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Die Annahme der Dissertation wurde empfohlen von:

1. Professor Dr. Delio Mugnolo (FernUniversität in Hagen)
2. Professor Dr. Jürgen Jost (MPI Leipzig)

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Leipzig, den 27. November 2015

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(Lavinia Roncoroni)

Abstract

In this thesis we analyze *lumpability* of infinite dimensional dynamical systems. *Lumping* is a method to project a dynamics by a linear reduction operator onto a smaller state space on which a self-contained dynamical description exists. This means that the dynamics of the reduced model does not depend anymore on the original state variable, but has its autonomous evolution. We consider the system $\dot{x} = F(x)$ defined on a Banach space X, together with a linear and bounded map $M : X \to Y$, where Y is another Banach space. The operator M is surjective but not an isomorphism and it represents a *reduction* of the state space. We investigate whether the variable $y = Mx$ also satisfies a well-posed and self-contained dynamics on Y. This happens if and only if an operator $\widehat{F}$ exists on Y in such a way that $MF = \widehat{F}M$ holds, and the system $\dot{y} = \widehat{F}(y)$ admits a unique solution for every initial condition. If possible, we implicitly define the reduced operator by $\widehat{F}(y) := MF(x)$, for $y = Mx$. We first discuss lumpability of linear systems in Banach spaces. In this context, we assume that F generates a C_0-semigroup $T(t)$ on X. We give conditions for the reduced operator to exist and to be itself the infinitesimal generator of a reduced C_0-semigroup on Y. We extend these results to the dual Banach space, and we describe the behaviour of the adjoint operator M^* in relation to a particular subspace of X^*, called the *sun dual space*. This is the largest closed subspace on which the adjoint semigroup $T^*(t)$ is itself strongly continuous (being in general only *weak-star continuous* on X^*). Next, we study lumpability of nonlinear evolution equations. We focus on semigroups of contractions, for which some interesting results exist, concerning the existence and uniqueness of solutions, both in the classical sense of smooth solutions and in the weaker sense of *strong solutions*. It is known that, under suitable hypotheses (e.g. dissipativity) F generates a semigroup of nonlinear contractions in the sense of Crandall-Liggett: F is not necessarily the infinitesimal generator of $T(t)$, but if the semigroup is differentiable for almost every $t \geq 0$, then it is the unique solution of the Cauchy problem $\dot{x} = F(x)$. We discuss in details under which conditions the operator $\widehat{F}$ can be again associated with a nonlinear, strongly continuous semigroup giving the solutions of the reduced system. We also investigate the regularity properties inherited by $\widehat{F}$ from the original operator F. Finally, we describe a particular kind of lumping in the context of $C^\star$-algebras. This lumping represents a different interpretation of the restriction operator from $C_0(S)$ to $C_0(\mathscr{C})$, S and $\mathscr{C}$ being a locally compact, Hausdorff space and a closed subset, respectively. We apply this lumping to *Feller semigroups*, which are important because they can be associated in a unique way to Markov processes. We show that the fundamental properties of Feller semigroups are preserved by this lumping. Using these ideas, we give a short proof of the classical Tietze extension theorem based on $C^\star$-algebras and Gelfand theory.

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Contents

Chapter 1

Introduction

In this thesis we analyze lumpability of dynamical systems on Banach spaces. *Lumping* is a method to project a dynamics by a linear reduction operator onto a smaller state space on which a self-contained dynamical description exists. The term *self-contained* means that the dynamics of the reduced model does not depend anymore on the original state variable, but has its autonomous evolution.

Let us consider a dynamical system defined on a Banach space X:

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0, \end{cases} \tag{1.1}$$

with $F : \mathscr{D}(F) \subseteq X \to X$. We assume that the dynamics (1.1) is well defined, in the sense that for every $x_0 \in \mathscr{D}(F)$ there exists a unique solution (classical, or another weaker kind of solution to be specified). In addition, let us consider a linear and bounded map $M : X \to Y$, where Y is another Banach space. We view the operator M as a *reduction* of the state space: it is surjective but not an isomorphism. The question we are interested in is whether the variable $y = Mx$ also satisfies a well-posed and self-contained dynamics on Y, say

$$\dot{y}(t) = \widehat{F}(y(t)), \quad y = Mx. \tag{1.2}$$

If this is the case, then we refer to M as a *reduction* or *lumping* operator.
According to the definition given by Wei and Kuo in [72], the system (1.1) is said to be *exactly lumpable* by the operator M if there exists an operator $\widehat{F} : \mathscr{D}(\widehat{F}) \subseteq Y \to Y$ such that the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, "\widehat{F}"] & Y \\ X \arrow[u, "M"] \arrow[r, "F"] & X \arrow[u, "M"'] \end{tikzcd} \tag{1.3}$$

that is,

$$MF = \widehat{F}M. \tag{1.4}$$

The term *lumping* originates from chemical reaction systems: the number of chemical components involved in the reactions is in general very high, and for this reason it is often

necessary to aggregate all the reagents into a few groups, called *lumps* [48,66,72]. A similar concept of *aggregation* of states has been used in the theory of Markov chains, where the question is whether the newly-formed aggregates also admit a Markovian description for the state transitions [15, 38, 47]. Indeed, Markovianity may be lost after the projection, and memory effects may appear in the reduced system. In [58, 59] these memory effects are quantified using concepts from information theory, such that *mutual information*. In particular, we have *informational closure* when there is no information flow from the original system to the reduced one.
Diagram (1.3), however, is more general, as the operator M can also represent other types of reduction, for example projections or averages. It can also be interpreted in the context of *multi-level* systems, where X and Y are sometimes called micro (lower) and macro (upper) levels, respectively. Here the question is the following: given some dynamics on the micro states X, to find the conditions on M such that Y represents a new *level* with its own autonomous dynamics.
The notion of lumping is related to the one of *conjugate dynamical systems*. We obtain two conjugate systems if we ask M to be not only surjective, but also invertible. In this case orbits of the original systems are homeomorphically mapped into orbits of the new system, so we don't obtain a reduction as in the case we analyze. Lumping can be rather associated with *semi-conjugacy*: two flows ϕ_t and ψ_t defined on two topological spaces, X and Y respectively, are semi-conjugate if there exists a surjection $h : X \to Y$ such that $h(\phi_t) = \psi_t(h)$ (see [11, 12]). In the context of *topological systems* (i.e. couples (K, ϕ) with $\phi : K \to K$ continuous and K compact space), semi-conjugacy can be identified with the existence of a *factor map*: given two topological systems (K, ϕ) and (L, ψ), a factor map is a surjection $h : K \to L$ satisfying $h\phi = \psi h$ (see [26]). This is also related to the notion of *factors* for *measure preserving systems* (i.e. measure spaces associated with a measure preserving transformation): a measure preserving system (Y, ν, ψ) is a factor of another system (X, μ, ϕ) if there exists a measurable factor map $h : X \to Y$ such that $\mu \circ h^{-1} = \nu$, being μ and ν the measures on X and Y, respectively.
However, the focus on diagram (1.3) is different, because in the context of semi-conjugacy and factor maps the flow ψ_t, as well as the system (L, ψ), is in general already known and the problem is to find a suitable surjection h, while in the case of lumping we start from a surjection and, if it is possible, we construct the reduced flow.
An interesting connection exists between lumpability and *factorization* of operators: given two linear operators E and D on a Banach space X, we say that D is a left multiple of E if there exists another linear operator C such that

$$D = CE. \tag{1.5}$$

Assuming $D = MF$ and $E = M$, (1.5) corresponds to the lumping relation (1.4), with $C = \widehat{F}$. Factorization is analyzed in [25] for bounded operators on Hilbert spaces, and in [9, 27] for bounded operators on Banach spaces. Some generalizations to unbounded operators can be found in [33], under the assumption of a *pseudoinverse operator* for E. The operator C in (1.5) exists if and only if E *majorizes* D [9, 27], i.e. there exists some $k > 0$ such that:

$$\|Dx\| \leq k\|Ex\| \quad \forall x \in X. \tag{1.6}$$

In this context the operator E need not to be surjective, and the operator C is then defined on the range of E. However, in the lumping analysis we consider mainly surjective lumping operators M, because all the reduction operators used in the lumping literature, like averages or projections, are indeed surjective. Moreover, it is worth noting that, if we relax our assumption on the range of M, we do not fall in the setting considered in [9, 27] and our analysis does not generalize straightforwardly.
For what concerns the case of unbounded operators, we don't assume the existence of a

pseudoinverse for the lumping operator, but we concentrate our attention on generators of strongly continuous semigroups, which may be unbounded but have some interesting spectral properties. Indeed, our aim is to study lumpability from the dynamical systems point of view, in order to obtain a well-posed reduced dynamics.

The need to study lumping in the infinite dimensional setting firstly arose in the context of chemical reactions. In reaction systems a mixture of vary many components can be involved, which may even not be distinguishable. In these cases it is convenient to describe the mixture by a distribution function, rather than a finite set of components: the state space becomes an infinite dimensional space of functions (see [3, 6]). Moreover, many interesting equations, used to describe some natural phenomena, have a very complete mathematical description in an infinite dimensional context (i.e. they can be written as *abstract Cauchy problems* in Banach spaces). This is the case of some *parabolic partial differential equations* (like the Heat equation, generated by the Laplace operator), and *delay differential equations*, whose initial date is not a single point, but a *history function* defined on an interval. The importance of these classes of equations drives us to develop some reduction techniques, like lumping, to be applied in the context of Banach spaces.
Previous analysis in the area of infinite-dimensional lumpability was done for bounded operators by Coxson [21], and by Rózsa and Tóth in the context of Hilbert spaces [63]. They both require the existence of a continuous *pseudoinverse* of the lumping operator. As we will explain in the present thesis, a pseudoinverse of $M : X \to Y$ is a linear operator $\overline{M} : Y \to X$ such that $M\overline{M}M = M$.
If this operator exists, it is possible to define the reduced operator explicitly as

$$\widehat{F}(y) := MF(\overline{M}y),$$

and this definition does not depend on the particular choice of the pseudoinverse (see [48]). In this way, provided that $\overline{M}$ is bounded, the regularity properties of $\widehat{F}$ follow easily by the regularity of F. This is the approach that characterizes the whole literature of lumping.
However, a bounded pseudoinverse does not necessarily exist in a general Banach space, unless we require that $\ker(M)$ is a topologically complemented subspace in X. It has been proved by Lindenstrauss and Tzafriri [49] that if $(X, \|\cdot\|)$ is a Banach space such that every closed subspace is complemented, then the norm is induced by a scalar product, i.e. $(X, \|\cdot\|)$ is a Hilbert space. For this reason it is important to approach lumping without assuming the existence of a topological complement for $\ker(M)$.
Indeed, the main contribution of this thesis is to develop a theory of lumping which doesn't make use of the pseudoinverse operator. We rather define the reduced operator $\widehat{F}$ in the following implicit way (when possible):

$$\widehat{F}(y) := MF(x), \quad y = Mx.$$

Here we study conditions for such an implicit definition to hold and, when this is possible, we study the properties of the reduced operator. In particular, we find conditions for $\widehat{F}$ to generate a semigroup of operators (i.e. a well-posed dynamics). Due to the indirect approach to the problem, we need to adapt and exploit different techniques with respect to those already used in the literature on lumping, such as results in quotient Banach space theory.

In the present work, we first consider systems generated by linear operators, and then we generalize our results to nonlinear dynamics. We use methods from the theory of strongly continuous semigroups to obtain conditions on the reduction operator for lumpability. We also indicate several applications to particular systems, including delay differential equations.

The following is a more detailed description of the structure of the thesis. In the second chapter we discuss lumpability of linear systems in Banach spaces. In this context, we assume that the original dynamics is well-posed in the sense of the Hille and Yosida theorem (i.e. F is the infinitesimal generator of a C_0-semigroup of linear operators $T(t)$). We analyze conditions to guarantee that the reduced operator $\widehat{F}$ exists and generates a new C_0-semigroup on the reduced state space. In particular, we show that a necessary and sufficient condition for lumpability is the invariance of $\ker(M)$ under the whole semigroup:

$$\ker(M) \subseteq \ker(MT(t)), \quad t \geq 0.$$

Since in general we don't know *a priori* the semigroup of solutions, we give necessary and sufficient conditions for lumpability directly on the infinitesimal generator. This generalization is non-trivial because the invariance of a closed subspace under a semigroup is not equivalent to its invariance under the infinitesimal generator.

Next, we extend these results to the dual Banach space. The dual approach to lumping, which is the topic of the third chapter, is interesting because the adjoint of a strongly continuous semigroup is not strongly continuous on the whole dual Banach space (indeed, it is *weak-star continuous*). However, there always exists a closed subspace of X^* on which the adjoint semigroup $T^*(t)$ is itself strongly continuous: this space is called the *Sun dual space* and we describe the behaviour of the adjoint operator M^* in relation to this subspace.

We extend our dual condition to the case of some particular nonlinear systems defined on compact Hausdorff spaces, by the associated *Koopman operators*, acting as composition operators on a space of continuous functions. In this way we obtain a linear condition for lumpability, even if the functions involved are nonlinear. In the fourth chapter we study in details lumpability of nonlinear evolution equations. As in the case of linear systems, our approach is still based on the theory of strongly continuous semigroups. For semigroups of nonlinear operators, the differentiability of $t \to T(t)x$ is not automatically guaranteed even if x belongs to the domain of the infinitesimal generator. We focus on semigroups of nonlinear contractions, for which some interesting results exist, concerning the existence and uniqueness of solutions, both in the classical sense of smooth solutions and in the weaker sense of *strong solutions*. Under suitable hypotheses, such as dissipativity, F generates a semigroup of nonlinear contractions in the sense of Crandall-Liggett [22]: F is not necessarily the infinitesimal generator of $T(t)$, but if $t \to T(t)x$ is differentiable for almost every $t \geq 0$, then it is the unique solution of the Cauchy problem (1.1).

In the nonlinear case, a necessary and sufficient condition for $\widehat{F}$ to be well-defined is that F preserves the *fibers* of M (in the sense of *level sets*). This means that for all $x_1, x_2 \in \mathscr{D}(F)$

$$Mx_1 = Mx_2 \Rightarrow MF(x_1) = MF(x_2).$$

If we assume $F(0) = 0$, when system (1.1) is lumpable, we can show that $\ker(M)$ is invariant under the semigroup $T(t)$, consistently with the linear case.

We discuss in details under which conditions the operator $\widehat{F}$ can be again associated with a nonlinear, strongly continuous semigroup giving the solutions of the reduced system (1.2).

Finally, the last chapter is dedicated to a particular kind of lumping in the context of $C^\star$-algebras. This lumping gives a different interpretation of the restriction operator from the space of continuous functions defined on a locally compact Hausdorff space S to the space of continuous functions defined on an arbitrary, closed subset $\mathscr{C} \subset S$. Let $I_{\mathscr{C}}$ be the closed ideal of functions in $C_0(S)$ vanishing on $\mathscr{C}$. We construct the lumping operator $M := \mathscr{G}\pi$, where π is the quotient projection and $\mathscr{G}$ is the Gelfand isomorphism of the quotient $C^\star$-algebra

$C_0(S)/I_{\mathscr{C}}$:

$$\begin{tikzcd}
C_0(S) \arrow[r, "M"] \arrow[d, "\pi"'] & C_0(\mathscr{C}) \\
C_0(S)/I_{\mathscr{C}} \arrow[ur, "\mathscr{G}"'] &
\end{tikzcd}$$

Using some interesting results in the theory of $C^\star$-algebras, we show that this lumping acts as the restriction of a function over the subset $\mathscr{C}$. Even if $\mathscr{C}$ is arbitrary (indeed, it can also be a singleton), the construction of M remains the same for any choice of $\mathscr{C}$.
We apply this lumping to a *Feller semigroup* $T(t)$, provided that $I_{\mathscr{C}}$ is invariant under $T(t)$ ($I_{\mathscr{C}}$ being the kernel of M). Feller semigroups are important because they can be associated in a unique way to Markov processes. In particular, we characterize all the Feller semigroups preserving the ideal $I_{\mathscr{C}}$ as those semigroups satisfying $P_t(x,\cdot) \in \mathrm{Ran(M^*)}$ for any $x \in \mathscr{C}$, where P_t is the transition probability function associated with $T(t)$. We prove that the lumping made by $\mathscr{G}\pi$ maps Feller semigroups into Feller semigroups: this means that Markovianity is preserved, unlike the general lumping of Markov processes.
Using the same ideas, we give a short proof of the classical Tietze extension theorem based on $C^\star$-algebras and Gelfand theory. In particular, showing the surjectivity of the lumping operator $\mathscr{G}\pi$, we prove that every function on $\mathscr{C}$ can be extended to a continuous function on the whole S.

Chapter 2

Lumpability of linear evolution equations

2.1 Introduction

We consider a linear dynamical system defined on a Banach space X:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0, \end{cases} \tag{2.1}$$

with $A : \mathscr{D}(A) \subseteq X \to X$. We assume that the dynamics (2.1) is well-defined, in the sense that for every $x_0 \in \mathscr{D}(A)$ there exists a unique classical solution $x(t) \in C^1([0, +\infty), \mathscr{D}(A))$ that depends continuously on the initial condition x_0. In addition, we consider a linear and bounded reduction map $M : X \to Y$ where Y is another Banach space.
According to Wei and Kuo [72] the definition of lumpability for linear systems is the following:

Definition 2.1. The system (2.1) is said to be *lumpable* by the operator M if there exists a linear operator $\widehat{A} : \mathscr{D}(\widehat{A}) \subseteq Y \to Y$ such that the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, "\widehat{A}"] & Y \\ X \arrow[u, "M"] \arrow[r, "A"] & X \arrow[u, "M"'] \end{tikzcd} \tag{2.2}$$

that is,

$$MA = \widehat{A}M. \tag{2.3}$$

Let us note that often in the literature the term *exact lumping* is used instead of *lumping*, to distinguish from the case of *approximated lumping*, when an error appears in the lumping relation (2.3). Since in general we don't consider lumping relations with errors, we use simply the term *lumping*, unless it is specified. Throughout this work, we denote the kernel of an operator with $\ker(\cdot)$ and the range with $\operatorname{ran}(\cdot)$.

2.1.1 Previous works on linear lumping

Before proceeding to operators on generic Banach spaces, it is instructive to look at the situation in finite-dimensional Euclidean spaces and to give an overview on the main previous works concerning linear lumping.
In the notation of diagram (2.2), let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^k$, let M be a $(k \times n)$ matrix with full row rank (i.e. the dimension of the space generated by the rows is k) and A be a $(n \times n)$ matrix. Thus if $k < n$, M represents a reduction of the state space dimension. In this finite dimensional setting the following result is known (e.g., [48]).

Proposition 2.1. *The following statements are equivalent:*

1. *there exists a $(k \times k)$ matrix $\widehat{A}$ such that $MA = \widehat{A}M$;*
2. $\ker(M)$ *is A-invariant;*
3. $\ker(M) \subseteq \ker(MA)$.

Lumpability of finite-dimensional linear systems has been firstly studied by Wei and Kuo [72] for chemical kinetics. In this context, the aim is to lump all the chemical reagents (which can be a very large number) into few groups for practical purposes. When the chemical species are partitioned into several classes that may be considered independent entities, they refer to *proper lumping*. In this case, the columns of the matrix M must be unit vectors. They analyze systems described by a *monomolecular reaction scheme*, i.e. a system of n chemical species described by:

$$\frac{da}{dt} = -Ka,$$

where K is a matrix satisfying the following:

(i) nonnegative rate constants: $k_{ij} \geq 0$, $\forall\, i \neq 0$;

(ii) mass conservation: $k_{ii} = -\sum_{j \neq i} k_{ji}$;

(iii) there exists a^* such that $a_i^* > 0$ for all $i = 1, \dots, n$, $Ka^* = 0$, $k_{ij}a_j^* = k_{ji}a_i^*$ for all $i \neq j$.

They prove that the lumped dynamics is again described by a monomolecular reaction scheme if the lumping is proper. When each chemical species is not necessarily assigned to a unique class, they distinguish between *semiproper* and *improper lumping*, depending on the behaviour of the lumped dynamics. In particular, an improper lumping does not lead to a monomolecular reaction system.
In [71], they analyze also the case of linear *approximately lumpable* systems. They define an error matrix:

$$E = MK - \widehat{K}M,$$

for a given $\widehat{K}$, which must generate a monomolecular reaction scheme. When the error matrix is non-zero (i.e. the lumping is not exact), they refer to an approximated lumping. The matrix E is in general not unique and depend both on M and K. They prove that choosing E such that $EAM^T = 0$, where A is the diagonal matrix of the equilibrium composition a^*, it is possible to obtain a lumped monomolecular reaction scheme. This scheme is given by the matrix $\widehat{K} = MKAM^T\widehat{A}^{-1}$, for $\widehat{A} = MAM^T$.

Another important branch of works about linear lumping concerns lumpability of Markov chains. In this context, the main goal is to find conditions for the lumped process to be still Markovian. In particular, Gurvits and Ledoux study aggregations of Markov chains given by equivalence relations [38]. They consider a discrete-time homogeneous Markov chain (X_n), such that every random variable has values in $\mathscr{X} := \{1, \dots, N\}$, and they obtain a partition of $\mathscr{X}$ in $M < N$ classes $C(1), \dots, C(M)$. The reduced state space is then $\mathscr{Y} := \{1, \dots, M\}$. They define a *lumping map* $\phi : \mathscr{X} \to \mathscr{Y}$, eventually nonlinear, such that

$$\phi(k) := l \Leftrightarrow k \in C(l), \qquad \forall k \in \mathscr{X},\, l \in \mathscr{Y}.$$

The lumped process is $\phi(X_n)$, defined by:

$$\phi(X_n) = l \Leftrightarrow X_n \in C(l).$$

The lumped process is generally not Markovian and the Markov property may depend on the initial distribution of (X_n). They distinguish between *strong lumpability*, which happens when the lumped process is Markovian for every distribution of X_0, and *weak lumpability*, when the lumped process is Markovian only for some initial distributions. They define a family of *lumping projectors* as the following $N \times N$-matrices:

$$\Pi_Y(X_i, X_j) := \begin{cases} 1 & \text{if } i = j \text{ and } X_i \in \phi^{-1}\{Y\} \\ 0 & \text{otherwise} \end{cases} \qquad Y \in \mathscr{Y}.$$

Then they consider an $M \times N$-matrix:

$$V_\phi(Y, X) := \begin{cases} 1 & \text{if } X \in \phi^{-1}\{Y\} \\ 0 & \text{otherwise} \end{cases}.$$

Let P be the transition matrix of the process (X_n). Given an initial probability distribution α (i.e. a N-dimensional stochastic vector), they show that the lumped process $\phi(X_n)$ is again an homogeneous Markov chain if and only if

$$P(\ker(V_\phi) \cap \mathscr{S}(\alpha)) \subset \ker(V_\phi),$$

where $\mathscr{S}(\alpha)$ is the minimal subspace of $\mathbb{R}^n$ that contains α and is invariant under P and all the lumping projectors Π_Y. In particular, (X_n) is strongly lumpable through the map ϕ if and only if $P\ker(V_\phi) \subset \ker(V_\phi)$.
They also prove that weak and strong lumpability coincide when (X_n) has a normal, irreducible transition matrix.
In [47], Ledoux generalizes the concept of lumpability for Markov chains on countably-infinite state spaces. He considers aggregations generated by partitions of the state space. The lumped process is then defined on a space of classes. He proves that the process (X_n) is strongly lumpable with respect to a particular partition if and only if for every couple of classes $C(i)$, $C(j)$, the probability of going from the state k to any state in the class $C(j)$, denoted with $P(k, C(j))$, has the same value for every $k \in C(i)$. This value represents the transition probability of going from state i to state j in the lumped process.
He generalizes this result for continuous-time Markov chains, and he analyzes weak lumpability for irreducible chains (i.e. there exists a unique row vector π such that $\pi P = \pi$), associated with uniform stochastic semigroups.

Previous work in the area of infinite-dimensional linear lumpability was carried out for bounded operators by Coxson [21], and by Rózsa and Tóth in the context of Hilbert spaces [63], both requiring the existence of a continuous pseudoinverse of the lumping operator. As we will explain later, the pseudoinverse of $M : X \to Y$ is a linear operator $\overline{M} : Y \to X$ such that $M\overline{M}M = M$. A bounded pseudoinverse does not necessarily exist in a general Banach space, unless we require that $\ker(M)$ is a topologically complemented subspace in X.

Rózsa and Tóth show some interesting properties of lumping in Hilbert spaces (where pseudoinverse operators always exist): in particular, they prove that lumping transforms invariant sets for the original system into invariant sets for the reduced systems, as well as equilibria are transformed into equilibria, preserving some stability properties [63]. They also prove that, if $\widehat{H} \subset H$ are Hilbert spaces and the lumping relation (2.3) holds between two bounded operators A and $\widehat{A}$, defined on H and $\widehat{H}$ respectively, then $\sigma(\widehat{A}) \subset \sigma(A)$, where σ denotes the spectrum of a linear operator.
It is also worth noting the relation of lumpability to the notion of observability in control theory. In [21] Coxson pointed out the possibility to view the action of the lumping operator M as yielding a *system observable* $y = Mx$, or the output of a linear time-invariant control system

$$\begin{cases} \dot{x}(t) = Ax(t), & A : \mathbb{R}^n \to \mathbb{R}^n \\ y(t) = Mx(t), & M : \mathbb{R}^n \to \mathbb{R}^k, \end{cases} \tag{2.4}$$

where typically $k < n$. Recall that the system is called *observable* if every initial state $x_0 \in \mathbb{R}^n$ can be uniquely reconstructed from the system output y. This happens if and only if the *observability matrix*

$$\mathscr{O} = \begin{pmatrix} M \\ MA \\ \vdots \\ MA^{n-1} \end{pmatrix}$$

has full rank n. It is easy to see that if the system is lumpable by M, then

$$\operatorname{Rank}(\mathscr{O}) = \operatorname{Rank}(M) = k < n.$$

Thus, in this case lumpability implies that the control system (2.4) is not observable.

2.1.2 Our achievements

Our aim in this chapter is to extend these results to infinite-dimensional systems involving both bounded and unbounded operators. We will obtain more general conditions for lumpability of infinite-dimensional systems in abstract Banach spaces, that apply also to dynamics generated by unbounded operators, such as partial and delay differential equations. Given a well-posed system generated by a linear operator A and a reduction operator M, the problem we are interested in is whether the reduced operator exists and generates a well-posed dynamics in the sense of the Hille and Yosida theory. This means that we find conditions for $\widehat{A}$ to be the generator of a strongly continuous semigroup of linear operators on the reduced state space.
We first discuss the case of bounded operators: in this context, the main goal is to show that, under suitable hypothesis, the reduced operator $\widehat{A}$ exists and it is still bounded. Indeed, boundedness is a necessary and sufficient condition to guarantee that $\widehat{A}$ generates an uniformly continuous semigroup. Previous work in this area has been done by Barnes in the context of operator factorization [9], with no connection to semigroup theory, and by Coxson, using the pseudoinverse of the lumping operator [21]. We follow mainly the approach of Barnes, where the pseudoinverse is not involved. We don't need the kernel of M to be complemented, we only ask M to be bounded and surjective.
Next, we explain our results about lumpability of systems generated by unbounded operators, i.e. linear operators that are not everywhere defined on X. In this case, the property of $\widehat{A}$ to generate a semigroup is no more guaranteed, even if $\widehat{A}$ is well-defined on Y. We first give conditions for lumpability on the semigroup generated by A (representing the family of solution operators), which apply when the solutions of the original system are known. Since the solution operators are generally unknown *a priori*, we also give equivalent conditions for

lumpability directly on the operator A. We also discuss a condition for lumpability in the case of a non surjective lumping operator M, in relation with operator factorization.

2.2 Lumpability for bounded operators

We first analyze the case of systems associated with linear and bounded operators, which are always generators of *uniformly continuous semigroups.* Before giving the details of lumpability, we report some basic definitions and results in the theory of bounded operators. For this preliminaries, we refers especially to [54], [57].

2.2.1 Bounded operators on Banach spaces

We recall that a vector space X (eventually infinite-dimensional) on a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is called a *normed space* if there exists a map $X \ni x \to \|x\| \in \mathbb{R}^+$ such that:

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|, \quad \lambda \in \mathbb{K}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

We call the map $x \to \|x\|$ a *norm* on X. Every norm induces a distance on X, given by: $d(x, y) := \|x - y\|$. The space X is said to be a *Banach space* if the metric induced by d is complete.

A linear operator $A : X \to Y$ between Banach spaces is *bounded* if and only if it maps bounded sets into bounded sets. This is equivalent to ask that there exists $\alpha > 0$ such that:

$$\|Ax\| \leq \alpha\|x\|, \qquad \forall\, x \in X.$$

The concepts of boundedness and continuity coincide for linear operators. In particular, the following holds:

Proposition 2.2. *Let $A : X \to Y$ be a linear operator between Banach spaces. Then, the followings are equivalent:*

(i) A is bounded,

(ii) A is continuous in a point,

(iii) A is uniformly continuous on X.

We denote with $\mathscr{B}(X, Y)$ ($\mathscr{B}(X)$) the space of linear and bounded operators from X to Y (X). It is itself a Banach space with the following *operatorial norm*:

$$\|A\| := \sup_{\|x\|\leq 1} \|Ax\|.$$

In particular, $\mathscr{B}(X, Y)$ is a Banach algebra, according to the following definition:

Definition 2.2. A Banach algebra is an associative algebra $\mathscr{A}$ over the real or complex numbers that is also a Banach space. The algebra multiplication and the Banach space norm are related by the following inequality:

$$\|xy\| \leq \|x\|\,\|y\| \;\; \forall x, y \in \mathscr{A}$$

(i.e., the norm of the product is less than or equal to the product of the norms, which ensures that the multiplication operation is continuous).

Definition 2.3. A map $f : X \to Y$ between Banach spaces is called a *homeomorphism* if it has the following properties:

(i) f is a bijection (one-to-one and onto),

(ii) f is continuous,

(iii) the inverse function $f^{-1} : Y \to X$ is continuous.

Note that the inverse of a linear operator, if exists, is itself a linear operator.
We recall that a map between topological spaces is said to be *open* if it maps open sets into open sets. The following result, which is fundamental in the theory of linear operators, is also known as the *open mapping Theorem*:

Theorem 2.3 (Banach-Schauder)**.** *Let X, Y be Banach spaces and let $A \in \mathscr{B}(X, Y)$. If $AX = Y$ (i.e. A is surjective), then A is open.*

it is known that a bijective continuous map is a homeomorphism if and only if it is open. As a consequence of the Banach-Schauder Theorem, if a linear map between Banach spaces is bounded and bijective, then it is also an homeomorphism.

In our lumping analysis we don't assume that the kernel of the reduction operator is complemented, however, we introduce here the notion of *topological complement*, that is fundamental to define the pseudoinverse. In this way, the reason for which we decide to use a different method and to work without the pseudoinverse becomes clear.

Definition 2.4. Given a Banach space X and two subspaces $U, V \subset X$, we define two different notions of *direct sum*:

1. X is the *algebraic direct sum* of U and V if $U \cap V = 0$ and $X = u + v$, i.e. every $x \in X$ has a unique representation as $x = u_1 + v_1$, where $u_1 \in U$, $v_1 \in V$;

2. X is the *topological direct sum* of U and V if it is the direct sum of U and V and the map
$$S : U \times V \ni (u, v) \to u + v \in X$$
is a homeomorphism, when U, V have their subspace topology and $U \times V$ has the product topology.

We recall that a linear operator P on X is said to be a *projector* if and only if it is *idempotent*, i.e. $P^2 = P$.
If X is the algebraic direct sum of U and V, then the map $P : X \ni u + v \to u \in U$ is called the projector on U along V. An important criteria to establish if an algebraic direct sum is also a topological direct sum is the following:

Proposition 2.4. *X is the topological direct sum of U and V if and only if the projector on U along V is continuous (i.e. bounded) on X.*

Note that, if the projector P on U along V is continuous, then the projector Q on V along U is also continuous, because $Q = I - P$.
Furthermore, it is known that if U and V are closed subspaces in X, then an algebraic direct sum is also a topological direct sum (if $U \cap V = 0$ and $X = u + v$, then the projector on U along V is automatically bounded).

Definition 2.5 (complemented subspace)**.** Let X_1 be a closed subspace of a Banach space X. We say that X_1 is complemented in X if and only if there exists a closed subspace X_2 such that:
$$X = X_1 \oplus X_2,$$
where $X_1 \oplus X_2$ denotes the direct sum of X_1 and X_2 (that is automatically a topological direct sum). In this case, X_2 is a *topological complement* for X_1.

The existence of a topological complement in a Banach space is then the same as the existence of a *closed algebraic complement*.
It can be proved that every finite dimensional subspace has a topological complement. Furthermore, if H is a Hilbert space every closed subspace $Y \subset H$ is complemented; indeed, the orthogonal complement $Y^\perp$ (i.e. $\langle y, y^\perp \rangle = 0$ for every $y \in Y$, $y^\perp \in Y^\perp$) is a closed subspace of H and we have $H = Y \oplus Y^\perp$. A famous theorem of Lindenstrauss and Tzafriri asserts that the converse is true as well [49]. More precisely, if $(X, \|.\|)$ is a Banach space such that every closed subspace is complemented then $\|.\|$ is induced by a scalar product, i.e. $(X, \|.\|)$ is a Hilbert space.
Two known examples of non-complemented subspaces in Banach spaces are $c_0(\mathbb{Z}) \subset l^\infty(\mathbb{Z})$, the closed subspace of null sequences in the Banach space of the bounded sequences, and $C([0,1])$ in $L^\infty([0,1])$. Moreover, in the space $X = L^1(S^1)$ of integrable functions on the unit circle, the closed subspace of all functions f whose Fourier coefficients $\widehat{f}(n) = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta$ vanish for $n < 0$ is not complemented (see e.g., [65]).

We are ready to define the pseudoinverse of a linear operator (see [1] for details):

Definition 2.6 (Pseudoinverse)**.** A pseudoinverse of $A \in \mathscr{B}(X, Y)$ is any operator $\overline{A} \in \mathscr{B}(Y, X)$ such that $A\overline{A}A = A$.

Proposition 2.5. *Let A be a linear operator between Banach spaces. Then the followings are equivalent:*

(i) *A admits a bounded pseudoinverse $\overline{A} : Y \to X$,*

(ii) $\ker(A)$ *and* $\operatorname{ran}(A)$ *are complemented subspaces in X and Y respectively,*

(iii) *there exist continuous projections P and Q such that* $\operatorname{ran}(P) = \ker(A)$ *and* $\operatorname{ran}(Q) = \operatorname{ran}(A)$ *respectively.*

Note that the pseudoinverse operator is in general not unique, because the topological complements of $\ker(A)$ and $\operatorname{ran}(A)$ are themselves not unique.
If A is a surjective operator, then $\overline{A}$ is also a right inverse, since $A\overline{A}A = A$ implies that $A\overline{A}y = y$ for every $y \in Y$. To obtain an example of a surjective operator without pseudoinverse, take the quotient projection $\pi : X/C \to Y$, where C is a non-complemented subspace of X (see [13]).
It is now clear that not every linear and bounded operator has a pseudoinverse, and for this reason it is useful to find another method for the analysis of lumping in Banach spaces.

We point out that a dynamical system given by a linear and bounded operator always admits a classical solution for every initial condition. Consider system (2.1) when the operator A belongs to $\mathscr{B}(X)$. Since A is bounded, the system (2.1) is well-defined and the solutions are given by $x(t) = e^{At}x(0)$ for the family of exponential operators

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

It follows from the properties of Banach algebras that the series is absolutely convergent, and hence convergent in the topology of $\mathscr{B}(X)$.
As we will explain later, the family e^{At} is a *uniformly continuous semigroup*.

The following result is related to *factorization of operators* and has an interesting connection to lumpability:

Theorem 2.6 (Thm. 1 [27]). *Let D and E be bounded linear operators from a Banach space X to itself. The following conditions are equivalent:*

(i) $D = CE$ *for some bounded operator* C *on* $\operatorname{ran}(E)$*,*

(ii) $\|Dx\| \leq k\|Ex\|$ *for some* $k > 0$ *and all* $x \in X$*,*

(iii) $\operatorname{ran}(D^*) \subset \operatorname{ran}(E^*)$*.*

Assuming $D = MA$ and $E = M$, (i) corresponds to the lumping relation (2.3), with $C = \widehat{A}$. Unlike the case of lumping, in the context of factorization the operator E need not to be surjective, and the operator C is then defined on the range of E. Here we mainly assume the surjectivity of M, but at the same time we relax (ii) and we only ask an invariance condition for the kernel of M. Lumpability in the case of a non surjective operator M is discussed in Section 2.3. of this thesis.

2.2.2 Lumpability for bounded operators

We consider the diagram (2.2) where $M : X \to Y$ is a linear, bounded and surjective operator between Banach spaces X, Y. In order to have a lumping it is necessary that the kernel of M is invariant under A:

Theorem 2.7. *There exists a linear, bounded operator $\widehat{A} \in \mathscr{B}(Y)$ satisfying $MA = \widehat{A}M$ if and only if* $\ker(M) \subseteq \ker(MA)$*.*

This result was proved by B. A. Barnes in [9], in the context of factorization of operators. A proof in the context of lumping can be found in [21], using the pseudoinverse of a bounded operator under the additional assumption that the kernel of M is topologically complemented in X. In other words, it was assumed that there exists a closed subspace $N \subset X$ such that

$$N \cap \ker(M) = 0, \quad N + \ker(M) = X. \tag{2.5}$$

However, as we pointed out in the previous section, in a generic Banach space not every closed subspace is complemented. We present in details the proof given in [9], because some interesting operators on a quotient Banach space are considered, which are widely used throughout this thesis. The following proof does not require the hypothesis (2.5) but uses only the continuity and surjectivity of M.

Proof of Theorem 2.7. If there exists an operator $\widehat{A}$ such that $MA = \widehat{A}M$, then the kernel of M is invariant under A because

$$Mx = 0 \quad \Rightarrow \quad MAx = \widehat{A}Mx = 0,$$

so that $\ker(M)$ is contained in $\ker(MA)$. For the inverse implication, consider the quotient space $X/\ker(M)$ of the set of the equivalence classes $[x]$, $x \in X$, given by the relation $y \in [x] \Leftrightarrow (x - y) \in \ker(M)$. Since the kernel is closed, the quotient is a Banach space with norm

$$\|[x]\| = \inf_{m \in \ker(M)} \|x - m\|. \tag{2.6}$$

Let π be the quotient projection $\pi(x) = [x]$, and consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "M"] \arrow[dr, "\pi"'] & Y \arrow[d, leftrightarrow, "\widetilde{M}"] \\ & X/\ker(M) \end{tikzcd} \tag{2.7}$$

Since one can take $m = 0$ in (2.6), the projection satisfies $\|\pi(x)\| \leq \|x\|$. Define a new operator on the quotient:

$$\widetilde{M} : \; X/\ker(M) \to Y, \quad \widetilde{M}[x] = Mx.$$

This definition is well posed in the sense that it does not depend on the choice of the particular element in the equivalence class. Furthermore, since $[x] = [x - m]$ $\forall m \in \ker(M)$,

$$\begin{aligned} \|\widetilde{M}[x]\| &= \inf_{m \in \ker(M)} \|\widetilde{M}[x - m]\| = \inf_{m \in \ker(M)} \|M(x - m)\| \\ &\leq \inf_{m \in \ker(M)} \|M\| \, \|x - m\| = \|M\| \, \|[x]\|, \end{aligned}$$

which shows that $\widetilde{M}$ is bounded. We observe the followings:

1. Since M is surjective, $\widetilde{M}$ is also a surjective operator between Banach spaces. The open mapping theorem implies that $\widetilde{M}$ is an open map on the quotient.
2. $\widetilde{M}$ is injective. Indeed, by definition, $\widetilde{M}[x] = 0$ if and only if $x \in \ker(M)$, and the zero element in the quotient space is exactly $[0] = \ker(M)$.
3. $\widetilde{M}$ is an open and continuous bijection (that is, a homeomorphism), so the inverse $\widetilde{M}^{-1} : Y \to X/\ker(M)$ is a bounded linear operator.

Now consider the diagram:

$$\begin{tikzcd} X \arrow[r, "MA"] \arrow[dr, "\pi"'] & Y \\ & X/\ker(M) \arrow[u, "\widetilde{MA}"'] \end{tikzcd} \tag{2.8}$$

where the operator $\widetilde{MA} : \; X/\ker(M) \to Y$ is defined by $\widetilde{MA}[x] = MAx$. This is well-defined: if $y \in [x]$, then $y - x = m$ for some element $m \in \ker(M)$ and the invariance of $\ker(M)$ under A implies $MAx = MAy$. The operator $\widetilde{MA}$ is not a homeomorphism but it is still bounded (which can be shown in the same way as for $\widetilde{M}$). We can now define the linear operator $\widehat{A}$ on Y by

$$\widehat{A}y = MAx, \quad y = Mx.$$

Again, the definition is well posed by the invariance hypothesis and the surjectivity of M. It only remains to show that $\widehat{A}$ is bounded. Referring to the diagrams (2.7) and (2.8), we have

$$\widehat{A}y = MAx = \widetilde{MA}[x] = (\widetilde{MA} \circ \widetilde{M}^{-1})y,$$

showing that $\widehat{A}$ is a composition of linear bounded operators, hence bounded. □

Remark 2.1. The quotient projection itself provides a simple example of lumping. Consider a closed subset $\mathscr{C} \subset X$ such that $A\mathscr{C} \subseteq \mathscr{C}$, and take $Y = X/\mathscr{C}$. By the invariance of $\mathscr{C}$ we can define the bounded linear operator $\widehat{A}[x] := [Ax]$. Then, for $x \in X$,

$$\pi A x = [Ax] = \widehat{A}[x] = \widehat{A}\pi x,$$

so that the following diagram commutes:

$$\begin{tikzcd} X/\mathscr{C} \arrow[r, "\widehat{A}"] & X/\mathscr{C} \\ X \arrow[u, "\pi"] \arrow[r, "A"] & X \arrow[u, "\pi"'] \end{tikzcd}$$

Example 2.1 (Convolution)**.** Consider the convolution operator A on $L^1(\mathbb{R}^N)$:

$$Af(x) = h * f(x) := \int_{\mathbb{R}^N} h(x-y) f(y)\, dy\, dx,$$

for some given function $h \in L^1(\mathbb{R}^N)$. Since $L^1(\mathbb{R}^N)$ is a Banach algebra with the convolution product, the operator A belongs to $\mathscr{B}(L^1(\mathbb{R}^N))$. Define the continuous and surjective functional $M : L^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$Mf = \int_{\mathbb{R}^N} f(x) dx.$$

By Fubini's theorem and the translation invariance of the Lebesgue measure,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h(x-y) f(y)\, dy dx = \left(\int_{\mathbb{R}^N} h(x)\, dx\right)\left(\int_{\mathbb{R}^N} f(x)\, dx\right).$$

This implies that $\ker(M)$ is invariant under A and we have a lumping made by M on the system associated to the convolution operator. The operator $\widehat{A}$ on the upper level is simply

$$\widehat{A}x = \lambda x, \quad \text{where } \lambda = \int_{\mathbb{R}^N} h(x)\, dx.$$

Remark 2.2. As in the finite-dimensional case, we can view the system

$$\begin{cases} \dot{x}(t) = Ax(t), & A \in \mathscr{B}(X) \\ y(t) = Mx(t) \end{cases} \tag{2.9}$$

as a control system with output $y = Mx$. In this context, (2.9) is said to be observable if

$$\bigcap_{k=0}^{+\infty} \ker(MA^k) = \{0\}$$

(see [68]). If the system is lumpable by M, by definition $\bigcap_{k=0}^{+\infty} \ker(MA^k) = \ker(M) \neq \{0\}$, so that it is non-observable [21].

2.3 Lumpability for unbounded operators

We now turn to the case when the operator generating the dynamics is unbounded. Indeed, in many applications one needs to deal with operators defined on a proper subset of the Banach space X, such as in partial or delay differential equations. We consider the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0, \qquad x_0 \in X, \end{cases} \tag{2.10}$$

where $A : \mathscr{D}(A) \subset X \to X$ is a linear unbounded operator. The existence and uniqueness of a smooth solution is no longer guaranteed as in the bounded case. Such problems have been extensively studied beginning with the works of Hille and Yosida at the end of the 1940s. We briefly recall the essential elements of the theory.

2.3.1 Background in semigroup theory

Let X be a Banach space. A one-parameter family of bounded operators $\{T(t)\}_{t\geq 0}$ in $\mathscr{B}(X)$ is called a *strongly continuous semigroup* if

1. $T(0) = I$,
2. $T(t+s) = T(t)T(s) \quad \forall t, s \geq 0$,
3. The map $t \mapsto T(t)x \in X$ is continuous for every $x \in X$.

