\right)^2 d\hat{\hat{z}} = 1\,, \tag{2.133}$$

Provided

$$\ell \;\le\; \frac{1}{4} \quad \text{and} \quad \hat{z}' \;\in\; \left(\frac{1}{4},\frac{3}{4}\right)\,, \tag{2.134}$$

then $w_{\ell,\hat{z}'}$ is, as desired, uniformly compactly supported in $\hat{z}\in(0,1]$. If we choose the length scale to be bounded away from zero, i. e.

$$\ell \;\ge\; \frac{1}{C}, \tag{2.135}$$

then the error term (2.130) is clearly finite, so that (2.89) follows from (2.128) via integration with respect to $\rho(dw)$ of (2.128).

It thus remains to construct a probability measure in ℓ and $\hat{z}'$ with (2.134) & (2.135) such that (2.129) is non-negative (and non-trivial). Note that $w_{\ell,\hat{z}'}$ in (2.131) is scaled such that the corresponding multipliers satisfy

$$\phi_{\ell,\hat{z}'}(\hat{z}) \;=\; \left(\frac{1}{\ell}w_0\frac{d^4}{d\hat{\hat{z}}^4}w_0 - 2\ell w_0\frac{d^2}{d\hat{\hat{z}}^2}w_0 + \ell^3 w_0^2\right)\left(\frac{\hat{z}-\hat{z}'}{\ell}\right)\,, \tag{2.136}$$

and w_0 is normalized in (2.133) in such a way that $\int_0^1 w_0 \frac{d^4}{d\hat{z}^4} w_0 \, d\hat{z} = 1$. Hence for all $\hat{z}' \in (\frac{1}{4}, \frac{3}{4})$ we have the convergence as $\ell \downarrow 0$

$$\phi_{\ell,\hat{z}'}(\hat{z}) \rightharpoonup \delta(\hat{z}' - \hat{z}) \quad \text{when tested against smooth functions of } \hat{z}' . \tag{2.137}$$

On the other hand we note the following: Writing $w_0 = \exp(I)$ with $I = -\frac{1}{(1-\hat{z}^2)^2}$, we have

$$\begin{aligned}
\frac{d^2 w_0}{d\hat{z}^2} &= \frac{1}{\sqrt{C_0}} \left[\left(\frac{dI}{d\hat{z}} \right)^2 + \frac{d^2 I}{d\hat{z}^2} \right] \exp(I), \\
\frac{d^4 w_0}{d\hat{z}^4} &= \frac{1}{\sqrt{C_0}} \left[\left(\frac{dI}{d\hat{z}} \right)^4 + 6 \left(\frac{dI}{d\hat{z}} \right)^2 \frac{d^2 I}{d\hat{z}^2} + 3 \left(\frac{d^2 I}{d\hat{z}^2} \right)^2 + 4 \frac{dI}{d\hat{z}} \frac{d^3 I}{d\hat{z}^3} + \frac{d^4 I}{d\hat{z}^4} \right] \exp(I) .
\end{aligned}$$

Since near the edges $\{-1, 1\}$ of the support $[-1, 1]$ of w_0, i. e. for $1 - |\hat{z}| \ll 1$, $\left(\frac{dI}{d\hat{z}}\right)^4 \gg 1$ dominates the other terms thanks to the *quadratic* blow up of I near the edges, we have, according to (2.136)

$$\phi_{\ell,\hat{z}'}(\hat{z}) \approx \frac{1}{C_0} \frac{1}{\ell} \left(\frac{d}{d\hat{z}} I \left(\frac{\hat{z} - \hat{z}'}{l} \right) \right)^4 \exp \left(2I \left(\frac{\hat{z} - \hat{z}'}{l} \right) \right) .$$

Hence in particular for $\ell = \frac{1}{4}$ and $\hat{z}' = \frac{1}{2}$, $w_{\frac{1}{4},\frac{1}{2}}$ and thus $\phi_{\frac{1}{4},\frac{1}{2}}$ are supported in $[\frac{1}{4}, \frac{3}{4}]$, $\phi_{\frac{1}{4},\frac{1}{2}}$ is positive near the edges of the support (and thus bounded away from zero at some small distance of the edges of the support), and trivially bounded away from $-\infty$ in the support. The universal constants $\delta_0 > 0$, $\delta_1 > 0$, and $0 < C_1 < \infty$ are to quantify this:

$$\phi_{\frac{1}{4},\frac{1}{2}} \left\{ \begin{array}{lll} = 0 & \text{for} & \hat{z} \notin (\frac{1}{4}, \frac{3}{4}) \\ > 0 & \text{for} & \hat{z} \in (\frac{1}{4}, \frac{1}{4} + \delta_0] \cup [\frac{3}{4} - \delta_0, \frac{3}{4}) \\ > \delta_1 & \text{for} & \hat{z} \in [\frac{1}{4} + \delta_0, \frac{1}{4} + 3\delta_0] \cup [\frac{3}{4} - 3\delta_0, \frac{3}{4} - \delta_0] \\ > -C_1 & \text{for} & \hat{z} \in [\frac{1}{4} + 3\delta_0, \frac{3}{4} - 3\delta_0] \end{array} \right\} . \tag{2.138}$$

We now choose a universal smooth $\rho_0(\hat{z}')$ with

$$\rho_0 = \left\{ \begin{array}{lll} 0 & \text{for} & \hat{z}' \notin (\frac{1}{4} + 2\delta_0, \frac{3}{4} - 2\delta_0) \\ 2C_1 & \text{for} & \hat{z}' \in [\frac{1}{4} + 3\delta_0, \frac{3}{4} - 3\delta_0] \end{array} \right\} . \tag{2.139}$$

Since ρ_0 is smooth in $\hat{z}'$ we have according to (2.137)

$$\int_{-\infty}^{\infty} \phi_{\ell,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}' \to \rho_0(\hat{z}) \quad \text{uniformly in } \hat{z} \text{ as } \ell \downarrow 0 .$$

In view of the properties (2.139), there exists (a possibly small) $\ell_0 > 0$ such that

$$\int_{-\infty}^{\infty} \phi_{\ell_0,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}' \left\{ \begin{array}{lll} = 0 & \text{for} & \hat{z} \notin (\frac{1}{4} + \delta_0, \frac{3}{4} - \delta_0) \\ \geq -\delta_1 & \text{for} & \hat{z} \in [\frac{1}{4} + \delta_0, \frac{3}{4} - \delta_0] \\ \geq C_1 & \text{for} & \hat{z} \in [\frac{1}{4} + 3\delta_0, \frac{3}{4} - 3\delta_0] \end{array} \right\} . \tag{2.140}$$

In view of (2.138), the properties (2.140) just ensure that

$$\phi_0(\hat{z}) := \phi_{\frac{1}{4},\frac{1}{2}}(\hat{z}) + \int_{-\infty}^{\infty} \phi_{\ell_0,\hat{z}'}(\hat{z}) \rho_0(\hat{z}') \, d\hat{z}' \left\{ \begin{array}{lll} = 0 & \text{for} & \hat{z} \notin (\frac{1}{4}, \frac{3}{4}) \\ > 0 & \text{for} & \hat{z} \in (\frac{1}{4}, \frac{3}{4}) \end{array} \right\}$$

defines a ϕ_0 that is strictly positive in its support and that is of the form (2.129) (after a gratuitous normalization to obtain a probability measure).

2.2.5.2 Approximate logarithmic growth: Proof of Lemma 2

In this subsection, we return to the approximate logarithmic growth of τ worked out in case of the reduced stability condition in Section 2.2.1. Compared to Section 2.2.1, we have to work with the mollified version $\hat{\xi}_0$ of $\hat{\xi}$, cf. (2.90), since only for the former we have approximate positivity in the bulk according to Subsection 2.2.5.1. As stated in Lemma 2, we shall show that for $S_1 \geq C$ we have

$$\int_{-1}^{0} \hat{\xi}_0 \, ds \;\leq\; C \left(\frac{1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds + 1 \right), \tag{2.141}$$

where $0 < C < \infty$ denotes a (possibly large) generic universal constant.

We start the proof recalling

- The starting point for Subsection 2.2.5.1, that is (2.128), which we rewrite as

$$\int_{-\infty}^{\infty} \hat{\xi}(s + s' + s'') w \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w \, ds$$
$$\geq - \exp(-3s' - 3s'') \int_0^1 \left(\left(\frac{d^5 w}{d\hat{z}^5} \right)^2 + \cdots + w^2 \right) d\hat{z},$$

  for all $s' \leq \ln H$, $s'' \leq 0$ and all smooth w compactly supported in $\hat{z} \in (0, 1]$.

- The outcome of Subsection 2.2.5.1, that is (2.125), which we rewrite as

$$\hat{\xi}_0(s') \;=\; \int_{-\infty}^{\infty} \hat{\xi}(s' + s'') \phi_0(s'') \, ds'' \;\geq\; -C \exp(-3s') \tag{2.142}$$

  for all $s' \leq \ln H$.

Since the kernel $\phi_0(s'')$ is non-negative and compactly supported in $s'' \in (-\infty, 0]$, we obtain by testing the inequality in (1) with $\phi_0(s'') \, ds''$:

$$\int_{-\infty}^{\infty} \hat{\xi}_0(s'') \phi(s'' - s') \, ds'' \;=\; \int_{-\infty}^{\infty} \hat{\xi}_0(s + s') \phi(s) \, ds \tag{2.143}$$
$$\geq\; -C \exp(-3s') \int_0^1 \left[\left(\frac{d^5 w}{d\hat{z}^5} \right)^2 + \cdots + w^2 \right] d\hat{z},$$

where we continue to use the abbreviation ϕ for the multiplier corresponding to the generic w:

$$\phi \;=\; w \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w.$$

The structure of the argument is similar to the one for (2.38) in Subsection 2.2.1. We seek

- a family $\mathfrak{F} = \{w_{s'}\}_{s'}$ of smooth functions $w_{s'}$ parametrized by $s' \in \mathbb{R}$ and compactly supported in $\hat{z} \in (0, 1]$, that is $s \in (-\infty, 0]$, and
- a probability measure $\rho(ds')$ supported in $s' \in (-\infty, \ln H]$,

such that the corresponding convex combination of multipliers shifted by s', i. e.

$$\phi_1(s'') \;:=\; \int_{-\infty}^{\infty} \phi_{s'}(s'' - s') \, \rho(ds'), \quad \text{where} \quad \phi_{s'} \;:=\; w_{s'} \left(-\frac{d^2}{d\hat{z}^2} + 1 \right)^2 w_{s'}, \tag{2.144}$$

is close to

$$\left\{ \begin{array}{lll} -1 & \text{for} & -1 \leq s'' \leq 0 \\ \frac{1}{S_1} & \text{for} & 0 \leq s'' \leq S_1 \\ 0 & \text{else} & \end{array} \right\}.$$

In fact, we just need the upper bound

$$\phi_1(s'') \quad \le \quad \left\{ \begin{array}{ccc} -1 & \text{for} & -1 \le s'' \le 0 \\ \frac{C_1}{S_1} & \text{for} & 0 \le s'' \le S_1 \\ 0 & \text{else} & \end{array} \right\} , \tag{2.145}$$

where $C_1(\le S_1)$ is some universal constant whose value we want to remember momentarily, and a much weaker lower bound of the form

$$\phi_1(s'') \quad \ge \quad -C \left\{ \begin{array}{ccc} \exp(6s'') & \text{for} & s'' \le 0 \\ 1 & \text{for} & s'' \ge 0 \end{array} \right\} , \tag{2.146}$$

where the exponential rate 6 could be replaced by any rate larger than 3. Furthermore, we need that the functions $w_{s'}$ do not degenerate too much such that the error term on the r. h. s. of (2.143) stays under control:

$$\int_{-\infty}^{\infty} \exp(-3s') \int_0^1 \left[\left(\frac{d^5 w_{s'}}{d\hat{z}^5} \right)^2 + \cdots + w_{s'}^2 \right] d\hat{z}\, \rho(ds') \;\le\; C . \tag{2.147}$$

For both (2.146) and (2.147) we need that ρ decays sufficiently fast for $s' \downarrow -\infty$.
It is almost obvious how (2.145), (2.146) & (2.147) allow to pass from (2.143) to (2.141) by substituting w with $w_{s'}$ and integrating in $\rho(ds')$. We just need to show how (2.145) & (2.146) yield

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1 \, ds'' \;\le\; -\int_{-1}^{0} \hat{\xi}_0 \, ds'' + \frac{C_1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds'' + C .$$

Indeed, we write

$$\begin{aligned} & \int_{-\infty}^{\infty} \hat{\xi}_0 \phi_1 \, ds'' + \int_{-1}^{0} \hat{\xi}_0 \, ds'' - \frac{C_1}{S_1} \int_0^{S_1} \hat{\xi}_0 \, ds'' \\ &= \int_{-\infty}^{\infty} (-\hat{\xi}_0) \left(\left\{ \begin{array}{ccc} -1 & \text{for} & -1 \le s'' \le 0 \\ \frac{C_1}{S_1} & \text{for} & 0 \le s'' \le S_1 \\ 0 & \text{else} & \end{array} \right\} - \phi_1 \right) ds'' \\ &\overset{(2.145),(2.142)}{\le} C \int_{-\infty}^{\infty} \exp(-3s'') \left(\left\{ \begin{array}{ccc} -1 & \text{for} & -1 \le s'' \le 0 \\ \frac{C_1}{S_1} & \text{for} & 0 \le s'' \le S_1 \\ 0 & \text{else} & \end{array} \right\} - \phi_1 \right) ds'' \\ &\overset{(2.146)}{\le} C \int_{-\infty}^{\infty} \exp(-3s'') \left\{ \begin{array}{ccc} \exp(6s'') & \text{for} & s'' \le 0 \\ 1 & \text{for} & s'' \ge 0 \end{array} \right\} ds'' \;\le\; C . \end{aligned}$$

Imitating the argument given in Section 2.2.1 for the reduced stability condition, we introduce the rescaled logarithmic variable $\hat{s}$ and also rescale the amplitude of $\hat{w}_{s'}$:

$$s \;=\; \lambda\, \hat{s} \quad \text{and} \quad \hat{w}_{s'} \;=\; \frac{1}{\sqrt{\lambda}} \hat{w}_0 . \tag{2.148}$$

The choice of the normalization of $\hat{w}_{s'}$ will become apparent in (2.151) below. In (2.154) below, we shall choose $\lambda > 0$ as a function of s', so that $\hat{w}_{s'}$ indeed depends on s'. At this stage, we just assume that the mask $\hat{w}_0$, that we assumed to be fixed, satisfies

$$\hat{w}_0 \quad \text{is supported in } \hat{s} \in \left[-\frac{1}{2}, 0 \right] \quad \text{and nonvanishing in } \left(-\frac{1}{2}, 0 \right) , \tag{2.149}$$

(why we restrict the support to this interval will become apparent in (2.156)), so that $\hat{w}_{s'}$ is indeed supported in $s \in (-\infty, 0]$. We note that (2.148) implies that the multiplier $\phi_{s'} = \hat{w}_{s'}(\frac{d}{ds} + 2)(\frac{d}{ds} + 1)\frac{d}{ds}(\frac{d}{ds} - 1)\hat{w}_{s'} = \hat{w}_{s'} \left(-2\frac{d}{ds} - \frac{d^2}{ds^2} + 2\frac{d^3}{ds^3} + \frac{d^4}{ds^4} \right) \hat{w}_{s'}$ is of the following form:

$$\phi_{s'} \;=\; -\frac{2}{\lambda^2} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}} - \frac{1}{\lambda^3} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}^2} + \frac{2}{\lambda^4} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}^3} + \frac{1}{\lambda^5} \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}^4} , \tag{2.150}$$

a form that highlights the desired dominance of the term $-\frac{2}{\lambda^2}\hat{w}_0\frac{d\hat{w}_0}{d\hat{s}} = -\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}$ for $\lambda \gg 1$. In (2.148), we normalized $\hat{w}_{s'}$ in such a way that the first moment of the multiplier $\phi_{s'}$ is independent of λ to leading order in $\lambda \gg 1$:

$$\int_{-\infty}^{\infty} s\phi_{s'}\,ds \approx \int_{-\infty}^{\infty} s\left(-\frac{1}{\lambda^2}\frac{d\hat{w}_0^2}{d\hat{s}}\right)ds = \int_{-\infty}^{\infty} \hat{s}\left(-\frac{d\hat{w}_0^2}{d\hat{s}}\right)d\hat{s} \;=\; \int_{-\infty}^{\infty} w_0^2\,d\hat{s}\,. \tag{2.151}$$

It is the second term in (2.150) that gives the leading order contribution to the *zeroth* moment of the multiplier $\phi_{s'}$:

$$\int_{-\infty}^{\infty} \phi_{s'}\,ds \;\approx\; \frac{1}{\lambda^2}\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s}\,. \tag{2.152}$$

Let us now motivate why we need to choose λ to be s'-dependent. Note that for an s'-independent λ, ϕ_1 is just the convolution of $\phi_{s'}$ and ρ. Hence for $\lambda \ll 1$ and a $\rho(s')$ that varies sufficiently slowly on scale λ we would obtain from (2.151) & (2.152)

$$\begin{aligned}\phi_1(s'') \;&=\; \int_{-\infty}^{\infty} \phi_{s'}(s''-s')\rho(s')\,ds' \\ &\approx\; -\frac{d\rho}{ds'}(s'')\int_{-\infty}^{\infty} w_0^2\,d\hat{s} + \frac{1}{\lambda^2}\rho(s'')\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s}\,.\end{aligned} \tag{2.153}$$

Hence in order to obtain a $\phi_1(s'') \sim -1$ over an s''-interval of length of the order 1 followed by $\phi_1(s'') \lesssim \frac{1}{S_1}$, the first term on the r. h. s. of (2.153) suggests to choose $\rho(s')$ as a function that increases from 0 to 1 over an s'-interval of order 1 followed by a decrease from 1 to 0 over an s'-interval of length of order S_1. Now for $S_1 \gg 1$, a dilemma becomes apparent:

- On the one hand, in order for ρ to vary slowly on the scale λ, we need $\lambda \lesssim 1$.
- On the other hand, in order for the second term on the r. h. s. of (2.153) not to destroy $\phi_1(s'') \lesssim \frac{1}{S_1}$, we need $\lambda^2 \gtrsim S_1$.

This argument shows that we have to choose λ to be an increasing function of s'. The simplest choice turns out to be

$$\lambda \;=\; s'\,, \tag{2.154}$$

for $s' \gg 1$ (in order to avoid negative values, for the time we think of ρ as being supported in $s' \in (0,\infty)$; later, we shall modify (2.154)). From (2.150) and (2.154) we obtain

$$\phi_{s'}(s) = -\frac{1}{(s')^2}\left(\frac{d\hat{w}_0^2}{d\hat{s}}\right)\left(\frac{s}{s'}\right) - \frac{1}{(s')^3}\left(\hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\left(\frac{s}{s'}\right) + \frac{2}{(s')^4}\left(\hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\left(\frac{s}{s'}\right) + \frac{1}{(s')^5}\left(\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\left(\frac{s}{s'}\right). \tag{2.155}$$

However, (2.154) in turn means that ϕ_1 is no longer a simple convolution of $\phi_{s'}$ and ρ, c. f. (2.144). Because of (2.149), the following change of variables

$$\frac{s''-s'}{\lambda} = \frac{s''-s'}{s'} = \frac{s''}{s'} - 1 \;=\; \hat{s} \quad\Longleftrightarrow\quad s' \;=\; \frac{s''}{1+\hat{s}}\,, \tag{2.156}$$

where s'' is considered as a parameter is invertible with $\frac{d}{d\hat{s}} = -\frac{s''}{(1+\hat{s})^2}\frac{d}{ds'} = -\frac{(s')^2}{s''}\frac{d}{ds'}$ and $ds' = \frac{s''}{(1+\hat{s})^2}d\hat{s}$. We thus get the first representation

$$\begin{aligned}\phi_1(s'') \;&=\; -\int_{-\infty}^{\infty}\frac{1}{(1+\hat{s})^2}\,\hat{w}_0^2\frac{d\rho}{ds'}\,d\hat{s} - \frac{1}{(s'')^2}\int_{-\infty}^{\infty}(1+\hat{s})\,\hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\rho\,d\hat{s} \\ &+\; \frac{2}{(s'')^3}\int_{-\infty}^{\infty}(1+\hat{s})^2\,\hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\rho\,d\hat{s} + \frac{1}{(s'')^4}\int_{-\infty}^{\infty}(1+\hat{s})^3\,\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\rho\,d\hat{s}\,,\end{aligned} \tag{2.157}$$

which will be used only at the end of the proof.

Arguing as in Subsection 2.2.1 (see (2.62)) we get the second representation (c.f. (2.63))

$$\begin{aligned}\phi_1 &= \int_{-\infty}^{\infty} \hat{w}_0^2 \left(-\frac{1}{(1+\hat{s})^2}\frac{d\rho}{ds'} + \frac{1}{2}\frac{1}{(1+\hat{s})^3}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s})^4}\frac{d^3\rho}{ds'^3} - \frac{1}{2}\frac{1}{(1+\hat{s})^5}\frac{d^4\rho}{ds'^4}\right) d\hat{s} \\ &+ \left(\frac{1}{(s'')^2} + \frac{6}{(s'')^3} - \frac{12}{(s'')^4}\right) \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \rho \, d\hat{s} \\ &- \left(\frac{3}{(s'')^2} - \frac{8}{(s'')^3}\right) \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \frac{d\rho}{ds'} \, d\hat{s} \\ &- \frac{2}{(s'')^2} \int_{-\infty}^{\infty} \frac{1}{1+\hat{s}} \left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \frac{d^2\rho}{ds'^2} \, d\hat{s} \\ &+ \frac{1}{(s'')^4} \int_{-\infty}^{\infty} (1+\hat{s})^3 \left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 \rho \, d\hat{s} \,. \end{aligned} \tag{2.158}$$

From this representation we learn the following: If $\rho(s')$ varies on large length scales only (so that e. g. $\frac{d\rho}{ds'}$ is negligible w. r. t. ρ) and for $s'' \gg 1$, we obtain to leading order from the above

$$\begin{aligned}\phi_1 &\approx -\int_{-\infty}^{\infty} \frac{1}{(1+\hat{s})^2} \hat{w}_0^2 \frac{d\rho}{ds'} \, d\hat{s} + \frac{1}{(s'')^2} \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \rho \, d\hat{s} \\ &+ \frac{1}{(s'')^4} \int_{-\infty}^{\infty} (1+\hat{s})^3 \left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 \rho \, d\hat{s} \,.\end{aligned}$$

If $\rho(s')$ varies slowly even on a logarithmic scale (so that e. g. $s'\frac{d\rho}{ds'}$ is negligible w. r. t. ρ), the above further reduces to

$$\phi_1 \approx -\frac{d\rho(s'')}{ds'} \int_{-\infty}^{\infty} \frac{1}{(1+\hat{s})^2} \hat{w}_0^2 \, d\hat{s} + \frac{\rho(s'')}{(s'')^2} \int_{-\infty}^{\infty} (1+\hat{s}) \left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 d\hat{s} \,.$$

As in Subsection (2.2.1) (see argument below (2.65)) we notice that the first, negative, term r. h. s. of (2.159) dominates the second positive term provided

$$\frac{d\rho}{ds'} \gg \frac{1}{(s')^2} \,.$$

This motivates the following Ansatz for ρ in the range $1 \ll s' \ll S_1$: We fix a smooth mask $\rho_0(\hat{s}')$ such

$$\rho_0 = 0 \text{ for } \hat{s}' \le 0, \quad \frac{d\rho_0}{d\hat{s}'} > 0 \text{ for } 0 < \hat{s}' \le 2, \quad \rho_0 = 1 - \frac{1}{\hat{s}'} \text{ for } 2 \le \hat{s}' \,. \tag{2.159}$$

For $S_0 \gg 1$, we consider the rescaled version

$$\rho(S_0(\hat{s}'+1)) = \rho_0(\hat{s}'), \quad \text{i. e. the change of variables } s' = S_0(\hat{s}'+1) \,. \tag{2.160}$$

As in the argument for (2.38) (Section 2.2.1) with the rescaling of s''

$$s'' = S_0 \hat{s}'' \,, \tag{2.161}$$

(2.158) turns into

$$
\begin{aligned}
\phi_1 \;=\;& -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\frac{d\rho_0}{d\hat{s}'}\,d\hat{s}-\frac{1}{2S_0^2}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^3}\frac{d^2\rho_0}{d\hat{s}'^2}\,d\hat{s}\\
&+\frac{1}{S_0^3}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^4}\frac{d^3\rho_0}{d\hat{s}'^3}\,d\hat{s}+\frac{1}{2S_0^4}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^5}\frac{d^4\rho_0}{d\hat{s}'^4}\,d\hat{s}\\
&+\left(\frac{1}{S_0^2}\frac{1}{(\hat{s}'')^2}+\frac{1}{S_0^3}\frac{6}{(\hat{s}'')^3}-\frac{1}{S_0^4}\frac{12}{(\hat{s}'')^4}\right)\int_{-\infty}^{\infty}(1+\hat{s})\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\rho_0\,d\hat{s}\\
&-\left(\frac{1}{S_0^4}\frac{3}{(\hat{s}'')^3}-\frac{1}{S_0^5}\frac{8}{(\hat{s}'')^4}\right)\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d\rho_0}{d\hat{s}'}\,d\hat{s}\\
&-\frac{1}{S_0^5}\frac{2}{(\hat{s}'')^3}\int_{-\infty}^{\infty}\frac{1}{1+\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2\frac{d^2\rho_0}{d\hat{s}'^2}\,d\hat{s}\\
&+\frac{1}{S_0^4}\frac{1}{(s'')^4}\int_{-\infty}^{\infty}(1+\hat{s})^3\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2\rho_0\,d\hat{s}\,.
\end{aligned}
\tag{2.162}
$$

Since in the integrals in formula (2.158), the argument of ρ was given by $s=\frac{s''}{1+\hat{s}}$, c. f. (2.156), it follows from (2.159) and (2.161) that the argument of ρ_0 is given by

$$
\hat{s}' \;=\; \frac{\hat{s}''}{1+\hat{s}}-1\,. \tag{2.163}
$$

Thus (2.162) just depends on $\hat{s}''$, not on S_0.

As for the proof of (2.48) in Section 2.2.1, our reduced goal is now to show that the constructions of w (c. f. (2.149) and (2.154)) and ρ (c. f. (2.159) and (2.160)) yield the bound

$$
\phi_1^*(s'') := \int \phi_{s'}(s''-s')\rho(s')\begin{cases}=0 & \text{for all } s''\le \frac{1}{2}S_0\,,\\ <0 & \text{for } s''>\frac{1}{2}S_0\,,\end{cases} \tag{2.164}
$$

for $S_0 \gg 1$. As in Section 2.2.1, we distinguish the regions of small, intermediate and large s'' Arguing as for (2.72) and (2.76) , it is easy to deduce that in the range $s''\ge 3S_0$ (small s'') and $s''\in\left[\frac{3}{4}S_0,3S_0\right]$ (intermediate s''), ϕ_1 (see (2.162)) is strictly negative for $S_0\gg 1$: indeed in Section 2.2.1 we proved

$$
\phi_1^* \;\approx\; -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(\hat{s}''-(1+\hat{s}))^2}\,d\hat{s}\quad \text{uniformly in } \hat{s}''\ge 3 \quad \text{for } S_0\gg 1 \tag{2.165}
$$

and

$$
\phi_1^* \;\approx\; -\frac{1}{S_0}\int_{-\infty}^{\infty}\frac{\hat{w}_0^2}{(1+\hat{s})^2}\frac{d\rho_0}{d\hat{s}'}\,d\hat{s}\quad \text{uniformly in } \hat{s}''\in\left[\frac{3}{4},3\right]\quad \text{for } S_0\gg 1\,. \tag{2.166}
$$

Noting that for $s''\in(\infty,\frac{S_0}{2}]$ all the integrals in (2.162) are vanishing because the supports of $\hat{w}$ and ρ_0 do not intersect, we are left to study the region of small s'', i.e. $s''\in\left(\frac{S_0}{2},\frac{3}{4}S_0\right)$. Also in the range of small s''

$$
s''\;\in\;\left(\frac{1}{2}S_0,\frac{3}{4}S_0\right)\quad\text{or equivalently}\quad \hat{s}''\;\in\;\left(\frac{1}{2},\frac{3}{4}\right).
$$

we would like ϕ_1^* to be strictly negative for $S_0\gg 1$. But here (exactly as in (2.79)), no matter how large $\lambda=s'$ in (2.150) is, the behavior of $\phi_{s'}$ near the left edge $-\frac{1}{2}$ of its support $[-\frac{1}{2},0]$ (and also at its right edge 0, but there we don't care) is dominated by the $\frac{1}{\lambda^5}\hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}$-term and thus automatically is *strictly positive*. Taking the $\rho(s')\,ds'$-average of the shifted $\phi_{s'}(s''-s')$ does not alter this behavior as long as ρ is compactly supported in $[S_0,\infty)$, c. f. (2.159): ϕ_1 is strictly positive near the left edge $\frac{S_0}{2}$ of its support. In Section 2.2.1, we solved this problem by *giving up smoothness* of $\hat{w}_0$ near the left $-\frac{1}{2}$ of its support $[-\frac{1}{2},0]$ and eventually using an approximation argument in $H^{2,2}$ (in (2.162) ϕ_1 is expressed in derivatives of $\hat{w}_0$ up to second order). In this case, this (approximation) argument does no longer work because we need to asses the error term (2.147), which involves higher derivatives of $\hat{w}_0$. The way out to this dilemma will take the remainder of this section and it consists of three steps (the first one is the same as is (2.79) and we report it just for the sake of clarity).

- In the first stage, we *give up smoothness* of $\hat{w}_0$ near the left $-\frac{1}{2}$ of its support $[-\frac{1}{2},0]$. In fact, we shall first assume that $\hat{w}_0$ satisfies in addition

$$\hat{w}_0 \;=\; \frac{1}{2}\left(\hat{s}+\frac{1}{2}\right)^2 \quad \text{for } \hat{s}\in\left[-\frac{1}{2},-\frac{1}{4}\right]. \tag{2.167}$$

This means that $\hat{w}_0$ has a bounded but discontinuous second derivative. It will be easy to see that the so defined ϕ_1^* is, as desired, strictly negative on $s''\in(\frac{S_0}{2},\frac{3}{4}S_0]$ for all S_0. Once this is done, we fix a sufficiently large but universal S_0 such that

$$-\phi_1^*(s'') \;\in\; \left\{\begin{array}{llcccc} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 & \le & s'' & \\ (0,C] & \text{for} & \frac{1}{2}S_0 & < & s'' & \le & S_0 \\ \{0\} & \text{for} & & & s'' & \le & \frac{1}{2}S_0 \end{array}\right\}, \tag{2.168}$$

for some generic universal constant C. In Subsection 2.2.1 the choice of (2.167) allows us to show (2.71), which implies (2.48). In this case instead the choice of (2.167) is clearly not admissible because we need to control the error term (2.147). In two further steps we show how to solve this issue.

- In the second stage, *we modify the definition (2.160) of* $\rho(s')$ by adding a small-amplitude and fast-decaying (exponential) tail for $s'\downarrow-\infty$. More precisely, we make the Ansatz

$$\widetilde{\rho} \;=\; \rho+\delta\rho \quad \text{with} \quad \delta\rho \;=\; \varepsilon\exp\left(\frac{s'}{S_2}\right)\eta_0\left(\frac{s'}{S_0}\right), \tag{2.169}$$

where η_0 is the mask of a smooth cut-off function $\eta_0(\hat{s}')$ with

$$\eta_0 \;=\; 1 \quad \text{for } \hat{s}'\le 2 \quad \text{and} \quad \eta_0 \;=\; 0 \quad \text{for } \hat{s}'\ge 3\,. \tag{2.170}$$

Here $0<S_2\ll 1$ is some small length scale and $\varepsilon\ll 1$ is some small amplitude to be chosen below. Recall that S_0 is the universal constant fixed in the first stage. Since $\tilde{\rho}$ is no longer supported on $s'\in[S_0,\infty)$ but is positive on the entire line, we need to extend our definition of the function $\hat{w}_{s'}$ from $s'\ge S_0$ to all s'. In view of (2.148) we just have to extend the definition (2.154) of the rescaling parameter $\lambda(s')$ to

$$\lambda \;=\; \left\{\begin{array}{lll} s' & \text{for} & s'\ge S_0 \\ S_0 & \text{for} & s'\le S_0 \end{array}\right\}. \tag{2.171}$$

We will show that we can first choose a universal $0<S_2\ll 1$ and then a universal $0<\varepsilon\ll 1$ such that we obtain for $\widetilde{\phi_1^*}(s''):=\int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\widetilde{\rho}(s')\,ds'=\int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\rho(s')\,ds'+\int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\delta\rho(s')\,ds'=:\phi_1^*(s'')+\delta\phi_1^*(s'')$ the following estimates

$$-\widetilde{\phi_1^*}(s'') \;\in\; \left\{\begin{array}{llccccc} [\frac{1}{C}\frac{1}{(s'')^2}, C\frac{1}{(s'')^2}] & \text{for} & S_0 & \le & s'' & \\ [\frac{1}{C},C] & \text{for} & \frac{1}{2}S_0 & < & s'' & \le & S_0 \\ [\frac{1}{C}\exp(\frac{s''}{S_2}), C\exp(\frac{s''}{S_2})] & \text{for} & & & s'' & \le & \frac{1}{2}S_0 \end{array}\right\}, \tag{2.172}$$

for some generic universal constant C. The gain with respect to ϕ_1^* is that $\widetilde{\phi_1^*}$ is strictly negative also for $s''\le\frac{1}{2}S_0$ which will allow us to pass to the third stage.

