# Unbounded operators on Hilbert $C^*$-modules: graph regular operators

Von der Fakultät für Mathematik und Informatik
der Universtät Leipzig

angenommene

DISSERTATION

zur Erlangung des akademischen Grades

DOKTOR der NATURWISSENSCHAFTEN
(Dr. rer. nat.)

im Fachgebiet

Mathematik

Vorgelegt

von Dipl.-Math. René Gebhardt

geboren am 07. Juni 1983 in Wernigerode.

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Die Verleihung des akademischen Grades erfolgt mit Bestehen der Verteidigung am 28.09.2016 mit dem Gesamtprädikat magna cum laude.

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# Introduction

A theory of *graph regular* operators is developed in this thesis guided by the case of commutative $C^*$-algebras and oriented towards Hilbert space theory. It is applied to a fraction algebra that is related to the Weyl algebra, as well as to the group $C^*$-algebra of the Heisenberg group and the Toeplitz algebra. Both the graph regular and the more general *orthogonally closed* operators are new operator classes that are considered on Hilbert $C^*$-modules the first time. There will be presented a theory of graph regular operators concerning an $(a, a_*, b)$*-transform* developed here, the *bounded transform* (also $z$*-transform*), the *polar decomposition* and the *functional calculus*.

*Essentially* instead of densely *defined* operators are considered, that is the operator's domain has a trivial orthogonal complement. The adjoint is well-defined in this case. An essentially defined operator $t$ is graph regular if its graph $\mathcal{G}(t)$ is orthogonally complemented and orthogonally closed if $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t)$.

As an analogon to the graph regular operators the *association relation* for concrete realised $C^*$-algebras is introduced and its correlation to the affiliation relation is discussed. Thereby *resolvent criteria* for both relations are given. As a corollary of this a *Kato-Rellich theorem* is deduced.

*Hilbert $C^*$-modules* are a well-established tool in the theory of $C^*$-algebras and their applications. They have been invented by I. Kaplansky [Kap53] for commutative $C^*$-algebras and by W. Paschke [Pas73] and M. Rieffel [Rie74] in the general case. Standard textbooks are [Lan95] and [MT05].

*Unbounded operators* on Hilbert $C^*$-modules play an important role for the study of noncompact quantum groups as S. L. Woronowicz has shown in [Wor91], in KK-theory [BJ83], [Kuc97] where S. Baaj, P. Julg and D. Kucerovski have applied them and in noncommutative geometry [GVF01] done by H. Figueroa, J. M. Gracia-Bondia and J. Varilly.

*Regular operators* form the most important class of unbounded operators on Hilbert $C^*$-modules. They were invented by Baaj [Baa81], [BJ83] and extensively studied by Woronowicz (and K. Napiórkowski) in a series of seminal papers [Wor91], [WN92] and [Wor95]; without lose of generality (see [Pal99]) Woronowicz considered only the case of operators on $C^*$-algebras itself and called them affiliated to the algebra.

Unbounded operators on Hilbert $C^*$-modules are studied in [Hil89], [Kuc97], [Pal99], [Kuc02], [Pie06], [FS10a], [FS10b], [KL12].

Only densely defined (closed) operators are considered in all of these fundamental works. This is correlated to the worse geometrical properties of Hilbert $C^*$-modules compared to Hilbert spaces. „Useful“ classes of operators are generated by further assumption on the operators, this is expressed in the quality of regularity. In this manner a closed operator $t$ is called *semi-regular*, if $\mathcal{D}(t)$ and $\mathcal{D}(t^*)$ are dense (Pal). If additionally the range of $1+t^*t$ is dense, $t$ is called regular.

Checking the density of a right ideal in a $C^*$-algebra is already as hard as computing all (irreducible) representations of this algebra and checking the corresponding density in all Hilbert spaces of those representations (see [Dix77] Lemma

2.9.4). (That was an occasion for Pal to investigate semi-regular operators.) Thus, regularity of an operator is a strong postulation: On *unital* $C^*$-algebras exist only bounded regular operators - the elements of the algebra itself.

The idea to quit the density of the operator's domain can already be found (for $C^*$-algebras explicitly) e.g. in the concepts of the *symmetric algebra of quotients* and the *local multiplier algebra* (see [Ell76], [Ped78], [AM03]). The adjoint is already well-defined, if the operator's domain is *essential*, that is, it has a trivial orthogonal complement. But from an operator point of view those concepts are "unnatural", since the domains are assumed to be two-sided ideals.

The origin of this thesis was the following question: Is it and how is it possible to relate the function $1/x$ to the $C^*$-algebra $C_0(\mathbb{R})$? Let me explain this problem very shortly. The concept of affiliation enabled Woronowicz in [Wor91] to relate all functions that are continuous on $\mathbb{R}$ via the bounded transform to the multiplier algebra of $C_0(\mathbb{R})$, that is, $C_b(\mathbb{R})$:

$$f \mapsto z_f := \frac{f}{\sqrt{1+|f|^2}} \in C_b(\mathbb{R}) \quad \text{for } f \in C(\mathbb{R}).$$

Now, one can apply the theory of multiplier algebras to study those unbounded operators. But for $t(x) = 1/x$ is

$$z_t(x) = \frac{sgn(x)}{\sqrt{1+x^2}} \quad \text{for } x \in \mathbb{R} \setminus \{0\},$$

hence $z_t$ can not be identified with some function in $C_b(\mathbb{R})$. An easy way out of this situation is to consider $t$ as function on $\mathbb{R} \setminus \{0\}$ and come back to the affiliation theory of Woronowicz. However, $t$ is the inverse of the operator acting as multiplication on $C_0(\mathbb{R})$ with the function $x$. But the latter is affiliated to $C_0(\mathbb{R})$ and a theory that includes $t$ and its inverse in one language seems to be useful. A first idea was to consider another transform:

$$a_t(x) := \frac{1}{1+|t|^2(x)} = \frac{x^2}{1+x^2}, \quad b_t(x) := \frac{t(x)}{1+|t|^2(x)} = \frac{x}{1+x^2} \text{ for } x \in \mathbb{R} \setminus \{0\}.$$

Now, $a_t$ and $b_t$ can be identified with some functions in the multiplier algebra $C_b(\mathbb{R})$ and $t$ can be recovered from the quotient $b_t a_t^{-1}$. This was the starting point for this thesis.

If we consider $1/x$ as multiplication operator on $C_0(\mathbb{R})$, it is clear that its domain can not be dense. Hence, a study of (unbounded) operators on commutative $C^*$-algebras seems to be a good guide already. Hilbert $C^*$-modules are generalisations of Hilbert spaces, hence a theory of (unbounded) operators on the Hilbert $C^*$-modules should be oriented at the well-known theory of Hilbert spaces. In particular, it should contain adjoints of operators.

A theory of operators avoiding those density criteria seems to be reasonable.

The purpose of this thesis was to study new examples of unbounded operators on $C^*$-algebras and to find a class of (unbounded) operators relaxing the assumptions on regularity, such that there is still a "typical" operator theory one can work with in applications. The ***graph regular*** operators on Hilbert $C^*$-modules achieve this aim inasmuch as

$\alpha$) they include the regular operators, but are in general not densely defined anymore; their domain are essential, so an adjoint is given,

$\beta$) they can be characterized by certain triples of adjointable operators via the so called $(a, a_*, b)$-transform,

$\gamma$) a bounded transform is available,
$\delta$) a regular operator can be associated to each graph regular one,
$\varepsilon$) the polar decomposition (with the restriction known from regular operators) is possible for those graph regular operators having an adjointable bounded transform,
$\varphi$) a functional calculus is available.

Moreover several *examples* of graph regular operators are described that are not regular.

$\aleph$) The graph regular operators on the $C^*$-algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space vanishing at infinity can be identified with those functions on $X$ being continuous on an open and dense subset of $X$ such that the discontinuities are "continuous poles". Note that this algebra is unital if $X$ is compact.
$\beth$) On an unital (!) fraction algebra associated to the Weyl-algebra the position and momentum operators are graph regular.
$\gimel$) On the group $C^*$-algebra of the Heisenberg group - in the basis $\{X, Y, Z\}$ the Lie-algebra is described by the commutation relations $[X, Y] = Z$, $[X, Z] = 0$, $[Y, Z] = 0$ - the inverse of $Z$ is graph regular.
$\daleth$) On the unital (!) Toeplitz algebra the Toeplitz operators with rational symbol where the denominator has no zero in the open unit disc are graph regular.

For concrete realised $C^*$-algebras the *association* and the *affiliation relation* are discussed and compared. The following results are given:

$A$) Affiliated operators are associated.
$B$) If a $C^*$-algebra contains the compact operators, the associated operators can be identified with the graph regular ones on the algebra. In particular, the graph regular operators on the algebra of bounded operators can be identified with the densely defined closed operators.
$\Gamma$) The multiplier algebra of a $C^*$-algebra consists precisely of the bounded associated operators.
$\Delta$) Criteria for association and affiliation via resolvents.
$E$) A Kato-Rellich theorem for affiliated operators.

The text is organised as follows. Technical subtleties are still avoided; precise definitions and further explanations will be given in the corresponding sections.

The *first* of either *parts* of this thesis addresses the idea of graph regularity in three sightings each one given an own chapter.

*Chapter 1*: The algebraic aspect of essential submodules is taken seriously here while introducing *unitary $*$-module spaces* as a generalisation of Hilbert $C^*$-modules. Thereby the algebraic arguments can be separated better from the topological ones; this thought is maintained until the transition to the second part. The focus is on the orthogonality relation: essential submodules, orthogonal closure (biorthogonal complement).

*Chapter 2*: Another sighting consists in a development of an elementary theory of essentially defined *orthogonally closed* operators (the graph coincides with its biorthogonal complement). To each (orthogonal) *projection* corresponds an orthogonally closed submodule and *vice versa*. The *essential core* is introduced.

*Chapter 3*: The third sighting deals with different types of regularity. The graph regular operators are introduced; also weak and strong types are mentioned - the latter one are just the regular operators in the context of Hilbert $C^*$-modules, so we won't give them a new name. All of them are orthogonally closed with further properties to be gathered from the following table:

| | $\mathcal{D}(t)$, $\mathcal{D}(t^*)$ | $\mathcal{R}(1+t^*t)$, $\mathcal{R}(1+tt^*)$ |
|---|---|---|
| weak regular | essential | essential |
| graph regular | essential | dense |
| regular | dense | dense |

In particular $\alpha$) is respected. Projections and partial isometries are weakly regular. From *the case* $C(X)$ we will gain a first insight to these regularities. A full description of the sets of orthogonally closed, weakly regular, graph regular and regular operators is given. Finally, we study graph regular operators in the general case; the decision to study this type of regularity is justified by the results of the second part and the achievement of $\beta$) already in this algebraic context. The term "graph" regular stems from the observation that for an orthogonally closed operator $t : E \to F$ the ranges $\mathcal{R}(1+t^*t)$ and $\mathcal{R}(1+tt^*)$ are dense if and only if the graph of $t$ is orthogonally complemented: $\mathcal{G}(t) \oplus v\mathcal{G}(t^*) = E \oplus F$.

In a *transition chapter* arguments for studying graph regularity on Hilbert $C^*$-modules rather then on unitary $*$-module spaces are given. The existence of graph regular operators depends on the existence of orthogonally complemented submodules and we will see that on $\mathbb{C}[X]$ only trivial operators are graph regular.

The *second part* actually contains the study of graph regular operators on Hilbert $C^*$-modules.

*Chapter 4*: A careful investigation of operators on commutative $C^*$-algebras shows where difficulties for a development of a general theory arises. The investigation of $C_0(X)$ can be still considered as a sighting but it is also a preparation for the functional calculus. We achieve $\aleph$) and show some phenomena of graph regular operators (unknown for regular ones) that occur even in this simple situation.

An *interjection* shows that *cum grano salis* bounded graph regular operators are already adjointable. Hence, graph regularity is addressed to unboundedness. From the beginning the main interest was on unbounded operators, and the above observation fits very nicely to this.

*Chapter 5*: Graph regularity behaves well with adjointable operators with respect to addition and composition. Closed quotients of adjointable operators are graph regular, which gives a source for examples at hand. Indeed, the graph regularity is verified this way in all three examples $\beth$), $\gimel$) and $\daleth$).

*Chapter 6*: For concrete realised $C^*$-algebras $\mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$ and closed operators $T$ on $\mathcal{H}$ the new *association relation* ($T\mu\mathcal{A}$) is introduced generalising the affiliation relation ($T\eta\mathcal{A}$) - aim $A$). Association is the analogon of graph regularity on this algebra (as affiliation is for regularity). It is deduced that the graph regular operators on the algebra of compact operators as well on the unital algebra of bounded operators on a Hilbert space can be identified with the closed operators - aim $B$). Further, a bounded operator is associated to $\mathcal{A}$ if and only if it belong to the corresponding multiplier algebra $\mathtt{M}(\mathcal{A})$ - aim $\Gamma$). If 0 belongs to resolvent set of $T$, $T\mu\mathcal{A}$ if and only if $T^{-1} \in \mathtt{M}(\mathcal{A})$. If $\lambda$ is in the resolvent set of $T$, then $T\eta\mathcal{A}$ if and only if $(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $(T-\lambda)^{-1}\mathcal{A}$ and $(T^*-\overline{\lambda})^{-1}\mathcal{A}$ are dense in $\mathcal{A}$. This is $\Delta$). As application of $\Delta$) one deduces $E$): For a self-adjoint $T\eta\mathcal{A}$ and a symmetric operator $S$ with $T$-bound less then 1 is $(T+S)\eta\mathcal{A}$, if $S(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ for some (any) $\lambda$ in the resolvent set of $T$.

*Chapter 7*: Here are the *examples* $\beth$), $\gimel$) and $\daleth$) presented.

*Chapter 8*: The *bounded transform* - aim $\gamma$) - assigns to each graph regular operator an adjointable one on an in general smaller Hilbert $C^*$-module such that one can *associate a regular operator* (on this Hilbert $C^*$-submodule) to this operator - aim $\delta$. The image of the set of regular operators from $E$ into $F$ under the (injective) bounded transform maps onto $\mathcal{Z}^d(E,F)$, a certain subset of the algebra $\mathcal{L}(E,F)$ of adjointable operators. The inverse of this transform is enlarged to

$\mathscr{Z}(E,F) \subseteq \mathcal{L}(E,F)$. This extension maps into the set of graph regular operators, but is not onto. In this sense some graph regular operators are said to have an adjointable bounded transform.

*Chapter 9*: A *polar decomposition* is possible for graph regular operators with adjointable bounded transform if the closures of the ranges of the operator and its adjoint are orthogonally closed - aim $\epsilon$). This generalises the polar decomposition known for regular operators in two directions. It applies even to graph regular operators, but also to a larger class of regular ones.

*Chapter 10*: A continuous *functional calculus* (with function in $C_0(\mathbb{C}) \dotplus 1\mathbb{C}$) is constructed for normal graph regular operators - aim $\phi$).

*Chapter 11*: Further counter examples are given to round out the theory developed so far. We study $C^*$-algebras constructed by matrices.

During my stipendiary scholarship at the International Max Planck Research School at the Max Planck Institut for Mathematics in the Science in Leipzig, together with K. Schmüdgen I completed a paper [GS15]. We already presented large parts of this thesis there, so it is necessary to clarify which content of this thesis is not alone up to me.

All chapters are done and invented by myself, except for what I have quoted, Lemma 90 and Theorem 95 in chapter 6, and chapter 7b) where the application of Proposition 58 to prove graph regularity was my only input.

Finally, there are some remarks to make. The reader should be familiar with $C^*$-algebras (and Hilbert $C^*$-modules), although everything will be defined and quoted. Definitions that are in no way new, won't be given a separate environment. General notations are collected in an appended section.

*Sightings*

# Unitary $*$-module spaces

## Algebraic essence of adjointability on Hilbert $C^*$-modules

After fixing some standard notations for Hilbert $C^*$-modules in the first section, we prepare the first sighting in the second section with a discussion of essential submodules and orthogonally closed ones. We characterize them for the commutative $C^*$-algebra $C_0(X)$. The simple but central observation of this chapter is:

> For a unique definition of an operator's adjoint it is only necessary to have an essential domain.

Since this enables us to define the adjoint without referring to some topology on the space a possibility shows up to leave the sphere of Hilbert $C^*$-modules for a moment; in the third section we study the more general unitary $*$-module spaces.

The adjoint was just a motivation for this chapter and will be investigated in the next chapter where we start a general study of operators on unitary $*$-module.

### *a) Operators on Hilbert $C^*$-modules - Notions*

The result of the following summary of Hilbert $C^*$-modules can be found in [Lan95]. Initially, we present the definition of *pre-Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$* while fixing the notations: They are pairs $(E, \langle .,. \rangle)$ consisting of a right $\mathcal{A}$-module $E$, where $E$ is also a vector space over $\mathbb{C}$, and a mapping $\langle .,. \rangle : E \times E \to \mathcal{A}$ - abusing the language this mapping is called an *scalar product* -, such that for $\alpha, \beta, \lambda \in \mathbb{C}$, $x, y, z \in E$ and $a \in \mathcal{A}$

| | |
|---|---|
| $\lambda(xa) = (\lambda x)a = x(\lambda a)$ | (connection of the vector space and the module structure), |
| $\langle z, \alpha x + \beta y \rangle = \alpha \langle z, x \rangle + \beta \langle z, y \rangle$ | ($\mathbb{C}$-linearity of $\langle .,. \rangle$ in the second entry), |
| $\langle x, ya \rangle = \langle x, y \rangle a$ | ($\mathcal{A}$-linearity of $\langle .,. \rangle$ in the second entry), |
| $\langle x, y \rangle = \langle y, x \rangle^*$ | (symmetry of $\langle .,. \rangle$), |
| $\langle x, x \rangle \geq 0$ | (positivity of $\langle .,. \rangle$), |
| $\langle x, x \rangle = 0 \Rightarrow x = 0$ | (non-degeneracy of $\langle .,. \rangle$). |

Obviously, for $\alpha, \beta \in \mathbb{C}$, $x, y, z \in E$, $a \in \mathcal{A}$ is

| | |
|---|---|
| $\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$ | ($\mathbb{C}$-antilinearity of $\langle .,. \rangle$ in the first entry), |
| $\langle xa, y \rangle = a^* \langle x, y \rangle$ | ($\mathcal{A}$-antilinearity of $\langle .,. \rangle$ in the first entry). |

A pre-Hilbert $C^*$-module $(E, \langle .,. \rangle)$ over $\mathcal{A}$ is a *Hilbert $C^*$-module over $\mathcal{A}$*, if $E$ is complete in the norm $\|.\|_E$ that is given by

$$\|x\|_E := \| \langle x, x \rangle \|_{\mathcal{A}}^{1/2} \qquad \text{for } x \in E.$$

For the sake of clarity the (pre)-Hilbert $C^*$-module $(E, \langle .,. \rangle)$ will be denoted just by $E$ or the symbol $\langle .,. \rangle$ will be extended to $\langle .,. \rangle_E$, whenever this is sensible.

We have the inequality

$$\| \langle x, y \rangle \|_{\mathcal{A}} \leq \|x\|_E \|y\|_E \qquad \text{for } x, y \in E,$$

and the equality

$$\|x\| = \sup\{\| \langle x, y\rangle \| : y \in E, \|y\| \leq 1\} \quad \text{for } x \in E.$$

Important examples of Hilbert $C^*$-modules are the $C^*$-algebras itself:

EXAMPLE 1. Each $C^*$-algebra $\mathcal{A}$ is a (right)-module over itself. Further, by defining $\langle ., .\rangle$ on $\mathcal{A} \times \mathcal{A}$ via

$$\langle a, b\rangle := a^*b \qquad\qquad \text{for } a, b \in \mathcal{A},$$

$(E = \mathcal{A}, \langle ., .\rangle)$ becomes obviously a pre-Hilbert $C^*$-module. In fact, $E$ is complete with respect to $\|.\|_E$, since $\|a\|_E = \| \langle a, a\rangle \|_{\mathcal{A}}^{1/2} = \|a^*a\|_{\mathcal{A}}^{1/2} = \|a\|_{\mathcal{A}}$ for $a \in E$ by the $C^*$-condition and $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is complete.

The direct sum of two Hilbert $\mathcal{A}$-modules is again complete, hence a Hilbert $\mathcal{A}$-module.

By an *operator* $t$ from a Hilbert $\mathcal{A}$-module $E$ into a Hilbert $\mathcal{A}$-module $F$ we mean a $\mathbb{C}$-linear and $\mathcal{A}$-linear mapping defined on a right $\mathcal{A}$-submodule $\mathcal{D}(t)$ of $E$ which is called the *domain* of $t$. The symbol $t : E \to F$ always denotes an operator from $E$ into $F$. The $\mathbb{C}$-linearity and $\mathcal{A}$-linearity of $t$ means that

$$t(\lambda x) = \lambda t(x) \quad \text{and} \quad t(xa) = t(x)a \quad \text{for } \lambda \in \mathbb{C}, x \in \mathcal{D}(t), a \in \mathcal{A}.$$

For an operator $t : E \to F$, its *null space* $\mathcal{N}(t) := \{x \in E | tx = 0\}$ is a right $\mathcal{A}$-submodule of $E$, its *range* $\mathcal{R}(t) := \{tx | x \in \mathcal{D}(t)\}$ is a right $\mathcal{A}$-submodule of $F$ and its *graph* $\mathcal{G}(t) := \{(x, tx) | x \in \mathcal{D}(t)\}$ is a right $\mathcal{A}$-submodule of $E \oplus F$. As in the case of ordinary Hilbert space operators we say $t$ is *closed* if $\mathcal{G}(t)$ is closed in $E \oplus F$ and $t$ is *closable* if there exists an operator $s$ which is a closed extension of $t$. In this case there exists a unique closed operator, denoted by $\bar{t}$ and called the *closure* of $t$ such that $\mathcal{G}(\bar{t}) = \overline{\mathcal{G}(t)}$.

There are three important classes of operators on Hilbert $C^*$-modules so far: adjointable, regular operators and semi-regular operators. Adjointable operators are regular and both of them are invented as analogues of the bounded respectively closed unbounded operators on Hilbert spaces. One should notice that by their definitions these operators classes compensate the geometrical defect of Hilbert $C^*$-modules (with respect to Hilbert spaces): Not all closed submodules are orthogonal complemented; this corresponds to the existence of closed proper submodules having a trivial orthogonal complement.

We repeat from [Lan95] p. 8 and p. 96: An operator $t : E \to F$ is *adjointable* if its domain is all of $E$ and the adjoint's domain is all of $F$. A closed operator $t$ is called *regular* if it domain is dense in $E$, the adjoint's domain is dense in $F$ and the range of $1 + t^*t$ is dense in $E$.

Adjointable operators are bounded and regular. Examples for bounded operator $t$ that are defined on the whole space such that the adjoint is not densely defined are e.g. given in chapter 11.

The set $\mathcal{L}(E)$ of adjointable operators on a Hilbert $C^*$-module $E$ becomes a $C^*$-algebra with the operator norm.

Pal studied in [Pal99] semi-regular operators, that is, “regular” operators for which the range of $1 + t^*t$ is not dense anymore. This thesis relaxes the definition of regular operators in another direction. For graph regular operators we relax the density of the operator's domains but hang on the idea to have a dense range of $1 + t^*t$.

## b) Essential submodules and adjointability

First, the orthogonality relation will be introduced together with definitions, that are concerned with situations that can not appear in the Hilbert space case. Thereafter the phenomena are exemplarily presented at the $C^*$-algebra $C_0(X)$.

Suppose that $E$ is a Hilbert $\mathcal{A}$-module. If $x, y \in E$ and $\langle x, y\rangle = 0$, we write $x \perp y$. For a subset $F$ of $E$, the set

$$F^\perp := \{x \in E | \forall y \in F : \langle x, y\rangle = 0\}$$

is called the *orthogonal complement* of $F$. A subset $F$ of $E$ is said to be *essential* if $F^\perp = \{0\}$. A submodule $F$ of $E$ is called *orthogonally closed* if $F = F^{\perp\perp}$ and *orthogonally complemented* if $F \oplus F^\perp = E$.

Example 2. For the $C^*$-algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff $X$ space vanishing at infinity each closed ideal is of the form

$$\mathcal{I}_O := \{f \in C_0(X) | \forall x \in X \setminus O : f(x) = 0\}$$

for some open subset $O \subseteq X$ (see [GJ76]). The mapping $O \to \mathcal{I}_O$ is a bijection of the open subsets of $X$ onto the closed ideals of $C_0(X)$. For $x \in X$ we have $x \in O$ if and only if there exists $f \in \mathcal{I}_O$ with $f(x) \neq 0$. The following facts are easily verified and will be proofed in Example 10:

- $\mathcal{I}_O^\perp = \mathcal{I}_{(X \setminus O)^\circ}$.
- $\mathcal{I}_O$ is essential in $C_0(X)$ if and only if $O$ is dense in $X$.
- $\mathcal{I}_O$ is orthogonally closed if and only if $O$ coincides with the interior of its closure.
- $\mathcal{I}_O$ is orthogonally complemented if $O$ is closed.

Now, we turn to the main observation of this chapter. Let $E, F$ be Hilbert $\mathcal{A}$-modules and $t : E \to F$ be an operator. Let $y \in F$ such that there exists a $z \in E$ with $\langle tx, y\rangle = \langle x, z\rangle$ for all $x \in \mathcal{D}(t)$. In this case, $z$ is unique if and only if $\mathcal{D}(t)$ has a trivial orthogonal complement, that is, $t$ is essentially defined. Hence, we have

Definition. Let $t : E \to F$ be an essentially defined operator. The *adjoint* $t^*$ of $t$ is given by

$$\begin{aligned} \mathcal{D}(t^*) &:= \{y \in F | \exists z \in E : \forall x \in \mathcal{D}(t) : \langle tx, y\rangle = \langle x, z\rangle\}, \\ t^* y &:= z \quad \text{for } y \in \mathcal{D}(t^*). \end{aligned}$$

We have

$$\langle tx, y\rangle = \langle x, t^* y\rangle \quad \text{for } x \in \mathcal{D}(t), y \in \mathcal{D}(t^*),$$

and $t^*$ is the largest operator with this property.

Having this definition established, it appropriate to focus the reader's attention to the following point: The above definition of the adjoint operator is sensible, only if the operator's domain is essential; hence its density is no longer necessary, that is, the Hilbert $C^*$-module's topology plays a subordinate role. (A dense submodule is still essential.) Therefore, it is advisable to leave the Hilbert $C^*$-module case for a moment: To be able to clarify later where topological arguments enter, we study an operator theory on a purely algebraic structure, the unitary $*$-module spaces to be defined below. Indeed, we will see, that our objects of interest, the graph regular operators, can already be characterized by a transform already in this purely algebraic setup. The so called $(a, a_*, b)$*-transform* will also be useful in the Hilbert $C^*$-module case later on.

### c) From Hilbert $C^*$-modules to unitary $*$-module spaces

But now the question is: To which extend do we want to generalize the Hilbert $C^*$-modules? At hand is to exchange the $C^*$-algebra by a $*$-algebra. To this idea a remarks can be made. In contrast to $C^*$-algebras which have only one appropriate cone of positive elements, this is not true for $*$-algebras. Hence this $*$-algebra should come with a associated cone. Secondly, we can weaken the conditions on the mapping $\langle .,. \rangle$, since the main theorem in the algebraic case is still valid if $\langle .,. \rangle$ maps into a $*$-bimodule over the $*$-algebra - now we have already in mind, that this $*$-bimodule comes with a cone. Let me now introduce precisely the structure of unitary $*$-module spaces.

Let $\mathcal{A}$ be a $*$-algebra. A *$*$-bimodule* $S$ over $\mathcal{A}$ is an $\mathcal{A}$-bimodule $S$ which is also a vector space over $\mathbb{C}$, where $\lambda(sa) = (\lambda s)a = s(\lambda a)$ for $\lambda \in \mathbb{C}, s \in S, a \in \mathcal{A}$, together with a mapping $s \mapsto s^*$ on $S$ called *involution*, that is,

$$\begin{aligned} (s^*)^* &= s && \text{for } s \in S, \\ (as)^* &= s^* a^*, \quad (sa)^* = a^* s^* && \text{for } a \in \mathcal{A}, s \in S, \\ (s+t)^* &= s^* + t^* && \text{for } s, t \in S, \\ (\lambda s)^* &= \overline{\alpha} s^* && \text{for } \lambda \in \mathbb{C}, s \in S. \end{aligned}$$

Note that the involution on the $*$-algebra is denoted in the same way; since they act on different sets, a single notation is still adequate. Clearly $(asb)^* = ((as)b)^* = b^*(as)^* = b^* s^* a^*$ for $a, b \in \mathcal{A}$, $s \in S$. Let $S_h := \{s \in S | s = s^*\}$ be the set of *hermitian* elements of $S$. A *cone* of $S$ is a set $\mathcal{P} \subseteq S_h$, such that

$$\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}, \quad \lambda \mathcal{P} \subseteq \mathcal{P} \quad \text{for } \lambda \in \mathbb{R}_+, \quad \mathcal{P} \cap (-\mathcal{P}) = \{0\}.$$

Together with a cone $\mathcal{P}$ the $*$-bimodule $S$ becomes a *ordered* $*$-bimodule: The order is given by $s \geq t$ for $s, t \in S$ if $s - t \in \mathcal{P}$.

Example 3. If $\mathcal{A}$ is a $*$-algebra, then $\mathcal{A}$ is in the obvious way a $*$-bimodule over any sub-$*$-algebra of $\mathcal{A}$.

Example 4. If $\mathcal{A}$ is a $*$-algebra, then any $*$-ideal of $\mathcal{A}$ is in the obvious way a $*$-bimodule over $\mathcal{A}$.

Definition. The triple $(E, (S, \mathcal{P}), \langle .,. \rangle)$ is an *unitary $*$-module space* over the $*$-algebra $\mathcal{A}$ if $E$ is a right $\mathcal{A}$-module which is also a vector space over $\mathbb{C}$ - again $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $\lambda \in \mathbb{C}$, $x \in E$ and $a \in \mathcal{A}$ -, $(S, \mathcal{P})$ is an ordered $*$-bimodule over $\mathcal{A}$ and $\langle .,. \rangle : E \times E \to S$ is a mapping such that for $\alpha, \beta \in \mathbb{C}$, $x, y, z \in E$ and $a \in \mathcal{A}$

$$\begin{aligned} \langle z, \alpha x + \beta y \rangle &= \alpha \langle z, x \rangle + \beta \langle z, y \rangle && (\mathbb{C}\text{-linearity of } \langle .,. \rangle \text{ in the second entry}), \\ \langle x, ya \rangle &= \langle x, y \rangle a && (\mathcal{A}\text{-linearity of } \langle .,. \rangle \text{ in the second entry}), \\ \langle x, y \rangle &= \langle y, x \rangle^* && (\text{symmetry of } \langle .,. \rangle), \\ \langle x, x \rangle &\geq 0 && (\text{positivity of } \langle .,. \rangle), \\ \langle x, x \rangle = 0 &\Rightarrow x = 0 && (\text{non-degeneracy of } \langle .,. \rangle). \end{aligned}$$

If it is convenient $\langle .,. \rangle$ will be denote by $\langle .,. \rangle_E$ to assure that it belongs to $E$; abusing terminology, the map $\langle .,. \rangle$ is called *scalar product*. Further, if no confusion can arise $(E, (S, \mathcal{P}), \langle .,. \rangle_E)$ will be shortened up to $E$. A *submodule* of $E$ is always assumed to be also a subspace. This definition is a generalization of unitary spaces (for $S = \mathcal{A}$ the complex numbers and $\mathcal{P}$ the non-negative reals).

We have the polarisation identity

$$4 \langle x, y \rangle = \sum\nolimits_{n=0}^{3} i^n \langle x + i^n y, x + i^n y \rangle$$

The set

$$\sum \langle E, E\rangle := \{\sum_{i=1}^{n} \langle x_i, x_i\rangle \,|n \in \mathbb{N}, x_1, \ldots, x_n \in E\}$$

is a cone of $S$ contained in $\mathcal{P}$. Moreover, it is even a pre-quadratic module, that is, we have additionally

$$a^* \sum \langle E, E\rangle\, a \subseteq \sum \langle E, E\rangle \quad \text{for } a \in \mathcal{A}.$$

This follows immediately from $a^* \langle x, x\rangle\, a = \langle xa, xa\rangle$ for $a \in \mathcal{A}$, $x \in E$.

But, why do we not assume directly this set to be a cone? This would be possible and recommendable, when dealing only with one unitary $*$-module space. If more are involved, we will always assume that all scalar products map into the same ordered $*$-module. This will assure the direct sum of those to be again an unitary $*$-module space. Otherwise the direct sum can become degenerated, which could be circumvented by assuming a compatibility for the different cones, say $\mathcal{P}_E$ and $\mathcal{P}_F$ in the case of two spaces: $\mathcal{P}_E \cap (-\mathcal{P}_F) = \{0\}$, as one can read off from Proposition 5 below. But we will have no need for this.

Assume $(E, (S, \mathcal{P}), \langle ., .\rangle_E)$ and $(F, (S, \mathcal{P}), \langle ., .\rangle_F)$ are unitary $*$-module spaces over the $*$-algebra $\mathcal{A}$. Let $E \oplus F$ be the direct sum of the $\mathcal{A}$-modules $E$ and $F$. Define the mapping $\langle ., .\rangle_{E\oplus F} : (E \oplus F) \times (E \oplus F) \to S$ via

$$\langle (e_1, f_1), (e_2, f_2)\rangle_{E\oplus F} := \langle e_1, e_2\rangle_E + \langle f_1, f_2\rangle_F \quad \text{for } e_1, e_2 \in E, f_1, f_2 \in F.$$

Proposition 5. $(E \oplus F, (S, \mathcal{P}), \langle ., .\rangle_{E\oplus F})$ is an unitary $*$-module space over $\mathcal{A}$, where $E \oplus F$ is the direct sum of the modules $E$ and $F$.

Proof. We will only prove the non-degeneracy of $\langle ., .\rangle_{E\oplus F}$., since the other statements are quiet easy and less instructive. Assume $\langle (x, y), (x, y)\rangle_{E\oplus F} = 0$ for $x \in E$, $y \in F$. Then $\langle x, x\rangle_E = - \langle y, y\rangle_F \in \mathcal{P} \cap (-\mathcal{P})$, so $\langle x, x\rangle_E = 0$ and $x = 0$. Analogously $y = 0$, so the assertion is shown. □

Before studying unitary $*$-modules and the operators on them in detail, we give two important examples. The first

Example 6. A Hilbert $C^*$-module $(E, \langle ., .\rangle_E)$ over $\mathcal{A}$ is indeed an unitary $*$-module space over $\mathcal{A}$, when we set $(E, (S, \mathcal{P}), \langle ., .\rangle_E) := (E, (\mathcal{A}, \mathcal{A}_+), \langle ., .\rangle_E)$.

The second

Example 7. Unitary $*$-module spaces over $\mathcal{A}$ being of the form $(E, (\mathcal{A}, \mathcal{P}), \langle ., .\rangle)$, that is, the $*$-algebra is at the same time the $*$-bimodule, so the scalar product maps $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, could be called *unitary $*$-modules*.

More special: The $*$-algebra itself could be considered as unitary $*$-module space over itself, but not for all of them is $\sum \langle \mathcal{A}, \mathcal{A}\rangle$ indeed a cone. If we set $\langle a, b\rangle := a^* b$ for $a, b \in \mathcal{A}$ and if $\mathcal{P}$ is a cone for $\mathcal{A}$ with $\sum \langle \mathcal{A}, \mathcal{A}\rangle \subseteq \mathcal{P}$, then $(\mathcal{A}, (\mathcal{A}, \mathcal{P}), \langle ., .\rangle)$ is an unitary $*$-modules spaces over $\mathcal{A}$. This is true e.g. for $O^*$-algebras, see [Sch90] Lemma 2.6.2 (i).

For the rest of this section let $E$ be an unitary $*$-module spaces over $\mathcal{A}$. The scalar product $\langle ., .\rangle$ enables us to consider an orthogonality relation.

Definition. For a subset $F \subseteq E$ let

$$F^\perp := \{x \in E | \forall y \in F : \langle x, y\rangle = 0\}$$

be the *orthogonal complement* of $F$ in $E$. If $\langle x, y\rangle = 0$ for some $x, y \in E$ let $x \perp y$ and call $x$ *orthogonal* to $y$.

Clearly, $x \perp y$ if and only if $y \perp x$, since $\langle x, y\rangle^* = \langle y, x\rangle$.

LEMMA 8. For $F, G \subseteq E$ is

$$F \subseteq F^{\perp\perp}, \quad F \subseteq G \implies G^{\perp} \subseteq F^{\perp}, \quad F^{\perp} = F^{\perp\perp\perp}.$$

Further, $F^{\perp}$ is a submodule of $E$.