The last property is called *strong continuity* as it corresponds to the continuity of the map $t \mapsto T(t) \in \mathscr{B}(X)$ when $\mathscr{B}(X)$ is endowed with the strong operator topology, namely, $T_n \to T$ if and only if $\lim_{n\to+\infty} \|T_n x - Tx\| = 0$, $\forall x \in X$.

Definition 2.7. A linear operator $A : \mathscr{D}(A) \to X$ is said to be *closed* if for every sequence $x_n \subset X$ such that x_n converges to $x \in X$ and Ax_n converges to $z \in X$, then $x \in \mathscr{D}(A)$ and $Ax = z$.
A non closed operator A is said to be *closable* if for every $x_n \subset X$ such that x_n converges to 0 and Ax_n converges to $z \in X$, then $z = 0$.

Definition 2.8. Let A be a closable operator. The *closure* of A is the smallest closed extension of A and it is denoted with $\overline{A}$. Its domain is the following set:

$$\mathscr{D}(\overline{A}) := \{x \in X \text{ such that } \exists\, x_n \subset \mathscr{D}(A),\, x_n \to x, \text{ and } Ax_n \to z \text{ for some } z \in X\}$$

The *generator* of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is the closed and densely defined operator $A : \mathscr{D}(A) \subset X \to X$ defined by $Ax = \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x)$ on the domain

$$\mathscr{D}(A) = \left\{ x \in X : \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x) \in X \right\},$$

The following results are basic in semigroup theory [28, 41, 56].

Theorem 2.8. *The dynamics (2.10) is well posed if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X, and in that case for every $u_0 \in \mathscr{D}(A)$ the unique classical solution of (2.10) is given by $t \mapsto T(t)u_0$.*

Theorem 2.9. *If A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ then the following hold:*

1. *$x \in \mathscr{D}(A) \Rightarrow T(t)x \in \mathscr{D}(A)$, and $\dfrac{d}{dt}T(t)x = T(t)Ax = AT(t)x$, $\forall t \geq 0$.*

2. $T(t)x - x = \int_0^t T(s)Ax\,ds$, $\forall x \in \mathscr{D}(A)$.

3. $T(t)x - x = A\int_0^t T(s)x\,ds$, $\forall x \in X$.

A strongly continuous semigroup $T(t)$ is characterized by a real number $\omega(T)$ called the *growth bound of the semigroup*, defined as

$$\omega(T) = \inf\left\{\omega_0 \in \mathbb{R} : \exists C > 0 \text{ with } ||T(t)|| \leq Ce^{\omega_0 t} \;\; \forall t > 0\right\}.$$

The growth bound is linked to the spectral properties of the generator A; in particular, $\sup_{\lambda\in\sigma(A)}\{\mathrm{Re}(\lambda)\} \leq \omega(T)$, where $\sigma(A)$ denotes the spectrum of A. Another useful property is the possibility to write the integral operator of a generator as the Laplace transform of the associated semigroup:

Proposition 2.10 (Integral representation of the resolvent operator)**.** *Let A be the generator of $\{T(t)\}_{t\geq 0}$ and $\rho(A)$ be the complementary set of $\sigma(A)$. Then the following hold:*

1. *If $\mathrm{Re}(\lambda) > \omega(T)$, then $\lambda \in \rho(A)$ and*

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda s}T(s)x\,ds, \quad \forall x \in X.$$

2. *If the integral*

$$\mathscr{R}(\lambda) = \int_0^\infty e^{-\lambda s}T(s)x\,ds$$

exists for every $x \in X$, then $\lambda \in \rho(A)$ and $(\lambda I - A)^{-1} = \mathscr{R}(\lambda)$.

Since not every operator generates a semigroup, it is useful to have conditions for a linear operator to be a generator. The following theorem was firstly proved by E. Hille and K. Yosida independently, in the case of contraction semigroups [40], [73]. Here we give the following more general version (see [50]):

Theorem 2.11 (Feller-Miyadera-Phillips,1952)**.** *Let $(A, \mathscr{D}(A))$ be a linear operator on a Banach space X and $\omega \in \mathbb{R}$, $C \geq 1$ constants. Then the following statements are equivalent.*

1. *A generates a strongly continuous semigroup satisfying $\|T(t)x\| \leq Ce^{\omega t}$ for every $t \geq 0$.*

2. *A is closed and densely defined, and for every $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\lambda) > \omega$ one has $\lambda \in \rho(A)$ and $\|(\lambda I - A)^{-n}\| \leq \dfrac{C}{(\mathrm{Re}(\lambda) - \omega)^n} \quad \forall n \in \mathbb{N}$.*

Example 2.2. A simple example of a closed and densely-defined operator that is not a generator is the differentiation operator on the Banach space $C[0,1]$:

$$Af = f', \quad f \in \mathscr{D}(A) = C^1[0,1].$$

This operator cannot be the generator of a strongly continuous semigroup since its spectrum is the whole complex plane; so it does not satisfy the second statement in Theorem 2.11.

If A is the generator of a strongly continuous semigroup, it is possible to recover A from its restriction to some particular subspaces $D \subset \mathscr{D}(A)$, in the sense that the closure of A with domain D is A itself with its original domain $\mathscr{D}(A)$. These subspaces must be *cores*. Recall that a subspace $D \subseteq \mathscr{D}(A)$ is called a *core* for a linear operator A if D is dense in $\mathscr{D}(A)$ for the *graph norm* [28]

$$||x||_A := ||x|| + ||Ax||.$$

A useful criterion to identify cores is the following (see [2]):

Proposition 2.12. *Let $T(t)$ be a strongly continuous semigroup generated by A. Let $D \subset X$ be a dense subspace such that $D \subseteq \mathscr{D}(A)$ and $T(t)D \subset D$ for every $t \geq 0$. Then D is a core for A.*

It is important to note that the invariance of a closed subspace under the infinitesimal generator A is not equivalent to the invariance under the generated semigroup, unless A is bounded. For instance, consider the left translation semigroup $T(t)f(x) := f(x+t)$ on $C_0(\mathbb{R})$. It is generated by $Af(x) = f'(x)$ defined on:

$$\mathscr{D}(A) := \{f \in C_0^1(\mathbb{R}) \,:\, f' \in C_0(\mathbb{R})\}.$$

We take the closed subspace

$$\mathscr{C} := \{f \in C_0(\mathbb{R}) \,:\, f(x) = 0 \ \forall x \leq 0\}.$$

Then, $\mathscr{C}$ is clearly A-invariant, but it is not invariant under the semigroup $T(t)$.
For this reason, we state a result on closed invariant subspaces (see [74]), which will be relevant in the analysis of lumpability for unbounded operators.

Theorem 2.13 ($T(t)$-invariance of a closed subspace)**.** *Let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ having growth bound ω. Let $\mathscr{V} \subset X$ be a closed subspace such that $A\left(\mathscr{D}(A) \cap \mathscr{V}\right) \subseteq \mathscr{V}$, and let $A|_{\mathscr{V}} : \mathscr{D}(A) \cap \mathscr{V} \to \mathscr{V}$ be the restriction of A to $\mathscr{V}$. Then the following are equivalent:*

1. *$\mathscr{V}$ is invariant under $T(t)$.*
2. *There exists $\lambda > \omega$ such that $\lambda \in \rho(A) \cap \rho(A|_{\mathscr{V}})$.*

Theorem 2.13 follows essentially from the fact that its condition 2 implies the invariance of $\mathscr{V}$ under the resolvent operators $\mathscr{R}(\lambda, A) = (\lambda I - A)^{-1}$ for every $\lambda \in \mathbb{C}$ with $\mathrm{re}[\lambda] > \omega$ (see [74]). This kind of invariance implies the invariance under the semigroup, which follows from an asymptotic exponential formula (see e.g. [56, Section 1.8])

$$T(t)x = \lim_{n\to+\infty} \left[\frac{n}{t}\mathscr{R}\left(\frac{n}{t}, A\right)\right]^n x.$$

2.3.2 Lumpability for unbounded operators

We are now ready to address the problem of lumpability in the unbounded case. The aim is to obtain the commutativity of the following diagram.

$$\begin{tikzcd}
M\left(\mathscr{D}(A)\right) \subset Y \arrow[r, "\widehat{A}"] & Y \\
\mathscr{D}(A) \subset X \arrow[r, "A"] \arrow[u, "M"] & X \arrow[u, "M"']
\end{tikzcd}$$

We assume that the linear operator $M : X \to Y$ is bounded and surjective, while A and $\widehat{A}$ will be defined on a proper subset of X and Y, respectively.

Suppose that A generates a strongly continuous semigroup on X, which we denote by $\{T(t)\}_{t\geq 0}$. We want the operator $\widehat{A}$ to be again the generator of a strongly continuous semigroup in order to obtain a well-defined dynamics on the upper level. Thus, we need the lumping relation $MA = \widehat{A}M$ to hold on $\mathscr{D}(A)$.

Theorem 2.14. *The following statements are equivalent.*

1. $\ker(M)$ *is invariant under* $T(t)$ *for every* $t \geq 0$.

2. *There exists a linear operator* $\widehat{A}$ *on* $M(\mathscr{D}(A))$ *such that* $\widehat{A}$ *generates a strongly continuous semigroup on* Y, *and* $\widehat{A}M = MA$.

Proof. $1 \Rightarrow 2$. Suppose that $\ker(M)$ is invariant under $T(t)$, $\forall t \geq 0$. Consider the family of linear operators $\{\widehat{T}(t)\}_{t\geq 0}$ on Y defined by

$$\widehat{T}(t)y = MT(t)x, \quad y = Mx. \tag{2.11}$$

For each $t \geq 0$, $\widehat{T}(t)$ is well-defined due to the invariance of the kernel, and, applying the same arguments as in the continuous case, one can see that it is bounded. Moreover, the family (2.11) is a strongly continuous semigroup on Y because:

1. $\widehat{T}(0)y = \widehat{T}(0)Mx = MT(0)x = Mx = y$;

2. for all $t, s \geq 0$,

$$\begin{aligned}\widehat{T}(t+s)y &= MT(t+s)x = MT(t)T(s)x \\ &= \widehat{T}(t)MT(s)x = \widehat{T}(t)\widehat{T}(s)Mx = \widehat{T}(t)\widehat{T}(s)y;\end{aligned}$$

3. $\lim_{h\to 0^+} \widehat{T}(h)y - y = \lim_{h\to 0^+} \|MT(h)x - Mx\| \leq \lim_{h\to 0^+} \|M\| \, \|T(h)x - x\| = 0.$

In particular, let us denote with $\widehat{\omega}$ the growth bound of $\widehat{T}(t)$. We show that $\widehat{\omega}$ is less or equal than the growth bound ω of $T(t)$. As we did in section 2.2.2, we define the following operators from $X/\ker(M)$ to Y:

(i) $\widetilde{M}[x] := Mx$,

(ii) $\widetilde{MT(t)}[x] := MT(t)x$, $t \geq 0$.

By the Banach-Schauder theorem, $\widetilde{M}$ is a homeomorphism. By the boundedness of $T(t)$, it follows that $\widetilde{MT(t)}$ is bounded, and:

$$\|\widetilde{MT(t)}[x]\| = \inf_{m\in\ker(M)} \|MT(t)(x-m)\| \leq \|M\|\|T(t)\|\|[x]\| \leq C\|M\|e^{\omega t}\,\|[x]\|.$$

From this we obtain:

$$\|\widehat{T}(t)y\| = \|\widetilde{MT(t)}\widetilde{M}^{-1}y\| \leq C\|M\|\|\widetilde{M}^{-1}\|e^{\omega t}\,\|y\|,$$

showing that $\widehat{\omega} \leq \omega$.
Let $\widehat{A}$ be the generator of the new semigroup $\widehat{T}(t)$. Consider an element $y = Mx$ in $M(\mathscr{D}(A))$. By the definition of a generator and the continuity of M on X,

$$\begin{aligned}\widehat{A}y = &\lim_{h\to 0^+} \frac{1}{h}\left(\widehat{T}(h)y - y\right) = \lim_{h\to 0^+} \frac{1}{h}\left(MT(h)x - Mx\right) \\ =&M\left(\lim_{h\to 0^+} \frac{1}{h}\left(T(h)x - x\right)\right) = MAx.\end{aligned}$$

This implies that $\widehat{A}$ is defined on $M(\mathscr{D}(A))$, which is a dense subset of Y since A is densely defined and M is bounded and surjective. On this subset the lumping relation holds also between the two generators: $\widehat{A}Mx = MAx$. We have thus obtained the inclusion

$M(\mathscr{D}(A)) \subset \mathscr{D}(\widehat{A})$. We next show that the domain of $\widehat{A}$ is exactly $M(\mathscr{D}(A))$. For this purpose, we take $\lambda \in \mathbb{C}$ that belongs both to the resolvent set of A and $\widehat{A}$ and use the integral representation of the resolvent operator. Given an arbitrary element y for which $\widehat{A}$ is defined, there exists $s = Mx \in Y$ such that $y = (\lambda I - \widehat{A})^{-1}s$. Hence we can write:

$$\begin{aligned} y &= \int_0^{+\infty} e^{-\lambda t}\widehat{T}(t)s\,dt = \int_0^{+\infty} e^{-\lambda t}\widehat{T}(t)Mx\,dt \\ &= \int_0^{+\infty} e^{-\lambda t}MT(t)x\,dt = M\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt \\ &= M(\lambda I - A)^{-1}x = Mz, \end{aligned}$$

where z belongs to $\mathscr{D}(A)$. This implies that $\mathscr{D}(A) = M(\mathscr{D}(A))$.

$2 \Rightarrow 1$. We will show that the invariance of $\ker(M)$ under the semigroup is a necessary condition to have a well-defined dynamics on Y. Suppose that the operator $\widehat{A}y := MAx$ defined on $M(\mathscr{D}(A))$ generates a strongly continuous semigroup on Y. Consider the following maps from $\mathbb{R}^+$ to Y:

1. $t \mapsto \widehat{T}(t)y_0$,
2. $t \mapsto MT(t)x_0$,

where $y_0 = Mx_0$, $x_0 \in \mathscr{D}(A)$. These two maps are both solutions of the abstract Cauchy problem

$$\begin{cases} \dot{y}(t) = \widehat{A}y(t), \\ y(0) = y_0. \end{cases} \tag{2.12}$$

In fact, the first map is a solution by definition, while for the second map we have

$$\frac{d}{dt}MT(t)x_0 = M\frac{d}{dt}T(t)x_0 = MAT(t)x_0 = \widehat{A}MT(t)x_0,$$

and $MT(0)x_0 = Mx_0 = y_0$, where we have used the continuity of M to interchange with the differentiation. Since the solution of the Cauchy problem (2.12) is unique, for all $t > 0$ we have

$$\widehat{T}(t)Mx_0 = MT(t)x_0,$$

and this equality holds for every $x_0 \in \mathscr{D}(A)$. The operators MT and $\widehat{T}M$ are equal on a dense subspace of Y, so they coincide on the whole space. The invariance of $\ker(M)$ under the semigroup follows then from the relation $MT = \widehat{T}M$, which proves the statement above. $\square$

We note that if a closed subspace is invariant under $T(t)$ for all $t \geq 0$, then by definition it is invariant under the infinitesimal generator A; however, the converse is not true. As a simple counterexample, let X be the Banach space $C_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ that tend to zero at infinity, endowed with the supremum norm. The operator

$$Af = f', \quad \mathscr{D}(A) = \left\{f \in C_0^1(\mathbb{R}) : f' \in X\right\},$$

generates the strongly continuous semigroup of left translations $T(t)f(s) = f(s+t)$. Clearly, the closed subspace $\mathscr{C} = \{f \in X : f(s) = 0, \forall s \leq 0\}$ is invariant under A but not invariant under translations.

It is typically the case in applications that one knows the generator A but not the associated semigroup. Therefore, it is necessary to find conditions on M that give the invariance of its kernel under the semigroup without knowing the semigroup itself. The next result gives conditions on the operator A for lumpability.

Theorem 2.15. *System (2.10) is lumpable by the linear, bounded, and surjective operator $M : X \to Y$ if and only if the following two conditions hold:*

1. $A\left(\ker(M) \cap \mathscr{D}(A)\right) \subset \ker(M)$*, and*
2. *there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective from $\ker(M) \cap \mathscr{D}(A)$ to $\ker(M)$.*

Proof. If (2.10) is lumpable by M, by definition there exists a linear operator $\widehat{A}$ such that $MA = \widehat{A}M$ on $\mathscr{D}(A)$ and $\widehat{A}$ generates a strongly continuous semigroup on Y. By Theorem 2.14 $\ker(M)$ is $T(t)$-invariant, and from this it follows that $\ker(M)$ is also A-invariant, i.e. condition 1 holds. By Theorem 2.13, there exists $\lambda > \omega$ such that $\lambda \in \rho(A) \cap \rho(A_{|_{\ker(M)}})$. This means that $(\lambda I - A)$ must be surjective from $\ker(M) \cap \mathscr{D}(A)$ onto $\ker(M)$, i.e. conditions 2 holds.
Conversely, condition 1 gives that $\ker(M)$ is invariant under A. Let us consider λ as in condition 2. Since $\lambda > \omega$, it follows that $\lambda \in \rho(A)$. This implies that $(\lambda I - A)$ is injective on the whole domain $\mathscr{D}(A)$, and in particular it is injective on the subspace $\ker(M) \cap \mathscr{D}(A)$. Indeed, since condition 2 holds, $(\lambda I - A)$ is invertible from $\ker(M) \cap \mathscr{D}(A)$ to $\ker(M)$, i.e. $\lambda \in \rho(A) \cap \rho(A_{|_{\ker(M)}})$. We can now apply Theorem 2.13 with $\mathscr{V} = \ker(M)$, to obtain that $\ker(M)$ is invariant under the semigroup $\{T(t)\}_{t \geq 0}$ generated by A. Lumpability then follows by Theorem 2.14. □

Remark 2.3. As a special case of condition 1 in Theorem 2.15, consider the case when

$$\ker(M) \subset \mathscr{D}(A) \quad \text{and} \quad A(\ker(M)) \subset \ker(M). \tag{2.13}$$

If (2.13) holds, the restricted operator $A|_{\ker(M)} : \ker(M) \to \ker(M)$ is bounded by the closed graph theorem; so its spectrum is compact in the complex plane, and one can find a $\lambda > \omega$ such that $\lambda \in \rho(A) \cap \rho(A|_{\ker(M)})$. This implies that there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective from $\ker(M) \cap \mathscr{D}(A)$ *to* $\ker(M)$, so that M makes a lumping by Theorem 2.15. However, condition (2.13) is usually too strong and is generally not satisfied, as we will see in the examples below.

Remark 2.4 (Observability with unbounded operators)**.** Let A be the unbounded generator of a strongly continuous semigroup with growth bound ω. It can be shown that the system

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Mx(t) \end{cases}$$

is observable if and only if, for any $\mu \in \rho(A)$ satisfying $\operatorname{re}(\mu) > \omega$, the following system is observable:

$$\begin{cases} \dot{x}(t) = \mathscr{R}(\mu, A)x(t), \\ y(t) = Mx(t), \end{cases}$$

where the resolvent operator $\mathscr{R}(\mu, A) = (\mu I - A)^{-1}$ is indeed bounded [32, 67]. Hence the condition for observability is reduced to

$$\bigcap_{k=0}^{\infty} \ker(M\mathscr{R}(\mu, A)^k) = 0. \tag{2.14}$$

If the system is lumpable by M then $\ker(M)$ is invariant under the semigroup; so it is invariant under the resolvent operators for $\operatorname{re}(\mu) > \omega$ [74]. Since $\ker(M) \neq 0$, this implies that (2.14) is not satisfied and the system is non-observable. Hence, the observation stated in [21] for bounded operators holds also in the unbounded case.

Example 2.3 (Quotient semigroup)**.** Let $\mathscr{C}$ be a closed subspace that is invariant under a semigroup $\{T(t)\}_{t\geq 0}$ (or, equivalently, satisfying statement 2 of Theorem 2.13). As in the bounded case, the quotient projection

$$\pi : X \to X/\mathscr{C}, \quad x \mapsto [x]$$

yields a lumping on the system associated with the generator A. The semigroup induced on the quotient space is

$$\widehat{T}(t)[x] = [T(t)x], \quad t \geq 0, \ x \in X,$$

generated by $\widehat{A}[x] = [Ax]$. (See [2] for more details on quotient semigroups).

Example 2.4. Consider the space $X = C_0(\mathbb{R})$, and let $h : \mathbb{R} \to \mathbb{C}$ be a continuous function. Define the multiplicative operator

$$Af(x) = h(x)f(x), \quad \mathscr{D}(A) = \{f \in X : hf \in X\},$$

(which is bounded if and only if h is a bounded function). One can show that A generates a strongly continuous semigroup if and only if $\sup_{x\in\mathbb{R}} \operatorname{Re}(h(x)) < \infty$, and in this case the semigroup is given by $T(t)f(x) = e^{th(x)}f(x)$, $\forall t \geq 0$. If h is nonzero, then for any positive integer k there exist k points $\{x_1, \dots, x_k\}$ on the real line at which h does not vanish. Define the linear bounded operator $M : C_0(\mathbb{R}) \to \mathbb{C}^k$ by $Mf = (f(x_1), \dots, f(x_k))^\top$, which simply evaluates a given function at the k points. We can write

$$\begin{aligned} MAf &= M(hf) = (h(x_1)f(x_1), \dots, h(x_k)f(x_k))^\top \\ &= \operatorname{diag}(h(x_1), \dots, h(x_k)) \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_k) \end{pmatrix} := \widehat{A}Mf, \end{aligned}$$

where "diag" denotes a diagonal matrix. Thus M yields a lumping on the system associated with A. Note that the kernel of M is invariant under A, but not fully contained in $\mathscr{D}(A)$; hence (2.13) is not satisfied. On the other hand, the resolvent condition given in statement 2 of Theorem 2.13 is satisfied. This can be easily seen considering that the resolvent set of A is the complementary set of

$$\sigma(A) = \{\lambda \in \mathbb{C} : h(x) = \lambda \text{ for some } x \in \mathbb{R}\}.$$

Taking $\lambda \in \rho(A)$, the operator $\lambda I - A$ is surjective from $\mathscr{D}(A) \cap \ker(M)$ to $\ker(M)$ if and only if for every $g \in \ker(M)$ the function f defined by $f(x) = \dfrac{g(x)}{\lambda - h(x)}$ belongs to $\mathscr{D}(A) \cap \ker(M)$. This is indeed verified because:

1. since $\lambda \in \rho(A)$, $\frac{h(x)}{\lambda - h(x)}$ is bounded, so that $h(x)f(x)$ tends to zero at infinity;

2. since g vanishes at the points x_i, and the previous property holds, f also vanishes on this set of points. Hence, we can take every element in $\rho(A)$ that is greater than ω as λ of statement 2 of Theorem 2.13.

Example 2.5 (Delay differential equations)**.** Given $r \geq 0$, let $X = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous vector-valued functions on the compact interval $[-r, 0]$ equipped with the supremum norm, and let $L : X \to \mathbb{R}^n$ be linear and continuous. A linear *delay differential equation (DDE)* is an equation of the form $\dot{x}(t) = Lx_t$, where $x_t \in X$ is the function given by

$$x_t(s) = x(t + s), \quad s \in [-r, 0].$$

The unbounded linear operator A defined by

$$Af = f', \quad \mathscr{D}(A) = \{f \in C^1([-r,0],\mathbb{R}^n) : f'(0) = Lf\}$$

generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ that gives the solutions of the DDE. In other words, the unique solution $x(t)$ of the Cauchy problem

$$\begin{cases} \dot{x}(t) = Lx_t & t \geq 0 \\ x(t) = f(t) & t \in [-r,0], \end{cases} \tag{2.15}$$

with initial condition $f \in X$, satisfies

$$x_t(s) = T(t)f(s), \quad s \in [-r,0],\, t \geq 0.$$

Given a set of non-zero real numbers a_i, $i = 1,\dots,n$, we define a linear, bounded and surjective operator $M : X \to Y := C([-r,0],\mathbb{R})$ such that

$$M(f)(s) = a_1 f_1(s) + \cdots + a_n f_n(s), \quad \forall f \in X.$$

Keeping the notation as above, the next result follows.

Proposition 2.16. *If there exists a linear and bounded functional $\widehat{L} : Y \to \mathbb{R}$ such that $ML = \widehat{L}M$, then system (2.15) is lumpable by the operator M. The upper level dynamics is described by a DDE on the space of scalar-valued functions $C([-r,0],\mathbb{R})$:*

$$\begin{cases} \dot{y}(t) = \widehat{L}y_t & t \geq 0 \\ y(t) = g(t) & t \in [-r,0]. \end{cases} \tag{2.16}$$

Proof. It is easy to verify that $\ker(M) \cap \mathscr{D}(A)$ is invariant under A. (Note that $\ker(M)$ is not fully contained in the domain of A, so the condition (2.13) does not hold). If ω is the growth bound of the semigroup generated by A, we need to prove that there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective from $\ker(M) \cap \mathscr{D}(A)$ *to* $\ker(M)$. To do this, we take $\lambda > 0$ in $\rho(A) \cap \rho(\widehat{L})$ (this number always exists because A is a generator and $\widehat{L}$ is bounded; so its spectrum is closed and bounded in $\mathbb{C}$). For every $g \in \ker(M)$ there exists $f \in \mathscr{D}(A)$ such that $(\lambda I - A)f = g$, that is $f'(x) = \lambda f(x) - g(x)$. Solving this differential equation, one can write f as

$$f(x) = \left(c_0 - \int_0^x g(s)e^{-\lambda s}\, ds\right) e^{\lambda x}$$

for $c_0 = f(0) \in \mathbb{R}^n$. We will prove that $f \in \ker(M)$. Since $g \in \ker(M)$ and by the linearity of M,

$$Mf(x) = e^{\lambda x} M c_0.$$

Therefore $Mf = 0$ if and only if $Mc_0 = 0$. We need to prove that $c_0 \in \ker(M)$.
Since $f \in \mathscr{D}(A)$, we have $f'(0) = Lf$, i.e.

$$\lambda c_0 - g(0) = Lf.$$

Applying M on both sides we obtain $\lambda M c_0 = MLf$. Using the hypothesis, we can write $\lambda M c_0 = \widehat{L} M f$, which leads to

$$\lambda M c_0 = e^{\lambda x} \widehat{L} M c_0, \quad \forall x \in [-r,0]. \tag{2.17}$$

Evaluating at $x = 0$ yields

$$\widehat{L} M c_0 = \lambda M c_0. \tag{2.18}$$

Since $\lambda \in \rho(\widehat{L})$, (2.18) holds if and only if $Mc_0 = 0$, that is $c_0 \in \ker(M)$.

We have proved that system (2.15) is lumpable by M. For every $h = Mf$, $f \in \mathscr{D}(A)$, the generator of the semigroup on the upper level is

$$\widehat{A}h(x) = MAf(x) = a_1 f_1'(x) + \cdots + a_n f_n'(x) = h'(x);$$

which is again the differentiation operator, but defined on

$$M\mathscr{D}(A) = \{h \in Y \,:\, h' \in Y \text{ and } h'(0) = \widehat{L}f\}.$$

This operator is exactly the generator of the semigroup associated to the delayed system (2.16). $\square$

To have an example of functionals L on X which satisfy the hypothesis of the previous proposition, take

$$Lf(x) := \sum_{i=1}^{k} q_i f(-\alpha_i)$$

where $q_i \in \mathbb{R}$ and $\alpha_i \in (0, r)$. It is easy to verify that $\widehat{L}$ acts the same way as L but on a space of scalar-valued functions,

$$\widehat{L}h(x) = \sum_{i=1}^{k} q_i h(-\alpha_i), \quad h \in C([-r, 0], \mathbb{R}).$$

Example 2.6 (Differential equations with distributed delays)**.** In this example we describe a reduction of a linear delay differential equation (DDE) to a finite dimensional system of ordinary differential equations. We consider a delay distribution $f(\tau) = \alpha e^{-\alpha\tau}$, $\alpha > 0$. We deal with the following DDE, with order higher than one:

$$\sum_{k=1}^{n} a_k x^{(k)}(t) = -a_0 x(t) + \int_0^\infty x(t-\tau) f(\tau)\, d\tau, \tag{2.19}$$

where the superscript (k) denotes the kth derivative, for some $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$. We write the DDE (2.19) in vector form

$$\dot{u}(t) = B_1 u(t) + \int_0^\infty f(\tau) B_2 u(t-\tau)\, d\tau, \tag{2.20}$$

where $u \in C_0((-\infty, 0], \mathbb{R}^n)$ is the the vector-valued function

$$u(t) = \big(u_1(t),\ u_2(t),\ \ldots,\ u_n(t)\big)^\top = \big(x(t),\ x'(t),\ \ldots,\ x^{(n-1)}(t)\big)^\top,$$

and B_1, B_2 are the $n \times n$ matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\dfrac{a_0}{a_n} & -\dfrac{a_1}{a_n} & -\dfrac{a_2}{a_n} & \ldots & \ldots & -\dfrac{a_{n-1}}{a_n} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let us define a bounded operator $L : C_0((-\infty, 0], \mathbb{R}^n) \to \mathbb{R}^n)$ such that

$$Lu = B_1 u(0) + \int_{-\infty}^0 f(-s) B_2 u(s)\, ds,$$

and consider the family of translated functions $u_t(\theta) = u(t+\theta)$, $\theta \in (-\infty, 0]$.
Given a vector-valued initial condition ϕ, (2.20) is equivalent to the following Cauchy problem on the Banach space $\mathscr{X} = C_0((-\infty,0],\mathbb{R}^n)$:

$$\begin{cases} \dot{u}_t = Au_t, & t \geq 0, \\ u_0 = \phi, \end{cases} , \tag{2.21}$$

where

$$Au(\theta) := u'(\theta), \quad \mathscr{D}(A) = \{u \in X : u' \in X \text{ and } u'(0) = Lu\}. \tag{2.22}$$

Consider the lumping operator $M : \mathscr{X} \to \mathbb{R}^{n+1}$ defined by

$$Mu = \left(x(0),\ x_1(0),\ \ldots,\ x_{(n-1)}(0),\ \textstyle\int_{-\infty}^0 x(s)f(-s)\,ds\right)^\top .$$

By the properties of the generator A, it is easy to calculate

$$MAu = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ -\dfrac{a_0}{a_n}x(0) - \cdots - \dfrac{a_{n-1}}{a_n}x_{n-1}(0) + \int_{-\infty}^0 x(s)f(-s)\,ds \\ \alpha x(0) - \alpha \int_{-\infty}^0 x(s)f(-s)\,ds \end{pmatrix}.$$

This leads to the lumping relation $MA = \widehat{A}M$, where

$$\widehat{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\dfrac{a_0}{a_n} & -\dfrac{a_1}{a_n} & -\dfrac{a_2}{a_n} & \ldots & \ldots & -\dfrac{a_{n-1}}{a_n} & 1 \\ \alpha & 0 & \ldots & \ldots & \ldots & \ldots & -\alpha \end{pmatrix}.$$

This matrix generates a dynamical system in $\mathbb{R}^{n+1}$. Therefore, we have obtained a finite dimensional dynamics on the upper level.

In these particular kinds of lumping we start from a time-delayed system, that is our lower level dynamics, and we gain a finite set of ordinary differential equations as the upper level dynamics. To obtain this reduction, we are forced to consider exponential distributions. However, we can generalize this method through a different choice of the distribution f. For more definiteness, let us consider a first order delay differential equation (taking in mind that the method can be extended to higher order systems). Let f be given by a Gamma distribution $f(\tau) = k\tau^n e^{-\alpha\tau}$, where $k \in \mathbb{R}$ and $n \in \mathbb{N}$. With this definition of f, let us describe how to obtain a $\mathbb{R}^{n+2}$-dimensional dynamics on the upper level.

We consider the following bounded operator:

$$Lu = -a_0 u(0) + \int_{-\infty}^0 f(-s)u(s)\,ds = -a_0 u(0) + \int_{-\infty}^0 k(-1)^n s^n e^{\alpha s} u(s)\,ds.$$

Inserting this choice of L in definition (2.22), we obtain the generator A of the Gamma-distributed delays dynamics. Define $M : C_0(-\infty, 0] \to \mathbb{R}^{n+2}$ by

$$Mu = \left(u(0),\ \int_{-\infty}^0 f(-s)u(s)\,ds,\ \int_{-\infty}^0 \frac{f(-s)}{s}u(s)\,ds,\ \ldots,\ \int_{-\infty}^0 \frac{f(-s)}{s^n}u(s)\,ds\right)^\top .$$

Considering the shape of the Gamma distribution and using integration by parts, one has, for $p < n$,

$$\int_{-\infty}^{0} \frac{f(-s)}{s^p} u'(s)\, ds = -(n-p) \int_{-\infty}^{0} \frac{f(-s)}{s^{p+1}} u(s)\, ds - \alpha \int_{-\infty}^{0} \frac{f(-s)}{s^p} u(s)\, ds,$$

and for $p = n$,

$$\int_{-\infty}^{0} \frac{f(-s)}{s^n} u'(s)\, ds = k(-1)^n u(0) - \alpha \int_{-\infty}^{0} k(-1)^n e^{\alpha s} u(s) ds.$$

Using these relations we can write

$$MAu = \begin{pmatrix} -a_0 u(0) + \int_{-\infty}^{0} f(-s)u(s)\, ds \\ -n \int_{-\infty}^{0} \frac{f(-s)}{s} u(s)\, ds - \alpha \int_{-\infty}^{0} f(-s) u(s)\, ds \\ -(n-1) \int_{-\infty}^{0} \frac{f(-s)}{s^2} u(s)\, ds - \alpha \int_{-\infty}^{0} \frac{f(-s)}{s} u(s)\, ds \\ \vdots \\ (-1)^n k u(0) - \alpha \int_{-\infty}^{0} \frac{f(-s)}{s^n} u(s) ds \end{pmatrix} = \widehat{A} M u$$

where

$$\widehat{A} = \begin{pmatrix} -a_0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & -\alpha & -n & 0 & \dots & \dots & 0 \\ 0 & 0 & -\alpha & -(n-1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^n k & 0 & \dots & \dots & \dots & \dots & -\alpha \end{pmatrix}.$$

Thus $MA = \widehat{A}M$ and we have a lumping through the operator M. The dynamics generated by $\widehat{A}$ is

$$\begin{cases} \dot{x_1}(t) &= -a_0 x_1(t) + x_2(t) \\ \dot{x_2}(t) &= -\alpha x_2(t) - n x_3(t) \\ \vdots & \qquad \vdots \\ \dot{x}_{n+2}(t) &= (-1)^n k x_1(t) - \alpha x_{n+2}(t). \end{cases} \tag{2.23}$$

Even if all the lumping operators appearing in the literature of lumping are surjective, it is interesting to discuss lumpability in the case of $\mathrm{ran}(M) \neq Y$. In this case, a condition for the existence of a reduced bounded operator is given by Theorem 2.6, in the context of operator factorization. Here we prove the following:

Proposition 2.17. *Let $T(t)$ be a strongly continuous semigroup on X generated by A. Let M be linear and continuous from X to Y such that the following condition holds:*

(j) For $x_n \subset X$, $\|Mx_n\| \to 0 \Rightarrow \|MT(t)x_n\| \to 0\ \forall t \geq 0$.

Then, there exists a strongly continuous semigroup $\widehat{T}(t)$ on $\overline{\mathrm{ran}(M)}$ such that $MT(t) = \widehat{T}(t)M$. Moreover, $\widehat{T}(t)$ is generated by the closure $\overline{\widehat{A}}$, where $\widehat{A}$ is the following operator:

$$\widehat{A}y = MAx, \quad y = Mx \in M\mathscr{D}(A).$$

Remark 2.5. Note that condition (j) is stronger than the following:

$$T(t)\ker(M) \subset \ker(M) \quad \forall t \geq 0. \tag{2.24}$$

If in addition to (2.24) we ask M to have closed range, then we obtain condition (j) (see [27]). However, we can't obtain (j) from (2.24) if the range of M is not closed. This means that proposition 2.17 does not generalize theorem 2.14, but it rather give another version of lumpability with a different assumption.

Proof. We first note that (j) is equivalent to the following:

$$\text{there exists } k_t > 0 \text{ such that } \|MT(t)x\| \leq k_t\|Mx\| \ \forall t \geq 0. \tag{2.25}$$

It is clear that (2.25) implies (j). On the converse, let us suppose by contradiction that (j) holds and there exists $x_n \subset X$ such that $\|MT(t)x_n\| > n\|Mx_n\|$ for some $t > 0$ and all n. Without loss of generality, we can assume:

$$y_n := \frac{x_n}{\|MT(t)x_n\|}, \text{ and } \|MT(t)y_n\| > n\|My_n\|.$$

This means that $1 > n\|My_n\|$, i.e. $\|My_n\|$ tends to zero for $n \to \infty$. But since $\|MT(t)y_n\| \equiv 1$, we have contradicted (j).
By (2.25) and theorem 2.6, for every $t \geq 0$ we can construct a family of linear and bounded operators on $\text{ran}(M)$:

$$\widehat{T}(y) := MT(t)x, \quad y = Mx. \tag{2.26}$$

By the boundedness of $\widehat{T}(y)$, we can extend these operators on $\overline{\text{ran}(M)}$ in the following way:

$$\widehat{T}(t)y := \lim_{n\to\infty} MT(t)x_n, \quad \text{for } Mx_n \to y.$$

It is possible to verify that $\widehat{T}(t)$ is a strongly continuous semigroup of operators on $\overline{\text{ran}(M)}$. Note also that the value of k_t in (2.25) can be controlled by an exponential function. Indeed, $\widehat{\omega}$ being the growth bound of $\widehat{T}(t)$, we have that $\|MT(t)x\| \leq Ke^{\widehat{\omega}t}\|Mx\|$, for some positive constant K and all $x \in X$.
Let us denote with $\widetilde{A}$ the infinitesimal generator of $\widehat{T}(t)$. Given $y = Mx \in M\mathscr{D}(A)$, we can write:

$$\lim_{h\to 0} \frac{1}{h}(\widehat{T}(t)y - y) = M \lim_{h\to 0} \frac{1}{h}(T(t)x - x) = MAx.$$

This means that $M\mathscr{D}(A) \subset \mathscr{D}(\widetilde{A})$ and $\widetilde{A}y = \widehat{A}y$ on $M\mathscr{D}(A)$, where $\widehat{A}y := MAx$. Now, consider $y \in \mathscr{D}(\widetilde{A})$. Since both A and $\widetilde{A}$ are infinitesimal generators, we can find some $\lambda > 0$ in $\rho(A) \cap \rho(\widetilde{A})$, such that $\lambda > \widehat{\omega}$. By the integral representation of the resolvent operator, for some $y_0 \in Y$ with $Mx_n \to y_0$ we have:

$$y = (\lambda I - \widetilde{A})^{-1} y_0 = \int_0^{+\infty} e^{-\lambda s} \lim_{n\to\infty} MT(s)x_n \, ds$$

$$= \lim_{n\to\infty} \int_0^{+\infty} e^{-\lambda s} MT(s)x_n \, ds = \lim_{n\to\infty} M \int_0^{+\infty} e^{-\lambda s} T(s)x_n \, ds = \lim_{n\to\infty} M(\lambda I - A)^{-1}x_n.$$

Note that we have applied the Lebesgue theorem in the following passage:

$$\int_0^{+\infty} e^{-\lambda s} \lim_{n\to\infty} MT(s)x_n \, ds = \lim_{n\to\infty} \int_{0y}^{+\infty} e^{-\lambda s} MT(s)x_n \, ds.$$

This is possible because $\|MT(t)x_n\| \leq Ke^{\widehat{\omega}t}\|Mx_n\|$ and Mx_n is convergent. Moreover, by assumption we have that $\lambda > \widehat{\omega}$.

Note that $M(\lambda I - A)^{-1}x_n$ belongs to $M\mathscr{D}(A)$. To prove that $y \in \mathscr{D}(\overline{\widehat{A}})$, we need to show that also $\widehat{A}(M(\lambda I - A)^{-1}x_n)$ is convergent to some element in $\overline{\text{ran}(M)}$. To this aim, we can write $\widetilde{A}$ in the following way:

$$\widetilde{A}x = \lambda x - (\lambda I - \widetilde{A})x. \tag{2.27}$$

Then, by (2.27) we obtain:

$$\widehat{A}(M(\lambda I-A)^{-1}x_n)=\widetilde{A}(M(\lambda I-A)^{-1}x_n)=\lambda M(\lambda I-A)^{-1}x_n-(\lambda I-\widetilde{A})M(\lambda I-A)^{-1}x_n$$
$$=\lambda M(\lambda I-A)^{-1}x_n-M(\lambda I-A)(\lambda I-A)^{-1}x_n=\lambda M(\lambda I-A)^{-1}x_n-Mx_n.$$

From this it follows that:

$$\lim_{n\to\infty}\widehat{A}(M(\lambda I-A)^{-1}x_n)=\lambda y-y_0.$$

We have proved that $\mathscr{D}(\widetilde{A})\subset\mathscr{D}(\overline{\widehat{A}})$. Since $\widetilde{A}$ is closed, by definition of the closure of a linear operator it follows that $\mathscr{D}(\widetilde{A})=\mathscr{D}(\overline{\widehat{A}})$. □

Remark 2.6. Let us consider the following assumption on the generator A:

(jj) For every $x_n\subset\mathscr{D}(A)$, $\|Mx_n\|\to 0\Rightarrow\|MAx_n\|\to 0$.

Condition (jj) implies that a reduced operator $\widehat{A}$ can be constructed on the dense subspace $M\mathscr{D}(A)$ in such a way that $\widehat{A}M=MA$. But from (jj) it follows that there exists $k>0$ such that $\|MAx\|\le k\|Mx\|$ for all $x\in\mathscr{D}(A)$ (this fact can be proved in the same way as for bounded operators, see the proof of proposition 2.17).
This means that the reduced operator $\widehat{A}$ can be extended to a bounded operator on $\overline{\mathrm{ran}(M)}$. Condition (jj) is stronger than the hypotheses of theorem 2.15. Indeed, not every lumping leads to a bounded reduced operator. Note also that, $T(t)$ being the semigroup generated by A, condition (jj) cannot be obtained from the analogous condition (j) on $T(t)$, unless we assume stronger hypotheses, such as the boundedness of A.

2.4 Spectrum of the reduced operator

It is interesting to investigate whether the spectrum of the reduced operator $\widehat{A}$ is in some way related to the spectrum of the original operator A. This problem was already approached in [66] for finite dimensional operators: in this case, every eigenvalue of $\widehat{A}$ is also an eigenvalue of A. We generalize this result for operators in abstract Banach spaces. In this context the problem is non trivial, for the following fundamental fact:

An infinite dimensional closed subspace that is invariant under an invertible linear operator, is not necessarily invariant under the inverse operator.

This fact can't happen if the subspace is finite dimensional: indeed, let A be a linear operator such that $A\mathscr{C}\subset\mathscr{C}$, and $\mathscr{C}$ is finite dimensional. Since A is invertible, it preserves linear independence. This implies that the subspaces $A\mathscr{C}$ and $\mathscr{C}$ have the same dimension. But for finite dimensional subspaces, this means $A\mathscr{C}=\mathscr{C}$. Applying the inverse operator we obtain $A^{-1}\mathscr{C}=\mathscr{C}$.

We recall that the spectrum of a linear operator is defined in the following way:

$$\sigma(A):=\{\lambda\in\mathbb{C}\text{ such that }(\lambda I-A)\text{ does not have a bounded inverse}\},$$

and the resolvent set of A is $\rho(A)=\mathbb{C}\setminus\sigma(A)$. If A is a matrix, the spectrum is made up by its eigenvalues only, but if A is an operator on a general Banach space, the spectrum can be a larger set. Indeed, cases in which $(\lambda I-A)$ is injective but not surjective may occur.
We assume that $A(\ker(\mathrm{M})\cap\mathscr{D}(A))\subseteq\ker(\mathrm{M})$, i.e. $\widehat{A}$ is well-defined. In this section we will show that the relation between the spectra of $\sigma(A)$ and $\sigma(\widehat{A})$ depends on the shape of the subset $\rho(A)\subset\mathbb{C}$. As usual, M is a bounded, surjective operator from X to Y. We first show the following important fact:

Lemma 2.18. *Let us suppose that, for a given $\lambda \in \rho(A)$, the operator $(\lambda I - A)$ is surjective from* $\ker(\mathrm{M}) \cap \mathscr{D}(A)$ *to* $\ker(\mathrm{M})$. *Then, $\lambda \in \rho(\widehat{A})$.*
Conversely, if $\lambda \in \rho(A) \cap \rho(\widehat{A})$, then $(\lambda I - A)$ is surjective from $\ker(\mathrm{M}) \cap \mathscr{D}(A)$ *to* $\ker(\mathrm{M})$.