- In a third stage, *we smoothen out* $\hat{w}_0$: We define a sequence of smooth functions $\{w_0^\alpha\}_{\alpha\downarrow 0}$ which approximate $\hat{w}_0$ in such a way that the corresponding $\widetilde{\phi_1^*}$ still satisfies (2.172).

We start with the first stage, that is, our non-smooth Ansatz (2.167). In view of (2.150) and (2.171), this implies

$$\phi_{s'} \;=\; -\frac{1}{(s')^2}\left(\frac{s}{s'}+\frac{1}{2}\right)^3-\frac{1}{2}\frac{1}{(s')^3}\left(\frac{s}{s'}+\frac{1}{2}\right)^2 \;<\; 0 \quad \text{for } s\in\left(-\frac{s'}{2},-\frac{s'}{4}\right]. \tag{2.173}$$

We now consider $\phi_1^*(s'')=\int_{-\infty}^{\infty}\phi_{s'}(s''-s')\rho(s')\,ds'$. In view of (2.159) & (2.160), $\rho\geq 0$ is strictly positive for $s'\in(S_0,\infty)$. On the other hand, it follows from (2.173) that $s'\mapsto\phi_{s'}(s''-s')$ is strictly negative for $s''-s'\in(-\frac{s'}{2},-\frac{s'}{4}]$, that is, for $s'\in[\frac{4}{3}s'',2s'')$ (and supported in $s'\in[s'',2s''])$. Hence ϕ_1^* is strictly negative for $S_0\in[\frac{4}{3}s'',2s'')$, that is, for $s''\in(\frac{1}{2}S_0,\frac{3}{4}S_0]$, as desired. This holds for any value of $S_0>0$. In conjunction with (2.165) and (2.166) this implies (2.168).

We now turn to the second stage, i. e. the effect of the modification $\widetilde{\rho}(s')$ of $\rho(s')$. Consider the perturbation $\delta\phi_1^*(s'')$ of the multiplier $\phi_1^*(s'')$:

$$\delta\phi_1^*(s'') \;=\; \int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\delta\rho(s')\,ds' \;\overset{(2.171)}{=}\; \varepsilon\int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\exp\left(\frac{s'}{S_2}\right)\eta_0\left(\frac{s'}{S_0}\right)ds'\,. \tag{2.174}$$

In order to show that the unperturbed (2.168) upgrades to (2.172), it is enough to establish

$$-\frac{1}{\varepsilon}\delta\phi_1^*(s'') \;\in\; \left\{\begin{array}{llcccc} \{0\} & \text{for} & 3S_0 & \leq & s'' & \\ [-C,C] & \text{for} & S_0 & \leq & s'' \leq & 3S_0 \\ [\frac{1}{C},C] & \text{for} & \frac{1}{2}S_0 & \leq & s'' \leq & S_0 \\ [\frac{1}{C}\exp(\frac{s''}{S_2}),C\exp(\frac{s''}{S_2})] & \text{for} & & & s'' \leq & \frac{1}{2}S_0 \end{array}\right\}, \tag{2.175}$$

for some sufficiently small *but fixed* S_2, where C denotes a universal constant. Indeed, choosing $\varepsilon\ll 1$, we see from $\widetilde{\phi_1^*}=\phi_1^*+\delta\phi_1^*$ that (2.175) upgrades (2.168) to (2.172).

We start with the large s'', i. e. $s''\geq 3S_0$ and consider the integral $\frac{1}{\varepsilon}\delta\phi_1^*(s'')=\int_{-\infty}^{\infty}\phi_{\lambda(s')}(s''-s')\exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})\,ds'$. Because of our choice (2.170) of η_0, the second factor $\exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})$ is thus supported in $s'\in(-\infty,3S_0]$. We note that in view of our choice (2.171) of the scaling factor λ, $\hat{w}_{s'}(s)$ and thus $\phi_{\lambda(s')}(s)$ is supported in $s\in(-\frac{1}{2}S_0,0)$ for $s'\leq S_0$ and $s\in(-\frac{1}{2}s',0)$ for $s'\geq S_0$. Hence $(s',s'')\mapsto\phi_{\lambda(s')}(s''-s')$ is supported in $s''\in(s'-\frac{1}{2}S_0,s')$ for $s'\leq S_0$ and $s''\in(\frac{1}{2}s',s')$ for $s'\geq S_0$, or — equivalently — in $s'\in(s'',s''+\frac{1}{2}S_0)$ for $s''\leq\frac{S_0}{2}$ and in $s'\in(s'',2s'')$ for $s''\geq\frac{S_0}{2}$. Since $s''\geq 3S_0$, we are in the latter case and $\phi_{\lambda(s')}(s''-s')$ is supported in $s'\in(s'',2s'')\subset[3S_0,\infty)$. Hence both factors $\exp(-\frac{s'}{S_2})\eta_0(\frac{s'}{S_0})$ and $\phi_{\lambda(s')}(s''-s')=\phi_{s'}(s''-s')$ have disjoint support in s' and thus the integral (2.174) in s' vanishes. This establishes the first line in (2.175).

We now turn to the very small s'', i. e. $s''\leq\frac{S_0}{2}$. By the above, $s'\mapsto\phi_{\lambda(s')}(s''-s')$ is supported in $s'\in(s'',s''+\frac{S_0}{2})\subset(-\infty,S_0]$, we have in this s'-range for the cut-off function $\eta_0(\frac{s'}{S_0})=1$. Moreover, in this range we have $\phi_{\lambda(s')}=\phi_{S_0}$. Hence the definition (2.174) simplifies to

$$\frac{1}{\varepsilon}\delta\phi_1^*(s'')=\int_{-\infty}^{\infty}\phi_{S_0}(s''-s')\exp\left(\frac{s'}{S_2}\right)ds'=\exp\left(\frac{s''}{S_2}\right)\int_{-\infty}^{\infty}\phi_{S_0}(s)\exp\left(-\frac{s}{S_2}\right)ds\,. \tag{2.176}$$

We note that by (2.173) we have

$$\phi_{S_0}\quad\text{is strictly negative for } s\in\left(-\frac{S_0}{2},-\frac{S_0}{4}\right)\quad\text{and supported in } s\in\left[-\frac{S_0}{2},0\right].$$

On the last integral, we can now use Laplace's method for $S_2 \ll 1$. We thus have

$$\begin{aligned}
&-\int_{-\infty}^{\infty} \phi_{S_0} \exp\left(-\frac{s}{S_2}\right) ds \\
\approx\quad & -\int_{-\infty}^{-\frac{S_0}{4}} \phi_{S_0} \exp\left(-\frac{s}{S_2}\right) ds \\
\overset{(2.173)}{=}\quad & \int_{-\frac{S_0}{2}}^{-\frac{S_0}{4}} \left(\frac{1}{S_0^2}\left(\frac{s}{S_0}+\frac{1}{2}\right)^3 + \frac{1}{2S_0^3}\left(\frac{s}{S_0}+\frac{1}{2}\right)^2\right) \exp\left(-\frac{s}{S_2}\right) ds \\
\approx\quad & \int_{-\frac{S_0}{2}}^{\infty} \frac{1}{2S_0^3}\left(\frac{s}{S_0}+\frac{1}{2}\right)^2 \exp\left(-\frac{s}{S_2}\right) ds \\
=\quad & \frac{1}{S_0^2}\int_{-\frac{1}{2}}^{\infty} \frac{1}{2}\left(\hat{s}+\frac{1}{2}\right)^2 \exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
=\quad & \exp\left(\frac{1}{2}\frac{S_0}{S_2}\right)\frac{S_2^3}{S_0^5}\,.
\end{aligned}$$

Plugging this into (2.176) yields

$$-\frac{1}{\varepsilon}\delta\phi_1^*(s'') \;\approx\; \frac{S_2^3}{S_0^5}\exp\left(\frac{1}{2}\frac{S_0}{S_2}\right)\exp\left(\frac{s''}{S_2}\right) \quad \text{uniformly in } s'' \le \frac{S_0}{2} \text{ for } S_2 \ll 1\,. \tag{2.177}$$

We now treat the intermediary small values $\frac{S_0}{2} \le s'' \le S_0$. This time, the function $s' \mapsto \phi_{\lambda(s')}(s''-s')$ is supported in $s' \in [s'', 2s''] \subset (-\infty, 2S_0]$, so that also in this s'-range we have for the cut-off function $\eta_0(\frac{s'}{S_0}) = 1$. Hence the representation simplifies to

$$\frac{1}{\varepsilon}\delta\phi_1^*(s'') \;=\; \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s''-s')\exp\left(\frac{s'}{S_2}\right) ds'\,.$$

On this integral, we can again use Laplace's method for $S_2 \ll 1$: By (2.173) we have for the continuous function $(s', s'') \mapsto \phi_{\lambda(s')}(s''-s')$

$$\phi_{\lambda(s')}(s''-s') \left\{ \begin{array}{ccl} < & 0 & \text{for } s' \in (\frac{3}{2}s'', 2s'') \\ = & 0 & \text{for } s' \notin (s'', 2s'') \end{array} \right\}.$$

Hence we obtain

$$\frac{1}{\varepsilon}\delta\phi_1^*(s'') \;<\; 0 \quad \text{uniformly in } s'' \in \left[\frac{S_0}{2}, S_0\right] \text{ for } S_2 \ll 1\,. \tag{2.178}$$

We finally address the remaining intermediary range, that is, $S_0 \le s'' \le 3S_0$. We clearly have by continuity of $(s', s'') \mapsto \phi_{\lambda(s')}(s''-s')$ and $\eta_0(\hat{s}')$:

$$\frac{1}{\varepsilon}\delta\phi_1^*(s'') \;=\; \int_{-\infty}^{\infty} \phi_{\lambda(s')}(s''-s)\exp\left(\frac{s'}{S_2}\right)\eta_0\left(\frac{s'}{S_0}\right) ds' \tag{2.179}$$

is uniformly bounded for $s'' \in [S_0, 3S_0]$. Estimate (2.175) now follows from (2.177), (2.178) & (2.179) for a choice of sufficiently small S_2.

We now turn to the third stage. We approximate $\hat{w}_0$, which is non-smooth at the left edge of its support, c. f. (2.167), by a sequence of smooth $\hat{w}_0^\alpha$ in such a way that the corresponding $\phi_{\lambda(s')}$ and $\phi^\alpha_{\lambda(s')}$ are close in L^1. More precisely, we select a smooth function $F(w)$ with

$$F(w) \;=\; 0 \quad \text{for } w \le 0 \quad \text{and} \quad F(w) \;=\; w \quad \text{for } w \ge 1\,.$$

For a small parameter $0<\alpha\ll 1$ we now define $\hat w_0^\alpha(\hat s)$ via

$$\hat w_0^\alpha \;:=\; \alpha^2 F\left(\frac{\hat w_0}{\alpha^2}\right) \overset{(2.167)}{=} \alpha^2 F\left(\frac{(\hat s+\frac12)^2}{2\alpha^2}\right) \quad \text{for } \hat s\in\left[-\frac12,-\frac14\right];$$

for $\hat s\notin[-\frac12,-\frac14]$, $\hat w_0^\alpha$ is set equal to $\hat w_0$. Clearly, the so defined $\hat w_0^\alpha$ is smooth on the whole line. Following (2.148), we consider the corresponding $\hat w^\alpha=\frac{1}{\sqrt\lambda}\hat w_0^\alpha$ in the variables $s=\lambda\hat s$ and its multiplier that analogously to (2.155) is given by

$$\phi^\alpha_{\lambda(s')}=-\frac{2}{(\lambda(s'))^2}\hat w_0^\alpha\frac{d\hat w_0^\alpha}{d\hat s}-\frac{1}{(\lambda(s'))^3}\hat w_0^\alpha\frac{d^2\hat w_0^\alpha}{d\hat s^2}+\frac{2}{(\lambda(s'))^4}\hat w_0^\alpha\frac{d^3\hat w_0^\alpha}{d\hat s^3}+\frac{1}{(\lambda(s'))^5}\hat w_0^\alpha\frac{d^4\hat w_0^\alpha}{d\hat s^4}. \tag{2.180}$$

We want to show that the convex combination of multipliers

$$\widetilde\phi_1^{*,\alpha}(s'')=\int_{-\infty}^{\infty}\phi^\alpha_{\lambda(s')}(s''-s')\widetilde\rho(s')\,ds',$$

still satisfies (2.172), that is

$$-\widetilde\phi_1^{*,\alpha}(s'')\;\in\;\left\{\begin{array}{llllll}[\frac1C\frac{1}{(s'')^2},C\frac{1}{(s'')^2}] & \text{for} & S_0 & \le & s'' & \\ [\frac1C,C] & \text{for} & \frac12S_0 & < & s'' & \le\; S_0\\ [\frac1C\exp(\frac{s''}{S_2}),C\exp(\frac{s''}{S_2})] & \text{for} & & & s'' & \le\; \frac12 S_0\end{array}\right\}, \tag{2.181}$$

for some choice of $0<\alpha\ll1$ and a generic universal constant. For this purpose we consider the difference of the combination of multipliers, that is, $\delta\phi_1^{*,\alpha}=\widetilde\phi_1^{*,\alpha}-\widetilde\phi_1^*$ and show that it is sufficiently small. For this purpose, we first observe that

$$\left|(\hat w_0^\alpha)^2-\hat w_0^2\right| \;\le\; C\alpha^4, \tag{2.182}$$

$$\left|\hat w_0^\alpha\frac{d\hat w_0^\alpha}{d\hat s}-\hat w_0\frac{d\hat w_0}{d\hat s}\right| \;\le\; C\alpha^3, \tag{2.183}$$

$$\left|\hat w_0^\alpha\frac{d^2\hat w_0^\alpha}{d\hat s^2}-\hat w_0\frac{d^2\hat w_0}{d\hat s^2}\right| \;\le\; C\alpha^2, \tag{2.184}$$

$$\left|\hat w_0^\alpha\frac{d^3\hat w_0^\alpha}{d\hat s^3}-\hat w_0\frac{d^3\hat w_0}{d\hat s^3}\right| \;\le\; C\alpha, \tag{2.185}$$

$$\left|\hat w_0^\alpha\frac{d^4\hat w_0^\alpha}{d\hat s^4}-\hat w_0\frac{d^4\hat w_0}{d\hat s^4}\right| \;\le\; C, \tag{2.186}$$

which follows from the fact that

$$\text{all these differences are supported on the interval} \qquad \hat s\in\left[-\frac12,-\frac12+\sqrt2\alpha\right], \tag{2.187}$$

and that on this interval, the two terms forming the difference are by themselves of the claimed size.

We first treat the case of large s''-values, that is, of $s''\ge 3S_0$. In this case, $s'\mapsto\phi_{\lambda(s')}(s''-s')$ and $s'\mapsto\phi^\alpha_{\lambda(s')}(s''-s')$ are supported in $s'\in(s'',2s'')$. In particular, $s'\ge S_0$ so that $\lambda\overset{(2.171)}{=}s'$. Hence, by (2.157), we obtain the representation

$$\begin{aligned}\delta\phi_1^{*,\alpha}(s'') \;=\; & -\int_{-\infty}^{\infty}\frac{1}{(1+\hat s)^2}((\hat w_0^\alpha)^2-\hat w_0^2)\frac{d\bar\rho}{ds'}\,d\hat s\\ &-\frac{1}{(s'')^2}\int_{-\infty}^{\infty}(1+\hat s)\left(\hat w_0^\alpha\frac{d^2\hat w_0^\alpha}{d\hat s^2}-\hat w_0\frac{d^2\hat w_0}{d\hat s^2}\right)\bar\rho\,d\hat s\\ &+\frac{2}{(s'')^3}\int_{-\infty}^{\infty}(1+\hat s)^2\left(\hat w_0^\alpha\frac{d^2\hat w_0^\alpha}{d\hat s^3}-\hat w_0\frac{d^3\hat w_0}{d\hat s^3}\right)\bar\rho\,d\hat s\\ &+\frac{1}{(s'')^4}\int_{-\infty}^{\infty}(1+\hat s)^3\left(\hat w_0^\alpha\frac{d^2\hat w_0^\alpha}{d\hat s^4}-\hat w_0\frac{d^4\hat w_0}{d\hat s^4}\right)\bar\rho\,d\hat s.\end{aligned}$$

In particular, we also have $s' \geq 3S_0$ so that $\widetilde{\rho}(s') \overset{(2.169)}{=} \rho(s') \overset{(2.159)}{=} 1-\frac{1}{\frac{s'}{S_0}-1} = 1-\frac{S_0}{s'-S_0}$, and thus $\frac{d\widetilde{\rho}}{ds'} = \frac{S_0}{(s'-S_0)^2}$. In terms of the variable $\hat{s}$ given by $s' = \frac{s''}{1+\hat{s}}$, this translates into $\widetilde{\rho} = 1-\frac{S_0(1+\hat{s})}{s''-S_0(1+\hat{s})}$ and $\frac{d\widetilde{\rho}}{ds'} = \frac{S_0(1+\hat{s})^2}{(s''-S_0(1+\hat{s}))^2}$. Hence the above representation specifies to

$$
\begin{aligned}
\delta\phi_1^{*,\alpha}(s'') \;=\; & -\int_{-\infty}^{\infty} \left((\hat{w}_0^\alpha)^2 - \hat{w}_0^2\right) \frac{S_0}{(s''-S_0(1+\hat{s}))^2}\, d\hat{s} \\
& -\frac{1}{(s'')^2}\int_{-\infty}^{\infty} (1+\hat{s}) \left(\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\left(1-\frac{S_0(1+\hat{s})}{s''-S_0(1+\hat{s})}\right) d\hat{s} \\
& +\frac{2}{(s'')^3}\int_{-\infty}^{\infty} (1+\hat{s})^2 \left(\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\left(1-\frac{S_0(1+\hat{s})}{s''-S_0(1+\hat{s})}\right) d\hat{s} \\
& +\frac{1}{(s'')^4}\int_{-\infty}^{\infty} (1+\hat{s})^3 \left(\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\left(1-\frac{S_0(1+\hat{s})}{s''-S_0(1+\hat{s})}\right) d\hat{s}\,.
\end{aligned}
$$

Using (2.187) and inserting the estimates (2.182), (2.184), (2.185), and (2.186) we obtain

$$
|\delta\phi_1^{*,\alpha}| \;\leq\; C\alpha\left(\frac{\alpha^4}{(s'')^2} + \frac{\alpha^2}{(s'')^2} + \frac{\alpha}{(s'')^3} + \frac{1}{(s'')^4}\right) \;\leq\; C\frac{\alpha}{(s'')^2} \quad \text{for } s'' \geq 3S_0. \tag{2.188}
$$

We now address the small s''-values, that is, $s'' \leq \frac{S_0}{2}$. In this case, $s' \mapsto \phi_{\lambda(s')}(s''-s')$ and $s' \mapsto \phi^\alpha_{\lambda(s')}(s''-s')$ are supported in $s' \in [s'', s''+\frac{S_0}{2}]$. In particular, $s' \leq S_0$ so that $\lambda \overset{(2.171)}{=} S_0$. Hence, by (2.150) and (2.144) we obtain the representation

$$
\begin{aligned}
\delta\phi_1^{*,\alpha}(s'') \;=\; & -\frac{2}{S_0^2}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}}\right)\widetilde{\rho}\, d\hat{s} \\
& -\frac{1}{S_0^3}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\rho}\, d\hat{s} \\
& +\frac{2}{S_0^4}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\rho}\, d\hat{s} \\
& +\frac{1}{S_0^5}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\rho}\, d\hat{s}\,.
\end{aligned}
$$

Moreover, $s' \leq 2S_0$ implies $\rho(s') = 0$, $\eta_0(\frac{s'}{S_0}) = 1$ and thus $\widetilde{\rho}(s') = \varepsilon\exp(\frac{s'}{S_2})$. In terms of $\hat{s}$ given by $s' = s'' - S_0\hat{s}$, this translates into $\widetilde{\rho}(s') = \exp(\frac{s''}{S_2})\exp(-\frac{S_0}{S_2}\hat{s})$. Hence the above representation specifies to

$$
\begin{aligned}
\delta\phi_1^{*,\alpha}(s'') \;=\; & -\frac{2\exp(\frac{s''}{S_2})}{S_0^2}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0 \frac{d\hat{w}_0}{d\hat{s}}\right)\exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& -\frac{\exp(\frac{s''}{S_2})}{S_0^3}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0 \frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& +\frac{2\exp(\frac{s''}{S_2})}{S_0^4}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0 \frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s} \\
& +\frac{\exp(\frac{s''}{S_2})}{S_0^4}\int_{-\infty}^{\infty} \left(\hat{w}_0^\alpha \frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0 \frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\exp\left(-\frac{S_0}{S_2}\hat{s}\right) d\hat{s}\,.
\end{aligned}
$$

Inserting the estimates (2.183),(2.184), (2.185), and (2.186) we obtain

$$
|\delta\phi_1^{*,\alpha}| \;\leq\; C\alpha\exp\left(\frac{s''}{S_2}\right)(\alpha^3+\alpha^2+\alpha^1+1) \;\leq\; C\alpha\exp\left(\frac{s''}{S_2}\right) \quad \text{for } s'' \leq \frac{S_0}{2}. \tag{2.189}
$$

We finally address the intermediate values of s'', that is, $\frac{S_0}{2} \leq s'' \leq 3S_0$. Splitting the ds'-integrals into $s' \in [S_0, \infty)$ and $s' \in (-\infty, S_0]$ in order to treat $\lambda \overset{(2.171)}{=} \max\{s', S_0\}$, we obtain

$$\begin{aligned}
\delta\phi_1^{*,\alpha}(s'') \;=\; & -\frac{2}{s''}\int_{-\infty}^{\frac{s''}{S_0}-1}\left(\hat{w}_0^\alpha\frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0\frac{d\hat{w}_0}{d\hat{s}}\right)\widetilde{\rho}\,d\hat{s} \\
& -\frac{1}{(s'')^2}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\rho}\,d\hat{s} \\
& +\frac{2}{(s'')^3}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})^2\left(\hat{w}_0^\alpha\frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\rho}\,d\hat{s} \\
& +\frac{1}{(s'')^4}\int_{-\infty}^{\frac{s''}{S_0}-1}(1+\hat{s})^3\left(\hat{w}_0^\alpha\frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\rho}\,d\hat{s} \\
& -\frac{2}{S_0^2}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d\hat{w}_0^\alpha}{d\hat{s}} - \hat{w}_0\frac{d\hat{w}_0}{d\hat{s}}\right)\widetilde{\rho}\,d\hat{s} \\
& -\frac{1}{S_0^3}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^2\hat{w}_0^\alpha}{d\hat{s}^2} - \hat{w}_0\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)\widetilde{\rho}\,d\hat{s} \\
& +\frac{2}{S_0^4}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^3\hat{w}_0^\alpha}{d\hat{s}^3} - \hat{w}_0\frac{d^3\hat{w}_0}{d\hat{s}^3}\right)\widetilde{\rho}\,d\hat{s} \\
& +\frac{1}{S_0^5}\int_{\frac{s''}{S_0}-1}^{\infty}\left(\hat{w}_0^\alpha\frac{d^4\hat{w}_0^\alpha}{d\hat{s}^4} - \hat{w}_0\frac{d^4\hat{w}_0}{d\hat{s}^4}\right)\widetilde{\rho}\,d\hat{s}\,.
\end{aligned}$$

Since $|\widetilde{\rho}| \leq 1$ and since $|\frac{1}{s''}| \leq \frac{2}{S_0}$, we obtain from inserting the estimates (2.183),(2.184), (2.185), and (2.186):

$$|\delta\phi_1^{*,\alpha}| \;\leq\; C\alpha(\alpha^3+\alpha^2+\alpha^1+1) \;\leq\; C\alpha \quad \text{for } s'' \in \left[\frac{S_0}{2}, 2S_0\right]. \tag{2.190}$$

Now (2.188), (2.189) and (2.190) show that one may pass from (2.172) to (2.181) by choosing a sufficiently small $\alpha > 0$.

As in Section 2.2.1, in order to prove (2.181) we need to cut-off the measure ρ (defined in (2.159)&(2.160)) in the region $\frac{S_1}{2} \leq s' \leq S_1$ so that (2.83) is an admissible multiplier.By the argument given at the end of the proof of (2.48), it is clear that the multiplication of the measure ρ by η affects only the region of big s'' (specifically $s'' \geq \frac{S_1}{4}$). Thus, appealing to the argument at the end of the proof of Proposition 1 the proof of (2.181) is concluded.

2.2.5.3 Approximate positivity in the boundary layers: Proof of Lemma 3

The approximate non-negativity of $\hat{\xi}_0$, cf. (2.90), is lost in the boundary layer $s \ll -1$, cf. (2.89). However, in this subsection we show that $\hat{\xi}_0$ cannot be too negative in the boundary layer *provided $\hat{\xi}_0$ it is sufficiently small in the transition region $|s| \lesssim 1$*. For all $S_2 \geq C$ and $\varepsilon \leq 1$ we have

$$\int_{-S_2}^{-1}\hat{\xi}_0 ds \geq -C\left(\frac{1}{\varepsilon}\int_{-1}^{0}\hat{\xi}_0 ds + \frac{1}{\varepsilon} + \int_{-S_2}^{-S_2+1}|\hat{\xi}_0|ds + \varepsilon\exp(5S_2)\right), \tag{2.191}$$

where $C < \infty$ denotes a (possibly large) generic constant. With the rescaling

$$s \;\rightsquigarrow\; s+S_0, \quad \hat{\xi} \;\rightsquigarrow\; \exp(-3S_0)\hat{\xi} \quad \text{and thus also} \quad \hat{\xi}_0 \;\rightsquigarrow\; \exp(-3S_0)\hat{\xi}_0\,,$$

it is enough to show for all $S_0 \geq C$ and $S_1 \geq S_0 + 3$

$$\int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds \geq -C\Big(\frac{1}{\varepsilon}\int_{-S_0-1}^{-S_0} \hat{\xi}_0\, ds + \frac{1}{\varepsilon}\exp(5S_0) + \int_{-S_1}^{-S_1+1} |\hat{\xi}_0|\, ds + \varepsilon \exp(5S_1)\Big)\,, \qquad (2.192)$$

where we indicate $S_1 = S_2 + S_0$.

Multiplying both sides of (2.128) by $\phi_0(s')$ (see definition (2.88)) and integrating in $(-\infty, \infty)$, we deduce

$$\int_{-\infty}^{\infty} \hat{\xi}_0\, \phi\, ds \geq -C\left\{\int_0^1 \left[\frac{d}{d\hat{z}}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right]^2 + \left[\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w\right]^2 d\hat{z},\right\}, \qquad (2.193)$$

for any smooth w, supported in $\hat{z} \in [0,1]$ and satisfying the boundary conditions $w = \frac{dw}{d\hat{z}} = \left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w = 0$ at $\hat{z} = 0$, where as before we use the abbreviation

$$\phi \;:=\; w\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 w \qquad (2.194)$$

for the multiplier. This time, w will *not* be compactly supported in $\hat{z} \in (0,1]$ so that the boundary conditions matters. Using the fact that the function $\hat{z}\sinh\hat{z}$ satisfies these boundary conditions, we enforce them for w by the Ansatz

$$w \;=\; (\hat{z}\sinh\hat{z})\hat{w} \quad \text{with} \quad \hat{w} = const \;\text{ for }\; \hat{z} \ll 1\,. \qquad (2.195)$$

As in the previous subsections, it is more telling to express (2.193) in terms of the s-variable. Appealing to the estimates

$$\begin{aligned}&(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \qquad (2.196)\\ &= \hat{z}^{-2}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 2)(\partial_s - 1) + 4\cosh\hat{z}(\partial_s - 1) + 4\hat{z}\sinh\hat{z}\Big) \times (\partial_s + 1)\partial_s\,.\end{aligned}$$

and

$$\begin{aligned}&\partial_{\hat{z}}(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z}\\ &= \hat{z}^{-3}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 3)(\partial_s - 2)(\partial_s - 1) + 5\cosh\hat{z}(\partial_s - 2)(\partial_s - 1)\\ &\quad + 8\hat{z}\sinh\hat{z}(\partial_s - 1) + 4\hat{z}^2\cosh\hat{z}\Big) \times (\partial_s + 1)\partial_s\,. \qquad (2.197)\end{aligned}$$

(their proves are reported in Appendix to Subsection 2.2.5.3 (see (2.196) and (2.270))) we obtain

$$\int_{-\infty}^{\infty} \hat{\xi}_0 \phi\, ds \geq -C\int_{-\infty}^{\infty} \exp(-5s)\left[\left(\frac{d^5\hat{w}}{ds^5}\right)^2 + \cdots + \left(\frac{d\hat{w}}{ds}\right)^2\right] ds\,, \qquad (2.198)$$

where according to the formula

$$\begin{aligned}&\hat{z}\sinh\hat{z}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \hat{z}\sinh\hat{z} \qquad (2.199)\\ &= \left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds} - 2\left(\frac{d}{ds}+1\right)\frac{d}{ds}\,,\end{aligned}$$

(the argument for the formula above is given at the end of this section, see (2.221)), the multiplier is given by

$$\phi \;=\; \hat{w}\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds} - 2\right]\hat{w}\,. \qquad (2.200)$$

We now make the following Ansatz for $\hat{w}$:

$$\hat{w} \;=\; \frac{1}{\sqrt{\varepsilon}}\hat{w}_0 + \sqrt{\varepsilon}\hat{w}_1\,, \tag{2.201}$$

with the constraints

$$\hat{w}_0 \;=\; \left\{ \begin{array}{ccc} 1 & \text{for} & s \le -S_0 - 1 \\ 0 & \text{for} & s \ge -S_0 \end{array} \right\}, \quad \hat{w}_1 \;=\; \left\{ \begin{array}{ccc} const & \text{for} & s \le -S_1 \\ 0 & \text{for} & s \ge -S_0 - 1 \end{array} \right\}, \tag{2.202}$$

so that (2.195) is satisfied. We don't want to specify the value of the constant appearing in the definition of w since it will not appear in the future estimates. The merit of the Ansatz (2.201) is that, because $\frac{d\hat{w}_0}{ds}$ and $\frac{d\hat{w}_1}{ds}$ have disjoint support, the multiplier ϕ, cf. (2.200), splits into three parts

$$\phi \;=\; \frac{1}{\varepsilon}\phi_0 + \phi_{01} + \varepsilon\phi_1\,, \tag{2.203}$$

where

$$\begin{aligned} \phi_0 \;&:=\; \hat{w}_0\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right]\hat{w}_0, \\ \phi_{01} \;&:=\; \hat{w}_0\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right]\hat{w}_1, \\ \phi_1 \;&:=\; \hat{w}_1\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right]\hat{w}_1\,. \end{aligned} \tag{2.204}$$

As a related side effect of the disjoint support of the functions $\frac{d\hat{w}_0}{ds}$ and $\frac{d\hat{w}_1}{ds}$, the error term in (2.198) splits into two parts:

$$\int_{-\infty}^{\infty}\exp(-5s)\left[\left(\frac{d^5\hat{w}}{ds^5}\right)^2+\cdots+\left(\frac{d\hat{w}}{ds}\right)^2\right]ds$$

$$= \; \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\exp(-5s)\left[\left(\frac{d^5\hat{w}_0}{ds^5}\right)^2+\cdots+\left(\frac{d\hat{w}_0}{ds}\right)^2\right]ds \tag{2.205}$$

$$+ \; \varepsilon\int_{-\infty}^{\infty}\exp(-5s)\left[\left(\frac{d^5\hat{w}_1}{ds^5}\right)^2+\cdots+\left(\frac{d\hat{w}_1}{ds}\right)^2\right]ds\,. \tag{2.206}$$

Hence in the sequel, we will have to consider five terms:

- Three multiplier terms: $\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\hat{\xi}_0\phi_0\,ds$, $\int_{-\infty}^{\infty}\hat{\xi}_0\phi_{01}\,ds$, and $\varepsilon\int_{-\infty}^{\infty}\hat{\xi}_0\phi_1\,ds$.
- Two error terms: the $\hat{w}_0$-error term (2.205) and the $\hat{w}_1$-error term (2.206).