PROOF. If $x \in F$, then $x \perp y$ for all $y \in F^{\perp}$, so $x \in F^{\perp\perp}$. Now assume $F \subseteq G$. If $x \in G^{\perp}$, then $x \perp y$ for all $y \in G$. In particular, $x \perp y$ for all $y \in F$, so $x \in F^{\perp}$. From these statements it follows $F^{\perp} \subseteq F^{\perp\perp\perp} \subseteq F^{\perp}$, so equality holds. To show that $F^{\perp}$ is a submodule, let $x, y \in F^{\perp}, a \in \mathcal{A}, \alpha \in \mathbb{C}$. Then $\langle z, x + \alpha y \rangle = \langle z, x \rangle + \alpha \langle z, y \rangle = 0$ for all $z \in F$, so $x + \alpha y \in F^{\perp}$, and $\langle z, xa \rangle = \langle z, x \rangle a = 0$, so $xa \in F^{\perp}$. □

It is easy to see, that $E^{\perp} = \{0\}$ and $\{0\}^{\perp} = E$.

For subsets $F, G \in E$ let $F + G := \{f + g | f \in F, g \in G\}$. If $F \subseteq G^{\perp}$ let $F \oplus G := F + G$. Since $F \subseteq G^{\perp}$ implies $G \subseteq G^{\perp\perp} \subseteq F^{\perp}$, it is $F \oplus G = G \oplus F$. In particular, $F \subseteq (F^{\perp})^{\perp}$, so it is justified to write $F \oplus F^{\perp}$.

DEFINITION. A submodule $F$ of $E$ is called *essential* if $F^{\perp} = \{0\}$. $F$ is called *orthogonally closed* if $F = F^{\perp\perp}$ and *orthogonally complemented* if $F \oplus F^{\perp} = E$.

In the next lines these concepts will be discussed to some detail. By Lemma 8, $F^{\perp}$ is always orthogonally closed and every proper essential submodule $F$ of $E$ is not orthogonally closed, since $F^{\perp} = \{0\}$ implies $F \subsetneq E = F^{\perp\perp}$.

In the following example a construction of submodules that are not orthogonally complemented is given using proper essential submodules.

EXAMPLE 9. Let $G' \subsetneq G$ be an essential submodule of a unitary $*$-module space $G$ and put $E := G \oplus G'$. Further let $F := \{(x, x) | x \in G\}$. Then $(x_1, x_2) \in G \oplus G'$ is orthogonal to $F$ if and only if $\langle x_1 + x_2, x \rangle = \langle (x_1, x_2), (x, x) \rangle = 0$ for all $x \in G'$. Since $G'$ is an essential submodule of $G$ this is equivalent to $x_1 = -x_2 \in G'$, that is, $F^{\perp} = \{(x, -x) | x \in G'\}$. Clearly, $F \oplus F^{\perp} = G' \oplus G' \neq E$. Moreover $(x_1, x_2) \in G \oplus G'$ is orthogonal to $F^{\perp}$ if and only if $\langle x_1 - x_2, x \rangle = \langle (x_1, x_2), (x, -x) \rangle = 0$ for all $x \in G'$. Again, since $G'$ is an essential submodule, $x_1 = x_2 \in G'$, so $F^{\perp\perp} = F$. Finally, this gives

$$F^{\perp\perp} \oplus F^{\perp} \neq E.$$

Now, we show how orthogonality acts in the commutative case of continuous function on a locally compact Hausdorff space.

EXAMPLE 10. Let $X$ be a locally compact Hausdorff space $X$. Let $\mathcal{C}$ be the $*$-algebra $C(X)$ or $C_b(X)$ or $C_0(X)$. The involution on $\mathcal{C}$ is just given by the pointwise complex conjugate.

Let $\mathcal{C}_{\geq}$ be the set of pointwise non-negative functions of $\mathcal{C}$. Clearly $\mathcal{C}_{\geq}$ is a cone. Via Example 7, we can consider $\mathcal{C}$ as unitary $*$-module space over itself. For a subset $O$ of $X$ let

$$\mathcal{I}_O := \{f \in \mathcal{C} | \forall x \in X \setminus O : f(x) = 0\}.$$

Since $\overline{X \setminus O} = X \setminus O^{\circ}$ and $\mathcal{I}_O$ consists only of continuous functions, without lose of generality $O$ can assumed to be open: $\mathcal{I}_O = \mathcal{I}_{O^{\circ}}$. For an open set $O$ we have $x \in O$ if and only if there exists an $f \in \mathcal{I}_O$ with $f(x) \neq 0$, since $X$ is locally compact Hausdorff. This implies that the mapping $O \to \mathcal{I}_O$ is injective on the set of open subsets of $X$, and that $\mathcal{I}_O \subseteq \mathcal{I}_{O'}$ if and only if $O \subseteq O'$ for open sets $O, O'$. In particular $\mathcal{I}_{\emptyset} = \{0\}$ and $\mathcal{I}_X = \mathcal{C}$.

Next we prove

$$\mathcal{I}_O^{\perp} = \mathcal{I}_{(X \setminus O)^{\circ}}. \tag{$\perp$}$$

If $f \in \mathcal{I}_O$ and $g \in \mathcal{I}_{(X\setminus O)^\circ} = \mathcal{I}_{X\setminus O}$, then $f \equiv 0$ on $X \setminus O$ and $g \equiv 0$ on $O$, hence $\langle f, g\rangle = \overline{f}g = 0$. So $\mathcal{I}_{(X\setminus O)^\circ} \subseteq \mathcal{I}_O^\perp$. Now, let $g \perp \mathcal{I}_O$. If $x \in O$, then there exists $f \in \mathcal{I}_O$ with $f(x) \neq 0$, so $\overline{f(x)}g(x) = \langle f, g\rangle(x) = 0$, so $g(x) = 0$. Therefore, $g \in \mathcal{I}_{X\setminus O} = \mathcal{I}_{(X\setminus O)^\circ}$. This proves ($\perp$).

In particular, this implies that if $O$ is open, then:

- $\mathcal{I}_O$ is essential if and only if $O$ is dense,
- $\mathcal{I}_O$ is orthogonally closed if and only if $O$ coincides with the interior of its closure: $(X \setminus (X \setminus O)^\circ)^\circ = (X \setminus (X \setminus \overline{O}))^\circ = \overline{O}^\circ$ implies $\mathcal{I}_O^{\perp\perp} = \mathcal{I}_{\overline{O}^\circ}$,

Finally is proven, that for open $O$:

- $\mathcal{I}_O$ is orthogonally complemented if $O$ is closed.

Assume $O$ is open and closed. If $f \in \mathcal{C}$, let $f_O$ be the function on $X$ that coincides with $f$ on $O$ and is 0 on $X \setminus O$. Since $f$ is continuous and $O$ is open and closed, $f_O$ is continuous. Clearly $f_O \in \mathcal{I}_O$. Similarly, $f_{X\setminus O} := f - f_O \in \mathcal{I}_{X\setminus O} = \mathcal{I}_{(X\setminus O)^\circ} = \mathcal{I}_O^\perp$. Therefore, $f \in \mathcal{I}_O \oplus \mathcal{I}_O^\perp$, so $\mathcal{I}_O$ is orthogonally complemented. To the contrary assume now $\mathcal{I}_O \oplus \mathcal{I}_O^\perp = \mathcal{C}$ for some open set $O$. To prove that $X \setminus O$ is open, assume $x \in X \setminus O$. Since $X$ is locally compact, there is a function $f \in \mathcal{C}$ with $f(x) \neq 0$. By assumption, $f = f_O + f_{X\setminus O}$ for some $f_O \in \mathcal{I}_O$ and $f_{X\setminus O} \in \mathcal{I}_O^\perp = \mathcal{I}_{(X\setminus O)^\circ}$. Since $x \in X \setminus O$, $f_O(x) = 0$, so $f_{X\setminus O}(x) \neq 0$. Since $f_{X\setminus O}$ is continuous, it does not vanish in some neighbourhood of $x$. Hence this neighbourhood is contained in the interior of $X \setminus O$; so does $x$.

The next few results are concerned with sums and intersections of submodules. Some ideas for the proof are taken from [MM70].

Lemma 11. Let $(F_i)_{i\in I}$ be a family of subsets of $E$. Then

$$\left(\bigcap\nolimits_{i\in I} F_i\right)^\perp \supseteq \mathtt{span}\{F_i^\perp | i \in I\}^{\perp\perp}, \quad (\mathtt{span}\{F_i | i \in I\})^\perp = \bigcap\nolimits_{i\in I} F_i^\perp.$$

If further all $F_i$ are orthogonally closed, then so is $\bigcap_{i\in I} F_i$ and

$$\bigcap\nolimits_{i\in I} F_i = (\mathtt{span}\{F_i^\perp | i \in I\})^\perp.$$

Proof. It is easy to see that $\mathtt{span}\{F_i^\perp | i \in I\} \subseteq \left(\bigcap_{i\in I} F_i\right)^\perp$. Since the latter is orthogonally closed, the first statement is established.

Now, $(\mathtt{span}\{F_i | i \in I\})^\perp \subseteq \bigcap_{i\in I} F_i^\perp$ is again directly proven. Therefore

$$(\mathtt{span}\{F_i | i \in I\})^{\perp\perp} \supseteq \left(\bigcap_{i\in I} F_i^\perp\right)^\perp \supseteq (\mathtt{span}\{F_i^{\perp\perp} | i \in I\})^{\perp\perp} \supseteq (\mathtt{span}\{F_i | i \in I\})^{\perp\perp},$$

hence equality holds and $(\mathtt{span}\{F_i | i \in I\})^\perp = \left(\bigcap_{i\in I} F_i^\perp\right)^{\perp\perp}$ follows. It remains to show that $\bigcap_{i\in I} F_i^\perp$ is already orthogonally closed: For all $i \in I$ is $\bigcap_{i\in I} F_i^\perp \subseteq F_i^\perp$, so $(\bigcap_{i\in I} F_i^\perp)^{\perp\perp} \subseteq F_i^\perp$ since $F_i^\perp$ is orthogonally closed. Therefore $(\bigcap_{i\in I} F_i^\perp)^{\perp\perp} \subseteq \bigcap_{i\in I} F_i^\perp$.

The last assertion follows from the second: Replace $F_i$ by $F_i^\perp$, taking $F_i^{\perp\perp} = F_i$ into account. $\square$

Lemma 12. Let $F$ be a subset of $E$. Then $F \oplus F^\perp$ is essential.

Proof. By Lemma 11 $(F \oplus F^\perp)^\perp = F^\perp \cap F^{\perp\perp}$, so any element in this set is orthogonal to itself, hence it is zero. $\square$

PROPOSITION 13. Let $F \subseteq G$ be submodules of $E$. Then

(1) $(F \oplus F^\perp) \cap (G \oplus G^\perp) = F \oplus G^\perp \oplus (F^\perp \cap G)$.
(2) If $G$ is orthogonally closed, then $(F \oplus F^\perp) \cap (G \oplus G^\perp) = F \oplus G \oplus (F \oplus G)^\perp$ is essential.
(3) If $F^\perp \cap G = \{0\}$, then $F^\perp \cap (G \oplus G^\perp) = G^\perp$.

PROOF. (1): Assume that $x = f + f^\perp = g + g^\perp$ with $f \in F$, $f^\perp \in F^\perp$, $g \in G$, $g^\perp \in G^\perp$. Then $f - g = g^\perp - f^\perp \in G \cap F^\perp$, since $G^\perp \subseteq F^\perp$. So $x = f + g^\perp + (f^\perp - g^\perp) \in F \oplus G \oplus (F^\perp \cap G)$. This shows one inclusion. To the contrary assume $x = f + g^\perp + h$ with $f \in F$, $g^\perp \in G^\perp$, $h \in F^\perp \cap G$. Then $x = f + (g^\perp + h) \in F \oplus F^\perp$ and $x = g^\perp + (f + h) \in G^\perp \oplus G$, so equality is shown.

(2): Since $G = G^{\perp\perp}$, it is $(F^\perp \cap G) = (F \oplus G^\perp)^\perp$ by Lemma 11. With (1) and Lemma 12 the assertion follows.

(3): Follows from (1) when intersected with $F^\perp$ and inserting $F^\perp \cap G = \{0\}$. $\square$

# 2. Operators on unitary $*$-module spaces

## Basic theory

Our second sighting is a look at an operator theory on unitary $*$-module spaces, that is, on the algebraic kernel of Hilbert $C^*$-modules as it is suitable for our aims. We study essentially defined operators and their adjoints (Theorem 15), introduce and study orthogonally closed operators (Theorem 17 and Proposition 18), projections (Proposition 22) and the essential core (Proposition 24).

In the sequel, the mappings $u, v : E \oplus F \to F \oplus E$ are defined via

$$u(x,y) := (y,x), \quad v(x,y) := (y,-x) \quad (x \in E, y \in F).$$

This mappings are unitary, which is justified by the following

Lemma 14. If $G \subseteq E \oplus F$, then $uG^\perp = (uG)^\perp$ and $vG^\perp = (vG)^\perp$.

Proof. Let $(x,y) \in G^\perp$ and $(r,s) \in G$, then

$$\langle u(x,y), u(r,s)\rangle = \langle (y,x),(s,r)\rangle = \langle (x,y),(r,s)\rangle = 0,$$

so $uG^\perp \subseteq (uG)^\perp$. Conversely, if $(x,y) \in (uG)^\perp$ and $(r,s) \in G$, then

$$\langle u(x,y),(r,s)\rangle = \langle (y,x),r,s\rangle = \langle (x,y),(s,r)\rangle = \langle (x,y),u(r,s)\rangle = 0,$$

so $u(x,y) \in G^\perp$ and finally $(x,y) \in uG^\perp$. Analogously, $vG^\perp = (vG)^\perp$. □

In this chapter, we let $E$, $F$ and $G$ be unitary $*$-module spaces over $\mathcal{A}$.

An *operator* $t$ from $E$ to $F$ is a $\mathbb{C}$-linear and $\mathcal{A}$-linear map $t$ defined on a submodule $\mathcal{D}(t) \subseteq E$, which is called the *domain*: $t(\lambda xa) = \lambda t(xa) = t(\lambda x)a$ for all $\lambda \in \mathbb{C}$, $x \in \mathcal{D}(t)$, $a \in \mathcal{A}$. For short $tx$ is the image of $x$. Let $\mathcal{N}(t) := \{x \in E | tx = 0\}$ the *null space* of $t$, which is a submodule of $E$, $\mathcal{R}(t) := \{tx | x \in \mathcal{D}(t)\}$ the *range* of $t$, which is a submodule of $F$ and $\mathcal{G}(t) := \{(x,tx) | x \in \mathcal{D}(t)\} \subseteq E \oplus F$ the *graph* of $t$, which is a submodule of $E \oplus F$. An operator $t$ is a restriction of the operator $s$ or $s$ is an extension of $t$, denoted $t \subseteq s$, if $\mathcal{D}(t) \subseteq \mathcal{D}(s)$ and $tx = sx$ for $x \in \mathcal{D}(t)$. If $\mathcal{D} \subseteq \mathcal{D}(t)$, then $t\restriction_{\mathcal{D}}$ denotes the restriction of $t$ to the domain $\mathcal{D}$.

Now, a first group of important definitions is given.

Definition. An operator $t : E \to F$ is called *orthogonally closed* if $\mathcal{G}(t)$ is orthogonally closed, that is, $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t)$. $t$ is called *orthogonally closable* if $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(s)$ for some (orthogonally closed) operator $s$.

Definition. An operator $t$ from $E$ into $F$ is called *essentially defined* if $\mathcal{D}(t)$ is an essential submodule of $E$.

Definition. For an essentially defined operator $t$ from $E$ to $F$ let

$$\mathcal{D}(t^*) := \{y \in F | \exists z \in E : \forall x \in \mathcal{D}(t) : \langle tx, y\rangle_F = \langle x, z\rangle_E\},$$
$$t^*y := z.$$

We call $t^*$ the *adjoint* of $t$.

Since $\mathcal{D}(t)$ is an essential submodule $z$ is unique and $t^*y$ is well-defined. It is easy to see that $t^*$ is indeed an operator from $F$ into $E$ (linearity in $\mathbb{C}$ and $\mathcal{A}$) with the property

$$\langle tx, y\rangle = \langle x, t^*y\rangle \quad (x \in \mathcal{D}(t), y \in \mathcal{D}(t^*)).$$

As in the case of unitary spaces, $\mathcal{D}(t^*)$ can reduce to $\{0\}$. For later use, we also give at this point

Definition. Two operators $t : E \to F$ and $s : F \to E$ are called *formally adjoint* to each other if

$$\langle tx, y\rangle = \langle x, sy\rangle \quad (x \in \mathcal{D}(t), y \in \mathcal{D}(s)).$$

Neither $t$ nor $s$ is assumed to be essentially defined. But if $t$ is, then $s$ is a restriction of $t^*$. On the other hand if $t$ is essentially defined but $t^*$ is not, then $t$ and $t^*$ are formally adjoint to each other; of course $t$ is not the adjoint of $t^*$ in this case, since the latter is not essentially defined.

Theorem 15. Let $t : E \to F$ be essentially defined. Then $\mathcal{G}(t^*) = v\mathcal{G}(t)^\perp$. In particular $t^*$ is orthogonally closed.

Proof. Let $(y, t^*y) \in \mathcal{G}(t^*)$. Then $\langle (y, t^*y), v(x, tx)\rangle = \langle y, tx\rangle - \langle t^*y, x\rangle = 0$ for $x \in \mathcal{D}(t)$ , so $(y, t^*y) \in (v\mathcal{G}(t))^\perp$. Conversely, let $(y, z) \in (v\mathcal{G}(t))^\perp$. For $x \in \mathcal{D}(t)$ is $\langle tx, y\rangle - \langle x, z\rangle = \langle v(x, tx), (y, z)\rangle = 0$, so $y \in \mathcal{D}(t^*)$ and $z = t^*y$. □

Corollary 16. Let $t : E \to F$ be essentially defined. Then $\mathcal{G}(t) \oplus v\mathcal{G}(t^*)$ is an essential submodule of $E \oplus F$.

Proof. This is just Theorem 15 combined with Lemma 12. □

Theorem 17. Let $t$ be an essentially defined operator from $E$ into $F$. Then $\mathcal{D}(t^*)$ is essential if and only if $t$ is orthogonally closable.

In this case, $t \subseteq t^{**}$ and $t$ is orthogonally closed if and only if $t = t^{**}$; more precisely, $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t^{**})$. Further, $t^* = t^{***}$.

Proof. Let $\mathcal{D}(t^*)$ be an essential submodule of $F$. Then $t^*$ has an adjoint and by applying Theorem 15 twice (to $t$ and $t^*$) is $\mathcal{G}(t^{**}) = v\mathcal{G}(t^*)^\perp = \mathcal{G}(t)^{\perp\perp}$, therefore $t \subseteq t^{**}$. That is $t$ is orthogonally closable and essentially defined. Clearly $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t)$ if and only if $t = t^{**}$. Using once again Theorem 15 it follows $\mathcal{G}(t^{***}) = v\mathcal{G}(t^{**})^\perp = v\mathcal{G}(t)^{\perp\perp\perp} = v\mathcal{G}(t)^\perp = \mathcal{G}(t^*)$.

Now let $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(s)$ for some (essentially defined) operator $s$. Assume $z \in \mathcal{D}(t^*)^\perp$. Then $\langle (-z, 0), (y, t^*y)\rangle = -\langle z, y\rangle + \langle 0, t^*y\rangle = 0$ for all $y \in \mathcal{D}(t^*)$, so $(z, 0) \in v\mathcal{G}(t^*)^\perp = \mathcal{G}(t)^{\perp\perp} = \mathcal{G}(s)$, so $z = 0$, since $s$ is an operator. □

Let

$$\mathcal{C}'_o(E, F) := \{t : E \to F | \mathcal{D}(t), \mathcal{D}(t^*) \text{ are essential}\},$$
$$\mathcal{C}_o(E, F) := \{t : E \to F | \mathcal{D}(t), \mathcal{D}(t^*) \text{ are essential}, \mathcal{G}(t) \text{ is orthogonally closed}\}.$$

By Theorem 17 $\mathcal{C}'_o(E, F)$ (resp. $\mathcal{C}_o(E, F)$) is the set of orthogonally closable (resp. closed) operators that are essentially defined. Let $\mathcal{C}'_o(E) := \mathcal{C}'_o(E, E)$, $\mathcal{C}_o(E) := \mathcal{C}_o(E, E)$. The operator $t^{**}$ can be seen as the *orthogonal closure* of the orthogonally closable operator $t$.

The operators behaving best when one is concerned with the adjoint will be defined in the next

Definition. An operator $t \in \mathcal{C}_o(E, F)$ is called *adjointable* if $\mathcal{D}(t) = E$ and $\mathcal{D}(t^*) = F$.

The set of adjointable operators from $E$ into $F$ by is denoted by $\mathcal{L}(E,F)$; set $\mathcal{L}(E) := \mathcal{L}(E,E)$.

Proposition 18. Let $t : E \to F$ be essentially defined. Then

(1) $\mathcal{N}(t^*) = \mathcal{R}(t)^\perp$,
(2) If $t$ is orthogonally closable, then $\mathcal{N}(t^{**})$ is orthogonally closed,
(3) If $t$ is injective and $\mathcal{R}(t)$ essential, then $t^*$ is injective and $(t^*)^{-1} = (t^{-1})^*$,
(4) If $t$ is injective and orthogonally closable, then $t^{-1}$ is orthogonally closable if and only if $t^{**}$ is injective. In this case $\mathcal{G}(t^{-1})^{\perp\perp} = \mathcal{G}((t^{**})^{-1})$. If further $\mathcal{R}(t)$ is essential, then $(t^{-1})^{**} = (t^{**})^{-1}$.

Proof. (1): This follows from $\langle tx, y\rangle = \langle x, t^*y\rangle$ for $x \in \mathcal{D}(t)$ and $y \in \mathcal{D}(t^*)$, because $\mathcal{D}(t)$ is an essential submodule.

(2): From (1) we conclude $\mathcal{N}(t^{**})^{\perp\perp} = \mathcal{R}(t^*)^{\perp\perp\perp} = \mathcal{R}(t^*)^\perp = \mathcal{N}(t^{**})$.

(3): Since $\mathcal{R}(t)$ is an essential submodule, $t^{-1}$ is essentially dense defined and $(t^{-1})^*$ exists. Further, $\mathcal{G}((t^{-1})^*) = v\mathcal{G}(t^{-1})^\perp = v(w\mathcal{G}(t))^\perp = wv\mathcal{G}(t)^\perp = w\mathcal{G}(t^*)$. Therefore $w\mathcal{G}(t^*)$ is graph of an operator, so $t^*$ is invertible and $(t^*)^{-1} = (t^{-1})^*$.

(4): By Theorem 17, it is $w\mathcal{G}(t^{**}) = w\mathcal{G}(t)^{\perp\perp} = (w\mathcal{G}(t))^{\perp\perp} = \mathcal{G}(t^{-1})^{\perp\perp}$. So, $(0,x) \in \mathcal{G}(t^{-1})^{\perp\perp}$ if and only if $(x,0) \in \mathcal{G}(t^{**})$ if and only if $x \in \mathcal{N}(t^{**})$ for $x \in E$. Therefore, $t^{-1}$ is orthogonally closable if and only if $\mathcal{G}(t^{-1})^{\perp\perp}$ is graph of an operator if and only if $\mathcal{N}(t^{**})$ is trivial; it is $\mathcal{G}(t^{-1})^{\perp\perp} = w\mathcal{G}(t^{**}) = \mathcal{G}((t^{**})^{-1})$. If additionally $t$ has essential range, then $t^{-1}$ is essentially defined. By Theorem 17, $\mathcal{G}((t^{-1})^{**}) = \mathcal{G}(t^{-1})^{\perp\perp}$, so finally $(t^{-1})^{**} = (t^{**})^{-1}$ is proven. $\square$

There are some basic results concerning the adjoints of the sum and the product of operators. The following statement prepares this and is interesting on its own.

Lemma 19. Let $r \in \mathcal{C}'_o(E,F)$ with $\mathcal{D}(r^*) = F$ and $\mathcal{R}(r)$ essential. Further, let $\mathcal{D} \subseteq \mathcal{D}(t)$ be essential. Then $\mathcal{R}(r \restriction_{\mathcal{D}})$ is still essential.

Proof. Let $x \in F$ with $\langle x, ry\rangle = 0$ for all $y \in \mathcal{D}$. Then $\langle r^*x, y\rangle = 0$ for all $y \in \mathcal{D}$. Since $\mathcal{D}^\perp = \{0\}$, $r^*x = 0$. That is, $x \in \mathcal{N}(r^*) = \mathcal{R}(r)^\perp = \{0\}$. $\square$

Proposition 20. Let $t, t_1, t_2$ be essentially defined operators from $E$ into $F$ and $s$ an essentially defined operator from $F$ into $G$. Then

(1) $t_1 \subseteq t_2$ implies $t_1^* \supseteq t_2^*$.
(2) If $t_1 + t_2$ is essentially defined, then $(t_1+t_2)^* \supseteq t_1^* + t_2^*$.
(3) If $\mathcal{D}(t_1) \subseteq \mathcal{D}(t_2)$ and $\mathcal{D}(t_2^*) = F$, then $t_1 + t_2$ is essentially defined and $(t_1+t_2)^* = t_1^* + t_2^*$.
(4) If $st$ is essentially defined, then $(st)^* \supseteq t^*s^*$.
(5) If $\mathcal{R}(t) \subseteq \mathcal{D}(s)$ and $\mathcal{D}(s^*) = G$, then $st$ is essentially defined and it is $(st)^* = t^*s^*$.
(6) If $t$ is injective with $\mathcal{D}(s) \subseteq \mathcal{R}(t)$ and $\mathcal{D}((t^{-1})^*) = F$, then $st$ is essentially defined and $(st)^* = t^*s^*$.

Proof. (1): Follows from Theorem 15, since $\perp$ reverses the inclusion.

(2): If $x \in \mathcal{D}(t_1+t_2)$ and $y \in \mathcal{D}(t_1^* + t_2^*)$, then

$$\langle (t_1+t_2)x, y\rangle = \langle t_1x, y\rangle + \langle t_2x, y\rangle = \langle x, t_1^*y\rangle + \langle x, t_2^*y\rangle = \langle x, (t_1^*+t_2^*)y\rangle \,,$$

so $y \in \mathcal{D}((t_1+t_2)^*)$ and $(t_1+t_2)^* \supseteq t_1^* + t_2^*$.

(3): By the assumptions, $t_1+t_2$ is essentially defined on $\mathcal{D}(t_1)$. Further, $t_1^*+t_2^*$ is defined on $\mathcal{D}(t_1^*)$. By (2) it is enough to prove $\mathcal{D}((t_1+t_2)^*) \subseteq \mathcal{D}(t_1^*+t_2^*) = \mathcal{D}(t_1^*)$, so let $y \in \mathcal{D}((t_1+t_2)^*) \subseteq F$ and $x \in \mathcal{D}(t_1)$. Then $y \in \mathcal{D}(t_1^*)$, since

$$\langle t_1x, y\rangle = \langle (t_1+t_2)x, y\rangle - \langle t_2x, y\rangle = \langle x, (t_1+t_2)^*y - t_2^*y\rangle \,.$$

(4): If $x \in \mathcal{D}(st)$ and $y \in \mathcal{D}(t^*s^*)$, then $\langle stx, y\rangle = \langle tx, s^*y\rangle = \langle x, t^*s^*y\rangle$, so $y \in \mathcal{D}((st)^*)$ and $(st)^* \supseteq t^*s^*$.

(5): By the assumptions, $st$ is essentially defined on $\mathcal{D}(t)$. By (4) it is enough to prove $\mathcal{D}((st)^*) \subseteq \mathcal{D}(t^*s^*)$, so let $z \in \mathcal{D}((st)^*) \subseteq G$ and $x \in \mathcal{D}(st) = \mathcal{D}(t)$. Then $s^*z \in \mathcal{D}(t^*)$, since $\langle tx, s^*z\rangle = \langle stx, z\rangle = \langle x, (st)^*z\rangle$. Hence $z \in \mathcal{D}(t^*s^*)$.

(6): By Lemma 19, $\mathcal{D}(st) = t^{-1}\mathcal{D}(s)$ is essential. By (4) it is enough to prove $\mathcal{D}((st)^*) \subseteq \mathcal{D}(t^*s^*)$, so let $z \in \mathcal{D}((st)^*)$ and $y \in \mathcal{D}(s) \subseteq \mathcal{R}(t)$. It follows $\langle sy, z\rangle = \big\langle (st)t^{-1}y, z\big\rangle = \big\langle t^{-1}y, (st)^*z\big\rangle = \big\langle y, (t^{-1})^*(st)^*z\big\rangle$. Therefore, $z \in \mathcal{D}(s^*)$ and $s^*z = (t^{-1})^*(st)^*z = (t^*)^{-1}(st)^*z \in \mathcal{D}(t^*)$. Hence, $z \in \mathcal{D}(t^*s^*)$. ☐

Corollary 21. Let $t \in \mathcal{C}_o'(E, F)$. Then:

(1) If $s \in \mathcal{L}(E, F)$, then $t + s \in \mathcal{C}_o'(E, F)$.
(2) If $s \in \mathcal{L}(F, G)$ is injective with $s^{-1} \in \mathcal{L}(G, F)$, then $st \in \mathcal{C}_o'(E, G)$.
(3) If $s \in \mathcal{L}(G, E)$ is injective with $s^{-1} \in \mathcal{L}(E, G)$, then $ts \in \mathcal{C}_o'(G, F)$.

All statements remain true if $\mathcal{C}_o'$ is replaced by $\mathcal{C}_o$.

Proof. (1): Clearly, $t + s$ is essentially defined and $(t + s)^* = t^* + s^*$ by Proposition 20 (3), since $s$ is adjointable. Now, $t^* + s^*$ is essentially defined, too. Hence, $t+s$ is orthogonally closable by Theorem 17. We compute $(t+s)^{**} = t^{**}+s$, again using Proposition 20 (3). So, $t + s$ is orthogonally closed if $t$ is.

(2): Clearly, $st$ is essentially defined and $(st)^* = t^*s^*$ by Proposition 20 (5), since $s$ is adjointable. By Proposition 20 (6), $t^*s^*$ is essentially defined, since $s^{-1}$ is adjointable. So, $st$ is orthogonally closable by Theorem 17; $(t^*s^*)^* = st^{**}$, implying that $st$ is orthogonally closed if $t$ is.

(3): This is similar ly proven to (2).

The last statement can easily be read off from the proofs of (1)-(3). ☐

Before introducing and characterizing the class of projections, we define symmetric and self-adjoint operators.

Definition. An operator $t : E \to E$ is called *symmetric* if $\langle tx, x\rangle = \langle x, tx\rangle$ for all $x \in \mathcal{D}(t)$.

If $t$ is essentially defined, this is equivalent to $t \subseteq t^*$.

Definition. An operator $t \in \mathcal{C}_o(E)$ is called *self-adjoint* if $t = t^*$.

As usual: If $t \subseteq s$ are both self-adjoint, then $s = s^* \subseteq t^* = t$, so $t = s$.

The simplest class of self-adjoint operators are the projections. It has to be emphasised that they will not be defined on the whole space, but its domain is still the direct sum of its kernel and range. By Proposition 18 (2) the kernel is orthogonally closed, but not orthogonally complemented in general. In fact the projection will be defined on the whole space only in the latter case.

Definition. $p \in \mathcal{C}_o(E)$ is called *(orthogonal) projection* if $p = p^2 = p^*$.

The following gives a characterisation of projections in form of a bijection onto the set of orthogonally closed submodules.

Proposition 22. For a submodule $G \subseteq E$, define an operator $p_G$ on $E$ via

$$\mathcal{D}(p_G) := G \oplus G^\perp, \quad p_G(x + y) := x \quad (x \in G, y \in G^\perp).$$

Then $p_G$ is essentially defined with $p_G = p_G^2 \subseteq p_G^* = p_{G^{\perp\perp}}$. In particular $p_G = p_G^*$ is a projection if and only if $G$ is orthogonally closed.

If $p$ is a projection, then $\mathcal{R}(p)$ is orthogonally closed and $p = p_{\mathcal{R}(p)}$.

Proof. Further $p := p_G$ is essentially defined by Lemma 12, since $E$ is anisotropic. Clearly $p = p^2$. From

$$\langle p(x+y), x'+y'\rangle = \langle x, x'\rangle = \langle x+y, x'\rangle \quad (x, x' \in G, y, y' \in G^\perp)$$

it follows that $\mathcal{D}(p) \subseteq \mathcal{D}(p^*)$. Now let $z \in \mathcal{D}(p^*)$, that is, there exists $w \in E$ with

$$\langle x, z\rangle = \langle p(x+y), z\rangle = \langle x+y, w\rangle \quad (x \in G, y \in G^\perp).$$

Equivalently $\langle x, z-w\rangle = \langle y, w\rangle$ for all $x \in G$ and $y \in G^\perp$; both sides are zero. Therefore this is equivalent to $z - w \in G^\perp$ and $w \in G^{\perp\perp}$, that is, $z \in G^\perp \oplus G^{\perp\perp}$. Since $w = p^*(z)$, $p^*$ is given by

$$\mathcal{D}(p^*) := G^{\perp\perp} \oplus G^\perp, \quad p^*(x+y) := x \quad (x \in G^{\perp\perp}, y \in G^\perp).$$

So, the first half is proven. Now assume $p = p^2 = p^*$ and let $G := \mathcal{R}(p)$. For $x = p(y) \in G$ is $x \in \mathcal{R}(p) \subseteq \mathcal{D}(p)$ and $p(x) = p^2(y) = p(y) = x$. For $x \in G^\perp$

$$\langle x, p(y)\rangle = 0 = \langle 0, y\rangle \quad (y \in \mathcal{D}(p)).$$

So $x \in \mathcal{D}(p^*) = \mathcal{D}(p)$ and $p(x) = p^*(x) = 0$. Further $\mathcal{R}(p) = \mathcal{N}(1-p)$: If $x = p(y) \in \mathcal{R}(p)$, then $(1-p)x = 0$. Conversely, $x = p(x)$ for $x \in \mathcal{N}(1-p)$. Hence,

$$G = \mathcal{R}(p) = \mathcal{N}(1-p) = \mathcal{N}(1-p^*) = \mathcal{R}(1-p)^\perp.$$

That is $G = G^{\perp\perp}$. Finally, $p_G \subseteq p$, so $p_G = p$ since $p$ and $p_G$ are self-adjoint. □

The idea behind the concept of an operator's core is to recover the original operator from a restriction. If this is possible, the restriction's domain is a core.

Definition. If $t$ is an operator from $E$ into $F$, then a submodule $\mathcal{D} \subseteq E$ is called *essential core* for $t$ if $\mathcal{G}(t\restriction_{\mathcal{D}})^\perp = \mathcal{G}(t)^\perp$.

Clearly, this definition fulfils the above requirements on $\mathcal{D}$ to be a core, since $\mathcal{G}(t) \subseteq \mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t\restriction_{\mathcal{D}})^{\perp\perp}$ in this case. On the other side, as long as $t$ is not orthogonally closed, there will only be a possibility of computing its orthogonal closure.

Note, $\mathcal{D} \subseteq \mathcal{D}(t)$ is an essential core if and only if $\mathcal{G}(t\restriction_{\mathcal{D}})^\perp \subseteq \mathcal{G}(t)^\perp$, since obviously $\mathcal{G}(t\restriction_{\mathcal{D}}) \subseteq \mathcal{G}(t)$. In other words, this is true if and only if $\mathcal{G}(t\restriction_{\mathcal{D}}) + \mathcal{G}(t)^\perp$ is a orthogonal sum $\mathcal{G}(t\restriction_{\mathcal{D}}) \oplus \mathcal{G}(t)^\perp$. If $\mathcal{D} \subseteq \mathcal{D}(t)$ is essential, then $(t\restriction_{\mathcal{D}})^* = t^*$ is equivalent to $\mathcal{D}$ being an essential core for $t$, since $\mathcal{G}(s^*) = v\mathcal{G}(s)^\perp$ for any essentially defined operator $s$. So, $t \subseteq (t\restriction_{\mathcal{D}})^{**}$ if $t$ is orthogonally closable.

Example 23. Let $b \in \mathcal{L}(E, F)$ and $\mathcal{D} \subseteq E$ be an essential submodule. Then $\mathcal{D}$ is an essential core for $b$, since $(b\restriction_{\mathcal{D}})^* \supseteq b^*$. But $b^*$ is everywhere defined, so equality holds.

Proposition 24. Let $t \in \mathcal{C}_o(E, F)$ and $\mathcal{D} \subseteq \mathcal{D}(t)$ be a submodule. If $\mathcal{D}$ is an essential core for $t$, then $\mathcal{D}$ is an essential submodule and

$$(t\restriction_{\mathcal{D}})^* = t^*, \quad t \subseteq (t\restriction_{\mathcal{D}})^{**} = t^{**}.$$

If $t$ is orthogonally closed, then $t = (t\restriction_{\mathcal{D}})^{**}$.