Proof. We prove that $\ker(\mathrm{M})$ is invariant under the resolvent operator $\mathscr{R}(\lambda) := (\lambda I - A)^{-1}$. Consider $x \in \ker(\mathrm{M})$. Since $(\lambda I - A)$ is surjective onto $\ker(\mathrm{M})$, we can write $x = (\lambda I - A)\widetilde{x}$, where $\widetilde{x} \in \ker(\mathrm{M})$. Then:

$$\mathscr{R}(\lambda)x = \mathscr{R}(\lambda)(\lambda I - A)\widetilde{x} = \widetilde{x} \ \in \ker(\mathrm{M}).$$

By the invariance of $\ker(\mathrm{M})$, the following operator from Y to $\mathscr{D}(A)$ is well-defined:

$$\widehat{\mathscr{R}(\lambda)}y := M\mathscr{R}(\lambda)x, \ y = Mx.$$

We know by our lumping analysis that $\widehat{\mathscr{R}(\lambda)}$ is bounded. Moreover, $\widehat{\mathscr{R}(\lambda)}$ is the inverse operator of $(\lambda I - \widehat{A})$, indeed, for $y = Mx$:

1. $(\lambda I - \widehat{A})\widehat{\mathscr{R}(\lambda)}y = (\lambda I - \widehat{A})M\mathscr{R}(\lambda)x = M(\lambda I - A)\mathscr{R}(\lambda)x = y;$
2. $\widehat{\mathscr{R}(\lambda)}(\lambda I - \widehat{A})y = M\mathscr{R}(\lambda)(\lambda I - A)x = Mx = y.$

This means that $\widehat{A} - \lambda I$ has a bounded inverse $\widehat{\mathscr{R}(\lambda)} = (\lambda I - \widehat{A})^{-1}$, i.e. $\lambda \in \rho(\widehat{A})$.
On the converse, take $\lambda \in \rho(A) \cap \rho(\widehat{A})$ and $x \in \ker(\mathrm{M})$. Then $x = (\lambda I - A)\widetilde{x}$ for a given $\widetilde{x} \in \mathscr{D}(A)$. We need to prove that $\widetilde{x} \in \ker(M)$. To this aim, we write:

$$0 = M(\lambda I - A)\widetilde{x} = (\lambda I - \widehat{A})M\widetilde{x}.$$

Since by assumption $(\lambda I - \widehat{A})$ is one to one, $M\widetilde{x}$ must be equal to 0, i.e. $\widetilde{x} \in \ker(M)$. □

Remark 2.7. We observe that, if $(\lambda I - A)^{-1}$ exists and maps $\ker(\mathrm{M})$ into $\ker(\mathrm{M})$, then, for every $x \in \ker(\mathrm{M})$, we can write:

$$(\lambda I - A)^{-1}x = \widetilde{x} \ \ \in \ker(\mathrm{M}).$$

Since $(\lambda I - A)$ is bijective from $\mathscr{D}(A)$ to X, we have $x = (\lambda I - A)\widehat{x}$ for some $\widehat{x} \in \mathscr{D}(A)$. Then:

$$\widetilde{x} = (\lambda I - A)^{-1}x = (\lambda I - A)^{-1}(\lambda I - A)\widehat{x} = \widehat{x} \ \ \in \ker(\mathrm{M}).$$

This means that $(\lambda I - A)$ is surjective from $\mathscr{D}(A) \cap \ker(\mathrm{M})$ to $\ker(\mathrm{M})$.

Let us define the following subset of the resolvent $\rho(A)$:

$$\rho_M(A) := \{\lambda \in \rho(A) \ : \ (\lambda I - A)^{-1}\ker(\mathrm{M}) \subseteq \ker(\mathrm{M})\}.$$

The Lemma above tells us that $\rho_M(A) \subset \rho(\widehat{A})$. However, in general $\rho_M(A)$ does not coincide with the whole resolvent set of A. More details about the invariance of a subspace for the resolvent operators can be found in [10], where the subset of $\rho(A)$ for which all the operators $(\lambda I - A)^{-1}$ map a given subspace $\mathscr{C}$ into itself is called the *rotationally invariant resolvent* with respect to $\mathscr{C}$.
Now, consider the case of a bounded operator A from X to itself. We know that for bounded operators $\sigma(A)$ is a compact subset of $\mathbb{C}$ and $\rho(A)$ is an open subset where the *resolvent function*

$$\rho(A) \ni s \to (sI - A)^{-1} \in \mathscr{B}(X)$$

is analytic. Moreover, for every $|\lambda| > \|A\|$, $\lambda \in \rho(A)$ and the *Neumann series expansion* holds:

$$\mathscr{R}(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n$$

(see, for instance, [1]).
Using this series expansion we can see that for these values of λ, when ker(M) is A-invariant, ker(M) is also invariant under $\mathscr{R}(\lambda)$, i.e. $\{\lambda : |\lambda| > \|A\|\} \subset \rho_M(A)$.
We recall that the resolvent function is analytic even if A is unbounded, provided that $\rho(A)$ is non-empty. For instance, if A generates a C_0-semigroup then $\rho(A)$ always contains an interval $[r, +\infty)$, $r \in \mathbb{R}$. In this case, if the system generated by A is lumpable through M, i.e. Theorem 2.15 holds, then by assumption for all $\lambda > \omega$ (being ω the growth bound of the semigroup $T(t)$ generated by A), $(\lambda I - A)$ is surjective from $\mathscr{D}(A) \cap \ker(\mathrm{M})$ to $\ker(\mathrm{M})$, i.e. $\lambda \in \rho_M(A)$ (see the proof of lemma 2.18).
This means that in both the case of bounded and unbounded operators, if the generated dynamics is lumpable by M, then there exists a connected components of $\rho(A)$ which contains an interval of the kind $[r, +\infty)$ and belongs to $\rho_M(A)$. Indeed, this connected component does not depend on M directly.
Let us denote with $\rho_\infty(A)$ the largest connected component of $\rho(A)$ containing an interval of the kind $[r, +\infty)$. We know that if A is the infinitesimal generator of a C_0-semigroup then $\rho_\infty(A) \neq \emptyset$.
We prove that $\rho_M(A) = \rho_\infty(A)$, following the idea given in [23, Lemma 2.5.6] for a general closed subspace of an Hilbert space.
Let A be the infinitesimal generator of a C_0-semigroup such that the generated dynamics is lumpable by M.
We have pointed out that there exists $\lambda_0 \in \rho_\infty(A) \cap \rho_M(A)$. By assumption, the resolvent function $s \to (sI - A)^{-1}$ is analytic in $\rho_\infty(A)$.
We recall that the *annihilator* of ker(M) is defined as the following subspace of X^* (we refer to the next chapter for an introduction on dual spaces and annihilators):

$$\ker(\mathrm{M})^\perp := \{f \in X^* \,:\, f(m) = 0 \,\forall m \in \ker(\mathrm{M})\}.$$

For fixed $m \in \ker(\mathrm{M})$ and $f \in \ker(\mathrm{M})^\perp$, we define the following map:

$$G(s) := f((sI - A)^{-1}m),$$

which is an holomorphic function from $\rho(A)$ to $\mathbb{C}$. It is known that for $|\lambda - \lambda_0|$ small enough (to be precise, $|\lambda - \lambda_0| < \frac{1}{\|\mathscr{R}(\lambda_0)\|}$), we have

$$\mathscr{R}(\lambda) = \sum_{n=0}^{\infty}\mathscr{R}(\lambda_0)^{n+1}(\lambda - \lambda_0)^n$$

(see for instance [57, Chapter 4,5] for details on the resolvent operator and its formalism).
In particular, this shows that all the derivatives of G vanish in the point λ_0. Using the properties of the holomorphic functions on the complex plane, we obtain that G vanishes in a neighborhood of λ_0. But $\rho_\infty(A)$ is a connected component of $\rho(A)$, and this implies that G must be identically zero on $\rho_\infty(A)$ (see [34, Thm III.3.2]).
Since f is arbitrary, we conclude that every functional in $\ker(\mathrm{M})^\perp$ vanishes on $(sI - A)^{-1}m$. As a consequence of the Hahn-Banach Theorem, $(sI - A)^{-1}m \in \ker(\mathrm{M})$ for all $s \in \rho_\infty(A)$. Since $m \in \ker(M)$ is also arbitrary, we conclude that $(sI - A)^{-1}\ker(\mathrm{M}) \subseteq \ker(\mathrm{M})$ for all $s \in \rho_\infty(A)$.
Combining this conclusion with Lemma 2.18, we arrive to the following result, which is an equivalent formulation of Theorem 2.15:

Proposition 2.19. *Let A be the generator of a strongly continuous semigroup on X. The dynamical system associated with A is lumpable by M if and only if $A(\ker(M) \cap \mathscr{D}(A)) \subset \ker(M)$ and $\rho_\infty(A) \subseteq \rho(\widehat{A})$.*

If A is not a generator, the resolvent set could be empty, and even if it is not empty, we don't know a priori if a complex number $\lambda \in \rho_M(A)$ exists. Using the analyticity of the resolvent function, we can say that, if $\lambda \in \rho_M(A)$ exists, then the maximal connected component of the resolvent set containing λ is entirely contained into $\rho_M(A)$.
Note that in some cases $\rho_\infty(A) = \rho(A)$. For instance, this happens when the spectrum of A is discrete and $\rho_\infty(A) \neq \emptyset$, or more generally when A is a generator and $\rho(A)$ is connected in $\mathbb{C}$. In theses cases, we can conclude that $\sigma(\widehat{A}) \subseteq \sigma(A)$. In general, we can write:

$$\sigma(\widehat{A}) \subseteq \mathbb{C} \setminus \rho_\infty(A).$$

In the previous section of the present chapter several examples of lumping are presented. It is easy to verify that, in those examples, the inclusion $\sigma(\widehat{A}) \subseteq \sigma(A)$ holds by construction. However, we have proved that in general this is not the case. For this reason, here we give an example in which $\sigma(\widehat{A})$ is larger than $\sigma(A)$, and $\rho(A)$ is indeed disconnected.

Example 2.7. We consider the Banach space $C_0(\mathbb{R})$ with the supremum norm and the left translations semigroup:

$$T(t)f(x) := f(x+t), \quad x \in \mathbb{R},\, t \geq 0.$$

We have already mentioned in Section 3.1 that $T(t)$ is generated by the derivative operator $Af(x) = f'(x)$ defined on:

$$\mathscr{D}(A) := \{f \in C_0^1(\mathbb{R}) \,:\, f' \in C_0(\mathbb{R})\}.$$

It has been shown (see [2, A-III,2.4]) that the spectrum of A is the imaginary axis $\sigma(A) = i\mathbb{R}$, i being the imaginary unit. Indeed, for every $\lambda = i\alpha$, $\alpha \in \mathbb{R}$, there exists a sequence $f_n(x) := e^{-\frac{|x|}{n}} e^{i\alpha x}$ such that $\|f_n\| = 1$ and $\lim_{n\to+\infty} \|Af_n - \lambda f_n\| = 0$. A sequence of this kind is called an *approximated eigenvector* and its existence tell us that $(\lambda I - A)$ is not bounded below, i.e. it is not invertible.
It follows that $\rho(A)$ is a disconnected subset of the complex plane.
We consider the following lumping operator:

$$M : C_0(\mathbb{R}) \to C_0(\mathbb{R}^+), \quad Mf := f_{|_{\mathbb{R}^+}},$$

which acts as the restriction operator to $\mathbb{R}^+$. This operator is linear, bounded and surjective by the Tietze extension Theorem (see Chapter 4 of the present thesis for a concrete construction of a lumping operator acting as a restriction operator).
For this choice of M, ker(M) is the ideal of functions vanishing on $\mathbb{R}^+$ and it is invariant under A. If $f \in \mathscr{D}(A)$, it is clear that $MAf = f'_{|_{\mathbb{R}^+}} = (f_{|_{\mathbb{R}^+}})'$.
This means that the reduced operator $\widehat{A}$ is again a derivative generating the left translations semigroup on $C_0(\mathbb{R}^+)$:

$$\widehat{T}(t)g(s) = g(s+t), \;\; s \in \mathbb{R}^+,\, t \geq 0,\, g \in C_0(\mathbb{R}^+).$$

It is known that the spectrum of $\widehat{A}$ is

$$\sigma(\widehat{A}) = \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda) \leq 0\}.$$

Indeed, the functions $e^{\lambda x}$ are eigenvectors for $\mathrm{Re}(\lambda) < 0$, and $f_n(x) := e^{-\frac{x}{n}} e^{i\alpha x}$ is an approximated eigenvector for $\mathrm{Re}(\lambda) = 0$ ([2]). In this case $\sigma(\widehat{A})$ is larger than the spectrum

of the original operator A.
Note that the growth bound of the semigroup $T(t)$ is $\omega = 0$ (indeed, $T(t)$ is a contractions semigroup). In this case $\sup_{\lambda\in\sigma(A)}\{\mathrm{Re}(\lambda)\} = \omega(T) = 0$. The larger connected component of $\rho(A)$ containing an interval $[r, +\infty)$ is

$$\rho_\infty(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}.$$

We can see that for all $\lambda \in \rho_\infty(A)$, $(\lambda I - \widehat{A})$ is invertible, according to the statement of Proposition 2.19.

Chapter 3

Dual conditions for lumpability of linear systems

3.1 Introduction

In this chapter we analyze the lumpability problem from a dual perspective. As a motivation, let us look at the case of euclidean spaces: Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^k$ with $k < n$. Transposing both sides of the lumping relation $MA = \widehat{A}M$ yields

$$MA = \widehat{A}M \Leftrightarrow A^T M^T = M^T \widehat{A}^T.$$

Moreover,

$$\ker(M) \subseteq \ker(MA) \Leftrightarrow \operatorname{ran}(A^T M^T) \subseteq \operatorname{ran}(M^T).$$

Since the matrix $\widehat{A}$ exists if and only if $\ker(M)$ is A-invariant [21,44], an equivalent condition for lumpability is the invariance of $\operatorname{ran}(M^T)$ under A^T.

The dual point of view on lumping has been widely used for finite-dimensional systems and Markov chains. Li and Rabitz have studied lumping in application to reaction systems with n species, described by a first order, ordinary differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. In [48] they show that a nonlinear reaction system is lumpable by a reduction matrix M only if the subspace spanned by the row vectors of M is invariant under $J^T x$ at any point, where J^T is the transpose of the Jacobian matrix of the function f. It is clear that, if the system is linear, this condition becomes also sufficient.
As we mentioned in the second chapter of the present thesis, Gurvits and Ledoux analyzed aggregations of Markov chains given by equivalence relations. In [38] they consider a discrete-time homogeneous Markov chain (X_n), with transition matrix P, such that every random variable has values in $\mathscr{X} := \{1, \dots, N\}$, and they obtain a partition of $\mathscr{X}$ in $M < N$ classes $C(1), \dots, C(M)$. They define a *lumping map* $\phi : \mathscr{X} \to \mathscr{Y}$, $\mathscr{Y} := \{1, \dots, M\}$, such that $\phi(k) := l \Leftrightarrow k \in C(l), \qquad \forall k \in \mathscr{X}, l \in \mathscr{Y}$.
The lumped process is then defined by $\phi(X_n) = l \Leftrightarrow X_n \in C(l)$, and it is in general not markovian. They distinguish between *weak lumpability*, when the reduced process is markovian for some initial distributions of (X_n), and *strong lumpability*, when the reduced process is markovian for every initial distributions of (X_n). In particular, they prove that (X_n) is strongly lumpable through the map ϕ if and only if $P(\ker(V_\phi)) \subset \ker(V_\phi)$ (see [38] and section 1.1 of the present thesis for the definition of V_ϕ).
To approach lumpability from a dual space point of view, they define the following inner product in $\mathbb{R}^n$:

$$< x, y >_v := x^T \operatorname{diag}(v)^{-1} y,$$

where v is a fixed positive stochastic vector and $\mathrm{diag}(v)$ is the diagonal matrix with entries v_i. With respect to this inner product, the adjoint of the transition matrix P is

$$P^* := \mathrm{diag}(v)P^T\mathrm{diag}(v)^{-1}.$$

They define the linear subspace $\mathscr{V} := \mathrm{span}\{v^{(y)},\, y \in \mathscr{Y}\}$, where $v^{(y)} := \frac{\Pi_y v}{1^T \Pi_y v}$ and Π_y are the *lumping projectors* (see section 1.1). Using $\mathscr{V} = (\ker(V_\phi))^*$, they show that $\mathscr{V}$ is P-invariant if and only if $\ker(V_\phi)$ is P^*-invariant and, if P is normal, $P(\ker(V_\phi)) \subset \ker(V_\phi) \Leftrightarrow P\mathscr{V} \subset \mathscr{V}$. They use this dual result to prove that weak and strong lumpability coincide when (X_n) has a normal, irreducible transition matrix.
A dual approach has been also considered in infinite dimensional Hilbert spaces. Using the pseudoinverse operator, Rózsa and Tóth have proved that a necessary and sufficient condition for lumpability is the invariance of $\ker(M)^\perp$ under the bounded operator A^*.

Our aim is to generalize these results to abstract Banach spaces, for both bounded and unbounded operators, without requiring the existence of a pseudoinverse for M. In particular, we establish a dual condition for lumpability in terms of the adjoint of the infinitesimal generator A.
The problem of dual lumpability in Banach space is non-trivial, because in general the adjoint of a strongly continuous semigroup $T(t)$ is not strongly continuous on the whole X^*, but only on a proper, closed subspace called the *sun dual space* and denoted with $X^\odot$ (note that this subspace depends on the semigroup $T(t)$, even if it is not explicitly written in the notation). The restriction of an adjoint semigroup $T^*(t)$ on the sun dual space is known as the *sun dual semigroup* $T^\odot(t)$. Furthermore, the adjoint operator A^* is a generator only in the sense of the weak* topology. As we will discuss in the present chapter, many properties of strongly continuous semigroups hold also for the adjoint semigroup where the same limits are considered in the weak* topology.
Furthermore, we suppose that the lumping relation occurs between two semigroups $T(t)$ and $\widehat{T}(t)$ through a reduction operator M. We show some interesting consequences of the lumping relation on the sun dual spaces $X^\odot$ and $Y^\odot$ of X and Y respectively, Y being the reduced state space. In particular, we prove that the restriction of M^* on the sun dual space $Y^\odot$, defined by $M^\odot := M^*_{Y^\odot}$, is injective and has closed range in X^*. Moreover, $\mathrm{ran}(M^\odot)$ is an invariant subspace for the sun dual semigroup $T^\odot(t)$. Finally, we construct a particular lumping for the semigroup $T(t)$ on X starting from an operator with closed range contained in $X^\odot$ and $T^\odot(t)$-invariant.
The last part of this chapter is dedicated to a generalization of our dual condition to nonlinear systems, by the associated *Koopman operators*. Given a nonlinear map $f : K \to L$, with K, L compact Hausdorff spaces, we study a kind of "adjoint map" for f, defined on $C(L)$. Instead of looking at $F \circ f$ with F linear functional (as in the case of the adjoint of a linear operator), we look at $h \circ f$, with $h \in C(L)$. The Koopman operator associated with f acts as the composition operator $h \to h \circ f$ and it is concerned with the evolution of the observables of a given system. We will show that, in the context of continuous functions between compact Hausdorff spaces, the lumpability of a system by a surjective lumping map m is equivalent to a linear condition on the associated Koopman operators. This represents a connection between linear lumping and lumpability of nonlinear systems, which will be the topic of Chapter 4.
We recall that throughout this work we use the term lumping with the meaning of *exact lumping* (i.e. the lumping relation holds without any error).

Before going into details of lumping analysis, we give some general definitions about duality in Banach spaces theory and adjoint operators (we refer especially to [55] and [36]).

3.2 Background in adjoint operators and semigroups

Given a Banach space X, its dual space X^*, i.e., the set of the linear and bounded functionals from X to $\mathbb{C}$, is also a Banach space that can be endowed with different notions of convergence:

1. Strong (norm) topology: $x_n^* \to x^*$ if and only if $||x_n^* - x^*|| := \sup_{||x|| \le 1} |x_n^*(x) - x^*(x)| \to 0$.
2. Weak topology: $x_n^* \to x^*$ if and only if $|x^{**}(x_n^* - x^*)| \to 0$ for every x^{**} in the double dual X^{**}.
3. Weak* topology: $x_n^* \to x^*$ if and only if $|x_n^*(x) - x^*(x)| \to 0$ for every $x \in X$.

The norm topology is the strongest one, and the weak* topology is the weakest among the three because in general $X \subset X^{**}$. The *canonical inclusion* j in the double dual X^{**} is the linear isometry defined by

$$j : X \to X^{**}, \quad j(x)(x^*) := x^*(x), \quad \forall x^* \in X^*, \, x \in X. \tag{3.1}$$

If $j(X) = X^{**}$, then X is called a *reflexive* space, in which case the weak and weak* topologies on its dual space coincide.

For two subspaces $\mathscr{C}$ and $\mathscr{S}$ of X and X^*, respectively, we denote the annihilators

$$\mathscr{C}^{\perp} = \{x^* \in X^* : x^*(x) = 0 \ \forall x \in \mathscr{C}\}, \quad \mathscr{S}^{\perp} = \{x \in X : x^*(x) = 0 \ \forall x^* \in \mathscr{S}\}.$$

If $\mathscr{C}$ is closed then $\mathscr{C} = \mathscr{C}^{\perp\perp}$, while $\mathscr{S}^{\perp\perp}$ coincides with the weak* closure of $\mathscr{S}$.

Let A be a linear operator between two Banach spaces X and Y, whose domain $\mathscr{D}(A)$ is dense in X. The adjoint operator $A^* : Y^* \to X^*$ is defined by $A^*(y^*)(x) = y^*(Ax)$ on the domain

$$\mathscr{D}(A^*) = \{y^* \in Y^* \text{ such that the composition } y^*A \text{ is continuous on } \mathscr{D}(A)\}.$$

The adjoint of a linear and densely-defined operator is a closed operator, but if X is not reflexive it may be not densely defined. If A is bounded, then so is its adjoint, $\mathscr{D}(A^*) = Y^*$, and $||A^*|| = ||A||$.

Proposition 3.1 ([14, 36]). *For a linear operator A between Banach spaces the following hold:*

1. $\mathrm{ran}(A)^{\perp} = \ker(A^*)$,
2. $\overline{\mathrm{ran}(A)} = \ker(A^*)^{\perp}$,
3. $\ker(A) \subset \mathrm{ran}(A^*)^{\perp}$,
4. *if A is bounded, then* $\ker(A) = \mathrm{ran}(A^*)^{\perp}$,
5. $\overline{\mathrm{ran}(A^*)} \subseteq \ker(A)^{\perp}$,
6. $\mathrm{ran}(A^*) = \ker(A)^{\perp} \Leftrightarrow \mathrm{ran}(A)$ *is closed,*
7. $\mathrm{ran}(A)$ *is closed* $\Leftrightarrow \mathrm{ran}(A^*)$ *is weak* closed.*

The following theorem, which can be found, e.g., in [36], is fundamental for the lumpability problem since in general we will deal with surjective maps as lumping operators.

Theorem 3.2. *For a closed, linear map $A : X \to Y$ between Banach spaces, the following hold:*

1. $\operatorname{ran}(A) = Y$ *if and only if* A^* *has a bounded inverse,*

2. $\operatorname{ran}(A^*) = X^*$ *if and only if* A *has a bounded inverse.*

Let T be a strongly continuous semigroup on X generated by an operator A. Consider the family of the adjoint operators $T^*(t) : X^* \to X^*$, $t \geq 0$. This family is again a semigroup of bounded operators on X^* and is a continuous semigroup with respect to the weak* topology: in fact, using the strong continuity of $T(t)\}_{t\geq 0}$, for every $x^* \in X^*$ and $x \in X$,

$$\begin{aligned}\lim_{h\to 0^+} |T^*(h)x^*(x) - x^*(x)| &= \lim_{h\to 0^+} |x^*(T(h)x) - x^*(x)| \\ &\leq \lim_{h\to 0^+} ||T(h)x - x|| \cdot ||x^*|| = 0.\end{aligned}$$

The adjoint semigroup is generated exactly by the operator A^*, which is closed and densely defined with respect to the weak* topology, and is given by

$$A^*x^* = \text{weak*-}\lim_{h\to 0+}\left(\frac{T^*(h)x^* - x^*}{h}\right).$$

However, the semigroup $\{T^*\}_{t\geq 0}$ may fail to be strongly continuous.

Example 3.1. Consider the C^0 semigroup of left translations on $L^1(\mathbb{R})$:

$$T(t)f(s) = f(s+t), \quad t \geq 0, \tag{3.2}$$

generated by the operator $Af(s) = f'(s)$ with domain

$$\mathscr{D}(A) = \{f \in L^1(\mathbb{R}) \text{ absolutely continuous such that } f' \in L^1(\mathbb{R})\}.$$

It is well known that by the Riesz representation theorem, $L^1(\mathbb{R})^*$ can be identified with $L^\infty(\mathbb{R})$. The adjoint semigroup acts as

$$T^*(t)\phi(s) = \phi(s-t), \quad \phi \in L^\infty(\mathbb{R}),\, t \geq 0. \tag{3.3}$$

It is generated (in the weak* sense) by the adjoint operator $A^*\phi := -\phi'$, with domain

$$\mathscr{D}(A^*) = \{\phi \in L^\infty(\mathbb{R}) \text{ absolutely continuous such that } \phi' \in L^\infty(\mathbb{R})\}$$

(note that this operator is not densely defined in $L^\infty(\mathbb{R})$). The adjoint semigroup (3.3) fails to be strongly continuous. This can be seen by considering the characteristic function $\chi_{[0,+\infty)}$ and observing that, for $t > 0$,

$$||T^*(t)\chi_{[0,+\infty)}(s) - \chi_{[0,+\infty)}(s)||_\infty = ||\chi_{[t,+\infty)}(s) - \chi_{[0,+\infty)}(s)||_\infty = 1.$$

Even when the adjoint of a C^0 semigroup is not strongly continuous, one can find a closed subspace of the dual space in which strong continuity holds.

Definition 3.1. The *sun dual* of X is the closed subspace $X^\odot \subset X^*$ defined by

$$X^\odot = \{x^* \in X^* \text{ such that } \lim_{h\to 0+} ||T^*(h)x^* - x^*|| = 0\}. \tag{3.4}$$

Note that, by definition, the sun dual space depends on the semigroup $T(t)$, even if, for neatness of the notation $X^\odot$, this dependence is never explicitly written in the literature. The *sun dual semigroup* of $\{T(t)\}_{t\geq 0}$ is the strongly continuous semigroup obtained by restricting the adjoint semigroup to the sun dual space,

$$T^\odot(t)x^* := T^*(t)x^*, \quad x^* \in X^\odot,\, t \geq 0. \tag{3.5}$$

We denote the generator of the sun dual semigroup by $A^{\odot}$. It is the restriction of the adjoint operator A^* to the domain

$$\mathscr{D}(A^{\odot}) = \{x^* \in \mathscr{D}(A^*) : A^*x^* \in X^{\odot}\}.$$

It is known that A^* is the weak* closure of $A^{\odot}$ and $\overline{\mathscr{D}(A^*)} = X^{\odot}$ [41]. As an example of a sun dual space we mention that, for the semigroup of left translations (3.2) on $X = L^1(\mathbb{R})$, $X^{\odot}$ is the space $C_{\mathrm{ub}}(\mathbb{R})$ of bounded and uniformly continuous functions on the real line [55].

One can iterate the construction of the sun dual space and define the double sun dual $X^{\odot\odot}$ as the closed subspace of $X^{\odot *}$ on which the adjoint semigroup $T^{\odot *}(t)$ is strongly continuous. We call X *sun-reflexive* if X is isomorphic to $X^{\odot\odot}$.

Finally, let us recall that there are some cases in which the passage to the adjoint semigroup preserves strong continuity. This always happens when X is a reflexive space: Because in this case the weak and the weak* topologies on the dual space coincide, the adjoint semigroup is weakly continuous and thus strongly continuous [41]. Similarly, if the semigroup is uniformly continuous, then its adjoint will be strongly continuous, because

$$\lim_{h\to 0+} ||T^*(h)x^* - x^*|| \leq \lim_{h\to 0+} \sup_{||x||\leq 1} ||T(h)x - x|| \cdot ||x^*|| = 0.$$

3.3 Dual conditions for lumpability of linear systems

We are ready to generalize the dual conditions for lumpability to abstract Banach spaces. We first consider the case of systems generated by bounded operators, and then we analyze the case of a possibly unbounded generator.

3.3.1 Dual lumpability for bounded operators

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0. \end{cases} \tag{3.6}$$

generated by a bounded operator $A \in \mathscr{B}(X)$. We have seen in Theorem 2.7 that a lumping of this system through a bounded, surjective map $M : X \to Y$ can be obtained if and only if $\ker(M)$ is invariant under A. Similar to the finite-dimensional case, we give an equivalent condition for lumpability in terms of adjoint operators.

Proposition 3.3. *Consider system (3.6) with $A \in \mathscr{B}(X)$ and a surjective map $M \in \mathscr{B}(X, Y)$. Then the following statements are equivalent.*

1. *There exists $\widehat{A} \in \mathscr{B}(Y)$ such that $MA = \widehat{A}M$ for all $x \in X$, so that system (3.6) is lumpable by the operator M.*

2. $\mathrm{ran}(M^*)$ *is invariant under A^*.*

Proof. $1 \Rightarrow 2$. By the properties of the adjoint of a bounded operator, we have the implication $(MA = \widehat{A}M) \Rightarrow (A^*M^* = M^*\widehat{A}^*)$. Given $x^* = M^*y^*$, we obtain $A^*x^* = A^*M^*y^* = M^*\widehat{A}^*y^* \in \mathrm{ran}(M^*)$; i.e., statement 2 holds.

$2 \Rightarrow 1$. Note that statement 2 is equivalent to $\mathrm{ran}(A^*M^*) \subseteq \mathrm{ran}(M^*)$. Thus,

$$\begin{aligned} \mathrm{ran}(A^*M^*) \subseteq \mathrm{ran}(M^*) &\Rightarrow \mathrm{ran}(M^*)^{\perp} \subseteq \mathrm{ran}(A^*M^*)^{\perp} \\ &\Rightarrow \ker(M) \subseteq \ker(MA), \end{aligned}$$

which is the condition for lumpability. $\square$

Example 3.2. Consider a lumping as in Observation 2.1. By definition, the adjoint of the quotient projection is

$$\pi^* : \left(\frac{X}{\mathscr{C}}\right)^* \to X^*, \qquad \pi^*\phi(x) := \phi([x]).$$

It is known that we can identify the range of π^* with the annihilator $\mathscr{C}^\perp$ (see [57]), and the map π^* with the inclusion of $\mathscr{C}^\perp$ in X^*. The annihilator $\mathscr{C}^\perp$ is invariant under A^*. This fact can be seen by taking $\phi \in \mathscr{C}^\perp$, applying A^*, and using the invariance of $\mathscr{C}$ to obtain $A^*\phi(x) = \phi(Ax) = 0$, $\forall x \in \mathscr{C}$. The reduction of A to $\widehat{A}$ through π can be identified with the restriction of A^* to the closed subspace $\mathscr{C}^\perp$. Indeed, we can write $\widehat{A}^* = (\pi^*)^{-1}A^*\pi^*$, and this operator acts on $\phi \in \mathscr{C}^\perp$ as $A^*\phi$. This means that a lumping through a quotient projection becomes a restriction from the point of view of the dual operators.

3.3.2 Dual lumpability for unbounded operators

We want to analyze the dual conditions for lumpability in the most general case of a dynamics generated by an unbounded operator A, namely:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0, \qquad x_0 \in X, \end{cases} \tag{3.7}$$

where $A : \mathscr{D}(A) \subset X \to X$.
Since the family $\{T(t)\}_{t\geq 0}$ is made up by bounded operators, we can obtain the following dual condition for lumpability:

Proposition 3.4. *The following statements are equivalent:*

1. *There exists an operator $\widehat{A}$ defined on $M(\mathscr{D}(A))$ such that $\widehat{A}$ generates a strongly continuous semigroup on Y and $\widehat{A}M = MA$ (i.e. the system is lumpable by the operator M);*

2. $\operatorname{ran}(M^*)$ *is invariant under $T^*(t)$ for every $t \geq 0$.*

Proof. 1 $\Rightarrow$ 2. Let $\widehat{T}(t)$ be the strongly continuous semigroup generated by $\widehat{A}$. We have shown (Proof of Theorem 2.14) that $\widehat{T}$ satisfies the lumping relation $\widehat{T}(t)Mx = MT(t)x$, $x \in X$. This implies that the kernel of M is $T(t)$-invariant. Statement 2 then follows through the following implications (considering that the surjectivity of M implies the range of its adjoint is star-weakly closed):

$$\ker(M) \subseteq \ker(MT(t)) \Rightarrow (\ker(MT(t)))^\perp \subseteq \ker(M)^\perp$$

$$\Rightarrow \operatorname{ran}(T(t)^*M^*) \subseteq \ker(MT(t))^\perp \subseteq \ker(M)^\perp = \operatorname{ran}(M^*).$$

2 $\Rightarrow$ 1. From the invariance of $\operatorname{ran}(M^*)$ under $T^*(t)$ we can write

$$\operatorname{ran}(T(t)^*M^*) \subseteq \operatorname{ran}(M^*) \Rightarrow \ker(M) \subseteq \ker(MT(t)),$$

which is the necessary and sufficient condition for lumpability. □

We now establish a dual condition for lumpability in terms of the adjoint of the generator. To do this, we recall that the adjoint operator A^* is a generator only in the sense of the weak* topology. Fortunately, many properties of strongly continuous semigroups hold also for the adjoint semigroup where the same limits are considered in the weak* topology; in

fact it is easy to verify that (see [55] for more details)
1) for every $x^* \in X^*$, $t > 0$,

$$T^*(t)x^* := \text{weak*-} \lim_{k\to\infty} \left[\frac{k}{t} \mathscr{R} \left(\frac{k}{t}; A^* \right) \right]^k x^*,$$

where $\mathscr{R}(\lambda; A^*)$, $\lambda \in \rho(A^*)$, is the resolvent operator of A^*, and
2) $\mathscr{R}(\lambda; A^*) = \text{weak*-} \int_0^\infty e^{-\lambda s} T^*(s)x^* ds$, where the right side is the *weak* integral*, defined as the unique element such that for every $x \in X$

$$\int_0^\infty e^{-\lambda s} T^*(s)x^* ds\ (x) = \lim_{k\to\infty} \int_0^k e^{-\lambda s} T^*(s)x^*(x) ds.$$

Since the range of M^* is weak* closed, by the above results it is easy to verify that $\operatorname{ran}(M^*)$ is invariant under the adjoint semigroup $T^*(t)$ if and only if it is invariant under the resolvent operators $\mathscr{R}(\lambda; A^*)$ for all $\lambda > \omega(T)$. Moreover, A being closed and densely defined, we have $\mathscr{R}(\lambda; A^*) = \mathscr{R}(\lambda; A)^*$, and

$$\operatorname{ran}(\mathscr{R}(\lambda; A^*) M^*) \subseteq \operatorname{ran}(M^*) \Leftrightarrow \ker(M) \subseteq \ker(M\mathscr{R}(\lambda; A)).$$

These facts allow us to write the dual condition of (2.15).

Proposition 3.5. *System (3.7) is lumpable by the bounded, surjective, linear map M if and only if both the following conditions hold:*

1. $A^* (\operatorname{ran}(M^*) \cap \mathscr{D}(A^*)) \subset \operatorname{ran}(M^*)$,
2. *There exists $\lambda > \omega$ such that $(\lambda I - A^*)$ is surjective from $\operatorname{ran}(M^*) \cap \mathscr{D}(A^*)$ to $\operatorname{ran}(M^*)$.*

Remark 3.1. Let us suppose that the lumping operator M is bounded but it is not surjective. Applying theorem 2.6 to strongly continuous semigroups, we obtain the following:

Proposition 3.6. *Given a strongly continuous semigroup $T(t)$ generated by A, there exists another strongly continuous semigroup $\widehat{T}(t)$ on $\overline{\operatorname{ran}(M)}$ such that $MT(t) = \widehat{T}(t)M$ if and only if $\operatorname{ran}(T^*(t)M^*) \subset \operatorname{ran}(M)^*$ for all $t \geq 0$.*

Applying proposition 2.17 we can show that $\widehat{T}(t)$ is generated by the closure $\overline{\widehat{A}}$, where $\widehat{A}$ is the following operator:

$$\widehat{A}y = MAx, \quad y = Mx \in M\mathscr{D}(A).$$

Note that $\operatorname{ran}(T^*(t)M^*) \subset \operatorname{ran}(M)^*$ does not imply $\ker(M) \subset \ker(MT(t))$, unless $\operatorname{ran}(M)$ is closed. This means that proposition 3.6 does not generalize proposition 3.4, but it rather gives a different version of dual lumpability.

3.4 Lumpability and Sun Dual Semigroups

In this section we investigate how the adjoint of the lumping operator acts on the sun dual semigroup of $T(t)$.
Assume we have a lumping by M, so that $\widehat{T}(t)Mx = MT(t)x$, $x \in X$, and $M^*\widehat{T}^*(t)y^* = T^*(t)M^*y^*$, $y^* \in Y^*$. Let $Y^\odot \subset Y^*$ be the sun dual space of Y with respect to $\widehat{T}(t)$. We define the following restriction:

$$M^\odot := M^*_{Y^\odot} : Y^\odot \to X^*,$$

which is bounded and injective. If $y^\odot \in Y^\odot$, we have

$$\begin{aligned}\lim_{h\to 0^+} ||T^*(h)M^*y^\odot - M^*y^\odot|| &= \lim_{h\to 0^+} ||M^*\widehat{T}^*(h)y^\odot - M^*y^\odot|| \\ &\leq \lim_{h\to 0^+} ||\widehat{T}^*(h)y^\odot - y^\odot|| \cdot \|M^*\| = 0,\end{aligned}$$

so that $M^\odot$ maps $Y^\odot$ in $X^\odot$. Even though $M^\odot$ is not necessarily surjective, its range is a closed subspace of $X^\odot$. To see this we use the boundedness of $(M^*)^{-1}$ from $\mathrm{ran}(M^*)$ to Y^*. Let $M^\odot y_n^\odot$ be a sequence in $\mathrm{ran}(M^\odot)$ such that $M^\odot y_n^\odot$ converges to x^* in X^*. Since both $X^\odot$ and $\mathrm{ran}(M^*)$ are closed subspaces, $x^* = x^\odot = M^*y^*$ belongs to $X^\odot \cap \mathrm{ran}(M^*)$. Furthermore,

$$\begin{aligned}\lim_{h\to 0^+} ||\widehat{T}^*(h)y^* - y^*|| &= \lim_{h\to 0^+} ||\widehat{T}^*(h)(M^*)^{-1}x^\odot - (M^*)^{-1}x^\odot|| \\ &= \lim_{h\to 0^+} ||(M^*)^{-1}T^*(h)x^\odot - (M^*)^{-1}x^\odot|| \\ &\leq \lim_{h\to 0^+} ||T^*(h)x^\odot - x^\odot|| \cdot ||(M^*)^{-1}|| = 0,\end{aligned}$$

implying that $y^* \in Y^\odot$. Thus $x^\odot = M^\odot y^* \in \mathrm{ran}(M^\odot)$. Note that $\mathrm{ran}(M^\odot)$ is closed in the norm topology but not in the weak* topology, since $X^\odot$ is not weak* closed (if it were, we would have $X^\odot = X^*$). Thus,

$$M^\odot \widehat{T}^\odot(t) = T^\odot(t)M^\odot;$$

therefore, $\mathrm{ran}(M^\odot)$ is invariant under $T^\odot(t)$.

Example 3.3. Let us consider again the semigroup $T(t)$ of left translations (3.2) on $L^1(\mathbb{R})$. The adjoint semigroup $T^*(t)$ is the semigroup of right translations on $L^\infty(\mathbb{R})$:

$$T^*(t)f(s) := f(s-t), \quad t \geq 0,\ f \in L^\infty(\mathbb{R}).$$

We have already mentioned that the sun dual semigroup of $T(t)$ is the space $C_{\mathrm{ub}}(\mathbb{R})$ of bounded and uniformly continuous functions on the real line [55]. We consider the following lumping operator:

$$Mf := f_{|_{\mathbb{R}^+}}, \ \ M : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}^+),$$

mapping f to its restriction on $\mathbb{R}^+$.
It is easy to verify that the reduced semigroup $\widehat{T}(t)$ satisfying the lumping relation $\widehat{T}(t)M = MT(t)$ is the semigroup of left translations on $L^1(\mathbb{R}^+)$:

$$\widehat{T}(t)g(s) := g(s+t), \quad t \geq 0,\ g \in L^1(\mathbb{R}^+),$$

and the adjoint semigroup $\widehat{T}^*$ acts as follows on $L^\infty(\mathbb{R}^+)$:

$$\widehat{T}^*(t)\phi(s) := \begin{cases} 0 & s < t \\ \phi(s-t) & s \geq t \end{cases}.$$

Similarly to the case of $T(t)$, it is possible to show that the sun dual space of $\widehat{T}(t)$, namely $Y^\odot$, is the following subspace of $L^\infty(\mathbb{R}^+)$:

$$Y^\odot := \{\phi \in C_{\mathrm{ub}}(\mathbb{R}^+) \ : \ \phi(0) = 0\}.$$

Let us consider the adjoint operator M^*. It maps a function $\phi \in L^\infty(\mathbb{R}^+)$ to the following function in $L^\infty(\mathbb{R})$:

$$\widetilde{\phi}(s) := M^*(t)\phi(s) = \begin{cases} 0 & s < 0 \\ \phi(s) & s \geq 0 \end{cases}.$$

It is easy to see that, $M^\odot$ being $M^*_{|_{Y^\odot}}$, $\operatorname{ran}(M^\odot) \subset X^\odot$. Indeed, if ϕ is a uniformly continuous function with $\phi(0) = 0$, then its extension $\widetilde{\phi}$ is again uniformly continuous on the whole $\mathbb{R}$. Moreover, $\operatorname{ran}(M^\odot)$ is invariant under the sun dual semigroup of $T(t)$, because the right translations preserve uniform continuity, as well as they preserve the ideal of functions vanishing on $\mathbb{R}^-$.