Below, we will construct $\hat{w}_1$ such that the mixed expression ϕ_{01}, cf. (2.204), in the multiplier ϕ gives rise to the left-hand side of (2.192). Before, we address the multiplier and the error term that only involve $\hat{w}_0$. Clearly, $\hat{w}_0$ can be chosen to satisfy S_0-independent bounds:

$$\sup_{s\in\mathbb{R}}|\hat{w}_0|,\;\cdots\;\sup_{s\in\mathbb{R}}\left|\frac{d^5\hat{w}_0}{ds^5}\right| \;\le\; C\,.$$

Hence in view of (2.202), we obtain for the $\hat{w}_0$-error term (2.205)

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\exp(-5s)\left[\left(\frac{d^5\hat{w}_0}{ds^5}\right)^2+\cdots+\left(\frac{d\hat{w}_0}{ds}\right)^2\right]ds \;\le\; C\frac{1}{\varepsilon}\exp(5S_0). \tag{2.207}$$

Moreover, in view of (2.202), we obtain

$$|\phi_0| \;\le\; \begin{cases} 0 & \text{for} \qquad\qquad\quad s \;\le -S_0-1\,, \\ C_0 & \text{for} \quad -S_0-1\le \;\; s \;\le -S_0\,, \\ 0 & \text{for} \qquad\qquad\quad s \;\ge -S_0\,, \end{cases} \tag{2.208}$$

where we momentarily want to remember the value of the universal constant C_0. From (2.208) we have

$$\begin{aligned} \int_{-S_0-1}^{-S_0} \hat\xi_0(\phi_0-C_0)\,ds \quad &= \quad \int_{-S_0-1}^{-S_0} (-\hat\xi_0)(-\phi_0+C_0)\,ds \\ &\overset{(2.208)\&(2.89)}{\le} \quad C\int_{-S_0-1}^{-S_0} \exp(-3s)(-\phi_0+C_0)\,ds \\ &\overset{(2.208)}{\le} \quad 2CC_0\int_{-S_0-1}^{-S_0} \exp(-3s)\,ds \\ &\le \quad C\exp(3S_0)\,, \end{aligned} \tag{2.209}$$

so that for the ϕ_0-multiplier term we obtain

$$\begin{aligned} \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \hat\xi_0\phi_0\,ds \quad &\overset{(2.208)}{=} \quad \frac{1}{\varepsilon}\int_{-S_0-1}^{-S_0} \hat\xi_0\phi_0\,ds \\ &\le \quad \frac{1}{\varepsilon}\left(C_0\int_{-S_0-1}^{-S_0}\hat\xi_0\,ds + C\exp(3S_0)\right) \\ &\le \quad C\frac{1}{\varepsilon}\left(\max\left\{\int_{-S_0-1}^{-S_0}\hat\xi_0\,ds, 0\right\} + \exp(3S_0)\right)\,. \end{aligned} \tag{2.210}$$

We now specify $\hat w_1$ with the goal that ϕ_{01}, cf. (2.204), gives rise to the l. h. s. of (2.192). This motivates the construction of a universal function $\hat w_2$ with the property that

$$\left(\frac{d}{ds}+1\right)\frac{d}{ds}\left[\left(\frac{\sinh\hat z}{\hat z}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds}-2\right]\hat w_2 \;=\; 1 \quad \text{for } s\le -C\,, \tag{2.211}$$

which will be carried out below in such a way that

$$\frac{|\hat w_2|}{|s|+1},\; \left|\frac{d\hat w_2}{ds}\right|,\; \cdots\; \left|\frac{d^5\hat w_2}{ds^5}\right| \;\le\; C\,. \tag{2.212}$$

Equipped with $\hat w_2$, we now make the Ansatz

$$\hat w_1(s) \;=\; \eta(s+S_1)\eta(-(s+S_0+1))\hat w_2(s) + (1-\eta(s+S_1))\hat w_2(-S_1)\,, \tag{2.213}$$

where η is a universal cut-off function with

$$\eta(s) = \begin{cases} 0 & \text{for } s\le 0\,, \\ 1 & \text{for } s\ge 1\,, \end{cases} \tag{2.214}$$

so that (2.202) is satisfied. The main merit of Ansatz (2.213) & (2.214) is that it makes use of (2.211) which for $S_0\ge C$ due to (2.204) yields

$$\phi_{01} = \begin{cases} 0 & \text{for} \qquad\qquad\quad s \;\le -S_1\,, \\ 1 & \text{for} \quad -S_1+1\le \;\; s \;\le -S_0-S_2\,, \\ 0 & \text{for} \qquad\qquad\quad s \;\ge -S_0-1\,. \end{cases} \tag{2.215}$$

Furthermore, the estimates (2.212) turn into

$$|\phi_{01}|,\ \frac{|\hat{w}_1|}{S_1},\ \left|\frac{d\hat{w}_1}{ds}\right|,\cdots,\left|\frac{d^5\hat{w}_1}{ds^5}\right| \ \le\ C\,. \tag{2.216}$$

In particular, we obtain for the ϕ_{01}-multiplier term

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{\xi}_0\phi_{01}\,ds \qquad (2.217)\\
\overset{(2.215)}{=}\ & \int_{-S_1}^{-S_0-1}\hat{\xi}_0\,ds+\int_{-S_1}^{-S_1+1}\hat{\xi}_0(\phi_{01}-1)\,ds+\int_{-S_0-2}^{-S_0-1}\hat{\xi}_0(\phi_{01}-1)\,ds\\
\overset{(2.216)}{\le}\ & \int_{-S_1}^{-S_0-1}\hat{\xi}_0\,ds+C\int_{-S_1}^{-S_1+1}|\hat{\xi}_0|\,ds+C\exp(3S_0)\,,
\end{aligned}$$

where for $\int_{-S_0-2}^{-S_0-1}\hat{\xi}_0(\phi_{01}-1)\,ds$, we have used the same argument as leading to (2.209). Because of $\phi_1=\hat{w}_1\phi_{01}$ another consequence of (2.216) and (2.215) is

$$|\phi_1|\ \le\ \begin{cases} 0 & \text{for} \quad s\ \le -S_1\,,\\ CS_1 & \text{for} \quad -S_1\le s\ \le -S_0-1\,,\\ 0 & \text{for} \quad s\ \ge -S_0-1\,.\end{cases}$$

By the same argument as leading to (2.210), this implies for the ϕ_1-multiplier term

$$\varepsilon\int_{-\infty}^{\infty}\hat{\xi}_0\phi_1\,ds\ \le\ C\varepsilon S_1\left(\max\left\{\int_{-S_1}^{-S_0-1}\hat{\xi}_0ds,0\right\}+\exp(3S_1)\right)\,. \tag{2.218}$$

We finally address the $\hat{w}_1$-error term (2.206). It follows from (2.202) and (2.216) that

$$\varepsilon\int_{-\infty}^{\infty}\exp(-5s)\left[\left(\frac{d^5\hat{w}_1}{ds^5}\right)^2+\cdots+\left(\frac{d\hat{w}_1}{ds}\right)^2\right]ds\ \le\ C\varepsilon\exp(5S_1)\,. \tag{2.219}$$

We now collect the five estimates (2.207), (2.210), (2.217), (2.218), and (2.219). Via (2.203) and (2.206) we obtain from (2.198) that

$$\begin{aligned}
&-\int_{-S_1}^{-S_0-1}\hat{\xi}_0\,ds\\
\le\quad & C\frac{1}{\varepsilon}\exp(5S_0)+C\frac{1}{\varepsilon}\left(\max\left\{\int_{-S_0-1}^{-S_0}\hat{\xi}_0\,ds,0\right\}+\exp(3S_0)\right)\\
+\quad & C\left(\int_{-S_1}^{-S_1+1}|\hat{\xi}_0|\,ds+\exp(3S_0)\right)\\
+\quad & C\varepsilon S_1\left(\max\left\{\int_{-S_1}^{-S_0-1}\hat{\xi}_0ds,0\right\}+\exp(3S_1)\right)+C\varepsilon\exp(5S_1)\\
\overset{S_1\ge S_0\ge 0,\varepsilon\le 1}{\le}\quad & C\frac{1}{\varepsilon}\left(\max\left\{\int_{-S_0-1}^{-S_0}\hat{\xi}_0\,ds,0\right\}+\exp(5S_0)\right)\\
+\quad & C\int_{-S_1}^{-S_1+1}|\hat{\xi}_0|\,ds+C\varepsilon S_1\max\left\{\int_{-S_1}^{-S_0-1}\hat{\xi}_0ds,0\right\}+C\varepsilon\exp(5S_1)\,,
\end{aligned}$$

which implies

$$\begin{aligned}
-\int_{-S_1}^{-S_0-1}\hat{\xi}_0\,ds\ \le\ & C\frac{1}{\varepsilon}\left(\int_{-S_0-1}^{-S_0}\hat{\xi}_0\,ds+\exp(5S_0)\right)\\
&+\ C\int_{-S_1}^{-S_1+1}|\hat{\xi}_0|\,ds+C\varepsilon S_1\max\left\{\int_{-S_1}^{-S_0-1}\hat{\xi}_0ds,0\right\}+C\varepsilon\exp(5S_1)
\end{aligned} \tag{2.220}$$

thanks to the (2.142). We now distinguish two cases: when $\int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds \geq 0$, then the estimate (2.192) is trivially true by the positivity of the terms on its right-hand side; indeed by the approximate positivity estimate (2.125) we have $\int_{-S_0-1}^{-S_0} \hat{\xi}_0\, ds + C_0 \exp(3S_0) \geq 0$ and in particular $\int_{-S_0-1}^{-S_0} \hat{\xi}_0\, ds + \exp(5S_0) \geq 0$. When $\int_{-S_1}^{-S_0-1} \hat{\xi}_0\, ds < 0$ then we run the argument above and from (2.220) we obtain (2.192).

We will now derive the operator-valued formula

$$\begin{aligned} &\hat{z}\sinh\hat{z}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \hat{z}\sinh\hat{z} \\ &= \left(\frac{d}{ds}+1\right)\frac{d}{ds}\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\left(\frac{d}{ds}+1\right)\frac{d}{ds} - 2\left(\frac{d}{ds}+1\right)\frac{d}{ds}\,, \end{aligned} \tag{2.221}$$

that is a non-homogeneous generalization of $\hat{z}^2\frac{d^4}{d\hat{z}^4}\hat{z}^2 = (\frac{d}{ds}+2)(\frac{d}{ds}+1)\frac{d}{ds}(\frac{d}{ds}-1)$ (cf. (2.41)). The fairly simple structure of this formula is not a surprise: Since the functions $\sinh\hat{z}$ and $\hat{z}\sinh\hat{z}$ are in the kernel of $(-\frac{d^2}{d\hat{z}^2}+1)^2$, the functions 1 and $\hat{z}^{-1}$ are in the kernel of $(-\frac{d^2}{d\hat{z}^2}+1)^2\hat{z}\sinh\hat{z}$. In s coordinates, these functions are 1 and $\exp(-s)$, respectively. This explains the *right* factor $(\frac{d}{ds}+1)\frac{d}{ds}$ on the r. h. s. of (2.221). On the other hand, the *adjoint* of the l. h. s. of (2.221) *with respect to the measure* $\frac{d\hat{z}}{\hat{z}} = ds$ is given by $\hat{z}\sinh\hat{z}(-\frac{d^2}{d\hat{z}^2}+1)^2\sinh\hat{z}$ and thus has a kernel containing 1 and $\hat{z} = \exp(s)$. Hence the adjoint of the r. h. s. of (2.221) w. r. t. ds has to contain the right factor $(\frac{d}{ds}-1)\frac{d}{ds}$, which means that the operator itself should contain the *left* factor $(\frac{d}{ds}+1)\frac{d}{ds}$.

We claim that the formula (2.221) can be split into the two formulas

$$\left(\frac{d^2}{d\hat{z}^2}-1\right)\hat{z}\sinh\hat{z} = \left(\frac{\sinh\hat{z}}{\hat{z}}\frac{d}{ds}+2\cosh\hat{z}\right)\left(\frac{d}{ds}+1\right), \tag{2.222}$$

$$\hat{z}\sinh\hat{z}\left(\frac{d^2}{d\hat{z}^2}-1\right) = \frac{d}{ds}\left[\left(\frac{d}{ds}+1\right)\frac{\sinh\hat{z}}{\hat{z}} - 2\cosh\hat{z}\right]. \tag{2.223}$$

Indeed, the composition of (2.222) and (2.223) yields

$$\begin{aligned} &\hat{z}\sinh\hat{z}\left(-\frac{d^2}{d\hat{z}^2}+1\right)^2 \hat{z}\sinh\hat{z} \\ =\ & \frac{d}{ds}\left(\frac{d}{ds}+1\right)\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right) \\ & -2\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\frac{d}{ds}\left(\frac{d}{ds}+1\right) + 2\frac{d}{ds}\left(\frac{d}{ds}+1\right)\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\left(\frac{d}{ds}+1\right) \\ & -4\frac{d}{ds}(\cosh\hat{z})^2\left(\frac{d}{ds}+1\right) \\ =\ & \frac{d}{ds}\left(\frac{d}{ds}+1\right)\left(\frac{\sinh\hat{z}}{\hat{z}}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right) \\ & +2\frac{d}{ds}\left[\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right) + \cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}} - 2(\cosh\hat{z})^2\right]\left(\frac{d}{ds}+1\right)\,, \end{aligned} \tag{2.224}$$

where $(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}})$ denotes the multiplication with the s-derivative of the function $\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}$. This implies (2.221) since because of

$$\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right) = \hat{z}\left(\frac{d}{d\hat{z}}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right) = (\sinh\hat{z})^2 + (\cosh\hat{z})^2 - \cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\,,$$

the factor in the last term of (2.224) simplifies to

$$\left(\frac{d}{ds}\cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}}\right) + \cosh\hat{z}\frac{\sinh\hat{z}}{\hat{z}} - 2(\cosh\hat{z})^2 = (\sinh\hat{z})^2 - (\cosh\hat{z})^2 = -1\,.$$

We now turn to the argument for (2.222) and (2.223). We first note that (2.222) and (2.223) reduce to

$$\left(\frac{d^2}{d\hat z^2}-1\right)\hat z\exp\hat z \;=\; \left(\frac{\exp\hat z}{\hat z}\frac{d}{ds}+2\exp\hat z\right)\left(\frac{d}{ds}+1\right) \tag{2.225}$$

$$= \left(\exp\hat z\frac{d}{d\hat z}+2\exp\hat z\right)\left(\hat z\frac{d}{d\hat z}+1\right)$$

$$= \exp\hat z\left(\frac{d}{d\hat z}+2\right)\frac{d}{d\hat z}\hat z \quad \text{and} \tag{2.226}$$

$$\hat z\exp\hat z\left(\frac{d^2}{d\hat z^2}-1\right) \;=\; \frac{d}{ds}\left[\left(\frac{d}{ds}+1\right)\frac{\exp\hat z}{\hat z}-2\exp\hat z\right] \tag{2.227}$$

$$= \hat z\frac{d}{d\hat z}\left[\left(\hat z\frac{d}{d\hat z}+1\right)\frac{\exp\hat z}{\hat z}-2\exp\hat z\right]$$

$$= \hat z\frac{d}{d\hat z}\left(\frac{d}{d\hat z}-2\right)\exp\hat z\,. \tag{2.228}$$

Indeed, replacing $\hat z$ by $-\hat z$ in (2.225), using the invariance of $\frac{d}{ds}=\hat z\frac{d}{d\hat z}$ under this change of variables, and adding both identities yields (2.222). Likewise, (2.227) yields (2.223). The identities (2.226) and (2.228) can easily be checked using the commutator relation $\frac{d}{d\hat z}\exp\hat z = \exp\hat z\left(\frac{d}{d\hat z}+1\right)$ on their left hand sides:

$$\left(\frac{d^2}{d\hat z^2}-1\right)\exp\hat z \;=\; \exp\hat z\left[\left(\frac{d}{d\hat z}+1\right)^2-1\right]=\exp\hat z\left(\frac{d}{d\hat z}+2\right)\frac{d}{d\hat z} \quad \text{and}$$

$$\exp\hat z\left(\frac{d^2}{d\hat z^2}-1\right) \;=\; \left[\left(\frac{d}{d\hat z}-1\right)^2-1\right]\exp\hat z=\frac{d}{d\hat z}\left(\frac{d}{d\hat z}-2\right)\exp\hat z\,.$$

We will now turn to the construction of the function $\hat w_2$ with (2.211) and (2.212).
We start by reducing (2.211) to a second-order problem with bounded right-hand side: It is enough to construct a universal smooth $\hat v_2$ with

$$\left[\frac{d}{ds}\left(\frac{\sinh\hat z}{\hat z}\right)^2\left(\frac{d}{ds}+1\right)-2\right]\hat v_2 \;=\; 1 \quad \text{for } s \;\le\; -S_0 \tag{2.229}$$

and

$$|\hat v_2|,\ \left|\frac{d\hat v_2}{ds}\right|,\ \cdots,\ \left|\frac{d^4\hat v_2}{ds^4}\right| \;\le\; C \quad \text{for all } s\,. \tag{2.230}$$

Indeed, consider the anti derivative $\hat w_2(s):=\int_0^s\hat v_2 ds'$. Since $\frac{d\hat w_2}{ds}=\hat v_2$, the estimates (2.230) turn into the estimates (2.212). Moreover, (2.229) yields

$$\left[\left(\frac{\sinh\hat z}{\hat z}\right)^2\frac{d}{ds}\left(\frac{d}{ds}+1\right)-2\right]\hat w_2 \;=\; s+const \quad \text{for } s \;\le\; -S_0\,,$$

for some constant of integration. Applying $\frac{d}{ds}\left(\frac{d}{ds}+1\right)$ to the last identity yields (2.211).

We now extend (2.229) to a problem on the entire line with nearly constant coefficients. Note that the coefficient $\left(\frac{\sinh\hat z}{\hat z}\right)^2$ is an entire, even function in $\hat z$ with value 1 at $\hat z=0$. Hence for every $S_0\gg 1$, we may write

$$\left(\frac{\sinh\hat z}{\hat z}\right)^2 \;=\; 1-a \quad \text{for all } s\le -S_0\,,$$

where

$$\sup_{s\in\mathbb{R}} |a|, \sup_{s\in\mathbb{R}} \left|\frac{da}{ds}\right|, \cdots, \sup_{s\in\mathbb{R}} \left|\frac{d^3a}{ds^3}\right| \;\le\; C\exp(-2S_0)\,. \tag{2.231}$$

We will thus construct a universal smooth $\hat{v}_2(s)$ with

$$\left[\frac{d}{ds}(1-a)\left(\frac{d}{ds}+1\right)-2\right]\hat{v}_2 \;=\; 1 \quad \text{for all } s \tag{2.232}$$

and

$$\sup_{s\in\mathbb{R}} |\hat{v}_2|, \sup_{s\in\mathbb{R}} \left|\frac{d\hat{v}_2}{ds}\right|, \cdots, \sup_{s\in\mathbb{R}} \left|\frac{d^4\hat{v}_2}{ds^4}\right| < \infty\,. \tag{2.233}$$

We finally reformulate (2.232) as a fixed point problem. Note that since $\frac{d}{ds}(\frac{d}{ds}+1)-2 = (\frac{d}{ds}-1)(\frac{d}{ds}+2)$, the bounded solution of $\left[\frac{d}{ds}(\frac{d}{ds}+1)-2\right]\hat{v} \;=\; \hat{f}$ for some bounded continuous $\hat{f}$ is given by

$$\begin{aligned}
\hat{v}(s) &= -\int_{-\infty}^{s}\exp(2(s'-s))\int_{s'}^{\infty}\exp(s'-s'')\hat{f}(s'')\,ds''\,ds' \\
&= -\frac{1}{3}\int_{-\infty}^{\infty}\exp(3\min\{s,s''\}-2s-s'')\hat{f}(s'')\,ds'' \\
&=: (T\hat{f})(s)\,,
\end{aligned} \tag{2.234}$$

defining an operator T. From its above representation with the Lipschitz-continuous kernel $\exp(3\min\{s,s''\}-2s-s'')$ we read off that T is a bounded operator from C^0 (the space of bounded continuous functions endowed with the sup norm) into C^1 and by the solution property of T thus also into C^2. Note that (2.232) can be reformulated as

$$\begin{aligned}
&\left[\frac{d}{ds}\left(\frac{d}{ds}+1\right)-2\right]\hat{v}_2 \\
&= 1+\frac{d}{ds}a\left(\frac{d}{ds}+1\right)\hat{v}_2 \\
&= 1+\left[\left(\frac{d^2}{ds^2}+\frac{d}{ds}\right)a-\frac{d}{ds}\frac{da}{ds}\right]\hat{v}_2 \\
&= 1+\left[\left(\frac{d}{ds}\left(\frac{d}{ds}+1\right)-2\right)a-\frac{d}{ds}\frac{da}{ds}+2a\right]\hat{v}_2\,.
\end{aligned} \tag{2.235}$$

An application of the translation-invariant operator T (formally) yields

$$\hat{v}_2 \;=\; T\,1+\left(a-\frac{d}{ds}T\frac{da}{ds}+2T\,a\right)\hat{v}_2\,. \tag{2.236}$$

We view this equation as a fixed-point equation for $\hat{v}_2$ in the Banach space C^0. As mentioned above, T and even the composition $\frac{d}{ds}T$ are bounded operators (in C^0). In view of (2.231), the multiplication with a and with $\frac{da}{ds}$ are operators with C^0-operator norm estimated by $C\exp(-2S_0)$. Hence for sufficiently large S_0, the operator $a-\frac{d}{ds}T\frac{da}{ds}+2T\,a$ has norm strictly less than one. Thus the contraction mapping theorem ensures the existence of a solution of (2.236), that is, a C^2-solution $\hat{v}_2$ of (2.232) with $\sup_{s\in\mathbb{R}}|\hat{v}_2|$, $\sup_{s\in\mathbb{R}}|\frac{d\hat{v}_2}{ds}|$, $\sup_{s\in\mathbb{R}}|\frac{d^2\hat{v}_2}{ds^2}|<\infty$. Finally, we obtain the rest of (2.233) from (2.231) by a booth-strap argument.

2.2.6 Finite lateral size

In Section 2.2 we assumed the stability condition to hold for all $k' \in \mathbb{R}$ (see (2.34)). In this subsection we go back to work with a container of finite lateral size, assuming $k' \in \frac{2\pi}{L}\mathbb{Z} \setminus \{0\}$. Therefore we can no longer pass to the limit $k' \to 0$ to infer the approximate logarithmic growth for τ (2.38) (see Proposition 1). In this case we show that Proposition 1 is still valid provided the height H and the lateral size L of the container satisfy a certain relation.

Proposition 2. *Let $\tau \in (0,H) \to \mathbb{R}$ satisfy the reduced stability condition, i.e. for all $k' \in \frac{2\pi}{L}\mathbb{Z} \setminus \{0\}$ and for all $w(z)$ satisfying (2.33), the condition (2.36) holds. If $H \ll L$, then we have*

-
$$\xi \geq 0\,, \tag{2.237}$$

-
$$\int_{1/e}^{1} \xi\, dz \lesssim \frac{1}{\ln H} \int_{1}^{H} \xi\, dz\,. \tag{2.238}$$

Proof of Proposition 2. The starting point is the reduced stability condition (2.36). Since the argument for (2.237) remains the same as the one for (2.37) in Proposition 1, we start directly with the proof of (2.238).
Argument for (2.238):
In order to infer the logarithmic growth we can no longer let $k' \to 0$ (that would contradict the fact that L is finite). Therefore we need to expand the multiplier in its three terms:

$$\phi := w \left(-\frac{d^4}{dz^2} + k'^2 \right)^2 \overline{w} = w \left(\frac{d^4}{dz^4} - 2k'^2 \frac{d^2}{dz^2} + k'^4 \right) \overline{w}. \tag{2.239}$$

Therefore the reduced stability condition can be rewritten as

$$\begin{aligned} &\int_0^H \xi w \left(-\frac{d^2}{dz^2} + k'^2 \right)^2 \overline{w}\, dz \\ &= \int_0^H \xi w \frac{d^4}{dz^4} \overline{w}\, dz - 2k'^2 \int_0^H \xi w \frac{d^2}{dz^2} \overline{w}\, dz + k'^4 \int_0^H w^2\, dz \geq 0 \end{aligned} \tag{2.240}$$

Focusing on the lower half of the container, i.e. let us think of $z \in \left(0, \frac{H}{2}\right)$, we make the following Ansatz

$$w = z^2 \hat{w}\,,$$

where $\hat{w}(z)$ is a real function with compact support in $(0,H)$.
The merit of this Ansatz is that in the new variable $\hat{w}$, the multiplier in (2.240) can be written in the scale invariant form

$$\begin{aligned} \phi &= w \left(\frac{d^4}{dz^4} - 2k'^2 \frac{d^2}{dz^2} + k'^4 \right) \overline{w} \\ &= \hat{w} \left(z\frac{d}{dz} + 2 \right) \left(z\frac{d}{dz} + 1 \right) \left(z\frac{d}{dz} \right) \left(z\frac{d}{dz} - 1 \right) \hat{w} \\ &- 2k'^2 \hat{w} z \left(z\frac{d}{dz} \right) \left(z\frac{d}{dz} + 1 \right) z\hat{w} + k'^4 z^4 |\hat{w}|^2\,. \end{aligned}$$

This suggests the introduction of the new variables

$$s = \ln z \;\; \text{and} \;\; \xi = z^{-1}\hat{\xi}, \tag{2.241}$$

for which the stability condition turns into

$$\int_{-\infty}^{\ln H} \hat{\xi}\, \phi\, ds \geq 0 \tag{2.242}$$

where

$$\phi = \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}-1\right)\hat{w}$$
$$-2k'^2\hat{w}\exp(s)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}+1\right)\exp(s)\hat{w}+k'^4\exp(4s)\hat{w}^2\,.$$

for all functions $\hat{w}$ with compact support in $z \in (0, H)$. Here it comes the following heuristic argument: For $H \gg 1$, we can think at test functions $\hat{w}$ that vary slowly in the logarithmic variables s. For these $\hat{w}$ we have

$$\begin{aligned}
\phi &= \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}-1\right)\hat{w} \qquad (2.243)\\
&\quad -2k'^2\hat{w}\exp(s)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}+1\right)\exp(s)\hat{w}+k'^4\exp(4s)\hat{w}^2\,.\\
&\approx -2\hat{w}\frac{d}{ds}\hat{w}-2k'^2\exp(s)\hat{w}\frac{d}{ds}\hat{w}\exp(s)+k'^4\exp(4s)\hat{w}^2 \qquad (2.244)\\
&= -\frac{d}{ds}\hat{w}^2-k'^2\frac{d}{ds}\left(\exp(s)\hat{w}\right)^2+k'^4\exp(4s)\hat{w}^2\,, \qquad (2.245)
\end{aligned}$$

and this in particular implies

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\ln H}\hat{\xi}\,\phi\,ds\\
&\approx \int_{-\infty}^{\ln H}\hat{\xi}\left(-\frac{d}{ds}\hat{w}^2-k'^2\frac{d}{ds}\left(\exp(s)\hat{w}\right)^2\right)ds+k'^4\int_{-\infty}^{\ln H}\hat{\xi}\exp(4s)\hat{w}^2\,ds\\
&= \int_{-\infty}^{\ln H}\frac{d\hat{\xi}}{ds}\left(\hat{w}^2+k'^2(\exp(s)\hat{w})^2\right)+k'^4\int_{-\infty}^{\ln H}\hat{\xi}\exp(4s)\hat{w}^2\,ds\\
&= \int_{-\infty}^{\ln H}\left((1+k'^2\exp(2s))\frac{d\hat{\xi}}{ds}+k'^4\exp(4s)\hat{\xi}\right)\hat{w}^2\,ds\,,
\end{aligned}$$

for all $\hat{w}(s)$ with compact support in $(-\infty, \ln H)$. Thus it follows that

$$\frac{d\hat{\xi}}{ds} \geq -\frac{k'^4\exp(4s)}{(1+k'^2\exp(2s))}\hat{\xi} \geq -k'^4\exp(4s)\hat{\xi}\,,$$

approximately on large $s-$scales.
This means that

$$\hat{\xi} \gtrsim \exp\left(-k'^4\exp(4s)\right).$$

The assumption $H \ll L$ implies $k'\exp(s) \ll 1$ and therefore we have

$$\hat{\xi} \geq \exp(-1). \qquad (2.246)$$

We can rewrite the last inequality in the original variables z and ξ

$$\xi \gtrsim \frac{1}{z},$$

which, recalling that $\xi := \frac{d\tau}{dz}$ and integrating in z, yields

$$\tau \gtrsim \ln(z).$$

We expect that (2.246) implies that for any $1 \ll S_1 \leq \ln H$:

$$\int_{-1}^{0}\hat{\xi}ds \lesssim \frac{1}{S_1}\int_{0}^{S_1}\hat{\xi}\,ds\,, \qquad (2.247)$$

which in the original variables (2.241), for $S_1 = \ln \frac{H}{2}$ turns into

$$\int_{1/e}^{1} \xi dz \lesssim \frac{1}{\ln H} \int_{1}^{H} \xi \, dz \, .$$

a behavior that corresponds to logarithmic growth on the level of the antiderivative τ of ξ. We now prove that (2.240) and (2.237) imply (2.247).

Argument for (2.238):

Let us rewrite the multiplier ϕ defined in (2.239) as follow:

$$\begin{aligned} \phi \;&=\; \hat{w}\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}-1\right)\hat{w} \\ &\quad -2k'^2\exp(2s)\hat{w}\left(\frac{d^2}{ds^2}+4\frac{d}{ds}+2\right)\hat{w}+k'^4\exp(4s)|\hat{w}|^2 \, . \end{aligned}$$

We observe that the operator

$$\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\left(\frac{d}{ds}\right)\left(z\frac{d}{ds}-1\right)-2k'^2\exp(2s)\left(\frac{d^2}{ds^2}+4\frac{d}{ds}+2\right)+k'^4\exp(4s)\, ,$$

is not translation invariant so that we cannot extend the integral (2.242) to the whole space $\mathbb{R}$. Therefore we define a new multiplier $\psi : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} \psi_{s'}(s) \;&=\; \hat{w}_{s'}(s)\Bigg[\left(\frac{d}{ds}+2\right)\left(\frac{d}{ds}+1\right)\left(\frac{d}{ds}\right)\left(\frac{d}{ds}-1\right) \\ &\quad -2k'^2\exp(2s)\left(\frac{d^2}{ds^2}+4\frac{d}{ds}+2\right)+k'^4\exp(4s)\Bigg]\hat{w}_{s'}(s), \end{aligned}$$

where $\hat{w}_{s'}$ is the translation of the function $\hat{w}$ of a parameter s'. so that we can write

$$\int_{-\infty}^{\infty} \hat{\xi}(s)\psi_{s'}(s)\, ds \,\geq 0. \tag{2.248}$$

We note that (2.247) follows from (2.248) once for given S_1 we construct

- a family $\mathfrak{F} = \{w_{s'}\}_{s'}$ of smooth functions $w_{s'}$ parameterized by $s' \in \mathbb{R}$ and compactly supported in $z \in (0,1)$ (i. e. $s \in (-\infty, 0]$) and
- a probability measure $\rho(ds')$ supported in $s' \in (-\infty, \ln H]$,

such that the corresponding convex combination of multipliers $\{\psi_{s'}\}_{s'}$ i. e.

$$\psi_1(s) \;:=\; \int_{-\infty}^{\infty} \psi_{s'}(s)\, \rho(s')ds' \, ,$$

satisfies

$$\psi_1(s) \leq \begin{cases} -1 & \text{for } \; -1 \leq s \leq 0, \\ \frac{C}{S_1} & \text{for } \quad 0 \leq s \leq S_1, \\ 0 & \text{else,} \end{cases} \tag{2.249}$$

for a (possibly large) universal constant C. Indeed, using (2.249) and the positivity (2.37) of the profile we have

$$0 \leq \int_{-\infty}^{\infty} \hat{\xi}\, \psi_1 \, ds \leq -\int_{-1}^{0} \hat{\xi}\, ds + \frac{C}{S_1}\int_{0}^{S_1} \hat{\xi}\, ds,$$

which implies

$$\int_{-1}^{0} \hat{\xi} ds \leq \frac{C}{S_1}\int_{0}^{S_1} \hat{\xi} ds.$$

We first address the form of the family $\mathfrak{F}$. We consider the change of variables

$$s = \lambda \hat{s} \quad \text{with} \quad \lambda \geq 1,$$

and fix a smooth "mask" $\hat{w}_0(\hat{s})$ compactly supported in

$$\operatorname{supp} \hat{w}_0 \subset \left[-\frac{1}{2}, 0\right] \tag{2.250}$$

and normalized, i.e. $\int \hat{w}_0^2 \, d\hat{s} = 1$. We choose

$$\hat{w}(\lambda \hat{s}) = \lambda^{-1/2} \hat{w}_0(\hat{s}). \tag{2.251}$$

With this change of variables, the multiplier can be estimated as follows

$$\begin{aligned}
\phi &= \hat{w} \left[\left(\frac{d}{ds} + 2\right)\left(\frac{d}{ds} + 1\right)\frac{d}{ds}\left(\frac{d}{ds} - 1\right) \right. \\
&\quad \left. -2k'^2 \exp(2s)\left(\frac{d^2}{ds^2} + 4\frac{d}{ds} + 2\right) + k'^4 \exp(4s) \right] \hat{w} \\
&= \frac{1}{\lambda}\hat{w}_0 \left[\left(\frac{1}{\lambda}\frac{d}{d\hat{s}} + 2\right)\left(\frac{1}{\lambda}\frac{d}{d\hat{s}} + 1\right)\frac{1}{\lambda}\frac{d}{d\hat{s}}\left(\frac{1}{\lambda}\frac{d}{d\hat{s}} - 1\right) \right. \qquad (2.252) \\
&\quad \left. -2k'^2 \exp(2\lambda\hat{s})\left(\frac{1}{\lambda^2}\frac{d^2}{d\hat{s}^2} + 4\frac{1}{\lambda}\frac{d}{d\hat{s}} + 2\right) + k'^4 \exp(4\lambda\hat{s}) \right] \hat{w}_0 \\
&= \hat{w}_0 \left[\left(\frac{1}{\lambda^5}\frac{d^4}{d\hat{s}^4} + \frac{2}{\lambda^4}\frac{d^3}{d\hat{s}^3} - \frac{1}{\lambda^3}\frac{d^2}{d\hat{s}^2} - \frac{2}{\lambda^2}\frac{d}{d\hat{s}}\right) \right. \qquad (2.253) \\
&\quad \left. -2k'^2 \exp(2\lambda\hat{s})\left(\frac{1}{\lambda^3}\frac{d^2}{d\hat{s}^2} + 4\frac{1}{\lambda^2}\frac{d}{d\hat{s}} + 2\frac{1}{\lambda}\right) + k'^4 \exp(4\lambda\hat{s}) \right] \hat{w}_0 \\
&\qquad (2.254)
\end{aligned}$$

and reordering the terms we have

$$\begin{aligned}
\phi &= -\frac{2}{\lambda^2}\hat{w}_0\frac{d}{d\hat{s}}\hat{w}_0 - \frac{1}{\lambda^3}\hat{w}_0\frac{d^2}{d\hat{s}^2}\hat{w}_0 + \frac{2}{\lambda^4}\hat{w}_0\frac{d^3}{d\hat{s}^3}\hat{w}_0 + \frac{1}{\lambda^5}\hat{w}_0\frac{d^4}{d\hat{s}^4}\hat{w}_0 \qquad (2.255) \\
&\quad -2k'^2 \exp(2\lambda\hat{s})\left(2\frac{1}{\lambda}\hat{w}_0^2 + 4\frac{1}{\lambda^2}\hat{w}_0\frac{d}{d\hat{s}}\hat{w}_0 + \frac{1}{\lambda^3}\hat{w}_0\frac{d^2}{d\hat{s}^2}\hat{w}_0\right) \\
&\quad +\frac{k'^4}{\lambda}\exp(4\lambda\hat{s})\hat{w}_0^2 = \\
&\quad \frac{1}{\lambda}\left(-4k'^2\exp(2\lambda\hat{s}) + k'^4\exp(4\lambda\hat{s})\right)\hat{w}_0^2 - \frac{1}{\lambda^2}(2 + 8k'^2\exp(2\lambda\hat{s}))\hat{w}_0\frac{d}{d\hat{s}}\hat{w}_0 \\
&\quad -\frac{1}{\lambda^3}(1 + 2k'^2\exp(2\lambda\hat{s}))\hat{w}_0\frac{d^2}{d\hat{s}^2}\hat{w}_0 + \frac{2}{\lambda^4}\hat{w}_0\frac{d^3}{d\hat{s}^3}\hat{w}_0 + +\frac{1}{\lambda^5}\hat{w}_0\frac{d^4}{d\hat{s}^4}\hat{w}_0 \,.
\end{aligned}$$