Proof. Define $p_E : E \oplus F \to E$ via $p_E(x, y) := x$ for $x \in E, y \in F$. Then it is easy to see that $p_E^* : E \to E \oplus F$ is given by $p_E^*(x) = (x, 0)$ for $x \in E$, so $p_E \in \mathcal{L}(E \oplus F, E)$. Now, using Proposition 20 (5) in the second equation, using Proposition 22 in the third equation and $\mathcal{G}(t\restriction_{\mathcal{D}})^{\perp\perp} = \mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t^{**})$ in the forth equation, it is

$$\mathcal{D}^\perp = \mathcal{R}(p_E p_{\mathcal{G}(t\restriction_{\mathcal{D}})})^\perp = \mathcal{N}(p^*_{\mathcal{G}(t\restriction_{\mathcal{D}})} p_E^*) = \mathcal{N}(p_{\mathcal{G}(t\restriction_{\mathcal{D}})^{\perp\perp}} p_E^*) = \mathcal{N}(p_{\mathcal{G}(t^{**})} p_E^*).$$

But $\mathcal{N}(p_{\mathcal{G}(t^{**})} p_E^*) = \{0\}$: Assume $x \in \mathcal{D}(p_{\mathcal{G}(t^{**})} p_E^*)$, then $(x, 0) \in \mathcal{G}(t^{**}) \oplus v\mathcal{G}(t^*)$, so there exists $u \in \mathcal{D}(t^{**})$, $v \in \mathcal{D}(t^*)$ with $x = u + t^* v$, $0 = t^{**}u - v$. If $p_{\mathcal{G}(t^{**})} p_E^* x =$

$p_{\mathcal{G}(t^{**})}(x, 0) = (u, t^{**}u) = (0, 0)$, then $u = 0$, so $v = 0$ and $x = 0$. This shows that $\mathcal{D}$ is essential. The other statements are easy now. $\square$

For later use we finally give

DEFINITION. An essentially defined operator $t : E \to E$ is called *normal* if $t^*t = tt^*$. If additionally $t$ is orthogonally closable, then $t$ is called *essentially self-adjoint* if $t^* = t^{**}$.

# 3. Graph regularity

## Pragmatism between weak and (strong) regularity

After discussing different types of regularity of operators in the first section, we study the situation on $C_0(X)$ carefully in the second one. A pragmatic choice, that will finally be justified by the theory developed in part two of this thesis, is graph regularity. The third section is concerned with the theory of graph regularity on an algebraic level; this is our last sighting and substantiate the decision to separate the topological arguments from the algebraic ones.

### *a) Types of regularity*

In which sense is it possible to denote an essentially defined orthogonally closed operator $t$ regular? We have an adjoint $t^*$, so it is a good idea to focus on the domains of $t$ and $t^*$ and the ranges of $1 + t^*t$ and $1 + tt^*$, as it was done in the Hilbert $C^*$-module case: for semi-regular operators the domains of $t$ and $t^*$ are assumed to be dense, for regular operators additionally the ranges of $1 + t^*t$ and $1 + tt^*$ are also assumed to be dense. This definitions could also be used in the situation for unitary $*$-module spaces. Following this line, combined with the concept of essential submodules, there are two further types of regularity for an (already) essentially defined orthogonally closed operator $t$:

(w) $t$ is *weakly regular*, if additionally the ranges of $1 + t^*t$ and $1 + tt^*$ are essential,

(gr) $t$ is *graph regular*, if additionally the ranges of $1+t^*t$ and $1+tt^*$ are dense.

We have:

$$\begin{array}{ccccc} \text{regular} & \Longrightarrow & \text{graph regular} & \Longrightarrow & \text{weakly regular} \\ \Downarrow & & & & \\ \text{semi-regular} & & & & \end{array}$$

From the case $C(X)$ we will learn, that non of the implications in the first line is invertible in general. But (essentially defined) orthogonally closed operators are already weakly regular in this situation.

### *b) The case $C(X)$*

The commutative $*$-algebra $C(X)$ teases us with the existence of all types of regularity. We compute $\mathcal{C}_o(C(X))$ and show that all operators in this class are weakly regular; we compute the graph regular and the regular operators on $C(X)$. As one could expect, no theory is needed to study these $*$-algebra. We use a few times the preceding section, but only to enlarge the text not any more. Further, some examples of operators are given to show some phenomena. But since the important ones are also valid in the $C^*$-algebra case $C_0(X)$, we will study those latter in chapter four.

Of course, unbounded operators on the commutative $*$-algebra $C(X)$ show up to be multiplication operators, as one would anticipate. For this reason, we study this operator class first and in detail. We use the following notation, that is inspired by [KL12] Section 6: For a function $m : X \to \mathbb{C}$ set

$$\begin{aligned}
\texttt{reg}(m) :=& \{x \in X | m \text{ is continuous in a neighborhood of } x\},\\
\texttt{reg}_b(m) :=& \Big\{x \in \partial\texttt{reg}(m) | \exists U \text{ open}, \tilde{m} : U \to \mathbb{C} \text{ continuous}\\
&\quad \text{with } x \in U \subseteq \overline{\texttt{reg}(m)}, \tilde{m} \equiv m \text{ on } U \cap \texttt{reg}(m)\Big\},\\
\texttt{reg}_\infty(m) :=& \Big\{x \in \partial\texttt{reg}(m) | \exists U \text{ open}, \tilde{m} : U \to \overline{\mathbb{C}} \text{ continuous}\\
&\quad \text{with } x \in U \subseteq \overline{\texttt{reg}(m)}, \tilde{m} \equiv m \text{ on } U \cap \texttt{reg}(m), \tilde{m}(x) = \infty\Big\},\\
\texttt{sing-supp}_\texttt{r}(m) :=& \partial\texttt{reg}(m) \setminus (\texttt{reg}_b(m) \cup \texttt{reg}_\infty(m)).
\end{aligned}$$

Clearly, $\texttt{reg}(m)$ is the largest open set on which $m$ is continuous. Further, $\texttt{reg}(m) \cup \texttt{reg}_b(m)$ is the largest open set contained in $\overline{\texttt{reg}(m)}$ on which $m$ restricted to $\texttt{reg}(m)$ has an (indeed unique) continuous $\mathbb{C}$-valued extension. Finally, $\texttt{reg}(m) \cup \texttt{reg}_b(m) \cup \texttt{reg}_\infty(m)$ is the largest open set contained in $\overline{\texttt{reg}(m)}$ on which $m$ restricted to $\texttt{reg}(m)$ has a (unique) continuous $\overline{\mathbb{C}}$-valued extension. For reasons of clarity and comprehensibility: We have the disjoint union

$$X = \texttt{reg}(m) \cup \underbrace{\texttt{reg}_b(m) \cup \texttt{reg}_\infty(m) \cup \texttt{sing-supp}_\texttt{r}(m)}_{\partial\texttt{reg}(m)} \cup \underbrace{(X \setminus \overline{\texttt{reg}(m)})}_{(X \setminus \texttt{reg}(m))^\circ}.$$

Speaking a little bit vague in this paragraph: $X \setminus \overline{\texttt{reg}(m)}$ could be called singular support of $m$: a set where every open subset contains points of discontinuity. This suggested the notation for the residual singular support $\texttt{sing-supp}_\texttt{r}(m)$ of $m$: a set where every neighbourhood of any of its points contains a point of discontinuity, but also an open subset of continuous points. On the other hand $m$ is regular on $\texttt{reg}(m)$, $\texttt{reg}_b(m)$ and $\texttt{reg}_\infty(m)$ in the sense that $m$ is already continuous, has a locally bounded continuous extension or could be extended when allowing (continuous) poles, respectively. At the end of this section some examples are given.

Summary: To an arbitrary function $m$ on $X$ a multiplication operator $t_m$ is associated and the first task is to describe an equivalence relation for those functions giving the same operator (Lemma 28). Then we show, that $t_m$ is essentially defined if and only if $\texttt{reg}(m)$ is dense in $X$. In this case, $t_m$ is orthogonally closed and its adjoint is $t_{\overline{m}}$ (Theorem 29). Preparing the discussion on regularity we compute $\mathcal{R}(1 + t_m^* t_m)$ and show that $\mathcal{D}(t_m^* t_m)$ is a core for $t_m$ (Proposition 32). The main theorem for this commutative case states: All essentially defined orthogonally closed operators are in fact multiplication operators (Theorem 34). We characterise the (essentially defined) orthogonally closed operators - all of them are weakly regular -, the graph regular and regular operators (Corollary 35) and projections(Proposition 36). Finally, some functions are discussed, to explain some subtleties and varieties occurring; the most notable is also valid for the $C^*$-situation, so we state it later as Example 69.

## *Theory*

To simplify notations, it is useful to associate to each function $m : X \to \mathbb{C}$ a function $\hat{m} : X \to \mathbb{C}$ defined via

$$\hat{m}(x) = \begin{cases} m(x) & , x \in X \setminus \texttt{reg}_b(m) \\ \tilde{m}(x) & , x \in \texttt{reg}_b(m) \end{cases},$$

where $\tilde{m}$ is one of those function appearing in the definition of $\texttt{reg}_b(m)$; in fact, two different of those functions will have the same value at $x$, since $x$ is in the boundary of $\texttt{reg}(m)$ and these two continuous functions coincide on this set. Hence, $\hat{m}$ is well-defined.

The following relation is needed to characterize those functions which define the same operators. For $m, \tilde{m} : X \to \mathbb{C}$ we write

$$m \simeq \tilde{m} \quad :\Leftrightarrow \quad \overline{\mathtt{reg}(m)} = \overline{\mathtt{reg}(\tilde{m})} \quad \text{and} \quad m \equiv \tilde{m} \text{ on } \mathtt{reg}(m) \cap \mathtt{reg}(\tilde{m}).$$

In Lemma 25 (1) it will be shown that $\simeq$ is indeed an equivalence relation; in (3) we proof $\hat{m} \simeq m$.

Lemma 25.

(1) $\simeq$ is an equivalence relation on the set of functions from $X$ to $\mathbb{C}$.

(2) Let $m_1, m_2 : X \to \mathbb{C}$. If $m_1 \simeq m_2$, then

$$\mathtt{reg}_\infty(m_1) = \mathtt{reg}_\infty(m_2), \quad \mathtt{sing\text{-}supp}_\mathtt{r}(m_1) = \mathtt{sing\text{-}supp}_\mathtt{r}(m_2).$$

(3) Let $m : X \to \mathbb{C}$. It is $\hat{m} \simeq m$. Further, $\mathtt{reg}(\hat{m}) = \mathtt{reg}(m) \cup \mathtt{reg}^b_\infty(m)$ and $\mathtt{reg}^b_\infty(\hat{m}) = \emptyset$.

Proof. We use several times that the intersection of open and dense sets is again (open and) dense.

(1): Obviously, $\simeq$ is reflexive and symmetric, so it remains to show the transitivity of $\simeq$. Let $m_1 \simeq m_2$ and $m_2 \simeq m_3$. Clearly, $\overline{\mathtt{reg}(m_1)} = \overline{\mathtt{reg}(m_2)} = \overline{\mathtt{reg}(m_3)}$ and arguing as above $\mathtt{reg}(m_1) \cap \mathtt{reg}(m_2) \cap \mathtt{reg}(m_3)$ is open and dense in the latter set. By continuity of $m_1$ and $m_3$ on $\mathtt{reg}(m_1) \cap \mathtt{reg}(m_3)$ these functions coincides on this set, since they do on $\mathtt{reg}(m_1) \cap \mathtt{reg}(m_2) \cap \mathtt{reg}(m_3)$. This shows $m_1 \simeq m_3$.

(2): From $R := \overline{\mathtt{reg}(m_1)} = \overline{\mathtt{reg}(m_2)}$ we conclude that $\mathtt{reg}(m_1) \cap \mathtt{reg}(m_2)$ is dense $R$. Since $m_1$ and $m_2$ are equal and continuous on this set, any $x$ in $R$ is in $\mathtt{reg}(m_1) \cup \mathtt{reg}_b(m_1)$ if and only if it is in $\mathtt{reg}(m_2) \cup \mathtt{reg}_b(m_2)$. By the same argument any $x$ in $R$ is in $\mathtt{reg}(m_1) \cup \mathtt{reg}_b(m_1) \cup \mathtt{reg}_\infty(m_1)$ if and only if it is in $\mathtt{reg}(m_2) \cup \mathtt{reg}_b(m_2) \cup \mathtt{reg}_\infty(m_2)$. This proves both $\mathtt{reg}_\infty(m_1) = \mathtt{reg}_\infty(m_2)$ and $\mathtt{sing\text{-}supp}_\mathtt{r}(m_1) = \mathtt{sing\text{-}supp}_\mathtt{r}(m_2)$.

(3): Clearly, $\hat{m}$ is continuous on $\mathtt{reg}(m) \cup \mathtt{reg}_b(m)$, so the latter is contained in $\mathtt{reg}(\hat{m})$. Since $m$ and $\hat{m}$ coincide on the open set $X \setminus \overline{\mathtt{reg}(m)}$, we even have $\mathtt{reg}(\hat{m}) \subseteq \overline{\mathtt{reg}(m)}$; in particular is $\overline{\mathtt{reg}(m)} = \overline{\mathtt{reg}(\hat{m})}$, and since $m$ and $\hat{m}$ are equal on $\mathtt{reg}(m) \cap \mathtt{reg}(\hat{m}) = \mathtt{reg}(m)$, it follows $m \simeq \hat{m}$. With (1) the proof is done if $\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(m) \cup \mathtt{reg}_b(m)$ is shown. Let $x \in \mathtt{reg}(\hat{m})$ and let $\tilde{m}$ be the restriction of $\hat{m}$ to the open set $\mathtt{reg}(\hat{m})$. Then either $x$ is already in $\mathtt{reg}(m)$ or in $\partial\mathtt{reg}(m)$, since the closures of $\mathtt{reg}(m)$ and $\mathtt{reg}(\hat{m})$ are identical. In the latter case $\tilde{m}$ serves for a function fulfilling the assumptions on $x$ to be in $\mathtt{reg}_b(m)$, since $\mathtt{reg}(m) \subseteq \mathtt{reg}(\hat{m})$. $\square$

In particular, changing $m$ on $\mathtt{reg}^\infty_\infty(m) \cup \mathtt{sing\text{-}supp}_\mathtt{r}(m)$ does not change any of the sets: $\mathtt{reg}(m)$, $\mathtt{reg}_b(m)$, $\mathtt{reg}_\infty(m)$, $\mathtt{sing\text{-}supp}_\mathtt{r}(m)$ and $X \setminus \overline{\mathtt{reg}(m)}$.

Proposition 26. Let $m, m' : X \to \mathbb{C}$ be given and assume $m \equiv m'$ on some open and dense set $O \subseteq X$. Then $m \equiv m'$. That is, the equivalence class of $m$ is already given by defining $m$ on an open and dense subset of $X$.

Proof. Since $\mathtt{reg}(m) \cap \mathtt{reg}(m')$ is open and $O$ is open and dense, the intersection of these sets is dense in $\mathtt{reg}(m) \cap \mathtt{reg}(m')$. The functions $m$ and $m'$ coincide on $O \cap \mathtt{reg}(m) \cap \mathtt{reg}(m')$ and are continuous there, hence $m \equiv m'$ on $\mathtt{reg}(m) \cap \mathtt{reg}(m')$. Since $m \equiv m'$ on $O \cap \mathtt{reg}(m)$ and $m$ is continuous there, the same is true for $m'$. That is, $O \cap \mathtt{reg}(m) \subseteq \mathtt{reg}(m')$, since $O \cap \mathtt{reg}(m)$ is open. So $\overline{\mathtt{reg}(m)} = \overline{O \cap \mathtt{reg}(m)} \subseteq \overline{\mathtt{reg}(m')}$, since $O \cap \mathtt{reg}(m)$ is dense in $\mathtt{reg}(m)$. Analogously is $\overline{\mathtt{reg}(m')} \subseteq \overline{\mathtt{reg}(m)}$. that is $m \simeq m'$. $\square$

Now, we define the multiplication operators $t_m$ associated to $m : X \to \mathbb{C}$. Note first, for a function $h : X \to \mathbb{C}$ is $\hat{h} \in E = C(X)$ if $\hat{h}$ is continuous everywhere, that is, $X = \mathtt{reg}(\hat{h}) = \mathtt{reg}(h) \cup \mathtt{reg}^b_\infty(h)$.

DEFINITION. For a function $m : X \to \mathbb{C}$ let

$$\mathcal{D}(t_m) := \{f \in E | \widehat{mf} \in E\}, \quad t_m f := \widehat{mf} \quad (f \in \mathcal{D}(t_m)).$$

We check that $t_m$ is indeed an operator on $E$: For $\alpha \in \mathbb{C}$ and $f \in \mathcal{D}(t_m)$ we clearly have $\alpha f \in \mathcal{D}(t_m)$. Let $f \in \mathcal{D}(t_m)$ and $g \in \mathcal{A} = C(X)$. Since both $\widehat{mf}g$ and $(mfg)^\wedge$ are at least continuous on the dense set $\mathtt{reg}(mf)$ and coincide there, they coincide on $X$, because $\widehat{mf}g$ is continuous on all of $X$; $(t_m f)g = t_m(fg)$ follows. Now, let $f, g \in \mathcal{D}(t_m)$. Since both $\widehat{mf} + \widehat{mg}$ and $(m(f+g))^\wedge$ are at least continuous on the dense set $\mathtt{reg}(mf) \cap \mathtt{reg}(mg)$ and coincide there, they coincide on $X$, because $\widehat{mf} + \widehat{mg}$ is continuous on all of $X$. Hence, $t_m(f+g) = t_m f + t_m g$.

The following argument will be used several times in the preceding, so it is given an own lemma.

LEMMA 27. Let $m, f : X \to \mathbb{C}$, $x \in X$ and assume $f$ is continuous at $x$ with $f(x) \neq 0$. Then $x \in \mathtt{reg}(m)$ if and only if $x \in \mathtt{reg}(mf)$.

PROOF. Since $f(x) \neq 0$, there is an open set $U_f$ containing $x$ such that $f(x') \neq 0$ for all $x' \in U_f$. By definition, $x \notin \mathtt{reg}(m)$ if for any neighbourhood $U$ of $x$, there is an $x' \in U$ such that $m$ is discontinuous at $x'$. Looking only for neighbourhoods contained in $U_f$, the latter holds if and only if for any neighbourhood $U$ of $x$, there is an $x' \in U$ such that $mf$ is discontinuous at $x'$. This means $x \notin \mathtt{reg}(mf)$. $\square$

We describe the action of $t_m$ and show that $t_m$ depends only on the equivalence class of $m$.

LEMMA 28. Let $m : X \to \mathbb{C}$ be given. Then

$$\mathcal{D}(t_m) = \{f \in E | f \equiv 0 \text{ on } X \setminus \overline{\mathtt{reg}(\hat{m})}, \partial\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})\},$$
$$(t_m f)(x) = \hat{m}(x) f(x) \quad (f \in \mathcal{D}(t_m), x \in X \setminus \mathtt{reg}_\infty(m)).$$

For $x \in X$, there is an $f \in \mathcal{D}(t_m)$ with $f(x) \neq 0$ if and only if $x \in \overline{\mathtt{reg}(\hat{m})}$.
If $\tilde{m} : X \to \mathbb{C}$, then $t_m = t_{\tilde{m}}$ if and only if $m \simeq \tilde{m}$. In particular, $t_m = t_{\hat{m}}$.

PROOF. $\mathcal{D}(t_m)$ consists exactly of those $f \in E$ with $\mathtt{reg}(\widehat{mf}) = X$. Clearly, $\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$ for all $f \in E$, so $\mathcal{D}(t_m)$ is the set of $f \in E$ with $X \setminus \mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$.

So, if $f \in E$ with $f \equiv 0$ on $X \setminus \overline{\mathtt{reg}(\hat{m})}$, then $mf \equiv 0$ on this set is implying continuity of $mf$ on the open set $X \setminus \overline{\mathtt{reg}(\hat{m})}$, hence the latter is contained in $\mathtt{reg}(\widehat{mf})$. If additionally $\partial\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$ is true, then $f \in \mathcal{D}(t_m)$ is shown.

Now assume $f \in \mathcal{D}(t_m)$; in particular $\mathtt{reg}(mf)$ is dense in $X$. As noted above, it is $\partial\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$. Assume $x$ is contained in the open set $X \setminus \overline{\mathtt{reg}(\hat{m})}$ and $f(x) \neq 0$. Hence, there is an open set $U$, contained in $X \setminus \overline{\mathtt{reg}(\hat{m})}$ such that $f$ does not vanish at some point in $U$. Since $\overline{\mathtt{reg}(\hat{m})} = \overline{\mathtt{reg}(m)}$, $U$ is also contained in $X \setminus \mathtt{reg}(m)$. By Lemma 27, it follows that $U$ is even contained in $X \setminus \mathtt{reg}(mf)$, since $f$ is not zero at any point of $U$. Finally, this gives a contradiction to the density of $\mathtt{reg}(mf)$ in $X$. Hence $f(x) = 0$ and the description of $\mathcal{D}(t_m)$ is proven.

For $f \in \mathcal{D}(t_m)$, $\mathtt{reg}(m) \cup \mathtt{reg}_b(m) = \mathtt{reg}(\hat{m})$ is a subset of $\mathtt{reg}(\widehat{mf})$ and $t_m f \equiv \hat{m} f$ on this set. It remains to prove $t_m f \equiv \hat{m} f$ on the sets $X \setminus \overline{\mathtt{reg}(m)}$ and $\mathtt{sing\text{-}supp}_\mathtt{r}(m)$, and since $f \equiv 0$ there, as we have seen above, it must be shown that $\widehat{mf} \equiv 0$ on these sets. This will be done indirectly by proving: If $x \in X$ and $f \in \mathcal{D}(t_m)$ with $f(x) = 0$ and $(t_m f)(x) \neq 0$, then $x \in \mathtt{reg}_\infty(m)$. Indeed, from the given assumptions it follows, that there is an open neighbourhood $U$ of $x$ with $\widehat{mf}(y) \neq 0$ for all $y \in U$. Letting $\tilde{m} : U \to \overline{\mathbb{C}}$ be the function $\widehat{mf}/f$ on $U$ we clearly have, that $\tilde{m}$ is continuous, since $\widehat{mf}$ does not vanish at any point of $U$, coincides with $m$ on $U \cap \mathtt{reg}(m)$, and $\tilde{m}(x) = \infty$, since $f(x) = 0$ and $\widehat{mf}(x) \neq 0$. Hence $x \in \mathtt{reg}_\infty(m)$.

If $x \in \mathtt{reg}(\hat{m})$, there is a function $f : X \to \mathbb{C}$ with (compact) support in $\mathtt{reg}(\hat{m})$ and $f(x) \neq 0$, since $X$ is locally compact Hausdorff. Clearly, this $f$ is in $\mathcal{D}(t_m)$, since $\hat{m}f$ is continuous everywhere. On the other side, if $x \notin \mathtt{reg}(\hat{m})$, then $f(x) = 0$ for all $f \in \mathcal{D}(t_m)$ by the above representation of $\mathcal{D}(t_m)$.

Using this, next we prove the equivalence of $t_m = t_{\tilde{m}}$ and $m \simeq \tilde{m}$. First, we assume $t_m = t_{\tilde{m}}$. By the above,

$$x \in \mathtt{reg}(\hat{m}) \Leftrightarrow \exists f \in \mathcal{D}(t_m) : f(x) \neq 0 \Leftrightarrow \exists f \in \mathcal{D}(t_{\tilde{m}}) : f(x) \neq 0 \Leftrightarrow x \in \mathtt{reg}(\hat{\tilde{m}}).$$

So, $\mathtt{reg}(\hat{m}) = \mathtt{reg}(\hat{\tilde{m}})$, which implies $\overline{\mathtt{reg}(m)} = \overline{\mathtt{reg}(\hat{m})} = \overline{\mathtt{reg}(\hat{\tilde{m}})} = \overline{\mathtt{reg}(\tilde{m})}$. For $x \in \mathtt{reg}(m) \cap \mathtt{reg}(\tilde{m})$, we have choose $f \in \mathcal{D}(t_m)$ with $f(x) \neq 0$: $m(x)f(x) = \widehat{mf}(x) = \widehat{\tilde{m}f}(x) = \tilde{m}(x)f(x)$, hence $m(x) = \tilde{m}(x)$. Thus $m \simeq \tilde{m}$.

Now assume $m \simeq \tilde{m}$. To show $t_m \subseteq t_{\tilde{m}}$, we let $f \in \mathcal{D}(t_m)$. It is $\widehat{mf} \equiv mf \equiv \tilde{m}f$ on $\mathtt{reg}(m) \cap \mathtt{reg}(\tilde{m})$, and the latter set is dense in $\overline{\mathtt{reg}(\hat{m})}$, since $m \simeq \tilde{m}$. On the other side $f \equiv 0$ on $X \setminus \overline{\mathtt{reg}(\hat{m})}$, since $f \in \mathcal{D}(t_m)$. Hence $\widehat{mf} \equiv 0 \equiv \tilde{m}f$ on $X \setminus \overline{\mathtt{reg}(\hat{m})}$. That is $\widehat{mf} \equiv \tilde{m}f$ on an open and dense set. Since $\widehat{mf}$ is continuous every where $\mathtt{reg}(\tilde{m}f) = X$ and $\widehat{\tilde{m}f} \equiv \widehat{mf}$ on $X$, hence $f \in \mathcal{D}(t_{\tilde{m}})$ and $t_{\tilde{m}}f = t_m f$. Since $\simeq$ is a symmetric relation, the inverse inclusion is also proven, so $t_m = t_{\tilde{m}}$. $\square$

Hence, by Proposition 26 and Lemma 28 the operator $t_m$ is already uniquely defined by $m$ if we describe $m$ on an open and dense subset of $X$. It is for example enough to define $m : \mathbb{R} \to \mathbb{C}$ by $m(x) = 1/x$ not caring about the definition at 0. This will be done in some cases.

Having understand the action of $t_m$, we will check now when the domain of $t_m$ is essential and compute the adjoint in this case. Note that all of the sets $\mathtt{reg}(m)$, $\mathtt{reg}_b(m)$, $\mathtt{reg}_\infty(m)$ and $\mathtt{sing\text{-}supp}_\mathtt{r}(m)$ remain unchanged if $m$ is replaced by its complex conjugate function $\overline{m}$; it is $\hat{\overline{m}} = \overline{\hat{m}}$.

Theorem 29. Let $m : X \to \mathbb{C}$. The operator $t_m$ is essentially defined if and only if $\mathtt{reg}(m)$ is dense in $X$. In this case, we have $t_m^* = t_{\overline{m}}$ and $t_m \in \mathcal{C}_o(E)$.

Proof. Assume $\mathtt{reg}(m)$ is dense in $X$ and let $g \perp \mathcal{D}(t_m)$. For all $x \in \mathtt{reg}(m)$, there is an $f \in \mathcal{D}(t_m)$ with $f(x) \neq 0$ by Lemma 28. From $\overline{g(x)}f(x) = \langle g, f\rangle(x) = 0$, we conclude $g(x) = 0$. By density of $\mathtt{reg}(m)$ and continuity of $g$, it follows $g = 0$. That is, $\mathcal{D}(t_m)$ is essential. Now suppose $\mathtt{reg}(m)$ is not dense. Since $X$ is locally compact Hausdorff, there is a non-zero function $g$ on $X$ that vanishes outside of $(X \setminus \mathtt{reg}(m))^\circ$. Then $\langle f, g\rangle(x) = \overline{f(x)}g(x) = 0$ for all $f \in \mathcal{D}(t_m)$ and $x \in X$ by Lemma 28. Hence, $\mathcal{D}(t_m)$ is not essential.

If $\mathtt{reg}(m)$ is dense, we will show $t_m^* = t_{\overline{m}}$: Let $f \in \mathcal{D}(t_m)$ and $g \in \mathcal{D}(t_{\overline{m}})$. For $x \in \mathtt{reg}(\hat{m}) = \mathtt{reg}(\hat{\overline{m}})$ we derive

$$\begin{aligned}\langle t_m f, g\rangle(x) &= \overline{(t_m f)(x)}g(x) = \overline{\hat{m}(x)f(x)}g(x) = \overline{f(x)}\,\overline{\hat{m}(x)}g(x)\\ &= \overline{f(x)}(t_{\overline{m}}g)(x) = \langle f, t_{\overline{m}}g\rangle(x).\end{aligned}$$

Since $\mathtt{reg}(m)$ is dense, $\mathtt{reg}(\hat{m})$ is dense as well. By continuity of $\langle t_m f, g\rangle$ and $\langle f, t_{\overline{m}}g\rangle$, we conclude $\langle t_m f, g\rangle = \langle f, t_{\overline{m}}g\rangle$. Thus $t_{\overline{m}} \subseteq t_m^*$. If $f \in \mathcal{D}(t_m^*)$, there exists $h \in E$ such that $\overline{\widehat{mg}(x)}f(x) = \langle t_m g, f\rangle = \langle g, h\rangle = \overline{g(x)}h(x)$ for all $g \in \mathcal{D}(t_m)$, $x \in X$. If $x \in \mathtt{reg}(m)$, there is a $g \in \mathcal{D}(t_m)$ with $g(x) \neq 0$, so $h \equiv \overline{m}f$ on $\mathtt{reg}(m)$. Since $\mathtt{reg}(m)$ is dense and $h$ is continuous, we get $\widehat{\overline{m}f} = h \in E$, hence $f \in \mathcal{D}(t_{\overline{m}})$.

Finally, from $\mathtt{reg}(\overline{m}) = \mathtt{reg}(m)$ and the previous it is easily concluded that $t_m^{**} = (t_{\overline{m}})^* = t_m \in \mathcal{C}_o(E)$. $\square$

The following two lemmas are concerned with sums, products and inverses of multiplication operators.

LEMMA 30. Let $m, n : X \to \mathbb{C}$ be two functions.

(1) It is $t_m + t_n \subseteq t_{m+n}$ and $t_m t_n \subseteq t_{mn}$.
(2) If $\mathtt{reg}(m)$ and $\mathtt{reg}(n)$ are dense in $X$, then $t_m + t_n, t_m t_n \in \mathcal{C}'_o(E)$ and $(t_m + t_n)^{**} = t_{m+n}$, $(t_m t_n)^{**} = t_{mn}$.

PROOF. We only show the statements about the product, since the proof for the sum is very similar and less involved.

(1): If $f \in \mathcal{D}(t_m t_n)$, then $\mathtt{reg}(nf)$ and $\mathtt{reg}(m\widehat{nf})$ are dense and open in $X$, so their intersection is also dense. It is $\mathtt{reg}(nf) \cap \mathtt{reg}(m\widehat{nf}) \subseteq \mathtt{reg}(mnf)$ and $mnf \equiv m\widehat{nf}$ on the latter one. Hence $[(mn)f]^\wedge = [m\widehat{nf}]^\wedge \in E$, that is, $f \in \mathcal{D}(t_{mn})$ and $t_m t_n f = t_{mn} f$.

(2): Firstly, we show that $\mathcal{D}(t_m t_n)$ is essential: Assume that $g \perp \mathcal{D}(t_m t_n)$ and $x \in \mathtt{reg}(n) \cap \mathtt{reg}(m)$. Since the latter set is open and $X$ is locally compact Hausdorff, there exists $f \in E$ with (compact) support in this set and $f(x) \neq 0$; clearly $nf \in E$ and $mnf \in E$, hence $f \in \mathcal{D}(t_m t_n)$. But then $\overline{f(x)} g(x) = \langle f, g \rangle (x) = 0$, so $g(x) = 0$. Since $\mathtt{reg}(n) \cap \mathtt{reg}(m)$ is dense, $g = 0$ by continuity of $g$. That is, $t_m t_n$ is essentially defined; analogously $t_{\overline{n}} t_{\overline{m}}$ is. Therefore, $(t_m t_n)^* \supseteq t_n^* t_m^* = t_{\overline{n}} t_{\overline{m}}$ is also essentially defined, that is, $t_m t_n$ is orthogonally closable. Moreover, $(t_m t_n)^* \supseteq t_{mn}^* = t_{\overline{mn}}$ follows from (1).

Secondly, we show $\mathcal{D}((t_m t_n)^*) \subseteq \mathcal{D}(t_{\overline{mn}})$, which implies the last assertion: $(t_m t_n)^{**} = t_{\overline{mn}}^* = t_{mn}$. If $f \in \mathcal{D}((t_m t_n)^*)$, there is an $g \in E$ such that $\langle f, t_m t_n h \rangle = \langle g, h \rangle$ for all $h \in \mathcal{D}(t_m t_n)$. Arguing as above, for $x \in \mathtt{reg}(n) \cap \mathtt{reg}(m)$, there exists $h \in \mathcal{D}(t_m t_n)$ with $h(x) \neq 0$, so $\overline{f} mn \equiv \overline{g}$ on the open and dense set $\mathtt{reg}(m) \cap \mathtt{reg}(n)$. Therefore, $\widehat{\overline{mn} f} = g \in E$ and $f \in \mathcal{D}(t_{\overline{mn}})$. □

LEMMA 31. Let $m : X \to \mathbb{C}$ be a function.

(1) $t_m$ is injective if and only if $\{x \in \mathtt{reg}(m) | m(x) \neq 0\}$ is dense in $\mathtt{reg}(m)$.
(2) If $t_m$ is injective an $m$ does not vanish at any point, then $t_m^{-1} = t_{1/m}$.
(3) $\mathcal{R}(t_m)$ is essential if and only if $\mathtt{reg}(m)$ is dense in X and $t_m$ is injective.

PROOF. (1): Set $N := \{x \in \mathtt{reg}(m) | m(x) = 0\}$. Assume that $N$ contains a nonempty open set $U$. Since $X$ is locally compact Hausdorff, there is a non-zero function $f \in E$ with (compact) support in $U$. Hence $mf = 0$ and $f \in \mathcal{D}(t_m)$: $t_m f = 0$. So $t_m$ is not injective.

On the other hand, assume $f \in \mathcal{D}(t_m)$ with $t_m f = 0$. For $x \in \mathtt{reg}(m)$ is $m(x) f(x) = (t_m f)(x) = 0$. So $f \equiv 0$ on $\mathtt{reg}(m) \setminus N$. If the latter is dense in $\mathtt{reg}(m)$, then $f \equiv 0$ on $\mathtt{reg}(m)$ by continuity of $f$. Finally, $f \equiv 0$ on $X \setminus \overline{\mathtt{reg}(m)}$ by Lemma 28, since $f \in \mathcal{D}(t_m)$. That is $f = 0$.

(2): For $f \in \mathcal{D}(t_{1/m})$ let $g := t_{1/m} f = (f/m)^\wedge$. It is $m(f/m)^\wedge \equiv f$ on the dense set $\mathtt{reg}(m/f)$, hence $\widehat{mg} = f \in E$. That is, $g \in \mathcal{D}(t_m)$ and $f = t_m g \in \mathcal{R}(t_m)$ with $t_m^{-1} f = g = t_{1/m} f$. Now let $f \in \mathcal{D}(t_m^{-1})$. There exists $g \in \mathcal{D}(t_m)$ with $f = t_m g = (mg)^\wedge$. It is $\frac{1}{m} f \equiv g$ on the dense set $\mathtt{reg}(mg)$, hence $(f/m)^\wedge = g \in E$. That is $f \in \mathcal{D}(t_{1/m})$. This proves $t_m^{-1} = t_{1/m}$.

(3): If $\mathtt{reg}(m)$ is dense in $X$, then $t_m$ is essentially defined and $t_m^* = t_{\overline{m}}$ by Theorem 29. $\mathcal{R}(t_m)^\perp = \mathcal{N}(t_m^*) = \mathcal{N}(t_{\overline{m}})$ is trivial if and only if $t_{\overline{m}}$ is injective, and the latter is true if and only if $t_{\overline{m}}$ is injective, which can be read off from (1).

Now assume $\mathtt{reg}(m)$ is not dense. By Lemma 28, the range of $t_m$ contains only functions that vanishes (at least) on the non-empty open set $X \setminus \overline{\mathtt{reg}(\hat{m})}$. Again, since $X$ is locally compact Hausdorff, it follows that such a set cannot be essential. □

We now have a look at the operator $1+t_m^*t_m$.

PROPOSITION 32. Let $m : X \to \mathbb{C}$ be a function with $\mathtt{reg}(m)$ dense in $X$.

(1) $t_m$ is normal and $t_m^*t_m$ is essentially self-adjoint.
(2) $\mathcal{R}(1+t_m^*t_m) = \{g \in E | \forall x \in \mathtt{sing\text{-}supp_r}(m) : g(x) = 0\}$; in particular, $\mathcal{R}(1+t_m^*t_m)$ is essential.
(3) $\mathcal{D}(t_m^*t_m)$ is an essential core for $t_m$.

PROOF. By Theorem 29, $t_m \in \mathcal{C}_o(E)$ and $t_m^* = t_{\overline{m}}$, and by Lemma 28, $t_m = t_{\hat{m}}$. So, without lose of generality we assume $m = \hat{m}$.

(1): Using Lemma 30 (2), it is $(t_m^*t_m)^* = (t_{\overline{m}}t_m)^{***} = t_{|m|^2}^* = t_{|m|^2}$, and the latter is self-adjoint, so $t_m^*t_m$ is essentially self-adjoint.