From the point of view of the generator, we obtain the following result:

Proposition 3.7. *1.* $\mathscr{D}(\widehat{A}^*) = (M^*)^{-1}[(\mathscr{D}(A^*) \cap \operatorname{ran}(M^*)]$*;*

2. $\mathscr{D}(\widehat{A}^\odot) = (M^\odot)^{-1}[(\mathscr{D}(A^\odot) \cap \operatorname{ran}(M^\odot)]$*.*

Proof. 1.) Consider $x^* = M^*y^*$, with $y^* \in \mathscr{D}(\widehat{A}^*)$. For every $x \in X$ we have

$$\begin{aligned}
&\lim_{h\to 0+} |T^*(h)x^*(x) - x^*(x)| = \lim_{h\to 0+} |T^*(h)M^*y^*(x) - M^*y^*(x)| \\
&= \lim_{h\to 0+} |M^*\widehat{T}^*(h)y^*(x) - M^*y^*(x)| = \lim_{h\to 0+} |\widehat{T}^*(h)y^*(Mx) - y^*(Mx)| \\
&= \widehat{A}^*y^*(Mx) = M^*\widehat{A}^*y^*.
\end{aligned}$$

This implies that x^* belongs to the domain of A^*; thus $M^*\mathscr{D}(\widehat{A}^*) \subseteq (\mathscr{D}(A^*) \cap \operatorname{ran}(M^*))$. For the reverse inclusion, consider $x^* = M^*y^* \in \mathscr{D}(A^*)$. Then, $\forall y \in Y$,

$$\begin{aligned}
&\lim_{h\to 0+} \frac{1}{h}|\widehat{T}^*(h)y^*(y) - y^*(y)| = \lim_{h\to 0+} |\widehat{T}^*(h)y^*(Mx) - y^*(Mx)| \\
&= \lim_{h\to 0+} \frac{1}{h}|y^*(\widehat{T}(h)Mx) - y^*(Mx)| = \lim_{h\to 0+} |y^*(MT(h)x) - y^*(Mx)| \\
&= \lim_{h\to 0+} \frac{1}{h}|M^*y^*(T(h)x) - M^*y^*(x)| \\
&= \lim_{h\to 0+} \frac{1}{h}|T^*(h)M^*y^*(x) - M^*y^*(x)| = A^*M^*y^*x,
\end{aligned}$$

implying that $(\mathscr{D}(A^*) \cap \operatorname{ran}(M^*)) \subseteq M^*(\mathscr{D}(\widehat{A}^\odot))$.
2.) The proof is similar to that of the first statement but now we work in the strong topology. If $x^\odot = M^\odot y^\odot$ where $y^\odot \in \mathscr{D}(\widehat{A}^\odot)$, then

$$\begin{aligned}
\lim_{h\to 0^+} \frac{1}{h}\left(T^\odot x^\odot - x^\odot\right) &= \lim_{h\to 0^+} \frac{1}{h}\left(M^\odot \widehat{T}^\odot y^\odot - M^\odot y^\odot\right) \\
&= M^\odot \lim_{h\to 0^+} \frac{1}{h}\left(\widehat{T}^\odot(h) y^\odot - M^\odot y^\odot\right) = M^\odot \widehat{A}^\odot y^\odot,
\end{aligned}$$

that is, $M^\odot\mathscr{D}(\widehat{A}^\odot) \subseteq (\mathscr{D}(A^\odot) \cap \operatorname{ran}(M^\odot))$. On the other hand, since $\operatorname{ran}(M^\odot)$ is invariant under $T^\odot(t)$, the restriction $A^\odot_{\operatorname{ran}(M^\odot)}$ is again a generator of a strongly continuous semigroup (see [2]). We take $\lambda \in \rho(A^\odot_{\operatorname{ran}(M^\odot)}) \cap \rho(\widehat{A}^\odot)$ and we use the integral representation of the resolvent operator: If $x^\odot \in (\mathscr{D}(A^\odot) \cap \operatorname{ran}(M^\odot))$, then

$$\begin{aligned}
x^\odot &= \int_0^{+\infty} e^{-\lambda t} T^\odot_{\operatorname{ran}(M^\odot)}(t) s^\odot \, dt = \int_0^{+\infty} e^{-\lambda t} T^\odot_{\operatorname{ran}(M^\odot)}(t) M^\odot y^\odot \, dt \\
&= \int_0^{+\infty} e^{-\lambda t} M^\odot \widehat{T}^\odot y^\odot \, dt = M^\odot \int_0^{+\infty} e^{-\lambda t} \widehat{T}^\odot y^\odot \, dt \\
&= M^\odot z^\odot, \quad z \in \mathscr{D}(\widehat{A}^\odot).
\end{aligned}$$

This implies $(\mathscr{D}(A^\odot) \cap \operatorname{ran}(M^\odot)) \subseteq M^\odot\mathscr{D}(\widehat{A}^\odot)$. □

It is interesting to note that, if we impose the condition $\operatorname{ran}(M^*) \subseteq X^\odot$, then $\widehat{T}^*(t)$ is strongly continuous on the whole Y^*. Indeed, for every y^*, we have:

$$\begin{aligned}\lim_{h\to 0+} ||\widehat{T}^*(h)(M^*)^{-1}M^*y^* - (M^*)^{-1}M^*y^*|| &= \lim_{h\to 0+} ||(M^*)^{-1}T^*(h)M^*y^* - (M^*)^{-1}M^*y^*|| \\ &\leq \lim_{h\to 0+} ||(M^*)^{-1}|| \, ||T^*(h)M^*y^* - M^*y^*|| = 0.\end{aligned}$$

Remark 3.2. Note that if the semigroup $T^*(t)$ is strongly continuous on the whole X^* (e.g., when X is a reflexive Banach space), then $\widehat{T}^*(t)$ is also strongly continuous, since $\widehat{T}^*(t) = (M^*)^{-1}T^*(t)M^*$. On the other hand, if $\widehat{T}^*(t)$ is strongly continuous on the whole Y^*, then $\operatorname{ran}(M^*)$ is contained in $X^\odot$. Since it is also invariant for the adjoint semigroup, the restriction $T^*|_{\operatorname{ran}(M^*)}(t)$ is a strongly continuous semigroup.

Our aim is to obtain a lumping on system (3.7) starting from some conditions on the sun dual semigroup. To do this, we need the following well-known results about sun dual spaces (see e.g., [41] or [55]). Recall that the canonical inclusion J of X in $X^{\odot\odot}$ is the linear and bounded map

$$J: X \to X^{\odot *}, \quad J(x)(x^\odot) = x^\odot(x).$$

Proposition 3.8. *J is an isomorphism from X to a closed subspace $J(X) \subseteq X^{\odot\odot}$. Moreover, the following hold:*

1. $T^{\odot\odot}(t)J(x) = J(T(t)x)$.
2. $J\mathscr{D}(A) = \mathscr{D}(A^{\odot\odot}) \cap J(X)$.
3. $A^{\odot\odot}J(x) = J(A(x))$.

Observe that J is also an isometry if we consider an equivalent norm on X given by $||x|| := \sup_{||x^\odot||\leq 1} |x^\odot(x)|$ (see [41]).

Consider now the following hypotheses:

1. $T(t)$ strongly continuous semigroup on X with generator A;
2. $\widehat{T}(t)$ strongly continuous semigroup on Y with generator $\widehat{A}$;
3. $M : X \to Y$ bounded and surjective;
4. $M^*\widehat{T}^\odot(t) = T^\odot(t)M^*$.

It is easy to verify that hypotheses 1–4 imply the lumping relation on X, i.e. $MT(t) = \widehat{T}(t)M$. In fact, (iv) implies that M^* maps $Y^\odot$ in $X^\odot$. Therefore, we can consider $M^\odot := M^*_{Y^\odot} : Y^\odot \to X^\odot$ satisfying $M^\odot\widehat{T}^\odot(t) = T^\odot(t)M^\odot$. The adjoint operator $M^{\odot *}$ maps $X^{\odot\odot}$ into $Y^{\odot\odot}$, thus we can define $M^{\odot\odot} := M^{\odot *}_{X^{\odot\odot}}$ such that

$$\widehat{T}^{\odot\odot}(t)M^{\odot\odot} = M^{\odot\odot}T^{\odot\odot}(t). \tag{3.8}$$

If $\widetilde{J}$ is the canonical inclusion of Y in $Y^{\odot\odot}$, for every $x \in X$ and $y^\odot \in Y^\odot$ we have

$$M^{\odot\odot}J(x)(y^\odot) = J(x)(M^\odot y^\odot) = M^\odot y^\odot(x) = y^\odot(Mx) = \widetilde{J}(Mx)y^\odot.$$

Hence, using Proposition 3.8 and relation (3.8), we obtain:

$$\widehat{T}^{\odot\odot}(t)M^{\odot\odot}J(x) = M^{\odot\odot}T^{\odot\odot}(t)J(x) \Rightarrow \widetilde{J}(\widehat{T}(t)Mx) = \widetilde{J}(MT(t)x).$$

Applying $\widetilde{J}^{-1}$ we obtain the lumping relation between $T(t)$ and $\widehat{T}(t)$, namely, $MT(t)x = \widehat{T}(t)Mx$.

3.4.1 A dual construction of a lumping

Finally, we want to obtain a lumping on a strongly continuous semigroup having an operator whose range is contained in the sun dual space, but without knowing the lumping operator M *a priori*. More precisely, we want to construct M starting from an operator defined on a Banach space that is isomorphic to a subspace of $X^{\odot}$. Let us consider a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ generated by A and a linear, bounded operator $\widetilde{M}:\widetilde{Y}\to X^{\odot}$, where $\widetilde{Y}$ is a Banach space. We assume the following hypotheses:

1. $\operatorname{ran}(\widetilde{M})$ is closed in $X^{\odot}$;
2. $\widetilde{M}$ is an isomorphism from $\widetilde{Y}$ to $\operatorname{ran}(\widetilde{M})$;
3. $\operatorname{ran}(\widetilde{M})$ is invariant under the sun dual semigroup $\{T^{\odot}(t)\}_{t\geq 0}$.

Define the family of bounded operators on $\widetilde{Y}$,

$$\widetilde{T}(t)\widetilde{y}=\widetilde{M}^{-1}T^{\odot}(t)\widetilde{M}\widetilde{y},\quad t\geq 0. \tag{3.9}$$

The family (3.9) is a strongly continuous semigroup. Indeed, for every $\widetilde{y}\in\widetilde{Y}$,

1. $\widetilde{T}(0)\widetilde{y}=\widetilde{M}^{-1}T^{\odot}(0)\widetilde{M}\widetilde{y}=\widetilde{M}^{-1}\widetilde{M}\widetilde{y}=\widetilde{y}$;
2. For every $t,s\geq 0$,
$$\widetilde{T}(t)\widetilde{T}(s)\widetilde{y}=\widetilde{M}^{-1}T^{\odot}(t)\widetilde{M}\widetilde{M}^{-1}T^{\odot}(s)\widetilde{M}\widetilde{y}=\widetilde{M}^{-1}T^{\odot}(t+s)\widetilde{M}\widetilde{y}=\widetilde{T}(t)\widetilde{y};$$
3. $\lim_{h\to 0+}||\widetilde{T}(h)\widetilde{y}-\widetilde{y}||=\lim_{h\to 0+}||\widetilde{M}^{-1}T^{\odot}(h)\widetilde{M}\widetilde{y}-\widetilde{y}||=\lim_{h\to 0+}||\widetilde{M}^{-1}T^{\odot}(h)\widetilde{M}\widetilde{y}-\widetilde{M}^{-1}\widetilde{M}\widetilde{y}||$ $\leq\lim_{h\to 0+}||T^{\odot}(h)\widetilde{M}\widetilde{y}-\widetilde{M}\widetilde{y}||\cdot||\widetilde{M}^{-1}||=0.$

By definition, $\widetilde{T}(t)$ satisfies on $\widetilde{Y}$,

$$\widetilde{M}\widetilde{T}(t)=T^{\odot}(t)\widetilde{M}. \tag{3.10}$$

Remark 3.3. Let $\widetilde{A}$ be the generator of $\widetilde{T}(t)$. Using the definition of the infinitesimal generator and relation (3.10) we can verify that $\widetilde{M}\mathscr{D}(\widetilde{A})=\mathscr{D}(A^{\odot})\cap\operatorname{ran}(\widetilde{M})$.

Taking the adjoint of both sides in (3.10) gives $\widetilde{T}^{*}(t)\widetilde{M}^{*}=\widetilde{M}^{*}T^{\odot *}(t)$. By Theorem 3.2, the adjoint operator $\widetilde{M}^{*}$ is bounded and surjective from $X^{\odot *}$ to $\widetilde{Y}^{*}$, but not injective since $\ker(\widetilde{M}^{*})=\operatorname{ran}(\widetilde{M})^{\perp}$.

Remark 3.4. By the weak* continuity of $\widetilde{M}^{*}$ and the properties of the weak* generator, it is easy to verify that $\widetilde{M}^{*}\mathscr{D}(A^{\odot *})=\mathscr{D}(\widetilde{A}^{*})$.

Consider the two strongly continuous semigroups:

$$T^{\odot\odot}(t):=T^{\odot *}_{X^{\odot\odot}}(t),\qquad \widetilde{T}^{\odot}(t):=\widetilde{T}^{*}_{\widetilde{Y}^{\odot}}(t).$$

$\widetilde{M}^{*}$ maps $X^{\odot\odot}$ in $\widetilde{Y}^{\odot}$. In fact, if $\widetilde{y}^{*}=\widetilde{M}^{*}x^{\odot\odot}$ with $x^{\odot\odot}\in X^{\odot\odot}$, we have

$$\begin{aligned}\lim_{h\to 0+}||\widetilde{T}^{*}(h)\widetilde{y}^{*}-\widetilde{y}^{*}||&=\lim_{h\to 0+}||\widetilde{T}^{*}(h)\widetilde{M}^{*}x^{\odot\odot}-\widetilde{M}^{*}x^{\odot\odot}||\\&=\lim_{h\to 0+}||\widetilde{M}^{*}T^{\odot *}(h)x^{\odot\odot}-\widetilde{M}^{*}x^{\odot\odot}||\\&\leq\lim_{h\to 0+}||\widetilde{M}^{*}||||T^{\odot *}(h)x^{\odot\odot}-x^{\odot\odot}||=0.\end{aligned}$$

Hence we can define the restricted operator

$$\widetilde{M}^{\odot} := (\widetilde{M}^{*})_{X^{\odot\odot}} : X^{\odot\odot} \to \widetilde{Y}^{\odot},$$

such that

$$\widetilde{T}^{\odot}(t)\widetilde{M}^{\odot} = \widetilde{M}^{\odot}T^{\odot\odot}(t). \tag{3.11}$$

Note that the range of $\widetilde{M}^{\odot}$ is not necessarily closed, because $\widetilde{M}^{*}$ is surjective but not one-to-one. Let us restrict (3.11) to $J(X)$. Using proposition 3.8 we obtain

$$\widetilde{M}^{\odot}JT(t) = \widetilde{T}^{\odot}(t)\widetilde{M}^{\odot}J. \tag{3.12}$$

This is a lumping relation with the original semigroup $T(t)$ and $\widetilde{T}^{\odot}(t)$; the new Banach space is $\widetilde{Y}^{\odot}$ and the lumping operator is $\widetilde{M}^{\odot}J$. Since the lumping operator is not surjective, (3.12) holds on the whole X but not on the whole $\widetilde{Y}^{\odot}$. This means that, when $\widetilde{y}^{\odot} = \widetilde{M}^{\odot}Jx \in \mathrm{ran}(\widetilde{M}^{\odot}J)$, we can define

$$\widetilde{T}^{\odot}(t)\widetilde{y}^{\odot} := \widetilde{M}^{\odot}JT(t), \tag{3.13}$$

but when $\widetilde{y}^{\odot} \in (\widetilde{Y}^{\odot} - \mathrm{ran}(\widetilde{M}^{\odot}J))$, we can only write $\widetilde{T}^{\odot}(t)\widetilde{y}^{\odot} = \widetilde{T}^{*}(t)\widetilde{y}^{\odot}$. At any rate, since we know *a priori* that $\widetilde{T}^{\odot}(t)$ is strongly continuous on the whole $\widetilde{Y}^{\odot}$, we obtain a closed dynamics on the upper level, in the sense that for every solution $t \to T(t)x$ of the system

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = x, \end{cases}$$

the reduced solution $t \mapsto \widetilde{M}^{\odot}JT(t)x$ is a solution of

$$\begin{cases} \dot{u}(t) = \widetilde{A}^{\odot}u(t) \\ u(0) = \widetilde{M}^{\odot}Jx, \end{cases}$$

where $\widetilde{A}^{\odot}$ is the generator of $\widetilde{T}^{\odot}(t)$.

We obtain some information about the domain of the infinitesimal generator $\widetilde{A}^{\odot}$. We know that

$$\mathscr{D}(\widetilde{A}^{\odot}) = \{\widetilde{y}^{\odot} \in \mathscr{D}(\widetilde{A}^{*}) : \widetilde{A}^{*}\widetilde{y}^{\odot} \in \widetilde{Y}^{\odot}\},$$

where $\mathscr{D}(\widetilde{A}^{*}) = \widetilde{M}^{*}\mathscr{D}(A^{\odot *})$. If $x^{\odot\odot} \in \mathscr{D}(A^{\odot\odot})$ and $\widetilde{y}^{\odot} = \widetilde{M}^{\odot}x^{\odot\odot}$, then

$$\begin{aligned} &\lim_{h\to 0+}\frac{1}{h}\left(\widetilde{T}^{\odot}(h)\widetilde{y}^{\odot} - \widetilde{y}^{\odot}\right) = \lim_{h\to 0+}\frac{1}{h}\left(\widetilde{M}^{\odot}T^{\odot\odot}(h)x^{\odot\odot} - \widetilde{M}^{\odot}x^{\odot\odot}\right) \\ &= \widetilde{M}^{\odot}\left(\lim_{h\to 0+}\frac{1}{h}\left(T^{\odot\odot}(h)x^{\odot\odot} - x^{\odot\odot}\right)\right) = \widetilde{M}^{\odot}A^{\odot\odot}x^{\odot\odot}. \end{aligned}$$

Therefore,

$$\widetilde{M}^{\odot}\mathscr{D}(A^{\odot\odot}) \subseteq \mathscr{D}(\widetilde{A}^{\odot}) \cap \mathrm{ran}(\widetilde{M}^{\odot}), \tag{3.14}$$

and

$$\widetilde{A}^{\odot}\widetilde{M}^{\odot} = \widetilde{M}^{\odot}A^{\odot\odot} \quad \text{on } \mathscr{D}(A^{\odot\odot}).$$

The inclusion (3.14) is in general a proper one because of the non-surjectivity of $\widetilde{M}^{\odot}$. Using Proposition 3.8 and the fact that the image of an intersection is contained in the intersection of the images, we obtain

$$\widetilde{A}^{\odot}\widetilde{M}^{\odot}J = \widetilde{M}^{\odot}JA,$$

and

$$\widetilde{M}^{\odot}J(\mathscr{D}(A)) = \widetilde{M}^{\odot}(\mathscr{D}(A^{\odot\odot}) \cap J(X)) \subseteq \widetilde{M}^{\odot}\mathscr{D}(A^{\odot\odot}) \cap \widetilde{M}^{\odot}J(X). \tag{3.15}$$

By (3.14) and (3.15),

$$\widetilde{M}^{\odot}J(\mathscr{D}(A)) \subseteq \mathscr{D}(\widetilde{A}^{\odot}) \cap \widetilde{M}^{\odot}J(X).$$

We can characterize more precisely the generator of the new semigroup restricting $\widetilde{A}^{\odot}$ to the subspace $Y := \overline{\mathrm{ran}(\widetilde{M}^{\odot}J)}$. It is easy to verify that Y is closed and invariant for the semigroup $\widetilde{T}^{\odot}(t)$, so that we can define the restricted semigroup

$$\widehat{T}(t) = \widetilde{T}^{\odot}_{Y}(t) : Y \to Y,$$

which is again strongly continuous with generator $\widehat{A} := \widetilde{A}^{\odot}_{Y \cap \mathscr{D}(\widetilde{A}^{\odot})}$. The lumping relation $\widehat{T}(t)\widetilde{M}^{\odot}J = \widetilde{M}^{\odot}JT(t)$ is verified on X because (3.12) holds. Since $\widetilde{M}^{\odot}J(\mathscr{D}(A))$ is dense in Y and $\widehat{T}(t)$-invariant, it is a core for $\widehat{A}$ (see e.g. [28] and [2]). Therefore, we can define $\widehat{A}$ exactly as the closure of the linear operator mapping every element of the shape $\widetilde{M}^{\odot}Jx$, $x \in \mathscr{D}(A)$, in $\widetilde{M}^{\odot}JAx$. Note that if $\mathrm{ran}(\widetilde{M}^{\odot}J)$ was closed, we could say that $\mathscr{D}(\widehat{A}) = \widetilde{M}^{\odot}J(\mathscr{D}(A))$.

Example 3.4. To conclude, we give an example of the construction described in subsection 4.5. We build a lumping on a Banach space X starting from an isomorphism between a subspace of X^* and a Euclidean space. Here this construction looks simpler than the general one, because we are in the particular case $X^{\odot} = X^*$ (even if the Banach space is not reflexive). We give a dual interpretation of the lumping through the evaluation operator we have described in example 2.4. Consider a continuous function $h : \mathbb{R} \to \mathbb{C}$ such that $\sup_{x\in\mathbb{R}} \mathrm{Re}(h(x)) < \infty$. Then the family of bounded operators $T(t)f(x) = e^{th(x)}f(x)$ is a strongly continuous semigroup on the Banach space $X = C_0(\mathbb{R})$, with generator $Af(x) = h(x)f(x)$ (see [2]). By the Riesz-Markov theorem, $C_0(\mathbb{R})^*$ can be identified with the Banach space $\mathscr{M}(\mathbb{R})$ of all complex, regular, Borel measures on the real line. If $\phi \in C_0(\mathbb{R})^*$ and μ_ϕ is the measure associated with ϕ, $\forall f \in C_0(\mathbb{R})$ we have $\phi(f) = \int f(x)\, d\mu_\phi(x)$. Given a finite set of Dirac measures $\delta(x_1), \dots, \delta(x_k)$ (such that the function h does not vanish on $x_1, \dots, x_k$), define the bounded operator

$$\widetilde{M} : \widetilde{Y} = \mathbb{C}^k \to \mathscr{M}(\mathbb{R}), \qquad \widetilde{M}(\alpha_1, \dots, \alpha_k) = \alpha_1\delta(x_1) + \cdots + \alpha_k\delta(x_k).$$

This is an injective operator whose range is the closed subspace of all linear combinations of $\delta(x_1), \dots, \delta(x_k)$ with complex coefficients (clearly, this range is isomorphic to $\mathbb{C}^k$). It is easy to obtain

$$T^*(t)\widetilde{M}(\alpha_1, \dots, \alpha_k) = \alpha_1 e^{th(x_1)}\delta(x_1) + \cdots + \alpha_k e^{th(x_k)}\delta(x_k),$$

which implies that the range of $\widetilde{M}$ is invariant under $T^*(t)$. Note that the operator $\widetilde{M}$ coincides with the adjoint of the evaluation operator considered in example 2.4. This dual construction shows the advantages of the dual approach, because $\widetilde{M}$ is indeed an invertible operator on a finite dimensional space.

Keeping the notation as in the construction presented in this section, we define the following semigroup:

$$\widetilde{T}(t)(\alpha_1, \dots, \alpha_k) = (\alpha_1, \dots, \alpha_k)\mathrm{Diag}\left(e^{th(x_1)}, \dots, e^{th(x_k)}\right),$$

whose adjoint semigroup is

$$\widetilde{T}^*(t)\begin{pmatrix}\beta_1\\ \vdots \\ \beta_k\end{pmatrix} = \mathrm{Diag}\left(\overline{e^{th(x_1)}}, \dots, \overline{e^{th(x_k)}}\right)\begin{pmatrix}\beta_1\\ \vdots \\ \beta_k\end{pmatrix}.$$

We consider the composition of the adjoint operator of $\widetilde{M}$ and the canonical inclusion $j : C_0(\mathbb{R}) \to \mathscr{M}(\mathbb{R})^*$ (note that $\mathbb{C}^k$ is isomorphic to its dual by the Riesz isomorphism):

$$\widetilde{M}^*j : C_0(\mathbb{R}) \to (\mathbb{C}^k)^*,$$

$$\widetilde{M}^*(j(f)) = \begin{pmatrix} \overline{f(x_1)} \\ \vdots \\ \overline{f(x_k)} \end{pmatrix}.$$

Finally, we obtain the lumping relation on the Banach space X between $\widetilde{T}^*(t)$ and the strongly continuous semigroup $T(t)$, trough the lumping operator $\widetilde{M}^* j$:

$$\widetilde{T}^*(t)\widetilde{M}^* j(f) = \widetilde{M}^* j(T(t)f) = \begin{pmatrix} \overline{e^{th(x_1)}} \cdot \overline{f(x_1)} \\ \vdots \\ \overline{e^{th(x_k)}} \cdot \overline{f(x_k)} \end{pmatrix}.$$

3.5 A linear condition for nonlinear lumpability by the Koopman operator

In this chapter we have proved that the invariance of $\operatorname{ran}(M^*)$ under the adjoint operator $A^\star$ (being A the generator of the dynamics) is a necessary condition for lumpability. If A is a bounded operator, it becomes also a sufficient condition. By definition of the adjoint operator, the subspace $\operatorname{ran}(M^*)$ is made up by all the elements in X^* of the form $x^* = y^* \circ M$ for some $y^* \in Y^*$. This means that we are looking at all the linear functionals on X which can be written as a composition between a functional on Y and the operator M. A natural question is the following: which conditions do we obtain looking at all the compositions $h \circ M$, where h is a continuous function, not necessarily linear? Do we obtain some informations about lumpability for nonlinear system, in analogy with the linear case? Of course, the linear and continuous functionals on X define a Banach space, the dual of X, and the adjoint operator of M is well-defined and linear on this space. This is not possible if the operator is nonlinear and the domain is not a linear subspace. However, in some particular cases the space of continuous functions over a topological space defines again a Banach space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For instance, it is well known that if X is a compact Hausdorff space then the space $C(X)$ is a Banach space with the supremum norm. Indeed, it is a Banach algebra (see Chapter 5 for more details).
In this section we describe a case in which we can associate to a continuous map f on a space X a linear operator acting on a function $h \in C(X)$ as the composition $h \circ f$. This operator is called the *Koopman operator* and can be seen as a "nonlinear version" of the adjoint operator. In the case of a dynamical system, the Koopman operator is concerned with the evolution of the *system observables* and can reveal informations about the global behaviour of the system itself. This point of view is very convenient because, even if the system is nonlinear, the set of all its observables has a vector space structure. However, in general topological spaces we don't have a linear structure and we cannot work with the classical theory of dynamical systems: throughout this section we will consider *topological systems*, i.e. couples (K, f) with f continuous and K compact. We will give a linear condition on Koopman operators to obtain lumpability of a nonlinear system, with an eventually nonlinear lumping map. This condition will be indeed analogous to the invariance condition we have given on $\operatorname{ran}(M^*)$.
Before looking at the lumpability problem, we introduce some definitions and results on *Koopman operators*. For all these results we refer to [26], where these operators are studied in relation to Ergodic theory. Even if ergodicity is not one of the topics of this thesis, we mention that, in this context, the Koopman operator is mainly analyzed from the space $L^1(X)$ to itself and it can be proved that a topological system is *ergodic* if and only if 1 is a simple eigenvalue of the associated Koopman operator (we recall that a topological system is ergodic if every invariant sets is essentially equal either to $\emptyset$ or to the whole X).

Throughout this section, K and L will be compact Hausdorff spaces (in this way, $C(K)$ and $C(L)$ are both Banach spaces).

Definition 3.2. Given $\phi : K \to L$ continuous, we define the associated Koopman operator as the composition operator:

$$T_\phi : C(L) \to C(K), \quad T_\phi(h)(x) := h(\phi(x)), \quad \forall\, h \in C(L),\, x \in K.$$

It is easy to verify that T_ϕ is linear and bounded. Moreover, it can be proved that ϕ is surjective if and only if T_ϕ is injective (in this case, T_ϕ is an isometry), and ϕ is injective if and only if T_ϕ is surjective. An interesting property of the Koopman operator concerns invariant subsets for the system associated with ϕ: a closed subset A is ϕ-invariant if and only if the ideal I_A is T_ϕ-invariant, where I_A is the subspace of functions vanishing on A. The operator T_ϕ is not only continuous, but it is also an *algebra homomorphism*, i.e.

$$T_\phi(h \cdot g) = T_\phi h \cdot T_\phi g, \quad T_\phi 1_L = 1_K,$$

for any $h, g \in C(L)$ ($1_L, 1_K$ are the identities in $C(L), C(K)$ respectively).
In particular, every algebra homomorphism between functions algebras is such a Koopman operator:

Theorem 3.9 (Thm 4.13 [26])**.** *Let $T : C(L) \to C(K)$ be a linear operator. The followings are equivalent:*

(i) T is an algebra homomorphism;

(ii) there exists a unique $\phi : K \to L$ continuous such that $T = T_\phi$.

This result essentially follows from the fact that the space of all the continuous, linear, multiplicative functionals over $C(K)$ is homeomorphic to K itself (see [45]). It is now clear that the Koopman operator is so relevant because, according to the previous result, we don't loose any information by passing from the nonlinear map ϕ to the associated linear operator T_ϕ. We now prove the following simple fact that will be helpful in our lumping analysis:

Proposition 3.10. *Let $\phi : K \to L$ be continuous and surjective. Then the Koopman operator T_ϕ has closed range in $C(K)$.*

Proof. Let us consider a sequence f_n in $\operatorname{ran}(T_\phi)$, such that f_n converges to f in $C(K)$. We can write $f_n = h_n \circ \phi$ for some h_n in $C(L)$. We want to show that f is also equal to $h \circ \phi$ for some $h \in C(L)$.
Knowing that f_n is a Cauchy sequence, let us prove that h_n is also a Cauchy sequence in $C(L)$. Since ϕ is surjective, every $y \in L$ can be written as $y = \phi(x)$, $x \in K$, and the operator T_ϕ is isometric. It follows that:

$$\begin{aligned}\|h_n - h_m\| &= \sup_{y \in L} |h_n(y) - h_m(y)| = \sup_{y \in L} |h_n(\phi(x)) - h_m(\phi(x))| \\ &= \sup_{x \in K} |h_n(\phi(x)) - h_m(\phi(x))| = \|f_n - f_m\| < \epsilon \quad \text{for } n, m \text{ large enough.}\end{aligned}$$

This means that h_n converges in $C(L)$ to some continuous $h : L \to \mathbb{K}$. By consequence, $T_\phi h_n = h_n \circ \phi$ converges to $T_\phi h = h \circ \phi$. By uniqueness of the limit, we obtain $f = T_\phi h$. $\square$

An important role in the literature of dynamical systems and Ergodic theory is played by the adjoint of the Koopman operator, known as the *Perron-Frobenious* operator P_ϕ. We know that $P_\phi := T_\phi^*$ acts as

$$P_\phi : C(K)^* \to C(L)^*, \quad [P_\phi(F)](h) := F(T_\phi h), \quad F \in C(K)^*, h \in C(L).$$

It is known (Riesz-Markov-Kakutani representation theorem) that $C(K)^* \equiv \mathscr{M}(K)$ and $C(L)^* \equiv \mathscr{M}(L)$, where $\mathscr{M}(K), \mathscr{M}(L)$ are the spaces of finite, regular, Borel measures on K, L respectively. This means that, being μ_F the measure associated with the functional F, for any $h \in C(L)$ we can write:

$$\int_L h(y) d(P_\phi \mu_F) = \int_K h(\phi(x)) d\mu_F.$$

Using the *change of variable* formula we obtain:

$$\int_L h(y) d(P_\phi \mu_F) = \int_L h(y) d[\mu_F \circ \phi^{-1}](y).$$

This means that P_ϕ maps a measure μ on K into the *Push-forward measure* $\mu \circ \phi^{-1}$ on L, usually denoted by $\phi_* \mu$. Note that the invariant measures for the map ϕ are exactly the fixed points of the operator P_ϕ.

3.5.1 Lumpability and Koopman operators

Throughout this section, we assume the following:

1. K,L are compact Hausdorff spaces;
2. $f : K \to K$ is a continuous function;
3. $m : K \to L$ is continuous and surjective;
4. $T_f : C(K) \to C(K)$ and $T_m : C(L) \to C(K)$ are the Koopman operators associated with f and m, respectively.

We consider the couple (K, f) as a *topological system*. In this context, (K, f) is lumpable by the surjective map $m : K \to L$ if a continuous function $\widehat{f}$ exists in such a way that the couple $(L, \widehat{f})$ is again a topological system and $\widehat{f} \circ m = m \circ f$.
We prove the following necessary and sufficient linear condition for lumpability, which can be seen as an extension of our dual condition to nonlinear systems on compact spaces:

Proposition 3.11. *There exists a continuous map $\widehat{f} : L \to L$ such that $\widehat{f} \circ m = m \circ f$ on K (i.e. the topological system associated with f is lumpable by the lumping map m) if and only if* $\operatorname{ran}(T_m)$ *is T_f-invariant.*

Proof. Let us suppose that there exists a continuous function $\widehat{f}$ such that $\widehat{f} \circ m = m \circ f$. Let us consider $g \in \operatorname{ran}(T_m) \subset C(K)$. This means that there exists $h \in C(L)$ such that $g = h \circ m$. Applying T_f we can write:

$$T_f g = T_f(h \circ m) = h \circ (m \circ f) = h \circ (\widehat{f} \circ m) = (h \circ \widehat{f}) \circ m = T_m(h \circ \widehat{f}),$$

i.e. $T_f g \in \operatorname{ran}(T_m)$.
Conversely, let us suppose that $\operatorname{ran}(T_m)$ is T_f-invariant. For every $h \in C(L)$, $T_m h$ belongs to $\operatorname{ran}(T_m)$, and we can write:

$$T_f T_m h = T_m \widetilde{h},$$

for some $\widetilde{h}$ in $C(L)$, depending on h. For every $h \in C(L)$, we can define the operator $\mathscr{F}(h) := \widetilde{h}$ from $C(L)$ to itself.
Since m is surjective, T_m is injective and by Proposition 3.10 it has closed range. As a consequence, T_m admits a bounded inverse operator $T_m^{-1} : \operatorname{ran}(T_m) \to C(L)$. The operator $\mathscr{F}$ is then explicitly defined by:

$$\mathscr{F}(h) := T_m^{-1} T_f T_m \, h, \tag{3.16}$$

and it is linear and bounded. Moreover, as a composition of algebra homomorphisms, $\mathscr{F}$ is itself an algebra homomorphism:

$$\mathscr{F}(1_L) = 1_L, \;\; \mathscr{F}(h \cdot g) = T_m^{-1}[T_f T_m(h) \cdot T_f T_m(g)] = T_m^{-1}[T_m \mathscr{F}(h) \cdot T_m \mathscr{F}(g)] = \mathscr{F}(h) \cdot \mathscr{F}(g).$$

By Theorem 3.9, there exists a unique continuous function $\widehat{f} : L \to L$ such that $\mathscr{F} = T_{\widehat{f}}$. This means that:

$$T_f T_m \, h = T_m \mathscr{F}(h) = T_m T_{\widehat{f}} \, h,$$

i.e. $h(m(f(x)) = h(\widehat{f}(m(x)))$ for all $h \in C(L)$, $x \in K$.
From this follows that $T_{mf} = T_{\widehat{f}m}$, and applying again Theorem 3.9 we obtain $m \circ f = \widehat{f} \circ m$ on K. □

3.5.2 A construction of the reduced map by quotient metric space

It is interesting to observe that, even if we don't have a linear structure on K and L, it is possible to deduce some properties of $\widehat{f}$ passing through a quotient space decomposition analogous to the one of diagrams 2.7 and 2.8. In particular, let us suppose that $m : (X, d_X) \to (Y, d_Y)$ is a continuous surjective map between compact metric spaces and f is continuous from (X, d_X) to itself. If the topological system associated with f is lumpable by m, then there exists $\widehat{f}$ continuous on Y such that $m \circ f = \widehat{f} \circ m$. Since m is surjective, we can define an equivalence relation R such that $x_1 \sim x_2$ if and only if $m(x_1) = m(x_2)$ (i.e. the equivalence classes are the fibers of m). We denote by X/R the space of all equivalence classes of R and by π the canonical quotient map $X \ni x \to [x] \in X/\mathrm{R}$. The natural topology on X/R is the *quotient topology*, for which a subset $A \subset X/\mathrm{R}$ is open if and only if $\pi^{-1}(A)$ is open in X. This is the finest topology such that π is continuous. Let us consider the following diagrams:

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[dr, "\pi"'] & Y \arrow[d, leftrightarrow, "\widetilde{m}"] \\ & X/\mathrm{R} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "mf"] \arrow[dr, "\pi"'] & Y \\ & X/\mathrm{R} \arrow[u, "\widetilde{mf}"'] \end{tikzcd}$$

The map $\widetilde{m}$ is naturally defined as $\widetilde{m}([x]) := m(x)$. If m is linear and X, Y are Banach spaces, we know that m is open and $\widetilde{m}$ is always an homeomorphism. However, if X and Y are general topological spaces, even if $\widetilde{m}$ is bijective, it is not necessarily an homeomorphism. More precisely, all the surjective maps such that $\widetilde{m}$ is an homeomorphism are known as *quotient maps*, and the following holds:

Theorem 3.12 (Prop.2.4.3 [29]). *Let $m : X \to Y$ be a surjective map between topological spaces. The followings are equivalent:*

(i) m is a quotient map;

(ii) C is closed in Y if and only if $m^{-1}(C)$ is closed in X.

Fortunately, in the case we are considering, m is indeed a quotient map. In fact, being m a continuous surjection between compact Hausdorff spaces, it is a closed map. But a closed surjection is always a quotient map, because for every set C we have $f(f^{-1}(C)) = C$. This means that, in this particular case, the map $\widetilde{m}$ is an homeomorphism.

The second map that we need to define is $\widetilde{mf}[x] := m(f(x))$. This map is well-defined because the existence of $\widehat{f}$ guarantees that, if $m(x_1) = m(x_2)$, then also $m(f(x_1)) = m(f(x_2))$ holds. Even if it is not surjective, $\widetilde{mf}$ is at least continuous: by definition of the quotient topology, $\widetilde{mf}^{-1}(U)$ is open in X/R if and only if $\pi^{-1}(\widetilde{mf}^{-1}(U))$ is open in X. But, if U is open, then

$$\pi^{-1}(\widetilde{mf}^{-1}(U)) = (\widetilde{mf} \circ \pi)^{-1}(U) = (m \circ f)^{-1}(U)$$

is also open because $m \circ f$ is continuous.
These two maps are helpful because we can write $\widehat{f}$ explicitly as the composition $\widehat{f}(y) = \widetilde{mf}(\widetilde{m}^{-1}(y))$.
We recall that a map $g : X \to Y$ is called *bi-Lipschitz* if and only if there exists $L > 0$ such that:

$$\frac{1}{L}\, d_X(x_1, x_2) \le d_Y(g(x_1), g(x_2)) \le L\, d_X(x_1, x_2).$$

It is possible to show that, if f and m are Lipschitz continuous and $\widetilde{m}$ is a bi-Lipschitz homeomorphism, then $\widehat{f}$ is also Lipschitz on Y.
To verify this, we need to define a metric on the quotient space itself. Firstly, let us observe that the quotient X/R is a Hausdorff compact space: let us consider two distinct points $[x_1]$, $[x_2]$ in the quotient space. Then, $m(x_1) = y_1$ and $m(x_2) = y_2$ with $y_1 \ne y_2 \in Y$ (being y_1 and y_2 fixed for any choice of $x_1 \in [x_1]$ and $x_2 \in [x_2]$). Since Y is Hausdorff, there exist two disjoint neighborhoods U_1 and U_2 of y_1 and y_2, respectively. $\widetilde{m}$ being an homeomorphism between X/R and Y, $\widetilde{m}^{-1}(U_1)$ and $\widetilde{m}^{-1}(U_2)$ are disjoint open neighborhoods of $[x_1]$ and $[x_2]$, respectively.
The compactness of the quotient space follows by the continuity and surjectivity of π.
We can define the following *quotient pseudometric* (see [16, 70]):

$$d_R([x], [z]) := \inf\{\, \sum_{i=1}^{k} d_X(x_i, z_i)\,\}, \qquad [x], [z] \in X/\mathrm{R}, \tag{3.17}$$

where the infimum is taken over all the finite sequences $(x_1, \dots, x_k)$, $(z_1, \dots, z_k)$ such that $x_1 \sim x$, $z_n \sim z$ and $z_{(i-1)} \sim x_i$ for all $i = 2, \dots, k$.
It is possible to verify that d_R is positive, symmetric, and the triangular inequality holds. Note that, since $d_R([x], [z]) = 0$ does not always imply $[x] = [z]$, in general d_R is not a metric but only a pseudometric. Moreover, the topology induced by this pseudometric does not always coincide with the quotient topology (being the latter finer).
However, since the quotient space X/R is Hausdorff and X is compact, the equivalence classes are also compact subsets of X. This is enough to guarantee that, in this case, d_R is indeed a metric over the quotient space, and the topology induced by this metric coincides with the quotient topology [16].
Now, we prove that, if f and m are Lipschitz with Lipschitz constants L_f and L_m respectively, then the map $\widetilde{mf}$ is Lipschitz. In fact, since $x_1 \in [x] \Rightarrow m(f(x_1)) = m(f(x))$, we can write:

$$d_Y(\widetilde{mf}([x]), \widetilde{mf}([z])) = d_Y(m(f(x)), m(f(z))) = \inf_{\{(x_1,\dots,x_n),(z_1,\dots,z_n)\}} \sum_{i=1}^{k} d_Y(m(f(x_i)), m(f(z_i))$$

$$\le L_m\, L_f \inf_{\{(x_1,\dots,x_n),(z_1,\dots,z_n)\}} \sum_{i=1}^{k} d_X(x_i, z_i) = L_m\, L_f\, d_R([x], [z]),$$

where the sequences $(x_1, \dots, x_n)$, $(z_1, \dots, z_n)$ are chosen as in the definition of d_R.
The same argument holds to show that $\widetilde{m}$ is Lipschitz.

Let us suppose that the inverse map $\widetilde{m}^{-1}$ is also Lipschitz, then:

$$d_Y(\widehat{f}(y_1), \widehat{f}(y_2)) = d_Y(\widetilde{mf}(\widetilde{m}^{-1}(y_1)), \widetilde{mf}(\widetilde{m}^{-1}(y_2))) \leq L_m\, L_f\, L_{\widetilde{m}^{-1}} d_Y(y_1, y_2),$$

i.e. $\widehat{f}$ is Lipschitz continuous on Y.
Some conditions on m to obtain the bi-Lipschitzianity of the homeomorphism $\widetilde{m}$ are proved in [52]. We recall that the *Hausdorff distance* between two closed subset A,B of a metric space X is defined as follows:

$$d_H(A, B) := \max \left\{ \sup_{a\in A} \inf_{b\in B} d(a, b)\, ,\, \sup_{b\in B} \inf_{a\in A} d(a, b) \right\}.$$

For completeness, we report the following result:

Proposition 3.13 (Corollary 2.3. [52])**.** *Let $m : (X, d_X) \to (Y, d_Y)$ be a surjective Lipschitz map between compact metric spaces. Then the homeomorphism $\widetilde{m}$ is bi-Lipschitz if one of the following equivalent conditions holds:*

(i) the map $f^{-1} : (Y, d_Y) \to (\mathscr{H}(X), d_H)$ is Lipschitz, being $(\mathscr{H}(X), d_H)$ the space of all the non-empty compact subsets of X with the Hausdorff metric;

(ii) there exists $L > 0$ such that $B(f(x), \frac{\epsilon}{L}) \subset f(B(x, \epsilon))$, being $B(x, \epsilon)$ the closed ball with radius ϵ and center x.

Chapter 4

Lumpability of nonlinear evolution equations

4.1 Introduction

In this chapter we generalize our lumping analysis to the case of nonlinear abstract Cauchy problems. Given a general Banach space X, we consider the following evolution problem:

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $F : \mathscr{D}(F) \subseteq X \to X$ is a nonlinear operator and $x_0 \in \mathscr{D}(F)$.
As in the case of linear operators, we say that system (4.1) is lumpable by a linear, bounded and surjective operator $M : X \to Y$ if and only if the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "\widehat{F}"] & Y \\ X \arrow[u, "M"] \arrow[r, "F"] & X \arrow[u, "M"'] \end{tikzcd} \tag{4.2}$$

where $\widehat{F}$ is another nonlinear operator defined on the reduced state space Y. This commutativity is indeed equivalent to the lumping relation:

$$MF(x) = \widehat{F}(Mx). \tag{4.3}$$

Let us suppose that the nonlinear operator F preserves the fibers of M, i.e. the following condition holds for every $x_1, x_2 \in X$:

$$Mx_1 = Mx_2 \Rightarrow MF(x_1) = MF(x_2). \tag{4.4}$$

With the term *fibers* we mean the level sets of M: we ask that the images of x_1 and x_2 through F belong to the same level set of M, whenever x_1 and x_2 belong to the same level set. Then the nonlinear map

$$\widehat{F}(y) := MF(x), \ y = Mx, \tag{4.5}$$

is well-defined on Y and the lumping relation (4.3) holds.
We want to investigate whether the abstract Cauchy problem induced by $\widehat{F}$ is well-defined on Y, in the sense that a unique solution can be found for every suitable initial condition.

4.1.1 Lumpability in the context of chemical kinetics

Exact lumpability of finite dimensional nonlinear systems has been firstly analyzed in application to chemical reaction systems: chemical systems frequently contain large number of species and reactions, making the prediction of chemical kinetics computationally expensive. For this reason, it is helpful to reduce the number of variables through mathematical techniques, including lumping.

In particular, Li and Rabitz has studied lumpability in application to reaction systems with n species, described by a first order, ordinary differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. In [48] they consider a lumping matrix M in $\mathbb{R}^n$, such that $\mathrm{ran}(M) < n$, and a pseudoinverse matrix $\overline{M}$ such that $M\overline{M} = I$. They show that a reaction system is lumpable if and only if the subspace spanned by the row vectors of M is invariant under $J^T x$ at any point, where J^T is the transpose of the Jacobian matrix of the function f, and $M[J(x) - J(\overline{M}Mx)] = 0$. Using the pseudoinverse, they show that the reduced system is generated by the function $\widehat{f}(y) := Mf(\overline{M}y)$, where $y = Mx$, and this lumping scheme does not depend on the choice of the pseudoinverse. Using particular matrix decompositions, they also describe methods to construct fixed invariant subspaces for $J^T(y)$.