We now consider the translation of the mask $w_0(\hat{s})$ in the parameter $s' \in (-\infty, \ln H)$, namely,

$$w_{0,s'} = w_0(\hat{s} - s')\,,$$

and we define the multiplier

$$\begin{aligned}
\psi_{\lambda,s'}(s) &= \frac{1}{\lambda}\left(-4k'^2\exp(2\lambda\hat{s}) + k'^4\exp(4\lambda\hat{s})\right)\hat{w}_0^2(\hat{s} - s') \\
&\quad -\frac{1}{\lambda^2}(2 + 8k'^2\exp(2\lambda s))\hat{w}_0(\hat{s} - s')\frac{d}{d\hat{s}}\hat{w}_0(\hat{s} - s') \\
&\quad -\frac{1}{\lambda^3}(1 + 2k'^2\exp(2\lambda s))\hat{w}_0(\hat{s} - s')\frac{d^2}{d\hat{s}^2}\hat{w}_0(\hat{s} - s') \\
&\quad +\frac{2}{\lambda^4}\hat{w}_0(\hat{s} - s')\frac{d^3}{d\hat{s}^3}\hat{w}_0(\hat{s} - s') \\
&\quad +\frac{1}{\lambda^5}\hat{w}_0(\hat{s} - s')\frac{d^4}{d\hat{s}^4}\hat{w}_0(\hat{s} - s')\,.
\end{aligned}$$

We choose $\lambda = s'$ (the motivation can be found in the argument for (2.59), Subsection 2.2.1) obtaining

$$
\begin{aligned}
\psi_{s'}(s) &= \frac{1}{s'}\left(-4k'^2\exp(2s'\hat{s}) + k'^4\exp(4s'\hat{s})\right)\hat{w}_0^2(\hat{s}-s') \\
&\quad -\frac{1}{s'^2}(2+8k'^2\exp(2s'\hat{s}))\hat{w}_0(\hat{s}-s')\frac{d}{d\hat{s}}\hat{w}_0(\hat{s}-s') \\
&\quad -\frac{1}{s'^3}(1+2k'^2\exp(2s'\hat{s}))\hat{w}_0(\hat{s}-s')\frac{d^2}{d\hat{s}^2}\hat{w}_0(\hat{s}-s') \\
&\quad +\frac{2}{s'^4}\hat{w}_0(\hat{s}-s')\frac{d^3}{d\hat{s}^3}\hat{w}_0(\hat{s}-s') \\
&\quad +\frac{1}{s'^5}\hat{w}_0(\hat{s}-s')\frac{d^4}{d\hat{s}^4}\hat{w}_0(\hat{s}-s')\,.
\end{aligned}
$$

We can now write

$$
\begin{aligned}
\psi_1(s) &= \int \psi_{s'}(s)\rho(s')\,ds' \\
&= \int \frac{1}{s'}\left(-4k'^2\exp(2s'\hat{s}) + k'^4\exp(4s'\hat{s})\right)\hat{w}_0^2(\hat{s}-s')\rho(s')\,ds' \\
&\quad -\int \frac{1}{s'^2}(2+8k'^2\exp(2s'\hat{s}))\hat{w}_0(\hat{s}-s')\frac{d}{d\hat{s}}\hat{w}_0(\hat{s}-s')\rho(s')\,ds' \\
&\quad -\int \frac{1}{s'^3}(1+2k'^2\exp(2s'\hat{s}))\hat{w}_0(\hat{s}-s')\frac{d^2}{d\hat{s}^2}\hat{w}_0(\hat{s}-s')\rho(s')\,ds' \\
&\quad +\int \frac{2}{s'^4}\hat{w}_0(\hat{s}-s')\frac{d^3}{d\hat{s}^3}\hat{w}_0(\hat{s}-s')\rho(s')\,ds' \\
&\quad +\int \frac{1}{s'^5}\hat{w}_0(\hat{s}-s')\frac{d^4}{d\hat{s}^4}\hat{w}_0(\hat{s}-s')\rho(s')\,ds'
\end{aligned}
$$

Since $\lambda = s'$ then we have $s = s'\hat{s}$ and define the variable $\hat{s}''$ as follows

$$
\hat{s}'' = \frac{s-s'}{s'} = \frac{s}{s'} - 1 \Rightarrow s' = \frac{s}{1+\hat{s}''}\,, \tag{2.256}
$$

which is invertible with $ds' = -\frac{s}{(1+\hat{s})^2}d\hat{s}''$. Note that, in particular, $\hat{s} = \hat{s}''+1$. We thus get the representation

$$
\begin{aligned}
\psi_1(s) &= \int \phi_{s'}(s)\rho(s')\,ds' \\
&= \left(-4k'^2\exp(2s) + k'^4\exp(4s)\right)\int \frac{1}{(1+\hat{s}'')}\hat{w}_0^2(\hat{s}'')\rho(s')\,d\hat{s}'' \\
&\quad -(2+8k'^2\exp(2s))\frac{1}{s}\int \hat{w}_0(\hat{s}'')\frac{d}{d\hat{s}''}\hat{w}_0(\hat{s}'')\rho(s')\,d\hat{s}'' \\
&\quad -(1+2k'^2\exp(2s))\frac{1}{s^2}\int (1+\hat{s}'')\hat{w}_0(\hat{s}'')\frac{d^2}{d\hat{s}''^2}\hat{w}_0(\hat{s}'')\rho(s')\,d\hat{s}'' \\
&\quad +\frac{2}{s^3}\int (1+\hat{s}'')^2\hat{w}_0(\hat{s}'')\frac{d^3}{d\hat{s}''^3}\hat{w}_0(\hat{s}'')\rho(s')\,d\hat{s}'' \\
&\quad +\frac{1}{s^4}\int (1+\hat{s}'')^3\hat{w}_0(\hat{s}'')\frac{d^4}{d\hat{s}''^4}\hat{w}_0(\hat{s}'')\rho(s')\,d\hat{s}''\,.
\end{aligned}
$$

We rewrite (2.256) in terms of the three quadratic quantities $\hat{w}_0^2$, $(\frac{d\hat{w}_0}{d\hat{s}})^2$, and $(\frac{d^2\hat{w}_0}{d\hat{s}^2})^2$:

$$\begin{aligned}
\psi_{s'} &= \frac{1}{s'}(k'^4\exp(4s) - 4k'^2\exp(2s))\hat{w}_0^2 - \frac{1}{(s')^2}(2+8k'^2\exp(2s))\frac{d\hat{w}_0^2}{d\hat{s}} \\
&+\frac{1}{(s')^3}(1+2k'^2\exp(2s))\left[\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 - \frac{1}{2}\frac{d^2\hat{w}_0^2}{d\hat{s}^2}\right] \\
&+\frac{1}{(s')^4}\left[-3\frac{d}{d\hat{s}}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{d^3\hat{w}_0^2}{d\hat{s}^3}\right] \\
&+\frac{1}{(s')^5}\left[\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 - 2\frac{d^2}{d\hat{s}^2}\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 + \frac{1}{2}\frac{d^4\hat{w}_0^2}{d\hat{s}^4}\right] \\
&= \left(\frac{1}{s'}(k'^4\exp(4s) - 4k'^2\exp(2s)) - \frac{1}{(s')^2}(2+8k'^2\exp(2s))\frac{d}{d\hat{s}}\right. \\
&\left.-\frac{1}{2}\frac{1}{(s')^3}(1+2k'^2\exp(2s))\frac{d^2}{d\hat{s}^2} + \frac{1}{(s')^4}\frac{d^3}{d\hat{s}^3} + \frac{1}{2}\frac{1}{(s')^5}\frac{d^4}{d\hat{s}^4}\right)\hat{w}_0^2 \\
&+\left(\frac{1}{(s')^3}(1+2k'^2\exp(2s)) - 3\frac{1}{(s')^4}\frac{d}{d\hat{s}} - 2\frac{1}{(s')^5}\frac{d^2}{d\hat{s}^2}\right)\left(\frac{d\hat{w}_0}{d\hat{s}}\right)^2 \\
&+\frac{1}{(s')^5}\left(\frac{d^2\hat{w}_0}{d\hat{s}^2}\right)^2 . \qquad (2.257)
\end{aligned}$$

Now in this formula, using the change of variables $s' = \frac{s}{1+\hat{s}''}$ (between s' and $\hat{s}''$, with s as a parameter), we want to substitute the derivations $\frac{1}{(s')^m}\frac{d^n}{d\hat{s}''^n}$ by linear combinations of derivations of the form $\frac{1}{(s)^{m-k}}\frac{d^k}{ds'^k}(1+\hat{s}'')^{m-n-k}$ for $k = 0, \cdots, n$. The formulas

$$\frac{1}{(s')^{n+1}}\frac{d^n}{d\hat{s}''^n} = (-1)^n\frac{1}{s}\frac{d^n}{ds'^n}\frac{1}{(1+\hat{s}'')^{n-1}} , \qquad (2.258)$$

$$\frac{1}{(s')^4}\frac{d}{d\hat{s}''} = -\frac{1}{(s)^3}\frac{d}{ds'}(1+\hat{s}'')^2 - \frac{2}{(s)^4}(1+\hat{s}'')^3 , \qquad (2.259)$$

and

$$\frac{1}{(s')^5}\frac{d^2}{d\hat{s}''^2} = \frac{1}{(s)^3}\frac{d^2}{ds'^2}(1+\hat{s}'') + \frac{4}{(s)^4}\frac{d}{ds'}(1+\hat{s}'')^2 + \frac{6}{(s)^5}(1+\hat{s}'')^3. \qquad (2.260)$$

which are proved in Appendix 2.2.7, allow us to rewrite (2.257) as follows

$$\begin{aligned}
\psi_{s'} &= \frac{1}{s}\left[(k'^4\exp(4s) - 4k'^2\exp(2s))(1+\hat{s}'') + (2+8k'^2\exp(2s))\frac{d}{ds'}\right. \\
&\left.-\frac{1}{2}(1+2k'^2\exp(2s))\frac{d^2}{ds'^2}\frac{1}{(1+\hat{s}'')} - \frac{d^3}{ds'^3}\frac{1}{(1+\hat{s}'')^2} + \frac{1}{2}\frac{d^4}{ds'^4}\frac{1}{(1+\hat{s})^3}\right]\hat{w}_0^2 \\
&+ \left[\left(\frac{1+2k'^2\exp(2s)}{s^3} + \frac{6}{s^4} - \frac{12}{s^5}\right)(1+\hat{s}'')^3 + \left(\frac{3}{s^3} - \frac{8}{s^4}\right)\frac{d}{ds'}(1+\hat{s}'')^2\right. \\
&\left.- \frac{2}{s^3}\frac{d^2}{ds'^2}(1+\hat{s}'')\right]\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 + \frac{1}{s^5}(1+\hat{s}'')^5\left(\frac{d^2\hat{w}_0}{d\hat{s}''^2}\right)^2 .
\end{aligned}$$

The advantage of this form is that integrations by part in s' become easy, so that we obtain

$$\begin{aligned}
\psi_1 &= \frac{1}{s}\int_{-\infty}^{\infty} \hat{w}_0^2 \left[(k'^4 \exp(4s) - 4k'^2 \exp(2s))(1+\hat{s}'')\rho - (2+8k'^2\exp(2s))\frac{d\rho}{ds'} \right. \\
& \left. -\frac{1}{2}(1+2k'^2\exp(2s))\frac{1}{1+\hat{s}''}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s}'')^2}\frac{d^3\rho}{ds'^3} + \frac{1}{2}\frac{1}{(1+\hat{s}'')^3}\frac{d^4\rho}{ds'^4}\right] ds' \\
&+ \left(\frac{1+2k'^2\exp(2s)}{s^3} + \frac{6}{s^4} - \frac{12}{s^5}\right)\int_{-\infty}^{\infty}(1+\hat{s}'')^3\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \rho\, ds' \\
&- \left(\frac{3}{s^3} - \frac{8}{s^4}\right)\int_{-\infty}^{\infty}(1+\hat{s}'')^2\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \frac{d\rho}{ds'}\, ds' \\
&- \frac{2}{s^3}\int_{-\infty}^{\infty}(1+\hat{s}'')\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \frac{d^2\rho}{ds'^2}\, ds' \\
&+ \frac{1}{s^5}\int_{-\infty}^{\infty}(1+\hat{s}'')^5\left(\frac{d^2\hat{w}_0}{d\hat{s}''^2}\right)^2 \rho\, ds' .
\end{aligned}$$

Finally, using the substitution $\frac{ds'}{s} = -\frac{d\hat{s}''}{(1+\hat{s}'')^2}$, the last formula turns into the desired representation

$$\begin{aligned}
\psi_1 &= \int_{-\infty}^{\infty} \hat{w}_0^2 \left[(k'^4 \exp(4s) - 4k'^2 \exp(2s))\frac{1}{(1+\hat{s}'')}\rho - (2+8k'^2\exp(2s))\frac{1}{(1+\hat{s}'')^2}\frac{d\rho}{ds'} \right. \\
& \left. -\frac{1}{2}(1+2k'^2\exp(2s))\frac{1}{(1+\hat{s}'')^3}\frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s}'')^4}\frac{d^3\rho}{ds'^3} + \frac{1}{2}\frac{1}{(1+\hat{s}'')^5}\frac{d^4\rho}{ds'^4}\right] d\hat{s}'' \\
&+ \left(\frac{(1+2k'^2\exp(2s))}{s^2} + \frac{6}{s^3} - \frac{12}{s^4}\right)\int_{-\infty}^{\infty}(1+\hat{s}'')\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \rho\, d\hat{s}'' \\
&- \left(\frac{3}{s^2} - \frac{8}{s^3}\right)\int_{-\infty}^{\infty}\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \frac{d\rho}{ds'}\, d\hat{s}'' \\
&- \frac{2}{s^2}\int_{-\infty}^{\infty}\frac{1}{1+\hat{s}''}\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 \frac{d^2\rho}{ds'^2}\, d\hat{s}'' \\
&+ \frac{1}{s^4}\int_{-\infty}^{\infty}(1+\hat{s}'')^3\left(\frac{d^2\hat{w}_0}{d\hat{s}''^2}\right)^2 \rho\, d\hat{s}'' . \qquad (2.261)
\end{aligned}$$

From this representation we learn the following: If we assume that $\rho(s')$ varies only on large length scales, so that

1. $\frac{d\rho}{ds'}, \frac{d^2\rho}{ds'^2}, \dots \ll \rho$
2. $\frac{d^2\rho}{ds'^2}, \frac{d^3\rho}{ds'^3}, \dots \ll \frac{d\rho}{ds'}$

then for $s \gg 1$, we obtain to leading order from the above

$$\begin{aligned}
\psi_1 &\approx (k'^4\exp(4s) - 4k'^2\exp(2s))\int_{-\infty}^{\infty}\frac{1}{(1+\hat{s}'')}\hat{w}_0^2\rho\, d\hat{s}'' \\
&-(2+8k'^2\exp(2s))\int_{-\infty}^{\infty}\frac{1}{(1+\hat{s}'')^2}\hat{w}_0^2\frac{d\rho}{ds'}\, d\hat{s}'' \\
&+\frac{(1+2k'^2\exp(2s))}{s^2}\int_{-\infty}^{\infty}(1+\hat{s}'')\left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2\rho\, d\hat{s}'' .
\end{aligned}$$

If $\rho(s')$ varies slowly even on a logarithmic scale (so that e. g. $s'\frac{d\rho}{ds'}$ is negligible with respect to

ρ), the above further reduces to

$$\begin{aligned}\psi_1 \;\approx\; & (k'^4 \exp(4s) - 4k'^2 \exp(2s))\rho \int_{-\infty}^{\infty} \frac{1}{(1+\hat{s}'')} \hat{w}_0^2 \, d\hat{s}'' \\ & -(2 + 8k'^2 \exp(2s)) \frac{d\rho}{ds'} \int_{-\infty}^{\infty} \frac{1}{(1+\hat{s}'')^2} \hat{w}_0^2 \; d\hat{s}'' \\ & + \frac{(1 + 2k''^2 \exp(2s))}{s^2} \rho \int_{-\infty}^{\infty} (1+\hat{s}'') \left(\frac{d\hat{w}_0}{d\hat{s}''}\right)^2 d\hat{s}'' \,. \end{aligned} \tag{2.262}$$

We observe that $k'^4 \exp(4s) - 4s^2 \exp(2s) < 0$, and that by the hypothesis $H \ll L$ (which translates into $k' \exp(s) \ll 1$), the second -negative- term on the right-hand side of (2.262) dominates the third positive term provided

$$\frac{d\rho}{ds'} \gg \frac{1}{(s')^2} \,.$$

This is satisfied if ρ is of the form

$$\rho(s') = 1 - \frac{S_0}{s' - S_0} \quad \text{for} \quad S_0 \gg 1$$

and motivates the choice of ρ in the range $1 \ll s' \ll S_1$: We fix a smooth mask $\rho_0(\hat{s}')$ such that

$$\rho_0 \;=\; 0 \;\; \text{for } \hat{s}' \leq 0, \quad \frac{d\rho_0}{d\hat{s}'} \;>\; 0 \;\; \text{for } 0 < \hat{s}' \leq 2, \quad \rho_0 \;=\; 1 - \frac{1}{\hat{s}'} \;\; \text{for } 2 \leq \hat{s}' \,. \tag{2.263}$$

By the condition $0 < k' \exp(s) \ll 1$ (coming from the assumption $H \ll L$), we have

$$-(2 + 8k'^2 \exp(2s)) \lesssim -2 \,, \; -\frac{1}{2}(1 + 2k'^2 \exp(2s)) \lesssim -\frac{1}{2} \,, \; (1 + 2k'^2 \exp(2s)) \leq 3 \,,$$

from which we deduce that

$$\begin{aligned}\psi_1 \;\lesssim\; & \phi_2 := \int_{-\infty}^{\infty} \hat{w}_0^2 \left[-\frac{2}{(1+\hat{s}'')^2} \frac{d\rho}{ds'} - \frac{1}{2} \frac{1}{(1+\hat{s}'')^3} \frac{d^2\rho}{ds'^2} + \frac{1}{(1+\hat{s}'')^4} \frac{d^3\rho}{ds'^3} + \frac{1}{2} \frac{1}{(1+\hat{s}'')^5} \frac{d^4\rho}{ds'^4} \right] d\hat{s}'' \\ + \;\; & \left(\frac{3}{s^2} + \frac{6}{s^3} - \frac{12}{(s)^4} \right) \int_{-\infty}^{\infty} (1+\hat{s}'') \left(\frac{d\hat{w}_0}{d\hat{s}''} \right)^2 \rho \, d\hat{s}'' \\ - \;\; & \left(\frac{3}{s^2} - \frac{8}{(s)^3} \right) \int_{-\infty}^{\infty} \left(\frac{d\hat{w}_0}{d\hat{s}''} \right)^2 \frac{d\rho}{ds'} \, d\hat{s}'' \\ - \;\; & \frac{2}{s^2} \int_{-\infty}^{\infty} \frac{1}{1+\hat{s}''} \left(\frac{d\hat{w}_0}{d\hat{s}''} \right)^2 \frac{d^2\rho}{ds'^2} \, d\hat{s}'' \\ + \;\; & \frac{1}{s^4} \int_{-\infty}^{\infty} (1+\hat{s}'')^3 \left(\frac{d^2\hat{w}_0}{d\hat{s}''^2} \right)^2 \rho \, d\hat{s}'' \,. \end{aligned} \tag{2.264}$$

The observation that the multiplier ϕ_2 ($\phi_2 \sim \phi_1$ up to constants) satisfies (2.71) (see Section 2.2.1), completes the proof of (2.38) in the case $k' \in \frac{1}{L}\mathbb{Z} \setminus \{0\}$.

□

2.2.7 Appendix

Appendix for Section 2.2.1 and Subsection 2.2.5.2

If p, $\tilde{p}$ denote generic polynomial of degree n, we have

$$\begin{aligned}
\frac{1}{(s')^m}\frac{d^n}{d\hat{s}^n} &= \frac{1}{(s'')^n}\frac{1}{(s')^{m-n}}(1+\hat{s})^n\frac{d^n}{d\hat{s}^n} \\
&= \frac{1}{(s'')^n}\frac{1}{(s')^{m-n}}p((1+\hat{s})\frac{d}{d\hat{s}}) \\
&= \frac{1}{(s'')^n}\frac{1}{(s')^{m-n}}p(-s'\frac{d}{ds'}) \\
&= \frac{1}{(s'')^n}\tilde{p}(s'\frac{d}{ds'})\frac{1}{(s')^{m-n}} \\
&= \frac{1}{(s'')^n}\sum_{k=0}^{n}a_n\frac{d^k}{ds'^k}\frac{1}{(s')^{m-n-k}} \\
&= \sum_{k=0}^{n}a_n\frac{1}{(s'')^{m-k}}\frac{d^k}{ds'^k}(1+\hat{s})^{m-n-k}.
\end{aligned}$$

It remains to determine the coefficients $a_0, \cdots, a_n$. We start with the case $m = n+1$ (which yields the shortest formula). To this purpose, we again use $(1+\hat{s})\frac{d}{d\hat{s}} = -s'\frac{d}{ds'}$, which we rewrite as $\frac{d}{d\hat{s}}(1+\hat{s}) = -(s')^2\frac{d}{ds'}\frac{1}{s'}$. The latter yields

$$\left(\frac{d}{d\hat{s}}(1+\hat{s})\right)^n = (-1)^n(s')\left(s'\frac{d}{ds'}\right)^n\frac{1}{s'} \quad \text{for every } n \in \mathbb{N},$$

that implies inductively

$$\frac{d^n}{d\hat{s}^n}(1+\hat{s})^n = (-1)^n(s')^{1+n}\frac{d^n}{ds'^n}\frac{1}{s'} \quad \text{for every } n \in \mathbb{N},$$

which we rewrite as (using again $s'' = s'(1+\hat{s})$)

$$\frac{1}{(s')^{n+1}}\frac{d^n}{d\hat{s}^n} = (-1)^n\frac{d^n}{ds'^n}\frac{1}{s'}\frac{1}{(1+\hat{s})^n} = (-1)^n\frac{1}{s''}\frac{d^n}{ds'^n}\frac{1}{(1+\hat{s})^{n-1}}. \tag{2.265}$$

In view of the first line on the r. h. s. of (2.62), we need the latter transformation formula for $n = 1,2,3,4$. In view of the second line, we also need:

$$\begin{aligned}
\frac{1}{(s')^4}\frac{d}{d\hat{s}} &\overset{(2.265)}{=} -\frac{1}{s''}\frac{1}{(s')^2}\frac{d}{ds'} \\
&= -\frac{1}{s''}(\frac{d}{ds'}+\frac{2}{s'})\frac{1}{(s')^2} \\
&= -\frac{1}{(s'')^3}\frac{d}{ds'}(1+\hat{s})^2 - \frac{2}{(s'')^4}(1+\hat{s})^3,
\end{aligned} \tag{2.266}$$

$$\begin{aligned}
\frac{1}{(s')^5}\frac{d^2}{d\hat{s}^2} &\overset{(2.265)}{=} \frac{1}{s''}\frac{1}{(s')^2}\frac{d^2}{ds'^2}\frac{1}{1+\hat{s}} \\
&= \frac{1}{s''}\left(\frac{d^2}{ds'^2}+4\frac{d}{ds'}\frac{1}{s'}+6\frac{1}{(s')^2}\right)\frac{1}{(s')^2}\frac{1}{1+\hat{s}} \\
&= \frac{1}{(s'')^3}\frac{d^2}{ds'^2}(1+\hat{s}) + \frac{4}{(s'')^4}\frac{d}{ds'}(1+\hat{s})^2 + \frac{6}{(s'')^5}(1+\hat{s})^3.
\end{aligned} \tag{2.267}$$

Appendix for Subsection 2.2.5.3

The starting point is

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \\ = \Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 2)(\partial_s - 1) + 4\cosh\hat{z}(\partial_s - 1) + 4\hat{z}\sinh\hat{z}\Big) \times (\partial_s + 1)\partial_s\,. \tag{2.268}$$

Motivation for formula (2.268): The factor $(\partial_s + 1)\partial_s$ has to be there since $\hat{z}^{-1} = e^{-s}$ and 1 are in the kernel of $(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z}$, since $\sinh\hat{z}$ and $\hat{z}\sinh\hat{z}$ are in the kernel of $\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1$. Note that for $\hat{z} \ll 1$,

$$\hat{z}^{-1}\sinh\hat{z} = 1 + O(\hat{z}^2), \quad \cosh\hat{z} = 1 + O(\hat{z}^2), \quad \hat{z}\sinh\hat{z} = O(\hat{z}^2)\,,$$

so that for $\hat{z} \ll 1$, (2.268) collapses to

$$\hat{z}^2\partial_{\hat{z}}^4\hat{z}^2 \;=\; (\partial_s + 2)(\partial_s + 1)\partial_s(\partial_s - 1)\,. \tag{2.269}$$

This identity is easily seen to be true because both differential operators are of fourth order and are homogeneous of degree zero in $\hat{z}$, because the four functions $\hat{z}^{-2} = e^{-2s}$, $\hat{z}^{-1} = e^{-s}$, 1, and $\hat{z} = e^s$ are in the kernel of both differential operators, and because on $\hat{z}^2 = e^{2s}$, both operators give $4!\hat{z}^2 = 4!e^{2s}$.

We derive two formulas from (2.268):

$$(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \\ = \hat{z}^{-2}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 2)(\partial_s - 1) + 4\cosh\hat{z}(\partial_s - 1) + 4\hat{z}\sinh\hat{z}\Big) \\ \times(\partial_s + 1)\partial_s\,.$$

and

$$\partial_{\hat{z}}(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \\ = \hat{z}^{-3}\Big(\hat{z}^{-1}\sinh\hat{z}\,(\partial_s - 3)(\partial_s - 2)(\partial_s - 1) + 5\cosh\hat{z}(\partial_s - 2)(\partial_s - 1) \\ +8\hat{z}\sinh\hat{z}(\partial_s - 1) + 4\hat{z}^2\cosh\hat{z}\Big) \times (\partial_s + 1)\partial_s\,. \tag{2.270}$$

Let us give the argument for (2.268). Because of the transformation properties under $\hat{z} \rightsquigarrow -\hat{z}$, it suffices to show

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z}) \\ = \left[\hat{z}^{-1}\exp(\hat{z})\,(\partial_s + 2)(\partial_s - 1) + 4(\exp(\hat{z}) - \hat{z}^{-1}\exp(\hat{z}))(\partial_s - 1) + 4\hat{z}\exp(\hat{z})\right] \\ \times(\partial_s + 1)\partial_s\,,$$

which we rearrange as

$$\begin{aligned} &\hat{z}^3(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z}) \\ &\quad= \exp(\hat{z})\left[(\partial_s + 2)(\partial_s - 1) + 4(\hat{z} - 1)(\partial_s - 1) + 4\hat{z}^2\right](\partial_s + 1)\partial_s \\ &\quad= \exp(\hat{z})\left[(\partial_s - 2)(\partial_s - 1) + 4\hat{z}(\partial_s - 1) + 4\hat{z}^2\right](\partial_s + 1)\partial_s\,. \end{aligned} \tag{2.271}$$

We note that because of $\partial_{\hat{z}}\exp(\hat{z}) = \exp(\hat{z})(\partial_{\hat{z}} + 1)$, we have

$$\begin{aligned} (\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\exp(\hat{z}) &= \exp(\hat{z})\left[(\partial_{\hat{z}} + 1)^4 - 2(\partial_{\hat{z}} + 1)^2 + 1\right] \\ &= \exp(\hat{z})(\partial_{\hat{z}}^4 + 4\partial_{\hat{z}}^3 + 4\partial_{\hat{z}}^2)\,, \end{aligned}$$

so that

$$\hat{z}^2(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\exp(\hat{z}) \quad = \quad \exp(\hat{z})\left[\hat{z}^3\partial_{\hat{z}}^4\hat{z} + 4\hat{z}(\hat{z}^2\partial_{\hat{z}}^3\hat{z}) + 4\hat{z}^2(\hat{z}\partial_{\hat{z}}^2\hat{z})\right] .$$

Now (2.271) follows by inserting the formulas

$$\begin{aligned} \hat{z}\partial_{\hat{z}}^2\hat{z} &= (\partial_s + 1)\partial_s, \\ \hat{z}^2\partial_{\hat{z}}^3\hat{z} &= (\partial_s + 1)\partial_s(\partial_s - 1), \\ \hat{z}^3\partial_{\hat{z}}^4\hat{z} &= (\partial_s + 1)\partial_s(\partial_s - 1)(\partial_s - 2) . \end{aligned} \tag{2.272}$$

These formulas can easily seen to be true; let us address (2.272): Both sides are differential operators of order 4 that are homogeneous of degree 0 in $\hat{z}$; the kernel of both operators is spanned by the four functions $\hat{z}^{-1} = e^{-s}$, 1, $\hat{z} = e^s$, and $\hat{z}^2 = e^{2s}$; On $\hat{z}^3 = e^{3s}$, both operators yield $4!\hat{z}^3 = 4!e^{3s}$.

Formula (2.196) is an immediate consequence of (2.268). Formula (2.270) follows from (2.196) using the identities $\partial_{\hat{z}} = \hat{z}^{-1}\partial_s$ and

$$\begin{aligned} \partial_{\hat{z}}\hat{z}^{-3}\sinh\hat{z} &= \hat{z}^{-3}(\cosh\hat{z} - 3\hat{z}^{-1}\sinh\hat{z}), \\ \partial_{\hat{z}}4\hat{z}^{-2}\cosh\hat{z} &= \hat{z}^{-3}(4\hat{z}\sinh\hat{z} - 8\cosh\hat{z}), \\ \partial_{\hat{z}}4\hat{z}^{-1}\sinh\hat{z} &= \hat{z}^{-3}(4\hat{z}^2\cosh\hat{z} - 4\hat{z}\sinh\hat{z}) , \end{aligned}$$

which leads as desired to

$$\begin{aligned} &\partial_{\hat{z}}(\partial_{\hat{z}}^4 - 2\partial_{\hat{z}}^2 + 1)\hat{z}\sinh\hat{z} \\ &= \hat{z}^{-3}\big[(\hat{z}^{-1}\sinh\hat{z}\ ((\partial_s - 2)(\partial_s - 1)\partial_s - 3(\partial_s - 2)(\partial_s - 1)) \\ &\quad + \cosh\hat{z}\ (4(\partial_s - 1)\partial_s + (\partial_s - 2)(\partial_s - 1) - 8(\partial_s - 1)) \\ &\quad + \hat{z}\sinh\hat{z}\ (4\partial_s + 4(\partial_s - 1) - 4) \\ &\quad + \hat{z}^2\cosh\hat{z}\, 4\big] \times (\partial_s + 1)\partial_s . \end{aligned}$$

Chapter 3

Finite Prandtl number convection

Differently from the previous section, here we reintroduce the inertial term of the Navier-Stokes equation allowing the Pr number to be finite. In the case of $\mathrm{Pr} < \infty$, the lack of instantaneous *slaving* of the velocity field to the temperature field increases the difficulty in bounding the convection term $\left\langle \int u^z T\, dz \right\rangle$ in the definition of the Nusselt number and the background field method turns out to be no longer fruitful.
The first rigorous bound on the Nusselt number for arbitrary Pr was obtained by Constantin and Doering in [11], by the following argument: Using (1.16) and (1.19) we can write

$$\mathrm{Nu} \leq \frac{1}{\delta}\left\langle \int_0^\delta (T-1)u^z\, dz \right\rangle + \frac{1}{\delta}.$$

Applying the Cauchy-Schwartz inequality and the Poincaré inequality we can bound the right-hand side, obtaining

$$\mathrm{Nu} \leq \delta\left(\left\langle \int_0^1 |\nabla T|^2\, dz\right\rangle \left\langle \int_0^1 |\nabla u|^2\, dz\right\rangle\right)^{\frac{1}{2}} + \frac{1}{\delta}.$$

From the representation (1.13) and the the bound (1.15) we immediately get

$$\mathrm{Nu} \leq \delta(\mathrm{Nu}\mathrm{Ra}(\mathrm{Nu}-1))^{\frac{1}{2}} + \frac{1}{\delta}.$$

The optimization in δ leads to the choice $\delta = (\mathrm{Nu}\mathrm{Ra}(\mathrm{Nu}-1)))^{-\frac{1}{4}}$ which yields the bound

$$\mathrm{Nu} \leq \mathrm{Ra}^{\frac{1}{2}}.$$

We notice that this estimate is valid for all values of Pr and it is oblivious to replacing the no-slip boundary conditions by a no-stress boundary condition (the incompressibility condition is not used). In particular this result does not catch the scaling $\mathrm{Nu} \sim \mathrm{Pr}^{\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}}$ for $\mathrm{Pr} \ll 1$ and, as our result implies (see Theorem 2), it is suboptimal for $\mathrm{Pr} \gg 1$. The incompressibility condition combined with the no-slip boundary condition yields (1.18). This suggest the idea to control the Nusselt number with the second derivatives of u^z and calls for a maximal regularity argument for the equation

$$\begin{aligned}
\frac{1}{\mathrm{Pr}}(\partial_t u + (u\cdot\nabla)u) - \Delta u + \nabla p =& \mathrm{Ra} T e_z && \text{for} \quad 0<z<1\,,\\
\nabla\cdot u =& 0 && \text{for} \quad 0<z<1\,,\\
u =& 0 && \text{for} \quad z\in\{0,1\}\,,\\
u =& u_0 && \text{for} \quad t=0\,,
\end{aligned}$$

as we explain in what follows.