Let $f \in \mathcal{D}(t_{\overline{m}}t_m)$, that is, $\widehat{mf} \in E$ and $[\overline{m}\widehat{mf}]^\wedge \in E$. Since $f$ is continuous on $X$ and

$$[m\widehat{\overline{m}f}]^\wedge(x) = |m(x)|^2 f(x) = [\overline{m}\widehat{mf}]^\wedge(x) \quad (x \in \mathtt{reg}(m)),$$

$[m\widehat{\overline{m}f}]^\wedge = [\overline{m}\widehat{mf}]^\wedge \in E$, since $\mathtt{reg}(m)$ is dense in $X$. To show $\widehat{\overline{m}f} \in E$, set

$$h(x) := \begin{cases} \overline{\widehat{mf}} \left(\widehat{|m|^2f}\right)^2 / \left|\widehat{|m|^2f}\right|^2 & \text{, for } x \in X \text{ with } \widehat{|m|^2f}(x) \neq 0, \\ 0 & \text{, for } x \in X \text{ with } \widehat{|m|^2f}(x) = 0 \end{cases}.$$

It is easy to check, that $h$ is continuous at $x \in X$ if $\widehat{|m|^2f}(x) \neq 0$ and $h = \overline{m}f$ on $\mathtt{reg}(m)$. If $h$ is continuous on all of $X$, then $\widehat{\overline{m}f} \in E$ is proven. Aiming this, it is clearly enough to show that $\widehat{|m|^2f}(x) = 0$ implies $\widehat{mf}(x) = 0$: From $\widehat{mf} \in \mathcal{D}(t_{\overline{m}})$ it always follows $\widehat{mf}(x) = 0$ for $x \in X \setminus \mathtt{reg}(\overline{m}) = X \setminus \mathtt{reg}(m)$ by Lemma 28. But if $x \in \mathtt{reg}(m)$, then $0 = \widehat{|m|^2f}(x) = |m(x)|^2 f(x)$, so $m(x) = 0$ or $f(x) = 0$. In both cases $\widehat{mf}(x) = m(x)f(x) = 0$.

(2): Let $x \in \mathtt{sing\text{-}supp_r}(m) = \mathtt{sing\text{-}supp_r}(\overline{m})$ and $f \in \mathcal{D}(t_m^*t_m) \subseteq \mathcal{D}(t_m)$. By Lemma 28, $f(x) = 0$, $(t_mf)(x) = 0$ and $(t_{\overline{m}}t_mf)(x) = 0$, so $((1+t_m^*t_m)f)(x) = 0$. One inclusion is shown therefore. Conversely, let $g \in E$ with $g(x) = 0$ for all $x \in \mathtt{sing\text{-}supp_r}(m)$. Further, the functions $[1/(1+|m|^2)]^\wedge$, $[m/(1+|m|^2)]^\wedge$ and $[|m|^2/(1+|m|^2)]^\wedge$ are easily seen to be bounded and continuous on $\mathtt{reg}(m) \cup \mathtt{reg}_\infty^\infty(m)$. Therefore, setting $f := [g/(1+|m|^2)]^\wedge$, it is again easy to see that,

$$f \in E, \quad \widehat{mf} = \left[\frac{gm}{1+|m|^2}\right]^\wedge \in E, \quad [\overline{m}\widehat{mf}]^\wedge = \left[\frac{g|m|^2}{1+|m|^2}\right]^\wedge \in E,$$

by using continuity of $g$ and $g \restriction_{\mathtt{sing\text{-}supp_r}(m)} \equiv 0$. That is, $f \in \mathcal{D}(t_m^*t_m)$ and by $(1+t_m^*t_m)f \restriction_{\mathtt{reg}(m)} = g \restriction_{\mathtt{reg}(m)}$ and continuity of those functions, it is $g \in \mathcal{R}(1+t_m^*t_m)$.

(3): We have to show $\mathcal{G}(t_m \restriction_{\mathcal{D}(t_m^*t_m)})^\perp \subseteq \mathcal{G}(t_m)^\perp$.

If $(g,h) \perp (f, t_mf)$ for each $f \in \mathcal{D}(t_m^*t_m)$. For $x \in \mathtt{reg}(m)$, there exists $f \in E \setminus \{0\}$ with $f(x) \neq 0$ and (compact) support in $\mathtt{reg}(m)$. Then $f \in \mathcal{D}(t_m^*t_m)$ and

$$0 = \langle (g,h), (f, t_mf) \rangle (x) = \overline{g(x)}f(x) + \overline{h(x)}m(x)f(x),$$

hence $g(x) + m(x)h(x) = 0$. This inverses to $\langle (g,h), (f, t_mf) \rangle \equiv 0$ on the dense set $\mathtt{reg}(m)$, hence $(g,h) \perp \mathcal{G}(t_m)$. □

In general $t_m^*t_m \subsetneq t_{|m|^2}$. Let $m : [0,1] \to \mathbb{C}$ be given via $m(x) := e^{i/x}$ for $x \neq 0$. Then $\mathtt{sing\text{-}supp_r}(m) = \{0\}$, so $\mathcal{D}(t_m^*t_m) \neq E$ by Lemma 28, but $t_{|m|^2}$ is the identity on $C(X)$.

EXAMPLE 33. If $m : X \to \mathbb{C}$, then in general $\mathcal{D}(t_m) \neq \mathcal{D}(t_{\overline{m}})$. To see this let $X := [0,1]$ and $m, f : X \to \mathbb{C}$ be given via

$$m(x) := \begin{cases} e^{i/x}/x & , x \neq 0 \\ 0 & , x = 0 \end{cases}, \quad f(x) := \begin{cases} e^{-i/x}x & , x \neq 0 \\ 0 & , x = 0 \end{cases}.$$

Then $\texttt{reg}(m) = (0,1]$ and $\lim_{x\to 0} f(x) = 0$, so $f \in C(X)$. Further, $(mf)(x) = 1$ for $x \in (0,1]$, so $\widehat{mf} = 1$ on $X$ and $f \in \mathcal{D}(t_m)$. But $(\overline{m}f)(x) = e^{-2i/x}$ for $x \in (0,1]$, so $\texttt{reg}(\widehat{\overline{m}f}) = (0,1]$ and $f \notin \mathcal{D}(t_{\overline{m}})$.

Now, we are able to state the main theorem: All essentially defined and orthogonally closable operators $t$ on $E$ are in fact multiplication operators.

THEOREM 34. If $t \in \mathcal{C}_o(E)$, there is a function $m : X \to \mathbb{C}$ such that $t = t_m$. We have

$$a_{t_m} = t_{\frac{1}{1+|m|^2}} \quad \text{and} \quad b_{t_m} = t_{\frac{m}{1+|m|^2}}.$$

PROOF. Let $t \in \mathcal{C}_o(E)$. We abbreviate $\mathcal{D} := \mathcal{D}(t)$ and $\mathcal{D}_* := \mathcal{D}(t^*)$. We set

$$\mathcal{O} := \cup_{f\in\mathcal{D}}\mathcal{O}_f, \quad \mathcal{O}_* := \cup_{f\in\mathcal{D}_*}\mathcal{O}_f \quad \text{with} \quad \mathcal{O}_f := \{x \in X | f(x) \neq 0\}.$$

Since $\mathcal{O}_f$ is open for each continuous function $f$, $\mathcal{O}$ and $\mathcal{O}_*$ are open, too. Further, since $\mathcal{D}$ and $\mathcal{D}_*$ are essential, $\mathcal{O}$ and $\mathcal{O}_*$ are dense in $X$. Hence $\mathcal{O}' := \mathcal{O} \cap \mathcal{O}_*$ is also dense. For $x \in X$, we have

$$\overline{g(x)}(tf)(x) = \langle g, tf\rangle (x) = \langle t^*g, f\rangle (x) = \overline{(t^*g)(x)}f(x) \quad \text{for} \quad f \in \mathcal{D}, g \in \mathcal{D}_*.$$

If $x \in O'$, there are $f \in \mathcal{D}$ with $f(x) \neq 0$ and $g \in \mathcal{D}_*$ with $g(x) \neq 0$ such that

$$m(x) := (tf)(x)/f(x) = \overline{(t^*g)(x)}/\overline{g(x)}.$$

In particular, this shows that $m(x)$ is independent of the particular $f$ and $g$ chosen and that $m$ is continuous on $O'$. Let $f \in \mathcal{D}$ and $x \in O'$. Then, there is a $g \in \mathcal{D}_*$ such that $g(x) \neq 0$, so $(tf)(x) = m(x)f(x)$. Similarly, $(t^*g)(x) = \overline{m(x)}g(x)$ for $g \in \mathcal{D}_*$ and $x \in O'$.

We extend $m$ arbitrarily to a function defined on the whole set $X$. It follows that $\widehat{mf} \in E$ for $f \in \mathcal{D}$ and $\widehat{\overline{m}g} \in E$ for $g \in \mathcal{D}_*$. We then have $f \in \mathcal{D}(t_m)$, $t_m f = tf$ for $f \in \mathcal{D}$ and $g \in \mathcal{D}(t_{\overline{m}})$ and $t_{\overline{m}}g = t^*g$ for $g \in \mathcal{D}_*$. Thus $t \subseteq t_m$ and $t^* \subseteq t_{\overline{m}}$. Hence $t_m = (t_{\overline{m}})^* \subseteq t^{**} = t \subseteq t_m$, that is, $t_m = t$.

With Lemma 30 we compute

$$a_{t_m} = (1 + t_{\overline{m}}t_m)^{-1} \subseteq (1 + t_{|m|^2})^{-1} = (t_{1+|m|^2})^{-1} = t_{\frac{1}{1+|m|^2}},$$

hence equality holds, since $a_{t_m}$ is defined on all of $E$. Further,

$$b_{t_m} = t_m a_{t_m} = t_m t_{\frac{1}{1+|m|^2}} \subseteq t_{\frac{m}{1+|m|^2}},$$

and again equality holds, since $b_{t_m}$ is defined on all of $E$. □

We can characterise the weakly and graph regular operators now.

COROLLARY 35. Each $t \in \mathcal{C}_o(C(X))$ is already weakly regular.

(1) The operator $t = t_m$ is graph regular if and only if $\texttt{sing-supp}_\texttt{r}(m)$ is empty.
(2) The operator $t = t_m$ is regular if and only if $\texttt{reg}(\hat{m}) = X$.

PROOF. The first statement follows from Theorem 34 and Proposition 32.

(1): For this statement we use additionally $\texttt{sing-supp}_\texttt{r}(m) = \texttt{sing-supp}_\texttt{r}(\overline{m})$.

(2): From Lemma 28 it follows that the domains of $t_m$ and $t_m^* = t_{\overline{m}}$ are dense if and only if $\texttt{sing-supp}_\texttt{r}(m) \cup \texttt{reg}_\infty(m) = \texttt{sing-supp}_\texttt{r}(\overline{m}) \cup \texttt{reg}_\infty(\overline{m})$ is empty, or equivalently $\texttt{reg}(\hat{m}) = \texttt{reg}(\hat{\overline{m}}) = X$. Hence, the assertion follows from (1) now. □

*Examples*

Finally we will have a look at several classes and concrete examples to understand the phenomena that arises.

We start by computing the projections.

PROPOSITION 36. Let $m = \hat{m}$. Then $t_m \in \mathcal{C}_o(E)$ is a projection if and only if there is an open set $\mathcal{O}$ that coincides with the interior of its closure such that $m \equiv 1$ on $\mathcal{O}$ and $m \equiv 0$ on $X \setminus \overline{\mathcal{O}}$; it is $\mathtt{reg}(m) = \mathcal{O} \cup (X \setminus \overline{\mathcal{O}})$.

PROOF. At first, let us check that indeed $m = \hat{m}$, that is, $\mathtt{reg}_b(m)$ is empty. For $x \in \mathtt{reg}_b(m)$, there is an open set $U$ containing $x$, and a continuous function $\tilde{m} : U \to \mathbb{C}$ that coincides with $m$ on $U \cap \mathtt{reg}(m)$. But it is not possible to have non-empty intersections of $U$ with $\mathcal{O}$ and $X \setminus \overline{\mathcal{O}}$ at the same time, since this contradicts the continuity of $m$. So $U$ is contained in $\overline{\mathcal{O}}$, hence in $\overline{\mathcal{O}}^\circ = \mathcal{O}$, or it is contained in $X \setminus \mathcal{O}$, hence in $(X \setminus \mathcal{O})^\circ = X \setminus \overline{\mathcal{O}}$. In both cases is $x \in \mathtt{reg}(m)$; a contradiction.

Let $t_m = t_{\overline{m}} = t_m^2$ and assume $m = \hat{m}$. Then $t_m = t_{\overline{m}} = t_m^* t_m \subseteq t_{|m|^2}$. For $x \in \mathtt{reg}(m) \subseteq \mathtt{reg}(|m|^2)$ there exists $f \in \mathcal{D}(t_m)$ with $f(x) \neq 0$. It follows

$$m(x)f(x) = (t_m f)(x) = (t_{|m|^2} f)(x) = |m(x)|^2 f(x),$$

so $m(x) = |m(x)|^2$, hence $m(x) = 0$ or $m(x) = 1$. Let

$$\mathcal{O} := \{x \in \mathtt{reg}(m) | m(x) = 1\},$$

and since $m$ is continuous on $\mathtt{reg}(m)$, $\mathcal{O}$ is open in $\mathtt{reg}(m)$. Further, $\mathtt{reg}(m)$ is open in $X$, so is $\mathcal{O}$. It remains to prove that this set is equal to the interior of its closure. So let $x \in \overline{\mathcal{O}}^\circ \subseteq \overline{\mathtt{reg}(m)}^\circ$. Setting $\tilde{m} \equiv 1$ on the open set $\overline{\mathcal{O}}^\circ$ we have $m \equiv \tilde{m}$ on $\overline{\mathcal{O}}^\circ \cap \mathtt{reg}(m) = \mathcal{O}$, so $x \in \mathtt{reg}(\hat{m}) = \mathtt{reg}(m)$, hence $x \in \mathcal{O}$ as $\hat{m}(x) = 1$.

Now assume there is an open set $\mathcal{O}$ that coincides with $\overline{\mathcal{O}}^\circ$ and a function $m : X \to \mathbb{C}$ such that $m \equiv 1$ on $\mathcal{O}$, $m \equiv 0$ on $X \setminus \overline{\mathcal{O}}$ and $\mathtt{reg}(m) = \mathcal{O} \cup (X \setminus \overline{\mathcal{O}})$.

Finally, let $f \in E$. We have

$$\widehat{m m f} \equiv m^2 f \equiv m f \equiv \overline{m} f \quad \text{on } \mathtt{reg}(m).$$

In particular, $f \in \mathcal{D}(t_m^2)$ if and only if $f \in \mathcal{D}(t_m)$ if and only if $f \in \mathcal{D}(t_{\overline{m}}) = \mathcal{D}(t_m^*)$, and $t_m^2 = t_m = t_m^*$. □

Some examples are discussed finally.

COROLLARY 37. Let $m : X \to \mathbb{C}$ be bounded with $\mathtt{reg}(m)$ dense in $X$. Then $t_m$ is graph regular if and only if $\mathtt{reg}(m) = X$.

PROOF. Since $m$ is bounded $\mathtt{reg}_\infty(m)$ is empty in any case. Hence, the statement follows directly from Corollary 35. □

The next one shows, that the domain of a multiplication operator can be trivial, even though the function is continuous on a dense set - the irrational numbers.

EXAMPLE 38 (Thomae's function). Let $X := \mathbb{R}$ and $m : \mathbb{R} \to \mathbb{R}$ given by

$$m(x) := \begin{cases} 1/q & , \text{ if } x \text{ is rational with } x = p/q \text{ cancelled and } q > 0 \\ 0 & , \text{ if } x \text{ is irrational} \end{cases}.$$

Hence $m$ is continuous precisely at the irrational points. Therefore $\mathtt{reg}(m)$ is empty, which implies $\mathcal{D}(t_m) = \{0\}$.

The next example gives a bounded function with only one point of discontinuity (a jump) such that the corresponding operator is not graph regular.

EXAMPLE 39. Let $X := \mathbb{R}$ and $m$ be the sign function. Then $\mathtt{reg}(m) = \mathbb{R} \setminus \{0\}$ is dense, so $t_m \in \mathcal{C}_o(E)$. Since $\mathtt{sing\text{-}supp_r} = \{0\}$, $t_m$ is not graph regular, even though $m$ is bounded and there is only one point of discontinuity.

The following example shown, that $\mathtt{sing\text{-}supp_r}$ can contain points of continuity.

EXAMPLE 40. Let $X := \mathbb{R}^2$ and $m : \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$m(x,y) := \begin{cases} 0 & ,x \le 0 \\ x & ,x > 0, y \ge 0 \\ -x & ,x > 0, y < 0 \end{cases}.$$

Then $\mathtt{reg}(m)=\{(x,y) \in \mathbb{R}^2 | x < 0 \vee y \neq 0\}$. Clearly $\{(x,0)|x > 0\} \subseteq \mathtt{sing\text{-}supp_r}(m)$. But $\mathtt{reg}(m) \cup \mathtt{reg}_\infty(m)$ is open, so $(0,0)$ is also an element of $\mathtt{sing\text{-}supp_r}(m)$, even though $m(x_n, y_n) \to 0$ for *all* sequences $(x_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ (not only $\mathtt{reg}(m)$) converging to $(0,0)$.

The next example illustrates the difference of $\mathtt{reg}_\infty(m)$ and $\mathtt{sing\text{-}supp_r}(m)$; this can decide graph regularity by Corollary 35.

EXAMPLE 41. Let $X := \mathbb{R}$ and define $m_1, m_2 : \mathbb{R} \to \mathbb{R}$ by

$$m_1(x) := \begin{cases} 1/x & ,x \neq 0 \\ 0 & ,x = 0 \end{cases}, \quad m_2(x) := \begin{cases} 1/x & ,x > 0 \\ 0 & ,x \le 0 \end{cases}.$$

Then $\mathtt{reg}(m_1) = \mathtt{reg}(m_2) = \mathbb{R} \setminus \{0\}$, $\mathtt{reg}_\infty(m_1) = \{0\}$ and $\mathtt{sing\text{-}supp_r}(m_2) = \{0\}$. Therefore, $t_{m_1}, t_{m_2} \in \mathcal{C}_o(E)$ and $t_{m_1}$ is graph regular, but $t_{m_2}$ is not.

## *c) Graph regularity*

Instead of studying $1 + t^*t$ for an essentially defined operator, we will consider for a moment the more general case of the operator $1+st$, where $s$ and $t$ are formally adjoint to each other. So the result can be applied to $1+t^*t$ and as well as to $1+tt^*$ ($t$ may not coincide with $t^{**}$).

LEMMA 42. Let $t : E \to F$ and $s : F \to E$ be formally adjoint to each other. Then $1 + st$ is injective and positive. Further,

$$\mathcal{R}(1 + st) = \{x \in E | (x, 0) \in \mathcal{G}(t) \oplus v\mathcal{G}(s) \subseteq E \oplus F\},$$
$$\mathcal{R}(1 + ts) = \{y \in F | (0, y) \in \mathcal{G}(t) \oplus v\mathcal{G}(s) \subseteq E \oplus F\}.$$

In particular,

$$\mathcal{R}(1 + st) \oplus \mathcal{R}(1 + ts) \subseteq \mathcal{G}(t) \oplus v\mathcal{G}(s) \subseteq E \oplus F.$$

PROOF. If $x \in \mathcal{D}(st)$, then

$$\langle x, (1 + st)x\rangle = \langle x, x + stx\rangle = \langle x, x\rangle + \langle x, stx\rangle = \langle x, x\rangle + \langle tx, tx\rangle$$

is in $\mathcal{P}$, so $1+st$ is positive. If $(1+st)x = 0$, then $\langle x, x\rangle + \langle tx, tx\rangle = 0$, so $\langle x, x\rangle = 0$ and $x = 0$. This proves injectivity of $1 + st$.

Now assume $x \in \mathcal{R}(1 + st)$. There exists $y \in \mathcal{D}(st) \subseteq \mathcal{D}(t)$ with $x = (1 + st)y$. Then $(x, 0) = (y + sty, ty - ty) \in \mathcal{G}(t) \oplus v\mathcal{G}(s) =: \mathcal{G}$. Let $x \in E$ with $(x, 0) \in \mathcal{G}$. There are $y \in \mathcal{D}(t)$ and $z \in \mathcal{D}(s)$ with $x = y + sy$ and $0 = ty - z$, so $ty \in \mathcal{D}(s)$ and $x = (1 + st)z \in \mathcal{R}(1 + st)$. This shows $\mathcal{R}(1 + st) \oplus \{0\} \subseteq \mathcal{G}$. Analogously is $\{0\} \oplus \mathcal{R}(1 + ts) \subseteq \mathcal{G}$ and the last assertion follows, since $\mathcal{G}$ is a vector space. □

LEMMA 43. Let $t : E \to F$ and $s : F \to E$ be formally adjoint to each other. Then $\mathcal{R}(1 + st) \cap \mathcal{D}(st)^\perp = \{0\}$ and $\mathcal{D}(st) \cap \mathcal{R}(1 + st)^\perp = \{0\}$.

Proof. Let $z = (1 + st)u$ for some $u \in \mathcal{D}(st)$ be orthogonal to $\mathcal{D}(st)$. Then $\langle u, u\rangle + \langle tu, tu\rangle = \langle (1 + st)u, u\rangle = 0$, hence $\langle u, u\rangle = 0$, since $\mathcal{P}$ is a quadratic module. Therefore $u = 0$ and $z = 0$. The second equation is similar. □

Lemma 44. Let $t : E \to F$ and $s : F \to E$ be formally adjoint to each other. Then $(1 + st)^{-1}$ is symmetric, and $t(1 + st)^{-1}$ and $s(1 + ts)^{-1}$ are formally adjoint to each other. Moreover,

$$t(1 + st)^{-2} \subseteq (1 + ts)^{-1}t(1 + st)^{-1},$$

and in particular,

$$\mathcal{D}((1 + st)^{-2}) \subseteq \mathcal{D}(s(1 + ts)^{-1}t(1 + st)^{-1}) \subseteq \mathcal{D}((1 + st)^{-1}),$$
$$(1 + st)^{-1}x - (1 + st)^{-2}x = s(1 + ts)^{-1}t(1 + st)^{-1}x \quad (x \in \mathcal{D}((1 + st)^{-2})).$$

Proof. Clearly, $(1+st)^{-1}$ is symmetric by Lemma 42. Now let $x = (1+st)x_0$ for $x_0 \in \mathcal{D}(st)$ and $y = (1 + ts)y_0 \in \mathcal{R}(1 + ts)$ for $y_0 \in \mathcal{D}(ts)$. Then

$$\begin{aligned}\left\langle t(1 + st)^{-1}x, y\right\rangle &= \langle tx_0, (1 + ts)y_0\rangle = \langle tx_0, y_0\rangle + \langle tx_0, tsy_0\rangle \\ &= \langle x_0, sy_0\rangle + \langle stx_0, sy_0\rangle = \langle (1 + st)x_0, sy_0\rangle = \left\langle x, s(1 + ts)^{-1}y\right\rangle,\end{aligned}$$

so $s(1 + ts)^{-1}$ is formally adjoint to $t(1 + st)^{-1}$.

Now let $x \in \mathcal{D}(t(1+st)^{-2})$, that is, $x = (1+st)x_0 = (1+st)^2x_1 \in \mathcal{D}(t)$ for some $x_0 \in \mathcal{D}(st)$ and $x_1 \in \mathcal{D}(stst)$. Then $tx_0 = t(1+st)x_1 = (1+ts)tx_1 \in \mathcal{R}(1+ts)$, and so $x \in \mathcal{D}((1 + ts)^{-1}t(1 + st)^{-1})$ and $(1 + ts)^{-1}t(1 + st)^{-1}x = tx_1 = t(1 + st)^{-2}x$.

The last statements follows easily by applying $s$ to the inclusion proved yet. □

The case $s = t^*$ will be considered now.

Corollary 45. If $t : E \to F$ is essentially defined and $\mathcal{R}(1 + t^*t) = E$, then $t^*t$ is essentially defined.

Proof. Just apply the first equation of Lemma 43 where $s = t^*$ is a formal adjoint of $t$. □

Corollary 46. If $t$ is adjointable, then $t$ is weakly regular.

Proof. $\mathcal{D}(t^*t) = E$ and $\mathcal{D}(tt^*) = F$, so the second equation of Lemma 43 proves the claim. □

Two transform of $t$ are introduced now, that in particular will be combined to the $(a, a_*, b)$-transform of $t$ (see Theorem 56). If $t$ is essentially defined, we set

$$a_t := (1 + t^*t)^{-1}, \qquad\qquad b_t := t(1 + t^*t)^{-1}.$$

Obviously $b_t = ta_t$ and $\mathcal{D}(a_t) = \mathcal{D}(b_t) = \mathcal{R}(1 + t^*t)$, since $\mathcal{R}(a_t) = \mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$.

This idea to consider this pair of operators is of course to use the invertibility of $a_t$ to compute $t \restriction_{\mathcal{D}(t^*t)} = b_t a_t^{-1}$. Hence, if $\mathcal{D}(t^*t)$ is a core for $t$ - for the commutative case we have Proposition 32 -, then $t$ can be recalculated from $a_t$ and $b_t$. This will be done later in Theorem 56.

To begin with, we study $a_t$ and $b_t$ in the case of an essentially defined and orthogonally closable operator $t$. With the previous work, we have

Corollary 47. If $t \in \mathcal{C}'_o(E, F)$, then $a_t$ and $a_{t^*}$ are symmetric, $b_t$ and $b_{t^*}$ are formal adjoints, and

$$b_t a_t \subseteq a_{t^*} b_t, \qquad a_t - a_t^2 \subseteq b_{t^*} b_t, \qquad a_{t^*} - a_{t^*}^2 \subseteq b_{t^{**}} b_{t^*}.$$

In particular, if $t \in \mathcal{C}_o(E, F)$, then $b_{t^*} a_{t^*} \subseteq a_t b_{t^*}$ and $a_{t^*} - a_{t^*}^2 \subseteq b_t b_{t^*}$.

Proof. This is just a corollary of Lemma 44 - we firstly insert $t$ for $t$ and $t^*$ for $s$, we secondly insert $t^*$ for $t$ and $t = t^{**}$ for $s$. □

We repeat now the definitions of weakly and graph regular operators, since those concepts were only introduced before to have a view on some types of regularity.

DEFINITION. An operator $t \in \mathcal{C}_o(E,F)$ is called *weakly regular* if $\mathcal{R}(1+t^*t)$ and $\mathcal{R}(1+tt^*)$ are essential.

Clearly $t$ is weakly regular if and only if $t^*$ is.

DEFINITION. An operator $t \in \mathcal{C}_o(E,F)$ is called *graph regular* if $\mathcal{R}(1+t^*t)$ and $\mathcal{R}(1+tt^*)$ are dense.

The graph's property of being orthogonally complemented in this case (see Theorem 55) suggested this term.

LEMMA 48. If $t$ is weakly regular, then $a_t$, $b_t$, $a_{t^*}$, $b_{t^*}$ are essentially defined:

$$a_t \subseteq a_t^*, \qquad a_{t^*} \subseteq a_{t^*}^*, \qquad b_t \subseteq b_{t^*}^*, \qquad b_{t^*} \subseteq b_t^*.$$

In particular $a_t$, $b_t$, $a_{t^*}$, $b_{t^*}$ are orthogonally closable.

PROOF. This is just a corollary of Corollary 47. □

The next example gives a first class of weakly regular operators behaving even better. Thereafter, further classes of weakly regular operators will be discussed.

EXAMPLE 49. Let $t \in \mathcal{L}(E,F)$. Then $t$ is weakly regular by Corollary 46. Moreover $a_t$ is self-adjoint, $b_t^* = b_{t^*}$ and $a_t - a_t^2 = b_t^* b_t$ by Corollary 47.

In fact, the range of $1+t^*t$ is in general not all of $E$ for an adjointable operator; a counter example can be given by using Theorem 59; indeed, there are a lot of.

PROPOSITION 50. Let $p : E \to E$ be a projection. Then $p$ is weakly regular, $1+p^*p$ is self-adjoint and $p = b_p a_p^{-1}$.

(1) If $\mathcal{D}(p) \subsetneq E$ then $p$ is not graph regular.
(2) If $\mathcal{D}(p) = E$ then $p$ is adjointable.

PROOF. It is $1+p^*p = 1+p$. Hence, for proving (1) and (2) it is enough to show $\mathcal{R}(1+p) = \mathcal{D}(p)$: Let $a \in \mathcal{D}(p)$ and set $b := a - p(a)/2$. Then $b \in \mathcal{D}(p)$ and

$$a = a - p(a)/2 + p(a - p(a)/2) = (1+p)b \in \mathcal{R}(1+p),$$

so $\mathcal{D}(p) \subseteq \mathcal{R}(1+p)$. Since the inverse inclusion is clear, equality holds. Finally, $1+p$ is self-adjoint by Proposition 20 (3). From $\mathcal{D}(p) = \mathcal{R}(1+p) = \mathcal{D}(a_p)$ it also follows $b_p a_p^{-1} = p$. □

EXAMPLE 51. Following Example 7, assume that $\mathcal{A}$ is an unitary $*$-module space over itself. Let $\mathcal{I}$ be a proper essential right ideal of the $*$-algebra $\mathcal{A}$ and consider $E := \mathcal{I} \oplus \mathcal{A}$. Let

$$\mathcal{D}(p) := \mathcal{I} \oplus \mathcal{I}, \quad p(a,b) := \frac{1}{2}(a-b, b-a) \quad (a, b \in \mathcal{I}).$$

Then $\mathcal{D}(p)$ is an essential submodule and $p = p^2$, $p \subseteq p^*$. Now let $(a,b) \in \mathcal{D}(p^*)$. That is, there exists $(c,d) \in \mathcal{I} \oplus \mathcal{A}$ such that for all $(e,f) \in \mathcal{D}(p) = \mathcal{I} \oplus \mathcal{I}$ we have

$$\frac{1}{2}\left((e-f)^*a + (f-e)^*b\right) = p(e,f)^*(a,b) = (e,f)^*(c,d) = e^*c + f^*d.$$

So $e^*(a/2 - b/2 - c) = f^*(a/2 - b/2 + d)$ for all $e, f \in \mathcal{I}$, which implies $2c = a - b$ and $2d = b - a$, since $\mathcal{I}$ is an essential ideal of $\mathcal{A}$. Now, $a, c \in \mathcal{I}$ implies $b \in \mathcal{I}$ and even $d \in \mathcal{I}$ therefore. So $\mathcal{D}(p^*) \subseteq \mathcal{D}(p)$, hence $p = p^*$.

By the way, the range of $p$ is $\{(a,-a) | a \in \mathcal{I}\}$ and its kernel is $\{(a,a) | a \in \mathcal{I}\}$; both are orthogonally closed. In fact, there is no projection with this range and being defined on the whole space $\mathcal{I} \oplus \mathcal{A}$.

The concept of partial isometries is presented now. We will come back to them in the second part when considering a more general polar decomposition.

Definition. An operator $v : E \to F$ is called a *partial isometry* if $\mathcal{D}(v) = \mathcal{N}(v) \oplus \mathcal{N}(v)^\perp$, $\mathcal{N}(v)$ and $\mathcal{R}(v)$ are orthogonally closed and $\langle vx, vy\rangle = \langle x, y\rangle$ for $x, y \in \mathcal{N}(v)^\perp$. $\mathcal{N}(v)^\perp$ is the *initial space* and $\mathcal{R}(v)$ is the *final space* of $v$.

That is, the restriction of an partial isometry to the orthogonal complement of its kernel is an isometry between orthogonally closed submodules. Before carrying about the weak regularity of partial isometries, the adjoint will be computed and shown to be a partial isometry, too; it changes the roles of initial and final space.

Proposition 52. Let $v : E \to F$ be a partial isometry. Then $v \in \mathcal{C}_o(E, F)$. Further $v^*$ is a partial isometry with initial space $\mathcal{N}(v^*)^\perp = \mathcal{R}(v)$, final space $\mathcal{R}(v^*) = \mathcal{N}(v)^\perp$ and $\mathcal{D}(v^*) = \mathcal{R}(v)^\perp \oplus \mathcal{R}(v)$. Further,

$$v^*v = p_{\mathcal{R}(v^*)}, \quad vv^* = p_{\mathcal{R}(v)}, \quad vv^*v = v, \quad v^*vv^* = v^*.$$

Proof. We prove $\mathcal{R}(v) \subseteq \mathcal{D}(v^*)$ first: For an element $z \in \mathcal{D}(v)$ consider the decomposition $z = z^\parallel + z^\perp$ with $z^\parallel \in \mathcal{N}(v)$ and $z^\perp \in \mathcal{N}(v)^\perp$. For $x, y \in \mathcal{D}(v)$ is

$$\langle vx, vy\rangle = \left\langle vx^\perp, vy^\perp\right\rangle = \left\langle x^\perp, y^\perp\right\rangle = \left\langle x^\perp, y\right\rangle,$$

hence $\mathcal{R}(v)$ is contained in the domain of $v^*$ and $v^*vx = x^\perp$. In particular, $\mathcal{N}(v)^\perp$ is contained in $\mathcal{R}(v^*)$.

Denote by $w : E \to F$ the restriction of $v$ to $\mathcal{N}(v)^\perp$; clearly the range of $w$ is the range of $v$. The mapping $w$ is injective, since $\langle wx, wx\rangle = \langle vx, vx\rangle = \langle x, x\rangle$ for $x \in \mathcal{N}(v)^\perp$. Now we set

$$\mathcal{D}(v') := \mathcal{N}(v^*) \oplus \mathcal{R}(v), \quad v'(y^\parallel + y^\perp) := w^{-1}y^\perp \quad (y^\parallel \in \mathcal{N}(v^*), y^\perp \in \mathcal{R}(v)).$$

We want to show that $v' \subseteq v^*$. By the above, $\mathcal{D}(v') \subseteq \mathcal{D}(v^*)$, and for $x \in \mathcal{N}(v^*)$ is already $v'x = 0 = v^*x$. On the other hand, for $x \in \mathcal{D}(v)$ we have shown $v^*vx = x^\perp$; further, $x^\perp = w^{-1}wx^\perp = w^{-1}vx^\perp = w^{-1}vx = v'vx$. That is $v' \subseteq v^*$.

Now we prove that $v'$ is also a partial isometry. The computation

$$\langle v^*vx, v^*vy\rangle = \left\langle x^\perp, y^\perp\right\rangle = \left\langle vx^\perp, vy^\perp\right\rangle = \langle vx, vy\rangle \quad (x, y \in \mathcal{D}(v))$$

implies that $\langle v^*x', v^*y'\rangle = \langle x', y'\rangle$ for $x', y' \in \mathcal{R}(v)$. Further is $\mathcal{N}(v') = \mathcal{N}(v^*)$, since $w^{-1}$ is injective, and $\mathcal{N}(v')^\perp = \mathcal{N}(v^*)^\perp = \mathcal{R}(v)^{\perp\perp} = \mathcal{R}(v)$. Hence $\mathcal{D}(v') = \mathcal{N}(v') \oplus \mathcal{N}(v')^\perp$; in particular $v'$ is essentially defined, hence $v$ is orthogonally closable by $v' \subseteq v^*$. Since $v^*$ is orthogonally closed, the kernel of $v^*$ and that one of $v'$ is also orthogonally closed by Proposition 18 (2). Therefore, $v'$ is indeed a partial isometry with initial space $\mathcal{R}(v)$ and final space $\mathcal{N}(v)^\perp$.

We already have $v \subseteq v^{**} \subseteq (v')^*$. $\mathcal{D}((v')^*) \subseteq \mathcal{D}(v)$ would prove orthogonal closeness of $v$. So assume $x \in \mathcal{D}((v')^*)$, that is, there exists $y \in F$ such that $\langle x, v'z\rangle = \langle y, z\rangle$ for all $z \in \mathcal{D}(v')$. In particular, $0 = \langle x, 0\rangle = \langle y, z\rangle$ for all $z \in \mathcal{N}(v^*)$, so $y \in \mathcal{N}(v^*)^\perp = \mathcal{R}(v)^{\perp\perp} = \mathcal{R}(v)$. Let $y = v\tilde{x}$ for some $\tilde{x} \in \mathcal{N}(v)^\perp$. Then, we also have $\langle x, v'z\rangle = \langle v\tilde{x}, z\rangle$ for all $z \in \mathcal{R}(v)$. That is, $\langle x, \tilde{z}\rangle = \langle x, v'v\tilde{z}\rangle = \langle v\tilde{x}, v\tilde{z}\rangle = \langle \tilde{x}, \tilde{z}\rangle$ for all $\tilde{z} \in \mathcal{N}(v)^\perp$. Hence $x - \tilde{x} \in \mathcal{N}(v)^{\perp\perp} = \mathcal{N}(v)$, so $x \in \mathcal{N}(v) \oplus \mathcal{N}(v)^\perp = \mathcal{D}(v)$.