In [66], Li, Rabitz, Tomlin and Tóth analyze lumpability of nonlinear systems on an euclidean space, with a lumping transformation $h : \mathbb{R}^n \to \mathbb{R}^m$, $m < n$. The function h is continuously differentiable but not necessarily linear. Furthermore, in order to guarantee the existence of a pseudoinverse function $\overline{h}$, h is assumed to satisfy the following:

(i) $h(0) = 0$,

(iii) there exists $u : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that

$$\begin{pmatrix} h'(x) \\ u'(x) \end{pmatrix}$$

is a nonsingular matrix for all $x \in \mathbb{R}^n$,

(iii)

$$\lim_{\|x\| \to \infty} \left\| \begin{pmatrix} h(x) \\ u(x) \end{pmatrix} \right\| = \infty.$$

With this choice of h, they prove that the system associated with f is lumpable by h if and only if $h^{'}f = \widehat{f}h$. The reduced dynamics is then generated by $\widehat{f} = h^{'}f\overline{h}$. They also show that invariant sets, equilibria and periodic solutions are lumped into invariant sets, equilibria and periodic solutions respectively, and that $\widehat{f}$ preserves some regularity properties of f, such as Lipschitzianity.

Among the applications, Huang, Fairweather, Griffiths, Tomlin, and Brad have used lumping to reduce the complexity of a model describing the oxidation of fuel-rich methane mixtures in a closed vessel, under isothermal conditions [42].

In [30] Ranzi, Dente, Goldaniga, Bozzano, and Faravelli apply lumping to kinetic modelling of gasification, pyrolysis, partial oxidation and combustion of hydrocarbon. The dimensions and complexity of these models justify the adoption of analogy rules and other simplifying assumptions within the different chemical reaction classes, in order to lump a large number of real components into a properly selected number of equivalent components. The corresponding elementary reactions are also grouped into equivalent or lumped reactions. They report several simulations and data analysis.

In the context of chemical reactions, a generalization to the infinite dimensional setting makes sense: often in reaction systems a mixture of vary many components is involved, which may even not be distinguishable. In these cases it is convenient to describe the mixture by a distribution function, rather than a finite set of components: the state space becomes an

infinite dimensional space of functions (see [3]).
In [6], Bailey analyzes lumping of linear reactions in continuous mixtures. Instead of dealing with distinct chemical species A_i, $i = 1, \dots, n$, a continuous distribution of components $A(x)$ is considered, for x varying in a given real interval I, bounded or unbounded. He considers the integro-differential equation

$$\frac{d}{dt}c(x,t) = -\int_I K(y,x)c(y,t)dy,$$

where $c(x,t)$ is the concentration of $A(x)$ at time t. The Kernel K is assumed to be a linear combination of piecewise continuous functions. The lumped variable is then expressed by

$$\widehat{c}(t) = \int_I M(x)c(x,t)dx, \quad \widehat{c}(t) \in \mathbb{R}^m,$$

for a suitable piecewise continuous function M with values in $\mathbb{R}^m$. The kinetics of $\widehat{a}(t)$ should be described by a set of ordinary differential equations $\dot{\widehat{c}} = -\widehat{K}\widehat{c}$, $\widehat{K} \in \mathbb{R}^m$. Bailey shows that this kind of lumping holds if and only if

$$\int_I M(x)K(y,x)dx = \widehat{K}M(y) \quad \text{a.e. } y \in I.$$

In particular, $H(x)$ being the Heaviside step function, he analyzes the case of the vector-valued lumping function $M(x)$ with components:

$$M_i(x) = H(x - s_{i-1}) - H(x - s_i),$$

where s_0 is the left extremal point of I, and $s_{i-1} < s_i$ for all $i = 1, \dots, m$. In this way the lumped species $\widehat{c}_i$ includes all components $A(x)$ with $x \in [s_{i-1}, s_i]$: $\widehat{c}_i(t) := \int_{s_{i-1}}^{s_1} a(x,t)dx$.
He discusses some examples involving monomolecular reaction systems and irreversible chemical reactions.
This kind of lumping is generalized by Astarita and Ocone in [4] to the case of nonlinear chemical reactions, described by

$$\frac{d}{dt}c(x,t) = -v(c(x,t);x),$$

where v is a nonlinear function of the value of c, which also depends parametrically on x. As in [6], the lumped variable is the weighted total concentration of the chemical components. They focus on the case:

$$v(c(x,t);x) := \frac{k(x)c(x,t)}{1 + \int_I K(x)c(x,t)dx},$$

where k and K are dimensional parameters with units of a frequency and of an inverse concentration respectively. This model is derived from the so called Langmuir isotherm dominated kinetics for the adsorption of species onto simple surfaces (namely, the adhesion of molecules from a liquid to a solid surface), with the hypothesis that all the reactants undergo the same reaction and compete for the same sites on the solid surface (see [61] for a description of the Langmuir isotherm).
They gives examples involving bimolecular and thermodynamics systems.
In the same years, Chou and Ho approach the problem of lumping nonlinear reactions. In particular, they describe a method to find a species distribution function in such a way that the lumped model in the continuous mixtures approximate the behaviour of the discrete model with a high number of species and total concentration $C(t) = \sum_{i=1}^n c_i(t)$ [20].

4.1.2 Our contribution for a theoretical description of lumpability

We generalize these concepts to nonlinear systems in abstract Banach spaces in such a way that our theoretical analysis can be applied to many different kinds of problems, such as nonlinear delay equations and partial differential equations.
In this work we deal with a linear and bounded lumping operator M. Since in infinite dimensional Banach spaces the existence of a pseudoinverse operator requires some restrictive hypotheses, such as the existence of a topological complement for $\ker(M)$, our approach to lumpability is different from the one of Li, Rabitz and Tomlin. Indeed, we don't make use of the pseudoinverse operator.
Let us point out that the problem of existence of solutions for nonlinear Cauchy problems in infinite dimension is non trivial. In general not even the classical Peano existence theorem is valid for infinite dimensional Banach spaces.
According to Peano's theorem, the Cauchy problem associated with a continuous function on an euclidean space always admits a solution, at least locally. Many counterexamples have been found for this result in infinite dimension. For instance, let us define the sequence y with components:

$$y_n := \sqrt{|x_n|} + \frac{1}{n+1}, \quad x := \{x_n\}_{n\in\mathbb{N}} \in c_0,$$

on the Banach space c_0 of real sequences converging to zero. We consider the function $f(x) = y$ from c_0 to itself. It has been proved (see [24]) that the Cauchy problem $\dot{x} = f(x)$ with initial condition $x(0) = 0$ has no solution even if f is a continuous function.
However, note that the Cauchy-Lipschitz-Picard theorem is still valid in infinite dimensional Banach spaces: we can guarantee existence and uniqueness of the solution for abstract Cauchy problems associated with Lipschitz operators.

As we did for the lumpability of linear systems, our approach is still based on the theory of strongly continuous semigroups.
In the case of infinite dimensional linear systems, if $T(t)$ is a C_0 semigroup of linear operators with infinitesimal generator A, then the map $t \to T(t)x$, $x \in \mathscr{D}(A)$, is differentiable for every $t \geq 0$ and it is the unique solution of the Cauchy problem associated with A (see [56]). But for semigroups of nonlinear operators, the domain of the infinitesimal generator is not necessarily invariant under the semigroup. This means that for general Banach spaces the differentiability of $t \to T(t)x$ is not automatically guaranteed. In this work we focus on semigroups of nonlinear contractions. For this kind of semigroups, some interesting results exist, concerning the existence and uniqueness of solutions, both in the classical sense of smooth solutions and in the weaker sense of *strong solutions* (see [8] and [51]).

In the present, we first prove that the reduced map $\widehat{F}$ inherits the regularity of F. Then we discuss in detail under which conditions the operator $\widehat{F}$ can be again associated with a nonlinear, strongly continuous semigroup giving the solutions of the reduced abstract Cauchy problem on Y. In particular, the uniqueness of solutions for the reduced system is a non trivial problem. A sufficient condition to guarantee the uniqueness of solutions is the *dissipativity* of the generator, but dissipativity is generally not preserved by the lumping. First, we consider the case of an everywhere defined operator F: working on quotient Banach spaces, we can obtain the uniqueness of solutions for the reduced Cauchy problem even if $\widehat{F}$ is not necessarily dissipative. Then, we deal with operators defined on a proper subset of the state space. Since in this case the quotient Banach space method is not valid anymore, we overcome the problem finding some conditions to guarantee the dissipativity of $\widehat{F}$ *a priori*. These conditions don't require any restrictions on the domain of F, but involve the adjoint of the lumping operator, namely M^*.

4.2 Preliminaries

In this section we give some basic results of functional analysis that are needed later. In particular, we focus on nonlinear dissipative operators and semigroups of contractions. For the results we are going to present we refer especially to [51], [8] and [39]. We will always consider single valued operators from a Banach space X to itself, eventually defined on a proper subset of X.
We first mention some basic facts about differentiability of functions in Banach spaces.

Definition 4.1. Let X and Y be two Banach spaces. A function $F : X \to Y$ is said to be *Gâteaux differentiable* in a point $x \in X$ if the following limit exists for every $h \in X$:

$$\frac{d}{dt}(F(x+th))_{|t=0} = \lim_{t\to 0} \frac{F(x+th) - F(x)}{t} =: DF_x(h)$$

and, for x fixed, DF_x is a bounded linear operator on X.

The Gâteaux differential is a generalization of the classical directional derivative (with the additional request that the directional derivative is a linear operator acting on the directions) but we can obtain a stronger notion of derivative if we ask that the convergence in the above limit is uniform in h, when h belongs to the unit ball of X. In particular, we define the following:

Definition 4.2. A function $F : X \to Y$ is said to be *Fréchet differentiable* in $x \in X$ if there exists a linear and bounded operator $\mathscr{D}_x \in \mathscr{B}(X, Y)$ such that

$$\lim_{h\to 0} \frac{F(x+h) - F(x) - \mathscr{D}_x(h)}{\|h\|} = 0.$$

If F is Fréchet differentiable, then it is also Gâteaux differentiable and the two differentials coincide: $\mathscr{D}_x(h) = DF_x(h)$ for all $h \in X$. For this reason we will always use the notation DF_x for the derivative of a Fréchet differentiable function in the point x.
Note that, if we consider a linear and bounded operator A on X, then its Fréchet differential clearly coincides with A itself.
As in the case of euclidean spaces, the *chain rule* holds for the derivative of functions between Banach spaces:

Proposition 4.1. *If $F : X \to Y$ is Fréchet differentiable in $x \in X$ and $G : Y \to W$ is Fréchet differentiable in $F(x) \in Y$ then also the composition $G \circ F$ is differentiable in x and:*

$$D(G \circ F)_x[h] = DG_{F(x)}[DF_x h], \quad \forall\, h \in X.$$

Furthermore, we can use the following criteria to verify the Fréchet differentiability of a function:

Proposition 4.2. *If $F : X \to Y$ is Gâteaux differentiable in $x \in X$ and the Gâteaux derivative DF is continuous from X to $\mathscr{B}(X, Y)$, then F is also Fréchet differentiable (and, in this case, it is said to be a C^1 function).*

4.2.1 Nonlinear dissipative operators on Banach spaces

Dissipative operators are fundamental in the theory of infinite dimensional evolution equations and they represent a generalization of monotonic maps in euclidean spaces.

Definition 4.3. The dual mapping is the nonlinear, eventually multi valued map $\mathscr{F}$ from X to the dual space X^*, defined as

$$\mathscr{F}(x) = \{f \in X^* \,:\, f(x) = \|x\|^2 = \|f\|^2\}.$$

The set $\mathscr{F}(x)$ is non empty by the Hahn-Banach theorem; besides, if X is reflexive, then $\mathscr{F}$ is onto, i.e. $\mathscr{F}(X) = X^*$.
Note that, if H is an Hilbert space, the duality mapping is the identity map, since by the Riesz representation theorem we can identify H with its dual space H^*.

Definition 4.4 (dissipative operator)**.** An operator $F : \mathscr{D}(F) \subset X \to X$ is said to be *dissipative* if for every $x_1, x_2 \in \mathscr{D}(F)$ there exists $f \in \mathscr{F}(x_1 - x_2)$ such that

$$\mathrm{re}(f(F(x_1) - F(x_2))) \leq 0.$$

It is interesting to look briefly at the case of euclidean spaces. Let F be a real function from $\mathbb{R}^n$ to itself. Then the definition of dissipativity reads as:

$$\langle x_1 - x_2, F(x_1) - F(x_2) \rangle \leq 0 \qquad \forall\, x_1, x_2 \in \mathscr{D}(F).$$

If $n = 1$, the inner product becomes a simple product between real numbers, and F is dissipative if and only if it is a decreasing map.
Let us consider two solutions x_1 and x_2 of the equation $\dot{x} = F(x)$. Then by the dissipativity of F we obtain:

$$\frac{d}{dt}\left(\frac{1}{2}\|x_1(t) - x_2(t)\|^2\right) \leq 0.$$

This means that, if a solution of the system generated by F exists, then it is unique. Indeed, if $x_1(0) = x_2(0)$, then the norm $\|x_1(t) - x_2(t)\|$ must be equal to 0. This property can be generalized for abstract Banach space, where the solution of a system given by a dissipative operator is necessarily unique.
Is interesting to observe that the study of dissipative operators was motivated by the analysis of hyperbolic partial differential equations in Hilbert spaces (see [60]). Originally they represented an abstract description of *dissipativity* in a physical sense: in a dissipative system, the energy is non-increasing in time. In the study of such kind of systems, the Hilbert space is often equipped with a norm which corresponds to the total energy of the system. For instance, consider a linear dynamics in $\mathbb{C}^n$:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases} .$$

We suppose that the inner product is defined in such a way that he total energy is represented by $E = \|x(t)\|^2$, where $x(t)$ is the state variable at time t. Then, if the complex matrix A is dissipative:

$$\frac{d}{dt}\|x(t)\|^2 = \langle Ax(t), x(t)\rangle + \langle x(t), Ax(t)\rangle = 2\mathrm{re}(Ax(t), x(t)) \leq 0,$$

i.e. the energy of the system does not increase.
Since a lot of important examples of systems generated by dissipative operators arose in different contexts, this theory was generalized to Banach spaces. In particular, dissipative operators are fundamental in the study of dynamical systems, since, under suitable hypotheses, they generate nonlinear semigroups of evolution operators.

Note that dissipativity can be also interpreted as a metric property:

Proposition 4.3. *An operator $F : \mathscr{D}(F) \subset X \to X$ is dissipative if and only if for all $\lambda > 0$, $x_1, x_2 \in \mathscr{D}(F)$,*

$$\|x_1 - x_2 - \lambda(F(x_1) - F(x_2))\| \geq \|x_1 - x_2\|.$$

The following proposition gives an important characterization of dissipativity:

Proposition 4.4. *Let I be the identity operator on X and let F be a nonlinear operator. Let us suppose that $(I - \lambda F)$ admits an inverse operator for $\lambda > 0$. We define:*

$$R_\lambda := (I - \lambda F)^{-1},$$

with domain $\mathrm{ran}(I - \lambda F)$ *and range* $\mathscr{D}(F)$*. Then, the operator F is dissipative if and only if R_λ is a contractive operator for every $\lambda > 0$, i.e. for all $x_1, x_2 \in \mathscr{D}(R_\lambda)$*

$$\|R_\lambda(x_1) - R_\lambda(x_2)\| \leq \|x_1 - x_2\|.$$

We recall that given two single valued operators F and G, we call G an extension of F if $\mathscr{D}(F) \subset \mathscr{D}(G)$ and $G(x) = F(x)$ on $\mathscr{D}(F)$.

Definition 4.5. Let S be a subset of X. A dissipative operator F is called *maximal dissipative* on S if all dissipative extensions of F coincide with F on S. This means that for any dissipative extension G of F we have $\mathscr{D}(G) \cap S = \mathscr{D}(F) \cap S$ and $G(x) = F(x)$ on $\mathscr{D}(F) \cap S$.

Definition 4.6. A dissipative operator $F : \mathscr{D}(F) \subset X \to X$ is said to be *m-dissipative* if for every $\lambda > 0$

$$\mathrm{ran}(I - \lambda F) = X.$$

Note that an m-dissipative operator is a maximal dissipative operator on $S = X$. Besides, the following characterization holds:

Proposition 4.5. *A necessary and sufficient condition for F to be an m-dissipative operator is that F is dissipative and for some $\lambda_0 > 0$*

$$\mathrm{ran}(I - \lambda_0 F) = X.$$

In particular, it is possible to prove that a continuous, everywhere defined, dissipative operator from a Banach space to itself is m-dissipative ([8]) .

Very often in the literature ω-dissipative operators are considered. An operator F is said to be ω-dissipative if $F - \omega I$ is dissipative, i.e. for all x_1, x_2 in $\mathscr{D}(F)$ there exists $f \in \mathscr{F}(x_1 - x_2)$ such that:

$$\mathrm{re}(f(F(x_1) - F(x_2))) \leq \omega \|x_1 - x_2\|^2.$$

Note that if F is a globally Lipschitz operator on its domain with Lipschitz constant C, then F is C-dissipative. Given $f \in \mathscr{F}(x_1 - x_2)$ we can write

$$\mathrm{re} f(F(x_1) - F(x_2)) \leq \|f\| \|F(x_1) - F(x_2)\| = \|x_1 - x_2\| \|F(x_1) - F(x_2)\| \leq C \|x_1 - x_2\|^2.$$

The following characterization of ω-dissipative operators holds:

Proposition 4.6 ([43, Proposition 1.8])**.** *Let F be an operator on X and $\omega \in \mathbb{R}$. Then, the following are equivalent:*

(i) the operator F is ω-dissipative;

(ii) $\|x_1 - x_2 - \lambda(F(x_1) - F(x_2))\| \geq (1 - \lambda\omega)\|x_1 - x_2\|$ *for all* $\lambda \in (0, \frac{1}{|\omega|})$.

The main part of our results about lumpability of nonlinear systems will be proved for dissipative operators, but we will point out that these results can be easily generalized to the case of ω-dissipative operators.

4.2.2 Nonlinear semigroups

We give the definition of a nonlinear semigroup of operators. This concept is fundamental because the solution operators of a well-posed dynamics form a semigroup defined on the state space of the system.

Definition 4.7. Let X_0 be a closed subset of X and let $\omega \in \mathbb{R}$. The family of nonlinear operators $\{T(t)\}_{t\geq 0}$ is a nonlinear semigroup of type ω if the following hold:

1. $T(0)x = x \ \forall x \in X_0$,

2. $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,

3. $[0, +\infty) \ni t \to T(t)x \in X_0$ is continuous for every $x \in X_0$,

4. For all $x_1, x_2 \in X_0$ and $t \geq 0$

$$\|T(t)x_1 - T(t)x_2\| \leq e^{\omega t}\|x_1 - x_2\|.$$

When $\omega = 0$, we call $T(t)$ a semigroup of contractions on X_0.

A well-known example of a contraction semigroup in the linear case is the *heat semigroup* in $L_1(\mathbb{R})$, acting as a convolution operator:

$$T(t)f := K_t * f, \qquad K_t(s) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}.$$

For a simple example in the nonlinear case, which can be found in [51], we consider the Banach space $C[0,1]$ and we define a function $\phi : \mathbb{R} \to \mathbb{R}$ in the following way:

$$\phi(s) := \begin{cases} s & \text{if } s \geq 0 \\ 2s & \text{if } s < 0 \end{cases}.$$

Then, the semigroup:

$$T(t)f(x) := \phi(t + \phi^{-1}(f(x))), \quad t \geq 0,$$

is a nonlinear contractions semigroup on $C[0,1]$.

Given a nonlinear semigroup of type ω, we define an operator $F_0 : \mathscr{D}(F_0) \subset X_0 \to X$ by:

$$\mathscr{D}(F_0) := \{x \in X_0 \text{ such that } \lim_{h \to 0^+} \frac{1}{h}(x - T(h)x) \text{ exists}\},$$

$$F_0(x) := \lim_{h \to 0^+} \frac{1}{h}(x - T(h)x).$$

F_0 is said to be the *infinitesimal generator* of $T(t)$.
For a semigroup of contractions, the infinitesimal generator is always a dissipative operator. By the Hille and Yosida theory, the infinitesimal generator of a linear semigroup has always a dense domain. This doesn't hold for nonlinear semigroups: in fact, the existence of the infinitesimal generator is closely related to the differentiability of the semigroup, since $T(t)x$ belongs to $\mathscr{D}(F_0)$ if and only if $s \to T(s)x$ is strongly right differentiable in $s = t$. As we show in the next example [22], it is possible to construct a nonlinear semigroup such that $\mathscr{D}(F_0) = \emptyset$.

Example 4.1. We consider the Banach space $X = C[-1,1]$. For every $x \in [-1,1]$ we choose a semigroup of contractions $\{S_x(t)\}_{t\geq 0}$ from $\mathbb{R}$ in such a way that the following semigroup maps X to itself:

$$S(t)f\,(x) := S_x(t)f(x),\ t \geq 0.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a decreasing function such that its discontinuities are dense in $\mathbb{R}$. We define $g_\pm(x) := \lim_{h\to 0} g(x \pm h)$ and :

$$g_0(x) = \begin{cases} g_-(x) & \text{if } g_-(x) < 0 \\ 0 & \text{if } g_-(x) \geq 0 \text{ and } g_+(x) \leq 0 \\ g_+(x) & \text{if } g_+(x) < 0. \end{cases}$$

We impose $S_x(t) = T(t)$ for all $x \in [-1,1]$ and $t \geq 0$, where $T(t)$ is a semigroup of contractions on $\mathbb{R}$ satisfying:

$$D^+T(t)(x) = g_0(T(t)(x)), \quad x \in \mathbb{R},$$

(D^+ being the right derivative).
Then, $S(t)$ is a semigroup of contractions on X such that

$$\lim_{h\to 0^+} \frac{1}{h}(x - S(h)x) = g_0(f(x)),$$

but $g_0 \circ f$ is continuous if and only if f is a constant function.
Now, with a particular definition of $S_x(t)$, we can eliminate also the differentiability of $S(t)f$ for f constant. We suppose that g is continuous in 0 and, for $x \in [-1,1]$, we define the family of functions:

$$g^x(y) := g(x^2y + x), \quad y \in \mathbb{R}.$$

Then we define a corresponding family of semigroups on $\mathbb{R}$ by the following relation:

$$D^+S_x(t)y := g_0^x(S_x(t)y), \quad y \in \mathbb{R},\, t \geq 0,\, x \in [-1,1].$$

If $T(t)$ satisfies $D^+T(t)(x) = g_0(T(t)(x))$, $x \in \mathbb{R}$, then we can write

$$S_x(t)y = \begin{cases} x^{-2}(T(tx^2)(x^2y + x) - x) & \text{if } x \neq 0 \\ y + tg(0) & \text{if } x = 0. \end{cases}$$

It is possible to prove that in this way $S(t)$ is a contraction semigroup on X and

$$\lim_{h\to 0^+} \frac{1}{h}(x - S(h)x) = g_0(x^2f(x) + x),$$

but $g_0(x^2f(x) + x)$ is never an element of X. This means that the infinitesimal generator of $S(t)$ has empty domain.

4.2.3 The nonlinear abstract Cauchy problem

We consider the Cauchy problem (4.1) associated with a nonlinear operator F from a given domain $\mathscr{D}(F) \subset X$ to X.
We deal with two different kinds of solution, following the terminology of [51] and [8].

Definition 4.8. Let us consider a continuous function $x(t) : [0,+\infty) \to X$ such that $x(0) = x_0$. Then:

(i) $x(t)$ is called a *classical solution* of the Cauchy problem (4.1) if $x(t)$ is continuously differentiable on $[0,+\infty)$ and it satisfies $\dot{x}(t) = F(x(t))$ for all $t \geq 0$;

(ii) $x(t)$ is called a *solution* of the Cauchy problem (4.1) if $x(t)$ is Lipschitz continuous on any compact subinterval of $[0,+\infty)$, it is differentiable a.e. $t \in (0,+\infty)$ and it satisfies $\dot{x}(t) = F(x(t))$ a.e. $t \in (0,+\infty)$.

We have already mentioned in the Introduction that Peano's theorem, which guarantees the existence of local solutions for the Cauchy problem associated with a continuous function, fails in infinite dimension. However, when we deal with Lipschitz continuous functions, we can still use the following classical result:

Theorem 4.7 (Picard-Lindelöf theorem)**.** *Let us consider a function $f : [0,+\infty) \times X \to X$. If $f(t,x)$ is Lipschitz continuous in x and continuous in a neighbourhood of $t_0 \in [0,+\infty)$, then, for some $\epsilon > 0$, there exists a unique classical solution of the following Cauchy problem defined in $[t_0-\epsilon, t_0+\epsilon] \times X$:*

$$\begin{cases} \dot{x}(t) = f(t,x(t)) & t \in [t_0-\epsilon, t_0+\epsilon] \\ x(t_0) = x_0 \end{cases}.$$

In particular, the Lipschitzianity of f guarantees the uniqueness of the solution, as a consequence of the Banach fixed point theorem, which holds for general complete metric spaces:

Theorem 4.8. *Let (X,d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$).*

However, we want to study the dynamics associated with operators that are not necessarily Lipschitz continuous.
Dissipative operators are indeed the best candidates to obtain a well-posed dynamics, because they can generate semigroups of contractions. First of all, the following property holds:

Lemma 4.9 ([51, Lemma 4.9])**.** *If F is a dissipative operator, then the Cauchy problem associated with F has at most one solution for a given initial condition.*

The following result is known as the *Crandall-Liggett generation theorem.*

Theorem 4.10 ([22])**.** *Let F be a dissipative operator satisfying the following condition:*

$$\mathscr{D}(F) \subset \operatorname{ran}(I - \lambda(F)) \qquad \forall\, \lambda > 0. \tag{4.6}$$

Then, there exists a semigroup of contractions $T(t)$ on $\overline{\mathscr{D}(F)}$ such that:

(i) For every $x \in \overline{\mathscr{D}(F)} \cap \bigcap_{\lambda>0} \operatorname{ran}(I-\lambda F)$, $[\frac{t}{\lambda}]$ being the largest integer $\leq \frac{t}{\lambda}$, the following limit holds for every $t \geq 0$:

$$T(t)x = \lim_{\lambda \to 0^+} (I - \lambda F)^{-[\frac{t}{\lambda}]} x. \tag{4.7}$$

The convergence is uniform on every bounded intervals of $[0,+\infty)$.

(ii) For every $x \in \mathscr{D}(F)$, $t, s \geq 0$

$$\|T(t)x - T(s)x\| \leq \|F(x)\||t-s|.$$

When the theorem above holds, we say that F is the generator of the semigroup $T(t)$ in the sense of Crandall-Liggett. Note that we cannot say that F is the infinitesimal generator of this semigroup because we don't know anything about the differentiability of the map $t \to T(t)x$. However, it is possible to prove that if F is single valued, continuous, and it has closed domain (in addition to the hypotheses of theorem (4.10)), then F is also the infinitesimal generator of $T(t)$ (see [51] [Thm 4.8]).

Theorem 4.11. *Let F be a closed, dissipative operator satisfying Condition (4.6) and let $T(t)$ be the semigroup of contractions on $\overline{\mathscr{D}(F)}$ given in the Crandall-Liggett theorem by (4.7). Let $x \in \mathscr{D}(F)$. If the semigroup $T(t)x$ is strongly differentiable for almost every $t \geq 0$, then the map $t \to T(t)x$ is the unique solution of the abstract Cauchy problem*

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0. \end{cases} \tag{4.8}$$

Note that if the Banach space X is reflexive and the Crandall-Liggett theorem holds, then the map $t \to T(t)x$ is automatically differentiable for almost every $t \geq 0$, because it is Lipschitz continuous. Furthermore, if F is everywhere defined and (4.6) holds, F is also m-dissipative, and then it is automatically closed (see [51]).

Theorem 4.12 ([8, Corollary 3.1])**.** *Let X be a real Banach space and let F be a continuous, everywhere defined, dissipative operator from X to itself. Then, for every $x_0 \in X$ there exists a unique classical solution $u \in C^1(0,\infty;X)$ of the nonlinear abstract Cauchy problem associated with F.*
Moreover, the family of the solution operators $T(t)x_0 := u(t, x_0)$ is a strongly continuous semigroup of nonlinear contractions on X, generated by F:

$$F(x) = \lim_{h\to 0} \frac{1}{h}(T(h)x - x).$$

Since an everywhere defined, continuous and dissipative operator is m-dissipative, condition (4.6) holds and F is the generator of $T(t)$ also in the Crandall-Liggett sense, i.e. the exponential formula (4.7) holds.
Under suitable hypotheses, the following result allows us to characterize all the solutions of a nonlinear abstract Cauchy problem.

Theorem 4.13 ([51, Thm 5.1])**.** *Let F be a closed, dissipative operator satisfying condition (4.6) and let $x \in \overline{\mathscr{D}(F)}$. If $u(t) : [0, +\infty) \to X$ is a solution of the abstract Cauchy problem associated by F, then*

$$u(t) = \lim_{h\to 0^+} (I - hF)^{-[\frac{t}{h}]}\,x, \quad t \geq 0.$$

The next results, proved in [22], concerns the generation of nonlinear semigroups of type ω and the Cauchy problem associated with ω-dissipative operators.

Theorem 4.14. *Let $\omega \in \mathbb{R}$ and F be an ω-dissipative operator satisfying the following condition:*

$$\overline{\mathscr{D}(F)} \subset \operatorname{ran}(I - \lambda(F)) \qquad \forall\, \lambda > 0, \lambda\omega < 1.$$

Then, there exists a semigroup $T(t)$ of type ω on $\overline{\mathscr{D}(F)}$ such that:

(i) For every $x \in \overline{\mathscr{D}(F)}$, $[\frac{t}{\lambda}]$ being the largest integer $\leq \frac{t}{\lambda}$, the following limit holds for every $t \geq 0$:

$$T(t)x = \lim_{\lambda\to 0^+} (I - \lambda F)^{-[\frac{t}{\lambda}]}\,x.$$

The convergence is uniform on every bounded intervals of $[0, +\infty)$.

(ii) If F_0 is the infinitesimal generator of $T(t)$, then for every $x \in \mathscr{D}(F_0)$, $t, s \geq 0$:

$$\|T(t)x - T(s)x\| \leq (e^{2|\omega|t+s} - e^{4|\omega|t})\|F_0(x)\||t - s|.$$

Theorem 4.15. *Let F be ω-dissipative and* $\operatorname{ran}(I - \lambda F) = X$ *for every $\lambda > 0, \lambda\omega < 1$. Let $T(t)$ be the semigroup of type ω generated by F in the Crandall-Liggett sense. Let $x_0 \in \mathscr{D}(F)$. Then the following are equivalent:*

(i) *$x(t)$ is the unique strong solution of (4.1) on $[0, +\infty)$,*

(ii) *$x(t) = T(t)x_0$ for every $t \geq 0$ and $x(t)$ is differentiable a.e. on $[0, +\infty)$.*

Since for a semigroup of nonlinear contractions the differentiability of the map $t \to T(t)x$ is not automatically guaranteed for any x in the domain of the generator, it is common in the literature to give a generalized concept of solution. With the following definition, when the Crandall-Liggett theorem holds, we can always associate a particular kind of solution with the semigroup of type ω generated by F:

Definition 4.9. Let $x \in X$ and $\omega \in \mathbb{R}$. The function $u(t) : [0, +\infty) \to X$ is an *integral solution of type* ω of the Cauchy problem (4.1) if $u(t)$ satisfies the following:

(i) $u(0) = x$,

(ii) $u(t)$ is continuous on $[0, +\infty)$,

(iii) for every $T > 0$, $0 < r < t < T$, $x_0 \in \mathscr{D}(F)$, $x = F(x_0)$:

$$e^{-2\omega t}\|u(t) - x_0\|^2 - e^{-2\omega r}\|u(r) - x_0\|^2 \leq 2\int_r^t e^{-2\omega h}\langle F(x_0), u(h) - x_0\rangle_s \, dh,$$

where $\langle x_1, x_2\rangle_s := \sup\{re(f(x_1)),\ f \in \mathscr{F}(x_2)\}$.

If F is a dissipative operator of type ω and a solution of (4.1) exists, then it is also an integral solution of type ω. Conversely, it is possible to prove that, if $F - \omega I$ is m-dissipative, an integral solution $u(t)$ that is also Lipschitz continuous on the bounded sets and weakly differentiable almost everywhere is also a solution of (4.1) (see [51]).
In particular, it is possible to prove the following:

Theorem 4.16. *Let F be an ω-dissipative operator. Let us suppose that* $\mathrm{ran}(I - \lambda F) = X$ *for every $\lambda > 0, \lambda\omega < 1$. Then, for every $x \in \overline{\mathscr{D}(F)}$, the Cauchy problem (4.1) has a unique integral solution of type ω. This solution is given by $T(t)x$, where $T(t)$ is the semigroup of type ω generated by F in the Crandall-Liggett sense.*

Remark 4.1. Note that if the solution operators of system (4.1) form a nonlinear semigroup of contractions and x_e is an equilibrium point for the system, i.e.
$F(x_e) = 0$, then:

$$\|T(t)x - T(t)x_e\| = \|T(t)x - x_e\| \leq \|x - x_e\|.$$

This means that the trajectory is bounded and the equilibrium is stable. Moreover, if $T(t)x$ is a semigroup of type $\omega < 0$, then

$$\|T(t)x - T(t)x_e\| \leq e^{\omega t}\|x - x_e\| \quad t \geq 0,$$

i.e. the trajectory tends to x_e for $t \to +\infty$ (x_e is then asymptotically stable).

4.3 Regularity of the operator $\widehat{F}$

In this section we discuss which regularity properties of the function F are preserved by the lumping. We first prove that continuity is maintained:

Proposition 4.17. *Let $F : \mathscr{D}(F) \subset X \to X$ be a nonlinear map satisfying condition (4.4). If F is continuous, then the nonlinear map $\widehat{F} : \mathscr{D}(\widehat{F}) \subset Y \to Y$ defined in (4.5) is also continuous.*

Proof. For neatness of notation, we first assume that $\mathscr{D}(F) = X$, and then we generalize to the case of a smaller domain. We consider an open set $\mathscr{A} \subset Y$. We need to prove that $\widehat{F}^{-1}(\mathscr{A})$ is also an open set in Y. To this purpose we write:

$$M^{-1}\widehat{F}^{-1}(\mathscr{A}) = (\widehat{F} \circ M)^{-1}(\mathscr{A}) = (MF)^{-1}(\mathscr{A}).$$

Since M is linear and bounded and F is continuous, $(MF)^{-1}(\mathscr{A})$ is an open set in X, so that $M^{-1}\widehat{F}^{-1}(\mathscr{A})$ is open.
Given that M is surjective, we obtain:

$$M(M^{-1}\widehat{F}^{-1}(\mathscr{A})) = \widehat{F}^{-1}(\mathscr{A}),$$

and this set is open in Y because M is an open map by the Banach-Schauder theorem (i.e. it maps open sets in open sets). In the case $\mathscr{D}(F) \subset X$, F is continuous if and only if, for any open set $\mathscr{A} \cap X$, $F^{-1}(\mathscr{A})$ is open with respect to the subspace topology induced by X on $\mathscr{D}(F)$, i.e it can be written as $\mathscr{B} \cap \mathscr{D}(F)$ for some open set $\mathscr{B}$ in X. Using the notation as above,

$$(MF)^{-1}(\mathscr{A}) = \mathscr{D}(F) \cap \mathscr{B}, \quad \mathscr{B} \text{ open in } X.$$

Then we obtain

$$M(M^{-1}\widehat{F}^{-1}(\mathscr{A})) = M(\mathscr{D}(F) \cap \mathscr{B}) \subset M\mathscr{D}(F) \cap \mathscr{D},$$

where $\mathscr{D}$ is the open set $M(\mathscr{B})$. But since MF is continuous, $M(\mathscr{D}(F) \cap \mathscr{B})$ is open in Y: if we write it as $M(\mathscr{D}(F) \cap \mathscr{B}) \cap \mathscr{D}$, it is clear that it is open with respect to the subspace topology. From this it follows that $\widehat{F}^{-1}(\mathscr{A})$ is also open with respect to the subspace topology on $\mathscr{D}(\widehat{F})$. □

As we did in the previous chapters, we make use of two particular operators defined on the quotient Banach space $X/\ker(M)$. First of all, we need to consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "M"] \arrow[dr, "\pi"'] & Y \arrow[d, leftrightarrow, "\widetilde{M}"] \\ & X/\ker(M) \end{tikzcd} \tag{4.9}$$

We have already pointed out that the operator $\widetilde{M} : X/\ker(M) \to Y$, defined by $\widetilde{M}[x] := M(x)$, is an homeomorphism.
Then, we look at:

$$\begin{tikzcd} X \arrow[r, "MF"] \arrow[dr, "\pi"'] & Y \\ & X/\ker(M) \arrow[u, "\widetilde{MF}"'] \end{tikzcd} \tag{4.10}$$

where the operator $\widetilde{MF} : X/\ker(M) \to Y$ is defined by

$$\widetilde{MF}[x] := MF(x). \tag{4.11}$$

This operator is well-defined even if F is nonlinear, provided that it satisfies condition (4.4). These two operators will be fundamental in the proof of the next results.

Proposition 4.18. *Let $F : X \to X$ be a nonlinear, everywhere defined map from a Banach space X to itself satisfying condition (4.4). If F is Lipschitz continuous, i.e. there exists a constant $K \in \mathbb{R}^+$ such that $\|F(x_1) - F(x_2)\| \leq K\|x_1 - x_2\|$ for every $x_1, x_2 \in X$, then the nonlinear map $\widehat{F} : Y \to Y$ defined as in (4.5) is also Lipschitz continuous.*

Proof. Let us consider diagram (2.8). Since F preserves the fibers of M, the map $\widetilde{MF}$ defined in (4.11) is well-defined. Given two elements $[x_1], [x_2] \in X/\ker(M)$, we obtain that:

$$\|\widetilde{MF}([x_1]) - \widetilde{MF}([x_2])\| = \inf_{m \in \mathrm{Ker(M)}} \|\widetilde{MF}([x_1 - m]) - \widetilde{MF}([x_2])\| =$$

$$\inf_{m \in \mathrm{Ker(M)}} \|MF(x_1 - m) - MF(x_2)\| \leq \inf_{m \in \mathrm{Ker(M)}} \|M\| \, \|F(x_1 - m) - F(x_2)\| \leq$$

$$\leq \inf_{m \in \mathrm{Ker(M)}} K \, \|M\| \, \|(x_1 - m) - x_2\| = K \, \|M\| \, \|[x_1] - [x_2]\|.$$

This means that $\widetilde{MF}$ is Lipschitz with Lipschitz constant $K \, \|M\|$. Now, if $y_1 = Mx_1$, $y_2 = Mx_2$ are points in Y, then:

$$\|\widehat{F}(y_1) - \widehat{F}(y_2)\| = \|MF(x_1) - MF(x_2)\| = \|\widetilde{MF}([x_1]) - \widetilde{MF}([x_2])\|$$

$$\leq K \, \|M\| \, \|[x_1] - [x_2]\| = K \, \|M\| \, \|\widetilde{M}^{-1} y_1 - \widetilde{M}^{-1} y_2\| \leq K \, \|M\| \, \|\widetilde{M}^{-1}\| \, \|y_1 - y_2\|,$$

i.e. $\widehat{F}$ is Lipschitz with constant $K \, \|M\| \, \|\widetilde{M}^{-1}\|$. □

Remark 4.2. Let us observe that, by definition of $\widetilde{M}$ and of the equivalence class $[x]$,

$$\|\widetilde{M}\| = \sup_{\|[x]\| \leq 1} \|\widetilde{M}[x]\| = \sup_{\|[x]\| \leq 1} \inf_{m \in \mathrm{Ker(M)}} \|M(x - m)\| \leq$$

$$\leq \sup_{\|[x]\| \leq 1} \inf_{m \in \mathrm{Ker(M)}} \|M\| \, \|x - m\| \leq \sup_{\|[x]\| \leq 1} \|M\| \|[x]\| \leq \|M\|.$$

On the other hand, since the quotient map π is a contraction operator, we can write

$$\|M\| = \sup_{\|x\| \leq 1} \|Mx\| = \sup_{\|x\| \leq 1} \|\widetilde{M}[x]\| \leq$$

$$\leq \sup_{\|x\| \leq 1} \|\widetilde{M}\| \, \|\pi(x)\| \leq \sup_{\|x\| \leq 1} \|\widetilde{M}\| \|(x)\| \leq \|\widetilde{M}\|.$$

This means that $\|M\| = \|\widetilde{M}\|$.
Let us call $\widehat{K}$ the Lipschitz constant of $\widehat{F}$; we have $\widehat{K} = K \, \|M\| \, \|\widetilde{M}^{-1}\| = K \, \|\widetilde{M}\| \, \|\widetilde{M}^{-1}\|$. Since in general $\|\widetilde{M}^{-1}\| \, \|\widetilde{M}\| \geq 1$, we have that $\widehat{K} \geq K$.
For this reason $\widehat{F}$ need not to be a contractive operator even if $K < 1$, unless we put additional conditions on the lumping operator M, such as $\|M\| \|\widetilde{M}^{-1}\| = 1$.

Next, we prove that under suitable hypotheses the function $\widehat{F}$ preserves the smoothness of F.
First of all, we suppose that F is Gâteaux differentiable on the whole X with Gâteaux derivative DF_x in the point $x \in X$. Then, we claim that $\widehat{F}$ is also Gâteaux differentiable in Y. Indeed, if $y = Mx$ and $z = Mh$:

$$\lim_{t \to 0} \frac{\widehat{F}(y + tz) - \widehat{F}(y)}{t} = \lim_{t \to 0} \frac{MF(x + th) - MF(x)}{t}$$

$$= M \left(\lim_{t \to 0} \frac{F(x + th) - F(x)}{t} \right) = MDF_x(h).$$

This means that we can define the Gâteaux derivative of $\widehat{F}$ as

$$D\widehat{F}_y(z) := MDF_x(h),$$

for every $y = Mx$ and $z = Mh$. If y is fixed, then $D\widehat{F}_y$ is a linear and bounded operator from Y to itself.

Proposition 4.19. *Let us suppose that $F : \mathscr{D}(F) \to X$ satisfies condition (4.4). Let us assume that F is C^1 on its domain, i.e. DF is a continuous operator from $\mathscr{D}(F)$ to $\mathscr{B}(X)$. Then, the reduced operator $\widehat{F}$ defined in (4.5) is also C^1 on $M\mathscr{D}(F)$.*

Proof. In the next passages we will use the notation A for a linear operator, to distinguish from the nonlinear function F. We consider the following subspace of $\mathscr{B}(X)$:

$$\widetilde{\mathscr{B}}(X) := \{A \in \mathscr{B}(X) \text{ such that } \ker(M) \subset \ker(MA)\}.$$

This is the space of all the linear and bounded operators A such that the reduced operator $\widehat{A}y := MAx$, $y = Mx$ is well-defined and belongs to $\mathscr{B}(Y)$ (see [5] and Chapter 1 of the present work).
It is easy to verify that $\widetilde{\mathscr{B}}(X)$ is a linear space containing 0 and I. Moreover, it is a closed subspace of $\mathscr{B}(X)$. Indeed, given A_n such that $A_n \to A$ in $\mathscr{B}(X)$, $A_n \in \widetilde{\mathscr{B}}(X)$, and x_1, x_2 such that $Mx_1 = Mx_2$, we obtain:

$$\begin{aligned}\|MAx_1 - MAx_2\| &\le \|MAx_1 - MA_nx_1\| + \|MA_nx_1 - MA_nx_2\| + \|MA_nx_2 - MAx_2\| \\ &= \|MAx_1 - MA_nx_1\| + \|MA_nx_2 - MAx_2\|,\end{aligned}$$

because $MA_nx_1 = MA_nx_2$ for all $n \in \mathbb{N}$.
If we take the limit $n \to +\infty$, we obtain $\|MAx_1 - MAx_2\| = 0$. This means $MAx_1 = MAx_2$, i.e. $A \in \widetilde{\mathscr{B}}(X)$.
We define the following linear operator between Banach spaces:

$$\mathscr{M} : \widetilde{\mathscr{B}}(X) \to \mathscr{B}(Y), \quad \mathscr{M}(A) := \widehat{A},$$

where $\widehat{A}y = MAx$ for all $y = Mx \in Y$.
We first prove that this operator is continuous, and then we will show the continuity of $D\widehat{F}$ by the continuity of $\mathscr{M}$.
We consider diagram (4.10) and the operator $\widetilde{MA}[x] := MAx$ from $X/\ker(M)$ to Y, which is well-defined, linear and bounded (see Chapter 1 of the present work). Let us suppose that $A_n \to A$ in $\mathscr{B}(X)$. We can write:

$$\|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| = \|MA_n(x-m) - \widetilde{MA}(x-m)\|$$

$$= \inf_{m\in\ker(M)} \|MA_n(x-m) - \widetilde{MA}(x-m)\| \le \inf_{m\in\ker(M)} \|MA_n - MA\|\|x-m\|$$

$$= \|MA_n - MA\|\|[x]\|,$$

and then:

$$\sup_{\|[x]\|\le 1} \|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| \le \|MA_n - MA\|.$$

Taking the limit $n \to +\infty$, we obtain $\sup_{\|[x]\|\le 1}\|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| \to 0$.
Now, using the properties of the operator $\widetilde{M}$ (see diagram (4.9)), we can write:

$$\|\mathscr{M}A_n - \mathscr{M}A\|_{\mathscr{B}(Y)} = \sup_{\|y\|\le 1} \|\widehat{A}_n - \widehat{A}\|$$

$$= \sup_{\|y\|\le 1} \|MA_n x - MAx\| = \sup_{\|y\|\le 1} \|\widetilde{MA_n}[x] - \widetilde{MA}[x]\|$$

$$\le \sup_{\|y\|\le 1} \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1} y\| \le \sup_{\|y\|\le 1} \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1}\| \|y\|$$

$$\le \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1}\|.$$

Since the norm $\|\widetilde{MA_n} - \widetilde{MA}\|$ (which is the operatorial norm in $\mathscr{B}(X/\ker(M), Y)$ tends to zero for $n \to +\infty$, then $\mathscr{M}A_n$ converges to $\mathscr{M}A$ in $\mathscr{B}(Y)$.
This means that $\mathscr{M}$ is a linear and bounded operator from $\widetilde{\mathscr{B}}(X)$ to $\mathscr{B}(Y)$.