3.1 Prerequisite: Maximal Regularity

Let us first explain the meaning of the term "maximal regularity estimate " in a simpler example. Let us consider a function $u \in C_0^\infty$ in $\mathbb{R}^d$, in the L^2-space we have

$$||\Delta u||_{L^2} = ||\nabla^2 u||_{L^2}$$

where $\nabla^2 u = \sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j}$. Indeed, by exchanging derivatives, we obtain

$$\int_{\mathbb{R}^d} (\Delta u)^2 \, dx = -\int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial^3 u}{\partial x_j \partial x_j \partial x_i} \, dx = \int_{\mathbb{R}^d} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \, dx = \int_{\mathbb{R}^d} |\nabla^2 u|^2 \, dx \, .$$

The Calderón-Zygmund theory [18]tells us that the estimate $||\nabla^2 u||_{L^p} \lesssim C(d,p)||\Delta u||_{L^p}$ is true in every L^p spaces with $1 < p < \infty$ but fails in the "simpler" spaces L^1 and L^∞. In L^1 a counterexample is given by the fundamental solution of the Poisson equation in $\mathbb{R}^2$, i.e. $u(x_1, x_2) = (x_1^2 - x_2^2)\log(x_1^2 + x_2^2)$. Indeed, in this case $||\Delta u||_{L^1} = 1$ (since $-\Delta u = \delta$) but $||\nabla^2 u||_{L^1} = \infty$. By duality one can show that also the $L^\infty-$norm is non-admissible for the Calderón-Zygmund theory. For the Stokes equation $-\Delta u + \nabla p = f$ it is well know that the estimate $||\nabla^2 u||_{L^p} + ||\nabla p||_{L^p} \lesssim ||f||_{L^p}$ holds true for every L^p-norm with $1 < p < \infty$ (see for instance [19], Chapter IV). The term "maximal regularity " comes from the fact that the terms $\nabla^2 u$ and ∇p have the same (maximal) regularity as the right-hand side f. Although the above estimate is not true in the $L^\infty-$norm, Constantin and Doering in [7] obtained a maximal regularity estimate up to logarithmic corrections. We will consider the non-stationary Stokes equation and show that a maximal regularity estimate in the interpolation between the *weighted*$-L^1$ and L^∞ norm holds under bandedness assumptions (i.e. restriction to a packet of wave numbers in Fourier space). [1]

We now want to show how to derive upper bounds for the Nusselt number from a (logarithmically failing) maximal regularity estimate. For preparation, we illustrate this idea in the simpler situation of $\mathrm{Pr} = \infty$, following Constantin & Doering 1999 paper's approach.
From the estimate (1.17) on the Nusselt number we can deduce the bound

$$\mathrm{Nu} \leq \delta^2 \left\langle \sup_z |\partial_z^2 u^z| \right\rangle + \frac{1}{\delta},$$

where we could apply two times the Poincaré estimate in the vertical direction thanks to the no-slip boundary condition for u^z and the condition (1.18). Despite the fact that the L^∞-norm is a critical norm for the Calderón-Zygmund theory, Constantin and Doering [7] proved that, up to logarithms, the following maximal regularity estimate holds

$$\left\langle \sup_z |\partial_z^2 u^z| \right\rangle \lesssim \left\langle \sup_z |\mathrm{Ra} T e_z| \right\rangle .$$

By the maximum principle for the temperature (1.9) applied to the right-hand side term of the last bound we have

$$\left\langle \sup_z |\partial_z^2 u^z| \right\rangle \lesssim \mathrm{Ra} \, .$$

Inserting this result back into the bound on Nusselt number, one obtains

$$\mathrm{Nu} \lesssim \delta^2 \mathrm{Ra} + \frac{1}{\delta}.$$

The minimum occurs for $\delta \sim \mathrm{Ra}^{-\frac{1}{3}}$ and the resulting bound, up to logarithms is

$$\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} \, .$$

[1] The reason to consider the (interpolation of) the above norms will appear clear at the end the paragraph Strategy, Subsection 3.2.2.

We will now give a more rigorous argument for this bound (which differs from the one of Constantin and Doering [7]) and aims at introducing the main tools and techniques that will we apply to prove the upper bound at finite Prandtl number (see Section 3.2). Let us suppose that $u(x',z), p(x',z), f(x',z)$ [2] satisfy the equation

$$\begin{aligned} -\Delta u + \nabla p &= f \qquad \text{for} \quad 0 < z < 1\,, && (3.2a)\\ \nabla \cdot u &= 0 \qquad \text{for} \quad 0 < z < 1\,, && (3.2b)\\ u &= 0 \qquad \text{for} \quad z \in \{0,1\}\,, && (3.2c)\end{aligned}$$

where we denoted with $f = \mathrm{Ra}Te_z$. Let f be horizontally band-limited , i.e.

$$\mathcal{F}' f(k',z) = 0 \qquad \text{unless} \qquad 1 \leq R|k'| \leq 4\,,$$

and let us suppose that, under this assumption , u satisfies the *maximal regularity estimate*

$$\sup_{0\leq z\leq 1} \langle |\nabla^2 u|\rangle' \lesssim \sup_{0\leq z\leq 1} \langle |f|\rangle'. \tag{3.3}$$

Then we have the bound

$$\mathrm{Nu} \lesssim \mathrm{Ra}^{\frac{1}{3}} (\ln \mathrm{Ra})^{\frac{1}{3}}. \tag{3.4}$$

We observe that the estimate (3.3) can be deduced with the same method applied to prove Theorem 3, therefore we omit its proof. The argument for the bound (3.4) proceeds as follows: Given $r \in (0,\infty)$,$R \in (0,\infty)$ and $N \in \mathbb{N}$ related by $R = 2^N r$, we introduce the operators "projections" $\mathbb{P}_<$, $\mathbb{P}_1$, $\cdots$,$\mathbb{P}_N$, $\mathbb{P}_>$ which decompose the Fourier space in three regions

$$\mathcal{F}'\mathbb{P}_< = \chi_{R|k|<1}\mathcal{F}' f, \quad \mathcal{F}'\mathbb{P}_> = \chi_{R|k|>4}\mathcal{F}' f, \quad \mathcal{F}'\mathbb{P}_j = \chi_{1<2^{j-1}R|k|<4}\mathcal{F}' f, \tag{3.5}$$

were χ_I denotes the characteristic function of the set I. By construction $\mathbb{P}_< + \sum_{j=1}^N \mathbb{P}_j + \mathbb{P}_> = \mathrm{Id}$ and $\mathbb{P}_<, \mathbb{P}_j, \mathbb{P}_>$ are symmetric with respect to $\langle\cdot\rangle'$. Hence the convection term in (1.16) can be written as the following sum

$$\frac{1}{\delta}\int_0^\delta \langle T u^z\rangle'\,dz = \frac{1}{\delta}\int_0^\delta \langle T(\mathbb{P}_< u)^z\rangle'\,dz + \sum_{j=1}^N \frac{1}{\delta}\int_0^\delta \langle T(\mathbb{P}_j u)^z\rangle'\,dz + \frac{1}{\delta}\int_0^\delta \langle \mathbb{P}_>(T) u^z\rangle'\,dz. \tag{3.6}$$

We start by bounding the term coming from the middle length scales: let us recall that $\mathbb{P}_j u$, $\mathbb{P}_j p$ and $\mathbb{P}_j T$ satisfy the Stokes equations

$$\begin{aligned} -\Delta \mathbb{P}_j u + \nabla \mathbb{P}_j p &= \mathbb{P}_j f \qquad \text{for} \quad 0 < z < 1\,,\\ \nabla \cdot \mathbb{P}_j u &= 0 \qquad \text{for} \quad 0 < z < 1\,,\\ \mathbb{P}_j u &= 0 \qquad \text{for} \quad z \in \{0,1\}\,,\end{aligned}$$

[2]For simplicity of notations, here we may assume that the functions are independent on time.

and that in this case we have the estimate (3.3). We have

$$
\begin{aligned}
|\langle T(\mathbb{P}_j u)^z\rangle'| &\overset{(1.9)}{\leq} \langle |(\mathbb{P}_j u)^z|\rangle' \\
&\leq \int_0^z \int_0^{z'} \langle |\partial_z^2 (\mathbb{P}_j u)^z|\rangle' \, dz''\, dz' \\
&\leq \frac{z^2}{2} \sup_{z'\in(0,1)} \langle |\partial_z^2 (\mathbb{P}_j u)^z|\rangle' \\
&\overset{(3.3)}{\lesssim} \frac{z^2}{2} \sup_{z'\in(0,1)} \langle |\mathrm{Ra}(\mathbb{P}_j T) e_z|\rangle' \\
&\lesssim z^2 \mathrm{Ra} \sup_{z'\in(0,1)} \langle |\mathbb{P}_j T|\rangle' \\
&\leq z^2 \mathrm{Ra} \sup_{z'} \langle |\mathbb{P}_j T|^2\rangle'^{\frac{1}{2}} \\
&\overset{(1.9)}{\lesssim} z^2 \mathrm{Ra}\,,
\end{aligned}
$$

thus

$$
\sum_{j=1}^N \frac{1}{\delta}\int_0^\delta \langle T(\mathbb{P}_j u)^z\rangle' \, dz \lesssim N\,\delta^2\,\mathrm{Ra}\,. \tag{3.8}
$$

We now turn to the argument for the small length-scales:

$$
\begin{aligned}
\frac{1}{\delta}\int_0^\delta \langle \mathbb{P}_>(T) u^z\rangle' \, dz &\lesssim \frac{1}{\delta}\left(\int_0^\delta \langle (\mathbb{P}_>(T-1)^2)\rangle' \, dz \int_0^\delta \langle (u^z)^2\rangle \, dz\right)^{\frac{1}{2}} \\
&\overset{(3.114)}{\lesssim} \frac{1}{\delta}\left(\int_0^1 r^2 \langle |\nabla' T|^2\rangle \, dz\, \delta^2 \int_0^1 \langle (\partial_z u^z)^2\rangle' \, dz\right)^{\frac{1}{2}} \\
&\leq r\left(\int_0^1 \langle |\nabla' T|^2\rangle \, dz \int_0^1 \langle (\partial_z u^z)^2\rangle' \, dz\right)^{\frac{1}{2}}\,,
\end{aligned} \tag{3.9}
$$

where we have used the Cauchy-Schwarz inequality, the bandedness condition (3.114) and the Poincaré inequality. Finally we address the case of large length-scales:

$$
\begin{aligned}
\frac{1}{\delta}\int_0^\delta \langle T(\mathbb{P}_< u)^z\rangle' \, dz &\lesssim \frac{1}{\delta}\left(\int_0^\delta \langle (T-1)^2\rangle' \, dz \int_0^1 \langle (\mathbb{P}_< u)^z\rangle' \, dz\right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{\delta}\left(\delta^2 \int_0^1 \langle (\partial_z T)^2\rangle' \, dz \int_0^1 \delta^2 \langle (\partial_z (\mathbb{P}_< u)^z)^2\rangle' \, dz\right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{\delta}\left(\delta^2 \int_0^1 \langle (\partial_z T)^2\rangle' \, dz \int_0^1 \delta^2 \langle (\nabla' \cdot (\mathbb{P}_< u)')^2\rangle' \, dz\right)^{\frac{1}{2}} \\
&\overset{(3.116)}{\lesssim} \frac{1}{\delta}\left(\delta^2 \int_0^1 \langle (\partial_z T)^2\rangle' \, dz \int_0^1 \frac{\delta^2}{R^2} \langle |u'|^2\rangle' \, dz\right)^{\frac{1}{2}}\,,
\end{aligned} \tag{3.10}
$$

where we have used the Cauchy-Schwarz inequality, the Poincaré inequality, the divergence free equation and the bandedness condition (3.116). Now, using the estimates (3.8),(3.9) and (3.10) we can bound the Nusselt number as follows:

$$\begin{aligned}
&\frac{1}{\delta}\left\langle\int_0^\delta Tu^z\,dz\right\rangle\\
&\quad=\frac{1}{\delta}\left\langle\int_0^\delta T(\mathbb{P}_<u)^z\,dz\right\rangle+\sum_{j=1}^N\frac{1}{\delta}\left\langle\int_0^\delta T(\mathbb{P}_ju)^z\,dz\right\rangle+\frac{1}{\delta}\left\langle\int_0^\delta \mathbb{P}_>(T)u^z\,dz\right\rangle\\
&\quad\overset{(3.8)\&(3.9)\&(3.10)}{\lesssim}\left(r+\frac{\delta^2}{R}\right)\left(\left\langle\int_0^1|\nabla T|^2\,dz\right\rangle\left\langle\int_0^1|\nabla u|^2\,dz\right\rangle\right)^{\frac{1}{2}}+N\delta^2\mathrm{Ra}+\frac{1}{\delta}\\
&\quad\overset{(1.13)\&(1.15)}{\lesssim}\left(r+\frac{\delta^2}{R}\right)\mathrm{Nu}\mathrm{Ra}^{\frac{1}{2}}+N\delta^2\mathrm{Ra}+\frac{1}{\delta}\,.
\end{aligned}$$

From the first term of the right-hand side we learn that $r\sim\frac{\delta^2}{R}$ and since, by construction $R=2^Nr$, the optimal r and R are

$$r=2^{-\frac{N}{2}}\delta\qquad\qquad R=2^{\frac{N}{2}}\delta.$$

Therefore we obtain

$$\mathrm{Nu}\lesssim 2^{-\frac{N}{2}}\delta\mathrm{Ra}^{\frac{1}{2}}\mathrm{Nu}+\mathrm{Nu}\mathrm{Ra}\delta^2+\frac{1}{\delta}.$$

Finally we optimize in $N\in\mathbb{N}$: on the one hand we want to absorb the first term of the right-hand side in the left-hand side, so that we impose $\delta\mathrm{Ra}^{\frac{1}{2}}=2^{\frac{N}{2}}$ and, on the other hand, we need the second and the third term of the right-hand side to be of the same size, therefore we require $\delta^3\sim\frac{1}{N\mathrm{Ra}}$. Putting these two conditions together we obtain $N2^{\frac{N}{2}}\sim\mathrm{Ra}^{\frac{1}{6}}$ which implies [3] $N\sim\ln\mathrm{Ra}$. In conclusion we have

$$\mathrm{Nu}\lesssim(\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}}.$$

3.2 Result: Upper bounds on Nusselt number

Joint work with Antoine Choffrut and Felix Otto

3.2.1 Main theorem

As anticipated in the Section 1.1, for the system of equations (1.3) we will establish the following result

Theorem 2 (Bounds on the Nusselt number)**.**
Provided the initial data satisfy $T_0\in[0,1]$, $\int|u_0|dx<\infty$ and for $\mathrm{Ra}\gg1$

$$\mathrm{Nu}\le C\begin{cases}(\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} & \text{for}\quad \mathrm{Pr}\ge(\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}},\\ \left(\mathrm{Pr}^{-1}\mathrm{Ra}\ln\mathrm{Ra}\right)^{\frac{1}{2}} & \text{for}\quad \mathrm{Pr}\le(\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}},\end{cases}\tag{3.11}$$

where C depends only on the dimension d.

3.2.2 Maximal regularity in the strip

Strategy

We perturb the non-stationary Stokes equations, bringing *only* the nonlinear term to the right-hand side

$$\frac{1}{\mathrm{Pr}}\partial_tu-\Delta u+\nabla p=\mathrm{Ra}Te_z-\frac{1}{\mathrm{Pr}}(u\cdot\nabla)u.\tag{3.12}$$

[3]Here we use that $x^\alpha\ln x=y\Rightarrow x\sim y^\alpha\ln^{-\frac{1}{\alpha}}y$

Before going into details of the proof of Theorem 2 we want to argue why

$$\mathrm{Pr} \gg \mathrm{Ra} \tag{3.13}$$

amounts to the classical perturbative regime for Navier-Stokes equations. The classical perturbative argument goes as follows: One seeks a norm $||\cdot||$ in which the maximal regularity estimate for the non-stationary Stokes equations (3.12) holds, yielding

$$||\nabla^2 u|| \lesssim ||\mathrm{Ra}Te_z - \frac{1}{\mathrm{Pr}}(u\cdot\nabla)u||.$$

This norm has to be strong enough to control the convective nonlinearity in the sense of

$$||(u\cdot\nabla)u|| \lesssim ||\nabla^2 u||^2.$$

In the application the norm should be also sufficiently weak so that (1.9) translates into

$$||\mathrm{Ra}Te_z|| \lesssim \mathrm{Ra}.$$

The combination yields the estimate

$$||\nabla^2 u|| \lesssim \mathrm{Ra} + \frac{1}{\mathrm{Pr}}||\nabla^2 u||^2,$$

which is nontrivial only when $\mathrm{Pr} \gg \mathrm{Ra}$. Our analysis does not attempt to buckle via the classical perturbation argument but instead uses the dissipation bound (1.15) to estimate the convective nonlinearity: By the Cauchy-Schwarz inequality, Hardy's inequality and capitalizing once more on the no-slip boundary condition we have

$$\left\langle \int_0^1 |(u\cdot\nabla)u|\,\frac{dz}{z}\right\rangle \lesssim \left\langle \int_0^1 |u|^2\frac{dz}{z^2}\right\rangle^{\frac{1}{2}} \left\langle \int_0^1 |\nabla u|^2\,dz\right\rangle^{\frac{1}{2}} \lesssim \left\langle \int_0^1 |\nabla u|^2 dz\right\rangle, \tag{3.14}$$

which, using (1.15), implies

$$\left\langle \int_0^1 |(u\cdot\nabla)u|\frac{dz}{z}\right\rangle \lesssim \mathrm{NuRa}. \tag{3.15}$$

It is this estimate that motivates the maximal regularity theory in the norm $||\cdot|| = \left\langle\int_0^\infty (\cdot)\,\frac{dz}{z}\right\rangle$. In order to bound the right-hand side of (1.17) we split the solution to the Navier-Stokes equations u as

$$u = u_{CD} + u_{NL} + u_{IV},$$

where u_{CD} satisfies the non-stationary Stokes equations with the buoyancy force as right-hand side [4]

$$\begin{cases} \frac{1}{\mathrm{Pr}}\partial_t u_{CD} - \Delta u_{CD} + \nabla p_{CD} &= \mathrm{Ra}Te_z \quad \text{for} \quad 0<z<1\,,\\ \nabla\cdot u_{CD} &= 0 \quad \text{for} \quad 0<z<1\,,\\ u_{CD} &= 0 \quad \text{for} \quad z\in\{0,1\}\,,\\ u_{CD} &= 0 \quad \text{for} \quad t=0\,, \end{cases} \tag{3.16}$$

u_{NL} satisfies the non-stationary Stokes equations with the nonlinear term as right-hand side [5]

$$\begin{cases} \frac{1}{\mathrm{Pr}}\partial_t u_{NL} - \Delta u_{NL} + \nabla p_{NL} &= -\frac{1}{\mathrm{Pr}}(u\cdot\nabla)u \quad \text{for} \quad 0<z<1\,,\\ \nabla\cdot u_{NL} &= 0 \quad \text{for} \quad 0<z<1\,.\\ u_{NL} &= 0 \quad \text{for} \quad z\in\{0,1\}\,,\\ u_{NL} &= 0 \quad \text{for} \quad t=0 \end{cases} \tag{3.17}$$

[4] The stationary version of this problem was already analyzed by Constantin and Doering in the seminal paper of 1999. This motivates the subscript CD.

[5] The subscript NL stands for non-linear. Indeed only in this equation the non-linear term of the Navier-Stokes equations is appearing as right-hand side.

and u_{IV} satisfies the non-stationary Stokes equations with zero forcing term and non-zero initial values [6]

$$\begin{cases} \frac{1}{\mathrm{Pr}}\partial_t u_{IV} - \Delta u_{IV} + \nabla p_{IV} &= 0 \qquad \text{for} \quad 0 < z < 1\,, \\ \nabla \cdot u_{IV} &= 0 \qquad \text{for} \quad 0 < z < 1\,, \\ u_{IV} &= 0 \qquad \text{for} \quad z = 0\,, \\ u_{IV} &= u_0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{3.18}$$

Inserting the decomposition $u = u_{CD} + u_{NL} + u_{IV}$ into the bound (1.17) for the Nusselt number, we have

$$\begin{aligned} \mathrm{Nu} &\leq \tfrac{1}{\delta}\left\langle \int_0^\delta |u^z|\,dz \right\rangle + \tfrac{1}{\delta} \\ &\leq \left\langle \sup_{z\in(0,\delta)} |u^z_{CD}| \right\rangle + \delta \left\langle \int_0^\delta |\partial_z^2 u^z_{NL}|\,dz \right\rangle + \delta^{-\frac{1}{2}} \left\langle \int_0^\delta |u^z_{IV}|^2\,dz \right\rangle^{\frac{1}{2}} + \tfrac{1}{\delta} \\ &\leq \delta^2 \left(\left\langle \sup_{z\in(0,\delta)} |\partial_z^2 u^z_{CD}| \right\rangle + \left\langle \int_0^\delta |\partial_z^2 u^z_{NL}| \tfrac{dz}{z} \right\rangle \right) + \delta^{-\frac{1}{2}} \left\langle \int_0^\delta |u^z_{IV}|^2\,dz \right\rangle^{\frac{1}{2}} + \tfrac{1}{\delta}. \end{aligned} \tag{3.19}$$

We notice that

$$\left\langle \int_0^\delta |u^z_{IV}|^2\,dz \right\rangle = 0\,. \tag{3.20}$$

Indeed testing the equation (3.18) with u_{IV} we find that

$$\int_0^{t_0}\int_{x'}\int_z |\nabla u_{IV}(x',z,t)|^2 dz dx' dt \leq \int_{x'}\int_z |u_0(x',z)|^2 dz dx'$$

and in turn by the Poincaré inequality and passing to limits we get

$$\left\langle \int_0^1 |u_{IV}|^2\,dz \right\rangle = 0,$$

which implies (3.20). On the one hand, for equation (3.16) we expect the following logarithmically failing maximal regularity bound

$$\left\langle \sup_{z\in(0,1)} |\partial_z^2 u^z_{CD}| \right\rangle \lesssim \mathrm{Ra}, \tag{3.21}$$

just as for the case of $\mathrm{Pr} = \infty$. On the other hand, the problem of bounding the term $\left\langle \int_0^1 |\partial_z^2 u^z_{NL}| \frac{dz}{z} \right\rangle$ in (3.19) requires new techniques. Nevertheless we expect

$$\left\langle \int_0^\delta |\partial_z^2 u^z_{NL}| \frac{dz}{z} \right\rangle \lesssim \frac{1}{\mathrm{Pr}} \left\langle \int_0^1 |(u\cdot\nabla)u| \frac{dz}{z} \right\rangle, \tag{3.22}$$

up to logarithmic corrections. Using (3.15) we obtain

$$\left\langle \int_0^\delta |\partial_z^2 u^z_{NL}| \frac{dz}{z} \right\rangle \lesssim \frac{1}{\mathrm{Pr}} \mathrm{Nu}\mathrm{Ra}\,. \tag{3.23}$$

Inserting (3.20), (3.21) and (3.23) into the bound (3.19) for the Nusselt number and ignoring logarithmic correction factors, we get

$$\mathrm{Nu} \lesssim \delta^2 \mathrm{Ra}(1 + \frac{1}{\mathrm{Pr}}\mathrm{Nu}) + \frac{1}{\delta}.$$

After choosing $\delta \sim \left(\mathrm{Ra}(1 + \frac{\mathrm{Nu}}{\mathrm{Pr}})\right)^{-\frac{1}{3}}$ and applying Young's inequality, we have

$$\mathrm{Nu} \sim \mathrm{Ra}^{\frac{1}{3}} + \left(\frac{\mathrm{Ra}}{\mathrm{Pr}}\right)^{\frac{1}{2}},$$

[6] The subscript IV stands for initial value.

which implies, up to logarithms,

$$\mathrm{Nu} \lesssim \begin{cases} \mathrm{Ra}^{\frac{1}{3}} & \text{for } \mathrm{Pr} \geq \mathrm{Ra}^{\frac{1}{3}}\,, \\ \mathrm{Pr}^{-\frac{1}{2}}\mathrm{Ra}^{\frac{1}{2}} & \text{for } \mathrm{Pr} \leq \mathrm{Ra}^{\frac{1}{3}}\,. \end{cases} \tag{3.24}$$

We remark that in our analysis, the crucial no-slip boundary condition is both a blessing and a curse, as we shall presently explain. The no-slip boundary condition is a *blessing* because, via Hardy's inequality, it gives us good control of the convective nonlinearity $(u\cdot\nabla)u$, in an L^1-type space with the weight $\frac{1}{z(1-z)}$, in terms of the average viscous dissipation $\left\langle \int_0^1 |\nabla u|^2\,dz \right\rangle$ (see (3.14)), which is the physically relevant quantity (and the only bound at hand for the Leray solution) see (1.15). Hence a *maximal regularity theory* for the non-stationary Stokes equations with respect to this norm is required. Since this norm is borderline for Calderón-Zygmund estimates (both because the exponent and the weight are borderline), maximal regularity "fails logarithmically" and can only be recovered under bandedness assumptions — this is the source of the logarithm. It is in this maximal regularity theory where the *curse* of the no-slip boundary condition appears: As opposed to the no-stress boundary condition in the half space, the no-slip boundary condition does not allow for an extension by reflection to the whole space, and thereby the use of simple kernels or Fourier methods also in the normal variable. The difficulty coming from the the no-slip boundary condition in the non-stationary Stokes equations when deriving maximal regularity estimates is of course well-known; many techniques have been developed to derive Calderón-Zygmund estimates despite this difficulty. In the *half space* Solonnikov in [20] has constructed a solution formula for (3.25) with zero initial data via the Oseen an Green tensors. An easier and more compact representation of the solution to the problem (3.25) with zero forcing term and non-zero initial value was later given by Ukai in [21] by using a different method. Indeed he could write an explicit formula of the solution operator as a composition of Riesz' operators and solutions operator for the heat and Laplace's equation. This formula is an effective tool to get $L^p - L^q$ $(1 < q, p < \infty)$ estimates for the solution and its derivatives. In the case of *exterior domains*, Maremonti and Solonnikov [22] derive $L^p - L^q$ $(1 < q, p < \infty)$ estimates for (3.25), going through estimates for the extended solution in the half space and in the whole space. In particular in the half space they propose a decomposition of (3.25) with non-zero divergence equation. The book of Galdi [19] provides with a complete treatment of the classical theory and results on the non-stationary Stokes equations and Navier-Stokes equations. The new element here is that we need maximal regularity in the *borderline* space $L^1\left(dtdx'\frac{1}{z(1-z)}dz\right)$, and in $L^\infty_z(L^1_{t,x'})$ (the latter space coming from the original argument in 1999 paper of Constantin & Doering and pertaining to the buoyancy term). As mentioned, these borderline Calderón-Zygmund estimates can only hold under bandedness assumption.

Theorem 3 (Maximal regularity in the strip)**.**
There exists $R_0 \in (0,\infty)$ depending only on d and L such that the following holds. Let u, p, f satisfy

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad 0 < z < 1\,, \\ \nabla\cdot u &= 0 \qquad \text{for} \quad 0 < z < 1\,, \\ u &= 0 \qquad \text{for} \quad z \in \{0,1\}\,, \\ u &= 0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{3.25}$$

Assume f is horizontally band-limited , i.e.

$$\mathcal{F}' f(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4 \textit{ where } R < R_0. \tag{3.26}$$

Then,

$$||(\partial_t - \partial_z^2)u'||_{(0,1)} + ||\nabla'\nabla u'||_{(0,1)} + ||\partial_t u^z||_{(0,1)} + ||\nabla^2 u^z||_{(0,1)} + ||\nabla p||_{(0,1)} \lesssim ||f||_{(0,1)}, \tag{3.27}$$

where $||\cdot||_{(0,1)}$ *denotes the norm*

$$||f||_{(0,1)} := ||f||_{(R,(0,1))} = \inf_{f=f_0+f_1}\left\{\left\langle \sup_{0<z<1}|f_0|\right\rangle + \left\langle \int_0^1 |f_1|\frac{dz}{(1-z)z}\right\rangle\right\}, \tag{3.28}$$

where f_0 *and* f_1 *satisfy the bandedness assumption (3.26).*

See Notations.

Our analysis offers two insights: It turns out that for the maximal regularity estimate for the quantity of interest, namely the second vertical derivative $\partial_z^2 u^z$ of the vertical velocity component $u^z = u\cdot e_z$, bandedness only in the *horizontal* variable x' is required. This is extremely convenient, since the horizontal Fourier transform (or rather, series), with help of which bandedness is expressed, is compatible with the lateral periodic boundary condition. The maximal regularity appears to be naturally expressed in terms of the *interpolation* between the two norms of interest $L^1\left(dtdx'\frac{1}{z(1-z)}dz\right)$ and $L^\infty_z(L^1_{t,x'})$. This way, one avoids the logarithm that is expect to be the price of the borderline weight $\frac{1}{z(1-z)}$. It is a pleasant coincidence that the norm $L^\infty_z(L^1_{t,x'})$ arises for two unrelated reasons: It is needed to estimate the buoyancy term Te_z driving the Navier-Stokes equations and it is the natural partner of $L^1\left(dtdx'\frac{1}{z(1-z)}dz\right)$ in the maximal regularity estimate. Aside from their application to this problem (see Section 3.2.5) all the estimates in Theorem 3 might have an independent interest since they show the full extent of what one can obtain under the horizontal bandedness assumption only.

3.2.3 Proof of main theorem

Without loss of generality we will assume $u_0 = 0$ since we have already seen that the contribution of u_{IV} to the Nusselt number is zero (see (3.20)).

Let us fix a smooth cut-off function ψ in Fourier space satisfying

$$\psi(k') = \begin{cases} 1 & 0 \le |k'| \le \frac{7}{2}\,, \\ 0 & |k'| \ge 4\,. \end{cases}$$

Consider the function $\zeta(k') = \psi(k') - \psi(\frac{7}{2}k')$ and define $\zeta_j(k') = \zeta(2^{-j}k')$. Notice that ζ_j is supported in $(2^j, 2^{j+2})$. Imitating a Littlewood-Paley-type decomposition we define a smooth equipartition in Fourier space by constructing three operators $\mathbb{P}_<$, $\mathbb{P}_j$ and $\mathbb{P}_>$, which act at the level of Fourier space by multiplication by cut off functions that localize to small, intermediate and large wavenumbers respectively:

$$\mathcal{F}'\mathbb{P}_< f = \zeta_<\mathcal{F}'f, \qquad \mathcal{F}'\mathbb{P}_j f = \zeta_j\mathcal{F}'f, \qquad \mathcal{F}'\mathbb{P}_> f = \zeta_>\mathcal{F}'f\,,$$

where $\zeta_< = \sum_{j<j_1}\zeta_j$ and $\zeta_> = \sum_{j>j_2}\zeta_j$ with $j_1 < j_2$ to be determined. We notice that the operator $\mathbb{P}_j : L^p \to L^p$ for $1 \le p < \infty$ is bounded.