With $\mathcal{N}(v)^\perp = \mathcal{R}(v^*)$ and $\mathcal{N}(v^*)^\perp = \mathcal{R}(v)$ the representations of $v^*v$ and $vv^*$ are easily computed; they imply $vv^*v = v$ and $v^*vv^* = v^*$. □

Proposition 53. Let $v : E \to F$ be a partial isometry. Then $v$ is weakly regular, $1 + v^*v$ is self-adjoint and $b_v a_v^{-1} = v$.

(1) If $\mathcal{D}(v) \subsetneq E$, then $v$ is not graph regular.
(2) If $\mathcal{D}(v) = E$ then $v$ is adjointable.

Proof. Starting with $1+v^*v = 1+p_{\mathcal{R}(v^*)}$, we get $\mathcal{R}(1+v^*v) = \mathcal{R}(1+p_{\mathcal{R}(v^*)}) = \mathcal{D}(p_{\mathcal{R}(v^*)}) = \mathcal{D}(v)$ - compare the proof of Proposition 50 -, and since $1 + p_{\mathcal{R}(v^*)}$ is self-adjoint; now everything follows. $\square$

Example 54. We use the same idea as in Example 51. Define

$$\mathcal{D}(v) := \mathcal{I} \oplus \mathcal{I}, \quad v(a,b) := \frac{1}{2}(a+b, -a-b) \quad (a, b \in \mathcal{I}).$$

Then $\mathcal{D}(v)$ is an essential submodule of $E$ and as it is easy to see:

- $\mathcal{N}(v) = \mathcal{R}(v) = \{(a, -a) | a \in \mathcal{I}\}$ is orthogonally closed.
- $\mathcal{N}(v)^\perp = \{(a,a) | a \in \mathcal{I}\}$, hence $\mathcal{D}(v) = \mathcal{N}(v) \oplus \mathcal{N}(v)^\perp$.
- $\langle v(a,a), v(a',a') \rangle = \langle (a,a), (a',a') \rangle$ for $a, a' \in \mathcal{I}$.

That is, $v$ is a partial isometry. The adjoint is given by $\mathcal{D}(v^*) = \mathcal{D}(v)$ and $v^*(a,b) = \frac{1}{2}(a-b, a-b)$ for $a, b \in \mathcal{I}$.

Finally, unitaries are considered. But as one would define an unitary operator $u : E \to F$ to be in particular a partial isometry having no kernel - this would be the definition of an isometry - we see that an unitary has to be defined on the whole space: $\mathcal{D}(u) = \mathcal{N}(u) \oplus \mathcal{N}(u)^\perp = \{0\} \oplus \{0\}^\perp = E$. Hence, unitaries as well as isometries, are already adjointable, so there is nothing more to say about this here.

We finish at this point the discussion of weakly regular operators and start with the main object of our interest: graph regular operators.

By $\mathcal{R}_{gr}(E,F)$ we denote the set of graph regular operators from $E$ into $F$ that are essentially defined; $\mathcal{R}_{gr}(E) := \mathcal{R}_{gr}(E,E)$.

Theorem 55.
1. For $t \in \mathcal{C}_o'(E,F)$ the following are equivalent:
   - (a) $t \in \mathcal{R}_{gr}(E,F)$.
   - (b) $\mathcal{G}(t) \oplus v\mathcal{G}(t^*) = E \oplus F$.
   - (c) $\mathcal{R}(1+t^*t) = E$ and $\mathcal{R}(1+tt^*) = F$.
2. If $t \in \mathcal{C}_o(E,F)$, then $t$ is graph regular if and only if $t^*$ is.
3. If $t \in \mathcal{R}_{gr}(E,F)$, then $\mathcal{D}(t^*t)$ is a core for $t$.

Proof. (1a) $\Rightarrow$ (1b) follows from $v\mathcal{G}(t^*) = \mathcal{G}(t)^\perp$ by Proposition 15.

(1b) $\Rightarrow$ (1a): Since $\mathcal{G}(t) \subseteq \mathcal{G}(t)^{\perp\perp}$, (1b) clearly implies that $\mathcal{G}(t) = \mathcal{G}(t)^{\perp\perp}$. Hence, $t$ is orthogonally closed and graph regular.

(1b) $\Leftrightarrow$ (1c): This is an easy application of Lemma 42.

(2): This follows from $t = t^{**}$ with (for example) (1a) $\Leftrightarrow$ (1c).

(3): We have $\langle x, y \rangle + \langle tx, ty \rangle = \langle x, (1+t^*t)y \rangle$ for $x \in \mathcal{D}(t)$ and $y \in \mathcal{D}(t^*t)$, so $\mathcal{G}(t \restriction_{\mathcal{D}(t^*t)})^\perp \cap \mathcal{G}(t) = \{0\}$. With Proposition 13 (3) it follows

$$\mathcal{G}(t)^\perp = \mathcal{G}(t \restriction_{\mathcal{D}(t^*t)})^\perp \cap (\mathcal{G}(t) \oplus v\mathcal{G}(t^*)) = \mathcal{G}(t \restriction_{\mathcal{D}(t^*t)})^\perp \cap E = \mathcal{G}(t \restriction_{\mathcal{D}(t^*t)})^\perp,$$

so $\mathcal{D}(t^*t)$ is a core for $t$. $\square$

Now we define the $(a, a_*, b)$-*transform*, that will be a bijection from the graph regular operators onto a set, that consists of triples of adjointable operators fulfilling some relations. This set is given in the following

Definition. Let $\mathcal{AB}(E,F)$ be the set of triples

$$(a, a_*, b) \in \mathcal{L}(E) \times \mathcal{L}(F) \times \mathcal{L}(E,F)$$

with $a, a_*$ positive, $\mathcal{N}(a) = \{0\}$, $\mathcal{N}(a_*) = \{0\}$ and

$$b^*b = a - a^2, \quad bb^* = a_* - a_*^2, \quad ab^* = b^*a_* \quad (\Leftrightarrow ba = a_*b).$$

We are ready to state now the theorem on the $(a, a_*, b)$-transform.

Theorem 56. If $t \in \mathcal{R}_{gr}(E,F)$, then $(a_t, a_{t^*}, b_t) \in \mathcal{AB}(E,F)$, where

$$a_t := (1+t^*t)^{-1}, \quad a_{t^*} := (1+tt^*)^{-1}, \quad b_t := t(1+t^*t)^{-1}.$$

Further, $\mathcal{N}(b_t) = \mathcal{N}(t)$, $b_{t^*} = b_t^*$ and the projection onto the graph of $t$ is given by

$$p = \begin{pmatrix} a_t & b_t^* \\ b_t & 1-a_{t^*} \end{pmatrix} \in \mathcal{L}(E \oplus F, E \oplus F).$$

If $(a, a_*, b) \in \mathcal{AB}(E,F)$, then $t_{a,a_*,b} \in \mathcal{R}_{gr}(E,F)$, where

$$t_{a,a_*,b} := (ba^{-1})^{**} = (b^* a_*^{-1})^*.$$

Further $t_{a,a_*,b}^* = t_{a_*,a,b^*}$. Finally, $t \mapsto (a_t, a_{t^*}, b_t)$ is a bijection from $\mathcal{R}_{gr}(E,F)$ onto $\mathcal{AB}(E,F)$ with inverse $(a, a_*, b) \mapsto t_{a,a_*,b}$.

Proof. Since $\mathcal{R}(1+t^*t) = E$ by the assumption $t \in \mathcal{R}_{gr}(E,F)$, $a_t$ is defined on the whole of $E$. Therefore, since $1+t^*t$ is positive and injective, $a_t$ is positive with trivial kernel. Further, $b_t$ is defined on $E$ and similar $b_{t^*}$ is defined on $F$. For $x := (1+t^*t)x' \in E$ and $y := (1+tt^*)y' \in F$, where $x' \in E$, $y' \in F$, we compute

$$\begin{aligned} \langle b_t x, y \rangle &= \langle tx', (1+tt^*)y' \rangle = \langle tx', y' \rangle + \langle tx', tt^*y' \rangle = \langle tx', y' \rangle + \langle t^*tx', t^*y' \rangle \\ &= \langle (1+t^*t)x', t^*y' \rangle = \langle x, b_{t^*} y \rangle . \end{aligned}$$

Hence $b_t = (b_{t^*})^* \in \mathcal{L}(E,F)$. From $b_{t^*} = (b_t)^* = (ta_t)^*$ we get $b_{t^*}b_t \supseteq a_t t^* t a_t = a_t(1-a_t)$. Since $a_t(1-a_t)$ is defined on the whole of $E$, the latter yields $b_{t^*}b_t = a_t - a_t^2$. Further, $(1+t^*t)t^* = t^*(1+tt^*)$ and $\mathcal{R}(a_{t^*}^2) = \mathcal{D}(tt^*tt^*) \subseteq \mathcal{D}(t^*tt^*)$ imply

$$b_{t^*}a_{t^*} = t^*a_{t^*}^2 = 1\restriction_{\mathcal{D}(t^*t)} t^*a_{t^*}^2 = a_t(1+t^*t)t^*a_{t^*}^2 = a_t t^*(1+tt^*)a_{t^*}^2 = a_t b_{t^*}.$$

This proves $(a_t, a_{t^*}, b_t) \in \mathcal{AB}(E,F)$. $\mathcal{R}(b_{t^*}) \subseteq \mathcal{R}(t^*)$, so $\mathcal{N}(t) \subseteq \mathcal{N}(b_t)$. Now suppose $b_t x = 0$ for some $x \in E$. Then we have $(a_t - a_t^2)x = b_t^* b_t x = 0$, so $x = a_t x \in \mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$. Further, $(1+t^*t)x = x$, so $t^*tx = 0$ and from $\langle tx, tx \rangle = \langle t^*tx, x \rangle = 0$ it follows $x \in \mathcal{N}(t)$. The statement concerning the projection is easily verified.

Now assume $(a, a_*, b) \in \mathcal{AB}(E,F)$ and set $t := ba^{-1}$ and $s := b^*a_*^{-1}$. Since $\mathcal{D}(t)^\perp = \mathcal{R}(a)^\perp = \mathcal{N}(a) = \{0\}$, $t$ is essentially defined. Similarly, $s$ is essentially defined. For $x \in E$, $y \in F$ we have

$$\langle t(ax), a_*y \rangle = \langle bx, a_*y \rangle = \langle a_*bx, y \rangle = \langle bax, y \rangle = \langle ax, b^*y \rangle = \langle ax, s(a_*y) \rangle ,$$

so $t \subseteq s^*$ and $s \subseteq t^*$. In particular, $t \in \mathcal{C}_o'(E,F)$.

Now we show that $\mathcal{R}(a)$ is a core for $s^*$. We have $s^* = a_*^{-1}b$ by Proposition 20. Thus we have to show that $\mathcal{G}(ba^{-1})^\perp \subseteq \mathcal{G}(a_*^{-1}b)^\perp$. Assume $(r,s) \in \mathcal{G}(ba^{-1})^\perp$. Then $\langle (r,s), (ax, bx) \rangle = 0$ for all $x \in E$, so $ar + b^*s = 0$.

Further, $a_*(br + (1-a_*)s) = a_*br + (a_* - a_*^2)s = bar + bb^*s = b(ar + b^*s) = 0$. Since $a^*$ is injective, this yields $s = a_*s - br$.

Now let $x \in \mathcal{D}(a_*^{-1}b)$. Then there exists a (unique) element $z \in F$ with $bx = a_*z$. Using the assumption $b^*a_*z = ab^*z$ we obtain

$$b^*z = a^{-1}b^*a_*z = a^{-1}b^*bx = a^{-1}(a-a^2)x = (1-a)x.$$

We compute

$$\begin{aligned} \big\langle (r,s), (x, a_*^{-1}bx) \big\rangle &= \langle r, x \rangle + \big\langle s, a_*^{-1}bx \big\rangle = \langle r, x \rangle + \langle s, z \rangle \\ &= \langle r, x \rangle + \langle a_*s - br, z \rangle = \langle r, x \rangle + \langle s, a_*z \rangle - \langle r, b^*z \rangle \\ &= \langle r, x \rangle + \langle s, bx \rangle - \langle r, (1-a)x \rangle = \langle b^*s, x \rangle + \langle ar, x \rangle = 0. \end{aligned}$$

Hence $(r,s) \perp \mathcal{G}(a_*^{-1}b)$. This proves $t^{**} = s^*$. Finally,

$$\begin{aligned} 1 + t^*t^{**} &\supseteq 1 + t^*t = 1 + a^{-1}b^*ba^{-1} = 1 + a^{-1}(a-a^2)a^{-1} = a^{-1}, \\ 1 + t^{**}t^* &\supseteq 1 + s^*s = 1 + a_*^{-1}bb^*a_*^{-1} = 1 + a_*^{-1}(a_* - a_*^2)a_*^{-1} = a_*^{-1}. \end{aligned}$$

Hence $a_{t^{**}} = a \in \mathcal{L}(E)$ and $a_{t^*} = a_* \in \mathcal{L}(F)$, so that $t \in \mathcal{R}_{gr}(E,F)$. Furthermore is $b_{t^{**}} = t^{**}a_{t^{**}} = t^{**}a \supseteq ta = b \in \mathcal{L}(E,F)$, so $b_{t^{**}} = b$. □

In particular, $t^*t$ is self-adjoint for $t \in \mathcal{R}_{gr}(E,F)$.

COROLLARY 57. Let $t \in \mathcal{R}_{gr}(E)$. Then $t$ is normal if and only if $a_t = a_{t^*}$. In this case, $b_t$ is normal and the operators $a_t$ and $b_t$ commute.

PROOF. Since $t \in \mathcal{R}_{gr}(E)$, we have $(a_t, a_{t^*}, b_t) \in \mathcal{AB}(E)$ by Theorem 56, so

$$b_t^* b_t = a_t - a_t^2, \quad b_t b_t^* = a_{t^*} - a_{t^*}^2, \quad b_t a_t = a_{t^*} b_t.$$

Clearly, $a_t = a_{t^*}$ if and only if $t^*t = tt^*$. In this case, $b_t^* b_t = b_t b_t^*$ and $b_t a_t = a_t b_t$. □

PROPOSITION 58. If $t \in \mathcal{C}_o(E,F)$ is injective and has an essential range, then $t$ is graph regular if and only if $t^{-1}$ is.

PROOF. By Proposition 18 (4), $t^{-1} \in \mathcal{C}_o(F,E)$, and from

$$\mathcal{G}(t) \oplus v\mathcal{G}(t^*) = u[\mathcal{G}(t^{-1}) \oplus v\mathcal{G}((t^*)^{-1})] = u[\mathcal{G}(t^{-1}) \oplus v\mathcal{G}((t^{-1})^*)]$$

the statement follows . □

# Transition

## Orthogonal complementability and topology

In this section we prepare the step from graph regularity on unitary $*$-module spaces to Hilbert $C^*$-modules, that is, the topology will be involved then. The usefulness of graph regularity depends on the existence of "enough" orthogonally complemented submodules. But before continuing this argument we study the graph regular operators on $\mathbb{C}[X]$, which gives an extreme example in this direction.

The $*$-algebra $\mathcal{A} = \mathbb{C}[X]$ in several hermitian variables $X = (X_1, \ldots, X_n)$ (for some $n \in \mathbb{N}$) is via Example 7 an unitary $*$-module space over itself, where the sums of squares $\sum \mathbb{C}[X]^2$ are indeed a quadratic module, which is denoted by $\mathcal{P}$. Its submodules are precisely the ideals of $\mathbb{C}[X]$. It will be shown that every operator on $E$ having a non-trivial domain is in fact essentially defined and orthogonally closable. All orthogonally closed operators (with non-trivial domain) are multiplication operators with rational symbol. They are graph regular if and only if the symbol is a constant.

For $f, g \in \mathbb{C}[X]$ the *greatest common divisor* is denoted by $\gcd(f, g)$. The symbol $f|g$ says that $f$ divides $g$. Before stating the main theorem, the multiplication operators will be defined. For $f, g \in \mathbb{C}[X]$ with $\gcd(f, g) = 1$ and $f \neq 0$ let $t_{g/f} : E \to E$ be given via

$$\mathcal{D}(t_{g/f}) := f\mathbb{C}[X], \quad t_{g/f}(fh) := gh \quad (h \in \mathbb{C}[X]).$$

It is easily checked that $t_{g/f}$ is indeed an operator (linearity in $\mathbb{C}$ and $\mathcal{A}$).

Theorem 59. Let $t : E \to E$ be an operator with $\mathcal{D}(t) \neq \{0\}$. Then $t \in \mathcal{C}_o'(E)$. More precisely, there are $m \in \mathbb{N}$, $f_1, \ldots, f_m \in E$ with

$$\mathcal{D}(t) = f_1\mathcal{A} + \ldots + f_m\mathcal{A}.$$

Setting $f := f_1/\gcd(f_1, t(f_1))$ and $g := t(f_1)/\gcd(f_1, t(f_1))$ it is

$$t \subseteq t^{**} = t_{g/f} \quad \text{and} \quad t^* = t_{\overline{g}/\overline{f}}.$$

The operator $t_{g/f}$ is graph regular if and only if $f$ and $g$ are constant.

Proof. Let $t : \mathbb{C}[X] \to \mathbb{C}[X]$ be an operator with $\mathcal{D}(t) \neq \{0\}$. Since $\mathbb{C}[X]$ has no zero-divisors, any non-trivial ideal is essential. Moreover by Hilbert's basis theorem (see [Lan02] Chapter IV Theorem 4.1), it is finitely generated, say

$$\mathcal{D}(t) = f_1\mathbb{C}[X] + \ldots + f_m\mathbb{C}[X].$$

for some $m \in \mathbb{N}$, $f_1, \ldots, f_m \in \mathbb{C}[X]$. The action of $t$ is given then by

$$t(f_1h_1 + \ldots + f_mh_m) = g_1h_1 + \ldots + g_mh_m \quad (h_1, \ldots, h_m \in \mathbb{C}[X]),$$

for some $g_i := t(f_i)$ $(i = 1, \ldots, m)$. By assumption, $t$ is an operator, that is, if $\sum_i f_ih_i = 0$, then $\sum_i g_ih_i = 0$. Hence, for $j = 1, \ldots, m$, from $f_1f_j = f_jf_1$ it follows $g_1f_j = t(f_1f_j) = t(f_jf_1) = g_jf_1$. Therefore $gf_j/\gcd(f_j, g_j) = fg_j/\gcd(f_j, g_j)$. Obviously, $\gcd(f, g) = 1$, so there is a non-zero $\alpha \in \mathbb{C}$ with $f_j/\gcd(f_j, g_j) = \alpha f$

and $g_j / \gcd(f_j, g_j) = \alpha g$. In particular, $g|g_j$ and $f|f_j$, so

$$f_j = \frac{f_j g_j \gcd(f_j, g_j)}{g_j \gcd(f_j, g_j)} = \frac{\alpha f g_j}{\alpha g} = f \frac{g_j}{g} \in f\mathbb{C}[X] \quad (j = 1, \dots, m).$$

This proves $\mathcal{D}(t) \subseteq f\mathbb{C}[X]$. If $\sum_j f_j h_j = f \sum_j \frac{f_j}{f} h_j \in \mathcal{D}(t)$, then

$$t(f \sum_j \frac{f_j}{f} h_j) = t(\sum_j f_j h_j) = \sum_j g_j h_j = g \sum_j \frac{g_j}{g} h_j = g \sum_j \frac{f_j}{f} h_j,$$

since $f_j g = f g_j$ for all $j = 1, \dots, m$. Hence $t \subseteq t_{g/f}$.

The adjoint of $t$ is computed now. Let $h \in \mathbb{C}[X]$ and $k \in \mathcal{D}(t) \subseteq f\mathbb{C}[X]$, so there is an $l \in \mathbb{C}[X]$ with $k = fl$. Then $\langle t(k), \overline{f}h \rangle = \langle gl, \overline{f}h \rangle = \overline{gl}fh = \langle fl, \overline{g}h \rangle = \langle k, \overline{g}h \rangle$. That is, $\overline{f}h \in \mathcal{D}(t^*)$, $t^*(\overline{f}h) = \overline{g}h$, hence $t_{\overline{f}/\overline{g}} \subseteq t^*$. Now suppose $k \in \mathcal{D}(t^*)$ is not zero. Then there exists $l \in \mathbb{C}[X]$ with $\overline{g_1}k = \langle g_1, k \rangle = \langle t f_1, k \rangle = \langle f_1, l \rangle = \overline{f_1}l$. So $\overline{g}k = \overline{f}l$. Since $\gcd(f, g) = 1$, it is $k|\overline{f}$, whence $k \in \overline{f}\mathbb{C}[X]$: $\mathcal{D}(t^*) = \mathcal{D}(t_{\overline{f}/\overline{g}})$. Since $\overline{f} \neq 0$, the domain of $t^*$ is essential; in particular, $t$ is orthogonally closable.

Applying the already proven to $t_{\overline{g}/\overline{f}}$, it follows $t^{**} = t^*_{\overline{g}/\overline{f}} = t_{g/f}$.

Next, the operator $1 + t_{\overline{g}/\overline{f}} t_{g/f}$ will be computed. It is $\mathcal{D}(t_{\overline{g}/\overline{f}} t_{g/f}) = |f|^2\mathbb{C}[X]$ and $(1 + t_{\overline{g}/\overline{f}} t_{g/f})|f|^2 h = |f|^2 h + |g|^2 h = (|f|^2 + |g|^2)h$ for $h \in \mathbb{C}[X]$. Hence, the range of $1 + t_{\overline{g}/\overline{f}} t_{g/f}$ is all of $\mathbb{C}[X]$ if and only if $|f|^2 + |g|^2$ is constant. Now, $t_{\overline{g}/\overline{f}} t_{g/f} = t_{g/f} t_{\overline{g}/\overline{f}}$ gives the assertion concerning the graph regularity by Theorem 55 (1). $\square$

As Theorem 59 insinuate and the proof indeed justifies, one could also choose $f := f_i / \gcd(f_i, t(f_i))$ and $g := t(f_i) / \gcd(f_i, t(f_i))$ for $i \neq 1$, since $g_1 f_j = g_j f_1$.

The last example shows that even adjointable operators - polynomial symbol - are not graph regular; another argument to leave the sphere of unitary $*$-module spaces.

Hilbert space: One guarantee to have "enough" orthogonally complemented subspaces is its completeness, since for an unitary space not all (even closed) submodules are of that kind any more. Now, the gap between unitary $*$-modules spaces and Hilbert $C^*$-modules is much larger: It could be filled with pre-Hilbert $C^*$-modules or unitary $*$-modules spaces over Banach-$*$-algebras. For Hilbert $C^*$-modules we do not expect such a vacant set of graph regular operators - and we know that this is not the case, since there is the set of already regular operators. That is, the main task is to uncover the gap between regular and graph regular operators.

From now on we leave the unitary $*$-module spaces and specialise to Hilbert $C^*$-modules (see Example 6).

At first, let me state a well-known relation of orthogonality and topology in Hilbert $C^*$-modules. (We have in mind that for subsets in Hilbert $C^*$-modules the adjective closed reads as complete.)

Lemma 60. If $E$ is a Hilbert $C^*$-module and $G \subseteq E$, then $G^\perp$ is closed. In particular each orthogonally closed submodule is closed.

Proof. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in $G^\perp$ with $x_n \to x \in E$, then $\langle x, y \rangle = \lim_{n \to \infty} \langle x_n, y \rangle = 0$, so $x \perp G$. $\square$

With this simple observation, we will have a look at closed (closable) operators and their adjoints now, and then study the operator $1 + t^*t$. Assume $E$ and $F$ are Hilbert $\mathcal{A}$-modules.

As in the Hilbert space case, an operator $t : E \to F$ is *closed* if its graph is a closed submodule of $E \oplus F$. If $\overline{\mathcal{G}(t)}$ is still a graph of an operator, $t$ is called *closable* and we denote by $\bar{t}$ the operator with $\mathcal{G}(\bar{t}) = \overline{\mathcal{G}(t)}$; for each closed operator $s$ that extends $t$, we have $\mathcal{G}(\bar{t}) \subseteq \mathcal{G}(s)$. For a closed operator $t$ a subset $\mathcal{D}$ of $\mathcal{D}(t)$ is called *core* for $t$ if $\overline{t \restriction_{\mathcal{D}}} = t$.

From Lemma 60 we deduce:

Corollary 61. Each orthogonally closed resp. closable operator $t : E \to F$ is closed resp. closable. If $t$ is closable, then

(1) $\mathcal{G}(t) \subseteq \mathcal{G}(\bar{t}) \subseteq \mathcal{G}(t)^{\perp\perp}$.
(2) If $t$ is essentially defined, then $t^* = \bar{t}^*$ is closed.
(3) If $t \in \mathcal{C}'_o(E, F)$, then $t \subseteq \bar{t} \subseteq t^{**}$.

Lemma 62. Let $t : E \to F$ be essentially defined. Then

(1) For $x \in \mathcal{D}(t^*t)$ is $\|x\| \leq \|(1 + t^*t)x\|$ and $\|tx\| \leq \|(1 + t^*t)x\|$.
(2) $a_t$ and $b_t$ are bounded by 1.
(3) If $t$ is closed, then $1 + t^*t$ is closed and $\mathcal{R}(1 + t^*t)$ is closed.
(4) If $t$ is closable, then $\mathcal{R}(\overline{a_t}) \subseteq \mathcal{D}(\bar{t})$ and $\overline{b_t} = \bar{t}\overline{a_t}$. $\overline{a_t}$ is injective as well.
(5) If $t$ is closed, then $a_t$ and $b_t$ are closed.

Proof. (1): For $x \in \mathcal{D}(t^*t)$ are $\|x\|^2$ and $\|tx\|^2$ less or equal $\| \langle x, (1+t^*t)x\rangle \|_{\mathcal{A}}$. With $\| \langle x, (1 + t^*t)x\rangle \|_{\mathcal{A}} \leq \|(1 + t^*t)x\|\|x\| \leq \|(1 + t^*t)x\|^2$ assertion (1) is proven.

(2): From $\|x\| \leq \|(1 + t^*t)x\|$ for $x \in \mathcal{D}(t^*t)$ it follows $\|a_t\| \leq 1$. Now let $x \in \mathcal{D}(t^*t)$. Then $\|b_t(1 + t^*t)x\| = \|tx\| \leq \|(1 + t^*t)x\|$, that is, $\|b_t\| \leq 1$.

(3): Let $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{D}(t^*t)$ with $x_n \to x$ and $(1+t^*t)x_n \to y$. Then from (1) it follows that $tx_n$ is a Cauchy-sequence, so it converges. Since $t$ is closed, $tx_n \to tx$. Now, $t^*$ is closed and $t^*tx_n \to y - x$, so $tx \in \mathcal{D}(t^*)$ and $t^*tx = y - x$. Hence $x \in \mathcal{D}(t^*t)$ and $(1 + t^*t)x = y$. Let $((1 + t^*t)x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}(1 + t^*t)$ converging to $y$. Using (1) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence and converges to an $x \in E$. Since $1 + t^*t$ is closed $x \in \mathcal{D}(t^*t)$ and $y = (1 + t^*t)x \in \mathcal{R}(1 + t^*t)$.

(4): $\mathcal{D}(\overline{a_t}) = \overline{\mathcal{D}(a_t)} = \overline{\mathcal{R}(1 + t^*t)}$ since $a_t$ is bounded. Let $x \in \mathcal{D}(\overline{a_t})$. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$ with $(1 + t^*t)z_n \to x$. Since $\overline{a_t}$ is continuous, $z_n \to \overline{a_t}x$. Since $\overline{b_t}$ is continuous, $tz_n \to \overline{b_t}x$. Since $t$ is closable $\overline{a_t}x \in \mathcal{D}(\bar{t})$ and $\bar{t}\overline{a_t}x = \overline{b_t}x$. Now $\overline{b_t} = \bar{t}\overline{a_t}$ follows from $\mathcal{D}(\overline{a_t}) = \mathcal{D}(\overline{b_t}) = \overline{\mathcal{R}(1 + t^*t)}$. From $\|x\| \leq \|(1+t^*t)x\|$, for $x \in \mathcal{D}(t^*t)$, we conclude that the closure of $a_t$ remains injective.

(5): This follows from (3) using that $a_t$ is closed as the inverse of the closed operator $1 + t^*t$: $\overline{b_t} = \bar{t}\overline{a_t} = ta_t = b_t$. □

Another possibility to prove Theorem 55 (1a) ⇒ (1c) can be given by using the following lemma.

Lemma 63. If $t : E \to F$ is graph regular, then $\mathcal{R}(1 + t^*t) = E$.

Proof. By Lemma 62 the range of $1 + t^*t$ is closed, since graph regular operators are orthogonally closed and in particular closed. The range of $1 + t^*t$ is dense in $E$ by the definition of graph regularity; hence it is all of $E$. □

Lemma 64. Let $t : E \to F$ be an essentially defined operator with $\mathcal{R}(1 + t^*t)$ dense in $E$. Then $\mathcal{D}(t^*t)$ is an essential submodule of $E$.

Proof. Let $x \perp \mathcal{D}(t^*t)$. Since the range of $1 + t^*t$ is dense, there exists a sequence $(x'_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(t^*t)$ with $x = \lim(1+t^*t)x'_n$. Since $a_t$ is bounded, $x'_n \to \overline{a_t}x$. By assumption, for $y \in \mathcal{D}(t^*t)$ is

$$\langle \overline{a_t}x, (1 + t^*t)y\rangle = \lim_n \langle x'_n, (1 + t^*t)y\rangle = \lim_n \langle (1 + t^*t)x'_n, y\rangle = \langle x, y\rangle = 0.$$

Hence $\overline{a_t}x = 0$, since $\mathcal{R}(1+t^*t)$ is dense in $E$. Hence $x = 0$, as $\overline{a_t}$ is injective by Lemma 62 (4). $\square$

LEMMA 65. Let $t : E \to F$ be bounded with $\mathcal{D}(t) = E$. Then $t^*$ is bounded.

PROOF. The norm of $t^*$ is given by

$$\|t^*\| = \sup_{\substack{x\in\mathcal{D}(t^*)\\ \|x\|\le 1}} \|t^*x\| = \sup_{\substack{x\in\mathcal{D}(t^*)\\ \|x\|\le 1}} \sup_{\substack{y\in E\\ \|y\|\le 1}} \| \langle t^*x, y\rangle \| = \sup_{\substack{x\in\mathcal{D}(t^*)\\ \|x\|\le 1}} \sup_{\substack{y\in E\\ \|y\|\le 1}} \| \langle x, ty\rangle \| \le \|t\|.$$

$\square$

The next lemma is inspired by [Lan95] Lemma 9.1.

LEMMA 66. Let $t : E \to F$ be an essentially defined operator with $\mathcal{R}(1+t^*t)$ dense in $E$. If $\mathcal{R}(t) \subseteq \overline{\mathcal{D}(t^*)}$, then $\mathcal{D}(t^*t)$ is dense in $\mathcal{D}(t)$.

PROOF. Lemma 62 implies that $a_t$ and $b_t = ta_t$ are bounded. We compute $b_t^* \supseteq a_t^*t^* \supseteq a_tt^*$. By Lemma 65, $b_t^* = \overline{b_t}^*$ is bounded, so $a_tt^*$ is bounded as well. Hence $\mathcal{D}(\overline{a_tt^*}) = \overline{\mathcal{D}(t^*)}$ and $\mathcal{R}(\overline{a_tt^*}) \subseteq \overline{\mathcal{D}(t^*t)}$, since $\mathcal{D}(a_tt^*) = \mathcal{D}(t^*)$ and $\mathcal{R}(a_tt^*) \subseteq \mathcal{D}(t^*t)$.

Now suppose $z \in \mathcal{D}(t)$. Since $\mathcal{R}(t) \subseteq \overline{\mathcal{D}(t^*)}$, $tz \in \overline{\mathcal{D}(t^*)} = \mathcal{D}(\overline{a_tt^*})$. For all $y \in \mathcal{R}(1+t^*t) = \mathcal{D}(a_t) = \mathcal{D}(b_t)$ is

$$\left\langle y, (\overline{a_tt^*}t + a_t)z\right\rangle = \langle t^*b_ty, z\rangle + \langle a_ty, z\rangle = \langle (t^*t+1)a_ty, z\rangle = \langle y, z\rangle\,,$$

since $\mathcal{R}(b_t) \subseteq \mathcal{D}(t^*)$. The density of $\mathcal{R}(1+t^*t)$ implies $z = (\overline{a_tt^*}t + a_t)z \in \overline{\mathcal{D}(t^*t)}$. That is, $\mathcal{D}(t) \subseteq \overline{\mathcal{D}(t^*t)}$, which is the assertion. $\square$

PROOF. Lemma 63 implies that $a_t$ and $b_t = ta_t$ are defined on $E$. With , we compute $b_t^* \supseteq a_t^*t^* \supseteq a_tt^*$. By Lemma 65, $b_t^*$ is bounded, so $a_tt^*$ is bounded as well. Hence $\mathcal{D}(\overline{a_tt^*}) = \overline{\mathcal{D}(t^*)}$ and $\mathcal{R}(\overline{a_tt^*}) \subseteq \overline{\mathcal{D}(t^*t)}$, since $\mathcal{D}(a_tt^*) = \mathcal{D}(t^*)$ and $\mathcal{R}(a_tt^*) \subseteq \mathcal{D}(t^*t)$.

Now suppose $z \in \mathcal{D}(t)$. Since $\mathcal{R}(t) \subseteq \overline{\mathcal{D}(t^*)}$, $tz \in \overline{\mathcal{D}(t^*)} = \mathcal{D}(\overline{a_tt^*})$. It is

$$\left\langle y, (\overline{a_tt^*}t + a_t)z\right\rangle = \langle b_ty, tz\rangle + \langle a_ty, z\rangle = \langle (t^*t+1)a_ty, z\rangle = \langle y, z\rangle \quad (y \in E),$$

since $\mathcal{R}(b_t) \subseteq \mathcal{D}(t^*)$. This implies $z = (\overline{a_tt^*}t + a_t)z \in \overline{\mathcal{D}(t^*t)}$. Therefore $\mathcal{D}(t) \subseteq \overline{\mathcal{D}(t^*t)}$, which is the assertion. $\square$

Assume $t$ is essentially defined orthogonally closed (so $t^*$ is essentially defined) and $\mathcal{R}(1+t^*t)$ dense in $E$. In general, $\mathcal{R}(1+tt^*)$ is not dense in $F$ as we will see later in Example 122. This is different to what is known about regular operators: If $t$ and $t^*$ are densely defined, $t$ is orthogonally closed ($t = t^{**}$) with $\mathcal{R}(1+t^*t)$ dense in $E$, then $\mathcal{R}(1+tt^*)$ is dense in $F$ (see [Lan95] Corollary 9.6[1]).

We finish our sighting at this point and start to study only graph regular operators on Hilbert $C^*$-modules. Since the next chapter is again concerned with the commutative case, but with the $C^*$-algebra $C_0(X)$ now, it could be thought of as another sighting, since the computations in this case could be done directly without referring to the theory developed so far.

[1]This corollary is false as noted in [Pal99] Remark 2.4 (ii). But if one strengthen the condition of closeness of $t$ to orthogonally closeness, the statement becomes true, since in this case $t = t^{**}$.

# *Graph regular operators on Hilbert $C^*$-modules*

# 4. Commutative case: Operators on $C_0(X)$
## Phenomena

At first, in this chapter we study operators on $C_0(X)$ (Theorem 67 and Theorem 68), the $C^*$-algebra of continuous functions on a locally compact Hausdorff space$X$, for two reasons. Some examples will show us the difficulties and phenomena that occur for orthogonally closed/graph regular operators - already in this simple situation. On the other hand, since the computations on this $C^*$-algebra are quite easy, we can anticipate the general theory of the absolute value and the polar decomposition (Theorem 71), the bounded transform (Theorem 72).

Secondly, we transfer some of these results to unital commutative $C^*$-algebras $\mathcal{C}$ with $C_0(X) \subseteq \mathcal{C} \subseteq C_b(X)$ (Theorem 75). This is interesting in its own but also needed for the functional calculus.

First, let us notice that all results that we know from the theory for operators on $C(X)$ are verbatim true except for the following point: A function $h : X \to \mathbb{C}$ is in $C_0(X)$ if and only if $\mathtt{reg}(h) = X$ *and* $h \in \mathcal{F}_0(X)$, where we set

$$\mathcal{F}_0(X) := \{h : X \to \mathbb{C} | h \text{ vanisches at infinity}\}.$$

Therefore, the characterisation of the operator's domain has to be modified slightly. All of the other statements and their proofs are verbatim true. We repeat them (in a comprehensive form) in the next two theorems, and alter the statement and proof of the characterisation of the operator's domain in Theorem 67 (1).

Let $E = C_0(X)$. In the first group the statements for the multiplication operators are presented. The second one is concerned with $\mathcal{C}_o(E)$.

Theorem 67. Let $m : X \to \mathbb{C}$ be given. Then

(1) For $x \in X$, there is an $f \in \mathcal{D}(t_m)$ with $f(x) \neq 0$ if and only if $x \in \mathtt{reg}(\hat{m})$.

$$\mathcal{D}(t_m) = \{f \in E | f \equiv 0 \text{ on } X \setminus \mathtt{reg}(\hat{m}), \partial\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf}), \widehat{m}f \in \mathcal{F}_0(X)\},$$
$$(t_m f)(x) = \hat{m}(x) f(x) \quad (f \in \mathcal{D}(t_m), x \in X \setminus \mathtt{reg}_\infty(m)).$$

If $\tilde{m} : X \to \mathbb{C}$, then $t_m = t_{\tilde{m}}$ if and only if $m \simeq \tilde{m}$. In particular, $t_m = t_{\hat{m}}$.