For neatness of notation, we first suppose that $\mathscr{D}(F) = X$. We know that $\widehat{F}$ is at least Gâteaux differentiable with Gâteaux derivative $D\widehat{F}_y(z) := MDF_x(h)$, for every $y = Mx$ and $z = Mh$. Furthermore, DF is also the Fréchet differential of F because F is C^1.
It is easy to see that, by definition,

$$D\widehat{F} \circ M = \mathscr{M} DF$$

as operators from X to $\mathscr{B}(Y)$.
We want to show that $D\widehat{F}$ is continuous from Y to $\mathscr{B}(Y)$. We take an open set $\mathscr{A} \subset \mathscr{B}(Y)$ and we write:

$$M^{-1}(D\widehat{F}^{-1})(\mathscr{A}) = (D\widehat{F} \circ M)(\mathscr{A})^{-1} = (\mathscr{M}DF)^{-1}(\mathscr{A}),$$

that is an open set in X because $\mathscr{M}$ and DF are continuous.
Since M is surjective and open we obtain:

$$D\widehat{F}^{-1}(\mathscr{A}) = MM^{-1}(D\widehat{F}^{-1})(\mathscr{A}) = M(\mathscr{M}DF)^{-1}(\mathscr{A}),$$

which is an open set in Y.
By the continuity of the map $y \to D\widehat{F}_y$, we obtain that $\widehat{F}$ is C^1 and $D\widehat{F}$ is its Fréchet differential.
This result is still true if $\mathscr{D}(F) \neq X$, provided that F is C^1 on its domain. Indeed, even in this case DF_x is a bounded operator on X for $x \in \mathscr{D}(F)$. We have $D\widehat{F} \circ M = \mathscr{M}DF$ as operators from $\mathscr{D}(F)$ to $\mathscr{B}(Y)$. In this case, for every open set $\mathscr{A} \subset \mathscr{B}(Y)$, $D\widehat{F}^{-1}(\mathscr{A})$ is open with respect to the subspace topology on $\mathscr{D}(\widehat{F})$. Then, $\widehat{F}$ is C^1 on $M\mathscr{D}(F)$. $\square$

4.3.1 Linearization and local lumping

We consider an important application of the results we obtained in the previous section about regularity of the reduced map.
Let us consider a point $x_0 \in X$ in which F is C^1, and the ball $\mathscr{B}_\alpha(x_0)$ centered in x_0 with radius $\alpha > 0$. Let us call $y_0 := Mx_0$.
Since M is an open map, it follows that for every $\alpha > 0$ there exists $\beta > 0$ such that:

$$\mathscr{B}_\beta(Mx_0) \subset M\mathscr{B}_\alpha(x_0). \tag{4.12}$$

This means that, for α fixed, we can find $\beta > 0$ such that all the points $y \in \mathscr{B}_\beta(y_0)$ can be written as $y = Mx$, with $x \in \mathscr{B}_\alpha(x_0)$.
Since Proposition 4.19 holds, we can write the following linearization for the reduced operator $\widehat{F}$:

$$\widehat{F}(y_0 + y) = \widehat{F}(y_0) + D\widehat{F}_{y_0} y + o(\|y\|),$$

that is:

$$\widehat{F}(y_0 + y) = \widehat{F}(y_0) + MD\widehat{F}_{x_0} x + o(\|y\|).$$

Now, let us suppose that $F(0) = 0$ and let us choose $x_0 = 0$. Since 0 is an equilibrium for the system, the linearization around 0 becomes:

$$\widehat{F}(y) = MDF_0 x + o(\|y\|). \tag{4.13}$$

Let us choose $\alpha << 1$, and let us consider β such that (4.12) holds.
For all $y \in \mathscr{B}_\beta(0)$ we can approximate $\widehat{F}$ with MDF_0 using (4.13).
We have proved in the previous section that the lumping relation holds between DF and $D\widehat{F}$:

$$D\widehat{F}_0(y) := MDF_0(x),$$

i.e. $\ker(M)$ is DF_0-invariant.
This means that, looking at F and $\widehat{F}$ in $\mathscr{B}_\alpha(0) \subset X$ and $\mathscr{B}_\beta(0) \subset Y$ respectively, we deal with a lumping of linear maps.
Since DF_0 and $D\widehat{F}_0$ are linear and bounded operators, they both generate well-posed dynamics. Let us call $T(t)$ and $\widehat{T}(t)$ the uniformly continuous semigroups generated by DF_0 and $D\widehat{F}_0$ respectively. By theorem 2.14 $\ker(M)$ is $T(t)$-invariant and the following lumping relation holds on X:

$$MT(t) = \widehat{T}(t)M.$$

In particular, the *growth bound* $\widehat{\omega}$ of the semigroup $\widehat{T}(t)$ is always less or equal than the growth bound ω of $T(t)$ [63] and, by boundedness of the operators involved:

$$\sup_{\lambda \in \sigma(\widehat{A})} \{\mathrm{Re}(\lambda)\} = \omega(\widehat{T}) \leq \sup_{\lambda \in \sigma(A)} \{\mathrm{Re}(\lambda)\} = \omega(T).$$

It is well known that a semigroup $T(t)$ is exponentially stable if and only if $\omega < 0$.
Using the linearized stability theorem in Banach spaces (see, for instance, [19]), we can study the local stability of the zero equilibrium for the nonlinear system associated with $\widehat{F}$ simply looking at the growth bound of the semigroup generated by $D\widehat{F}_0$.
In particular, if 0 is exponentially stable for the system associated with DF_0, then it is locally exponentially stable for the nonlinear system associated with $\widehat{F}$. Indeed, it has been proved that stability of equilibria is preserved by lumping ([63, 66]).

4.4 Lumpability of a nonlinear system

Let us consider $F : \mathscr{D}(F) \subset X \to X$ such that the associated Cauchy problem (4.1) admits a unique solution for every initial condition in $\mathscr{D}(F)$. Provided that the nonlinear operator $\widehat{F}$ is well-defined, we investigate whether also the following reduced Cauchy problem admits a unique solution for every initial condition in $\mathscr{D}(\widehat{F})$:

$$\begin{cases} \dot{y}(t) = \widehat{F}(y(t)) \\ y(0) = y_0. \end{cases} \tag{4.14}$$

In the case of $0 \in \mathscr{D}(F)$ and $F(0) = 0$, if the system is lumpable by M (i.e., the reduced system (4.14) is well-defined and admits a unique solution for every initial condition in its domain), then $\ker(M)$ is invariant under the flow generated by F. This fact is true also in finite dimension: if $x_0 \in \ker(M)$ and $x(t, x_0)$ is the solution of system (4.1) starting in x_0, then $Mx(t)$ is a solution of the reduced Cauchy problem with initial condition $y_0 = Mx_0 = 0$. But since F maps zero into zero and preserves the fibers of M, we can write $\widehat{F}(y_0) = MF(x_0) = MF(0) = 0$. This means that $y(t) \equiv 0$ is an equilibrium for the reduced system. By the uniqueness of the solution with null initial condition, we have that $Mx(t) = 0$ for all $t \geq 0$.

Proposition 4.20. *Let $M : X \to Y$ be linear, bounded and surjective and let $F : X \to X$ be a single valued, everywhere defined, dissipative nonlinear operator preserving the fibers of M, i.e. satisfying condition (4.4). Let $\widehat{F}$ be defined as in (4.5). Let us suppose that a solution of the Cauchy problem (4.14) exists for a given $y_0 \in Y$. Then, this solution is unique.*

Proof. Let us define the following map on the quotient space $X/\ker(M)$:

$$\widetilde{F} : X/\ker(M) \to X/\ker(M), \qquad \widetilde{F}[x] := [F(x)].$$

This map is well-defined, because if $x_1 \in [x]$ then $Mx_1 = Mx$; since condition (4.4) holds and F is everywhere defined, $MF(x_1) = MF(x)$, that is $[F(x_1)] = [F(x)]$.
In particular, $\widetilde{F}\pi = \pi F$, where π is the quotient projection $\pi(x) = [x]$.
Since F is a dissipative operator, we can show that $\widetilde{F}$ is dissipative on Y. In fact, for every $\lambda > 0$:

$$\|[x_1] - [x_2]\| = \inf_{m \in \ker(M)} \|x_1 - x_2 - m\| \leq \inf_{m \in \ker(M)} \|(x_1 - m) - x_2 - \lambda(F(x_1 - m) - F(x_2))\|$$

$$= \|[x_1 - x_2 - \lambda(F(x_1 - m) - F(x_2))]\| = \|[x_1] - [x_2] - \lambda([F(x_1)] - [F(x_2)])\| =$$
$$= \|[x_1] - [x_2] - \lambda(\widetilde{F}[x_1] - \widetilde{F}[x_2])\|.$$

Note that we have made use of the linearity of π, and that, since $m \in \ker(M)$, $[F(x_1 - m)]$ is the same as $[F(x_1)]$.
Now, taking in mind that the operator $\widetilde{M} : X/\ker(M) \to Y$, $\widetilde{M}[x] = Mx$, is an homeomorphism (see diagram 4.9), we obtain that for all $y = Mx \in Y$:

$$\widetilde{M}^{-1}\widehat{F}(y) = \widetilde{M}^{-1} MF(x) = [F(x)] =$$

$$= \pi F(x) = \widetilde{F}(\pi x) = \widetilde{F}(\widetilde{M}^{-1} y),$$

i.e. $\widetilde{M}^{-1} \circ \widehat{F} = \widetilde{F} \circ \widetilde{M}^{-1}$ on Y.
Let us consider two different solutions (not necessarily classical solutions) of system (4.14), say $y_1(t)$ and $y_2(t)$, both with the same value at $t = 0$:
$y_1(0) = y_2(0) = y_0$.
We have that:

$$\frac{d}{dt}\widetilde{M}^{-1} y_1(t) = \widetilde{M}^{-1} \frac{d}{dt} y_1(t) =$$
$$= \widetilde{M}^{-1}\widehat{F}(y_1(t)) = \widetilde{F}(\widetilde{M}^{-1} y_1(t)),$$

where the equalities above hold for every $t \geq 0$ or almost everywhere in the case of a classical solution and a solution respectively.
Since the same passages hold for $\widetilde{M}^{-1} y_2(t)$, we obtain that both the maps
$t \to \widetilde{M}^{-1} y_1(t)$ and $t \to \widetilde{M}^{-1} y_2(t)$ are solutions of the Cauchy problem associated with the map $\widetilde{F}$, with the same initial condition $\widetilde{M}^{-1} y_0 \in X/\ker(M)$, namely

$$\begin{cases} \dot{v}(t) = \widetilde{F}(v(t)) \\ v(0) = \widetilde{M}^{-1} y_0. \end{cases} \tag{4.15}$$

Since $\widetilde{F}$ is dissipative, by theorem 4.9 this Cauchy problem can have at most one solution for a given initial condition. This means that $\widetilde{M}^{-1} y_2(t) = \widetilde{M}^{-1} y_2(t)$ for every $t \geq 0$.
$\widetilde{M}^{-1}$ being an homeomorphism, we conclude that $y_1(t) = y_2(t)$ for every $t \geq 0$
(note that the uniqueness theorem 4.9 holds for a.e. differentiable solutions, not necessarily classical, and that both classical solutions and solutions are continuous functions). ∎

Remark 4.3. In general, the dissipativity of F does not guarantee the dissipativity of $\widehat{F}$. Indeed, we can write

$$\|y_1 - y_2\| = \|Mx_1 - Mx_2\| = \|\widetilde{M}[x_1] - \widetilde{M}[x_2]\|$$

$$\leq \|\widetilde{M}\| \, \|[x_1] - [x_2]\| \leq \|\widetilde{M}\| \, \|[x_1] - [x_2] - \lambda(\widetilde{F}[x_1] - \widetilde{F}[x_2]\|$$

$$\leq \|\widetilde{M}\| \, \|[x_1] - [x_2] - \lambda([F(x_1)] - [F(x_2)])\| \leq \|\widetilde{M}\| \, \|\widetilde{M}^{-1}\| \|Mx_1 - Mx_2 - \lambda(MF(x_1) - MF(x_2))\|$$

$$= \|\widetilde{M}\| \, \|\widetilde{M}^{-1}\| \|y_1 - y_2 - \lambda(\widehat{F}(y_1) - \widehat{F}(y_2))\|.$$

Since $\|\widetilde{M}\| \, \|\widetilde{M}^{-1}\| \geq 1$, we don't obtain any dissipativity. However, we have that for every $\lambda > 0$, $(I - \lambda\widehat{F})$ is injective and the inverse operator $(I - \lambda\widehat{F})^{-1}$ is Lipschitz continuous on $\operatorname{ran}(I - \lambda\widehat{F})$.
Note that, without imposing some restrictions on $\|\widetilde{M}\| \, \|\widetilde{M}^{-1}\|$, not even the ω-dissipativity of $\widehat{F}$ for some real nonzero ω can be obtained.

Theorem 4.21 (Continuous, dissipative operators)**.** *Let $M : X \to Y$ be a linear, bounded and surjective operator between Banach spaces and let $F : X \to X$ be a single valued, continuous and dissipative nonlinear operator defined on the whole X. Let us suppose that F preserves the fibers of M, i.e. condition (4.4) holds. Then, system (4.1) is lumpable by M, i.e. there exists a nonlinear, continuous operator $\widehat{F} : Y \to Y$ such that the Cauchy problem (4.14) has a unique, classical solution for every $y_0 = Mx_0 \in Y$.*

Proof. We define the reduced operator from Y to itself as in (4.5): $\widehat{F}(y) := MF(x)$, for $y = Mx$. This operator is well-defined on the whole Y because F itself is well-defined and condition (4.4) holds. Since F is continuous, $\widehat{F}$ is also continuous by Proposition 4.17.
We know that theorem 4.12 holds, so that F is the infinitesimal generator of a strongly continuous semigroup of contractions on X; we denote this semigroup by $T(t)$. For every $x_0 \in X$, $T(t)x_0$ is the unique classical solution of the problem (4.1).
Let us consider the map $t \to MT(t)x$ for a given $y = Mx$. By the differentiability of $T(t)x$ and the continuity of M we obtain:

$$\frac{d}{dt}MT(t)x = M\frac{d}{dt}T(t)x = MF(T(t)x) = \widehat{F}(MT(t)x),$$

and $MT(0)x = Mx = y$. This means that $MT(t)x$ is a solution of system (4.14) with initial condition y, i.e. system (4.14) admits a classical solution for every initial condition $y \in Y$. Since Proposition 4.20 holds, and $t \to MT(t)x$ is differentiable for every $t \geq 0$, we can say that $y(t) := MT(t)x$ is the unique classical solution of the Cauchy problem associated with $\widehat{F}$.
In particular, we obtain that the semigroup $T(t)$ preserves the fibers on M. In fact, given $x_1, x_2 \in X$ such that $y_0 = Mx_1 = Mx_2$, both the maps $t \to MT(t)x_1$ and $t \to MT(t)x_2$ are classical solutions of system (4.14) with the same initial condition y_0. Thus, this solution being unique, we must have $MT(t)x_1 = MT(t)x_2$ for every $t \geq 0$.
Consequently, we can define the following family of operators:

$$\widehat{T}(t)y := MT(t)x, \qquad y = Mx,\, t \geq 0. \tag{4.16}$$

Let us verify that $\widehat{T}(t)$ is a strongly continuous, nonlinear semigroup of Lipschitz operators on Y:

(i) $\widehat{T}(t)$ is a semigroup of nonlinear operators:
$\widehat{T}(0)y = MT(0)x = Mx = y$, and $\widehat{T}(t+s)y = MT(t+s)x = MT(t)(T(s)x) = \widehat{T}(t)(MT(s)x) = \widehat{T}(t)(\widehat{T}(s)y)$;

(ii) $\widehat{T}(t)$ is strongly continuous:
$\lim_{h\to 0}\|\widehat{T}(h)y - y\| = \lim_{h\to 0}\|M(T(t)x - x)\| \leq \lim_{h\to 0}\|M\|\|T(t)x - x\| = 0$;

(iii) For every $t \geq 0$, $\widehat{T}(t)$ is Lipschitz continuous, because by proposition 4.18, for all $y_1, y_2 \in Y$ we have:
$\|\widehat{T}(t)y_1 - \widehat{T}(t)y_2\| \leq \|M\|\,\|\widetilde{M}^{-1}\|\|y_1 - y_2\|$.

Furthermore, being F itself a generator, the infinitesimal generator of $\widehat{T}(t)$ is the operator $\widehat{F}$:

$$\lim_{h\to 0}\frac{1}{h}\left(\widehat{T}(t)y - y\right) = \lim_{h\to 0}\frac{1}{h}M\left(T(t)x - x\right) = M\lim_{h\to 0}\frac{1}{h}\left(T(t)x - x\right)$$
$$= MF(x) = \widehat{F}(y).$$

□

Remark 4.4. Using the notation as above, we can verify that, if $t \to \widehat{T}(t)y$ is continuous in $t = 0$, then it is continuous on the whole $[0, +\infty)$. This follows by the semigroup property and the Lipschitz continuity of $\widehat{T}(t)$.

We know that for all $\epsilon > 0$ there exists δ such that, if $|t| < \delta$, then $\|\widehat{T}(t)y - y\| < \frac{\epsilon}{\|M\|\,\|\widetilde{M}^{-1}\|}$.
Given an arbitrary $t \geq 0$ and writing $t_0 + t_s = t$ we obtain:

$$\|\widehat{T}(t)y - \widehat{T}(t_0)y\| = \|\widehat{T}(t_0 + t_s)y - \widehat{T}(t_0)y\| = \|\widehat{T}(t_0)\widehat{T}(t_s)y - \widehat{T}(t_0)y\|$$
$$\leq \|M\|\,\|\widetilde{M}^{-1}\|\|\widehat{T}(t_s)y - y\|.$$

Then, if $|t - t_0|$ is small enough, $|t_s| < \delta$ and

$$\|M\|\,\|\widetilde{M}^{-1}\|\|\widehat{T}(t_s)y - y\| < \epsilon.$$

i.e. $t \to \widehat{T}(t)y$ is continuous on the whole $[0, +\infty)$.

Next, we consider the case in which F is not continuous. We cannot obtain classical solutions because the semigroup generated in the Crandall-Liggett sense is not necessarily everywhere differentiable. But under suitable hypotheses we can construct a semigroup on the new Banach space Y and we can find some relations between its infinitesimal generator and the infinitesimal generator of the original semigroup $T(t)$. Moreover, we show that the asymptotic formula (4.7) given in the Crandall-Liggett generation theorem is inherited by the reduced semigroup.

Theorem 4.22 (Everywhere defined, m-dissipative operators)**.** *Let $M : X \to Y$ be linear, bounded and surjective. Let F be a nonlinear, everywhere defined, m-dissipative operator from X to itself such that condition (4.4) holds. Let $T(t)$ be the semigroup of contractions on X given in the Crandall-Liggett theorem by (4.7) and let F_0 be its infinitesimal generator. Let us suppose that for every $x \in X$ the semigroup $T(t)x$ is strongly differentiable for almost every $t \geq 0$. Then, the following hold:*

(i) system (4.1) is lumpable by M, i.e. there exists a nonlinear, continuous operator $\widehat{F} : Y \to Y$ such that system (4.14) has a unique solution (differentiable a.e. $t \geq 0$) for every $y_0 = Mx_0 \in Y$. The family $\widehat{T}(t)$ of its solution operators is a semigroup of Lipschitz operators on Y.

(ii) For all $\lambda > 0$ and $y = Mx \in Y$, $(I - \lambda\widehat{F})^{-1}(y) = M(I - \lambda F)^{-1}(x)$. Consequently, for every $y \in Y$ the following formula holds:

$$\widehat{T}(t)y = \lim_{\lambda \to +0} \left(I - \lambda\widehat{F}\right)^{-[\frac{t}{\lambda}]} y. \tag{4.17}$$

(iii) If for some $\lambda_0 > 0$ the operator $M(I - \lambda_0 F_0)$ is surjective from $\mathscr{D}(F_0) \subset X$ to Y, then $M\mathscr{D}(F_0) = \mathscr{D}(\widehat{F}_0)$, where $\widehat{F}_0$ is the infinitesimal generator of $\widehat{T}(t)$.

Proof. [(i)]: By theorem 4.11 we know that the semigroup of contractions $T(t)x$ is the unique solution of the Cauchy problem (4.1). As we did in the proof of theorem 4.21, for a given $x \in X$ we consider the map $t \to MT(t)x$. Being $T(t)x$ differentiable a.e. $t \geq 0$ and being M linear and bounded, we obtain:

$$\frac{d}{dt}MT(t)x = M\frac{d}{dt}T(t)x = MF(T(t)x) = \widehat{F}(MT(t)x)$$

for almost every positive t, and $MT(0)x = Mx = y$. This means that $MT(t)x$ is a solution of system (4.14) with initial condition y, i.e. system (4.14) admits a solution (almost everywhere with respect to time) for every initial condition $y \in Y$.
Since Proposition 4.20 holds also in case of non-classical solutions, and $t \to MT(t)x$ is differentiable a.e. $t \geq 0$, we can say that $y(t) := MT(t)x$ is the unique solution of the Cauchy problem associated with $\widehat{F}$. In particular, the semigroup $T(t)$ preserves the fibers on M and we can define

$$\widehat{T}(t)y := MT(t)x, \qquad y = Mx,\, t \geq 0,$$

that is a strongly continuous semigroup of nonlinear Lipschitz operators on Y (see the proof of theorem 4.21).
Furthermore, we know that for all $x \in \mathscr{D}(F) = X$,

$$\|T(t)x - T(s)x\| \leq \|F(x)\||t - s|.$$

Then, if $y = Mx$:

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \leq \|M\|\|T(t)x - T(s)x\| \leq \|M\|\|F(x)\||t - s|.$$

If we put $K_y := \inf_{x \in \mathscr{D}(F): Mx = y}\|F(x)\|$, then

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \leq \|M\|\|T(t)x - T(s)x\| \leq \|M\|K_y|t - s|,$$

so that for $y \in M\mathscr{D}(F) = Y$, $t \to \widehat{T}(t)y$ is Lipschitz continuous on every bounded interval of $[0, +\infty)$.

[(ii)]: We first prove that for $\lambda > 0$, the operators $(I - \lambda F)^{-1}$ preserve the fibers of M. Let us consider $x_1, x_2 \in \mathrm{ran(I - \lambda F)} = X$ such that $Mx_1 = Mx_2 = y$ and $x_1 = (I - \lambda F)(\widetilde{x}_1)$, $x_2 = (I - \lambda F)(\widetilde{x}_2)$. We can write:

$$M(I - \lambda F)^{-1}x_1 = M(I - \lambda F)^{-1}(I - \lambda F)(\widetilde{x}_1) = M\widetilde{x}_1;$$

$$M(I - \lambda F)^{-1}x_2 = M(I - \lambda F)^{-1}(I - \lambda F)(\widetilde{x}_2) = M\widetilde{x}_2.$$

On the other hand, from the dissipativity of F we know that $(I - \lambda\widehat{F})$ is injective on Y (see Remark 4.3), from which:

$$M(I - \lambda F)(\widetilde{x}_2) = M(I - \lambda F)(\widetilde{x}_1) \Leftrightarrow (I - \lambda\widehat{F})M\widetilde{x}_1 = (I - \lambda\widehat{F})M\widetilde{x}_2 \Leftrightarrow M\widetilde{x}_2 = M\widetilde{x}_1.$$

From this we obtain $M(I-\lambda F)^{-1}x_1 = M(I-\lambda F)^{-1}x_2$.
For $\lambda > 0$ we can define the following operator on Y:

$$R_\lambda y := M(I-\lambda F)^{-1}x, \quad y = Mx.$$

Then, for $y = Mx$,

$$R_\lambda(I-\lambda\widehat{F})y = R_\lambda M(I-\lambda F)x = M(I-\lambda F)^{-1}(I-\lambda F)x = Mx = y;$$

$$(I-\lambda\widehat{F})R_\lambda(y) = (I-\lambda\widehat{F})M(I-\lambda F)^{-1}x = M(I-\lambda F)(I-\lambda F)^{-1}x = Mx = y.$$

this means that R_λ coincides with $(I-\lambda\widehat{F})^{-1}$ on Y and

$$M(I-\lambda F)^{-1}x = (I-\lambda\widehat{F})^{-1}y, \quad y = Mx. \tag{4.18}$$

Since, for $\lambda > 0$, $[\frac{t}{\lambda}]$ is a natural number, by (4.18) we can iterate the following passages:

$$M(I-\lambda F)^{-[\frac{t}{\lambda}]}(x) = M(I-\lambda F)^{-1}((I-\lambda F)^{-[\frac{t-\lambda}{\lambda}]}x) = (I-\lambda\widehat{F})^{-1}(M((I-\lambda F)^{-[\frac{t-\lambda}{\lambda}]}x)),$$

from which we obtain:

$$M(I-\lambda F)^{-[\frac{t}{\lambda}]}x = (I-\lambda\widehat{F})^{-[\frac{t}{\lambda}]}y, \quad y = Mx.$$

Let us consider the semigroup $T(t)$ given by (4.7). Then, we obtain an exponential formula for $\widehat{T}(t)$:

$$\widehat{T}(t)y = MT(t)x = \lim_{\lambda\to 0^+}(I-\lambda F)^{-[\frac{t}{\lambda}]}x = \lim_{\lambda\to 0^+} M(I-\lambda F)^{-[\frac{t}{\lambda}]}x =$$

$$= \lim_{\lambda\to 0^+}\left(I-\lambda\widehat{F}\right)^{-[\frac{t}{\lambda}]}y.$$

By the boundedness of M, the convergence in (4.17) is uniform on the bounded interval of $[0,+\infty)$, being uniform for the semigroup $T(t)$.

[(iii)]: Let us observe that the operator F_0 is dissipative (because it generates a semigroup of contractions) and its domain is non empty (because all the maps $T(t)x$ are a.e. differentiable), but it is not necessarily m-dissipative. The operator $\widehat{F_0}$ has also a non empty domain but it is not necessarily dissipative (unless $\widehat{T}(t)$ is itself contractive).
We first show that $M\mathscr{D}(F_0) \subset \mathscr{D}(\widehat{F_0})$. Given $y_0 = Mx_0$ with $x_0 \in \mathscr{D}(F_0)$, by the boundedness of M and the differentiability of $T(t)x_0$ in $t = 0$, we obtain that

$$\lim_{h\to 0}\frac{1}{h}\left(\widehat{T}(h)y_0 - y_0\right) = \lim_{h\to 0}\frac{1}{h}\left(MT(h)x_0 - Mx_0\right) =$$

$$= M\lim_{h\to 0}\frac{1}{h}\left(T(h)x_0 - x_0\right) = MF_0(x_0),$$

that is, $y_0 \in \mathscr{D}(\widehat{F_0})$ and $\widehat{F_0}(y_0) = MF_0(x_0)$.
For the inverse inclusion, let us consider $y \in \mathscr{D}(\widehat{F_0})$. We recall that $(I-\lambda_0\widehat{F_0})$ is injective, by the injectivity of $(I-\lambda_0\widehat{F})$. We put $y = (I-\lambda_0\widehat{F_0})^{-1}y_1$, where $y_1 \in \mathrm{ran}(I-\lambda\widehat{F_0})$. Since $M(I-\lambda_0F_0)$ is surjective from $\mathscr{D}(F_0) \subset X$ to Y, we can write $y_1 = M(I-\lambda_0F_0)x_1$ with $x_1 \in \mathscr{D}(F_0)$. Then:

$$y = (I-\lambda_0\widehat{F_0})^{-1}y_1 = (I-\lambda_0\widehat{F_0})^{-1}M(I-\lambda_0F_0)x_1 = (I-\lambda_0\widehat{F_0})^{-1}(I-\lambda_0\widehat{F_0})Mx_1 = Mx_1.$$

This means that $y \in M\mathscr{D}(F_0)$. □

Note that, since $F(x_0) = F_0(x_0)$ on $\mathscr{D}(F_0)$, F is the (unique) m-dissipative extension of F_0 on X, while $\widehat{F}$ is an extension of $\widehat{F}_0$ on Y.

Next, we generalize our result in the case of a densely defined operator. Since we can no more define the map $\widetilde{F}$ as in (4.20), we want to give some conditions in order to obtain the dissipativity of the reduced operator $\widehat{F}$ *a priori*. In this way, we could obtain the uniqueness of solution for the reduced system directly by the dissipativity of $\widehat{F}$.

Lemma 4.23. *Let F be a nonlinear operator from $\mathscr{D}(F)$ to X such that condition (4.4) holds and let $\widehat{F}$ be the nonlinear operator defined as in (4.5) on $M\mathscr{D}(F) \subset Y$. Let $M^* : Y^* \to X^*$ be the adjoint operator of M. The following conditions are equivalent:*

(i) $\widehat{F}$ is a dissipative operator,

(ii) for all y_1, $y_2 \in Y$ there exists $\widehat{f} \in Y^*$ such that $\widehat{f}(y_1 - y_2) = \|y_1 - y_2\|^2 = \|\widehat{f}\|^2$ and

$$\mathrm{re}(M^*\widehat{f}[F(x_1) - F(x_2)]) \leq 0, \qquad y_1 = Mx_1,\, y_2 = Mx_2. \tag{4.19}$$

Proof. If condition (4.19) holds for some $\widehat{f} \in Y^*$, then by definition $\widehat{f} \in \mathscr{F}(y_1 - y_2)$, where $\mathscr{F}$ is the duality mapping of the space Y. Then:

$$\mathrm{re}(\widehat{f}[\widehat{F}(y_1) - \widehat{F}(y_2)]) = \mathrm{re}(\widehat{f}[MF(x_1) - MF(x_2)]) =$$

$$= \mathrm{re}(M^*\widehat{f}[F(x_1) - F(x_2)]) \leq 0,$$

i.e. $\widehat{F}$ is dissipative.
Conversely, let us suppose that $\widehat{F}$ is dissipative. Then, there exists $\widehat{f} \in Y^*$ such that $\widehat{f}(y_1 - y_2) = \|y_1 - y_2\|^2$ and $\mathrm{re}(\widehat{f}[\widehat{F}(y_1) - \widehat{F}(y_2)]) \leq 0$. We can write:

$$0 \geq \mathrm{re}(\widehat{f}[\widehat{F}(y_1) - \widehat{F}(y_2)]) = \mathrm{re}(\widehat{f}[MF(x_1) - MF(x_2)]) = \mathrm{re}(M^*\widehat{f}[F(x_1) - F(x_2)]),$$

i.e. condition (4.19) holds. □

Remark 4.5. Let us discuss in brief the case of $F : H_1 \to H_1$ and $M : H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces. Then, condition (4.19) reads as:

$$\mathrm{re}(\langle y_1 - y_2, MF(x_1) - MF(x_2)\rangle) = \mathrm{re}(\langle Mx_1 - Mx_2, MF(x_1) - MF(x_2)\rangle) \leq 0,$$

that is:

$$\mathrm{re}(\langle x_1 - x_2, M^*M(F(x_1) - F(x_2))\rangle) \leq 0.$$

This means that in this case the dissipativity of $\widehat{F}$ on H_2 coincide with the dissipativity of M^*MF on H_1. Of course, if $M^*M = I$ (i.e. M is an isometry), then the dissipativity of $\widehat{F}$ is the same as the dissipativity of F, but in this case we don't have a lumping because M is injective, and the two maps are conjugate.

Remark 4.6. Given $\omega \in \mathbb{R}$, we can modify Lemma (4.23) in order to obtain the ω-dissipativity of $\widehat{F}$. In particular, it is easy to verify that $\widehat{F}$ is ω-dissipative if the following condition holds: for all $y_1, y_2 \in Y$ there exists $\widehat{f} \in Y^*$ such that
$\widehat{f}(y_1 - y_2) = \|y_1 - y_2\|^2 = \|\widehat{f}\|^2$ and

$$\mathrm{re}(M^*\widehat{f}[F(x_1) - F(x_2)]) \leq \omega\|Mx_1 - Mx_2\|^2,$$

where $y_1 = Mx_1$, $y_2 = Mx_2$.

Theorem 4.24 (Densely defined, m-dissipative operators)**.** *Let $M : X \to Y$ be linear, bounded and surjective. Let F be a nonlinear, densely defined operator from $\mathscr{D}(F) \subset X$ to X such that the following hold:*

(j) F is m-dissipative,

(jj) F satisfies condition (4.4),

(jjj) F satisfies condition (4.19).

Let $T(t)$ be the semigroup of contractions on $X = \overline{\mathscr{D}(F)}$ given in the Crandall-Liggett theorem by (4.7) and let F_0 be its infinitesimal generator. Let us suppose that for every $x \in \mathscr{D}(F)$ the semigroup $T(t)x$ is strongly differentiable for almost every $t \geq 0$. Then, the following hold:

(i) system (4.1) is lumpable by M, i.e. there exists a nonlinear, continuous operator $\widehat{F} : M\mathscr{D}(F) \subset Y \to Y$ such that system (4.14) has a unique solution (differentiable a.e. $t \geq 0$) for every $y_0 = Mx_0 \in M\mathscr{D}(F)$. The family $\widehat{T}(t)$ of its solution operators is a strongly continuous semigroup on Y.

(ii) $\widehat{T}(t)$ is a semigroup of contractions and, for every $y = Mx, y \in Y$, (4.17) holds, i.e.:

$$\widehat{T}(t)y = \lim_{\lambda \to 0^+} \left(I - \lambda \widehat{F}\right)^{-[\frac{t}{\lambda}]} y.$$

(iii) If for some $\lambda_0 > 0$ the operator $M(I - \lambda_0 F_0)$ is surjective from $\mathscr{D}(F_0) \subset X$ to Y, then $M\mathscr{D}(F_0) = \mathscr{D}(\widehat{F}_0)$, where $\widehat{F}_0$ is the infinitesimal generator of $\widehat{T}(t)$.

Proof. [(i)]: By theorem 4.11, for $x \in \mathscr{D}(F)$ the semigroup of contractions $T(t)x$ is the unique solution of the Cauchy problem (4.1).
Note that, for every $t > 0$, the operator $T(t)$ can be extended on the whole $\overline{\mathscr{D}(F)} = X$. Indeed, if $x_n \to x$, $x_n \in \mathscr{D}(F)$, by the contractivity of $T(t)$, $T(t)x_n$ is a Cauchy sequence and we can define $T(t)x = \lim_{n \to +\infty} T(t)x_n$.
As we did in the proof of theorem 4.22, for a given $x \in \mathscr{D}(F)$ we consider the map $t \to MT(t)x$. Being $T(t)x$ differentiable a.e. $t \geq 0$ and being M linear and bounded, we can write:

$$\frac{d}{dt} MT(t)x = M \frac{d}{dt} T(t)x = MF(T(t)x) = \widehat{F}(MT(t)x) \qquad a.e.\ t \geq 0,$$

where $\widehat{F}$ is the reduced operator on $M\mathscr{D}(F)$ defined as in (4.5). This means that $MT(t)x$ is a solution of system (4.14) with initial condition $y = Mx$. Consequently, (4.14) admits a solution (almost everywhere with respect to time) for every initial condition $y \in M\mathscr{D}(F)$.
Since condition (4.19) holds, the operator $\widehat{F}$ is dissipative (see Lemma 4.23). By the dissipativity of $\widehat{F}$, $y(t) := MT(t)x$ is the unique solution of the reduced Cauchy problem for an initial condition $y = Mx \in M\mathscr{D}(F)$.
In particular, the semigroup $T(t)$ preserves the fibers of M on $\mathscr{D}(F)$, and we can define

$$\widehat{T}(t)y := MT(t)x, \qquad y = Mx,\ x \in \mathscr{D}(F),\ t \geq 0.$$

Note that $M\mathscr{D}(F)$ is a dense subspace in Y. Since the semigroup $T(t)$ is defined on the whole X, for every $t \geq 0$, $\widehat{T}(t)$ is Lipschitz continuous (see Proposition 4.18). In particular, for all $y_1, y_2 \in M(\mathscr{D}(F))$ we have:

$$\|\widehat{T}(t)y_1 - \widehat{T}(t)y_2\| \leq \|M\| \, \|\widetilde{M}^{-1}\| \|y_1 - y_2\|.$$

This means that we can extend the family $\widehat{T}(t)$ on the whole $Y = M(\overline{\mathscr{D}(F)})$. In fact, for $x = \lim_{n\to+\infty} x_n$, with $x_n \in \mathscr{D}(F)$, and $y = Mx$, we can define

$$\widehat{T}(t)y = \lim_{n\to+\infty} MT(t)x_n$$

(for neatness, we use the same notation $\widehat{T}(t)$ for both the semigroup on $M(\mathscr{D}(F))$ and its extension on Y).
By the Lipschitz continuity of $\widehat{T}(t)$, $\widehat{T}(t)Mx_n$ converges (indeed, it is a Cauchy sequence). Similarly, it is easy to verify that $\widehat{T}(t)$ is well-defined, because it is well-defined on a dense subspace of Y.
Furthermore, for all $y \in M(\overline{\mathscr{D}(F)})$, the following holds:

$$\widehat{T}(t+s)y = \lim_{n\to+\infty} MT(t+s)x_n = \lim_{n\to+\infty} MT(t)T(s)x_n = \widehat{T}(t)(MT(s)x) =$$

$$= \widehat{T}(t)\left(\lim_{n\to+\infty} MT(s)x_n\right) = \widehat{T}(t)\widehat{T}(s)y.$$

We have used the fact that, if x_n tends to x, $T(s)x$ is by definition equal to $\lim_{n\to+\infty} T(s)x_n$. Of course, if $y = Mx$,

$$\widehat{T}(0)y = \lim_{n\to+\infty} MT(0)x_n = \lim_{n\to+\infty} Mx_n = Mx = y.$$

This means that $\widehat{T}(t)$ is a semigroup of nonlinear, Lipschitz operators on Y.

Furthermore, we know that for all $x \in \mathscr{D}(F)$, $\|T(t)x - T(s)x\| \le \|F(x)\||t-s|$. Then, if $y = Mx$:

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \le \|M\|\|T(t)x - T(s)x\| \le \|M\|\|F(x)\||t-s|.$$

If we put $K_y := \inf_{x\in\mathscr{D}(F):Mx=y}\|F(x)\|$, then

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \le \|M\|\|T(t)x - T(s)x\| \le \|M\|K_y|t-s|,$$

so that for $y \in M\mathscr{D}(F)$, $t \to \widehat{T}(t)y$ is Lipschitz continuous on every bounded interval of $[0, +\infty)$.
From this follows that the map $t \to \widehat{T}(t)y$ is continuous for every positive time and $\widehat{T}(t)$ is a strongly continuous semigroup on Y.

[(ii)] Since condition (4.19) holds, the operator $\widehat{F}$ is dissipative. Being $\operatorname{ran}(I - \lambda F)) = X$, we obtain $\operatorname{ran}(I - \lambda\widehat{F}) = M(\operatorname{ran}(I - \lambda F)) = Y$, i.e. $\widehat{F}$ is also m-dissipative.
This means that we can apply the Crandall-Liggett theorem to say that $\widehat{F}$ generates a contraction semigroup given exactly by (4.17). Furthermore, if this semigroup is strongly differentiable for almost every positive time on $\mathscr{D}(\widehat{F})$, then it gives the unique solution of the reduced abstract Cauchy problem (4.14) for an initial condition in $\mathscr{D}(\widehat{F})$. By dissipativity, this solution is unique, i.e. the semigroup given by the Crandall-Liggett theorem coincides with $\widehat{T}(t)$ for every $y \in \mathscr{D}(\widehat{F})$.
But thanks to the Lipschitz continuity, both semigroups are defined on the whole Y and, since they coincide on a dense subset, they coincide everywhere.
This implies that $\widehat{T}(t)$ is itself a semigroup of contractions and the exponential formula (4.17) holds.

[(iii)] As we did in the proof of theorem 4.22, we first show that $M\mathscr{D}(F_0) \subset \mathscr{D}(\widehat{F}_0)$. Given $y_0 = Mx_0$ with $x_0 \in \mathscr{D}(F_0)$, being M bounded and $T(t)x_0$ differentiable in $t = 0$, we have that:

$$\lim_{h\to 0}\frac{1}{h}\left(\widehat{T}(h)y_0 - y_0\right) = \lim_{h\to 0}\frac{1}{h}\left(MT(h)x_0 - Mx_0\right) =$$

$$= M \lim_{h\to 0} \frac{1}{h} (T(h)x_0 - x_0) = MF_0(x_0),$$

That is, $y_0 \in \mathscr{D}(\widehat{F}_0)$ and $\widehat{F}_0(y_0) = MF_0(x_0)$.
For the inverse inclusion, let us consider $y_0 \in \mathscr{D}(\widehat{F}_0)$. Then $y = (I - \lambda_0 \widehat{F_0})^{-1} y_1$, where $y_1 \in \operatorname{ran}(I - \lambda \widehat{F}_0)$. Since $M(I - \lambda_0 F_0)$ is surjective from $\mathscr{D}(F_0) \subset X$ to Y, we can write $y_1 = M(I - \lambda_0 F_0)x_1$ with $x_1 \in \mathscr{D}(F_0)$. Then,

$$y = (I - \lambda_0 \widehat{F_0})^{-1} y_1 = (I - \lambda_0 \widehat{F_0})^{-1} M(I - \lambda_0 F_0)x_1$$

$$= (I - \lambda_0 \widehat{F_0})^{-1} (I - \lambda_0 \widehat{F_0}) M x_1 = M x_1;$$

this means that $y_1 \in M\mathscr{D}(F_0)$. □

Remark 4.7. We use the same notation as in the theorem above but we assume that $\mathscr{D}(F)$ is not dense in X. Provided that condition (4.19) holds, for every $y = Mx$ in $M\mathscr{D}(F)$ we can find a unique solution for the abstract Cauchy problem (4.14), given by $t \to MT(t)x$. However, we are not able to say that $\widehat{T}(t)y := MT(t)x$ is a family of Lipschitz operators, but only that the map $t \to \widehat{T}(t)y$ is strongly continuous on $M\mathscr{D}(F)$. The family of the solution operators $\widehat{T}(t)$ is a strongly continuous semigroup in a non-standard sense, namely it is a strongly continuous semigroup from $M\mathscr{D}(F)$ to $\overline{M\mathscr{D}(F)}$.

In the case of F densely defined, we can still obtain an existence result for the reduced abstract Cauchy problem without assuming the dissipativity of $\widehat{F}$. However, the uniqueness of solution is not guaranteed:

Theorem 4.25. *Let F be a densely defined, m-dissipative operator from $\mathscr{D}(F)$ to X such that $F(0) = 0$ and condition (4.4) holds. Let $T(t)$ be the semigroup of contractions given by (4.7). Let us suppose that the following condition holds for every $\lambda > 0$:*

$$x_1 - x_2 - \lambda(F(x_1) - F(x_2)) \in ker(M) \Rightarrow x_1 - x_2 \in ker(M) \qquad \forall x_1 \neq x_2 \in X. \quad (4.20)$$

If for every $x \in \mathscr{D}(F)$ the semigroup $T(t)x$ is strongly differentiable for almost every $t \geq 0$, then the reduced system (4.14) has a solution for every $y = Mx,\ x \in \mathscr{D}(F)$. Consequently, we can build a family $\widehat{T}(t)$ of solution operators of the reduced Cauchy problem by the following formula:

$$\widehat{T}(t)y = \lim_{\lambda \to 0^+} \left(I - \lambda \widehat{F}\right)^{-[\frac{t}{\lambda}]} y.$$

Proof. It is easy to verify that condition (4.20) implies the injectivity of $(I - \lambda\widehat{F})$, because, given $y_1 = Mx_1$ and $y_2 = Mx_2$ with $x_1, x_2 \in \mathscr{D}(F)$:

$$(I - \lambda\widehat{F})y_1 = (I - \lambda\widehat{F})y_2 \Leftrightarrow M(I - \lambda F)x_1 = M(I - \lambda F)x_2$$

$$\Leftrightarrow M(x_1 - \lambda F(x_1) - x_2 + \lambda F(x_2)) = 0 \Leftrightarrow x_1 - x_2 - \lambda(F(x_1) - F(x_2)) \in \ker(M).$$

Since this implies $x_1 - x_2 \in ker(M)$, we obtain $Mx_1 = Mx_2$, i.e. $(I - \lambda\widehat{F})$ is injective for every $\lambda > 0$.