Inserting the decomposition into the bound for the Nusselt number (1.16) we get

$$\begin{aligned} \mathrm{Nu} &\le \tfrac{1}{\delta}\left\langle \int_0^\delta Tu^z\,dz\right\rangle + \tfrac{1}{\delta} \\ &= \tfrac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_< u^z\,dz\right\rangle + \sum_{j=j_1}^{j_2}\tfrac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_j u^z\,dz\right\rangle + \tfrac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_> u^z\,dz\right\rangle + \tfrac{1}{\delta}\,. \end{aligned} \tag{3.29}$$

At first, let us focus on the second term in (3.29) arising from the intermediate wavelengths. In order to bound this term we will need the maximal regularity estimate stated in Theorem 3. For this purpose, rewrite the Navier-Stokes equations in (1.3b) as non-stationary Stokes equations with the nonlinear term and the buoyancy term in the right-hand side

$$\left\{\begin{array}{rcll} \frac{1}{\mathrm{Pr}}\partial_t\mathbb{P}_j u - \Delta\mathbb{P}_j u + \nabla\mathbb{P}_j p &=& \mathrm{Ra}\mathbb{P}_j Te_z - \frac{1}{\mathrm{Pr}}\mathbb{P}_j(u\cdot\nabla)u & \text{for}\quad 0<z<1\,, \\ \nabla\cdot\mathbb{P}_j u &=& 0 & \text{for}\quad 0<z<1\,, \\ \mathbb{P}_j u &=& 0 & \text{for}\quad z\in\{0,1\}\,, \\ \mathbb{P}_j u &=& 0 & \text{for}\quad t=0\,. \end{array}\right. \tag{3.30}$$

Observe that the application of the operator "horizontal filtering", namely $\mathbb{P}_j$, preserves the no-slip boundary condition at $z = 0, 1$ and it commutes with ∂_z and all the differential operators that act in the vertical direction. Using the maximum principle for the temperature (1.9), the Poincaré inequality in the $z-$variable and considering a generic decomposition of $\partial_z^2 \mathbb{P}_j u^z = h_0 + h_1$ we have

$$\begin{aligned}
\frac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_j u^z \, dz \right\rangle &\leq \frac{1}{\delta}\left\langle \int_0^\delta |\mathbb{P}_j u^z| \, dz \right\rangle \\
&\leq \delta \left\langle \int_0^\delta |\partial_z^2 \mathbb{P}_j u^z| \, dz \right\rangle \\
&\leq \delta \left(\left\langle \int_0^\delta |h_0| \, dz \right\rangle + \left\langle \int_0^\delta |h_1| \, dz \right\rangle \right) \\
&\leq \delta^2 \left(\left\langle \sup_{0<z<1} |h_0| \right\rangle + \left\langle \int_0^1 |h_1| \frac{dz}{z(1-z)} \right\rangle \right) .
\end{aligned}$$

Passing to the infimum over all the possible decompositions of $\partial_z^2 \mathbb{P}_j u^z$ we get

$$\frac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_j u^z \, dz \right\rangle \leq \delta^2 ||\partial_z^2 \mathbb{P}_j u^z||_{(0,1)} .$$

We notice that $\mathbb{P}_j u$ satisfies the linear Stokes equations (3.30) and for $j > j_1$ it satisfies the bandedness assumption provided

$$j_1 \gtrsim \log_2 R_0^{-1} . \tag{3.31}$$

Therefore by the maximal regularity estimate (3.27) applied to $\mathbb{P}_j u$ we have

$$||\partial_z^2 \mathbb{P}_j u^z||_{(0,1)} \leq \left\langle \sup_{0<z<1} |\mathrm{Ra}\mathbb{P}_j T e_z| \right\rangle + \left\langle \int_0^1 \left| \frac{1}{\mathrm{Pr}} \mathbb{P}_j (u\cdot\nabla)u \right| \frac{dz}{z(1-z)} \right\rangle .$$

We observe that, by the smooth equipartition in Fourier space, we have

$$\langle |\mathrm{Ra}\mathbb{P}_j T e_z| \rangle' \leq \langle |\mathrm{Ra} T e_z| \rangle' . \tag{3.32}$$

and , by the maximum principle (1.9), we obtain $\sup_z \langle |\mathrm{Ra} T e_z| \rangle' \leq \mathrm{Ra}$. So that we find

$$\left\langle \sup_{0<z<1} |\mathrm{Ra}\mathbb{P}_j T e_z| \right\rangle \leq \mathrm{Ra} . \tag{3.33}$$

For the nonlinear part we first observe that by the smooth partition of unity we have

$$\langle |\mathbb{P}_j (u\cdot\nabla)u| \rangle' \leq \langle |(u\cdot\nabla)u| \rangle'$$

so that

$$\left\langle \int_0^1 |\frac{1}{\mathrm{Pr}} \mathbb{P}_j (u\cdot\nabla)u| \frac{dz}{z(1-z)} \right\rangle \leq \left\langle \int_0^1 \left| \frac{1}{\mathrm{Pr}} (u\cdot\nabla)u \right| \frac{dz}{z(1-z)} \right\rangle .$$

To estimate the nonlinear part we apply the Cauchy-Schwarz inequality and Hardy's inequality

$$
\begin{aligned}
&\left\langle \int_0^1 \left| \frac{1}{\Pr}(u\cdot\nabla)u \right| \frac{dz}{z(1-z)} \right\rangle \\
=\quad & \frac{1}{\Pr}\left\langle \int_0^1 |(u\cdot\nabla)u| \frac{dz}{z}\right\rangle + \frac{1}{\Pr}\left\langle \int_0^1 |(u\cdot\nabla)u| \frac{dz}{1-z}\right\rangle \\
\lesssim\quad & \frac{1}{\Pr}\left(\left\langle \int_0^1 \frac{1}{z^2}|u|^2\,dz\right\rangle + \left\langle \int_0^1 \frac{1}{(1-z)^2}|u|^2\,dz\right\rangle\right)^{\frac12} \left\langle \int_0^1 |\nabla u|^2\,dz\right\rangle^{\frac12} \\
\lesssim\quad & \frac{1}{\Pr}\left\langle \int_0^1 |\partial_z u|^2\,dz\right\rangle^{\frac12} \left\langle \int_0^1 |\nabla u|^2\,dz\right\rangle^{\frac12} \\
\lesssim\quad & \frac{1}{\Pr}\left\langle \int_0^1 |\nabla u|^2\,dz\right\rangle \\
\overset{(1.15)}{\leq}\quad & \frac{1}{\Pr}\mathrm{Ra}(\mathrm{Nu}-1)\,.
\end{aligned}
$$

Summing up over all the intermediate wavelengths we obtain

$$
\sum_{j=j_1}^{j_2} \frac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_j u^z\,dz\right\rangle \lesssim (j_2-j_1)\delta^2\left(\mathrm{Ra}+\frac{1}{\Pr}\mathrm{Ra}(\mathrm{Nu}-1)\right)\,. \tag{3.34}
$$

We now turn to the first term appearing on the right-hand side of (3.29), contribution of the large wavelengths. By using the Cauchy-Schwarz inequality, the divergence-free condition, the horizontal bandedness assumption in form of (3.116) and the Poincaré inequality in the $z-$variable, we obtain

$$
\begin{aligned}
\frac{1}{\delta}\left\langle \int_0^\delta T\mathbb{P}_< u^z\,dz\right\rangle &= \frac{1}{\delta}\left\langle \int_0^\delta (T-1)\mathbb{P}_< u^z\,dz\right\rangle \qquad (3.35)\\
&\lesssim \frac{1}{\delta}\left\langle \int_0^\delta |T-1|^2\,dz\right\rangle^{\frac12}\left\langle \int_0^\delta |\mathbb{P}_< u^z|^2\,dz\right\rangle^{\frac12}\\
&\lesssim \delta\frac{1}{\delta}\left\langle \int_0^\delta |\partial_z T|^2\,dz\right\rangle^{\frac12}\delta\left\langle \int_0^\delta |\partial_z\mathbb{P}_< u^z|^2\,dz\right\rangle^{\frac12}\\
&\lesssim \delta\left\langle \int_0^\delta |\nabla T|^2dz\right\rangle^{\frac12}\left\langle \int_0^\delta |\mathbb{P}_<\nabla'\cdot u'|^2\,dz\right\rangle^{\frac12}\\
&\lesssim \delta\left\langle \int_0^\delta |\nabla T|^2\,dz\right\rangle^{\frac12} 2^{j_1}\left\langle \int_0^\delta |u'|^2\,dz\right\rangle^{\frac12}\\
&\lesssim 2^{j_1}\delta^2\left\langle \int_0^1 |\nabla T|^2\,dz\right\rangle^{\frac12}\left\langle \int_0^1 |\nabla u|^2dz\right\rangle^{\frac12}\\
&\overset{(1.13)\&(1.15)}{\lesssim} 2^{j_1}\delta^2\mathrm{Ra}^{\frac12}\mathrm{Nu}\,, \qquad (3.36)
\end{aligned}
$$

where in (3.35) we used (1.19). Finally, we turn to the third term in (3.29), which represents the contribution from the small wavelengths. In order to estimate this term we use the Cauchy-Schwarz inequality, the Poincaré inequality in the $z-$variable and the horizontal bandedness assumption in form of (3.114) applied to T

$$\begin{aligned}
\frac{1}{\delta}\left\langle\int_0^\delta \mathbb{P}_> u^z T\,dz\right\rangle &\le \frac{1}{\delta}\left\langle\int_0^\delta |\mathbb{P}_> T|^2 dz\right\rangle^{\frac{1}{2}}\left\langle\int_0^\delta |u^z|^2\,dz\right\rangle)^{\frac{1}{2}}\\
&\le \frac{1}{\delta}\frac{1}{2^{j_2}}\left\langle\int_0^\delta |\nabla' T|^2\,dz\right\rangle^{\frac{1}{2}}\delta\left\langle\int_0^\delta |\partial_z u^z|^2\,dz\right\rangle^{\frac{1}{2}}\\
&\le \frac{1}{2^{j_2}}\left\langle\int_0^1 |\nabla T|^2 dz\right\rangle^{\frac{1}{2}}\left\langle\int_0^\delta |\nabla u|^2 dz\right\rangle^{\frac{1}{2}}\\
&\overset{(1.13)\&(1.15)}{\lesssim} \frac{1}{2^{j_2}}\mathrm{Ra}^{\frac{1}{2}}\mathrm{Nu}\,.
\end{aligned} \tag{3.37}$$

Putting the three estimates (3.34),(3.36) and (3.37) together, we have the following bound on the Nusselt number

$$\mathrm{Nu}\lesssim (j_2-j_1)\delta^2\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}+\left(\delta^2 2^{j_1}+\frac{1}{2^{j_2}}\right)\mathrm{Ra}^{\frac{1}{2}}\mathrm{Nu}+\frac{1}{\delta}\,.$$

In the last inequality we impose $2^{-j_2}=2^{j_1}\delta^2$. In turn, observe that $2^{-j_2}=2^{-\frac{(j_2-j_1)}{2}}\delta$ and therefore

$$\mathrm{Nu}\lesssim (j_2-j_1)\delta^2\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}+2^{-\frac{(j_2-j_1)}{2}}\delta\mathrm{Ra}^{\frac{1}{2}}\mathrm{Nu}+\frac{1}{\delta}\,. \tag{3.38}$$

Observe that, on the one hand, we want the second term of the right-hand side to be absorbed in the left-hand side, therefore we impose

$$1\sim 2^{-\frac{(j_2-j_1)}{2}}\delta\mathrm{Ra}^{\frac{1}{2}}$$

and, on the other hand, we require all the terms in the right-hand side to be of the same size

$$(j_2-j_1)\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}\sim\frac{1}{\delta^3}. \tag{3.39}$$

From these two conditions we deduce

$$(j_2-j_1)2^{\frac{3}{2}(j_2-j_1)}\sim\mathrm{Ra}^{\frac{1}{2}}\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)^{-1},$$

which is of the form $x\log_a x\sim y$ with $x=a^{(j_2-j_1)}$ and $y=\mathrm{Ra}^{\frac{1}{2}}\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)^{-1}$ for $a>1$. This implies that, asymptotically, $x\sim\frac{y}{\log_a y}$ and therefore

$$j_2-j_1\sim\log_a\left(\frac{\mathrm{Ra}^{\frac{1}{2}}\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)^{-1}}{\log_a(\mathrm{Ra}^{\frac{1}{2}}\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)^{-1})}\right)\sim\ln\mathrm{Ra}.$$

Inserting this back into (3.39), we are led to the natural choice of δ

$$\delta=\left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}\ln\mathrm{Ra}\right)^{-\frac{1}{3}},$$

which give us the bound

$$\mathrm{Nu}\lesssim\left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}+1\right)\mathrm{Ra}\ln\mathrm{Ra}\right)^{\frac{1}{3}}.$$

Applying the triangle inequality [7]

$$\mathrm{Nu} \lesssim (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} + \left(\left(\frac{\mathrm{Nu}}{\mathrm{Pr}}\right)\mathrm{Ra}\ln\mathrm{Ra}\right)^{\frac{1}{3}}$$

and Young's inequality, we finally obtain

$$\mathrm{Nu} \lesssim (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} + \left(\frac{\mathrm{Ra}\ln\mathrm{Ra}}{\mathrm{Pr}}\right)^{\frac{1}{2}} .$$

In conclusion we get the following bound on the Nusselt number

$$\mathrm{Nu} \lesssim \begin{cases} (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} & \text{for } \mathrm{Pr} \geq (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} , \\ \left(\frac{\mathrm{Ra}}{\mathrm{Pr}}\ln\mathrm{Ra}\right)^{\frac{1}{2}} & \text{for } \mathrm{Pr} \leq (\mathrm{Ra}\ln\mathrm{Ra})^{\frac{1}{3}} . \end{cases}$$

3.2.4 Proof of the maximal regularity in the strip

From the strip to the half space

Let us consider the non-stationary Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad 0 < z < 1 , \\ \nabla \cdot u &= 0 \qquad \text{for} \quad 0 < z < 1 , \\ u &= 0 \qquad \text{for} \quad z \in \{0, 1\} , \\ u &= 0 \qquad \text{for} \quad t = 0 . \end{cases}$$

In order to prove the maximal regularity estimate in the strip we extend the problem (3.25) in the half space. By symmetry, it is enough to consider for the moment the extension to the upper half space.
Consider the localization $(\tilde{u}, \tilde{p}) := (\eta u, \eta p)$ where

$$\eta(z) \text{ is a cut-off function for } \left[0, \frac{1}{2}\right) \text{ in } [0, 1) . \tag{3.40}$$

Extending $(\tilde{u}, \tilde{p})$ by zero they can be viewed as functions in the upper half space. The couple $(\tilde{u}, \tilde{p})$ satisfies

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} &= \tilde{f} \qquad \text{for} \quad z > 0 , \\ \nabla \cdot \tilde{u} &= \tilde{\rho} \qquad \text{for} \quad z > 0 , \\ \tilde{u} &= 0 \qquad \text{for} \quad z = 0 , \\ \tilde{u} &= 0 \qquad \text{for} \quad t = 0 , \end{cases} \tag{3.41}$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta)\partial_z u - (\partial_z^2 \eta) u + (\partial_z \eta) p e_z, \qquad \tilde{\rho} := (\partial_z \eta) u^z . \tag{3.42}$$

3.2.5 Maximal regularity in the upper half space

In the half space, taking advantages from the explicit representation of the solution via Green functions, we prove the regularity estimates which will be crucial in the proof of Theorem 3.

[7] Note that for $0 < p < 1$ we have

$$||f + g||_p \leq 2^{\frac{1}{p} - 1}(||f||_p + ||g||_p)$$

Proposition 3 (Maximal regularity in the upper half space)**.**
Consider the non-stationary Stokes equations in the upper half-space

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{for} \quad z > 0\,, \\ \nabla \cdot u &= \rho \qquad \text{for} \quad z > 0\,, \\ u &= 0 \qquad \text{for} \quad z = 0\,, \\ u &= 0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{3.43}$$

Suppose that f and ρ are horizontally band-limited , i.e.

$$\mathcal{F}' f(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4 \text{ where } R \in (0, \infty)\,, \tag{3.44}$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \text{ unless } 1 \le R|k'| \le 4 \text{ where } R \in (0, \infty)\,. \tag{3.45}$$

Then

$$\begin{aligned} &||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||\nabla p||_{(0,\infty)} + ||(\partial_t - \partial_z^2)u'||_{(0,\infty)} + ||\nabla'\nabla u'||_{(0,\infty)} \\ \lesssim\ & ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t \rho||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 \rho||_{(0,\infty)} + ||\nabla \rho||_{(0,\infty)}, \end{aligned}$$

where $||\cdot||_{(0,\infty)}$ denotes the norm

$$||f||_{(0,\infty)} := ||f||_{R;(0,\infty)} \inf_{f = f_0 + f_1} \left\{ \left\langle \sup_{0<z<\infty} |f_0| \right\rangle + \left\langle \int_0^\infty |f_1| \frac{dz}{z} \right\rangle \right\}\,, \tag{3.46}$$

where f_0 and f_1 satisfy the bandedness assumption (3.44).

The first ingredient to establish Proposition 3 is a suitable representation of the solution operator $(f = (f', f^z), \rho) \to u = (u', u^z)$ of the Stokes equations with the no-slip boundary condition. In the case of no-slip boundary condition the Laplace operator has to be factorized as $\Delta = \partial_z^2 + \Delta' = (\partial_z + (-\Delta')^{\frac{1}{2}})(\partial_z - (-\Delta')^{\frac{1}{2}})$. In this way the solution operator to the Stokes equations with the no-slip boundary condition (3.43) can be written as the fourfold composition of solution operators to three more elementary boundary value problems:

- Backward fractional diffusion equation (3.47):

$$\begin{cases} (\partial_z - (-\Delta')^{\frac{1}{2}})\phi &= \nabla \cdot f - (\partial_t - \Delta)\rho \qquad \text{for} \quad z > 0\,, \\ \phi &\to 0 \qquad \text{for} \quad z \to \infty. \end{cases} \tag{3.47}$$

- Heat equation (3.48):

$$\begin{cases} (\partial_t - \Delta)v^z &= (-\Delta')^{\frac{1}{2}}(f^z - \phi) - \nabla' \cdot f' + (\partial_t - \Delta)\rho \qquad \text{for} \quad z > 0, \\ v^z &= 0 \qquad \text{for} \quad z = 0\,, \\ v^z &= 0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{3.48}$$

- Forward fractional diffusion equation (3.49):

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})u^z &= v^z \qquad \text{for} \quad z > 0\,, \\ u^z &= 0 \qquad \text{for} \quad z = 0\,. \end{cases} \tag{3.49}$$

- Heat equation (3.50):

$$\begin{cases} (\partial_t - \Delta)v' &= (1 + \nabla'(-\Delta')^{-1}\nabla' \cdot)f' \qquad \text{for} \quad z > 0\,, \\ v' &= 0 \qquad \text{for} \quad z = 0\,, \\ v' &= 0 \qquad \text{for} \quad t = 0\,. \end{cases} \tag{3.50}$$

Finally set

$$u' = v' - \nabla'(-\Delta')^{-1}(\rho - \partial_z u^z)\,. \tag{3.51}$$

In order to prove the validity of the decomposition we need to argue that

$$(\partial_t - \Delta)u - f \text{ is irrotational },$$

which reduces to prove that

$$(\partial_t - \Delta)u' - f' \text{ is irrotational in } x'$$

and

$$\partial_z((\partial_t - \Delta)u' - f') = \nabla'((\partial_t - \Delta)u^z - f^z)\,. \tag{3.52}$$

Let us consider for simplicity $\rho = 0$. The first statement follows easily from the definition. Indeed by definition (3.51) and equation (3.50),

$$(\partial_t - \Delta)u' - f' = \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}\partial_z u^z).$$

Let us now focus on (3.52), which by using (3.51) and (3.50) can be rewritten as

$$\partial_z \nabla'((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = \nabla'((\partial_t - \Delta)u^z - f^z)\,.$$

Because of the periodic boundary condition in the horizontal direction, the latter is equivalent to

$$\partial_z(-\Delta')((-\Delta')^{-1}\nabla' \cdot f' + (-\Delta')^{-1}(\partial_t - \Delta)\partial_z u^z) = (-\Delta')((\partial_t - \Delta)u^z - f^z),$$

that, after factorizing $\Delta = (\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_z + (-\Delta')^{\frac{1}{2}})$, turns into

$$(\partial_z - (-\Delta')^{\frac{1}{2}})(\partial_t - \Delta)(\partial_z + (-\Delta')^{\frac{1}{2}})u^z = (-\Delta')f^z - \partial_z \nabla' \cdot f'\,.$$

One can easily check that the identity holds true by applying (3.49), (3.48) and (3.47). The no-slip boundary condition is trivially satisfied, indeed by (3.49) we have $u^z = 0$ and $\partial_z u^z = 0$. The combination of (3.51) with $\partial_z u^z = 0$ gives $u' = 0$.
For each step of the decomposition of the Navier-Stokes equations we will derive maximal regularity-type estimates. These are summed up in the following

Proposition 4.

1. *Let ϕ, f, ρ satisfy the problem (3.47) and assume f, ρ are horizontally band-limited, i.e.*

$$\mathcal{F}' f(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4.$$

Then,

$$||\phi||_{(0,\infty)} \lesssim ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t \rho||_{(0,\infty)} + ||\nabla \rho||_{(0,\infty)}\,.$$

2. *Let v^z, f, ϕ, ρ satisfy the problem (3.48) and assume f, ϕ, ρ are horizontally band-limited, i.e.*

$$\mathcal{F}' f(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4\,,$$
$$\mathcal{F}' \phi(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4$$

and

$$\mathcal{F}' \rho(k', z, t) = 0 \textit{ unless } 1 \leq R|k'| \leq 4\,.$$

Then,

$$\begin{aligned} & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} \\ \lesssim\ & ||f||_{(0,\infty)} + ||\phi||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t \rho||_{(0,\infty)} \\ +\ & ||(-\Delta)^{-\frac{1}{2}}\partial_z^2 \rho||_{(0,\infty)} + ||\nabla \rho||_{(0,\infty)}\,. \end{aligned} \tag{3.53}$$

3. Let u^z, v^z satisfy the problem (3.49) and assume v^z is horizontally band-limited, i.e.

$$\mathcal{F}'v^z(k',z,t) = 0 \text{ unless } 1 \le R|k'| \le 4\,.$$

Then,

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\ \lesssim\ & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)}\,. \end{aligned} \tag{3.54}$$

4. Let v', f', satisfy the problem (3.50) and assume f' is horizontally band-limited, i.e.

$$\mathcal{F}'f(k',z,t) = 0 \text{ unless } 1 \le R|k'| \le 4\,.$$

Then,

$$||\nabla'\nabla v'||_{(0,\infty)} + ||(\partial_t - \partial_z^2)v'||_{(0,\infty)} \lesssim ||f'||_{(0,\infty)}\,. \tag{3.55}$$

Proof of Proposition 3

By an easy application of Proposition 4, we will now prove the maximal regularity estimate on the upper half space.

Proof of Proposition 3.
From Proposition 4 we have the following bound for the vertical component of the velocity u

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\ \overset{(3.54)}{\lesssim}\ & ||\nabla v^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} \\ \overset{(3.53)}{\lesssim}\ & ||f||_{(0,\infty)} + ||\phi||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)} \\ \overset{(1)}{\lesssim}\ & ||f||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,. \end{aligned}$$

Instead for the horizontal components of the velocity u' we have

$$\begin{aligned} & ||(\partial_t - \partial_z^2)u'||_{(0,\infty)} + ||\nabla'\nabla u'||_{(0,\infty)} \\ \overset{(3.51)}{\lesssim}\ & ||(\partial_t - \partial_z^2)v'||_{(0,\infty)} + ||\nabla'\nabla v'||_{(0,\infty)} \\ +\ & ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)} \\ +\ & ||(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z||_{(0,\infty)} + ||\partial_z\nabla u^z||_{(0,\infty)} \\ \overset{(3.53),(3.54),(3.55)}{\lesssim}\ & ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,. \end{aligned}$$

Summing up we obtain

$$\begin{aligned} & ||\partial_t u^z||_{(0,\infty)} + ||\nabla^2 u^z||_{(0,\infty)} + ||(\partial_t - \partial_z^2)u'||_{(0,\infty)} + ||\nabla'\nabla u'||_{(0,\infty)} \\ \lesssim\ & ||f||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||(-\Delta)^{-\frac{1}{2}}\partial_z^2\rho||_{(0,\infty)} + ||\nabla\rho||_{(0,\infty)}\,. \end{aligned} \tag{3.56}$$

The bound for the ∇p follows by equations (3.43) and applying (3.56). □

Proof of Proposition 4

This section is devoted to the proof of Proposition 4, which rely on a series of lemmas (Lemma 4, Lemma 5 and Lemma 6) that we state here and prove in Section 3.2.5. The following lemmas contain the basic maximal regularity estimates for the three auxiliary problems. These estimates, together with the bandedness assumption in the form of (3.117), (3.118) and (3.119) will be the main ingredients for the proof of Proposition 4.

Lemma 4.
Let u, f satisfy the problem

$$\left\{\begin{array}{rclll} (\partial_z - (-\Delta')^{\frac{1}{2}})u & = & f & \text{for} & z>0\,, \\ u & \to & 0 & \text{for} & z \to \infty \end{array}\right. \tag{3.57}$$

and assume f to be horizontally band-limited, i.e.

$$\mathcal{F}' f(k',z,t) = 0 \quad \textit{unless} \quad 1 \le R|k'| \le 4\,.$$

Then,

$$||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{3.58}$$

Lemma 5.
Let $u, f, g = g(x',t)$ satisfy the problem

$$\left\{\begin{array}{rclll} (\partial_z + (-\Delta')^{\frac{1}{2}})u & = & f & \text{for} & z>0\,, \\ u & = & g & \text{for} & z = 0 \end{array}\right. \tag{3.59}$$

and define the constant extension $\tilde{g}(x',z,t) := g(x',t)$. Assume f and g to be horizontally band-limited, i.e.

$$\mathcal{F}' f(k',z,t) = 0 \quad \textit{unless} \quad 1 \le R|k'| \le 4$$

and

$$\mathcal{F}' g(k',z,t) = 0 \quad \textit{unless} \quad 1 \le R|k'| \le 4\,.$$

Then

$$||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)} + ||\nabla' \tilde{g}||_{(0,\infty)}\,. \tag{3.60}$$

Remark 1. *Clearly if $g = 0$ in Lemma 5, then we have*

$$||\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{3.61}$$

Lemma 6.
Let u, f satisfy the problem

$$\left\{\begin{array}{rclll} (\partial_t - \Delta)u & = & f & \text{for} & z>0\,, \\ u & = & 0 & \text{for} & z=0\,, \\ u & = & 0 & \text{for} & t=0 \end{array}\right. \tag{3.62}$$

and assume f to be horizontally band-limited, i.e.

$$\mathcal{F}' f(k',z,t) = 0 \quad \textit{unless} \quad 1 \le R|k'| \le 4\,.$$

Then,

$$||(\partial_t - \partial_z^2)u||_{(0,\infty)} + ||\nabla'\nabla u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}\,. \tag{3.63}$$

Now we are ready to prove Proposition 4.

Proof of Proposition 4.

1. Subtracting the quantity $(\partial_z - (-\Delta')^{\frac{1}{2}})(f^z + \partial_z\rho)$ from both sides of equation (3.47) and then multiplying the new equation by $(-\Delta)^{-\frac{1}{2}}$ we get

$$\begin{aligned} & (\partial_z - (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z\rho) \\ = \quad & \nabla' \cdot (-\Delta')^{-\frac{1}{2}} f' + f^z - (-\Delta')^{-\frac{1}{2}}\partial_t\rho + \partial_z\rho - (-\Delta')^{\frac{1}{2}}\rho \,. \end{aligned}$$

From the basic estimate (3.58) we obtain

$$\begin{aligned} & ||\nabla'(-\Delta')^{-\frac{1}{2}}(\phi - f^z - \partial_z\rho)||_{(0,\infty)} \lesssim ||\nabla' \cdot (-\Delta')^{-\frac{1}{2}} f'||_{(0,\infty)} \\ + \quad & ||f^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||\partial_z\rho||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\rho||_{(0,\infty)} \,. \end{aligned}$$

Thanks to the bandedness assumption in the form of (3.117) and (3.118) we have

$$\begin{aligned} & ||\phi - f^z - \partial_z\rho||_{(0,\infty)} \\ \lesssim \quad & ||f'||_{(0,\infty)} + ||f^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\rho||_{(0,\infty)} + ||\partial_z\rho||_{(0,\infty)} + ||\nabla'\rho||_{(0,\infty)} \end{aligned}$$

and from this we obtain easily the desired estimate (1).

2. After multiplying the equation (3.48) by $(-\Delta')^{-\frac{1}{2}}$, the application of (3.63) to $(-\Delta')^{-\frac{1}{2}}v^z$ yields

$$\begin{aligned} & ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\nabla'\nabla v^z||_{(0,\infty)} \\ \lesssim \quad & ||f^z||_{(0,\infty)} + ||\phi||_{(0,\infty)} + ||\nabla' \cdot (-\Delta')^{-\frac{1}{2}} f'||_{(0,\infty)} \\ + \quad & ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)\rho||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\rho||_{(0,\infty)} \,. \end{aligned}$$

The estimate (3.53) follows after observing (3.118) and applying the triangle inequality to the second to last term on the right-hand side.

3. We need to estimate the the three terms on the right-hand side of (3.54) separately. We start with the term $\nabla^2 u^z$: since $||\nabla^2 u^z||_{(0,\infty)} \leq ||\nabla'\nabla u^z||_{(0,\infty)} + ||\partial_z^2 u^z||_{(0,\infty)}$, we tackle the term $\nabla'\nabla u^z$ and $\partial_z^2 u^z$ separately. First multiply by ∇' the equation (3.49). An application of the estimate (3.61) to $\nabla' u^z$ yields

$$||\nabla\nabla' u^z||_{(0,\infty)} \lesssim ||\nabla' v^z||_{(0,\infty)}. \tag{3.64}$$

Now multiplying the equation (3.49) by ∂_z^2

$$\partial_z^2 u^z = -(-\Delta')^{\frac{1}{2}}\partial_z u^z + \partial_z v^z = -\Delta' u^z - (-\Delta')^{\frac{1}{2}} v^z + \partial_z v^z \tag{3.65}$$

and using the bandedness assumption in the form (3.118) we have

$$\begin{aligned} ||\partial_z^2 u^z||_{(0,\infty)} \quad & \leq \quad ||\nabla'^2 u^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} \\ & \overset{(3.64)}{\leq} \quad ||\nabla v^z||_{(0,\infty)} \,. \end{aligned} \tag{3.66}$$

The second term of (3.54), i.e. $(-\Delta')^{-\frac{1}{2}}\partial_z(\partial_t - \partial_z^2)u^z$, can be bounded in the following way: We multiply the equation (3.49) by $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)$

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}})(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z = (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z & \text{for} \quad z > 0, \\ (-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z = (-\Delta')^{-\frac{1}{2}}\partial_z v^z & \text{for} \quad z = 0, \end{cases}$$

where we have used that at $z = 0$

$$(\partial_t - \partial_z^2)u^z = -\partial_z^2 u^z \overset{(3.65)}{=} \partial_z v^z.$$

Applying (3.60) to $(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z$ and using the bandedness assumption in the form of (3.117),

$$||\nabla(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \lesssim ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\partial_z v^z||_{(0,\infty)} . \tag{3.67}$$

Finally we can bound the last term of (3.54), i.e. $\partial_t u^z$: We observe that $\partial_t u^z = (\partial_t - \partial_z^2)u^z + \partial_z^2 u^z$ thus

$$||\partial_t u^z||_{(0,\infty)} \leq ||(\partial_t - \partial_z^2)u^z||_{(0,\infty)} + ||\partial_z^2 u^z||_{(0,\infty)} . \tag{3.68}$$

For the first term in the right-hand side of (3.68) we notice that

$$\begin{aligned} ||(\partial_t - \partial_z^2)u^z||_{(0,\infty)} &\overset{(3.117)}{\leq} ||(-\Delta')^{-\frac{1}{2}}\nabla'(\partial_t - \partial_z^2)u^z||_{(0,\infty)} \\ &\overset{(3.67)}{\lesssim} ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\partial_z v^z||_{(0,\infty)} \\ &\lesssim ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} . \end{aligned}$$

The second term on the right-hand side of (3.68) is bounded in (3.66). Thus we have the following bound for $\partial_t u$

$$||\partial_t u^z||_{(0,\infty)} \leq ||(-\Delta')^{-\frac{1}{2}}(\partial_t - \partial_z^2)v^z||_{(0,\infty)} + ||\nabla v^z||_{(0,\infty)} . \tag{3.69}$$

Putting together all the above we obtain the desired estimate.