(2) The operator $t_m$ is essentially defined if and only if $\mathtt{reg}(m)$ is dense in $X$. In this case, we have $t_m^* = t_{\overline{m}}$ and $t_m \in \mathcal{C}_o(E)$.

(3) Let $n : X \to \mathbb{C}$ be another function.
   (a) It is $t_m + t_n \subseteq t_{m+n}$ and $t_m t_n \subseteq t_{mn}$.
   (b) If $\mathtt{reg}(m)$ and $\mathtt{reg}(n)$ are dense in $X$, then $t_m + t_n, t_m t_n \in \mathcal{C}_o'(E)$ and $(t_m + t_n)^{**} = t_{m+n}$, $(t_m t_n)^{**} = t_{mn}$.

(4) (a) $t_m$ is injective if and only if $\{x \in \mathtt{reg}(m) | m(x) \neq 0\}$ is dense in $\mathtt{reg}(m)$.
   (b) If $t_m$ is injective an $m$ does not vanish at any point, then $t_m^{-1} = t_{1/m}$.
   (c) $\mathcal{R}(t_m)$ is essential if and only if $\mathtt{reg}(m)$ is dense in X and $t_m$ is injective.

(5) Let $\mathtt{reg}(m)$ be dense in $X$.
   (a) $t_m$ is normal and $t_m^* t_m$ is essentially self-adjoint.
   (b) $\mathcal{R}(1 + t_m^* t_m) = \{g \in E | \forall x \in \mathtt{sing\text{-}supp_r}(m) : g(x) = 0\}$. In particular, $\mathcal{R}(1 + t_m^* t_m)$ is essential.

(c) $\mathcal{D}(t_m^* t_m)$ is an essential core for $t_m$.

Proof. By Definition, $\mathcal{D}(t_m)$ consists of those $f \in E$ for which $\mathtt{reg}(\widehat{mf}) = X$ and $\widehat{mf} \in F_0(X)$. Since $\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$ for $f \in E$, $\mathcal{D}(t_m)$ is the set of all $f \in E$ such that $X \setminus \mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$ and $\widehat{mf} \in F_0(X)$.

Let $f \in E$ with $\widehat{m}f \in F_0(X)$. Suppose that $\partial\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\widehat{mf})$ and $f \equiv 0$ on $X \setminus \mathtt{reg}(\hat{m})$. In particular, $mf \equiv 0$ on $X \setminus \overline{\mathtt{reg}(\hat{m})}$. Hence $X \setminus \mathtt{reg}(\hat{m}) = X \setminus \overline{\mathtt{reg}(\hat{m})} \cup \partial\mathtt{reg}(\hat{m})$ is contained in $\mathtt{reg}(\widehat{mf})$. To show $\widehat{mf} \in F_0(X)$, let $\epsilon > 0$. Since $\widehat{m}f \in F_0(X)$, there exists a compact set $K \subseteq X$ such that $|\widehat{m}f| \leq \epsilon$ on $X \setminus K$. By continuity of $\widehat{mf}$ on $X$, the same is true for this function, since $\widehat{m}f$ and $\widehat{mf}$ coincide on the dense set $X \setminus \partial\mathtt{reg}(m)$. That is, $\widehat{mf} \in F_0(X)$, hence $f \in \mathcal{D}(t_m)$.

We suppose now that $f \in \mathcal{D}(t_m)$. In particular, $\mathtt{reg}(mf)$ is dense in $X$, hence $\partial\mathtt{reg}(\hat{m}) \subseteq X = \mathtt{reg}(\widehat{mf})$. Assume that $x \in X \setminus \mathtt{reg}(\hat{m})$ and $f(x) \neq 0$. By the continuity of $f$, there exists an open set $U \subseteq X \setminus \mathtt{reg}(\hat{m})$ such that $f(y) \neq 0$ for $y \in U$. From Lemma 27 it follows that $U$ is contained even in $X \setminus \mathtt{reg}(\hat{m}f)$. But this contradicts the density of $\mathtt{reg}(mf) \subseteq \mathtt{reg}(\hat{m}f)$ in $X$, hence $f(x) = 0$. In particular, $\widehat{mf}$ coincides with $\widehat{m}f$ on $\mathtt{reg}(\hat{m})$, and $\widehat{m}f \equiv 0$ on $X \setminus \mathtt{reg}(\hat{m})$. So, $\widehat{mf} \in F_0(X)$ implies $\widehat{m}f \in F_0(X)$ and the description of $\mathcal{D}(t_m)$ is proven. □

Theorem 68.

(1) If $t \in \mathcal{C}_o(E)$, then there is a function $m : X \to \mathbb{C}$ such that $t = t_m$.

(2) Each $t \in \mathcal{C}_o(C(X))$ is already weakly regular.

(a) The operator $t = t_m$ is graph regular if and only if $\mathtt{sing\text{-}supp_r}(m)$ is empty. In this case,

$$a_{t_m} = t_{\frac{1}{1+|m|^2}} \quad \text{and} \quad b_{t_m} = t_{\frac{m}{1+|m|^2}}.$$

(b) The operator $t = t_m$ is regular if and only if $\mathtt{reg}(\hat{m}) = X$.

We are able to discuss the operators on $C_0(X)$ now. First, a strange behaviour of normal operators is presented, even if they are graph regular.

Example 69 (Continuing Example 33). We have seen that $\mathcal{D}(t_m) \neq \mathcal{D}(t_m^*)$. But since $\mathtt{reg}(m) = (0, 1]$ and $0 \in \mathtt{reg}_\infty(m)$, $t_m$ is graph regular by Theorem 68. By Theorem 67 $t_m$ is normal (as all operators on $C_0(X)$ are). Thus, even for graph regular operators $t$ the statements:

(1) $t^*t = tt^*$ (that is, $t$ is normal).

(2) $\mathcal{D}(t) = \mathcal{D}(t^*)$ and $\langle tf, tf\rangle = \langle t^*f, t^*f\rangle$ for all $f \in \mathcal{D}(t)$.

are *not* equivalent! We only have (2) $\Rightarrow$ (1) as the standard proof of the Hilbert spaces case shows: By the polarisation identity, (2) implies

$$\langle tf, tg\rangle = \langle t^*f, t^*g\rangle \quad (f, g \in \mathcal{D}(t) = \mathcal{D}(t^*)).$$

Hence,

$$\mathcal{D}(t^*t) = \{f \in \mathcal{D}(t) | \exists g \in E : \forall h \in \mathcal{D}(t) : \langle g, h\rangle = \langle tg, th\rangle = \langle t^*g, t^*h\rangle\} = \mathcal{D}(tt^*)$$

and $t^*tf = g = tt^*f$ for $f \in \mathcal{D}(t^*t) = \mathcal{D}(tt^*)$. That is, $t^*t = tt^*$.

We have already seen in Theorem 56 that in general the kernels of $t$ and $b_t$ are equal for a graph regular operator $t$; in particular, the biorthogonal complements of its ranges are equal, too: $\mathcal{R}(t)^{\perp\perp} = \mathcal{N}(t^*)^\perp = \mathcal{N}(b_{t^*})^\perp = \mathcal{R}(b_t^*)^\perp = \mathcal{R}(b_t)^{\perp\perp}$. The following example shows that this is not true for the closures of the ranges. Obviously, the range of $b_t = ta_t$ coincides with that one of $t$ when restricted to $\mathcal{D}(t^*t)$. Hence $\overline{\mathcal{R}(b_t)} \subseteq \overline{\mathcal{R}(t)}$. The other inclusion is false in general as the next example shows.

EXAMPLE 70. Consider the operator $t_m$ on $C_0(\mathbb{R})$ with $m(x) := 1/x$ on $\mathbb{R} \setminus \{0\}$ and $m(0)$ arbitrary but not zero. $\overline{\mathcal{R}(t_m)}$ is all of $C_0(\mathbb{R})$: $t_m = (t_{1/m})^{-1}$ by Theorem 67, and $t_{1/m}$ is densely defined since it is regular by Theorem 68. But $\overline{\mathcal{R}(b_{t_m})}$ consists precisely of those functions of $C_0(\mathbb{R})$ that vanishes at 0: This is easily seen, since $b_{t_m} = t_{\frac{x}{1+x^2}}$. But for our purpose it is enough to state that any function in the range of $b_{t_m}$ vanishes at 0 by Theorem 67 (1).

We define and study the absolute value of (unbounded) graph regular operators below. Here, in the commutative case, we have a natural candidate for $|t_m|$, namely the self-adjoint operator $t_{|m|}$, and we will see that both are the same. (For bounded $m$ this is easily checked.) At this point we do not want to state the definition of the absolute value in the general case, but refer to Theorem 114. It characterises the absolute value $|t_m|$, which is also self-adjoint and graph regular, by its $(a, a_*, b)$-transform: $a_{|t_m|} = a_{t_m}$ and $b_{|t_m|} = |b_{t_m}|$. Since the absolute value is a continuous function on the complex numbers, the graph regularity of $m$ implies that one of $|m|$ via $\texttt{sing-supp}_\texttt{r}(|m|) \subseteq \texttt{sing-supp}_\texttt{r}(m)$. From Theorem 34 it follows

$$a_{t_{|m|}} = t_{\frac{1}{1+|m|^2}} = a_{t_m}, \quad b_{t_{|m|}} = t_{\frac{|m|}{1+|m|^2}} = |b_{t_m}|.$$

That is, $|t_m|$ and $t_{|m|}$ are equal, since their $(a, a_*, b)$-transforms coincide.

We will use the natural $t_{|m|}$ for the absolute value in the remaining section.

For a function $m : X \to \mathbb{C}$ we let $m = u|m|$ be its polar decomposition and choose the phase $u$ to be zero whenever $m$ vanishes; in particular, this ensures $\texttt{reg}(u)$ to be dense if $\texttt{reg}(m)$ is. By Lemma 30, we immediately obtain $t_m \supseteq t_u t_{|m|}$ and even a *weak form of the polar decomposition* for the corresponding operator:

$$t_m = (t_u t_{|m|})^{**}.$$

Indeed, a *strong form of the polar composition* is not available in general. If $m$ is bounded, there is no problem to prove

$$t_m = t_u t_{|m|}.$$

THEOREM 71. Let $m : X \to \mathbb{C}$ be a function such that $t_m$ is graph regular. It is $t_m = t_u t_{|m|}$ if and only if

$$\texttt{reg}_\infty(m) \cap \texttt{sing-supp}_\texttt{r}(u) = \emptyset,$$

that is, the phase $u$ of $m$ has to be continuous at points of infinity.

PROOF. We prove this by constructing a function $f : X \to \mathbb{C}$ that belongs to the domain of $t_m$ but not to that one of $t_{|m|}$, whenever there exists $x \in \texttt{reg}_\infty(m)$ not belonging to $\texttt{reg}(\hat{u})$.

In this case, it is possible to choose a neighbourhood $U$ of $x$ such that $m$ does not vanish on any of its points. Since $X$ is locally compact Hausdorff, there exists a function $h \in C_0(X)$ and a compact neighbourhood $V \subseteq U$ of $x$ with $h \equiv 1$ on $V$ and $h \equiv 0$ outside of $U$. Indeed, the non-zero function $g := h/m$ belongs to $C_0(X)$. Further, $\widehat{mg} = (mh/m)^\wedge = h$, so $g \in \mathcal{D}(t_m)$. But $\widehat{|m|g} = (|m|h/m)^\wedge = h/\hat{u}$ on $V$, where $m$, $u$ and $h$ do not vanish. So $x \in \texttt{sing-supp}_\texttt{r}(h/\hat{u})$, since $h(x) \neq 0$ and $x \notin \texttt{reg}(\hat{u})$ (see Lemma 27). That is, $g \notin \mathcal{D}(t_{|m|})$ and $t_m \supsetneq t_u t_{|m|}$ is shown.

Now assume $\texttt{reg}_\infty(m) \cap \texttt{sing-supp}_\texttt{r}(u)$ is empty, that is, $\texttt{reg}_\infty(m) \subseteq \texttt{reg}(\hat{u})$. We prove $\mathcal{D}(t_m) \subseteq \mathcal{D}(t_{|m|})$ and $\mathcal{R}(t_{|m|}) \subseteq \mathcal{D}(t_u)$, from which $t_m = t_u t_{|m|}$ follows, since $t_m \supseteq t_u t_{|m|}$ was already established. Since $t_m$ is graph regular, we have $X = \texttt{reg}(\hat{m}) \cup \texttt{reg}_\infty(m)$. Moreover, since $|.|$ is continuous, $\texttt{reg}(\hat{m}) \subseteq \texttt{reg}(\widehat{|m|})$ and $\texttt{reg}_\infty(m) \subseteq \texttt{reg}_\infty(|m|)$. Actually equality holds for both cases, since $\texttt{reg}(\widehat{|m|})$ and $\texttt{reg}_\infty(|m|)$ are disjoint. Further, $\widehat{|m|} = |\hat{m}|$ on $\texttt{reg}(\hat{m})$.

Let $f \in \mathcal{D}(t_m)$. Then $f \equiv 0$ on $X \setminus \mathtt{reg}(\hat{m}) = X \setminus \mathtt{reg}(\widehat{|m|})$ by Theorem 67 (1). Let $x \in \mathtt{reg}_\infty(m)$, so $\subseteq \mathtt{reg}(\hat{u})$ by assumption. In a neighbourhood $U$ of $x$, where $m$ does not vanish and $\hat{u}$ is continuous, we compute (noting that $\hat{u}$ does not vanish there as well)

$$\widehat{|m|f} = (mf/\hat{u})^\wedge = \widehat{mf}/\hat{u}.$$

Therefore, $\widehat{|m|f}$ is continuous on $U$ and $x \in \mathtt{reg}(|m|f)$. That is, we have shown $\partial\mathtt{reg}(\widehat{|m|}) = \mathtt{reg}(\widehat{|m|f})$. At last, we show $\widehat{|m|}f \in \mathcal{F}_0(X)$. By Theorem 67 (1), $\hat{m}f \in \mathcal{F}_0(X)$, that is, for each $\epsilon > 0$, there exists $K \subseteq X$ compact with $|\hat{m}f| \leq \epsilon$ on $X \setminus K$. On $\mathtt{reg}(\hat{m})$ is $|\widehat{|m|}f| = ||\hat{m}|f| = |\hat{m}f|$, and on $\mathtt{reg}_\infty(m)$ is $f \equiv 0$ by Theorem 67 (1), so $\widehat{|m|}f \equiv 0$ there. Hence $|\widehat{|m|}f| \leq \epsilon$ on $X \setminus K$ as well. Using Theorem 67 (1) again, we conclude $f \in \mathcal{D}(t_{|m|})$. □

Note that the polar decomposition $t_m = t_u t_{|m|}$ is true for regular $t_m$, since $\mathtt{reg}_\infty(m)$ is empty in this case. This is related to the fact that $t_u$ is in general not a partial isometry as we have defined them before although $u$ is a phase, because the range of $t_u$ can fail to be orthogonally closed (e.g. if $u$ is the sign function on $\mathbb{R}$). But in the general (non-commutative) case this partial isometry will be needed for the polar decomposition. Even for regular operators further assumptions are needed (see Theorem 119) to construct this decomposition.

Having understood the polar decomposition so far, we study the bounded transform now. Remember, its idea is to connect unboundedness with boundedness via the mapping $z : \mathbb{C} \to \mathbb{C}$ given by

$$z(x) := x(1 + |x|^2)^{-1/2}.$$

So we consider for $m : X \to \mathbb{C}$ with graph regular operator $t_m$ the operator

$$z(t_m) := t_m((1 + |t_m|^2)^{-1})^{1/2}.$$

In chapter 8 we will define the bounded transform of $t_m$ slightly different. But it is also shown that $z(t) := t a_t^{1/2}$ belongs to $\mathcal{C}_o(C_0(X))$. Hence, we are allowed to compute

$$z(t_m) = z(t_m)^{**} = (t_m a_{t_m}^{1/2})^{**} = (t_m t^{1/2}_{\frac{1}{1+|m|^2}})^{**} = t_{\frac{m}{\sqrt{1+|m|^2}}},$$

where Theorem 67 (3b) was used. Whereas for a regular operator $t_m$ - $m \in C(X)$ - this transform is always an adjointable operator (see [Lan95] Theorem 10.4), for graph regular ones $z(t_m)$ can fail to be defined on the hole space.

THEOREM 72. If $m : X \to \mathbb{C}$ is a function such that $t_m$ is graph regular, then

$$\mathcal{D}(z(t_m)) = \{f \in C_0(X) | f \equiv 0 \text{ on } \mathtt{reg}_\infty(m) \cap \mathtt{sing\text{-}supp_r}(u)\}.$$

In particular, $z(t_m) \in \mathcal{L}(C_0(X))$ if and only if

$$\mathtt{reg}_\infty(m) \cap \mathtt{sing\text{-}supp_r}(u) = \emptyset.$$

PROOF. We present only a sketch of proof, because the techniques are exactly the same as in the proof of Theorem 71. We set $n := m/\sqrt{1 + |m|^2}$.

It is $\mathtt{reg}(\hat{m}) \subseteq \mathtt{reg}(\hat{n})$. For $x \in \mathtt{reg}_\infty(m)$, we have two cases. The first: $x \in \mathtt{reg}(\hat{u})$. One proves that this is the case, if and only if $x \in \mathtt{reg}(\hat{n})$. The second: $x \in \mathtt{sing\text{-}supp_r}(u)$. Here, one proves that this is the case, if and only if $x \in \mathtt{sing\text{-}supp_r}(n)$. We summarise this:

$$\mathtt{reg}(\hat{n}) = \mathtt{reg}(\hat{m}) \cup (\mathtt{reg}_\infty(m) \cap \mathtt{reg}(\hat{u})),$$
$$\mathtt{sing\text{-}supp_r}(n) = \mathtt{reg}_\infty(m) \cap \mathtt{sing\text{-}supp_r}(u).$$

Since $n$ is bounded one easily deduces the assertion using Theorem 67 (1). $\square$

EXAMPLE 73. The strong form of the polar decomposition is not valid for the function $m(x) := 1/x$ on $X = \mathbb{R}$, since $\mathtt{reg}_\infty(m) \cap \mathtt{sing\text{-}supp_r}(u) = \{0\}$. The function $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) := x$ for $|x| \leq 1$ and $f(x) := 1/x$ for $|x| > 1$ belongs to $\mathcal{D}(t_m)$ but not to $\mathcal{D}(t_{|m|})$. For the same reason, $z(t_m)$ is not adjointable. It is $m(x)/\sqrt{1 + |m(x)|^2} = \mathtt{sgn}(x)/\sqrt{1 + x^2}$, which can not be identified with some function in $C_b(\mathbb{R})$.

We finish the discussion of the commutative case by transferring the results to unital commutative $C^*$-algebras $\mathcal{C}$ with

$$C_0(X)^\sim \subseteq \mathcal{C} \subseteq C_b(X),$$

where $X$ is not compact and $C_0(X)^\sim$ denotes the unitisation of $C_0(X)$.

It is well known that in this case there exists an compact Hausdorff space $Y$ containing $X$ as open and dense subset such that $\mathcal{C}$ is isometrically $*$-isomorphic to $C(Y)$; the isomorphism from $C(Y)$ onto $\mathcal{C}$ is given by $f \mapsto \iota(f) := f\restriction_X$.[2]

We can apply the theory developed above to $C(Y)$ now. But first we generalise the definition of multiplication operators acting on $\mathcal{C}$ now; in the preceding for the function $m : X \to \mathbb{C}$ the operator $t_m$ acts by definition on $C_0(X)$.

DEFINITION. For a function $m : X \to \mathbb{C}$ let

$$\mathcal{D}(t_m^{\mathcal{C}}) := \{f \in \mathcal{C} | \widehat{mf} \in \mathcal{C}\}, \quad t_m f := \widehat{mf} \quad (f \in \mathcal{D}(t_m)).$$

PROPOSITION 74. Let $n : Y \to \mathbb{C}$ and $m := n\restriction_X$. Then

$$\mathcal{D}(t_m^{\mathcal{C}}) = \{\iota(f) | f \in \mathcal{D}(t_n)\}, \quad t_m \iota(f) = \iota(t_n f) \quad (f \in \mathcal{D}(t_n)).$$

PROOF. Let $g \in \mathcal{D}(t_m) \subseteq \mathcal{C}$, that is $\widehat{mg} \in C(Y)$. Hence there exists an unique $f \in C(Y)$ with $f\restriction_X = g$ and an unique $h \in C(Y)$ with $h\restriction_X = \widehat{mg}$. We have to check that $f \in \mathcal{D}(t_n)$ and $h = \widehat{t_n f}$. But since $nf \equiv mg$ on $X$ and $h$ is already continuous on $Y$ we deduce $h = \widehat{nf} \in C(Y)$.

It remains to prove $\{\iota(f) | f \in \mathcal{D}(t_n)\} \subseteq \mathcal{D}(t_m)$. Assume $f \in \mathcal{D}(t_n) \subseteq C(Y)$, that is $\iota(f) = f\restriction_X \in \mathcal{C}$ and $\widehat{nf} \in C(Y)$. Then $m \cdot (f\restriction_X) = (nf)\restriction_X$. Hence $\widehat{m \cdot \iota(f)} \in \mathcal{C}$, since $\widehat{nf} \in C(Y)$. $\square$

Hence we can not only identify operators on $C(Y)$ with operators on $\mathcal{C}$ but also have a concrete description of the operators domain in $\mathcal{C}$. Proposition 74 can also be applied in the other direction. Remember that the operator $t_n$ on $C(Y)$ is already uniquely given by the values of $n : Y \to \mathbb{C}$ on an open and dense subset of $Y$, for example $X$. That is, if $m : X \to \mathbb{C}$ is given, let $n : Y \to \mathbb{C}$ be any function coinciding with $m$ on $X$. Proposition 74 proves that we can identify $t_m^{\mathcal{C}}$ with $t_n$.

THEOREM 75. Let $n : Y \to \mathbb{C}$ be a function and $m := n\restriction_X$. Then

(1) $t_m^{\mathcal{C}} \in \mathcal{C}_o(\mathcal{C})$ if and only if $\mathtt{reg}(m)$ is dense in $X$.
(2) $t_m^{\mathcal{C}} \in \mathcal{R}_{gr}(\mathcal{C})$ if and only if $\mathtt{reg}(m)$ is dense in $X$ and $\mathtt{sing\text{-}supp_r}(n) = \emptyset$.
(3) $t_m^{\mathcal{C}} \in \mathcal{R}(\mathcal{C})$ if and only if $m \in \mathcal{C}$.

If $t := t_m^{\mathcal{C}} \in \mathcal{R}_{gr}(\mathcal{C})$, then

$$a_t = t^{\mathcal{C}}_{\frac{1}{1+|m|^2}}, \quad b_t = t^{\mathcal{C}}_{\frac{m}{1+|m|^2}}.$$

PROOF. From Proposition 74 it follows that $t_m^{\mathcal{C}}$ belongs to $\mathcal{C}_o(\mathcal{C})$ resp. $\mathcal{R}_{gr}(\mathcal{C})$ resp. $\mathcal{R}(\mathcal{C})$ if and only if $t_n$ belongs to $\mathcal{C}_o(C(Y))$ resp. $\mathcal{R}_{gr}(C(Y))$ resp. $\mathcal{R}(C(Y))$. By Theorem 67 (2) and Theorem 68 (2) the latter conditions are equivalent to $\mathtt{reg}(n)$ is dense in $Y$ resp. $\mathtt{reg}(n)$ is dense in $Y$ and $\mathtt{sing\text{-}supp_r}(n) = \emptyset$ resp.

[2]See [Kha09] Section 1.1 for example.

$\mathtt{reg}(n) = Y$. Since $X$ is open and dense in $Y$ the density of $\mathtt{reg}(m)$ in $X$ is equivalent to that one of $\mathtt{reg}(n)$ in $Y$. Finally, $\mathtt{reg}(n) = X$ if and only if $n \in C(Y)$, that is $m = n \upharpoonright_X \in \mathcal{C}$.

If $t := t_m^{\mathcal{C}} \in \mathcal{R}_{gr}(\mathcal{C})$, then $t$ is identified with $t_n$, hence $a_t$ and $b_t$ are identified with $a_{t_n}$ and $b_{t_n}$ respectively. We have $a_{t_n} = t_{\frac{1}{1+|n|^2}}$ and $b_{t_n} = t_{\frac{n}{1+|n|^2}}$ by Theorem 68. Finally, the latter operators can be identified with $t^{\mathcal{C}}_{\frac{1}{1+|m|^2}}$ and $t^{\mathcal{C}}_{\frac{m}{1+|m|^2}}$ respectively. That is the assertion. $\square$

# Interjection

## Unboundedness and graph regularity

We will see that *cum grano salis* the extension of the theory of regular operator to the graph regular ones is addressed to unbounded operators. This reads inversely: Each graph regular and bounded operators is already regular.

Proposition 76. Let $t \in \mathcal{C}_o(E,F)$ such that $t$ and $t^*$ are bounded. Then $t$ is graph regular if and only if $t$ is regular. In this case $t \in \mathcal{L}(E,F)$.

Proof. Each regular operator is graph regular, hence it remains to prove: If $t$ is graph regular (so $t^*$ is graph regular) and both $t$ and $t^*$ are bounded, then $\mathcal{D}(t) = E$ and $\mathcal{D}(t^*) = F$; this would imply $t \in \mathcal{L}(E,F)$.

Since $t$ and $t^*$ are orthogonally closed they are closed, hence both $\mathcal{D}(t)$ and $\mathcal{D}(t^*)$ are closed. This implies the closeness of $\mathcal{D}(t^*t)$ as well and as the range of the adjointable operator $a_t$ it is even orthogonally complemented by [Lan95] Theorem 3.2. But $\mathcal{R}(a_t)^\perp = \mathcal{N}(a_t)$ is trivial, so the range of $a_t$ is all of $E$. In particular $E = \mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$. Analogously is $\mathcal{D}(t^*) = F$. $\square$

Let me emphasis that both $t$ and $t^*$ are assumed to be bounded. Although no counter example is known to me, the question if one can do without the boundedness of $t^*$ (*grano salis*) remains open. But we know from Lemma 65 that $t^*$ is already bounded provided that the bounded operator $t$ is defined on all of $E$. Hence, we have the following corollary of Proposition 76.

Corollary 77. Let $t \in \mathcal{C}_o(E,F)$ such that $t$ is bounded and $\mathcal{D}(t) = E$. Then $t$ is graph regular if and only if $t \in \mathcal{L}(E,F)$.

# 5. Relation to adjointable operators

## Sources of graph regularity

Graph regularity behaves well with adjointable operators (Proposition 78); in the same way as closed operators behaves well with bounded ones in Hilbert spaces. Quotients of adjointable operators are graph regular, provides they are already closed (Theorem 79 and Theorem 83). A class of closed quotients is given in Corollary 82. We apply Theorem 83 in two cases. One of them prepares the bounded transform, the other prepares the absolute value.

Let $E, F, G, H$ be Hilbert $\mathcal{A}$-modules.

Proposition 78. Let $t \in \mathcal{R}_{gr}(E, F)$ and $q \in \mathcal{L}(E, F)$. Suppose $r \in \mathcal{L}(G, E)$ and $s \in \mathcal{L}(F, G)$ are invertible with $r^{-1} \in \mathcal{L}(E, G)$ and $s^{-1} \in \mathcal{L}(G, F)$ as well. Then the operators $t + q$, $tr$ and $st$ are essentially defined and graph regular.

Proof. We denote by $p_E$ and $p_F$ the projections from $E \oplus F$ onto $E$ and $F$, respectively. Clearly, $t+q$, $tr$ and $st$ are essentially defined and orthogonally closed by Proposition 20. In particular, their graphs are closed. Since $t$ is graph regular, $\mathcal{G}(t)$ is orthogonally complemented, hence there is a projection $p \in \mathcal{L}(E \oplus F)$ with $\mathcal{R}(p) = \mathcal{G}(t)$. We obtain

$$\begin{aligned}
\mathcal{G}(t+q) &= \{(x, tx+qx) | x \in \mathcal{D}(t)\} = \{(p_E pv, p_F pv + q p_E pv) | v \in E \oplus F\} \\
&= \mathcal{R}((p_E, p_F + q p_E)p), \\
\mathcal{G}(tr) &= \{(r^{-1}x, tx) | x \in \mathcal{D}(t)\} = \{(r^{-1} p_E pv, p_F pv) | v \in E \oplus F\} \\
&= \mathcal{R}((r^{-1} p_E, p_F)p), \\
\mathcal{G}(st) &= \{(x, stx) | x \in \mathcal{D}(t)\} = \{(p_E pv, s p_F pv) | v \in E \oplus F\} \\
&= \mathcal{R}((p_E, s p_F)p).
\end{aligned}$$

Thus the closed subspaces $\mathcal{G}(t+q)$, $\mathcal{G}(tr)$, and $\mathcal{G}(st)$ are ranges of adjointable operators, hence they are orthogonally complemented by [Lan95] Theorem 3.2. □

A large class of examples of unbounded graph regular operators can be obtained by quotients of adjointable operators. Following some ideas developed in [Izu89] we get the follows result.

Theorem 79. Let $a \in \mathcal{L}(G, E)$ and $b \in \mathcal{L}(G, F)$. Suppose that $\mathcal{N}(a) \subseteq \mathcal{N}(b)$ and $\mathcal{N}(a^*) = \{0\}$. If the operator $t : E \to F$, which is defined by

$$\mathcal{D}(t) := \mathcal{R}(a), \quad t(ax) := bx \quad (x \in G),$$

is closed, then $t \in \mathcal{R}_{gr}(E, F)$ and $t^* = (a^*)^{-1} b^*$.

Proof. Since $\mathcal{N}(a) \subseteq \mathcal{N}(b)$ and $\mathcal{R}(a)^\perp = \mathcal{N}(a^*) = \{0\}$, $t$ is well-defined and essentially defined. Since the graph of $t$ is $\{(ax, bx) | x \in G\}$, it is the range of the adjointable operator $q : G \to E \oplus F$ defined by $q(x) := (ax, bx)$. By assumption this range is closed, so [Lan95] Theorem 3.2 applies and evinces that this range is orthogonally complemented, hence $t$ is graph regular. The adjoint of $t$ is computed by Proposition 20 (6). □

Corollary 80. Let $x \in \mathcal{L}(F, E)$ and assume that $\mathcal{N}(x) = \mathcal{N}(x^*) = \{0\}$. Then $x^{-1} \in \mathcal{R}_{gr}(F, E)$ and $(x^{-1})^* = (x^*)^{-1}$.

Proof. Since $x^{-1}$ is closed, the assertion follows from Theorem 79 by letting $b$ be the identity on $F$.[3] □

Corollary 81. If $t \in \mathcal{R}_{gr}(E,F)$, then $t^*t \in \mathcal{R}_{gr}(E)$.

Proof. We have $a_t \in \mathcal{L}(E)$, since $t$ is graph regular. But the kernel of the self-adjoint operator $a_t$ is trivial, so $1+t^*t = a_t^{-1}$ is graph regular by Corollary 80. Finally, Proposition 78 implies $t^*t \in \mathcal{R}_{gr}(E)$. □

Corollary 82. Let $a \in \mathcal{L}(E)$ and $p,q \in \mathbb{C}[X]$ be relatively prime. Assume $\mathcal{R}(q(a))$ is essential and $\mathcal{N}(q(a)) \subseteq \mathcal{N}(p(a))$. The operator $t : E \to E$ given by

$$\mathcal{D}(t) := \mathcal{R}(q(a)), \quad t(q(a)x) := p(a)x \quad (x \in E)$$

is graph regular.

Proof. In order to apply Theorem 79 we only have to check that $t$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E$ such that $p(a)x_n \to x_p \in E$ and $q(a)x_n \to x_q \in E$. Since $p$ and $q$ are relatively prime, there are polynomials $\tilde{p},\tilde{q} \in \mathbb{C}[X]$ such that $\tilde{p}p + \tilde{q}q = 1$. To show that $(x_n)_{n\in\mathbb{N}}$ converges we compute

$$x_n = [\tilde{p}(a)p(a) + \tilde{q}(a)q(a)]x_n \to \tilde{p}(a)x_p + \tilde{q}(a)x_q =: x_r \in E,$$

As a result $x_p = p(a)x_r$ and $x_q = q(a)x_r$, which implies the closeness of $t$. □

Theorem 83. Let $a \in \mathcal{L}(G,E)$, $b \in \mathcal{L}(G,F)$, $a_* \in \mathcal{L}(H,F)$ and $b_* \in \mathcal{L}(H,E)$ such that $b^*a_* = a^*b_*$. If the kernels of $a^*$ and $a_*^*$ are trivial, then $\mathcal{N}(a) \subseteq \mathcal{N}(b)$ and $\mathcal{N}(a_*) \subseteq \mathcal{N}(b_*)$. The operators $t : E \to F$ and $t' : F \to E$ given by

$$\mathcal{D}(t) := \mathcal{R}(a), \quad t(ax) := bx \quad (x \in G),$$
$$\mathcal{D}(t') := \mathcal{R}(a_*), \quad t'(a_*y) := b_*y \quad (y \in H),$$

are essentially defined, orthogonally closable and they satisfy $(t')^{**} \subseteq t^*$, $t^{**} \subseteq (t')^*$. If in addition $t$ and $t'$ are closed and $ab^* = b_*a_*^*$, then $t \in \mathcal{R}_{gr}(E,F)$ and $t^* = t'$.

Proof. Suppose that $a_*x = 0$ for some $x \in H$. Then $0 = b^*a_*a = a^*b_*x$, hence $b_*x = 0$, since $a^*$ is injective; this shows $\mathcal{N}(a_*) \subseteq \mathcal{N}(b_*)$. Analogously, the injectivity of $a_*^*$ implies $\mathcal{N}(a) \subseteq \mathcal{N}(b)$. Thus the operators $t$ and $t'$ are well-defined. Since the ranges of $a$ and $a_*$ are essential, $t$ and $t'$ are essentially defined.

Comparing $(a^*)^{-1}b^*a_* = b_*$ and $t^* = (a^*)^{-1}b^*$ (see Theorem 79), we get $t' \subseteq t^*$. Hence $t^*$ is essentially defined, since $t'$ is. Therefore, $t$ is orthogonally closable. Interchanging the roles of $t$ and $t'$, we conclude that $t \subseteq (t')^*$ and $t'$ is orthogonally closable. Applying the involution to the relations $t \subseteq (t')^*$ and $t' \subseteq t^*$, we obtain $(t')^{**} \subseteq t^*$ and $t^{**} \subseteq (t')^*$ which proves the first half.

Now additionally suppose that $t$ and $t'$ are closed and $ab^* = b_*a_*^*$. We could directly conclude the graph regularity of $t$ and $t'$ by Theorem 79, but this will follow from the stronger statement $t' = t^*$ which we prove now. Since $t$ and $t'$ are closed,

$$\mathcal{G} := \mathcal{G}(t) \oplus v\mathcal{G}(t') = \{(ax + b_*y, bx - a_*y) | x \in E, y \in F\}$$

is a closed submodule of $E \oplus F$. We define $q(x,y) := (ax + b_*y, bx - a_*y)$ for $(a,b) \in E \oplus F$. Then $q \in \mathcal{L}(E \oplus F)$ and $\mathcal{R}(q) = \mathcal{G}$. By [Lan95] Theorem 3.2, $\mathcal{G}$ is orthogonally complemented. A simple calculation shows that

$$q^*(x',y') = (a^*x' + b^*y', b_*^*x' - a_*^*y') \quad \text{for} \quad (x',y') \in E \oplus F.$$

The kernel of $q^*$ is trivial: Suppose that $q^*(x',y') = 0$. Then $a^*x' + b^*y' = 0$ and $a_*^*y' - b_*^*x' = 0$ and we obtain

$$\langle a^*x', a^*x'\rangle + \langle a_*^*y', a_*^*y'\rangle = -\langle b^*y', a^*x'\rangle + \langle a_*^*y', b_*^*x'\rangle = \langle (-ab^* + b_*a_*^*)y', x'\rangle = 0.$$

[3] Corollary 80 is also a corollary to Proposition 58.