As we did in the proof of theorem 4.22, we prove that for $\lambda > 0$, the operators $(I - \lambda F)^{-1}$ preserve the fibers of M. Let us consider $x_1, x_2 \in \mathrm{ran(I - \lambda F)} = X$ such that $Mx_1 = Mx_2 = y$ and $x_1 = (I - \lambda F)(\widetilde{x}_1)$, $x_2 = (I - \lambda F)(\widetilde{x}_2)$, $\widetilde{x}_1, \widetilde{x}_2 \in \mathscr{D}(F)$. We can write:

$$M(I - \lambda F)^{-1} x_1 = M(I - \lambda F)^{-1}(I - \lambda F)(\widetilde{x}_1) = M\widetilde{x}_1;$$

$$M(I - \lambda F)^{-1} x_2 = M(I - \lambda F)^{-1}(I - \lambda F)(\widetilde{x}_2) = M\widetilde{x}_2.$$

By condition (4.20) we know that $(I - \lambda\widehat{F})$ is injective on Y, from which:

$$M(I-\lambda F)(\widetilde{x}_2) = M(I-\lambda F)(\widetilde{x}_1) \Leftrightarrow (I-\lambda\widehat{F})M\widetilde{x}_1 = (I-\lambda\widehat{F})M\widetilde{x}_2 \Leftrightarrow M\widetilde{x}_2 = M\widetilde{x}_1.$$

We obtain $M(I-\lambda F)^{-1}x_1 = M(I-\lambda F)^{-1}x_2$ and

$$M(I-\lambda F)^{-1}x = (I-\lambda\widehat{F})^{-1}y, \quad \text{for } y = Mx.$$

Then:

$$MT(t)x = M\left(\lim_{\lambda\to 0^+}(I-\lambda F)^{-[\frac{t}{\lambda}]}x\right) = \lim_{\lambda\to 0^+} M\,(I-\lambda F)^{-[\frac{t}{\lambda}]}\,x =$$
$$= \lim_{\lambda\to 0^+}\left(I-\lambda\widehat{F}\right)^{-[\frac{t}{\lambda}]} y =: \widehat{T}(t)y,$$

where $T(t)$ is the contraction semigroup on $\overline{\mathscr{D}(F)} = X$ given by (4.7).
By the Crandall-Ligget theorem for F and by the boundedness of M, the convergence of $\left(I-\lambda\widehat{F}\right)^{-[\frac{t}{\lambda}]} y$ is uniform on the bounded interval of $[0,+\infty)$.
It is possible to verify that the family of operators $\widehat{T}(t)$ is also a strongly continuous semigroup on $Y = M\mathscr{D}(F)$. Furthermore, $\widehat{T}(t)$ is a family of Lipschitz operators (we can work as in the proof of theorem 4.21 because the semigroup is everywhere defined) and, if $y \in \mathscr{D}(\widehat{F})$, the map $t \to \widehat{T}(t)y$ is Lipschitz continuous on the bounded intervals of $[0,+\infty)$.
In fact, by the Crandall-Liggett theorem, for all $x \in \mathscr{D}(F)$,

$$\|T(t)x - T(s)x\| \le \|F(x)\||t-s|.$$

As we did in the proof of theorem 4.22, we write:

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \le \|M\|\|T(t)x - T(s)x\| \le \|M\|\|F(x)\||t-s|.$$

From this we obtain:

$$\|\widehat{T}(t)y - \widehat{T}(s)y\| \le \|M\|\|T(t)x - T(s)x\| \le \|M\|K_y|t-s|,$$

where $K_y := \inf_{x\in\mathscr{D}(F):Mx=y}\|F(x)\|$.

Since for every $x \in X$ the semigroup $T(t)x$ is strongly differentiable for almost every $t \ge 0$ (so it is the unique solution of the Cauchy problem associated to F with initial condition x), also $\widehat{T}(t)y$ is strongly differentiable almost everywhere, with $y = Mx$. This means that

$$\frac{d}{dt}\widehat{T}(t)y = \frac{d}{dt}MT(t)x = M\frac{d}{dt}T(t)x =$$
$$= MF(T(t)x) = \widehat{F}(\widehat{T}(t)y), \qquad \text{for a.e. } t \ge 0.$$

However, without assuming the dissipativity of $\widehat{F}$, we are not able to obtain the uniqueness of the solution. ☐

Remark 4.8. We observe that, since F is m-dissipative, $\mathrm{ran}(I-\lambda F) = X$, and so $\mathrm{ran}(I - \lambda\widehat{F}) = Y$. This means that $(I-\lambda\widehat{F})^{-1}$ is everywhere defined on Y. Furthermore, since F is dissipative, we know that $(I-\lambda F)^{-1}$ is contractive.
Let us define:

$$G_\lambda : X/\ker(M) \to Y, \qquad G_\lambda[x] = M(I-\lambda F)^{-1}x.$$

It is well-defined because $(I-\lambda F)^{-1}$ preserves the fibers of M. Furthermore,

$$\|G_\lambda([x_1]) - G_\lambda([x_2])\| = \inf_{m\in\mathrm{Ker(M)}}\|G_\lambda([x_1-m]) - G_\lambda([x_2])\|$$

$$= \inf_{m\in \mathrm{Ker(M)}} \|M(I-\lambda F)^{-1}(x_1-m) - M(I-\lambda F)^{-1}(x_2)\|$$
$$\le \inf_{m\in \mathrm{Ker(M)}} \|M\|\,\|(I-\lambda F)^{-1}(x_1-m) - (I-\lambda F)^{-1}(x_2)\|$$
$$\le \inf_{m\in \mathrm{Ker(M)}} \|M\|\,\|(x_1-m) - x_2\| = \|M\|\|[x_1]-[x_2]\|.$$

From this we obtain:

$$\|(I-\lambda\widehat{F})^{-1}(y_1) - (I-\lambda\widehat{F})^{-1}(y_2)\| = \|M(I-\lambda F)^{-1}(x_1) - M(I-\lambda F)^{-1}(x_2)\| =$$
$$\|G_\lambda([x_1]) - G_\lambda([x_2])\| \le \|M\|\,\|[x_1]-[x_2]\|$$
$$= \|M\|\,\|\widetilde{M}^{-1}y_1 - \widetilde{M}^{-1}y_2\| \le \|M\|\,\|\widetilde{M}^{-1}\|\,\|y_1-y_2\|.$$

This means that $(I-\lambda\widehat{F})^{-1}$ is a Lipschitz operator for every $\lambda > 0$.

Under suitable hypotheses, the operators $(I-\lambda\widehat{F})^{-1}$ turn to be everywhere defined on the reduced state space Y (i.e. $(I-\lambda\widehat{F})$ are surjective), even if $\widehat{F}$ is not necessarily m-dissipative. In particular, the following fact holds:

Proposition 4.26. *Let F be a nonlinear, dissipative operator from $\mathscr{D}(F) \subset X$ to itself satisfying conditions (4.4) and (4.6). Let $\widehat{F}$ be defined as in (4.5). Let us suppose that, for every $\lambda > 0$, the operators $(I-\lambda\widehat{F})^{-1}$ exist and satisfy:*

$$\|(I-\lambda\widehat{F})^{-1}y_1 - (I-\lambda\widehat{F})^{-1}y_2\| \le K\|y_1-y_2\|, \qquad y_1, y_1 \in \mathrm{ran}(I-\lambda\widehat{F}).$$

If $M(I-\lambda_0 F)X = Y$ for a given $\lambda_0 > 0$, then $M(I-\lambda F)X = Y$ for every $\lambda > 0$.

Proof. We observe that, since F satisfies (4.6), $\mathscr{D}(\widehat{F}) \subset \mathrm{ran}(I-\lambda\widehat{F})$. Thus, if $K \le 1$, $\widehat{F}$ is a dissipative operator. Then, we obtain the thesis by Proposition 4.5.
Let us prove the case $K > 1$. We modify the idea employed in [51, Lemma 2.13]. Given an arbitrary $\lambda > 0$, the following representation holds on $M\mathscr{D}(F)$:

$$(I-\lambda\widehat{F}) = \frac{\lambda}{\lambda_0}[I-(1-\frac{\lambda_0}{\lambda})(I-\lambda_0\widehat{F})^{-1}](I-\lambda_0\widehat{F}). \tag{4.21}$$

Let us consider $\lambda = \widetilde{\lambda}K$ for an arbitrary $\widetilde{\lambda} > 0$. Given an element $y \in Y$, since $(I-\lambda_0\widehat{F})$ is surjective, we can define the following operator on Y:

$$S(\widetilde{y}) := y + \left(1-\frac{\lambda_0}{\widetilde{\lambda}K}\right)\left(I-\lambda_0\widehat{F}\right)^{-1}\widetilde{y}.$$

We obtain that:

$$\|S(\widetilde{y}_1) - S(\widetilde{y}_2)\| = \left\|\left(1-\frac{\lambda_0}{\widetilde{\lambda}K}\right)[(I-\lambda_0\widehat{F})^{-1})\widetilde{y}_1 - (I-\lambda_0\widehat{F})^{-1}\widetilde{y}_2]\right\|$$
$$\le \left|1-\frac{\lambda_0}{\widetilde{\lambda}K}\right| K\|\widetilde{y}_1-\widetilde{y}_2\| = \left|K-\frac{\lambda_0}{\widetilde{\lambda}}\right|\|\widetilde{y}_1-\widetilde{y}_2\|.$$

Let us impose the condition

$$\left|K-\frac{\lambda_0}{\widetilde{\lambda}}\right| < 1. \tag{4.22}$$

In this way, the operator S is a strict contraction on Y. By the Banach fixed point theorem 4.8, S has a unique fixed point $\widehat{y}$, i.e.:

$$\widehat{y} = y + \left(1-\frac{\lambda_0}{\widetilde{\lambda}K}\right)\left(I-\lambda_0\widehat{F}\right)^{-1}\widehat{y}.$$

From this, using the surjectivity of $(I - \lambda_0 F)$, we obtain:

$$y = \left[I - \left(1 - \frac{\lambda_0}{\widetilde{\lambda} K}\right)\left(I - \lambda_0 \widehat{F}\right)^{-1}\right]\widehat{y} =$$

$$= \left[I - \left(1 - \frac{\lambda_0}{\widetilde{\lambda} K}\right)\left(I - \lambda_0 \widehat{F}\right)^{-1}\right] M(I - \lambda_0 F)x =$$

$$= \left[I - \left(1 - \frac{\lambda_0}{\widetilde{\lambda} K}\right)\left(I - \lambda_0 \widehat{F}\right)^{-1}\right](I - \lambda_0 \widehat{F})\overline{y},$$

where $Mx = \overline{y} \in M\mathscr{D}(F)$.
Now, we use the representation (4.21) with $\lambda = K\widetilde{\lambda}$ in order to write:

$$y = \frac{\lambda_0}{\widetilde{\lambda} K}\left(I - \widetilde{\lambda} K \widehat{F}\right)\overline{y}.$$

Since y was arbitrary in Y, provided that condition (4.22) holds, we obtain the surjectivity of $(I - \widetilde{\lambda} K \widehat{F})$ (i.e. the surjectivity of $M(I - \widetilde{\lambda} K F)$).
We want to iterate this method. To this aim, we distinguish two cases:

1. Case $\widetilde{\lambda} < \frac{\lambda_0}{K-1}$: we iterate the passages described above putting $\lambda_0 = \widetilde{\lambda} K$ and $\lambda = \widetilde{\lambda}_1 K$, with the condition:
$$\widetilde{\lambda}_1 < \frac{\widetilde{\lambda} K}{K-1} < \frac{\lambda_0 K}{(K-1)^2}.$$
In this way we obtain the surjectivity of $M(I - \widetilde{\lambda}_1 K\, F)$. If we iterate n times, our condition on $\widetilde{\lambda}_n$ becomes the following:
$$\widetilde{\lambda}_1 < \frac{\widetilde{\lambda}_{n-1}}{K-1} < \cdots < \frac{\lambda_0 K^n}{(K-1)^{n+1}}.$$
The right side of this inequality tends to infinity, so with these iterations we cover all the positive values of λ.

2. Case $\widetilde{\lambda} > \frac{\lambda_0}{K+1}$: again we iterate the method putting $\lambda_0 = \widetilde{\lambda} K$ and $\lambda = \widetilde{\lambda}_1 K$. The condition on $\widetilde{\lambda}_1$ is:
$$\widetilde{\lambda}_1 > \frac{\widetilde{\lambda} K}{K+1} > \frac{\lambda_0 K}{(K-1)^2},$$
and after n iterations we have:
$$\widetilde{\lambda}_n > \frac{\widetilde{\lambda}_{n-1} K}{K+1} > \cdots > \frac{\lambda_0 K^n}{(K+1)^{n+1}}.$$
Since the right side of this inequality tends to zero, also in this case we cover all the positive values of λ.

□

It is important to observe that our lumping analysis can be easily generalized to the case of ω-dissipative operators. In this case, we need to take into account that the operators $(I - \lambda F)^{-1}$ are well-defined only for $\lambda\omega < 1$.
For completeness, we show the following:

Theorem 4.27. *Let $M : X \to Y$ be linear, bounded and surjective. Let F be a nonlinear, everywhere defined, ω-dissipative operator from X to itself such that condition (4.4) holds. Let $T(t)$ be the semigroup of contractions on X given in the Crandall-Liggett theorem (see theorem 4.14) and let F_0 be its infinitesimal generator. Let us suppose that for every $x \in X$ the semigroup $T(t)x$ is strongly differentiable for almost every $t \geq 0$. Then, the following hold:*

(i) system (4.1) is lumpable by M, i.e. there exists a nonlinear, continuous operator $\widehat{F} : Y \to Y$ such that system (4.14) has a unique solution (differentiable a.e. $t \geq 0$) for every $y_0 = Mx_0 \in Y$. The family $\widehat{T}(t)$ of its solution operators is a semigroup of nonlinear operators on Y.

(ii) For all $\lambda > 0$, $\lambda\omega < 1$ and $y = Mx \in Y$, $(I - \lambda\widehat{F})^{-1}(y) = M(I - \lambda F)^{-1}(x)$. Consequently, for every $y \in Y$, the following formula holds:

$$\widehat{T}(t)y = \lim_{\lambda \to 0^+} \left(I - \lambda\widehat{F}\right)^{-\left[\frac{t}{\lambda}\right]} y. \tag{4.23}$$

(iii) If for some $\lambda_0 > 0$, $\lambda_0\omega < 1$ the operator $M(I - \lambda_0 F_0)$ is surjective from $\mathscr{D}(F_0) \subset X$ to Y, then $M\mathscr{D}(F_0) = \mathscr{D}(\widehat{F}_0)$, where $\widehat{F}_0$ is the infinitesimal generator of $\widehat{T}(t)$.

Proof. Using the same method as in Proposition 4.20, we can prove that, if a solution of (4.14) exists, then it is unique. Indeed, with the same notations, we can prove that $\widetilde{F} - \omega I$ is dissipative on $X/\ker(M)$, i.e. $\widetilde{F}$ is ω-dissipative. Of course, if $(I - \lambda F)$ is surjective for a given λ, then $(I - \lambda\widetilde{F})$ is also surjective from $X/\ker(M)$ to itself. This means that the Cauchy problem associated to $\widetilde{F}$ has a unique integral solution of type ω.
If a strong or classical solution exists, it is also unique (indeed, strong and classical solutions are themselves integral solutions of type ω).
Then, we work with the isomorphism $\widetilde{M}^{-1}$, which maps solution of (4.14) into solutions of the Cauchy problem associated to $\widetilde{F}$ (see Proposition 4.20).
Knowing that the solution of (4.14) is unique, we can prove [(i)] as in the proof of theorem 4.22. We obtain a semigroup of nonlinear operators on Y: $\widehat{T}(t)(y) = MT(t)x$ for $y = Mx$. This semigroup satisfies:

$$\|\widehat{T}(y_1)\widehat{T}(y_2)\| \leq e^{\omega t}\|M\|\|\widetilde{M}^{-1}\|\|y_1 - y_2\|,$$

and for every $y \in Y$, $\widehat{T}(y)$ is the unique solution of the reduced Cauchy problem (as usual, the reduced operator is defined as $\widehat{F}y := MF(x)$).
For point [(ii)], consider $\lambda\omega < 1$. For these values of λ, we can work as in the proof of theorem 4.22, in order to obtain:

$$M(I - \lambda F)^{-1}x = (I - \lambda\widehat{F})^{-1}y, \quad y = Mx.$$

Since we need $\lambda \to 0^+$, this is enough to obtain the exponential formula for $\widehat{T}(t)$:

$$\widehat{T}(t)y = MT(t)x = \lim_{\lambda \to 0^+} (I - \lambda F)^{-\left[\frac{t}{\lambda}\right]} x = \lim_{\lambda \to 0^+} M\,(I - \lambda F)^{-\left[\frac{t}{\lambda}\right]} x =$$

$$= \lim_{\lambda \to 0^+} \left(I - \lambda\widehat{F}\right)^{-\left[\frac{t}{\lambda}\right]} y.$$

By the boundedness of M, the convergence is uniform on the bounded intervals of $[0, +\infty)$, being uniform for the semigroup $T(t)$.
Also point [(iii)] is the same as in theorem 4.22, provided that $\lambda_0\omega < 1$ (for these values of λ_0, we can guarantee the injectivity of $I - \lambda_0\widehat{F}_0$. □

We observe that this result can be generalized to the case of F densely defined, as we did for dissipative operators, provided that $\widehat{F}$ is itself ω-dissipative (see Remark 4.6).

4.5 Examples

1. Let us consider a continuous and decreasing function $\phi : \mathbb{R} \to \mathbb{R}$. Then ϕ is itself an m-dissipative operator on $\mathbb{R}$, since

$$(\phi(x_1) - \phi(x_2))(x_1 - x_2) \leq 0 \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

Let us consider the Banach space $X := \mathscr{B}(\mathbb{R})$ of the continuous, bounded, real valued functions from $\mathbb{R}$ to itself with the supremum norm:

$$\|f\| := \sup_{x\in\mathbb{R}} |f(x)|.$$

We define the following operator:

$$F(f)(x) := \phi(f(x)).$$

Since f is bounded and ϕ is continuous, the operator F maps X to X . Moreover, by the dissipativity of ϕ, for every $x \in \mathbb{R}$, $f, g \in X$ and $\lambda > 0$ we have:

$$\|f(x) - g(x)\| \leq \|f(x) - g(x) - \lambda(\phi(f(x)) - \phi(g(x)))\|.$$

Taking the supremum over $x \in \mathbb{R}$, we obtain that F is also a dissipative operator. Being continuous, everywhere defined and dissipative, F is m-dissipative. This means that the dynamics generated by F has a unique classical solution for every initial condition on X.
Given a sequence of n points $y_1, \ldots, y_n$ in $\mathbb{R}$, we consider the following lumping operator:

$$Mf := \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}, \quad M : X \to \mathbb{R}^n.$$

Of course F preserves the fibers of M, since

$$\begin{pmatrix} g(y_1) \\ \vdots \\ g(y_n) \end{pmatrix} = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix} \Rightarrow \begin{pmatrix} \phi(g(y_1)) \\ \vdots \\ \phi(g(y_n)) \end{pmatrix} = \begin{pmatrix} \phi(f(y_1)) \\ \vdots \\ \phi(f(y_n)) \end{pmatrix}.$$

If we define the function

$$\widehat{\phi}\left[\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right] := \begin{pmatrix} \phi(v_1) \\ \vdots \\ \phi(v_n) \end{pmatrix},$$

then the lumped dynamics is given by the following n-dimensional system:

$$\begin{cases} \dot{v}(t) = \widehat{\phi}(v(t)) \\ v(0) = v_0, \end{cases}$$

where v is a vector with n components $v_1, \ldots, v_n$.
Note that in this case is easy to verify that condition (4.19) holds, so it is possible to show the dissipativity of $\widehat{F}$ a priori.
Given v_1, v_2 in $\mathbb{R}^n$ such that:

$$v_1 = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}, \quad v_2 = \begin{pmatrix} g(y_1) \\ \vdots \\ g(y_n) \end{pmatrix},$$

we want to verify that:

$$M^*\widehat{f}(\phi(f(x)) - \phi(g(x))) \leq 0,$$

where the functional $\widehat{f}$ acting on $\mathbb{R}^n$ is given exactly by the inner product with the vector $(v_1 - v_2)$ itself.
Indeed, we can write:

$$M^*\widehat{f}(\phi(f(x)) - \phi(g(x))) = \widehat{f}(M\phi(f(x)) - M\phi(g(x))) =$$

$$\langle \begin{pmatrix} \phi(f(y_1)) - \phi(g(y_1)) \\ \vdots \\ \phi(f(y_n)) - \phi(g(y_n)) \end{pmatrix}, \begin{pmatrix} f(y_1) - g(y_1) \\ \vdots \\ f(y_n) - g(y_n) \end{pmatrix} \rangle = (\phi(f(y_1)) - \phi(g(y_1)))(f(y_1) - g(y_1))$$

$$+ \cdots + (\phi(f(y_n)) - \phi(g(y_n)))(f(y_n) - g(y_n)) \leq 0.$$

In fact, since ϕ is a decreasing function, we are adding only negative terms, from which condition (4.19) holds.

2. **(Delay differential equation)** We consider the Banach space $X = C([-r, 0]; \mathbb{R}^2)$ with the supremum norm. Given $r > 0$, we define the following delay differential system:

$$\begin{cases} \dot{x}(t) = G(x_t) & t \geq 0 \\ x(t) = \phi(t) & -r \leq t \leq 0, \end{cases} \tag{4.24}$$

where:

1. $G : X \to \mathbb{R}^2$ is a nonlinear operator acting as $G(\phi) := f(\phi(-r))$ for a given function $f : \mathbb{R}^2 \to \mathbb{R}^2$,
2. $\phi \in X$ is the history function (initial condition of the system),
3. $x(t) : [-r, +\infty) \to \mathbb{R}^2$ is an unknown function (solution of the system),
4. $x_t : [-r, 0] \to \mathbb{R}^2$, $x_t(s) := x(t + s)$.

This system can be formulated as an abstract Cauchy problem

$$\begin{cases} \dot{v}(t) = F(v(t)) \\ v(0) = v_0, \end{cases}$$

where F is the following nonlinear operator:

$$\mathscr{D}(F) := \{\phi \in X \,:\, \frac{d\phi}{dt} \in X,\ \frac{d\phi}{dt}(0) = G(\phi)\},$$

$$F(\phi) := \frac{d\phi}{dt}$$

(here the derivative is considered component by component).

Theorem 4.28 ([39]). *Assume that there exists a constant $\beta > 0$ such that*

$$\|G(\phi_1) - G(\phi_2)\| \leq \beta \|\phi_1 - \phi_2\|.$$

Then, for every $\alpha \geq \beta$, the operator F is α-dissipative, densely defined, and $\operatorname{ran}(I - \lambda F) = X$ *for $\lambda > 0, \lambda\alpha < 1$. Let $T(t)$ be the nonlinear semigroup of type α generated by F on X and let $x(t, \phi)$ be the solution of the delay differential system (4.24) with initial date ϕ. Then:*

$$T(t)\phi = x_t, \quad t \geq 0,$$

and

$$x(t, \phi) = \begin{cases} \phi(t) & -r \leq t \leq 0 \\ (T(t)\phi)(0) & t \geq 0 \end{cases}.$$

Note that if f is Lipschitz from $\mathbb{R}^2$ to itself, then the operator G is also Lipschitz on X. We consider system (4.24) with the following periodic Lipschitz continuous function:

$$f\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] := \begin{pmatrix} \sin(x_1)\cos(x_2) \\ \sin(x_2)\cos(x_1) \end{pmatrix}.$$

We define the following lumping operator:

$$M : X \to C([-r,0],\mathbb{R}), \quad M\begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix} = \phi_1(s) + \phi_2(s), \quad s \in [-r,0].$$

By the linearity of the derivative operator it follows that, for $\phi \in \mathscr{D}(F)$:

$$MF(\phi) = M\frac{d\phi}{dt} = M\begin{pmatrix} \frac{d\phi_1}{dt} \\ \frac{d\phi_2}{dt} \end{pmatrix} = \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} = \frac{d(\phi_1+\phi_2)}{dt} = \frac{dM(\phi)}{dt}.$$

Since $\frac{d\phi}{dt} \in X$ and M is bounded, $\frac{dM(\phi)}{dt}$ is continuous. Moreover, using the trigonometric addition formula $\sin(\alpha+\beta) = \sin(\alpha)\cos\beta + \cos(\alpha)\sin\beta$, we obtain:

$$\frac{dM(\phi)}{dt}(0) = M\frac{d\phi}{dt}(0) = MG(\phi) = Mf(\phi(-r)) =$$

$$M\begin{pmatrix} \sin(\phi_1(-r))\cos(\phi_2(-r)) \\ \cos(\phi_1(-r))\sin(\phi_2(-r)) \end{pmatrix} = \sin(\phi_1(-r))\cos(\phi_2(-r)) + \cos(\phi_1(-r))\sin(\phi_2(-r))$$

$$= \sin(\phi_1(-r) + \phi_2(-r)) = \widehat{f}(M\phi(-r)),$$

where $\widehat{f}(x) := \sin(x)$.
Let $Y := C([-r,0],\mathbb{R})$ and $\widehat{G}(\psi) := \widehat{f}(\psi(-r))$, then we can define our lumped operator:

$$\mathscr{D}(\widehat{F}) := \{\psi \in Y \,:\, \frac{d\psi}{dt} \in Y, \; \frac{d\psi}{dt}(0) = \widehat{G}(\psi)\} = M\mathscr{D}(F),$$

$$\widehat{F}(\psi) = \frac{d\psi}{ds}.$$

With this lumping we pass from a space of vector-valued functions to a state of real-valued functions. The lumped system is a delay differential system with initial state $\psi \in Y$:

$$\begin{cases} \dot{y}(t) = \sin(y(t-r)) & t \geq 0 \\ y(t) = \psi(t) & -r \leq t \leq 0 \end{cases}.$$

Note that, also in this case, it is possible to verify a priori that the reduced operator $\widehat{F}$ is 1-dissipative. Let us consider ψ_1 and ψ_2 in Y. We call s the maximum point of $|\psi_1 - \psi_2|$, i.e.:

$$\psi_1(s) - \psi_2(s) = \|\psi_1 - \psi_2\| = \max_{x\in[-r,0]} |\psi_1(x) - \psi_2(x)|.$$

Then, we define the linear and continuous functional $\widehat{f} \in Y^*$:

$$\widehat{f}(\psi) := (\psi_1(s) - \psi_2(s))\psi(s), \quad \psi \in Y.$$

This functional satisfies the following properties:

1. $\widehat{f}(\psi_1 - \psi_2) = |\psi_1(s) - \psi_2(s)|^2 = \|\psi_1 - \psi_2\|^2$,

2. $\|\widehat{f}\| = \sup_{\|\psi\|\leq 1} |\widehat{f}(\psi)| = \|\psi_1 - \psi_2\|$.

This means that $\widehat{f}$ is a good candidate to verify the dissipativity of $\widehat{F}$. Let us consider ϕ and φ in X such that $M\phi = \psi_1$ and $M\varphi = \psi_2$. Then, if $s \in (-r, 0)$:

$$M^*(\widehat{f})(F(\phi) - F(\varphi)) = \widehat{f}(MF(\phi) - MF(\varphi))$$

$$= \widehat{f}(\phi_1' + \phi_2' - \varphi_1' - \varphi_2') = (\psi_1(s) - \psi_2(s))(\phi_1'(s) + \phi_2'(s) - \varphi_1'(s) - \varphi_2'(s)) =$$

$$(\psi_1(s) - \psi_2(s))(\psi_1'(s) - \psi_2'(s)) = \frac{d}{dt}\left(\frac{1}{2}|\psi_1(t) - \psi_2(t)|^2\right)_{|t=s} = 0,$$

because s is a maximum point of $|\psi_1 - \psi_2|^2$ also.
We need to verify the case in which s is an extremal point of the interval $[-r, 0]$. If $s = -r$, then $|\psi_1 - \psi_2|^2$ must be non increasing in an open interval with extremal point $-r$, and since it is differentiable in $-r$:

$$M^*(\widehat{f})(F(\phi) - F(\varphi)) = \frac{d}{dt}\left(\frac{1}{2}|\psi_1(t) - \psi_2(t)|^2\right)_{|t=s} \leq 0.$$

Finally, if $s = 0$, we use the initial condition given in $\mathscr{D}(\widehat{F})$:
$\frac{d\psi_i}{dt}(0) = \sin(\psi_i(-r))$ for $i = 1, 2$.. Since the function $\sin(x)$ is Lipschitz continuous with Lipschitz constant 1, we obtain:

$$M^*(\widehat{f})(F(\phi) - F(\varphi)) = (\psi_1(0) - \psi_2(0))(\psi_1'(0) - \psi_2'(0))$$

$$= (\psi_1(0) - \psi_2(0))(\sin(\psi_1(-r)) - \sin(\psi_2(-r))) \leq \|\psi_1 - \psi_2\|^2,$$

i.e. $\widehat{F}$ is 1-dissipative.

3. (Langmuir equation) The following example is non-standard, because we cannot solve it using the theory of semigroups and dissipative operators. However, it is an interesting case of infinite dimensional lumping, which has some applications in the context of chemical kinetics (see [4]). We consider again the Langmuir model we mentioned in the Introduction:

$$\frac{d}{dt}c(x,t) = -\frac{\alpha c(x,t)}{1 + \int_0^1 K(x)c(x,t)dx}, \tag{4.25}$$

where K is a dimensional parameter with units of an inverse concentration, and $\alpha > 0$. This model arose in the analysis of the adsorption of species onto simple surfaces. It can describe the adhesion of particles from a liquid substance to a solid surface, when all the reactants undergo the same reaction and compete for the same sites on the solid surface (we refer to [61] for the meaning of this model in chemical kinetics).
Our state variable is a continuous function $c \in C[0, 1]$, representing the concentration of a continuum of chemical species.
In order to obtain an everywhere defined operator on $C[0, 1]$, we modify the equation as follows:

$$\frac{d}{dt}c(x,t) = -\frac{\alpha c(x,t)}{1 + G(\int_0^1 K(x)c(x,t)dx)}, \tag{4.26}$$

where

$$G(z) = \begin{cases} z & \text{if } z \geq 0 \\ \frac{e^{2z-1}}{2} & \text{if } z < 0. \end{cases}$$

We will show that this modification doesn't effect our lumping analysis. Indeed, from the chemical kinetics point of view, we are interested in functions c with positive values

(since c represents a concentration), and we will see that if we start from a positive value, then the solution of the reduced system will stay positive for every $t \geq 0$.
The operator $F : C[0,1] \to C[0,1]$ generating the dynamics is then defined as follows:

$$F(c)(x) := -\frac{\alpha c(x)}{1 + G(\int_0^1 K(s)c(s)ds)}.$$

This operator is locally Lipschitz, so that we can guarantee the existence of a unique local solution for every initial condition. Indeed, knowing that G is Lipschitz with Lipschitz constant 1, and $G(z) \geq -\frac{1}{2}$ for every z, we can write:

$$\|F(c_1) - F(c_2)\| = \|\frac{-\alpha c_1(1 + G(\int_0^1 K(x)c_2(x,t)dx)) + \alpha c_2(1 + G(\int_0^1 K(x)c_1(x,t)dx))}{[1 + G(\int_0^1 K(x)c_1(x,t)dx)][1 + G(\int_0^1 K(x)c_2(x,t)dx)]}\|$$

$$= \|\frac{-\alpha c_1 + \alpha c_2}{[1 + G(\int_0^1 K(x)c_1(x,t)dx)][1 + G(\int_0^1 K(x)c_2(x,t)dx)]} +$$

$$+ \frac{-\alpha G(\int_0^1 K(x)c_2(x,t)dx)c_1 + \alpha G(\int_0^1 K(x)c_1(x,t)dx)c_2}{[1 + G(\int_0^1 K(x)c_1(x,t)dx)][1 + G(\int_0^1 K(x)c_2(x,t)dx)]}\|$$

$$= \|\frac{\alpha(c_2 - c_1)}{[1 + G(\int_0^1 K(x)c_1(x,t)dx)]} + \frac{\alpha c_2(G(\int_0^1 K(x)c_1(x,t)dx) - G(\int_0^1 K(x)c_2(x,t)dx)}{[1 + G(\int_0^1 K(x)c_1(x,t)dx)][1 + G(\int_0^1 K(x)c_2(x,t)dx)]}\|$$

$$\leq 2\alpha\|c_1 - c_2\| + 4\alpha\|c_2\|\|K\|_{L^1}\|c_1 - c_2\|.$$

Note that $c \equiv 0$ is an equilibrium point for the system. Let us fix a point $\overline{x}$ in $[0,1]$ and a positive initial condition $c_0 \in C[0,1]$. Let us call $c(t,\overline{x})$ the solution of the Langmuir model with fixed value of x and initial date $c_0(\overline{x})$. This solution will be strictly positive for every $t \geq 0$. Furthermore, its derivative is negative, so that $c(t,\overline{x})$ is decreasing. Since $\overline{x}$ is arbitrary, we can take the maximum over $\overline{x} \in [0,1]$ to obtain the following maximum principle for the Langmuir model (4.26):

$$\|c(t,x)\| \leq \|c(0,x)\|.$$

This implies that we can't have any blow up phenomena, and the solution is defined globally for any initial condition.
We define the following lumping operator:

$$M : C[0,1] \to \mathbb{R}, \qquad Mc := \int_0^1 K(x)c(x)dx.$$

It is easy to see that F preserves the fibers of M, and:

$$MF(c) = -\frac{\alpha \int_0^1 K(x)c(x)dx}{1 + G(\int_0^1 K(s)c(s)ds)} = -\frac{\alpha Mc}{1 + G(Mc)}.$$

Let us call $y := Mc$ for $c \in C[0,1]$. The reduced model is then the following one-dimensional system:

$$\begin{cases} \dot{y}(t) = -\frac{\alpha y}{1+G(y)} \\ y(0) = y_0 \end{cases}.$$

The reduced operator $\widehat{F}(y) := -\frac{\alpha y}{1+G(y)}$ is a locally Lipschitz map from $\mathbb{R}$ to itself, and it is easy to verify that if we choose a positive initial condition, the solution of the reduced system is positive for every $t \geq 0$ and tends to zero for $t \to +\infty$.

4. (Nonlinear diffusion: Porous media equation) We analyse a nonlinear diffusion equation with a quadratic nonlinearity. This is another non-standard example because we can't analyze it using semigroups theory, but it is interesting because, as the reduced model, we obtain a quadratic *porous media equation*, which is an important nonlinear equation of parabolic type. The porous media equations can describe the flow of a gas through a porous medium, and in the particular case of a quadratic nonlinearity, they are known as *Boussinesq's equations* and have an interpretation in ground-water infiltration (see [69] for a detailed description of these equations).
Let us consider a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary, and the space $L^1(\Omega, \mathbb{R}^2)$ of the integrable functions from Ω to $\mathbb{R}^2$. We define the following nonlinear function from $\mathbb{R}^2$ to itself:

$$g(u) := \begin{pmatrix} u_1^2 + u_1 u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}.$$

We consider a Cauchy problem with a nonlinear diffusion:

$$\begin{cases} \dot{u}(t) = \Delta(g(u)) \\ u(0) = u_0, \end{cases} \tag{4.27}$$

where Δ is the Laplace operator acting on a vector-valued function:

$$u(x,y) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}, \qquad \Delta u(x,y) := \begin{pmatrix} \frac{d^2}{dx^2} u_1(x,y) + \frac{d^2}{dy^2} u_1(x,y) \\ \frac{d^2}{dx^2} u_2(x,y) + \frac{d^2}{dy^2} u_2(x,y) \end{pmatrix}.$$

Let us define the linear and bounded operator $M : L^1(\Omega, \mathbb{R}^2) \to L^1(\Omega, \mathbb{R})$:

$$Mu(x,y) := u_1(x,y) + u_2(x,y).$$

Then:

$$M\Delta(g(u)) = \Delta[Mg(u)] = \Delta(u_1^2 + u_1u_2 + u_2^2 + u_1u_2) = \Delta(u_1 + u_2)^2 = \Delta[(Mu)^2].$$

Let $v := Mu$ be the lumped variable. Then the reduced model is the following Boussinesq's system on $L^1(\Omega, \mathbb{R})$:

$$\begin{cases} \dot{v}(t) = \Delta(v^2) \\ v(0) = v_0 \end{cases}.$$

Since we focus on the lumping analysis, we don't give the analytical details about this equation. We just mention that, under suitable hypotheses, a unique *weak solution* of the following system can be obtained for any $T > 0$ [69]:

$$\begin{cases} \dot{v}(t) = \Delta(v^2) & \text{on } Q_T \\ v(0,x) = v_0 & \text{on } \Omega \\ v(x,t) = 0 & \text{on } \Sigma_T \end{cases}, \qquad Q_T = \Omega \times (0,T),\ \Sigma_T = \delta\Omega \times [0,T].$$

We recall that a *weak solution* in Q_T is a locally integrable function $u \in L^1_{\text{loc}}(Q_T)$ such that:

$$\int\int_{Q_T} (\nabla u^2 \nabla \mu - u \frac{d}{dt}\mu)\, dx\, dt = 0,$$

for every test function $\mu \in C_c^\infty(Q_T)$.

Note that this kind of lumping can be generalized using Newton's binomial formula:

$$(a+b)^m = \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} a^{m-k} b^k.$$

Let us consider system (4.27) with the function g modified as follows:

$$g(u) := \begin{pmatrix} u_1^m + \frac{1}{2}\sum_{k=1}^{m-1} \begin{pmatrix} m \\ k \end{pmatrix} u_1^{m-k} u_2^k \\ u_2^m + \frac{1}{2}\sum_{k=1}^{m-1} \begin{pmatrix} m \\ k \end{pmatrix} u_1^{m-k} u_2^k \end{pmatrix}.$$

Then

$$M\Delta g(u) = \Delta[Mg(u)] = \Delta[(u_1 + u_2)^m],$$

and we obtain the general porous media equation as the reduced model:

$$\begin{cases} \dot{v}(t) = \Delta(v^m) \\ v(0) = v_0 \end{cases}.$$

Chapter 5

Lumping of Feller semigroups: a $C^\star$-algebra approach

5.1 Introduction

In the present chapter we describe a particular kind of lumping, making use of some concepts in the theory of Banach algebras. This lumping represents a different and unusual interpretation of the restriction operator over an arbitrary closed subset of a locally compact Hausdorff space. Even if this subset is arbitrary (indeed, it could be even a single point), the construction of our lumping operator, made by a quotient projection and a Gelfand transform, remains the same for any closed subset.
We deal with a class of semigroups called Feller semigroups, which are relevant because they can be associated in a unique way to strong Markov processes on some particular spaces (see [18, 35]). Unlike the general case, the lumping we describe preserves markovianity.
We always consider a locally compact Hausdorff space X, and we denote by $C_0(X)$ the space of continuous complex valued functions on X vanishing at infinity, while $C(K)$ is the space of the continuous functions defined on K, both endowed with the supremum norm. We consider a Feller semigroup $T(t)$ defined on $C_0(X)$ (see next section for the precise definition of Feller semigroup) and we consider the restriction operator $R : C_0(X) \ni f \mapsto f_K \in C(K)$, that maps a function in $C_0(X)$ into its restriction f_K to the set K. The operator R is obviously linear and bounded. Furthermore, it is surjective because of the Tietze Extension Theorem (see [17]). Note that the kernel of R is the closed ideal I_K of continuous functions vanishing on K. Moreover, provided that $\ker(R)$ is invariant under $T(t)$, the family

$$\widehat{T}(t)g := (T(t)f)_K \tag{5.1}$$

is again a strongly continuous semigroup, and R can be interpreted as a lumping operator. In the present chapter we prove the following two results: that the lumping induced by R can be obtained by the use of the Gelfand transform on a quotient space, and that this lumping can be extended to the case of general closed subsets of X. In the first result, we consider a compact subset $K \subset X$, while the second result is a generalization to the case of a closed subset $\mathscr{C}$, not necessarily compact.

Theorem 5.1. *Assume that I_K is invariant under $T(t)$. Then the lumping obtained by R is equivalent to the one obtained through the lumping operator $\widehat{\mathscr{G}}\pi$, where π is the canonical quotient projection $\pi : C_0(X) \to C_0(X)/I_K$ and $\widehat{\mathscr{G}}$ is the Gelfand transform of the Banach algebra $C_0(X)/I_K$.*

This means that the semigroup (5.1) coincides with

$$\widehat{T}(t)g = \widehat{\mathscr{G}}\pi(T(t)f), \quad g = \widehat{\mathscr{G}}\pi f \in C_0(K). \tag{5.2}$$

Both the semigroups (5.2) and (5.1) are again strongly continuous, positive and contractive on $C(K)$ (i.e. they are Feller semigroups).

Theorem 5.2. *Let $\mathscr{C}$ be a closed subset of X, not necessarily compact. Suppose that the closed ideal $I_{\mathscr{C}} := \{f \in C_0(X) : f(x) = 0 \, \forall x \in \mathscr{C}\}$ is invariant under the Feller semigroup $T(t)$. Then $\widehat{\mathscr{G}}\pi : C_0(X) \to C_0(\mathscr{C})$ is a lumping operator for $T(t)$ and the family of operators*

$$\widehat{T}(t)g = \widehat{\mathscr{G}}\pi(T(t)f), \quad g = \widehat{\mathscr{G}}\pi f \in C_0(\mathscr{C}) \tag{5.3}$$

is again a Feller semigroup on $C_0(\mathscr{C})$.
The lumping operator $\widehat{\mathscr{G}}\pi$ acts as the restriction map $R(f) = f_{\mathscr{C}}$.

$$\begin{tikzcd}
C_0(X) \arrow[r, "R"] \arrow[d, "\pi"'] & C_0(\mathscr{C}) \\
C_0(X)/I_{\mathscr{C}} \arrow[ur, "\widehat{\mathscr{G}}"'] &
\end{tikzcd}$$

These results give a different algebraic interpretation of the lumping obtained through R and guarantee that the main properties of Feller Semigroups, like contractivity and positivity, are preserved by the lumping.
During this chapter, we discuss the relation between the transition probability functions associated to the original semigroup $T(t)$ and the reduced semigroup $\widehat{T}(t)$ respectively, dealing with the adjoint of the lumping operator M^*.
We also discuss a particular application of the lumping operator $\widehat{\mathscr{G}}\pi$ to the *Diffusion semigroup*, passing through the Fourier transform. We apply M with respect to a closed subset in the space of frequencies and then we apply the inverse Fourier transform. In this way, we obtain an equation for the integral average of the state variable. We also show some applications of M to nonlinear evolution equations.

5.2 Preliminaries

Before going into details of Lumping, we give some background on Banach and $C^\star$-algebras, ideals, and Gelfand transform. We refer especially to [65], [57], [45] and [2].

Definition 5.1. A Banach space E is called a $C^\star$-algebra if the followings hold:

1. E is a Banach algebra, i.e. a normed and complete algebra on the complex field such that $||xy||_E \leq ||x||_E ||y||_E \quad \forall x, y \in E$,

2. E is endowed with an involution $\star : E \to E$ satisfying the $C^\star$-property: $||x^\star x||_E = ||x||_E^2 \quad \forall x \in E$.

It is well known that the space $C_0(X)$ is a $C^\star$-algebra with the pointwise product, the involution given by the complex conjugation and the supremum norm
$||f|| := \sup_{x \in X} |f(x)|$. $C_0(X)$ is also a complex Banach lattice, in the sense of the following definitions:

Definition 5.2. A real Banach lattice is a Banach space E endowed with an order relation $\leq$ such that:

1. $\forall\, x, y \in E$ there exist $\sup(x, y)$ and $\inf(x, y)$,
2. $\forall\, x, y, s \in E$, $x \leq y \Rightarrow x + s \leq y + s$,
3. $\forall\, x \in E$, $t \in \mathbb{R}_+$, $0 \leq x \Rightarrow 0 \leq tx$,
4. $\forall\, x, y \in E$, $|x| \leq |y| \Rightarrow ||x||_E \leq ||y||_E$.