4. From the defining equation (3.50), the basic estimate (3.63) and the bandedness assumption in form of (3.119), we get

$$||(\partial_t - \partial_z^2)v'||_{(0,\infty)} + ||\nabla'\nabla v'||_{(0,\infty)} \lesssim ||f'||_{(0,\infty)} .$$

□

Proof of Theorem 3

Let u, p, f be the solutions of the non-stationary Stokes equations in the strip $0 < z < 1$ (3.25). Then $\tilde{u} = \eta u, \tilde{p} = \eta p$ (with η defined in (3.40) satisfy (3.41), namely

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \tilde{f} & \text{for} \quad z > 0 , \\ \nabla \cdot \tilde{u} = \tilde{\rho} & \text{for} \quad z > 0 , \\ \tilde{u} = 0 & \text{for} \quad z = 0 , \\ \tilde{u} = 0 & \text{for} \quad t = 0 , \end{cases}$$

where

$$\tilde{f} := \eta f - 2(\partial_z \eta)\partial_z u - (\partial_z^2 \eta)u + (\partial_z \eta)p e_z, \qquad \tilde{\rho} := (\partial_z \eta)u^z . \tag{3.70}$$

Since, by assumption f, ρ are horizontally band-limited , then also $\tilde{f}$ and $\tilde{\rho}$ satisfy the horizontal bandedness assumption (3.44) and (3.45) respectively. We can therefore apply Proposition 3 to the upper half space problem (3.41) and get

$$\begin{aligned} &||(\partial_t - \partial_z^2)\tilde{u}'||_{(0,\infty)} + ||\nabla'\nabla\tilde{u}'||_{(0,\infty)} + ||\partial_t \tilde{u}^z||_{(0,\infty)} + ||\nabla^2 \tilde{u}^z||_{(0,\infty)} + ||\nabla \tilde{p}||_{(0,\infty)} \\ \lesssim\ &||\tilde{f}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t \tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 \tilde{\rho}||_{(0,\infty)} + ||\nabla \tilde{\rho}||_{(0,\infty)} . \end{aligned}$$

By symmetry, we also have the same maximal regularity estimates in the lower half space. Indeed, let $\tilde{\tilde{u}}, \tilde{\tilde{p}}$ satisfy the equation

$$\begin{cases} \partial_t \tilde{\tilde{u}} - \Delta \tilde{\tilde{u}} + \nabla \tilde{\tilde{p}} = \tilde{\tilde{f}} & \text{for} \quad z < 1 , \\ \nabla \cdot \tilde{\tilde{u}} = \tilde{\tilde{\rho}} & \text{for} \quad z < 1 , \\ \tilde{\tilde{u}} = 0 & \text{for} \quad z = 1 , \\ \tilde{\tilde{u}} = 0 & \text{for} \quad t = 0 , \end{cases} \tag{3.71}$$

where

$$\tilde{\tilde{f}} := (1-\eta)f - 2(\partial_z(1-\eta))\partial_z u - (\partial_z^2(1-\eta))u + (\partial_z(1-\eta))pe_z, \qquad \tilde{\tilde{\rho}} := (\partial_z(1-\eta))u^z\,. \quad (3.72)$$

Again by Proposition 4 we have

$$\begin{aligned} & ||(\partial_t - \partial_z^2)\tilde{\tilde{u}}'||_{(-\infty,1)} + ||\nabla'\nabla\tilde{\tilde{u}}'||_{(-\infty,1)} + ||\partial_t\tilde{\tilde{u}}^z||_{(-\infty,1)} + ||\nabla^2\tilde{\tilde{u}}^z||_{(-\infty,1)} + ||\nabla\tilde{\tilde{p}}||_{(-\infty,1)} \\ \lesssim\ & ||\tilde{\tilde{f}}||_{(-\infty,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\tilde{\rho}}||_{(-\infty,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\tilde{\rho}}||_{(-\infty,1)} + ||\nabla\tilde{\tilde{\rho}}||_{(-\infty,1)}, \end{aligned}$$

where $||\cdot||_{(-\infty,1)}$ is the analogue of (3.46) (see Notations). Since $u = \tilde{u} + \tilde{\tilde{u}}$ in the strip $[0,L)^{d-1} \times (0,1)$, by the triangle inequality and using the maximal regularity estimates above, we get

$$\begin{aligned} & ||(\partial_t - \partial_z^2)u'||_{(0,1)} + ||\nabla'\nabla u'||_{(0,1)} + ||\partial_t u^z||_{(0,1)} + ||\nabla^2 u^z||_{(0,1)} + ||\nabla p||_{(0,1)} \\ \lesssim\ & ||(\partial_t - \partial_z^2)\tilde{u}'||_{(0,\infty)} + ||(\partial_t - \partial_z^2)\tilde{\tilde{u}}'||_{(-\infty,1)} + ||\nabla'\nabla\tilde{u}'||_{(0,\infty)} + ||\nabla'\nabla\tilde{\tilde{u}}'||_{(-\infty,1)} \\ +\ & ||\partial_t\tilde{u}^z||_{(0,\infty)} + ||\partial_t\tilde{\tilde{u}}^z||_{(-\infty,1)} + ||\nabla^2\tilde{u}^z||_{(0,\infty)} + ||\nabla^2\tilde{\tilde{u}}^z||_{(-\infty,1)} \\ +\ & ||\nabla\tilde{p}||_{(0,\infty)} + ||\nabla\tilde{\tilde{p}}||_{(-\infty,1)} \\ \lesssim\ & ||\tilde{f}||_{(0,\infty)} + ||\tilde{\tilde{f}}||_{(-\infty,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\tilde{\rho}}||_{(-\infty,1)} \\ +\ & ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\tilde{\rho}}||_{(-\infty,1)} + ||\nabla\tilde{\rho}||_{(0,\infty)} + ||\nabla\tilde{\tilde{\rho}}||_{(-\infty,1)}\,. \end{aligned}$$

By the definitions of $\tilde{f}$ and $\tilde{\tilde{f}}$ we get

$$||\tilde{f}||_{(0,\infty)} + ||\tilde{\tilde{f}}||_{(-\infty,1)} \lesssim ||f||_{(0,1)} + ||\partial_z u||_{(0,1)} + ||u||_{(0,1)} + ||p||_{(0,1)}$$

and similarly for $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ we have

$$||\nabla\tilde{\rho}||_{(0,\infty)} + ||\nabla\tilde{\tilde{\rho}}||_{(-\infty,1)} \lesssim ||\nabla u||_{(0,1)} + ||u||_{(0,1)}$$

$$||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_t\tilde{\tilde{\rho}}||_{(-\infty,1)} \lesssim ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)}$$

and

$$\begin{aligned} & ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\rho}||_{(0,\infty)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2\tilde{\tilde{\rho}}||_{(-\infty,1)} \\ \lesssim\ & ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Therefore, collecting the estimates, we have

$$\begin{aligned} & ||(\partial_t - \partial_z^2)u'||_{(0,1)} + ||\nabla'\nabla u'||_{(0,1)} + ||\partial_t u^z||_{(0,1)} + ||\nabla^2 u^z||_{(0,1)} + ||\nabla p||_{(0,1)} \\ \lesssim\ & ||f||_{(0,1)} + ||p||_{(0,1)} + ||\nabla u||_{(0,1)} + ||u||_{(0,1)} \\ +\ & ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} + ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Incorporating the horizontal bandedness assumption we find

$$\begin{aligned} ||\partial_z u||_{(0,1)} &\leq R||\nabla'\partial_z u||_{(0,1)}\,, \\ ||u||_{(0,1)} &\leq R^2||(\nabla')^2 u||_{(0,1)}\,, \\ ||p||_{(0,1)} &\leq R||\nabla' p||_{(0,1)}, \\ ||\nabla u||_{(0,1)} &\leq R||\nabla'\nabla u||_{(0,1)}, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_t u||_{(0,1)} &\leq R||\partial_t u||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}u^z||_{(0,1)} &\leq R^3||\nabla'^2 u^z||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_z u^z||_{(0,1)} &\leq R^2||\nabla'\partial_z u^z||_{(0,1)}\,, \\ ||(-\Delta')^{-\frac{1}{2}}\partial_z^2 u^z||_{(0,1)} &\leq R||\partial_z^2 u^z||_{(0,1)}\,. \end{aligned}$$

Thus, for $R < R_0$ where R_0 is sufficiently small, all the terms in the right-hand side, except f can be absorbed into the left-hand side and the conclusion follows.

Proof of main technical lemmas

Remark 2. *In the proof of Lemma 4, Lemma 5 and Lemma 6 we will derive inequalities between quantities where t is integrated between 0 and ∞. From the proof it is clear that the same inequalities are true with t integrated between 0 and t_0 with constants that are not depending on t_0. Therefore dividing by t_0 and taking $\limsup_{t_0\to\infty}$ (see (3.127)) we shall obtain the desired estimates in terms of the interpolation norm (3.46).*

Proof of Lemma 4.
In order to simplify the notations, in what follows we will omit the dependency of the functions from the time variable. It is enough to show

$$||\nabla' u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)},$$

since, by equation (3.57) $\partial_z u = (-\Delta')^{\frac{1}{2}}u + f$. We claim that, in order to prove (3.2.5), it is enough to show

$$\sup_z \langle|\nabla' u|\rangle' \lesssim \sup_z \langle|f|\rangle' \tag{3.73}$$

and

$$||\nabla' u||_{(0,\infty)} \lesssim \int \langle|f|\rangle' \frac{dz}{z}\,. \tag{3.74}$$

Indeed, by definition of the norm $||\cdot||_{(0,\infty)}$ (see (3.46)) if we select an arbitrary decomposition $\nabla' u = \nabla' u_1 + \nabla' u_2$, where u_1 and u_2 are solutions of the problem (3.57) with right-hand sides f_1 and f_2 respectively, we have

$$\begin{aligned} ||\nabla' u||_{(0,\infty)} &\le ||\nabla' u_1||_{(0,\infty)} + \sup_z \langle|\nabla' u_2|\rangle' \\ &\le \int \langle|f_1|\rangle' \frac{dz}{z} + \sup_z \langle|f_2|\rangle'\,. \end{aligned}$$

Passing to the infimum over all the decompositions of f we obtain

$$||\nabla' u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}.$$

We recall that by Duhamel's principle we have the following representation

$$u(x',z) = \int_z^\infty u_{x',z_0}(z)dz_0, \tag{3.75}$$

where u_{z_0} is the harmonic extension of $f(\cdot,z_0)$ onto $\{z<z_0\}$, i.e. it solves the boundary value problem

$$\begin{cases} (\partial_z - (-\Delta')^{\frac{1}{2}})u_{z_0} = 0 & \text{for } z<z_0\,, \\ u_{z_0} = f & \text{for } z = z_0\,. \end{cases} \tag{3.76}$$

Argument for (3.73):
Using the representation of the solution of (3.76) via the Poisson kernel, i.e.

$$u_{z_0}(x',z) = \int \frac{z_0 - z}{(|x'-y'|^2 + (z_0-z)^2)^{\frac{d}{2}}} f(x',z_0)dy'$$

we obtain the following bounds

$$\langle|\nabla' u_{z_0}(\cdot,z)|\rangle' \lesssim \begin{cases} & \langle|\nabla' f(\cdot,z_0)|\rangle', \\ \frac{1}{(z_0-z)} & \langle|f(\cdot,z_0)|\rangle', \\ \frac{1}{(z_0-z)^2} & \langle|\nabla'(-\Delta')^{-1} f(\cdot,z_0)|\rangle'. \end{cases} \tag{3.77}$$

By using the bandedness assumption in the form of (3.114) and (3.116), we have

$$\langle|\nabla' u_{z_0}(\cdot,z)|\rangle' \lesssim \min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle',$$

hence

$$\begin{aligned}\langle|\nabla' u(\cdot,z)|\rangle' &\lesssim \int_z^\infty \min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle' dz_0\\ &\lesssim \sup_{z_0\in(0,\infty)}\langle|f(\cdot,z_0)|\rangle'\int_z^\infty \min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}dz_0\\ &\lesssim \sup_{z_0\in(0,\infty)}\langle|f(\cdot,z_0)|\rangle',\end{aligned}$$

which, passing to the supremum in z, implies (3.73).
From the above and applying Fubini's rule, we also have

$$\begin{aligned}\int_0^\infty\langle|\nabla' u(\cdot,z)|\rangle' dz &\le \int_0^\infty\int_z^\infty \min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}\langle|f(\cdot,z_0)|\rangle' dz_0 dz\\ &\le \int_0^\infty\int_0^{z_0} \min\left\{\frac{1}{R},\frac{R}{(z_0-z)^2}\right\}dz\langle|f(\cdot,z_0)|\rangle' dz_0\\ &\lesssim \int_0^\infty\langle|f(\cdot,z)|\rangle' dz\,.\end{aligned} \tag{3.78}$$

Argument for (3.74):
Let us consider $\chi_{2H\le z\le 4H}f$ where $\chi_{2H\le z\le 4H}$ is the characteristic function on the interval $[2H,4H]$ and let u_H be the solution to

$$(\partial_z-(-\Delta')^{\frac12})u_H=\chi_{2H\le z\le 4H}f.$$

We claim

$$\sup_{z\le H}\langle|\nabla' u_H|\rangle' \le \int_0^\infty\langle|\chi_{2H\le z\le 4H}f|\rangle'\frac{dz}{z} \tag{3.79}$$

and

$$\int_H^\infty\langle|\nabla' u_H|\rangle'\frac{dz}{z} \le \int_0^\infty\langle|\chi_{2H\le z\le 4H}f|\rangle'\frac{dz}{z}\,. \tag{3.80}$$

From estimate (3.79) and (3.80) the statement (3.74) easily follow. Indeed, choosing $H=2^{n-1}$ and summing up over the dyadic intervals, we have

$$\begin{aligned}||\nabla' u|| &\le \sum_{n\in\mathbb{Z}}||\nabla' u_{2^{n-1}}||_{(0,\infty)}\\ &\le \sup_{z\le 2^{n-1}}\langle|\nabla' u_{2^{n-1}}|\rangle' + \int_{2^{n-1}}^\infty\langle|\nabla' u_{2^{n-1}}|\rangle'\frac{dz}{z}\\ &\le \sum_{n\in\mathbb{Z}}\int_0^\infty\langle|\chi_{2^n\le z\le 2^{n+1}}f|\rangle'\frac{dz}{z}\\ &= \int_0^\infty\langle|f|\rangle'\frac{dz}{z}\,.\end{aligned}$$

Argument for (3.79): Fix $z \leq H$. Then, we have

$$\begin{aligned}
\langle|\nabla' u_H|\rangle' &\overset{(3.77)}{\leq} \int_z^\infty \frac{1}{(z_0 - z)} \langle|\chi_{2H \leq z \leq 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\lesssim \int_{2H}^{4H} \frac{1}{(z_0 - z)} \langle|\chi_{2H \leq z \leq 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\lesssim \frac{1}{H} \int_{2H}^{4H} \langle|\chi_{2H \leq z \leq 4H} f(\cdot, z_0)|\rangle' dz_0 \\
&\leq \int_{2H}^\infty \langle|\chi_{2H \leq z \leq 4H} f(\cdot, z_0)|\rangle' \frac{dz_0}{z_0} \\
&\leq \int_0^\infty \langle|\chi_{2H \leq z \leq 4H} f(\cdot, z_0)|\rangle' \frac{dz_0}{z_0} \,.
\end{aligned}$$

Taking the supremum over all z proves (3.79).
Argument for (3.80): For $z \geq H$ we have

$$\begin{aligned}
\int_H^\infty \langle|\nabla' u_H|\rangle' \frac{dz}{z} &\lesssim \frac{1}{H} \int_0^\infty \langle|\nabla' u_H|\rangle' dz \\
&\overset{(3.78)}{\lesssim} \frac{1}{H} \int_0^\infty \langle|\chi_{2H \leq z \leq 4H} f|\rangle' dz \\
&= \frac{1}{H} \int_{2H}^{4H} \langle|\chi_{2H \leq z \leq 4H} f|\rangle' dz \\
&\lesssim \int_0^\infty \langle|\chi_{2H \leq z \leq 4H} f|\rangle' \frac{dz}{z} \,.
\end{aligned}$$

□

Proof of Lemma 5.
Let us first assume $g = 0$. It is enough to show

$$\sup_z \langle|\nabla' u|\rangle' \lesssim \sup_z \langle|f|\rangle' \tag{3.81}$$

and

$$\int_0^\infty \langle|\nabla' u|\rangle' \frac{dz}{z} \lesssim \int_0^\infty \langle|f|\rangle' \frac{dz}{z} \,. \tag{3.82}$$

Recall that by Duhamel's principle we have the following representation

$$u(z) = \int_0^z u_{z_0}(\cdot, z) dz_0, \tag{3.83}$$

where u_{z_0} is the harmonic extension of $f(z_0)$ onto $\{z > z_0\}$, i.e. it solves the boundary value problem

$$\begin{cases} (\partial_z + (-\Delta')^{\frac{1}{2}}) u_{z_0} = 0 & \text{for} \quad z > z_0 \,, \\ u_{z_0} = f & \text{for} \quad z = z_0 \,. \end{cases} \tag{3.84}$$

From the Poisson's kernel representation we learn that

$$\langle|\nabla' u_{z_0}(\cdot, z)|\rangle' \lesssim \begin{cases} \langle|\nabla' f(\cdot, z_0)|\rangle' \,, \\ \frac{1}{(z - z_0)^2} \langle|\nabla'(-\Delta')^{-1} f(\cdot, z_0)|\rangle' \,. \end{cases}$$

Using the bandedness assumption in the form of (3.114) and (3.116)

$$\langle|\nabla' u_{z_0}(\cdot, z)|\rangle' \lesssim \min\left\{\frac{1}{R}, \frac{R}{(z - z_0)^2}\right\} \langle|f(\cdot, z_0)|\rangle'$$

and observing (3.83), we obtain

$$
\begin{aligned}
\langle|\nabla' u(\cdot,z)|\rangle' &\lesssim \int_0^z \min\left\{\tfrac{1}{R},\tfrac{R}{(z-z_0)^2}\right\}\langle|f(\cdot,z_0)|\rangle' dz_0\\
&\leq \sup_{z_0}\langle|f(\cdot,z_0)|\rangle' \int_0^z \min\left\{\tfrac{1}{R},\tfrac{R}{(z-z_0)^2}\right\} dz_0\\
&\lesssim \sup_{z_0}\langle|f(\cdot,z_0)|\rangle' .
\end{aligned}
\tag{3.85}
$$

Estimate (3.81) follows from (3.85) by passing to the supremum in z.
From the above (3.85), multiplying by the weight $\frac{1}{z}$ and observing that $z > z_0$ we have

$$
\langle|\nabla' u(\cdot,z)|\rangle'\frac{1}{z} \lesssim \int_0^z \min\left\{\frac{1}{R},\frac{R}{(z-z_0)^2}\right\}\langle|f(\cdot,z_0)|\rangle'\frac{dz_0}{z_0} . \tag{3.86}
$$

After integrating in $z \in (0,\infty)$ and applying Young's estimate we get (3.82). Let's assume now the general case, with $g \neq 0$. We want to prove (3.60). Recall that by definition $\tilde{g}(x',z) := g(x')$ and consider $u-\tilde{g}$. By construction it satisfies

$$
\begin{cases}
(\partial_z + (-\Delta')^{-\frac{1}{2}})(u-\tilde{g}) = f - (-\Delta')^{-\frac{1}{2}} g & \text{for} \quad z>0 ,\\
u-\tilde{g} = 0 & \text{for} \quad z=0 .
\end{cases}
$$

Using the first part of the proof of (3.61) and triangle inequality, we have

$$
||\nabla u||_{(0,\infty)} \lesssim ||\nabla\tilde{g}||_{(0,\infty)} + ||f||_{(0,\infty)} + ||(-\Delta')^{\frac{1}{2}}\tilde{g}||_{(0,\infty)} .
$$

Therefore by the bandedness assumption in the form of (3.118) we can conclude (3.60). □

Proof of Lemma 6.
We will show that, for the non-homogeneous heat equation with Dirichlet boundary condition

$$
\begin{cases}
(\partial_t - \Delta)u = f & \text{for} \quad z>0 ,\\
u = 0 & \text{for} \quad z=0 ,\\
u = 0 & \text{for} \quad t=0 ,
\end{cases}
\tag{3.87}
$$

we have the following estimates

$$
\left\langle \int |(\partial_t-\partial_z^2)u(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle + \left\langle \int |\nabla'^2 u(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle \lesssim \left\langle \int |f(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle , \tag{3.88}
$$

$$
\langle|\nabla'\partial_z u(\cdot,z,\cdot)|_{z=0}\rangle \lesssim \left\langle \int |f(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle , \tag{3.89}
$$

$$
\left\langle \sup_z |\nabla'^2 u(\cdot,z,\cdot)|\right\rangle \lesssim \left\langle \sup_z |f(\cdot,z,\cdot)|\right\rangle , \tag{3.90}
$$

$$
\left\langle \sup_z |\nabla'\partial_z u(\cdot,z,\cdot)|\right\rangle \lesssim \left\langle \sup_z |f(\cdot,z,\cdot)|\right\rangle . \tag{3.91}
$$

In order to bound the off-diagonal components of the Hessian, we consider the decomposition

$$
u = u_N + u_C, \tag{3.92}
$$

where u_N solves

$$
\begin{cases}
(\partial_t - \Delta)u_N = f & \text{for} \quad z>0 ,\\
\partial_z u_N = 0 & \text{for} \quad z=0 ,\\
u_N = 0 & \text{for} \quad t=0 ,
\end{cases}
\tag{3.93}
$$

and u_C solves

$$
\begin{cases}
(\partial_t - \Delta)u_C = 0 & \text{for} \quad z>0 ,\\
\partial_z u_C = \partial_z u & \text{for} \quad z=0 ,\\
u_C = 0 & \text{for} \quad t=0 .
\end{cases}
\tag{3.94}
$$

The splitting (3.92) is valid by the uniqueness of the Neumann problem. For the auxiliary problems (3.93) and (3.94) we have the following bounds

$$\left\langle \int |\nabla'\partial_z u_N(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle \lesssim \left\langle \int |f(\cdot,z,\cdot)|\frac{dz}{z}\right\rangle, \tag{3.95}$$

$$\left\langle \sup_z |\nabla'\partial_z u_C(\cdot,z,\cdot)|\right\rangle \lesssim \langle |\nabla'\partial_z u(\cdot,z,\cdot)|_{z=0}\rangle. \tag{3.96}$$

We claim that estimates (3.88), (3.89),(3.90), (3.91), (3.95) and (3.96) yield (3.63).
Let us first consider the bound for ∇'^2. Consider $u = u_1 + u_2$, where u_1 and u_2 satisfy (3.87) with right-hand side f_1 and f_2 respectively. We have

$$\begin{aligned} ||\nabla'^2 u||_{(0,\infty)} &\lesssim \left\langle \sup_z |\nabla'^2 u_1|\right\rangle + \left\langle \int |\nabla'^2 u_2|\frac{dz}{z}\right\rangle \\ &\overset{(3.88)\&(3.90)}{\lesssim} \left\langle \sup_z |f_1|\right\rangle + \left\langle \int |f_2|\frac{dz}{z}\right\rangle, \end{aligned}$$

which implies, upon taking infimum over all decompositions $f = f_1 + f_2$

$$||\nabla'^2 u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}. \tag{3.97}$$

We now consider a further decomposition of u_2 , i.e. $u_2 = u_{2C} + u_{2N}$ where u_{2C} satisfies (3.94) and u_{2N} satisfies (3.93). Therefore $u = u_1 + u_{2C} + u_{2N}$ and we can bound the off-diagonal components of the Hessian

$$\begin{aligned} ||\nabla'\partial_z u||_{(0,\infty)} &\lesssim \left\langle \sup_z |\nabla'\partial_z u_1|\right\rangle + \left\langle \sup_z |\nabla'\partial_z u_{2C}|\right\rangle + \left\langle \int |\nabla'\partial_z u_{2N}|\frac{dz}{z}\right\rangle \\ &\overset{(3.89),(3.96),(3.95)\&(3.91)}{\lesssim} \left\langle \sup_z |f_1|\right\rangle + \left\langle \int |f_2|\frac{dz}{z}\right\rangle. \end{aligned}$$

From the last inequality, passing to the infimum over all the possible decompositions of f we get

$$||\nabla'\partial_z u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}. \tag{3.98}$$

On one hand estimate (3.97) and (3.98) imply

$$||\nabla\nabla' u||_{(0,\infty)} \lesssim ||\nabla'^2 u||_{(0,\infty)} + ||\nabla'\partial_z u||_{(0,\infty)},$$

on the other hand equation (3.62) and estimate (3.97) yield

$$||(\partial_t - \partial_z^2)u||_{(0,\infty)} \lesssim ||f||_{(0,\infty)}.$$

<u>Argument for (3.88)</u>
Let u be a solution of problem of (3.87). Keeping in mind Remark (2) it is enough to show

$$\int_0^\infty \int_0^\infty \langle |\nabla'^2 u|\rangle' \frac{dz}{z} dt \lesssim \int_0^\infty \int_0^\infty \langle |f|\rangle' \frac{dz}{z} dt.$$

By the Duhamel's principle we have

$$u(x', z, t) = \int_{s=0}^t u_s(x', z, t) ds, \tag{3.99}$$

where u_s is the solution to the homogeneous, initial value problem

$$\begin{cases} (\partial_t - \Delta)u_s = 0 & \text{for} \quad z > 0, t > s, \\ u_s = 0 & \text{for} \quad z = 0, t > s, \\ u_s = f & \text{for} \quad z > 0, t = s. \end{cases} \tag{3.100}$$

Extending u and f to the whole space by odd reflection [8], we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_s &= 0 \qquad \text{for} \quad z \in \mathbb{R}, t > s\,, \\ u_s &= f \qquad \text{for} \quad z \in \mathbb{R}, t = s\,, \end{cases}$$

the solution of which can be represented via heat kernel as

$$\begin{aligned} u_s(x', z, t) &= \int_{\mathbb{R}} \Gamma(\cdot, z - \tilde{z}, t - s) *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z} \\ &= \int_0^\infty [\Gamma(\cdot, z - \tilde{z}, t - s) - \Gamma(\cdot, z + \tilde{z}, t - s)] *_{x'} f(\cdot, \tilde{z}, s) d\tilde{z}\,. \end{aligned} \tag{3.101}$$

The application of ∇'^2 to the representation above yields

$$\begin{aligned} &\nabla'^2 u_s(x', z, t) \\ = &\begin{cases} \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla' \Gamma_{d-1}(x' - \tilde{x}', t - s) \left(\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)\right) \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}\,, \\ \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'^3 \Gamma_{d-1}(x' - \tilde{x}', t - s) \left(\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)\right) (-\Delta')^{-1} \nabla' f(\tilde{x}', \tilde{z}, s) d\tilde{x}' d\tilde{z}\,. \end{cases} \end{aligned}$$

Averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned} &\langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\nabla' \Gamma_{d-1}(\cdot, t - s)| \rangle' |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \langle |\nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ &\overset{(3.122)\&(3.116)}{\lesssim} \int_0^\infty \frac{1}{(t - s)^{\frac{1}{2}}} |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \frac{1}{R} \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \end{aligned}$$

and, on the other hand

$$\begin{aligned} &\langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \\ &\lesssim \int_0^\infty \langle |\nabla'^3 \Gamma_{d-1}(\cdot, t - s) \rangle' |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \langle |(-\Delta')^{-1} \nabla' f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z} \\ &\overset{(3.122)\&(3.114)}{\lesssim} \int_0^\infty |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)| \frac{1}{(t - s)^{\frac{3}{2}}} R \langle |f(\cdot, \tilde{z}, s)| \rangle' d\tilde{z}\,. \end{aligned}$$

Multiplying by the weight $\frac{1}{z}$ and integrating in $z \in (0, \infty)$ we get

$$\int_0^\infty \langle |\nabla'^2 u_s(\cdot, t)| \rangle' \frac{dz}{z} \lesssim \left(\sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \right) \begin{cases} \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}\,, \\ \frac{R}{(t-s)^{\frac{3}{2}}} \int_0^\infty \langle |f(x', \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}\,, \end{cases}$$

where we called $K_{t-s}(z, \tilde{z}) = \frac{\tilde{z}}{z} |\Gamma_1(z - \tilde{z}, t - s) - \Gamma_1(z + \tilde{z}, t - s)|$.
From Lemma 8 we infer

$$\sup_{\tilde{z}} \int_0^\infty K_{t-s}(z, \tilde{z}) dz \overset{(3.121)}{\lesssim} \int_{\mathbb{R}} |\Gamma_1(z, t - s)| dz + \sup_{z \in \mathbb{R}} (z^2 |\partial_z \Gamma_1(z, t - s)|) \overset{(3.123)\&(3.126)}{\lesssim} 1$$

and therefore we have

$$\int_0^\infty \langle |\nabla'^2 u_s(\cdot, z, t)| \rangle' \frac{dz}{z} \lesssim \begin{cases} \frac{1}{(t-s)^{\frac{1}{2}}} \frac{1}{R} \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}\,, \\ \frac{1}{(t-s)^{\frac{3}{2}}} R \int_0^\infty \langle |f(\cdot, \tilde{z}, s)| \rangle' \frac{d\tilde{z}}{\tilde{z}}\,. \end{cases}$$

[8] with abuse of notation we will call again u and f these extensions.

Finally, inserting the previous estimate into the Duhamel formula (3.99) and integrating in time we get

$$\begin{aligned}
&\int_0^\infty \langle|\nabla'^2 u(\cdot,z,t)|\rangle' \frac{dz}{z} dt \\
\overset{(3.99)}{\lesssim}\ & \int_0^\infty \int_0^t \langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \frac{dz}{z} ds dt \\
\lesssim\ & \int_0^\infty \int_s^\infty \min\left\{\frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}}\right\} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' \frac{d\tilde{z}}{\tilde{z}} dt ds \\
\lesssim\ & \int_0^\infty \int_s^\infty \min\left\{\frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}}\right\} dt \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' \frac{d\tilde{z}}{\tilde{z}} ds \qquad (3.102)\\
\lesssim\ & \int_0^\infty \int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} d\tau \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' \frac{d\tilde{z}}{\tilde{z}} ds, \qquad (3.103)\\
\lesssim\ & \int_0^\infty \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' \frac{d\tilde{z}}{\tilde{z}} ds,
\end{aligned}$$

where in the second to last inequality we used

$$\int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} d\tau \lesssim 1\,. \qquad (3.104)$$

Argument for (3.89):
Let u be a solution of problem of (3.87). Recall that we need to prove

$$\int_0^\infty \langle|\nabla'\partial_z u|_{z=0}(\cdot,z,t)|\rangle' dt \lesssim \int_0^\infty \int_0^\infty \langle|f(\cdot,z,t)|\rangle' dt \frac{dz}{z}\,. \qquad (3.105)$$

The solution of the equation (3.100) extended to the whole space by odd reflection can be represented by (3.101) (see argument for (3.88)). Therefore

$$\begin{aligned}
&\nabla'\partial_z u_s(x',z,t)|_{z=0} \\
=\ & \begin{cases} -2\int_{\mathbb{R}^{d-1}}\int_0^\infty \Gamma_{d-1}(x'-\tilde{x}',t-s)\partial_z\Gamma_1(\tilde{z},t-s)\nabla' f(\tilde{x}',\tilde{z},s)d\tilde{x}'d\tilde{z}\,, \\ -2\int_{\mathbb{R}^{d-1}}\int_0^\infty \nabla'\Gamma_{d-1}(x'-\tilde{x}',t-s)\partial_z\Gamma_1(\tilde{z},t-s)\nabla'(-\Delta')^{-1}\nabla' f(\tilde{x}',\tilde{z},s)d\tilde{x}'d\tilde{z}\,. \end{cases}
\end{aligned}$$

Taking the horizontal average we get, on the one hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}|\rangle' \\
\lesssim\ & \int_0^\infty \langle|\Gamma_{d-1}(\cdot,t-s)|\rangle' |\partial_z\Gamma_1(\tilde{z},t-s)| \langle|\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\overset{(3.122)}{\lesssim}\ & \int_0^\infty |\partial_z\Gamma_1(\tilde{z},t-s)| \langle|\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\overset{(3.116)}{\lesssim}\ & \frac{1}{R}\int_0^\infty |\partial_z\Gamma_1(\tilde{z},t-s)| \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\lesssim\ & \frac{1}{R}\sup_{\tilde{z}} |\tilde{z}\partial_z\Gamma_1(\tilde{z},t-s)| \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' \frac{d\tilde{z}}{\tilde{z}}
\end{aligned}$$

and on the other hand

$$
\begin{aligned}
& \langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}|\rangle' \\
\lesssim\ & \int_0^\infty \langle|(\nabla')^2\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(3.122)}{\lesssim}\ & \frac{1}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(3.114)}{\lesssim}\ & \frac{R}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\lesssim\ & \frac{R}{(t-s)}\sup_{\tilde z}|\tilde z\partial_z\Gamma_1(\tilde z,t-s)|\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,.
\end{aligned}
$$

Using the estimate (3.125) we get

$$
\langle|\nabla'\partial_z u_s(x',z,t)|_{z=0}|\rangle' \lesssim \begin{cases} \frac{1}{(t-s)^{1/2}R}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,, \\ \frac{R}{(t-s)^{3/2}}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,. \end{cases}
$$

Finally, inserting into Duhamel's formula and integrating in time we have

$$
\begin{aligned}
& \int_0^\infty \langle|\nabla'\partial_z u(\cdot,z,t)|_{z=0}\rangle' dt \\
\overset{(3.99)}{\lesssim}\ & \int_0^\infty\int_0^t \langle|\nabla'\partial_z u_s(\cdot,z,t)|_{z=0}\rangle' ds dt \\
\lesssim\ & \int_0^\infty\int_s^\infty \min\{\frac{1}{R(t-s)^{\frac12}},\frac{R}{(t-s)^{\frac32}}\}\int_0^\infty \langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}dtds \\
\overset{(3.102)\&(3.103)}{\lesssim}\ & \int_0^\infty\int_0^\infty \langle|f(x',z,s)|\rangle'\frac{d\tilde z}{\tilde z}ds.
\end{aligned}
$$

<u>Argument for (3.90):</u>
<u>Let u be the solution</u> of problem (3.87). We recall that we want to prove

$$
\sup_z\int_0^\infty \langle|\nabla'^2 u(\cdot,z,t)|\rangle' dt \lesssim \sup_z\int_0^\infty \langle|f(\cdot,z,t)|\rangle' dt\,. \tag{3.106}
$$

The solution of equation (3.100) extended to the whole space can be represented by (3.101) (see argument for (3.88)). Therefore applying ∇'^2 to (3.101) and considering the horizontal average we have, on the one hand

$$
\begin{aligned}
& \langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\
\lesssim\ & \int_{\mathbb{R}} \langle|\nabla'\Gamma_{d-1}(\cdot,t-s)|\rangle'|\Gamma_1(z-\tilde z,t-s)|\langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(3.122)\&(3.116)}{\lesssim}\ & \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac12}}|\Gamma_1(z-\tilde z,t-s)|\frac{1}{R}\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
& \langle|\nabla'^2 u_s(\cdot,z,t)|\rangle' \\
\lesssim\ & \int_{\mathbb{R}} \langle|\nabla'^3\Gamma_{d-1}(\cdot,t-s)|\rangle'|\Gamma_1(z-\tilde z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
\overset{(3.122)\&(3.114)}{\lesssim}\ & \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac32}}|\Gamma_1(z-\tilde z,t-s)|R\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\,.
\end{aligned}
$$

Inserting the above estimates in the Duhamel's formula (3.99), we have

$$\begin{aligned}
&\int_0^\infty \int_0^t \langle|\nabla'^2 u_s(z,\cdot)|\rangle' ds dt \\
\lesssim\ &\int_0^\infty \int_s^\infty \min\left\{\frac{1}{R(t-s)^{\frac{1}{2}}}, \frac{R}{(t-s)^{\frac{3}{2}}}\right\} \int_{\mathbb{R}} |\Gamma_1(z-\tilde{z},t-s)| \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} ds dt \\
\lesssim\ &\int_{\mathbb{R}} \left(\int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} |\Gamma_1(z-\tilde{z},\tau)| d\tau\right) \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds d\tilde{z} \\
\lesssim\ &\sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds \int_{\mathbb{R}} \int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} |\Gamma_1(z-\tilde{z},\tau)| d\tau d\tilde{z} \\
\overset{(3.123)}{\lesssim}\ &\sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds \int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} d\tau \int_{\mathbb{R}} |\Gamma_1(z-\tilde{z},\tau)| d\tilde{z} \\
\overset{(3.104)}{\lesssim}\ &\sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds\,.
\end{aligned}$$

Taking the supremum in z we obtain the desired estimate. Argument for (3.91): Let u be the solution of problem (3.87). We claim

$$\sup_z \int_0^\infty \langle|\nabla'\partial_z u|\rangle' dt \lesssim \sup_z \int_0^\infty \langle|f|\rangle' dt\,. \tag{3.107}$$

The solution of the equation (3.100) extended to the whole space can be represented by (see argument for (3.88))

$$u_s(x',z,t) = \int_{\mathbb{R}} \Gamma(\cdot, z-\tilde{z}, t-s) *_{x'} f(\cdot,\tilde{z},s) d\tilde{z}\,.$$

Applying $\nabla'\partial_z$ and considering the horizontal average we obtain, on the one hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_s(\cdot,z,t)|\rangle' \\
\lesssim\ &\int_{\mathbb{R}} \langle|\Gamma_{d-1}(\cdot,t-s)|\rangle' |\partial_z \Gamma_1(z-\tilde{z},t-s)| \langle|\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\overset{(3.116)}{\lesssim}\ &\int_{\mathbb{R}} |\partial_z \Gamma_1(z-\tilde{z},t-s)| \frac{1}{R} \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z}
\end{aligned}$$

and, on the other hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_s(\cdot,z,t)|\rangle' \\
\lesssim\ &\int_{\mathbb{R}} \langle|\nabla'^2 \Gamma_{d-1}(\cdot,t-s)|\rangle' |\partial_z \Gamma_1(z-\tilde{z},t-s)| \langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} \\
\overset{(3.114)}{\lesssim}\ &\int_{\mathbb{R}} \frac{1}{(t-s)} |\partial_z \Gamma_1(z-\tilde{z},t-s)| R \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z}\,.
\end{aligned}$$

Inserting the above estimates in the Duhamel's formula (3.99), we have

$$\begin{aligned}
&\int_0^\infty \int_0^t \langle|\nabla'\partial_z u_s(z,\cdot)|\rangle' ds dt \\
\lesssim\ & \int_0^\infty \int_s^\infty \min\left\{\frac{1}{R}, \frac{R}{(t-s)}\right\} \int_{\mathbb{R}} |\partial_z\Gamma_1(z-\tilde{z},t-s)| \langle|f(\cdot,\tilde{z},s)|\rangle' d\tilde{z} dt ds \\
\lesssim\ & \int_{\mathbb{R}} \left(\int_0^\infty \min\left\{\frac{1}{R}, \frac{R}{\tau}\right\} |\partial_z\Gamma_1(z-\tilde{z},\tau)| d\tau\right) \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds d\tilde{z} \\
\lesssim\ & \sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds \int_{\mathbb{R}} \int_0^\infty \min\left\{\frac{1}{R}, \frac{R}{\tau}\right\} |\partial_z\Gamma_1(z-\tilde{z},\tau)| d\tau d\tilde{z} \\
\overset{(3.123)}{\lesssim}\ & \sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds \int_0^\infty \min\left\{\frac{1}{R\tau^{\frac{1}{2}}}, \frac{R}{\tau^{\frac{3}{2}}}\right\} d\tau \\
\overset{(3.104)}{\lesssim}\ & \sup_{\tilde{z}} \int_0^\infty \langle|f(\cdot,\tilde{z},s)|\rangle' ds\,.
\end{aligned}$$

Taking the supremum in z we obtain the desired estimate.