Thus $a^*x' = 0$ and $a_*^*y' = 0$, $x' = 0$ and $y' = 0$, since $a^*$ and $a_*^*$ are injective. That is, $\mathcal{N}(q^*) = \{0\}$. Hence, $\mathcal{G}^\perp = \mathcal{R}(q)^\perp = \mathcal{N}(q^*) = \{0\}$. We conclude that $\mathcal{G}$ is all of $E \oplus F$, it is orthogonally complemented and essential. This proves $t' = t^*$ and the graph regularity of $t$ and at the same time that one of $t' = t^*$. □

In preparation for the (inverse of the) bounded transform we have

EXAMPLE 84. Let $z \in \mathcal{L}(E, F)$ with $\|z\| \leq 1$. Consider Theorem 83 with

$$a := (1 - z^*z)^{1/2}, \quad b := z, \quad a_* := (1 - zz^*)^{1/2}, \quad b_* := z^*.$$

Then $b^*a_* = a^*b_*$ and $ab^* = b_*a_*^*$. If $\mathcal{N}(1 - z^*z) = \mathcal{N}(1 - zz^*) = \{0\}$, then $\mathcal{N}(a^*) = \mathcal{N}(a_*^*) = \{0\}$ and an easy computation shows that $t = z[(1 - z^*z)^{1/2}]^{-1}$ and $t' = z^*[(1 - zz^*)^{1/2}]^{-1}$ are closed: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E$ with $ax_n \to x_a \in E$ and $bx_n \to x_b \in E$. It follows

$$x_n = (1 - z^*z)x_n + z^*zx_n \to (1 - z^*z)^{1/2}x_a + z^*x_b,$$

so $t$ is closed by the continuity of $a$ and $b$. Analogously $t'$ is closed. Hence, $t$ and $t'$ are graph regular with $t' = t^*$:

$$[z(1 - z^*z)^{-1/2}]^* = z^*(1 - zz^*)^{-1/2}.$$

Another example prepares the absolute value for graph regular operators.

EXAMPLE 85. Let $c \in \mathcal{L}(E)$ be positive, injective and of norm less or equal 1. Setting

$$a = a_* := c^{1/2}, \quad b = b_* := (1 - c)^{1/2},$$

we clearly have $b^*a_* = a^*b_*$ and $ab^* = b_*a_*^*$. The operator $t = t' = (1-c)^{1/2}(c^{1/2})^{-1}$ is closed: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E$ with $ax_n \to x_a \in E$ and $bx_n \to x_b \in E$. It follows

$$x_n = cx_n + (1 - c)x_n \to c^{1/2}x_a + (1 - c)^{1/2}x_b,$$

hence $t$ is closed, by the continuity of $a$ and $b$. By Theorem 83 $t = (1-c)^{1/2}c^{-1/2}$ is graph regular and self-adjoint. The domain of $t$ is essential: $\mathcal{D}(t)^\perp = \mathcal{R}(c^{1/2})^\perp = \mathcal{N}(c^{1/2}) = \{0\}$. Hence $t \in \mathcal{R}_{gr}(E)$.

# 6. Concrete $C^*$-algebras

## Association relation and affiliation relation

In two steps we connect the abstract case of general $C^*$-algebra (and Hilbert $C^*$-modules over them) with the concrete situation of (non-degenerated) $C^*$-algebras realised on Hilbert spaces. After investigating the possibility of extending an homomorphism between the $C^*$-algebras $\mathcal{L}(E)$ and $\mathcal{L}(F)$ of adjointable operators of Hilbert $C^*$-modules $E$ and $F$ to the set of graph regular operators (Proposition 86), we specialize to the case where $E = \mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$ is the algebra itself and $F = \mathcal{H}$ is the corresponding Hilbert space on which the $C^*$-algebra is realized (Corollary 87). We define and study the association relation ($T\mu\mathcal{A}$) and relate it to the affiliation relation ($T\eta\mathcal{A}$) in Proposition 91. In Proposition 93 and Proposition 94 we give criteria for $T\mu\mathcal{A}$; in Theorem 95 an resolvent criterion for $T\eta\mathcal{A}$ is given. A number of corollaries is given concerning addition and composition affiliated operators. The most notable may be Theorem 100: The sum $T+S$ considered in the Kato-Rellich theorem ($S$ is $T$-bounded) is affiliated with $\mathcal{A}$ if $T$ is affiliated and $S(T-\lambda)^{-1}$ belongs to $\mathtt{M}(\mathcal{A})$ for some $\lambda \in \rho(T)$.

Proposition 86. Suppose that $t \in \mathcal{R}_{gr}(E)$ and $\phi \in \mathtt{Hom}(\mathcal{L}(E), \mathcal{L}(F))$. Further suppose that the kernels of $\phi(a_t)$ and $\phi(a_{t^*})$ are trivial. Then there exists an operator $\phi(t) \in \mathcal{R}_{gr}(F)$ such that $\phi(a_t)F$ is an essential core for $\phi(t)$ and

$$\phi(t)(\phi(a_t)x) = \phi(b_t)x \quad (x \in F).$$

Moreover, $\phi(t)^* = \phi(t^*)$, and $a_{\phi(t)} = \phi(a_t)$, $a_{\phi(t)^*} = \phi(a_{t^*})$ and $b_{\phi(t)} = \phi(b_t)$.

Proof. Clearly, $0 \leq \phi(a_t) \leq 1$, $0 \leq \phi(a_{t^*}) \leq 1$, and

$$\phi(b_t)^*\phi(b_t) = \phi(a_t) - \phi(a_t)^2, \quad \phi(b_t)\phi(b_t)^* = \phi(a_{t^*}) - \phi(a_{t^*})^2,$$
$$\phi(a_t)\phi(b_t)^* = \phi(b_t)^*\phi(a_{t^*}),$$

since $\phi$ is a $*$-homomorphism. Hence $(\phi(a_t), \phi(a_{t^*}), \phi(b_t)) \in \mathcal{AB}(F)$ and all statements follow from Theorem 56. $\square$

Clearly, $\mathcal{L}(\mathcal{H}) = \mathbf{B}(\mathcal{H})$. Further, $\mathtt{M}(\mathcal{A})$ can be identified with $\mathcal{L}(\mathcal{A})$ via $a \mapsto L_a$ for $a \in \mathtt{M}(\mathcal{A})$, where $L_a(b) := ab$ for $b \in \mathcal{A}$ (see [Lan95] Chapter 2).

Let us further note that $\mathcal{R}_{gr}(\mathcal{H}) = \mathcal{C}(\mathcal{H})$ for the Hilbert space $\mathcal{H}$, since each closed subspace of $\mathcal{H}$ is orthogonally complemented.

Corollary 87. Let $\mathcal{A}$ be a non-degenerated concrete $C^*$-algebra on $\mathcal{H}$. Let $\phi$ be the embedding of $\mathcal{L}(\mathcal{A}) = \mathtt{M}(\mathcal{A})$ into $\mathcal{L}(\mathcal{H}) = \mathbf{B}(\mathcal{H})$.

(1) For any $T \in \mathcal{C}(\mathcal{H})$ with $a_T, a_{T^*}, b_T \in \mathtt{M}(\mathcal{A})$ there exists a unique operator $t \in \mathcal{R}_{gr}(\mathcal{A})$ such that $\phi(t) = T$.

(2) If $\mathcal{A}$ contains the compact operators, then we have $T := \phi(t) \in \mathcal{C}(\mathcal{H})$ and $a_T, a_{T^*}, b_T \in \mathtt{M}(\mathcal{A})$ for $t \in \mathcal{R}_{gr}(\mathcal{A})$. In particular, $\mathcal{R}_{gr}(\mathcal{A})$ can be identified with those $T \in \mathcal{C}(\mathcal{H})$ for which $a_T, a_{T^*}, b_T \in \mathtt{M}(\mathcal{A})$.

Proof. (1): Since $\mathcal{C}(\mathcal{H}) = \mathcal{R}_{gr}(\mathcal{H})$, we have $(a_T, a_{T^*}, b_T) \in \mathcal{AB}(\mathcal{H})$ by Theorem 56. By assumption, $a_T, a_{T^*}$, and $b_T$ are elements of $\mathtt{M}(\mathcal{A})$. To show that $(a_T, a_{T^*}, b_T) \in \mathcal{AB}(\mathcal{A})$ it suffices to prove that $a_T$ and $a_{T^*}$ are injective as operators on $\mathcal{A}$; they are injective as operators on $\mathcal{H}$. Assume that $a_T a = 0$ for some

$a \in \mathcal{A}$. Then, $a_T a\xi = 0$ for $\xi \in \mathcal{H}$. Hence $a\xi = 0$ for all $\xi \in \mathcal{H}$, so $a = 0$. Thus $a_T$ is injective on $\mathcal{A}$. Analogously, $a_{T^*}$ is injective on $\mathcal{A}$. Using once more Theorem 56, it follows that there exists an operator $t \in \mathcal{R}_{gr}(\mathcal{A})$ such that $a_t = a_T$, $a_{t^*} = a_{T^*}$, $b_t = b_T$. Further, $\phi(t) = T$, since

$$T(\phi(a_t)\xi) = T(a_t\xi) = Ta_T\xi = b_T\xi = b_t\xi = \phi(b_t)\xi = \phi(t)(\phi(a_t)\xi) \quad (\xi \in \mathcal{H}),$$

and $\mathcal{R}(\phi(a_t)) = \mathcal{R}(a_T) = \mathcal{D}(T^*T)$ is a core for $T$.

(2): If $t \in \mathcal{R}_{gr}(\mathcal{A})$, then $(a_t, a_{t^*}, b_t) \in \mathcal{AB}(\mathcal{A})$. We show that the kernels of $\phi(a_t)$ and $\phi(a_{t^*})$ are trivial. Assume that $\phi(a_t)\xi = 0$ for some nonzero vector $\xi \in \mathcal{H}$. Since $\mathcal{A}$ contains all compact operators, the rank one projection $p_\xi$ onto $\mathbb{C}\cdot\xi$ is in $\mathcal{A}$. Therefore, $a_t p_\xi = \phi(a_t)p_\xi = 0$ which contradicts the injectivity of $a_t$ as operator on $\mathcal{A}$. Hence $a_t$, analogously $a_{t^*}$, is injective on $\mathcal{H}$. Therefore, $T \in \mathcal{R}_{gr}(\mathcal{H}) = \mathcal{C}(\mathcal{H})$ by Proposition 86. □

Example 88. From Corollary 87 it follows immediately that

$$\mathcal{R}_{gr}(\mathcal{K}(\mathcal{H})) = \mathcal{C}(\mathcal{H}) \quad \text{and} \quad \mathcal{R}_{gr}(\mathbf{B}(\mathcal{H})) = \mathcal{C}(\mathcal{H}),$$

since $\mathtt{M}(\mathcal{K}(\mathcal{H})) = \mathtt{M}(\mathbf{B}(\mathcal{H})) = \mathbf{B}(\mathcal{H})$.

On $\mathcal{K}(\mathcal{H})$ each essential submodule is dense, since each closed submodule is orthogonally complemented (see [Mag97] or [Sch99]). Therefore, all graph regular operators are regular. On the other hand, dense submodules (right ideals) of the unital $C^*$-algebra $\mathbf{B}(\mathcal{H})$ are already all of $\mathbf{B}(\mathcal{H})$, so each regular operator on $\mathbf{B}(\mathcal{H})$ is bounded by the closed graph theorem.

$$\mathcal{R}(\mathcal{K}(\mathcal{H})) = \mathcal{C}(\mathcal{H}) \quad \text{and} \quad \mathcal{R}(\mathbf{B}(\mathcal{H})) = \mathbf{B}(\mathcal{H}).$$

Throughout the remaining chapter we assume that the Hilbert $\mathcal{A}$-module $E$ is the $C^*$-algebra $\mathcal{A}$ itself equipped with the $\mathcal{A}$-valued scalar product $\langle a, b\rangle := a^*b$, $a, b \in \mathcal{A}$, and that $\mathcal{A}$ is realized as a non-degenerate $C^*$-algebra on a Hilbert space $\mathcal{H}$. Then $\mathcal{L}(E)$ is the multiplier algebra $\mathtt{M}(\mathcal{A}) = \{x \in \mathbf{B}(\mathcal{H}) : x\mathcal{A} \subseteq \mathcal{A}, \mathcal{A}x \subseteq \mathcal{A}\}$.

Corollary 87 gives rise to the following

Definition. An operator $T \in \mathcal{C}(\mathcal{H})$ is *associated with* $\mathcal{A}$ and we write $T\mu\mathcal{A}$, if $a_T, a_{T^*}, b_T \in \mathtt{M}(\mathcal{A})$. The set of associated operators with $\mathcal{A}$ is denoted by $\mathcal{A}^\mu$.

With this notion Corollary 87 (2) reads as: If $\mathcal{A}$ contains the compact operators, then each $t \in \mathcal{R}_{gr}(\mathcal{A})$ acting on the algebra can be identified with a $T\mu\mathcal{A}$ acting on the Hilbert space and vice versa. For the rest of this chapter we study the association relation $\mu$ - the counter part of graph regularity for concrete $C^*$-algebras; by Corollary 87 (1) this is more particular case.

Note that $T = \overline{T \restriction_{\mathcal{D}(T^*T)}} = \overline{b_T a_T^{-1}}$ for each $T \in \mathcal{C}(\mathcal{H})$, and we always have that $a_T$ and $a_{T^*}$ are self-adjoint, $b_{T^*} = b_T^*$ and

$$b_T a_T = a_{T^*} b_T, \qquad a_T - a_T^2 = b_{T^*} b_T, \qquad a_{T^*} - a_{T^*}^2 = b_T b_{T^*},$$

by Corollary 47 (with $E = F = \mathcal{H}$).[4]

Let us repeat the concept of the affiliation relation $\eta$. From [Wor91] Example 4, that is, for a concrete (non-degenerated) $C^*$-algebra $\mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$, we get: An operator $T \in \mathcal{C}(\mathcal{H})$ is *affiliated to* $\mathcal{A}$, $T\eta\mathcal{A}$, if and only if

$$z_T := T(I + T^*T)^{-1/2} \in \mathtt{M}(\mathcal{A}) \quad \text{and} \quad (I + T^*T)^{-1/2}\mathcal{A} \text{ is dense in } \mathcal{A}.$$

By the second section in [Wor91] the affiliated operators are shown to be the regular ones on the $C^*$-algebra by the graph criterion (see [Lan95] Theorem 9.3 and Proposition 9.5).

[4] Since each closed subspace of a Hilbert space is orthogonally complemented, $\mathcal{C}_o(\mathcal{H}) = \mathcal{R}(\mathcal{H}) = \mathcal{C}(\mathcal{H})$. That is, $a_T, a_{T^*}$ and $b_T$ are defined on all of $\mathcal{H}$.

We use two auxiliary lemmas to study the relationship between the affiliation and the association relation.

LEMMA 89. Let $s \in \mathbf{B}(\mathcal{H})$ and $x, y \in \mathtt{M}(\mathcal{A})$. Suppose that $x\mathcal{A}$ and $y\mathcal{A}$ are dense in $\mathcal{A}$. If $sx \in \mathtt{M}(\mathcal{A})$ and $s^*y \in \mathtt{M}(\mathcal{A})$, then $s \in \mathtt{M}(\mathcal{A})$.

PROOF. Let $a \in \mathcal{A}$. Since $x\mathcal{A}$ is dense in $\mathcal{A}$, there are elements $a_n \in \mathcal{A}$, $n \in N$, such that $xa_n \to a$ in $\mathcal{A}$. Hence $sxa_n \to sa$ in $\mathcal{A}$. Since $sx \in \mathtt{M}(\mathcal{A})$ by assumption, $sxa_n \in \mathcal{A}$ and so $sa \in \mathcal{A}$. Replacing $x$ by $y$ and $s$ by $s^*$ it follows that $s^*a \in \mathcal{A}$. Therefore, $s \in \mathtt{M}(\mathcal{A})$. $\square$

LEMMA 90. Let $\mathcal{A}$ be a $C^*$-algebra and $x, y \in \mathtt{M}(\mathcal{A})$. Suppose that $\lambda y \geq xx^*$ for some $\lambda > 0$. If $x\mathcal{A}$ is dense in $\mathcal{A}$, so is $y\mathcal{A}$. In particular, $x\mathcal{A}$ is dense in $\mathcal{A}$ if and only if $xx^*\mathcal{A}$ is.

PROOF. Assume to the contrary that $\overline{y\mathcal{A}} \neq \overline{x\mathcal{A}} = \mathcal{A}$. Then the closure of $(y\mathcal{A})^*$ is a proper left ideal. Hence there exists a state $\omega$ of $\mathcal{A}$ that annihilates $(y\mathcal{A})^*$ (see [Dix77] Lemma 2.9.4). Let $\pi_\omega$ be the GNS representation of $\mathcal{A}$ associated with the state $\omega$ and let $\varphi_\omega$ be the corresponding cyclic vector $\varphi_\omega$. We denote the extension of $\pi_\omega$ to the multiplier algebra $\mathtt{M}(\mathcal{A})$ also by the symbol $\pi_\omega$. Then

$$0 = \omega((ya)^*) = \langle \pi_\omega(a^*y)\varphi_\omega, \varphi_\omega \rangle = \langle \pi_\omega(y)\varphi_\omega, \pi_\omega(a)\varphi_\omega \rangle \tag{1}$$

for all $a \in \mathcal{A}$, so that $\pi_\omega(y)\varphi_\omega = 0$. Therefore,

$$\begin{aligned} |\omega(xa)|^2 &= |\langle \pi_\omega(a)\varphi_\omega, \pi_\omega(x^*)\varphi_\omega \rangle|^2 \leq \|\pi_\omega(a)\varphi_\omega\|^2 \|\pi_\omega(x^*)\varphi_\omega\|^2 \\ &= \|\pi_\omega(a)\varphi_\omega\|^2 \langle \pi_\omega(xx^*)\varphi_\omega, \varphi_\omega \rangle \leq \|\pi_\omega(a)\varphi_\omega\|^2 \lambda \langle \pi_\omega(y)\varphi_\omega, \varphi_\omega \rangle = 0 \end{aligned}$$

for $a \in \mathcal{A}$, that is, $\omega$ annihilates $x\mathcal{A}$. Hence $x\mathcal{A}$ is not dense in $\mathcal{A}$ which is a contradiction, since we assumed that $\overline{x\mathcal{A}} = \mathcal{A}$.

Applying this to the case $y = xx^*$ we conclude that $xx^*\mathcal{A}$ is dense provided that $x\mathcal{A}$ is dense. Since the converse implication is trivial, it follows that $xx^*\mathcal{A}$ is dense if and only if $x\mathcal{A}$ is dense. $\square$

PROPOSITION 91. If $T\eta\mathcal{A}$, then $T\mu\mathcal{A}$. For $T\mu\mathcal{A}$ are equivalent:

(1) $T\eta\mathcal{A}$,
(2) $a_T\mathcal{A} = (I + T^*T)^{-1}\mathcal{A}$ and $a_{T^*}\mathcal{A} = (I + TT^*)^{-1}\mathcal{A}$ are dense in $\mathcal{A}$,
(3) $a_T^{1/2}\mathcal{A} = (I + T^*T)^{-1/2}\mathcal{A}$ and $a_{T^*}^{1/2}\mathcal{A} = (I + TT^*)^{-1/2}\mathcal{A}$ are dense in $\mathcal{A}$.

PROOF. Let $T\eta\mathcal{A}$. From $a_T = I - z_T^* z_T \in \mathtt{M}(\mathcal{A})$, $a_{T^*} = I - z_T z_T^* \in \mathtt{M}(\mathcal{A})$ and $b_T = z_T a_T^{1/2} \in \mathtt{M}(\mathcal{A})$ we deduce $T\mu\mathcal{A}$.

Now let $T\mu\mathcal{A}$, that is, $a_T, a_{T^*}, b_T \in \mathtt{M}(\mathcal{A})$. Clearly, (3) is equivalent to (2) by Lemma 90. (1) $\Rightarrow$ (3): By [Wor91] Theorem 1.4 we also have $T^*\eta\mathcal{A}$ , so both density conditions are fulfilled. (3) $\Rightarrow$ (1): Using Lemma 89 with $s = z_T = T(I + T^*T)^{1/2}$, $x = a_T^{1/2} \in \mathtt{M}(\mathcal{A})$ and $y = a_{T^*}^{1/2} \in \mathtt{M}(\mathcal{A})$, we get $sx = z_T a_T^{1/2} = b_T \in \mathtt{M}(\mathcal{A})$ and $s^*y = z_{T^*} a_{T^*}^{1/2} = b_{T^*} \in \mathtt{M}(\mathcal{A})$. By (3) all assumption of Lemma 90 are fulfilled and $z_T = s \in \mathtt{M}(\mathcal{A})$ follows. Hence $T\eta\mathcal{A}$. $\square$

In Lemma 123 we give an example for an operator $T\mu\mathcal{A}$ fulfilling only one of the density conditions and still is not affiliated.

The following proposition is the counter part of Proposition 76 in the concrete situation.

PROPOSITION 92. $\mathtt{M}(\mathcal{A}) = \{T\mu\mathcal{A} | T \in \mathbf{B}(\mathcal{H})\}$.

PROOF. If $T \in \mathtt{M}(\mathcal{A})$, then $I + T^*T \in \mathtt{M}(\mathcal{A})$, so $a_T \in \mathtt{M}(\mathcal{A})$ and $b_T = Ta_T \in \mathtt{M}(\mathcal{A})$, hence $T\mu\mathcal{A}$. Conversely, suppose that $T$ is bounded. Then $I + T^*T$ is bounded and $a_T \in \mathtt{M}(\mathcal{A})$, hence $a_T^{-1} = I + T^*T \in \mathtt{M}(\mathcal{A})$. Therefore, $T = b_T a_T^{-1} \in \mathtt{M}(\mathcal{A})$. $\square$

It is natural to ask whether or not $a_T \in \mathtt{M}(\mathcal{A})$ and $b_T \in \mathtt{M}(\mathcal{A})$ already imply that $T\mu\mathcal{A}$, that is, $a_{T^*} \in \mathtt{M}(\mathcal{A})$. This is true if $T \in \mathcal{C}(\mathcal{H})$ is normal, since then $a_T = a_{T^*}$. Proposition 93 below contains an number of other sufficient conditions. In Example Example 122 we will show that this is not true in general.

Proposition 93. Suppose that $a_T, b_T \in \mathtt{M}(\mathcal{A})$. Each of the following conditions imply that $a_{T^*} \in \mathtt{M}(\mathcal{A})$ and so $T\mu\mathcal{A}$.

(1) $0 \in \rho(T)$.
(2) $\|a_{T^*}\| < 1$, or equivalently, $TT^* \geq \varepsilon$ for some $\epsilon > 0$.
(3) $\mathtt{M}(\mathcal{A})_h$ is closed under strong convergence of monotone sequences.
(4) $TT^* = qT^*T$ for some $q > 0$.

Proof. Clearly, from (1) it follows that $0 \in \rho(T^*)$ which in turn implies (2).

(2), (3): By $a_T, b_T \in \mathtt{M}(\mathcal{A})$ we have

$$a_{T^*} - a_{T^*}^{n+1} = b_T(I + \ldots + a_T^{n-1})b_T^* \in \mathtt{M}(\mathcal{A})_h.$$

If (2) is fulfilled, then $a_{T^*}^{n+1} \to 0$ in $\mathtt{M}(\mathcal{A})$, hence $a_{T^*} \in \mathtt{M}(\mathcal{A})$. On the other side, $a_{T^*}^{n+1} \in \mathtt{M}(\mathcal{A})_h$ is monotone decreasing and strongly converging. Hence, by assumption (3) it follows $a_{T^*} \in \mathtt{M}(\mathcal{A})$, again. (4): This follows from the relations

$$a_{T^*} = (I + TT^*)^{-1} = (I + qT^*T)^{-1} = q^{-1}(I + (q^{-1} - 1)a_T)^{-1}a_T \in \mathtt{M}(\mathcal{A}).$$

Since $\|a_T\| \leq 1$, $\|(q^{-1} - 1)a_T\| \leq (q - 1)/q < 1$ for $q \geq 1$, so $I + (q^{-1} - 1)a_t$ is indeed invertible in $\mathtt{M}(\mathcal{A})$. For $q < 1$, $I + (q^{-1} - 1)a_T \geq I$. Hence, $I + (q^{-1} - 1)a_t$ is invertible, again. □

The next two theorems are concerned with resolvents - the first one for the association relation, the second for the affiliation relation.

Proposition 94. Suppose that $T \in \mathcal{C}(\mathcal{H})$ and $0 \in \rho(T)$. Then $T\mu\mathcal{A}$ if and only if $T^{-1} \in \mathtt{M}(\mathcal{A})$.

Proof. Since $0 \in \rho(T)$, $(T^*)^{-1} = (T^{-1})^* \in \mathbf{B}(\mathcal{H})$. Simple computations show

$$I\text{-}a_T = (I{+}T^{-1}(T^{-1})^*)^{-1}, \quad I\text{-}a_{T^*} = (I{+}(T^{-1})^*T^{-1})^{-1}, \quad b_T = (T^{-1})^*(I\text{-}a_T).$$

We conclude from these identities that $T^{-1} \in \mathtt{M}(\mathcal{A})$, so $(T^{-1})^* \in \mathtt{M}(\mathcal{A})$, and this implies $a_T, b_T, a_{T^*} \in \mathtt{M}(\mathcal{A})$, that is, $T\mu\mathcal{A}$.

Conversely, suppose that $T\mu\mathcal{A}$. We have $b_{T^*} = (b_T)^* \in \mathtt{M}(\mathcal{A})$ and $a_{T^*} \in \mathtt{M}(\mathcal{A})$. Since $0 \in \rho(T)$, $\|a_{T^*}\| < 1$. Therefore, $T^{-1} = b_{T^*}(I - a_{T^*})^{-1} \in \mathtt{M}(\mathcal{A})$. □

The following theorem appeared in [Sch05].

Theorem 95. Suppose $T \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \rho(T)$. Then $T\eta\mathcal{A}$ if and only if $(T - \lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $(T - \lambda)^{-1}\mathcal{A}$ and $(T^* - \overline{\lambda})^{-1}\mathcal{A}$ are dense in $\mathcal{A}$.

Proof. Since $T\eta\mathcal{A}$ is equivalent to $(T - \lambda)\eta\mathcal{A}$ (see [Wor91] p. 412, Example 1), we can assume without restriction of generality that $\lambda = 0$. Then $t^{-1}$ and $(t^*)^{-1}$ are in $\mathbf{B}(\mathcal{H})$.

First we suppose that $T\eta\mathcal{A}$. Set $x := (I{+}(TT^*)^{-1})^{-1}$ and $s := T^{-1}$. Since $T\eta\mathcal{A}$ implies $T^*\eta\mathcal{A}$ and $z_T^* = z_{T^*}$ (see [Wor91] Theorem 1.4), $z_{T^*} = (z_T)^* \in \mathtt{M}(\mathcal{A})$ follows. Therefore, we obtain $(I + TT^*)^{-1} = I - z_T z_T^* \in \mathtt{M}(\mathcal{A})$, hence $(I + TT^*)^{-1/2} \in \mathtt{M}(\mathcal{A})$. These relations imply that

$$\begin{aligned} sx &= T^{-1}(I + (TT^*)^{-1})^{-1} = T^*(TT^*)^{-1}(I + (TT^*)^{-1})^{-1} \\ &= T^*(I + TT^*)^{-1} = z_{T^*}(I + TT^*)^{-1/2} \in \mathtt{M}(\mathcal{A}). \end{aligned}$$

Since $x := (I + (TT^*)^{-1})^{-1} = I - (I + TT^*)^{-1} \in \mathtt{M}(\mathcal{A})$ and $x^{-1}$ is also bounded, we have $x^{-1} \in \mathtt{M}(\mathcal{A})$ and hence $x\mathcal{A} = \mathcal{A}$. Recall that $sx \in \mathtt{M}(\mathcal{A})$. Now we interchange the roles of $T$ and $T^*$ and set $y := (I + (T^*T)^{-1})^{-1}$. By an analogue reasoning as above we derive $s^*y \in \mathtt{M}(\mathcal{A})$; further, $y \in \mathtt{M}(\mathcal{A})$ and $y\mathcal{A} = \mathcal{A}$. Hence the assumptions of Lemma 89 are satisfied, so we obtain $T^{-1} = s \in \mathtt{M}(\mathcal{A})$.

Recall that $(I + T^*T)^{-1}\mathcal{A}$ is dense in $\mathcal{A}$, because $T\eta\mathcal{A}$. Therefore, since

$$(I+T^*T)^{-1}\mathcal{A} = (T^*T)^{-1}(I+(T^*T)^{-1})^{-1}\mathcal{A} \subseteq (T^*T)^{-1}\mathcal{A} = T^{-1}(T^*)^{-1}\mathcal{A} \subseteq T^{-1}\mathcal{A},$$

$T^{-1}\mathcal{A}$ is dense in $\mathcal{A}$. Replacing $T$ by $T^*$, it follows that $(T^*)^{-1}\mathcal{A}$ is dense in $\mathcal{A}$. This completes the proof of the only if part.

Conversely, let us assume that $T^{-1} \in \mathtt{M}(\mathcal{A})$ and that $T^{-1}\mathcal{A}$ and $(T^*)^{-1}\mathcal{A}$ are dense in $\mathcal{A}$. Then

$$I - z_T^* z_T = (I + T^*T)^{-1} = T^{-1}(T^{-1})^*(I + T^{-1}(T^{-1})^*)^{-1} \in \mathtt{M}(\mathcal{A}),$$
$$z_T(I - z_T^* z_T)^{1/2} = T(I + T^*T)^{-1} = (T^{-1})^*(I + T^{-1}(T^{-1})^*)^{-1} \in \mathtt{M}(\mathcal{A}).$$

Therefore, setting $x := (I - z_T^* z_T)^{1/2}$ and $s := z_T$, we have $x \in \mathtt{M}(\mathcal{A})$ and $sx \in \mathtt{M}(\mathcal{A})$. Since $T$ has a bounded inverse, there exists $\epsilon \in (0, 1/4)$ such that $T^*T \geq 2\epsilon I$. Then $I+T^*T \leq \frac{1}{2\epsilon}T^*T+T^*T \leq \frac{1}{\epsilon}T^*T$ and hence $(I+T^*T)^{-1} \geq \epsilon T^{-1}(T^{-1})^*$. Therefore, since $T^{-1}\mathcal{A}$ is dense in $\mathcal{A}$ by assumption, $(I + T^*T)^{-1}\mathcal{A} = (I - z_T^* z_T)\mathcal{A} = x^2\mathcal{A}$ is dense in $\mathcal{A}$ by Lemma 90. Since $x \geq 0$, $x\mathcal{A}$ dense in $\mathcal{A}$ again by Lemma 90. By the assumptions we can interchange the roles of $T$ and $T^*$. Then we obtain $y := (I - z_T z_T^*)^{1/2} \in \mathtt{M}(\mathcal{A})$ and $s^*y = z_T^* y \in \mathtt{M}(\mathcal{A})$. Further, $(I + TT^*)^{-1}\mathcal{A} = (I - z_T z_T^*)\mathcal{A} = y^2\mathcal{A}$ in $\mathcal{A}$ and hence $y\mathcal{A}$ are dense in $\mathcal{A}$. Thus $z_T \in \mathtt{M}(\mathcal{A})$ by Lemma 89 and hence $T\eta\mathcal{A}$. □

The operator $T$ constructed in Lemma 123 will give a counter example again, this time for the following: Although $T^{-1} \in \mathtt{M}(\mathcal{A})$, that is, $T\mu\mathcal{A}$ by Proposition 94, and $T^{-1}\mathcal{A}$ is dense in the algebra, $T$ is not affiliated. It fails to fulfil the density of $(T^*)^{-1}\mathcal{A}$ in $\mathcal{A}$.

The preceding theorem has a number of interesting corollaries. All of them are concerned with sums and product of affiliated operators. From [Wor91] Section 2, Example 1-3 we know: If $T\eta\mathcal{A}$, $Q, V \in \mathtt{M}(\mathcal{A})$ with $V^{-1} \in \mathtt{M}(\mathcal{A})$, then

$$(T + Q)\eta\mathcal{A}, \quad TV\eta\mathcal{A}, \quad VT\eta\mathcal{A}.$$

Corollary 96. Suppose that $T, S\eta\mathcal{A}$, $\lambda \in \rho(T)$ and $\kappa \in \rho(S)$. Then we have $-\lambda\kappa \in \rho(TS - \lambda S - \mu T)$ and $(TS - \lambda S - \kappa T)\eta\mathcal{A}$.

Proof. By some straightforward arguments one verifies that

$$(TS - \lambda S - \kappa T + \lambda\kappa)^{-1} = (S - \kappa)^{-1}(T - \lambda)^{-1}, \tag{2}$$
$$((TS - \lambda S - \kappa T)^* + \overline{\lambda\kappa})^{-1} = (T^* - \overline{\lambda})^{-1}(S^* - \overline{\lambda})^{-1}. \tag{3}$$

Hence $-\lambda\kappa \in \rho(TS - \lambda S - \kappa T)$. From the only if part of Theorem 95 it follows that the operators in (2) and in (3) belong to $\mathtt{M}(\mathcal{A})$ and that they map $\mathcal{A}$ densely into $\mathcal{A}$. Therefore, by the if part of Theorem 95, $(TS - \lambda S - \kappa T)\eta\mathcal{A}$. □

Proposition 97. Let $T, S \in \mathcal{C}(\mathcal{H})$. Suppose that $\lambda \in \rho(T)$, $S(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $\|S(T - \lambda)^{-1}\| < 1$. Then $(T + S)\eta\mathcal{A}$.

Proof. By Theorem Theorem 95, $(T - \lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $(T - \lambda)^{-1}\mathcal{A}$ and $(T^* - \overline{\lambda})^{-1}\mathcal{A}$ are dense in $\mathcal{A}$. By the assumptions we have $R := S(T - \lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $\|R\| < 1$. Therefore $(I + R)^{-1}$ is bounded and an element of $\mathtt{M}(\mathcal{A})$, since $I + R \in \mathtt{M}(\mathcal{A})$. Further, since $T - \lambda$ and $I + R$ are bijective and $(T + S - \lambda)\varphi = (I + R)(T - \lambda)\varphi$ for $\varphi \in \mathcal{D}(T) \subseteq \mathcal{D}(S)$, the map $T + S - \lambda : \mathcal{D}(T) \to \mathcal{H}$ is bijective. Hence $\lambda \in \rho(T + S)$ and $(T + S - \lambda)^{-1} = (T - \lambda)^{-1}(I + R)^{-1} \in \mathtt{M}(\mathcal{A})$. Because $I+R$ is an invertible element of $\mathtt{M}(\mathcal{A})$, the density of $(T-\lambda)^{-1}\mathcal{A}$ implies the density of $(T + S - \lambda)^{-1}\mathcal{A}$ in $\mathcal{A}$. Finally, $(T^* + S^* - \overline{\lambda})^{-1} = (I + R^*)^{-1}(T^* - \overline{\lambda})^{-1}$ maps

$\mathcal{A}$ densely into $\mathcal{A}$, since $\|R^*\| < 1$. Now applying again Theorem 95 we obtain $(T+S)\eta\mathcal{A}$. □

COROLLARY 98. Suppose that $T, S\eta\mathcal{A}$. If $\lambda \in \rho(T)$, $\|\lambda(T-\lambda)^{-1}\| < 1$ and $0 \in \rho(S)$, then $TS\eta\mathcal{A}$.

PROOF. By Corollary 96 we have $0 \in \rho(TS - \lambda S)$ and $(TS - \lambda S)\eta\mathcal{A}$. Since $T\eta\mathcal{A}$ and $\lambda \in \rho(T)$, it follows from Theorem 95 that $(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$. Therefore,

$$\lambda S(TS - \lambda S)^{-1} = \lambda S((T-\lambda)S)^{-1} = \lambda(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A}).$$

Hence, since $\|\lambda(T-\lambda)^{-1}\| < 1$ by assumption, Proposition 97 applies to the operators $X := TS - \lambda S$ and $Y := \lambda S$ and implies that $X + Y = TS\eta\mathcal{A}$. □

COROLLARY 99. Suppose that $T\eta\mathcal{A}$ and $\lambda, \kappa \in \rho(T)$. For an operator $S$ on $\mathcal{H}$ we have $S(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ if and only if $S(T-\kappa)^{-1} \in \mathtt{M}(\mathcal{A})$.

PROOF. Since $(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ and $(T-\kappa)^{-1} \in \mathtt{M}(\mathcal{A})$ by Theorem 95, the assertion follows from the identities

$$\begin{aligned} S(T-\lambda)^{-1} - S(T-\kappa)^{-1} &= (\lambda-\kappa)S(T-\kappa)^{-1}(T-\lambda)^{-1} \\ &= (\lambda-\kappa)S(T-\lambda)^{-1}(T-\kappa)^{-1}. \end{aligned}$$

□

Let $T \in \mathcal{C}(\mathcal{H})$ be self-adjoint, $S \in \mathcal{C}(\mathcal{H})$ be symmetric and $T$-bounded with $T$-bound less then 1, that is, $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and for some $0 \le a < 1$ and $b > 0$ it is

$$\|S\phi\| \le a\|T\phi\| + b\|\phi\| \quad (\phi \in \mathcal{H}).$$

By the Kato-Rellich theorem $T + S$ is self-adjoint (see [RS75] Theorem X.12). We proof

THEOREM 100. Suppose that $T\eta\mathcal{A}$ is a self-adjoint operator and $S$ is a symmetric $T$-bounded operator on $\mathcal{H}$ with $T$-bound less than 1. If $S(T-\lambda)^{-1} \in \mathtt{M}(\mathcal{A})$ for some $\lambda \in \rho(T)$, then $(T+S)\eta\mathcal{A}$.