Using this notion we can also define a complex Banach lattice, that is the complexification of a real Banach lattice. In this way we can obtain a notion of positivity in a complex lattice: in the case $C_0(X)$ the real part of the lattice is the subspace of real valued functions $C_0(X, \mathbb{R})$, and a positive element of the lattice is a positive element in $C_0(X, \mathbb{R})$ (see for instance [2] for details on complex Banach lattices).
We give some notions about ideals of commutative Banach algebras (i.e. subsets closed under multiplications), especially in the case of our state space $C_0(X)$. These results allow us to define the spectrum of an algebra and the Gelfand transform, which is involved in the lumping process we are going to describe.

Theorem 5.3 ([45])**.** *Let X be a locally compact Hausdorff space. For every subset $\mathscr{C} \in X$ let us define $I_{\mathscr{C}} := \{f \in C_0(X) \,:\, f(x) = 0\, \forall\, x \in \mathscr{C}\}$. Then the map $\mathscr{C} \to I_{\mathscr{C}}$ is a bijection between the collection of nonempty closed subsets of X and the proper closed ideals of $C_0(X)$.*

Proposition 5.4. *Let $\mathscr{A}$ be a $C^\star$-algebra and $I \subset \mathscr{A}$ a closed ideal. Then the quotient space $\frac{\mathscr{A}}{I}$ is again a $C^\star$-algebra with the quotient norm $|||[f]||| := \inf_{h \in I} ||f - h||_{\mathscr{A}}$.*

Definition 5.3. Given a complex algebra $\mathscr{A}$ and an ideal $I \subset \mathscr{A}$, we say that I is modular if the quotient $\frac{A}{I}$ is an algebra with an identity.

Proposition 5.5. *Let X be a locally compact Hausdorff space. For a subset $\mathscr{C} \in X$ let us consider the ideal $I_{\mathscr{C}} := \{f \in C_0(X) \,:\, f(x) = 0\, \forall\, x \in \mathscr{C}\}$. Then $I_{\mathscr{C}}$ is a modular ideal if and only if $\mathscr{C}$ is compact, and it is a maximal closed ideal if and only if $\mathscr{C}$ is made up of a single point.*

Definition 5.4. Let $\mathscr{A}$ be a commutative Banach algebra. We define the spectrum of $\mathscr{A}$ as the following space:

$$\sigma(\mathscr{A}) := \{\phi : \mathscr{A} \to \mathbb{C} \,:\, \phi \text{ is a nonzero multiplicative linear functional}\}$$

5.4 is equivalent to $\sigma(\mathscr{A}) := \{I \subset \mathscr{A} \,:\, I \text{ is a maximal modular ideal}\}$, because there is a bijection between the set of nonzero multiplicative functionals and the set of maximal modular ideals, given by the map $\phi \to \ker(\phi)$.
Since the maximal modular ideals of $C_0(X)$ are $I_x := \{f \in C_0(X) \,:\, f(x) = 0\}$, $x \in X$, we can say that $\sigma(C_0(X))$ can be identified with X itself.
All the nonzero multiplicative linear functionals on $C_0(X)$ are indeed evaluations of the kind

$$\phi_x(f) := f(x), \quad x \in X. \tag{5.4}$$

The spectrum as in definition (5.4), endowed with the weak-$\star$ topology of the dual algebra, is a locally compact Hausdorff space. This topology is given by the following notion of convergence:

Definition 5.5. If $\mathscr{A}^\star$ is the dual space of a Banach algebra $\mathscr{A}$, a sequence $\phi_n \subseteq \mathscr{A}^\star$ converges to ϕ in the weak-$\star$ topology if and only if $\phi_n(A) \to \phi(A) \quad \forall A \in \mathscr{A}$.

It is possible to show that the map $x \to \phi_x$ that allows us to identify X and $\sigma(C_0(X))$ is an homeomorphism with respect to the weak-$\star$ topology on the spectrum ([45]).

Definition 5.6 (Gelfand map)**.** Let $\mathscr{A}$ be a commutative Banach algebra. For $x \in \mathscr{A}$ we define the linear and continuous map $\widehat{x} : \sigma(\mathscr{A}) \to \mathbb{C}, \quad \widehat{x}(\phi) := \phi(x)$.
The Gelfand map of $\mathscr{A}$ is the linear continuous homomorphism

$$\mathscr{G} : \mathscr{A} \to C(\sigma(\mathscr{A})), \quad \mathscr{G}(x) := \widehat{x}.$$

Theorem 5.6. *If $\mathscr{A}$ is a commutative $C^\star$-algebra then the Gelfand map $\mathscr{G}$ is an isometric $\star$-isomorphism from $\mathscr{A}$ into $C_0(\sigma(\mathscr{A}))$.*
If $\mathscr{A}$ is unital, then $\sigma(\mathscr{A})$ is compact and $C_0(\sigma(\mathscr{A}))$ can be identified with $C(\sigma(\mathscr{A}))$, thus the Gelfand map is an isometric $\star$-isomorphism from $\mathscr{A}$ into $C(\sigma(\mathscr{A}))$.

From these results it follows that, if $\mathscr{A} = C_0(X)$, then for a given $x \in X$ $\widehat{f}(\phi) = \phi(f) = \phi_x(f) = f(x)$, and the Gelfand map $\mathscr{G}(f) = \{\phi_x \to \phi_x(f)\}$ can be identified with the Identity map $\mathscr{G}(f) = \{x \to f(x)\}$.
The following result allows us to characterize the spectrum of quotient algebra [45] :

Proposition 5.7. *If $\mathscr{A}$ is a commutative Banach algebra and $I \subset \mathscr{A}$ is a closed ideal, let us define $h(I) := \{\phi \in \sigma(\mathscr{A}) \,:\, \phi(I) = 0\}$.*
If π is the canonical quotient projection, then the map $\{\psi \in \sigma\left(\frac{\mathscr{A}}{I}\right) \to \psi \circ \pi \in h(I)\}$ is a homeomorphism.

The next theorem guarantees that the restriction operator R, being bounded and surjective, can be interpreted as a lumping operator (see section 5.4 for more details on this theorem):

Theorem 5.8 (Tietze extension theorem for $C_0(X)$)**.** *Let $g \in C_0(\mathscr{C})$, where $\mathscr{C}$ is a closed subset of a locally compact Hausdorff space X and $C_0(\mathscr{C})$ denotes the space of continuous functions from $\mathscr{C}$ to $\mathbb{C}$ vanishing at infinity. Then there exists $f \in C_0(X)$ which extends g. The extension f can be chosen such that*

$$\|f\|_{C_0(X)} = \|g\|_{C_0(\mathscr{C})}.$$

Finally, we give the definition of Feller semigroups:

Definition 5.7 (Feller semigroup)**.** Let X be a locally compact Hausdorff space. A family of bounded linear operators $\{T(t)\}_{t\geq 0}$ is said to be a Feller semigroup if it is a strongly continuous semigroup on $C_0(X)$ satisfying the following properties:

1. $\{T(t)\}_{t\geq 0}$ is contractive: $||T(t)f|| \leq ||f|| \quad \forall t \geq 0,\, f \in C_0(X)$,
2. $\{T(t)\}_{t\geq 0}$ is positive: $T(t)f(x) \geq 0 \; \forall t \geq 0,\, x \in X,\, f$ positive function in $C_0(X)$.

We are ready to deal with lumpability of Feller Semigroups and to prove our main results.

5.3 Proof of the main results

Let us consider a Feller semigroup $\{T(t)\}_{t\geq 0}$ in $C_0(X)$. We want to interpret the restriction R as a lumping operator, in order to restrict our state space to a space of functions defined on a compact subset $K \subset X$. In particular we want to show that this lumping coincides with the lumping obtained through a particular application of the Gelfand transform. This equivalent lumping give us a way to show that the upper level semigroup on $C(K)$ is again positive and contractive. Let us consider the restriction map R. Thanks to the Tietze Extension Theorem every element $f \in C(K)$ can be seen as the restriction of some $F \in C_0(X)$, so that

R is surjective.
In order to have a lumping of the semigroup through R we need to apply theorem 2.14, thus we require the kernel of R to be invariant under all the operators $\{T(t)\}_{t\geq 0}$, where $\ker(R) = \{f \in C_0(X) : f(x) = 0 \,\forall x \in K\}$.
Note that $\ker(R)$ is a closed ideal in the algebra $C_0(X)$, so we will use equivalently the notations I_K and $\ker(R)$ for the subspace of the functions vanishing on K. As we have pointed out in the preliminaries, every closed ideal of $C_0(X)$ has the same shape of I_K. Let $T(t)$ be a Feller semigroup on $C_0(X)$ such that I_K is invariant under $T(t)$ for every $t \geq 0$. We consider the lumping of this semigroup made by the restriction map, which give rise to the new strongly continuous semigroup on $C(K)$ (5.1). Our aim is to show that the same lumping can be obtained through the composition of the Gelfand transform of a quotient algebra and the quotient projection. We extend this kind of lumping to $C_0(\mathscr{C})$ for an arbitrary closed subset $\mathscr{C}$ and we show that our lumping operator preserve the Feller semigroup properties. Before giving the proof of results (5.1) and (5.2), whose statement is in the introduction of the present chapter, we prove the following important fact:

Proposition 5.9. *$\sigma\left(C_0(X)/I_{\mathscr{C}}\right)$ is homeomorphic to the set of linear multiplicative functionals on $C_0(X)$ acting as:*

$$\phi_x(f) := f(x),\, x \in \mathscr{C}.$$

Proof. From Proposition 5.7 we know that $\sigma\left(C_0(X)/I_{\mathscr{C}}\right)$ is homeomorphic to

$$h(I_{\mathscr{C}}) = \{\phi \in \sigma(C_0(X)) : \phi(g) = 0,\ \forall g \in I_{\mathscr{C}}\}.$$

We need to characterize the functionals in $h(I_{\mathscr{C}})$. All the nonzero multiplicative linear functionals on $C_0(X)$ act as in (5.4). By definition, given a point $x \in \mathscr{C}$, ϕ_x is a multiplicative linear functional vanishing on $I_{\mathscr{C}}$, thus the set of functionals $\{\phi_x,\, x \in \mathscr{C}\}$ belongs to $h(I_{\mathscr{C}})$. Let us verify that the inverse inclusion is also true. If the functional ϕ_x vanishes on $I_{\mathscr{C}}$, then every element in $I_{\mathscr{C}}$ vanishes on $\mathscr{C} \cup \{x\}$, so that

$$I_{\mathscr{C}} \subset I_{\mathscr{C}\cup\{x\}}. \tag{5.5}$$

Since the inverse inclusion of (5.5) is obviously true, we have $I_{\mathscr{C}} = I_{\mathscr{C}\cup\{x\}}$. Moreover, by the one to one correspondence between closed subsets $\mathscr{C} \subset X$ and proper closed ideals $I_{\mathscr{C}}$ (Theorem 5.3) follows that $x \in \mathscr{C}$.
This means that $\{\phi_x,\, x \in \mathscr{C}\} \simeq h(I_{\mathscr{C}})$, from which we obtain the thesis. $\square$

Proof of Theorem 5.1. Let us consider the unital $C^\star$-algebra $C_0(X)/I_K$, whose elements are the equivalence classes of the kind $[f] := \{f + h,\, h \in I_K\}$, $f \in C_0(X)$.
All the functions in the same equivalence class coincide on the subset K, and the identity is obviously the class of the functions identically equal to 1 on K.
We already know (see [2]) that, thanks to the invariance of the considered ideal, the quotient projection $\pi : C_0(X) \to C_0(X)/I_K$ makes a lumping on the semigroup $T(t)$, and the new semigroup on the quotient is defined as

$$\widetilde{T}(t)[f] := [T(t)f]. \tag{5.6}$$

We observe that this semigroup is again contractive, in fact, since π is bounded and $T(t)$ is contractive, we can write:

$$||\widetilde{T}(t)[f]|| = \inf_{h\in I_K} ||\widetilde{T}(t)[f-h]|| = \inf_{h\in I_K} ||[T(t)(f-h)]|| \leq$$

$$\leq \inf_{h\in I_K} ||T(t)(f-h)|| \leq \inf_{h\in I_K} ||f-h|| = ||[f]||.$$

We have observed in our Preliminaries that the quotient $C_0(X)/I_K$ is isometrically isomorphic to $C(K)$. The isomorphism is given by the Gelfand transform $\widehat{\mathscr{G}}$, acting as

$$\widehat{\mathscr{G}} : [f] \in C_0(X)/I_K \to \widehat{\mathscr{G}}([f]) \in C(K),$$

$$\widehat{\mathscr{G}}([f]) := \{x \in K \to f(x)\}.$$

Now, let us consider again our quotient semigroup (5.6). For any $t \geq 0$ we define the bounded linear operator on $C(K)$:

$$\widehat{T}(t)g := \widehat{\mathscr{G}}(\widetilde{T}(t)[f]) = \widehat{\mathscr{G}}[T(t)f] = \{x \in K \to (T(t)f)(x)\},$$

where $g = \widehat{\mathscr{G}}[f]$ for some $[f]$ in the quotient space.
Since $\widehat{\mathscr{G}}$ is linear and isometric it is easy to see that $\widehat{T}(t)$ is again a strongly continuous semigroup. Furthermore $\widehat{T}(t)$ is contractive, because $\widetilde{T}(t)$ itself is contractive and the Gelfand map is an isometry; given $g = \widehat{\mathscr{G}}([f])$ we can write

$$||\widehat{T}(t)g|| = ||\widehat{\mathscr{G}}(\widetilde{T}(t)[f])|| = ||\widetilde{T}(t)[f])|| \leq$$

$$\leq ||[f]|| = ||\widehat{\mathscr{G}}([f])|| = ||g||.$$

Besides, $\widehat{T}(t)$ is positive, in fact, given a positive $g \in C(K)$, we have
$g = \widehat{\mathscr{G}}([f]) \geq 0$ for a given equivalence class; since $\widehat{\mathscr{G}}([f]) := \{x \in K \to f(x)\}$, all the elements in $[f]$ must be positive in K. We just need to choose an element in $[f]$ such that $f(x) \geq 0$ for every $x \in X$. For instance we can choose $|\mathrm{Re}(f(x))|$, which can be written as $f(x) - i\mathrm{Im}(f(x)) + 2\mathrm{Re}(f^-(x))$ and coincides with f on K (f^- being the negative part of f). Using the positivity of the original semigroup $T(t)$ we have that $\{x \in K \to (T(t)f)(x)\}$ is a positive map.
To conclude, the semigroup (5.1) coincides with the semigroup $\widehat{T}(t)$ and we have a lumping:

$$\widehat{T}(t)\widehat{\mathscr{G}}\pi(f) = \widehat{\mathscr{G}}\pi(T(t)f), \quad f \in C_0(X).$$

The lumping operator is $\widehat{\mathscr{G}} \circ \pi$ (acting in the same way of R), that is bounded, linear and surjective.
The new reduced semigroup is again contractive and positive on a space of functions defined on a compact set. □

Proof of Theorem 5.2. The same kind of lumping can be applied in the case of a quotient algebra without identity. Let us suppose that the closed ideal $I_{\mathscr{C}}$, where $\mathscr{C}$ is a closed but not compact subset of X, is invariant under $T(t)$. The quotient $C_0(X)/I_{\mathscr{C}}$ is a $C^\star$-algebra without identity, however we can still obtain a new Feller semigroup through the lumping operator $\widehat{\mathscr{G}} \circ \pi$. Indeed by Proposition 5.9 the Gelfand map is again an isometric $\star$-isomorphism from $C_0(X)/I_{\mathscr{C}}$ to $C_0(\mathscr{C})$.
Using the same arguments as before we can say that $\widehat{T}(t)g = \widehat{\mathscr{G}}(\widetilde{T}(t)[f])$, $g = \mathscr{G}[f]$, is a Feller semigroup in $C_0(\mathscr{C})$, so that it can be still associated to the transition function of a Markov Process. The lumping operator $\widehat{\mathscr{G}} \circ \pi$ acts again as the restriction operator R. □

Since in both the cases of K compact and $\mathscr{C}$ closed the reduced semigroup is a Feller semigroup, the lumping operator $M = \widehat{\mathscr{G}} \circ \pi$ preserves markovianity.

Remark 5.1 (Lumpability with respect to the infinitesimal generator)**.** We already know that, in order to have a lumping by $M = \widehat{\mathscr{G}} \circ \pi$, theorem 2.15 must hold with respect to $\ker(M) = I_{\mathscr{C}}$. Let A be the infinitesimal generator of the Feller semigroup $T(t)$ and let us suppose that $I_{\mathscr{C}}$ is A-invariant. In addiction we want the operator $(\lambda I - A)$ to be surjective from $I_{\mathscr{C}} \cap \mathscr{D}(A)$ to $I_{\mathscr{C}}$, for $\mathrm{Re}(\lambda) > 0$ (indeed, the growth bound of a contractive semigroup is zero). These values of λ belong to the resolvent set of A, so that, given $g \in I_{\mathscr{C}}$, there exists a unique $f \in C_0(X)$ such that $(\lambda I - A)f = g$. We want the restricted function $f_{\mathscr{C}}$ to be identically zero. Since $g \in I_{\mathscr{C}}$ if and only if $(\lambda I - A)f = 0$ over $\mathscr{C}$, condition 2 of theorem 2.15 is equivalent to the following:

$$[(\lambda I - A)f](x) = 0 \quad \forall x \in \mathscr{C} \Rightarrow f(x) = 0 \quad \forall x \in \mathscr{C}.$$

Remark 5.2 (A generalization to the nonlinear case)**.** We observe that the lumping operator $M = \widehat{\mathscr{G}}\pi$ can be also applied to some nonlinear equations in $C_0(X)$, even if these equations can't be associated with a standard strongly continuous semigroup. For instance, we consider the equation:

$$\dot{u}(t,x) = g(x)\, u(x)^n, \quad g, u \in C_0(X),\ n \in \mathbb{N}.$$

Given a closed subset $\mathscr{C} \subset X$, if $u_{|\mathscr{C}} \equiv v_{|\mathscr{C}}$, then also $(Fu)_{|\mathscr{C}} := (g(x)\, u(x)^n)_{|\mathscr{C}} \equiv (g(x)\, v(x)^n)_{|\mathscr{C}} \equiv (Fv)_{|\mathscr{C}}$, so F preserves the fibers of our lumping operator. We can define an operator on the quotient algebra $C_0(X)/I_{\mathscr{C}}$:

$$\widetilde{F}[u] := [F(u)].$$

It is well-defined because if $v = u + h$, where $h \in I_{\mathscr{C}}$, (i.e. $v \in [u]$) then:

$$F(v) = F(u+h) = g(x)(u(x) + h(x))^n =$$

$$= g(x)u(x)^n + g(x)\sum_{k=1}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} u(x)^k h(x)^{n-k} + g(x)h(x)^n = F(u) + \widetilde{h}(x),$$

where $\widetilde{h}(x)$ is an element of $I_{\mathscr{C}}$. This means that $F(v) \in [F(u)]$. By definition, $\pi F = \widetilde{F}\pi$. Then, we apply to $\widetilde{F}$ the Gelfand transform $\widehat{\mathscr{G}}$ of the quotient algebra to obtain the following nonlinear operator on $C_0(\mathscr{C})$:

$$\widehat{F}(u_{|\mathscr{C}}) = \widehat{F}(\widehat{\mathscr{G}})[u] := \widehat{\mathscr{G}}\pi(F(u)) = \{x \in \mathscr{C} \to g(x)u(x)^n\}.$$

Another nonlinear case that can be reduced through $\widehat{\mathscr{G}}\pi$ is the following:

$$\dot{u} = \phi(u),$$

where ϕ is a continuous, decreasing function from $\mathbb{R}$ to itself such that $\phi(0) = 0$. Our state space is the Banach space $X := \mathscr{B}(\mathbb{R})$ of the continuous, bounded, real valued functions from $\mathbb{R}$ to itself with the supremum norm.
This case can be treated using our theory for nonlinear lumping, indeed the operator $G(u) := \phi(u)$ is dissipative and generates a nonlinear semigroup (see the examples of Chapter 3). Let us apply M with respect to a closed subspace $\mathscr{C} \subset \mathbb{R}$. If we take $u, v \in X$ such that $u = v$ on $\mathscr{C}$, then also $\phi(u) = \phi(v)$ on $\mathscr{C}$, i.e. $MG(u) = MG(v)$.
Let us define the reduced variable $f := u_{|\mathscr{C}}$. As expected, this lumping leads to the equation

$$\dot{f} = \phi(f), \quad f \in \mathscr{B}(\mathscr{C}).$$

According to our theory, the solution operators of this equation still form a nonlinear C_0-semigroup on $\mathscr{B}(\mathscr{C})$.

5.3.1 An application to Markov processes

Let X be a locally compact, second countable Hausdorff space and $\mathscr{B}(X)$ be the Borel σ-algebra of X. We recall that a transition probability function on $\mathscr{B}(X)$ is a family of maps $P_t : X \times \mathscr{B}(X) \to [0,1]$ such that $\forall t \geq 0$ the following properties hold:

(i) for all $t \geq 0$ and for all $x \in X$, $P_t(x,\cdot)$ is a probability measure on $\mathscr{B}(X)$;

(ii) for all $x \in X$, $P_0(x,\cdot) = \delta_x$, being δ_x the Dirac measure centered in x;

(iii) $\forall t \geq 0$ and for all Borel subsets $A \in \mathscr{B}(X)$, $P_t(\cdot, A)$ is a measurable function;

(iii) The *Chapman-Kolmogorov equation* holds, i.e. for all $t, s \geq 0$, $x \in X$, $A \in \mathscr{B}(X)$:

$$P_{t+s}(x,A) = \int_X P_t(y,A)P_s(x,dy).$$

Given a Markov Process $\mathscr{X}(t)$, we say that it admits P_t as transition probability function if for all $A \in \mathscr{B}(X)$ and $t \geq s \geq 0$ we can write the conditional probabilities in the following way:

$$\mathbb{P}(\mathscr{X}(t) \in A | \mathscr{X}(s)) = P_{t-s}(\mathscr{X}(s), A).$$

The following result has been proved by R.M. Blumenthal and R.K. Getoor [35]:

Theorem 5.10 (R.M.Blumenthal, R.K.Getoor)**.** *Let X be a locally compact, second countable Hausdorff space and let $T(t)$ be a Feller Semigroup on $C_0(X)$. Then there exists a Markov Process $\mathscr{X}(t)$ such that*

$$T(t)f(x) = \mathbb{E}_x(f(\mathscr{X}(t))) = \int_X f(y)P_t(x,dy),$$

where P_t is the transition probability function associated to $\mathscr{X}(t)$. P_t is defined on $\mathscr{B}(X)$.

Conversely, it is proved in [18](Thm 2.10) that under suitable regularity hypotheses *strong Markov processes* can be associated to Feller semigroups.

Let us apply Theorem 5.10 to the Feller semigroup $T(t)$ and to the Feller semigroup $\widehat{T}(t)$ obtained by our lumping operator $\widehat{\mathscr{G}} \circ \pi$. We obtain the following relations:

$$T(t)f(x) = \int_X f(y)P_t(x,dy), \quad \forall f \in C_0(X);$$

$$\widehat{T}(t)g(x) = \int_{\mathscr{C}} g(y)\widehat{P}_t(x,dy), \quad \forall g \in C_0(\mathscr{C}),$$

where $P_t(x,dy)$ is a transition probability function defined on $X \times \mathscr{B}(X)$, while $\widehat{P}_t(x,dy)$ is a transition probability function defined on $\mathscr{C} \times \mathscr{B}(\mathscr{C})$. By the lumping relation $MT(t)f(x) = \widehat{T}(t)Mf(x)$, which holds for all $f \in C_0(X)$, we obtain the following identity:

$$\int_X f(y)P_t(x,dy) = \int_{\mathscr{C}} f_{|\mathscr{C}}(y)\widehat{P}_t(x,dy), \quad \forall\, x \in \mathscr{C}.$$

This means that for all $x \in \mathscr{C}$ and for every Borel subset $A \subset X$:

$$P_t(x,A) = \widehat{P}_t(x, A \cap \mathscr{C}).$$

In particular, being $\mathscr{M}(X)$ and $\mathscr{M}(\mathscr{C})$ the spaces of the complex measures on X and $\mathscr{C}$ respectively, let us consider the adjoint of our lumping operator

$$M^* : \mathscr{M}(\mathscr{C}) \to \mathscr{M}(\mathscr{X}).$$

This operator acts on a measure μ defined on $\mathscr{B}(\mathscr{C})$ as the extension ν of the measure over $\mathscr{B}(X)$ in such a way that $\nu(A) = 0$ if A and $\mathscr{C}$ are disjoint. Indeed, if we denote with ϕ_μ and ψ_ν the functionals associated with the measures μ and ν respectively:

$$M^*(\phi_\mu)(f) = \int_{\mathscr{C}} f_{|\mathscr{C}}(y)\, d\mu(y) = \int_X f(y)\, d\nu(y) =: \psi_\nu(f).$$

It is easy to verify that this operator is injective and has closed range, thus

$$[M^*]^{-1} : \mathrm{Ran}(M^*) \to \mathscr{M}(\mathscr{C})$$

is well defined. In particular, for all $x \in \mathscr{C}$ the following relations hold between the two probability measures:

$$P_t(x, \cdot) = M^* \widehat{P}_t(x, \cdot), \quad \widehat{P}_t(x, \cdot) = [M^*]^{-1} P_t(x, \cdot). \tag{5.7}$$

By the definition of M^*, the support of the measure $P_t(x, \cdot)$ is contained in the subset $\mathscr{C}$. Conversely, if for all $x \in \mathscr{C}$ the probability measure $P_t(x, \cdot)$ belongs to the range of M^*, then its support is contained in $\mathscr{C}$, thus the ideal $I_{\mathscr{C}}$ is invariant under the associated semigroup $T(t)$. This means that all the Feller semigroups preserving the ideal $I_{\mathscr{C}}$ satisfy $P_t(x, \cdot) \in \mathrm{Ran(M^*)}$.

5.3.2 Example: lumping of the diffusion semigroup and the Fokker-Planck equation in the Schwartz space

We describe an application of the operator $M = \mathscr{G}\pi$ acting on the Schwartz space and composed with the Fourier transform. This lumping gives an alternative interpretation of the *average operator*.
Let us define the diffusion semigroup on $C_0(\mathbb{R}^2)$:

$$T(t)f(s) := \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{|s-r|^2}{4t}} f(r)\, dr = \mu_t * f(s),$$

where $\mu_t(s) = \frac{1}{4\pi t} e^{-\frac{|s|^2}{4t}}$.
This semigroup gives the family of the solution operators for the heat equation $\dot{u} = \Delta u$.
For the diffusion semigroup $T(t)$ written as above it is not possible to find an invariant ideal of functions. For this reason, we exploit the properties of the Fourier transform to pass from a convolution semigroup to a multiplication semigroup in the space of frequencies.
We consider the space $\mathscr{S}(\mathbb{R}^2) \subset C_0(\mathbb{R}^2)$ (known as the *Schwartz space*) of functions all of whose derivatives are rapidly decreasing, i.e.:

$$\mathscr{S}(\mathbb{R}^2) := \{f \in C^\infty(\mathbb{R}^2) \,:\, \sup_{x \in \mathbb{R}^2} |x^\alpha D^\beta f(x)| < \infty \;\; \forall \alpha, \beta > 0\}.$$

As an example, the Gaussian function $\mu_t(s)$ is itself an element of $\mathscr{S}(\mathbb{R}^2)$.
We recall that the *Fourier transform* is the following linear operator from $L^1(\mathbb{R}^2)$ to $C_0(\mathbb{R}^2)$:

$$[\mathscr{F} f](\omega) = \int_{\mathbb{R}^2} e^{-i(\omega, s)} f(s) ds.$$

In particular, $\mathscr{F}$ is an isomorphism from $\mathscr{S}(\mathbb{R}^2)$ to itself, with inverse operator:

$$[\mathscr{F}^{-1} g](x) = \int_{\mathbb{R}^2} e^{i(\omega, x)} g(\omega) d\omega, \quad g \in \mathscr{S}(\mathbb{R}^2).$$

Applying the Fourier transform on the diffusion semigroup we obtain a multiplication semigroup (indeed, the Fourier transform of a convolution is the product of the Fourier transforms):

$$[\mathscr{F}T(t)f](\omega) = [\mathscr{F}\mu_t](\omega)[\mathscr{F}f](\omega) =$$

$$= e^{-t|\omega|^2}[\mathscr{F}f](\omega) =: \widetilde{T}(t)[\mathscr{F}f](\omega).$$

Let us consider $\mathscr{C} = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}$. Of course, if $[\mathscr{F}f](\omega)$ vanishes on $\mathscr{C}$, then $\widetilde{T}(t)[\mathscr{F}f](\omega)$ also vanishes on $\mathscr{C}$. We can apply to $\widetilde{T}(t)$ our lumping operator $M = \widehat{\mathscr{G}}\pi$ with respect to the closed ideal $I_{\mathscr{C}}$. We obtain the following:

$$M(\widetilde{T}(t)[\mathscr{F}f]) = M\left(e^{-t(\omega_1^2+\omega_2^2)}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(\omega_1 s_1+\omega_2 s_2)} f(s_1,s_2)\, ds_1 ds_2\right) =$$

$$= e^{-t\omega_1^2}\int_{\mathbb{R}} e^{-i(\omega_1 s_1)}\left(\int_{\mathbb{R}} f(s_1,s_2)\, ds_2\right) ds_1 = [\mathscr{F}_1\nu_t](\omega_1)\, M[\mathscr{F}f](\omega_1)$$

$$= [\mathscr{F}_1\nu_t](\omega_1)\, [\mathscr{F}_1\widetilde{f}](\omega_1),$$

where

$$\nu_t(s_1) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{s_1^2}{4t}}$$

is the one-dimensional Gaussian function, $\mathscr{F}_1$ is the one-dimensional Fourier transform and $\widetilde{f}$ can be interpreted as the *average* of f with respect to the second variable s_2:

$$\widetilde{f}(s_1) := \int_{\mathbb{R}} f(s_1,s_2)\, ds_2.$$

This happens because, contrary to expectations, $M[\mathscr{F}f]$ turns to be the one-dimensional Fourier transform $\mathscr{F}_1$ of the average $\widetilde{f}$ and it does not coincide with the Fourier transform of the restriction $f_{|\mathscr{C}}$.
Applying the one-dimensional inverse Fourier transform we obtain:

$$\mathscr{F}_1^{-1}[M\mathscr{F}[T(t)f]] = \mathscr{F}_1^{-1}[M(\widetilde{T}(t)[\mathscr{F}f])] =$$

$$= \mathscr{F}_1^{-1}[[\mathscr{F}_1\nu_t](\omega_1)\, [\mathscr{F}_1\widetilde{f}](\omega_1)] = \nu_t * \widetilde{f}(s_1) = T_1(t)\widetilde{f}(s_1),$$

where

$$T_1(t)f(s_1) := \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(s-r)^2}{4t}} f(r)\, dr.$$

From a solution $T(t)f$ of the two-dimensional heat equation $\dot{u} = \Delta u$ with initial condition $f \in \mathscr{S}(\mathbb{R}^2)$ we obtain a solution $T_1(t)\widetilde{f}$ of the one-dimensional heat equation $\dot{u} = \frac{\partial}{\partial s_1^2}u$ with initial condition $\widetilde{f}(s_1) := \int_{\mathbb{R}} f(s_1,s_2)\, ds_2$. We can say that this lumping gives the evolution of the average of f with respect to the second variable s_2.

The same method can be applied to a simple case of the *Fokker-Planck equation*. This equation describes the evolution of the probability density function of a stochastic process in $\mathbb{R}^d$. The heat equation is itself the Fokker-Planck equation for the Brownian motion. In this example we choose the *drift vector* $\sigma(x_1,x_2) = c(x_1,x_2),\ c > 0$ and the constant *diffusion coefficient* $\mu > 0$:

$$\dot{\rho} = c\sum_{i=1,2}\frac{d}{dx_i}(x_i\rho) + \mu\Delta(\rho) + 2\mu\frac{d}{dx_1 dx_2}(\rho), \tag{5.8}$$

where $\rho = \rho(x_1,x_2)$ is again a function in the Schwartz space and represents the density function of a stochastic process. In particular, we can interpret the equation as a Fokker-Planck equation for a two-dimensional *Ornstein-Uhlenbeck process*, which is a Markov process used to describe the velocity of a massive Brownian particle under the influence of friction. In this case we work directly on the partial differential equation, rather then the semigroup. First, we apply the Fourier transform to both sides of equation (5.8), to obtain:

$$\frac{d}{dt}[\mathscr{F}\rho](\omega) = c\sum_{i=1,2}\omega_i\frac{d}{d\omega_i}([\mathscr{F}\rho](\omega)) - \mu\,|\omega|^2[\mathscr{F}\rho](\omega) - 2\mu\,\omega_1\omega_2[\mathscr{F}\rho](\omega), \tag{5.9}$$

where $[\mathscr{F}\rho]$ is the Fourier transform of ρ and $\omega = (\omega_1, \omega_2)$.
As before, we want to apply the lumping operator $M = \widehat{\mathscr{G}}\pi$ with respect to the closed subspace $\mathscr{C} = \mathbb{R}\times\{0\} \subset \mathbb{R}\times\mathbb{R}$, where $\mathbb{R}\times\mathbb{R}$ is the space of frequencies, i.e. we restrict to the set $\{\omega_2 = 0\}$. We recall that:

$$M[\mathscr{F}\rho] = [\mathscr{F}\rho]_{|\{\omega_2=0\}} = \int_{\mathbb{R}} e^{-i\omega_1 x_1}\left(\int_{\mathbb{R}}\rho(x_1,x_2)dx_2\right)dx_1 = \mathscr{F}_1\widetilde{\rho},$$

where $\mathscr{F}_1$ is the one-dimensional Fourier transform and $\widetilde{\rho}(x_1) = \int_{\mathbb{R}}\rho(x_1,x_2)dx_2$ is the integral average of ρ with respect to x_2. The relation $M[\mathscr{F}\rho] = \mathscr{F}_1\widetilde{\rho}$ is fundamental. Indeed, if $M[\mathscr{F}\rho] \equiv 0$, then $\widetilde{\rho} \equiv 0$ because $\mathscr{F}_1$ is an isomorphism on $\mathscr{S}(\mathbb{R})$ and $\widetilde{\rho} \in \mathscr{S}(\mathbb{R})$. This means that

$$\left[\frac{d}{d\omega_1}[\mathscr{F}\rho](\omega)\right]_{|\{\omega_2=0\}} = -\int_{\mathbb{R}} ix_1 e^{-i\omega_1 x_1}\left(\int_{\mathbb{R}}\rho(x_1,x_2)dx_2\right)dx_1$$

is again equal to zero.
Let us apply M to both the sides of (5.9). We obtain the following equation:

$$\frac{d}{dt}[\mathscr{F}_1\widetilde{\rho}](\omega_1) = c\,\omega_1\frac{d}{d\omega_1}[\mathscr{F}_1\widetilde{\rho}](\omega_1) - \mu\,\omega_1^2[\mathscr{F}_1\widetilde{\rho}](\omega_1). \tag{5.10}$$

Finally, applying the one-dimensional inverse Fourier transform $\mathscr{F}_1^{-1}$ to both the sides of (5.10) we obtain a Fokker-Planck equation of the Ornstein-Uhlenbeck kind for the average function $\widetilde{\rho}$:

$$\frac{d}{dt}\widetilde{\rho} = -c\,\frac{d}{dx_1}(x_1\widetilde{\rho}(x_1)) + \mu\,\frac{d^2}{dx_1^2}\widetilde{\rho}.$$

To obtain this equation, it is necessary to work in the Schwartz space because the Fourier transform must be invertible. This kind of lumping is non-standard, because the Schwartz space is not a Banach space but is a dense subspace contained in C_0. However, it is an interesting example because, even if the lumping acts as a restriction, after the lumping we obtain an equation for the average of the original state variable, rather than an equation for its restriction. This happens because we pass through the Fourier transform and we apply the lumping in the space of frequencies.

5.4 A $C^\star$-algebra approach to the Tietze extension theorem

Let X and Y be two topological spaces, we say that a function $f : X \to Y$ *extends* $g : \mathscr{A} \subset X \to Y$ if f coincides with g over the subset $\mathscr{A}$. The *Tietze extension theorem* gives sufficient conditions for the existence of an extension of a given function g. This is one of the fundamental theorems in classical topology, and in this section we are going to prove the following version of this theorem using only functional analytical arguments.

Theorem 5.11 (Tietze extension theorem for $C_0(X)$)**.** *Let $g \in C_0(\mathscr{C})$, where $\mathscr{C}$ is a closed subset of a locally compact Hausdorff space X and $C_0(\mathscr{C})$ denotes the space of continuous functions from $\mathscr{C}$ to $\mathbb{C}$ vanishing at infinity. Then there exists $f \in C_0(X)$ which extends g. The extension f can be chosen such that*

$$\|f\|_{C_0(X)} = \|g\|_{C_0(\mathscr{C})}.$$

We recall that the Tietze extension theorem was originally proved for metric spaces by H. Tietze and then generalized by P. Urysohn to normal topological spaces. The following theorem collects these results. A classical proof can be found in [53, 64], and a shorter one in [37].

Theorem 5.12 (Tietze-Urysohn)**.** *Let X be a topological space, $\mathscr{C} \subset X$ a closed subset and $g : \mathscr{C} \to [-1,1]$ a continuous function. Then, there exists a continuous extension $f : X \to [-1,1]$ in the following cases:*

(i) (Tietze) X is a metric space;

(ii) (Urysohn) X is a normal space, i.e. every couple of disjoint closed sets can be separated by two disjoint open sets;

(iii) (Urysohn) X is a locally compact Hausdorff space and $\mathscr{C}$ is compact.

Moreover, the extension f can be chosen in such a way to satisfy

$$\|f\|_{C_0(X)} = \|g\|_{C_0(\mathscr{C})}$$

and in case (iii) the support of f is compact.

Note that (ii) generalizes (i), while Theorem 5.11 generalizes (iii) to the case of $\mathscr{C}$ non-compact. It is worth mentioning that Theorem 5.11 can be deduced from Theorem 5.12 using classical topological arguments for locally compact Hausdorff spaces (cp. for instance [64]). However, we propose a different proof, based on functional analysis arguments. We introduced the method that we are going to describe in a work about *lumpability* of abstract Cauchy problems in the context of dynamical systems ([62]). We remark that the connection between the Tietze extension Theorem and the theory of $C^\star$-algebras was previously exploited in order to generalize this theorem to surjective morphisms between $C^\star$-algebras, see for example the *non commutative Tietze extension Theorem* [46] (asserting that a surjective morphism between $C^\star$-algebras admits an extension to the corresponding multiplier algebras) and its generalizations to Hilbert bimodules [31] and Hilbert $C^\star$-modules [7]. Nevertheless we are not aware of any reference pointing out the interesting connection between the classical Tietze extension theorem and a proof based on results in Banach algebras theory, although we expect this to be known to the experts. Finally we also stress that our proof implicitly makes use of Urysohn's Lemma (in particular, it is used to show that the map $x \to \phi_x$ from X into $\sigma(C_0(X))$ is an homeomorphism with respect to the weak-$\star$ topology on the spectrum [45]), this not being avoidable for what we know.

Next we explain briefly the idea of the proof. Let $I_{\mathscr{C}}$ denote the closed ideal of continuous functions vanishing on $\mathscr{C}$. We consider the following two maps:

(1) the quotient map $\pi : C_0(X) \to Q$, where $Q := C_0(X)/I_{\mathscr{C}}$ is the quotient algebra;

(2) the Gelfand transform $\widehat{\mathscr{G}} : Q \to C_0(\sigma(Q))$, where $\sigma(Q)$ denotes the spectrum of the algebra Q.

By some classical results in Banach algebra theory, we know that $C_0(\sigma(Q))$ can be identified with $C_0(\mathscr{C})$, and the composition of the Gelfand transform with the quotient projection coincides with the restriction operator $R : C_0(X) \to C_0(\mathscr{C})$ which sends a map $f \in C_0(X)$ to its restriction over the set $\mathscr{C}$, i.e. $R(f) = f|_{\mathscr{C}}$.

Since both the quotient projection π and the Gelfand transform $\widehat{\mathscr{G}}$ are surjective, we deduce that every function in $C_0(\mathscr{C})$ has an extension in $C_0(X)$, that is the Tietze extension theorem 5.11.

5.4.1 Proof of the Tietze extension theorem

Let $\mathscr{C}$ be a closed subset in X. The closed ideal $I_{\mathscr{C}}$ is always a nonempty proper ideal. Indeed, for any closed subset $\mathscr{C}$ the null function belongs to $I_{\mathscr{C}}$. Moreover, given an arbitrary $x \in \mathscr{C}$ there exists an element in $C_0(X)$ which doesn't vanish on x, because by the Urysohn Lemma we can always construct a continuous function g such that $g(x) = 1$ and its support is contained in an open neighborhood of x.
We recall that the quotient $C^\star$-algebra $C_0(X)/I_{\mathscr{C}}$ is made up by the equivalence classes $[f] := \{f + g, g \in I_{\mathscr{C}}\}$, and the quotient projection

$$\pi : C_0(X) \to C_0(X)/I_{\mathscr{C}}, \qquad \pi(f) = [f]$$

is continuous and surjective.
Here we give the proof of Theorem 5.11:

Theorem (Tietze Extension Theorem for $C_0(X)$)**.** *Let $g \in C_0(\mathscr{C})$, where $\mathscr{C}$ is a closed subset of a locally compact Hausdorff space X. Then there exists $f \in C_0(X)$ which extends g. The extension f can be chosen in such a way to satisfy*

$$\|f\|_{C_0(X)} = \|g\|_{C_0(\mathscr{C})}.$$

Proof. Let us observe that there is a bijection between the set $h_{I_{\mathscr{C}}}$ and $\mathscr{C}$ itself, given by the map $\mathscr{C} \ni x \mapsto \phi_x \in h_{I_{\mathscr{C}}}$. Thus by Proposition 5.9 we can identify $\sigma\left(C_0(X)/I_{\mathscr{C}}\right)$ exactly with $\mathscr{C}$.
We want to apply the Gelfand Transform to the quotient algebra $C_0(X)/I_{\mathscr{C}}$.
By the identification of $\sigma\left(C_0(X)/I_{\mathscr{C}}\right)$ with $\mathscr{C}$, we obtain that the Gelfand transform $\widehat{\mathscr{G}}$ of the quotient algebra, which is an isometric $\star$-isomorphism by Theorem 5.6, is the operator:

$$\widehat{\mathscr{G}} : C_0(X)/I_{\mathscr{C}} \ni [f] \to \widehat{\mathscr{G}}([f]) \in C_0(\mathscr{C}), \qquad \widehat{\mathscr{G}}([f]) := \{\mathscr{C} \ni x \mapsto f(x) \in \mathbb{C}\}.$$

Finally, let us consider the composition of the Gelfand map with the quotient projection π: the operator $\widehat{\mathscr{G}}\pi$ maps f to its restriction $f|_{\mathscr{C}}$. This operator is a composition of an isomorphism with a surjective map, thus it is surjective.
From this follows that every $g \in C_0(\mathscr{C})$ is the restriction of some $f \in C_0(X)$, i.e. g can be extended.
Once we know that every $g \in C_0(\mathscr{C})$ admits an extension $f \in C_0(X)$, we can define:

$$F(x) := \begin{cases} \sup_{x\in\mathscr{C}} |g(x)| & \text{if } |f(x)| \geq \sup_{x\in\mathscr{C}} |g(x)| \\ f(x) & \text{if } |f(x)| < \sup_{x\in\mathscr{C}} |g(x)| \end{cases}.$$

The function F belongs to $C_0(X)$ and it is still a continuous extension of g, with the property $||F|| = \sup_{x\in X} |F(x)| = \sup_{x\in\mathscr{C}} |g(x)| = ||g||$. ☐

Remark 5.3. Let us consider the case of a compact subset $\mathscr{K} \subset X$. We can identify $C_0(\mathscr{K})$ with $C(\mathscr{K})$, so that by Proposition 5.9 the Gelfand transform $\widehat{\mathscr{G}}$ maps $\frac{C_0(X)}{I_{\mathscr{K}}}$ into $C(\mathscr{K})$. The composition $\widehat{\mathscr{G}}\pi$ is still surjective, thus for every $g \in C(\mathscr{K})$ there exists an extension

$f \in C_0(X)$. Besides, given an open subset $\mathscr{V}$ containing $\mathscr{K}$, by the Urysohn Lemma we can find a continuous function ϕ such that $\phi(x) = 1\, \forall x \in \mathscr{K}$ and it has compact support contained in $\mathscr{V}$. If we define

$$F(x) := \phi(x)\, f(x)$$

we obtain a continuous extension of g with compact support. This means that every function in $C(\mathscr{K})$ admits an extension in $C_c(X)$.

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