Argument for (3.95)

We recall that we want to show

$$\int_0^\infty \int_0^\infty \langle|\nabla'\partial_z u_N|\rangle' \frac{dz}{z} dt \lesssim \int_0^\infty \int_0^\infty \langle|f|\rangle' \frac{dz}{z} dt,$$

where u_N be the solution to the non-homogeneous heat equation with Neumann boundary conditions (3.93). By the Duhamel's principle we have

$$u_N(x',z,t) = \int_{s=0}^t u_{N_s}(x',z,t) ds,$$

where u_{N_s} is solution to

$$\begin{cases} (\partial_t - \Delta)u_{N_s} = 0 & \text{for } z>0, t>s\,, \\ \partial_z u_{N_s} = 0 & \text{for } z=0, t>s\,, \\ u_{N_s} = f & \text{for } z>0, t=s\,, \end{cases}$$

is the solution of problem (3.87). Extending this equation to the whole space by even reflection [9], we are left to study the problem

$$\begin{cases} (\partial_t - \Delta)u_{N_s} = 0 & \text{for } z\in\mathbb{R}, t>s\,, \\ u_{N_s} = f & \text{for } t=s\,, \end{cases}$$

the solution of which can be represented via heat kernel as

$$\begin{aligned}
u_{N_s}(x',z,t) &= \int_{\mathbb{R}} \Gamma(\cdot,z-\tilde{z},t-s) *_{x'} f(\cdot,\tilde{z},s) d\tilde{z} \\
&= \int_0^\infty [\Gamma(\cdot,\tilde{z}+z,t-s) + \Gamma(\cdot,\tilde{z}-z,t-s)] *_{x'} f(\cdot,\tilde{z},s) d\tilde{z}\,.
\end{aligned}$$

Applying $\nabla'\partial_z$ to the representation above

$$\begin{aligned}
&\nabla'\partial_z u_{N_s}(x',z,t) \\
=\ & \begin{cases} \int_0^\infty \int_{\mathbb{R}^{d-1}} \Gamma_{d-1}(x'-\tilde{x'},t-s)\left(\partial_z\Gamma_1(\tilde{z}+z,t-s) - \partial_z\Gamma_1(\tilde{z}-z,t-s)\right) \nabla' f(\tilde{x'},\tilde{z},s) d\tilde{x'} d\tilde{z}\,, \\ \int_0^\infty \int_{\mathbb{R}^{d-1}} \nabla'^2\Gamma_{d-1}(x'-\tilde{x'},t-s)\left(\partial_z\Gamma_1(\tilde{z}+z,t-s) - \partial_z\Gamma_1(\tilde{z}-z,t-s)\right)(-\Delta')^{-1}\nabla' f(\tilde{x'},\tilde{z},s) d\tilde{x'} d\tilde{z} \end{cases}
\end{aligned}$$

[9] With abuse of notation we will denote with u_{N_s} and f their even reflection

and averaging in the horizontal direction we obtain, on the one hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle' \\
&\lesssim \int_0^\infty \langle|\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|\nabla' f(\cdot,\tilde z,s)|\rangle' d\tilde z \\
&\overset{(3.122)\&(3.116)}{\lesssim} \frac{1}{R}\int_0^\infty |\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z
\end{aligned}$$

and, on the other hand

$$\begin{aligned}
&\langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle' \\
&\lesssim \int_0^\infty \langle|\nabla'^2\Gamma_{d-1}(\cdot,t-s)|\rangle'|\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|(-\Delta')^{-1}\nabla' f(\cdot,\tilde z,s)\rangle' d\tilde z \\
&\overset{(3.122)\&(3.114)}{\lesssim} \frac{R}{(t-s)}\int_0^\infty |\partial_z\Gamma_1(\tilde z+z,t-s)-\partial_z\Gamma_1(\tilde z-z,t-s)|\langle|f(\cdot,\tilde z,s)|\rangle' d\tilde z\,.
\end{aligned}$$

Multiplying by the weight $\frac{1}{z}$ and integrating in $z\in(0,\infty)$ we get

$$\int_0^\infty \langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\frac{dz}{z} \lesssim \sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz \begin{cases} \frac{1}{R}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,, \\ \frac{1}{(t-s)}R\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}\,, \end{cases}$$

where we called $K_{t-s}(z,\tilde z)=\frac{\tilde z}{z}|\partial_z\Gamma_1(\tilde z-z,t-s)-\partial_z\Gamma_1(z+\tilde z,t-s)|$.
Recalling

$$\sup_{\tilde z}\int_0^\infty K_{t-s}(z,\tilde z)dz \overset{(3.121)}{\lesssim} \int_{\mathbb{R}}|\partial_z\Gamma_1(z,t-s)|dz+\sup_{z\in\mathbb{R}}(z^2|\partial_z^2\Gamma_1(z,t-s)|)$$

and observing that, in this case

$$\int_{\mathbb{R}}|\partial_z\Gamma_1(z,t-s)|dz+\sup_{z\in\mathbb{R}}(z^2|\partial_z\Gamma_1(z,t-s)|) \overset{(3.123)\&(3.126)}{\lesssim} \frac{1}{(t-s)^{\frac{1}{2}}},$$

we can conclude that

$$\int_0^\infty \langle|\nabla'\partial_z u_{N_s}(\cdot,t)|\rangle'\frac{dz}{z} \lesssim \begin{cases} \frac{1}{(t-s)^{\frac{1}{2}}}\frac{1}{R}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z} \\ \frac{1}{(t-s)^{\frac{3}{2}}}R\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}. \end{cases}$$

Finally, inserting (3.99) and integrating in time we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\langle|\nabla'\partial_z u_{N_s}(\cdot,z,t)|\rangle'\frac{dz}{z}dt \\
\overset{(3.99)}{\lesssim}\; &\int_0^\infty\int_0^\infty\int_0^t\langle|\nabla'\partial_z u_{N_s}(\cdot,\tilde z,t)|\rangle'\frac{dz}{z}dsdt \\
\lesssim\; &\int_s^\infty\int_0^\infty \min\{\frac{1}{R(t-s)^{\frac{1}{2}}},\frac{R}{(t-s)^{\frac{3}{2}}}\}\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}dsdt \\
\overset{(3.102)\&(3.103)}{\lesssim}\; &\int_0^\infty\int_0^\infty\langle|f(\cdot,\tilde z,s)|\rangle'\frac{d\tilde z}{\tilde z}ds\,.
\end{aligned}$$

Argument for (3.96):
Recall that we need to prove

$$\sup_z\int_0^\infty|\nabla'\partial_z u_C|dt \lesssim \langle|\nabla'\partial_z u|_{z=0}\rangle'\,.$$

By equation (3.94), the even extension $\overline{u_C}$ satisfies

$$(\partial_t - \Delta)\overline{u_C} = -[\partial_z \overline{u_C}]\delta_{z=0} = -2\partial_z u_C \delta_{z=0} = -2\partial_z u|_{z=0}\delta_{z=0} \tag{3.108}$$

and therefore we study the following problem on the whole space

$$\left\{ \begin{array}{rcll} (\partial_t - \Delta)\overline{u_C} & = & -2\partial_z u|_{z=0}\delta & \text{for} \quad z \in \mathbb{R}, t > 0\,, \\ \overline{u_C} & = & 0 & \text{for} \quad t = 0\,. \end{array} \right. \tag{3.109}$$

By Duhamel's principle

$$\overline{u_C}(x', z, t) = \int_{s=0}^{t} \overline{u_{C_s}}(x', z, t) ds, \tag{3.110}$$

where $\overline{u_{C_s}}$ solves the initial value problem

$$\left\{ \begin{array}{rcll} (\partial_t - \Delta)\overline{u_{C_s}} & = & 0 & \text{for} \quad z \in \mathbb{R}, t > s\,, \\ \overline{u_{C_s}} & = & -2\partial_z u|_{z=0}\delta & \text{for} \quad z \in \mathbb{R}, t = s\,. \end{array} \right. \tag{3.111}$$

The solution of problem (3.111) can be represented via the heat kernel as

$$\begin{aligned} \overline{u_{C_s}}(x', z, t) &= \int \Gamma(z - \tilde{z}, t - s) *_{x'} (-2\partial_z u|_{z=0}\delta)(\tilde{z}, s) d\tilde{z}, \\ &= -2\Gamma(z, t - s) *_{x'} \partial_z u(z, s)|_{z=0}\,. \end{aligned}$$

We apply $\nabla'\partial_z$ to the representation above

$$\nabla'\partial_z \overline{u_{C_s}}(x', z, t) = \int_{\mathbb{R}^{d-1}} -2\Gamma_{d-1}(x' - \tilde{x'}, t - s)\partial_z \Gamma_1(z, t - s)\nabla'\partial_z u(\cdot, z, s)|_{z=0} d\tilde{x'}$$

and then average in the horizontal direction,

$$\begin{aligned} & \langle|\nabla'\partial_z \overline{u_{C_s}}(x', z, t)|\rangle' \\ \lesssim\; & \langle|\Gamma_{d-1}(x', t - s)|\rangle' |\partial_z \Gamma_1(z, t - s)| \langle|\nabla'\partial_z u(\cdot, z, s)|_{z=0}|\rangle' \\ \overset{(3.122)}{\lesssim}\; & |\partial_z \Gamma_1(z, t - s)| \langle|\nabla'\partial_z u(\tilde{x'}, z, s)|_{z=0}|\rangle'\,. \end{aligned}$$

Inserting the previous estimate in the Duhamel formula 3.110 and integrating in time we get

$$\begin{aligned} & \int_0^\infty \langle|\nabla'\partial_z \overline{u_C}(x', z, t)|\rangle' dt \\ \leq\; & \int_0^\infty \int_0^t \langle|\nabla'\partial_z \overline{u_{C_s}}(x', z, t)|\rangle' ds dt \\ \lesssim\; & \int_0^\infty \int_s^\infty |\partial_z \Gamma_1(z, t - s)| dt \langle|\nabla'\partial_z u(\tilde{x'}, z, s)|_{z=0}|\rangle' ds \\ \overset{(3.124)}{\lesssim}\; & \int_0^\infty \langle|\nabla'\partial_z u(\tilde{x'}, z, s)|_{z=0}|\rangle' ds\,. \end{aligned} \tag{3.112}$$

The estimate (3.96) follows immediately after passing to the supremum in (3.112). □

3.2.6 Appendix

Prerequisites We start this section by proving some elementary bounds and equivalences, coming directly from the definition of horizontal bandedness (3.128). These will turn to be crucial in the proof of the main result.

Lemma 7.

a) If

$$\mathcal{F}'r(k',z,t) = 0 \quad \text{unless} \quad R|k'| \geq 4 \tag{3.113}$$

then

$$\langle |r(\cdot,z,t)|\rangle' \leq R\langle |\nabla' r(\cdot,z,t)|\rangle' \,. \tag{3.114}$$

In particular

$$||r||_{(0,\infty)} \leq R||\nabla' r||_{(0,\infty)} \,.$$

b) If

$$\mathcal{F}'r(k',z,t) = 0 \quad \text{unless} \quad R|k'| \leq 1 \tag{3.115}$$

then

$$\langle |\nabla' r(\cdot,z,t)|\rangle' \leq \frac{1}{R}\langle |r(\cdot,z,t)|\rangle' \,. \tag{3.116}$$

In particular

$$||\nabla' r||_{(0,\infty)} \leq \frac{1}{R}||r||_{(0,\infty)} \,.$$

c) If

$$\mathcal{F}'r(k',z,t) = 0 \quad \text{unless} \quad 1 \leq R|k'| \leq 4$$

then

$$||\nabla'(-\Delta')^{-\frac{1}{2}} r||_{(0,\infty)} \sim ||r||_{(0,\infty)} \,, \tag{3.117}$$

and

$$||(-\Delta')^{\frac{1}{2}} r||_{(0,\infty)} \sim ||\nabla' r||_{(0,\infty)} \,. \tag{3.118}$$

Remark 3. *All the results stated in Lemma 7 are valid with the norm* $||\cdot||_{(0,\infty)}$ *replaced with* $||\cdot||_{(0,1)}$.

Remark 4. *Notice that from (3.117) and (3.118), it follows*

$$||\nabla'(-\Delta')^{-1}\nabla'\cdot r||_{(0,\infty)} \lesssim ||r||_{(0,\infty)} \,. \tag{3.119}$$

Proof.

a) By rescaling we may assume $R = 1$.
Let $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$ be a Schwartz function such that

$$\mathcal{F}'\phi(k') = \begin{cases} 0 & \text{for } |k'| \geq 1 \\ 1 & \text{for } |k'| \leq 1 \end{cases}$$

and such that $\int_{\mathbb{R}^{d-1}} \phi(x')dx' = 1$.
We claim that, under assumption (3.113), there exists $\psi \in L^1(\mathbb{R}^{d-1})$ such that

$$(\mathrm{Id} - \phi *')\mathrm{r} = \psi *' \nabla \mathrm{r} \,. \tag{3.120}$$

Since $r = r - \phi * r$, if we assume (3.120) the conclusion follows from Young's inequality

$$\int_{\mathbb{R}^{d-1}} |r(x',z)|dx' \leq \int_{\mathbb{R}^{d-1}} |\psi(x')|dx' \int_{\mathbb{R}^{d-1}} |\nabla r(x',z)|dx' \,.$$

Argument for (3.120):
Using the assumptions on ϕ and performing suitable change of variables, we find

$$\begin{aligned}
& r(x',z) - \int \phi(x'-y')r(y',z)dy' \\
= & \int \phi(x'-y')(r(x',z)-r(y',z))dy' \\
= & \int_{\mathbb{R}^{d-1}} \phi(x'-y') \int_0^1 (x'-y')\nabla' r(tx'+(t-1)(x'-y'),z)dy'dt \\
= & \int_0^1 \int_{\mathbb{R}^{d-1}} \phi(\xi)\nabla' r(x'+(t-1)\xi,z)\cdot \xi d\xi dt \\
= & \int_0^1 \int_{\mathbb{R}^{d-1}} \phi\left(\frac{\hat{y}'-x'}{t}\right) \nabla r(\hat{y}',z)\cdot \frac{\hat{y}'-x'}{t} dt \frac{1}{t^{d-1}} d\hat{y}' \\
= & \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}',z)\cdot \left(\int_0^1 \phi\left(\frac{\hat{y}'-x'}{t}\right)\frac{\hat{y}'-x'}{t^d}dt\right) d\hat{y}' \\
= & \int_{\mathbb{R}^{d-1}} \nabla' r(\hat{y}',z)\psi\left(\frac{\hat{y}'-x'}{t}\right) d\hat{y}',
\end{aligned}$$

where

$$\psi(x') = \int_0^1 \phi\left(\frac{-x'}{t}\right)\frac{x'}{t^d}dt\,.$$

We notice that $\psi \in L^1(\mathbb{R}^{d-1})$, in fact

$$\int_{\mathbb{R}^{d-1}} |\psi(x')|dx' \le \int_0^1 \int_{\mathbb{R}^{d-1}} \left|\phi(x'/t)\frac{x'}{t^d}\right| dx'dt = \int_{\mathbb{R}^{d-1}} |\phi(\xi)\xi|d\xi\,.$$

b) In Fourier space we have

$$\mathcal{F}'\nabla' r(k',z) = ik'\mathcal{F}'r(k',z) = R^{-1}\mathcal{F}'G(Rk')\mathcal{F}'r(k',z) = R^{-1}\mathcal{F}'G_R(k')\mathcal{F}'r(k',z),$$

where G is a Schwartz function and $G_R(x') = R^{-d}\mathcal{F}'G(x'/R)$. Since $\int |G_R|dx' = \int |G|dx'$ is independent of R, we may conclude by Young

$$\int |\nabla' r|dx' \le \frac{1}{R}\int |G_R|dx' \int |r|dx' \lesssim \frac{1}{R}\int |r|dx'\,.$$

□

Here we prove an elementary estimate that will be applied in the argument for (3.88) and (3.95), Lemma 6

Lemma 8.
Let $K = K(z)$ be a real function and define

$$\overline{K}(z,\tilde{z}) = \frac{\tilde{z}}{z}|K(\tilde{z}-z) - K(z+\tilde{z})|\,.$$

Then

$$\sup_{\tilde{z}} \int_0^\infty \overline{K}(z,\tilde{z})dz \lesssim \int_{\mathbb{R}} |K(z)|dz + \sup_{z\in\mathbb{R}}(z^2|\partial_z K(z)|)\,. \tag{3.121}$$

Proof. Let us distinguish two regions: $\frac{1}{2}\left|\frac{\tilde{z}}{z}\right| < 1$ and $\frac{1}{2}\left|\frac{\tilde{z}}{z}\right| > 1$.
For $|z| \geq \frac{1}{2}|\tilde{z}|$ we have

$$
\begin{aligned}
& \sup_{\tilde{z}} \int_{|z|\geq \frac{1}{2}|\tilde{z}|} |\overline{K}(z,\tilde{z})| dz \\
\leq \quad & \max_{\tilde{z}} \int_{|z|\geq \frac{1}{2}|\tilde{z}|} |K(\tilde{z}-z) - K(z+\tilde{z})| dz \lesssim \int |K(z)| dz \,.
\end{aligned}
$$

While for the region $|z| \leq \frac{1}{2}|\tilde{z}|$ we have,

$$
\begin{aligned}
& \max_{\tilde{z}} |\tilde{z}| \int_{|z|\leq \frac{1}{2}|\tilde{z}|} \frac{1}{|z|} |K(\tilde{z}-z) - K(z+\tilde{z})| dz \\
= \quad & \max_{\tilde{z}} |\tilde{z}| \int_{|z|\leq \frac{1}{2}|\tilde{z}|} \frac{1}{|z|} \left| \int_{-1}^{1} K'(\tilde{z}+tz) z dt \right| dz \\
\leq \quad & \max_{\tilde{z}} |\tilde{z}| \int_{-1}^{1} \frac{1}{t} \int_{|z|\leq \frac{t}{2}|\tilde{z}|} |K'(\tilde{z}+z)| dz dt \\
\overset{\frac{1}{2}|\tilde{z}|\leq|\tilde{z}+z|}{\leq} \quad & \max_{\tilde{z}} \int_{-1}^{1} \frac{1}{t} \int_{|z|\leq \frac{t}{2}|\tilde{z}|} 2|\tilde{z}+z||K'(\tilde{z}+z)| dt dz \\
\leq \quad & \max_{\tilde{z}} \int_{-1}^{1} \frac{2}{t} \max_{|z|\leq \frac{t}{2}|\tilde{z}|} \left\{ |\tilde{z}+z||K'(\tilde{z}+z)| \right\} \left(\int_{|z|\leq \frac{t}{2}|\tilde{z}|} dz \right) dt \\
= \quad & \max_{\tilde{z}} \int_{-1}^{1} \frac{1}{t} \max_{|z|\leq \frac{t}{2}|\tilde{z}|} \{ |\tilde{z}+z||K'(\tilde{z}+z)| \} t|\tilde{z}| dt \\
= \quad & 2 \max_{\tilde{z}} |\tilde{z}| \max_{|z|\leq \frac{t}{2}|\tilde{z}|} \{ |\tilde{z}+z||K'(\tilde{z}+z)| \} \\
\overset{\frac{1}{2}|\tilde{z}|\leq|\tilde{z}+z|}{\leq} \quad & 4 \max_{\tilde{z}} \max_{|z|\leq \frac{t}{2}|\tilde{z}|} \{ |z+\tilde{z}|^2 |K'(\tilde{z}+z)| \} \,.
\end{aligned}
$$

In conclusion we have

$$
\max_{z} \int |\bar{K}(z,\tilde{z})| dz \lesssim \int |K(z)| dz + \max_{z} |z|^2 |K'(z)| \,.
$$

□

Heat kernel: elementary estimates In this section we recall the definition of the heat kernel and some properties and estimates that we will use throughout the paper.
The function $\Gamma : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is defined as

$$
\Gamma(x,t) = \frac{1}{t^{d/2}} \exp\left(-\frac{|x|^2}{4t} \right)
$$

and we can rewrite it as

$$
\Gamma(x,t) = \Gamma_1(z,t)\Gamma_{d-1}(x',t) \qquad x' \in \mathbb{R}^{d-1}, z \in \mathbb{R},
$$

where

$$
\Gamma_1(z,t) = \frac{1}{t^{1/2}} \exp\left(-\frac{z^2}{4t} \right)
$$

and

$$
\Gamma_{d-1}(z,t) = \frac{1}{t^{(d-1)/2}} \exp\left(-\frac{|x'|^2}{4t} \right) \,.
$$

Here we list the bounds on the derivatives of Γ that are used in Section 3.2.6, Lemma 6:

1.
$$\langle|(\nabla')^n\Gamma_{d-1}|\rangle' \approx \frac{1}{t^{\frac{n}{2}}}\,. \tag{3.122}$$

2.
$$\int_{\mathbb{R}} |\partial_z^n\Gamma_1|dz \lesssim \frac{1}{t^{\frac{n}{2}}}\,. \tag{3.123}$$

3.
$$\int_0^\infty |\partial_z\Gamma_1(z,t)|dt = \int_0^\infty \left|\frac{1}{\hat{t}^{3/2}}\exp\left(-\frac{1}{4\hat{t}}\right)\right| d\tilde{t} \lesssim 1\,, \tag{3.124}$$

where we have used the change of variable $\hat{t} = \frac{t}{z^2}$.

4.
$$\sup_{z\in\mathbb{R}}\left(z|\partial_z\Gamma_1(z,t)|\right) = \sup_{\xi}\left|\frac{1}{t^{\frac{1}{2}}}\xi^2\exp^{-\xi^2}\right| \lesssim \frac{1}{t^{\frac{1}{2}}}\,, \tag{3.125}$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

5.
$$\sup_{z\in\mathbb{R}}\left(z^2|\partial_z\Gamma_1(z,t)|\right) = \sup_{\xi}\left|\xi^3\exp^{-\xi^2}\right| \lesssim 1\,, \tag{3.126}$$

where we have used the change of variable $\xi = \frac{z}{t^{\frac{1}{2}}}$.

Notations

The $(d-1)-$dimensional torus:

We denote with $[0,L)^{d-1}$ the $(d-1)-$dimensional torus of lateral size L.
The spatial vector:

$$x = (x', z) \in [0,L)^{d-1} \times [0,1]\,.$$

where L denotes the lateral horizontal cell-size.
Velocity vector:

$$u = u(x', z, t) \text{ where } u = (u', u^z)$$

$$u^z := u \cdot e_z \quad \text{ and } \quad u' = (u \cdot e_1, \cdots, u \cdot e_{d-1})$$

where e_z is the unit normal vector in the $z-$ direction and $e_1, \cdots, e_{d-1}$ are the unit normal vectors in the $x_1, \cdots, x^{d-1}-$directions respectively.
Gradient:

$$\nabla f = \begin{pmatrix} \nabla_{x'} \\ \partial_z \end{pmatrix} f$$

Laplacian:

$$\Delta f = \Delta_{x'} f + \partial_z^2 f$$

The horizontal average:

$$\langle \cdot \rangle' = \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} \cdot \quad dx'\,.$$

Long-time and horizontal average:

$$\langle \cdot \rangle = \limsup_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} \langle \,\cdot\, \rangle' dt\,. \tag{3.127}$$

Horizontal Fourier transform:

$$\mathcal{F}' f(k', z) = \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} e^{-ik' \cdot x'} f(x', z) dx'\,.$$

where k' is the dual variable of x'.
Real part of an imaginary number :
Re stands for the real part of a complex number.
Complex conjugate :
$\overline{\mathcal{F}' u^z}$ and $\overline{\mathcal{F}' \theta}$ are the complex conjugates of the (complex valued) functions $\mathcal{F}' u^z$ and $\mathcal{F}' \theta$.
Background profile:

$$\tau =: [0, H] \to \mathbb{R} \quad \text{ such that } \quad \tau(0) = 1 \text{ and } \tau(H) = 1$$

$$\tau = \tau(z)\,, \qquad\qquad \xi := \frac{d\tau}{dz}\,.$$

Universal constant:
We call *universal constant* a constant C such that $0 < C < \infty$ and it only depends on d. Throughout the paper we will denote with $\lesssim$ the inequality up to universal constants.
Convolution in the horizontal direction:

$$f *_{x'} g(x') = \int_{[0,L)^{d-1}} f(x' - \widetilde{x'}) g(\widetilde{x'}) d\widetilde{x'} \, .$$

Convolution in the whole space:

$$f * g(x) = \int_{\mathbb{R}} \int_{[0,L)^{d-1}} f(x' - \widetilde{x'}, z - \tilde{z}) g(\widetilde{x'}, \tilde{z}) d\widetilde{x'} d\tilde{z} \, .$$

Horizontally band-limited function:
A function $g = g(x', z, t)$ is called *horizontally band-limited* with bandwidth R if it satisfies the *bandedness assumption*

$$\mathcal{F}' g(k', z, t) = 0 \text{ unless } 1 \leq R|k'| \leq 4 \text{ where } R < R_0. \tag{3.128}$$

Interpolation norms:

$$||f||_{(0,1)} = ||f||_{R;(0,1)} = \inf_{f = f_1 + f_2} \left\{ \left\langle \sup_{z \in (0,1)} |f_1| \right\rangle + \left\langle \int_{(0,1)} |f_2| \frac{dz}{z(1-z)} \right\rangle \right\} ,$$

$$||f||_{(0,\infty)} = ||f||_{R;(0,\infty)} = \inf_{f = f_1 + f_2} \left\{ \left\langle \sup_{z \in (0,\infty)} |f_1| \right\rangle + \left\langle \int_{(0,\infty)} |f_2| \frac{dz}{z} \right\rangle \right\} ,$$

$$||f||_{(-\infty,1)} = ||f||_{R;(-\infty,1)} = \inf_{f = f_1 + f_2} \left\{ \left\langle \sup_{z \in (-\infty,1)} |f_1| \right\rangle + \left\langle \int_{(-\infty,1)} |f_2| \frac{dz}{1-z} \right\rangle \right\} .$$

where f_0, f_1 satisfy the bandedness assumption (3.128).

Bibliography

[1] David J Tritton. *Physical fluid dynamics*, volume 1. 1988.

[2] Willem VR Malkus. The heat transport and spectrum of thermal turbulence. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 225(1161):196–212, 1954.

[3] Edward A Spiegel. Convection in stars: I. Basic Boussinesq convection. *Annual review of astronomy and astrophysics*, 9:323, 1971.

[4] Robert H Kraichnan. Turbulent thermal convection at arbitrary Prandtl number. *Physics of Fluids (1958-1988)*, 5(11):1374–1389, 1962.

[5] Siegfried Grossman and Detlef Lohse. Scaling in thermal convection: a unifying theory. *Journal of Fluid Mechanics*, 407:27–56, 3 2000.

[6] Klaus Petschel. Near-wall kinetic energy dissipation in Rayleigh-Bénard convection: the role of velocity boundary conditions. *PhD Thesis*, 2014.

[7] Peter Constantin and Charles R Doering. Infinite Prandtl number convection. *Journal of Statistical Physics*, 94(1-2):159–172, 1999.

[8] Charles R Doering, Felix Otto, and Maria G Reznikoff. Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh-Bénard convection. *Journal of Fluid Mechanics*, 560:229–242, 2006.

[9] Felix Otto and Christian Seis. Rayleigh–Bénard convection: improved bounds on the Nusselt number. *Journal of Mathematical Physics*, 52(8):083702, 2011.

[10] Camilla Nobili and Felix Otto. Lower bound on the temperature background field method for Rayleigh-Bénard convection. *to be submitted*, 2015.

[11] Charles R Doering and Peter Constantin. Variational bounds on energy dissipation in incompressible flows. III. Convection. *Physical Review E*, 53(6):5957, 1996.

[12] Xiaoming Wang. Asymptotic behavior of the global attractors to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. *Communications on Pure and Applied Mathematics*, 60(9):1293–1318, 2007.

[13] Antoine Choffrut, Camilla Nobili, and Felix Otto. Upper bounds on Nusselt number at finite Prandtl number. *arXiv preprint arXiv:1412.4812*, 2014.

[14] Peter Constantin and Jiahong Wu. Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 26:159–180, 2009.

[15] Roberto Verzicco and Roberto Camussi. Numerical experiments on strongly turbulent thermal convection in a slender cylindrical cell. *Journal of Fluid Mechanics*, pages 19–49, 2 2003.

[16] Richard J. Goldstein and S. Tokuda. Heat transfer by thermal convection at high rayleigh numbers. *International Journal of Heat and Mass Transfer*, 23(5):738–740, 1980.

[17] Charles R. Doering and Peter Constantin. On upper bounds for infinite Prandtl number convection with or without rotation. *Journal of Mathematical Physics*, 42(2):784–795, 2001.

[18] Elias M Stein. *Singular integrals and differentiability properties of functions*, volume 2. Princeton university press, 1970.

[19] Giovanni P Galdi. *An introduction to the mathematical theory of the Navier-Stokes Equations: steady-state problems*, volume 501. Springer, 2011.

[20] Vsevolod Alekseevich Solonnikov. Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations. *Trudy Matematicheskogo Instituta im. VA Steklova*, 70:213–317, 1964.

[21] Seiji Ukai. A solution formula for the Stokes equation in $\mathbb{R}^n_+$. *Communications on Pure and Applied Mathematics*, 40(5):611–621, 1987.

[22] Paolo Maremonti and Vsevolod A Solonnikov . On nonstationary Stokes problem in exterior domains. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 24(3):395–449, 1997.

Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

Leipzig, den 9.11.2015

. . . .
(Camilla Nobili)

Daten zum Autor

Name:	Camilla Nobili
Geburtsdatum:	03/10/1987 in Rom
10/2006 - 07/2009	Bachelor degree in Mathematics at La Sapienza Università di Roma
10/2009 - 10/2011	Master degree in Mathematics at La Sapienza Università di Roma
seit 10/2011	PhD student at Max-Planck-Institut for Mathematics in the Sciences