PROOF. By Corollary 99 $S(T-i\kappa)^{-1} \in \mathtt{M}(\mathcal{A})$ for each $\kappa > 0$, since $i\kappa \in \rho(T)$. We have

$$\|S(T-i\kappa)^{-1}\phi\| \le a\|\underbrace{T(T-i\kappa)^{-1}}_{\|\cdot\|\le 1}\phi\| + b\|\underbrace{(T-i\kappa)^{-1}}_{\|\cdot\|\le 1/\kappa}\phi\| \le (a + b/\kappa)\|\phi\|.$$

Hence $\|S(T-i\kappa)^{-1}\| < 1$ for $\kappa$ large enough. Applying Proposition 97 to such an $\kappa$ we conclude $(T+S)\eta\mathcal{A}$. □

# 7. Examples

## Graph regular operators that are not regular

In this chapter we study three examples:

a) position and momentum operator
b) the inverse of $Z$ on the group $C^*$-algebra of the Heisenberg group
c) unbounded Toeplitz operators with rational symbol

We focus on presenting graph regular operators, that are not regular. In all three cases we construct inverses or quotients of adjointable operators.

### *a) Position and momentum operators as graph regular operators on a fraction algebra related to the Weyl algebra*

Let $P = -i\frac{d}{dt}$ and $Q = t$ be the momentum and position operators acting as self-adjoint operators on the Hilbert space $L^2(\mathbb{R})$. They are regular as operators on $\mathcal{K}(L^2(\mathbb{R}))$ by Example 88. But we want to consider them on a more involved $C^*$-algebra carrying a structure that is more related to this operators. Of course, we want $P$ and $Q$ to be graph regular; they wont be regular anymore.

We construct a $C^*$-algebra as completion of a fraction algebra where the fractions are generated by the resolvents

$$x := (Q - \alpha i I)^{-1} \quad \text{and} \quad y := (P - \beta i I)^{-1}$$

for some fix nonzero reals $\alpha$ and $\beta$. It is not difficult to verify that these operators satisfy the commutation relations

$$x - x^* = 2\alpha i x^* x = 2\alpha i x x^*,\ y - y^* = 2\beta i y^* y = 2\beta i y y^*, \tag{4}$$

$$xy - yx = -ixy^2x = -iyx^2y,\ xy^* - y^*x = -ix(y^*)^2x = -iy^*x^2y^*. \tag{5}$$

Let $\mathcal{X}$ be the unital $*$-subalgebra of $\mathbf{B}(L^2(\mathbb{R}))$ generated by $x$ and $y$.

From the equations (4) we deduce

$$x = x^*(1 + 2\alpha i x), \quad x^* = x(1 - 2\alpha i x^*),$$

hence

$$x\mathcal{X} = x^*(1 + 2\alpha i x)\mathcal{X} \subseteq x^*\mathcal{X} = x(1 - 2\alpha i x^*)\mathcal{X} \subseteq x\mathcal{X}; \quad x\mathcal{X} = x^*\mathcal{X}.$$

Analogously is $y\mathcal{X} = y^*\mathcal{X}$. The equations (5) can be reorganized as

$$xy = yx(1 - ixy) = (1 - ixy)yx, \quad xy^* = y^*x(1 - ixy^*) = (1 - ixy^*)y^*x. \tag{6}$$

We compute

$$xy\mathcal{X} = yx(1 - ixy)\mathcal{X} \subseteq yx\mathcal{X} = xy(1 - iyx)\mathcal{X} \subseteq xy\mathcal{X}; \quad xy\mathcal{X} = yx\mathcal{X}.$$

From (6) and $x^*x = xx^*$ we deduce $x\mathcal{X} = \mathcal{X}x$. A similar reasoning gives $y\mathcal{X} = \mathcal{X}y$.

Putting all together it is $(xy\mathcal{X})^* = \mathcal{X}y^*x^* = \mathcal{X}yx^* = y\mathcal{X}x^* = y\mathcal{X}x = yx\mathcal{X}$ and

$$\mathcal{J}_0 := \mathcal{X}xy = xy\mathcal{X} = yx\mathcal{X} = \mathcal{X}yx, \tag{7}$$

and $\mathcal{J}_0$ is a two-sided $*$-ideal of the $*$-algebra $\mathcal{X}$.

Since $\mathcal{X}$ is a $*$-subalgebra of $\mathbf{B}(L^2(\mathbb{R}))$ its norm-closure $\overline{\mathcal{X}}$ is a $C^*$-algebra. It can be shown (see [GS15] Section 6.2) that these closure actually is the universal $C^*$-algebra $\mathcal{X}_{un}$ of $\mathcal{X}$; it is the completion of $\mathcal{X}$ in the norm

$$\|a\|_{un} := \sup_{\pi} \|\pi(a)\|, \quad a \in \mathcal{X},$$

where the supremum is taken over all $*$-representation of $\mathcal{X}$.[5]

We show that $\mathcal{I}_0$ is an essential ideal of $\overline{\mathcal{X}}$: Let $a \in \overline{\mathcal{X}}$ be such that $a \perp \mathcal{J}_0$. Then $a^*xy = 0$ in $L^2(\mathbb{R})$. Since $x$ and $y$ are bijection, this implies $a^* = 0$; $a = 0$.

Let $\mathcal{I}$ be the norm-closure of $\mathcal{I}_0$. The operator $x^*y^*yx$ is an integral operator on $L^2(\mathbb{R})$ with kernel

$$K(t,s) := (2|\beta|)^{-1}(t+\alpha i)^{-1}(s-\alpha i)^{-1}e^{-|\beta||t-s|}.$$

Since $K \in L^2(\mathbb{R}^2)$, the operator $x^*y^*yx = |yx|^2$ is compact, so are $|yx|$ and hence $yx$. Hence $\mathcal{J}_0 \subseteq \mathbf{K}(L^2(\mathbb{R}))$ by (7) and therefore $\mathcal{J} = \mathbf{K}(L^2(\mathbb{R}))$.

Now we define two operators, denoted by $q$ and $p$, on the $C^*$-algebra $\overline{\mathcal{X}}$ by

$$q := \alpha i + x^{-1}, \quad \mathcal{D}(q) := x\overline{\mathcal{X}} \quad \text{and} \quad p := \beta i + y^{-1}, \quad \mathcal{D}(p) := y\overline{\mathcal{X}},$$

that is, $q(xa) = \alpha i x a + a$ and $p(ya) = \beta i y a + a$, for $a \in \overline{\mathcal{X}}$.

THEOREM 101. The operators $q$ and $p$ are essentially defined, graph regular and self-adjoint on the $C^*$-algebra $\overline{\mathcal{X}}$.

PROOF. We carry out the proof for $q$; a similar reasoning yields the assertions for $p$. Since $x\overline{\mathcal{X}}$ and $x^*\overline{\mathcal{X}}$ contain the essential ideal $\mathcal{J}_0$, they are essential, too. Therefore $\mathcal{N}(x) = \mathcal{R}(x^*)^\perp$ and $\mathcal{N}(x^*) = \mathcal{R}(x)^\perp$ are trivial and with Corollary 80 it follows that $x^{-1}$ is graph regular. So $q$ is graph regular too, by Proposition 78.

The first equation of (4) implies $-\alpha i x x^* + x = \alpha i x x^* + x^*$. As above one proves $x\overline{\mathcal{X}} \subseteq x^*\overline{\mathcal{X}} \subseteq x\overline{\mathcal{X}}$, hence $x$ and $x^*$ have equal ranges. Therefore, multiplying the above equation from the left with $x^{-1}$ and from the right with $(x^*)^{-1}$ we get $-\alpha i + (x^*)^{-1} = \alpha i + x^{-1}$, so

$$q^* = -\alpha i + (x^{-1})^* = -\alpha i + (x^*)^{-1} = \alpha i + x^{-1} = q$$

is self-adjoint. □

The operators $q$ and $p$ are not regular on $\overline{\mathcal{X}}$, since neither $x\overline{\mathcal{X}}$ nor $y\overline{\mathcal{X}}$ is dense in $\overline{\mathcal{X}}$, since $\overline{\mathcal{X}}$ is unital. Note that the corresponding restrictions of $q$ and $p$ are regular operators on the essential ideal $\mathcal{I} = \mathcal{K}(L^2(\mathbb{R}))$ of the $C^*$-algebra $\overline{\mathcal{X}}$.

[5]Also in [GS15] the structure of the $C^*$-algebra $\mathcal{X}_{un}$ is computed. Let $\mathcal{F}_x$ be the unital $*$-subalgebra of $\mathcal{X}$ generated by $x$, that is, $\mathcal{F}_x$ is the commutative $*$-algebra of polynomials $f(x,x^*)$ in $x$ and $x^*$ with complex coefficients. Likewise, $\mathcal{F}_y$ denotes the unital $*$-subalgebra of $\mathcal{X}$ generated by $y$. Each element $a \in \mathcal{X}$ can be written as

$$a = f_1(x,x^*) + g_1(y,y^*) + yxb_1 = f_2(x,x^*) + g_2(y,y^*) + xyb_2,$$

where $f_1, g_1, f_2, g_2$ are polynomials with $g_1(0,0) = f_2(0,0) = 0$ and $b_1, b_2 \in \mathcal{X}$. These triples $\{f_1, g_1, b_1\}$ and $\{f_2, g_2, b_2\}$ are uniquely determined by $a$. Two circles are given by

$$K_\alpha := \{(z,0) \in \mathbb{C}^2 | z - \overline{z} = 2\alpha i |z|^2\}, \quad K_\beta := \{(0,w) \in \mathbb{C}^2 | w - \overline{w} = 2\beta i |w|^2\}.$$

The irreducible $*$-representations of $\mathcal{X}$ are

$$\pi_{x,z}(a) := f_1(z,\overline{z}) \quad \text{for } (z,0) \in K_\alpha,$$
$$\pi_{y,w}(a) := g_2(w,\overline{w}) \quad \text{for } (0,w) \in K_\beta, \pi_0(a) := a \in \mathbf{B}(L^2(\mathbb{R})).$$

The closures of $\mathcal{F}_x$ and $\mathcal{F}_y$ are the commutative $C^*$-subalgebras $C(K_\alpha)$ and $C(K_\beta)$ of $\mathcal{X}_{un}$, respectively. The closure of $xy\mathcal{X}$ is $\mathcal{K}(L^2(\mathbb{R}))$.

*b) A graph regular but not regular operator on the group $C^*$-algebra of the Heisenberg group*

Let $H$ be the 3-dimensional Heisenberg group, that is, $H$ is the Lie group whose differential manifold is the vector space $\mathbb{R}^3$ and whose multiplication is given by

$$(x_1, x_2, x_3)(x'_1, x'_2, x'_3) := (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + \frac{1}{2}(x_1 x'_2 - x'_1 x_2)).$$

The $C^*$-algebra $C^*(H)$ of the Lie group $H$ was described in [LT11]. We briefly repeat this result.

First recall that $C^*(H)$ is defined as the completion of $L^1(H)$ with respect to the norm

$$\|f\| = \sup\{\|\pi^U(f)\| : U \text{ unitary representation of } H\}.$$

where $\pi^U$ is the $*$-representation of $L^1(H)$ associated with $U$ and given by

$$\pi^U(f) := \int_{\mathbb{R}^3} U(x_1, x_2, x_3) f(x_1, x_2, x_3) dx_1 dx_2 dx_3, \quad f \in L^1(H).$$

The irreducible unitary representations of $H$ consist of a series $U_\lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$, of infinite dimensional representations acting on $L^2(\mathbb{R})$ and of a series $U_a$, $a \in \mathbb{R}^2$, of one dimensional representations. For $(x_1, x_2, x_3) \in H$, these representations act as

$$(U_\lambda(x_1, x_2, x_3)\xi)(s) = e^{-2\pi i \lambda (x_3 + \frac{1}{2} x_1 x_2 + s x_2)} \xi(s - x_1), \quad \xi \in L^2(\mathbb{R}),\ \lambda \in \mathbb{R} \setminus \{0\},$$
$$U_a(x_1, x_2, x_3) = e^{-2\pi i (a_1 x_1 + a_2 x_2)}, \quad a = (a_1, a_2) \in \mathbb{R}^2.$$

The Lie algebra of $H$ has a basis $\{X, Y, Z\}$ with commutation relations

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0$$

and we have $dU_\lambda(iZ) = 2\pi\lambda I$ and $dU_a(iZ) = 0$.

We determine the structure of $C^*(H)$ in terms of operators fields: Let $\mathcal{F}$ be the $C^*$-algebra of all operator fields $F = (F(\lambda); \lambda \in \mathbb{R})$ satisfying the following conditions:

(i) $F(\lambda)$ is a compact operator on $L^2(\mathbb{R})$ for each $\lambda \in \mathbb{R}^\times$,
(ii) $F(0) \in C_0(\mathbb{R}^2)$,
(iii) $\mathbb{R}^\times \ni \lambda \to F(\lambda) \in \mathbf{B}(L^2(\mathbb{R}))$ is norm continuous,
(iv) $\lim_{\lambda\to\infty} \|F(\lambda)\| = 0$.

Let $\eta$ be a fixed function of the Schwartz space $\mathcal{S}(\mathbb{R})$ of norm one in $L^2(\mathbb{R})$. For $\xi \in L^2(\mathbb{R})$, let $P_\xi$ denote the projection on the one dimensional subspace $\mathbb{C} \cdot \xi$. Then for $h \in C_0(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}^\times := \mathbb{R}\setminus\{0\}$, the operator $\nu_\lambda(h)$ is defined by

$$\nu_\lambda(h) := \int_{\mathbb{R}^2} \hat{h}(x_1, x_2) P_{\eta(\lambda; x_1, x_2)} |\lambda|^{-1} dx_1 dx_2, \tag{8}$$

where $\hat{h}$ denotes the Fourier transform of $h$ and

$$\eta(\lambda; x_1, x_2)(s) := |\lambda|^{1/4} e^{2\pi i x_1 s}\ \eta\big(|\lambda|^{1/2}(s + x_2\lambda^{-1})\big), \quad x_1, x_2, s \in \mathbb{R}.$$

By Proposition 2.14 in [LT11], we have

$$\lim_{\lambda\to 0} \|\nu_\lambda(h)\| = \|\hat{h}\|_\infty \quad \text{for} \quad h \in C_0(\mathbb{R}^2). \tag{9}$$

Then, according to Theorem 2.16 in [LT11], the $C^*$-algebra $C^*(H)$ is the $C^*$-subalgebra of $C^*(H)$ formed by all operator fields $F \in \mathcal{F}$ such that

$$\lim_{\lambda\to 0} \|F(\lambda) - \nu_\lambda(F(0))\| = 0, \tag{10}$$

where $\nu_\lambda : C_0(\mathbb{R}^2) \to \mathcal{F}$ is defined by (8), and for $c \in C^*(H)$, we have the identities $F(c)(\lambda) = \pi_{U_\lambda}$, $\lambda \in \mathbb{R}^\times$, and $F(c)(0)(a) = \pi_{U_a}(c)$, $a \in \mathbb{R}^2$.

It was proved in [WN92] that the Lie algebra generators $X, Y, Z$ act as skew-adjoint regular operators on the $C^*$-algebra $C^*(H)$. We show

THEOREM 102. $(iZ)^{-1}$ is a graph regular self-adjoint operator on $C^*(H)$.

PROOF. Since $iZ$ is self-adjoint and regular, we only have to show that the range of $iZ$ is essential, since then $(iZ)^{-1}$ exists and is graph regular by Proposition 58. Clearly, $iZ$ is self-adjoint, so is its inverse.

Assume $G(\lambda) \in C^*(H)$ and $G(\lambda) \in \mathcal{R}(iZ)^\perp$. Since we have $dU_\lambda(iZ) = 2\pi\lambda I$ for $\lambda \in \mathbb{R}^\times$, $\mathcal{R}(iZ)$ contains all vector fields $F(\lambda) \in \mathcal{F}$ of compact support contained in $\mathbb{R}^\times$. This implies that $G(\lambda) = 0$ on $\mathbb{R}^\times$. Therefore, $\lim_{\lambda\to 0} \nu_\lambda(G(0)) = 0$ by (10) and hence $\widehat{G}(0) = 0$ by (9), so $G(0) \in C_0(\mathbb{R}^2)$ is zero. Thus $G = 0$ in $C^*(H)$ which proves that $\mathcal{R}(iZ)$ is essential. $\square$

Note that $(iZ)^{-1}$ is not regular, because $dU_a(iZ) = 0$ for $a \in \mathbb{R}^2$ and hence $(iZ)^{-1}$ is not densely defined.

## *c) Unbounded Toeplitz operators*

Let $L^2(\mathbb{T})$ be the Hilbert space of square integrable functions on the unit circle $\mathbb{T}$ with scalar product given by

$$\langle f, g\rangle := \int_0^1 \overline{f(e^{2\pi it})} g(e^{2\pi it})\, dt \quad f, g \in L^2(\mathbb{T}),$$

and let $P$ denote the projection of $L^2(\mathbb{T})$ on the closed subspace $H^2(\mathbb{T})$ generated by $\{z^n := e^{2\pi itn} | n \in \mathbb{N}_0\}$. For $\phi \in L^\infty(\mathbb{T})$ the Toeplitz operator $T_\phi$ is the bounded operator on the Hilbert space $H^2(\mathbb{T})$ is defined by

$$T_\phi f := P\phi f, \quad f \in H^2(\mathbb{T}).$$

The $C^*$-algebra generated by the unilateral shift $S := T_z$ is the Toeplitz algebra

$$\mathcal{T} := \{T_\phi | \phi \in C(\mathbb{T})\} \dotplus \mathcal{K}(\mathcal{H}^2(\mathbb{T})).$$

Our aim is to construct a class of examples of graph regular (unbounded) Toeplitz operators on the $C^*$-algebra $\mathcal{T}$. Let $p, q \in \mathbb{C}[z]$ be relatively prime polynomials such that $q$ has no zeros in the open unit disc $\mathbb{D}$. Then the Toeplitz operator with rational symbol $p/q$ is defined by

$$\mathcal{D}(T_{p/q}) := \{f \in \mathcal{H}^2(\mathbb{T}) | \frac{p}{q} f \in \mathcal{H}^2(\mathbb{T})\}, \quad T_{p/q} f := \frac{p}{q} f \quad (f \in \mathcal{D}(T_{p/q})),$$

Since $T_{p/q}$ is a multiplication operator, it is a closed densely defined operator on the Hilbert space $\mathcal{H}^2(\mathbb{T})$.

THEOREM 103. Suppose that $p, q$ are relatively prime polynomials such that $q$ has no zero in the open unit disc. Then the Toeplitz operator $T_{p/q}$ is associated with the Toeplitz algebra $\mathcal{T}$. Further, $T_{p/q}$ is affiliated with the Toeplitz algebra if and only if in addition $q$ has no zero on the unit circle.

PROOF. Since $q$ has no zero in $\mathbb{D}$, we can write $q$ as $q(z) = \alpha \prod_{j=1}^n (z - \alpha_j)$, where $\alpha, \alpha_j \in \mathbb{C}$ and $|\alpha_j| \geq 1$. Then $S^* - \lambda I$ is injective for $|\lambda| \geq 1$ and hence

$$\mathcal{N}(q(S)^*) = \mathcal{N}\big(\alpha \prod_{j=1}^n (S^* - \overline{\alpha_j})\big) = \{0\}.$$

Hence, $\overline{\mathcal{R}(q(S))} = \mathcal{N}(q(S)^*)^\perp = H^2(\mathbb{T})$, so $q$ is an outer function (see e.g. [RR85]).

Now we use an argument from [Sar08] Section 3. Since $p$ and $q$ are relatively prime, we have $|p|^2 + |q|^2 > 0$ on the closed unit disc $\overline{\mathbb{D}}$. Therefore, by the Riezs-Fejér Theorem, there exists a polynomial $r \in \mathbb{C}[z]$ such that $r$ has no zero in $\overline{\mathbb{D}}$ and $|p|^2 + |q|^2 = |r|^2$ on $\mathbb{T}$. Let $f := q/r$ and $g := p/r$. Then $f$ and $g$ are continuous and in the unit ball of $H^\infty(\mathbb{T})$, $f$ is outer, $|f|^2 + |g|^2 = 1$ on $\mathbb{T}$. Upon multiplying $r$ by some constant of modulus one we can assume that $f(0) > 0$.

From [Sar08] Proposition 5.3 it follows $\mathcal{D}(T_{p/q}) = fH^2(\mathbb{T})$ and $T_{p/q} = T_g T_f^{-1}$. Moreover, $T_{p/q}^* = (T_f^{-1})^* T_g^* = T_{\overline{f}}^{-1} T_{\overline{g}}$. Using these facts we compute

$$\begin{aligned}
1 + T_{p/q}^* T_{p/q} &= 1 + T_{\overline{f}}^{-1} T_{\overline{g}} T_g T_f = T_{\overline{f}}^{-1} (T_{\overline{f}} T_f + T_{\overline{g}} T_g) T_f^{-1} = T_{\overline{f}}^{-1} (T_{|f|^2} + T_{|g|^2}) T_f^{-1} \\
&= T_{\overline{f}}^{-1} T_f^{-1} = (T_f T_{\overline{f}})^{-1}, \\
1 + T_{p/q} T_{p/q}^* &= 1 + T_g T_f^{-1} T_{\overline{f}}^{-1} T_{\overline{g}} = 1 + T_g (T_{\overline{f}} T_f)^{-1} T_{\overline{g}} = 1 + T_g (1 - T_{\overline{g}} T_g)^{-1} T_{\overline{g}} \\
&= 1 + (1 - T_g T_{\overline{g}})^{-1} T_g T_{\overline{g}} = (1 - T_g T_{\overline{g}})^{-1}.
\end{aligned}$$

Hence $a_{T_{p/q}} = T_f T_{\overline{f}}$ and $a_{T_{p/q}^*} = I - T_g T_{\overline{g}}$ are in $\mathcal{T}$. Further,

$$b_{T_{p/q}} = T_{p/q} a_{T_{p/q}} = T_g T_f^{-1} T_f T_{\overline{f}} = T_g T_{\overline{f}} \in \mathcal{T}.$$

Since $a_{T_{p/q}}, a_{T_{p/q}^*}, b_{T_{p/q}} \in \mathcal{T}$, $T_{p/q}$ is associated with the $C^*$-algebra $\mathcal{T}$.

Suppose now $q$ has a zero at some $\lambda \in \mathbb{T}$. Then $f$ has a zero at $\lambda$ as well. For $z \in \mathbb{T}$ let $\omega_z$ be the character on $\mathcal{T}$ given by

$$\omega_z(T_\phi + K) = \phi(z) \quad (\phi \in C(\mathbb{T}), K \in \mathcal{K}(H^2(\mathbb{T})). \tag{11}$$

If $T_\phi + K \in \mathcal{T}$, then $T_f T_{\overline{f}}(T_\phi + K) = T_{|f|^2\phi} + \tilde{K}$ for some $\tilde{K} \in \mathcal{K}(\mathcal{H}^2(\mathbb{T}))$. Hence

$$\omega_\lambda(a_{T_{p/q}}(T_\phi + K)) = \omega_\lambda(T_{|f|^2\phi} + \tilde{K}) = |f(\lambda)|^2 \phi(\lambda) = 0.$$

Therefore, $a_{T_{p/q}} \mathcal{T}$ is not dense in $\mathcal{T}$ and hence $T_{p/q}$ is not affiliated with $\mathcal{T}$.

On the other hand, if $q$ has no zero on $\mathbb{T}$, then $p/q \in C(\mathbb{T})$ and hence $T_{p/q} \in \mathcal{T}$, so in particular, $T_{p/q}$ is affiliated with $\mathcal{T}$. □

The simplest interesting example is the following.

Example 104. Set $p(z) = 1$ and $q(z) = 1 - z$, so that $p/q = 1/(1-z)$. Then, by Theorem 103, $T_{1/(1-z)}$ associated with $\mathcal{T}$, but $T_{1/(1-z)}$ is not affiliated with $\mathcal{T}$. In fact, $T_{1/(1-z)} = (I - S)^{-1}$.

# 8. Bounded transform

## The canonical regular operator associated to a graph regular operator

In this chapter we generalize the bounded transform known from [Lan95] chapter 10 or [Wor91] where it was called z-transform. More precisely: Let $E$ and $F$ be Hilbert $\mathcal{A}$-modules. We denote by $\mathcal{Z}(E,F)$ the set of all $z \in \mathcal{L}(E,F)$ such that $\|z\| \leq 1$ and $\mathcal{N}(I - z^*z) = \{0\}$; in particular the range of $1 - z^*z$ is essential in $E$. By $\mathcal{Z}^d(E,F)$ we denote those $z \in \mathcal{Z}(E,F)$ for which the range of $1 - z^*z$ is dense in $E$. In Theorem 10.4 of [Lan95] the regular operators are mapped bijectively onto $\mathcal{Z}^d(E,F)$ via the bounded transform

$$t \mapsto z_t := t(1 + t^*t)^{-1/2} \quad \text{with inverse} \quad t_z := z(1 - z^*z)^{-1/2}.$$

It is $z_{t^*} = z_t^*$ and $t_{z^*} = t_z^*$. The inverse of the bounded transform will be extended to a mapping from $\mathcal{Z}(E,F)$ into the set of graph regular operators (Theorem 106); it will not be surjective anymore. We extend the bounded transform to the class of graph regular operators (Theorem 108); its range will not be contained in $\mathcal{Z}(E,F)$. But we can consider the bounded transform as operator between the submodules $E' = \overline{\mathcal{D}(t^*t)}$ and $F' = \overline{\mathcal{D}(tt^*)}$ and there exists an regular operator $t' : E' \to F'$ with the same bounded transform. In this sense, $t$ is an extension of $t'$ and $t'$ is associated to $t$ (Theorem 108). In Corollary 111, Theorem 112 and Corollary 113 we clarify the relation between the bounded transform and its inverse by introducing and characterizing the concept of the adjointable bounded transform.

In the following chapter we will prove that the bounded transform $z_{|t|}$ of the absolute value of any $t \in \mathcal{R}_{gr}(E,F)$ is adjointable (in the above sense!) (Corollary 116).

Let $E$ and $F$ be two Hilbert $\mathcal{A}$-modules.

Lemma 105. Let $z \in \mathcal{L}(E,F)$ with $\|z\| \leq 1$. Then $\mathcal{N}(I - z^*z) = \{0\}$ if and only if $\mathcal{N}(I - zz^*) = \{0\}$. In particular, $z \in \mathcal{Z}(E,F)$ if and only if $z^* \in \mathcal{Z}(F,E)$.

Proof. It suffices to show one direction, since $z$ can be replaced by $z^*$: it is $\|z\| = \|z^*\|$. Assume that $x \in \mathcal{N}(I - zz^*) \setminus \{0\}$. From $\|z^*x\|^2 = \langle x, zz^*x\rangle = \langle x, x\rangle = \|x\|^2$ we conclude $z^*x \neq 0$. But $(I - z^*z)z^*x = z^*(I - zz^*)x = 0$, so $z^*x \in \mathcal{N}(I - z^*z) \setminus \{0\}$. □

In [Lan95] Lemma 10.3 the analogous statement $\mathcal{Z}^d(E,F)$ was shown: Let $z \in \mathcal{L}(E,F)$ with $\|z\| \leq 1$. Then $\mathcal{R}(I - z^*z)$ is dense in $E$ if and only if $\mathcal{R}(I - zz^*)$ is dense in $F$. In particular, $z \in \mathcal{Z}^d(E,F)$ if and only if $z^* \in \mathcal{Z}^d(F,E)$.

We turn to the *first half* of the *bounded transform* now - correctly, its inverse: For $z \in \mathcal{Z}(E,F)$ we define an operator $t_z : E \to F$ by

$$t_z := z((I - z^*z)^{1/2})^{-1}.$$

Since $\mathcal{N}(I - z^*z)$ is trivial, $(I - z^*z)^{1/2}$ is injective and the domain $\mathcal{D}(t_z)$, which coincides with $\mathcal{R}((I - z^*z)^{1/2})$, is essential by Proposition 18 (1).

According to [Lan95] Theorem 10.4, the mapping $z \mapsto t_z$ is a bijection from the set $\mathcal{Z}^d(E,F)$ onto the set $\mathcal{R}(E,F)$ of regular operators. For the extended mapping acting on $\mathcal{Z}(E,F)$ the situation is more subtle.

THEOREM 106. The mapping $z \mapsto t_z$ is injective from $\mathcal{Z}(E,F)$ into $\mathcal{R}_{gr}(E,F)$, which respects the adjoint: $t_z^* = t_{z^*}$. It is $(I + t_z^* t_z)^{-1} = I - z^* z$ and

$$z = t_z((I + t_z^* t_z)^{-1})^{1/2}. \tag{12}$$

The kernels and ranges of $z$ and $t_z$ coincides respectively.

PROOF. By Example 84 $z_t$ is graph regular and $t_z^* = t_{z^*}$.
Finally, by Proposition 20(4), $t_z^* = ((I - z^*z)^{1/2})^{-1} z^*$, so that

$$\begin{aligned} t_z^* t_z &= ((1 - z^*z)^{1/2})^{-1} z^* z ((1 - z^*z)^{1/2})^{-1} \\ &= ((1 - z^*z)^{1/2})^{-1} (1 - z^*z)((1 - z^*z)^{1/2})^{-1} - (((1 - z^*z)^{1/2})^{-1})^2 \\ &= 1 - ((1 - z^*z)^{1/2})^{-1}((1 - z^*z)^{1/2})^{-1} = 1 - (1 - z^*z)^{-1}. \end{aligned}$$

Therefore $(1 + t_z^* t_z)^{-1} = 1 - z^*z$. In particular $t_z((1 + t_z^* t_z)^{-1})^{1/2} = z$.

Let $x \in E$ and assume that $zx = 0$. Then $x = (1 - z^*z)x \in \mathcal{D}(t_z^* t_z) \subseteq \mathcal{D}(t_z)$ and hence $(1 + t_z^* t_z)x = x$. This implies $\langle t_z x, t_z x \rangle = \langle x, t_z^* t_z x \rangle = 0$, so $x \in \mathcal{N}(t_z)$. Conversely, let $t_z x = 0$ for some $x \in \mathcal{D}(t_z)$. Then $(1 + t_z^* t_z)x = x$, so $(1 - z^*z)x = x$ and $z^*zx = 0$. So $\langle zx, zx \rangle = \langle x, z^*zx \rangle = 0$ and $x \in \mathcal{N}(z)$. That is $\mathcal{N}(z) = \mathcal{N}(t_z)$.

We deduce $\mathcal{R}(t_z) \subseteq \mathcal{R}(z)$ from the definition of $t_z$. From equation (12) it follows $\mathcal{R}(z) \subseteq \mathcal{R}(t_z)$; hence equality holds. □

The following example illustrates that the mapping $z \mapsto t_z$ is not onto in general (even in the commutative case).

EXAMPLE 107. Consider the operator $t := t_m \in \mathcal{C}_o(C_0(\mathbb{R}))$, where $m(x) := 1/x$ for $x \neq 0$. Then $t$ is self-adjoint and graph regular by Theorem 67 and Theorem 68. We show that $t \neq t_z$ for all $z \in \mathcal{Z}(\mathcal{A})$: If $t = t_z$ for some $z \in \mathcal{Z}(\mathcal{A})$, then $z = t a_t^{1/2}$ by Theorem 106. If the right-hand side is not defined on all of $\mathcal{A}$ this will be a contradiction. We have $a_t = t_{\frac{1}{1+|m|^2}} \in \mathcal{L}(C_o(\mathbb{R}))$ by Theorem 68, so $a_t$ acts as multiplication with the function $x^2/(1 + x^2)$. Taking into account that the square root has to be a positive operator, $a_t^{1/2}$ acts as multiplication with the function $|x|/\sqrt{1 + x^2}$. For any $g \in C_0(\mathbb{R}) = \mathcal{D}(a_t^{1/2})$ with $g(0) \neq 0$, the function $\frac{1}{x}\frac{|x|g(x)}{\sqrt{1+x^2}} = \frac{\mathbf{sgn}(x)g(x)}{\sqrt{1+x^2}}$ is continuous on $\mathbb{R} \setminus \{0\}$, but has no continuous extension on $\mathbb{R}$. Therefore $a_t^{1/2} g \notin \mathcal{D}(t)$, and this gives the argument for the above contradiction.

The *second half* of the *bounded transform* is more involved. We have already seen in Example 107, that $z \mapsto t_z$ is not onto and that this is related to the fact, that for a graph regular operator $t$ the range of $((1 + t^*t)^{-1})^{1/2}$ is not necessarily contained in $\mathcal{D}(t)$.

We can not expect the bounded transform of an graph regular operator $t$ to act on the whole space, but it would be natural for this purpose to consider the operator $t a_t^{1/2} = t((1 + t^*t)^{-1})^{1/2}$. In fact, this operator belongs to $\mathcal{C}_o(E,F)$: The essential range of $a_t^{1/2}$ is contained in the domain of $t a_t^{1/2}$, whereas the latter one is the adjoint of the essentially defined operator $a_t^{1/2} t^*$ by Proposition 20 (5), hence it is orthogonally closed. Further, $t a_t^{1/2}$ is bounded on $\mathcal{R}(a_t^{1/2})$ by 1, since

$$\begin{aligned} \|t a_t^{1/2} a_t^{1/2} x\|^2 &= \| \langle t a_t x, t a_t x \rangle \| = \| \langle t^* t a_t x, a_t x \rangle \| = \| \langle (1\text{-}a_t) x, a_t x \rangle \| \\ &= \| \left\langle (1\text{-}a_t) a_t^{1/2} x, a_t^{1/2} x \right\rangle \| \le \|1\text{-}a_t\| \|a_t^{1/2} x\|^2 \le \|a_t^{1/2} x\|^2 \quad (x \in E). \end{aligned}$$

Since $ta_t^{1/2}$ is closed, $\overline{\mathcal{R}(a_t^{1/2})} = \overline{\mathcal{R}(a_t)} = \overline{\mathcal{D}(t^*t)}$ is contained in $\mathcal{D}(ta_t^{1/2})$. We define the bounded transform by the restriction of $ta_t^{1/2}$ to this domain.

Definition. For $t \in \mathcal{R}_{gr}(E,F)$ let $E_t := \overline{\mathcal{D}(t^*t)}$. The *bounded transform* is given by

$$z_t := ta_t^{1/2} \restriction_{E_t} .$$

Note that $\mathcal{D}(t^*t)$ is essential by Proposition 24, since it is a core for $t$ by Theorem 55 (3). Further, $\mathcal{D}(t^*t) = \mathcal{R}(a_t) \subseteq \mathcal{R}(a_t^{1/2})$ is clearly contained in $\mathcal{D}(z_t)$, hence $z_t$ is densely defined in $E_t$. By definition $F_{t^*} = \overline{\mathcal{D}(tt^*)}$. Since the next theorem deals with operators between $E$ and $F$ and between $E_t$ and $F_{t^*}$ we have to choose two signs for the corresponding adjoint relation. For the first case - larger spaces $E$ and $F$ - we take the asterix $*$ with six rays and for the second case - smaller spaces $E_t$ and $F_{t^*}$ - the star $\star$ with only five rays is chosen.

Theorem 108. The bounded transform $z_t$ of $t \in \mathcal{R}_{gr}(E,F)$ is an element of $\mathcal{Z}^d(E_t, F_{t^*})$. The adjoints are respected via $z_t^\star = z_{t^*}$. It is $\mathcal{N}(z_t) = \mathcal{N}(t)$, $\mathcal{R}(z_t) \subseteq \mathcal{R}(t)$ and we have the representation

$$z_t = \overline{a_{t^*}^{1/2} t} \restriction_{E_t} .$$

We can compute $t$ from $z_t$ back via $t = (t_{z_t})^{**}$, and via the $(a, a_*, b)$-transform with

$$a_t = (1 - z_t^\star z_t)^*, \quad a_{t^*} = (1 - z_t z_t^\star)^*, \quad b_t = z_t a_t^{1/2}.$$

Especially, $t$ is an extension of the regular operator $t_{z_t} \in \mathcal{R}(E_t, F_{t^*})$.

Proof. From Theorem 56 we already know that $t = a_{t^*}^{-1} b_t$. By Theorem 56 we have $a_t b_t^* = b_t^* a_{t^*}$. Hence $a_t^n b_t^* = b_t^* a_{t^*}^n$ for $n \in \mathbb{N}_0$. Since the square root is continuous on $[0, 1]$, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials that converges uniformly on $[0, 1]$ to the square root. It is $\|a_t\|, \|a_{t^*}\| \leq 1$, so

$$a_t^{1/2} b_{t^*} = \lim_{n\to\infty} p_n(a_t) b_{t^*} = \lim_{n\to\infty} b_{t^*} p_n(a_{t^*}) = b_{t^*} a_{t^*}^{1/2}.$$

Since $a^{3/2}a^{-1} \subseteq a^{1/2}$, hence $a^{1/2}a^{-1} \subseteq a^{-1}a^{1/2}$. With the above it follows now

$$a_{t^*}^{1/2} t = a_{t^*}^{1/2} a_{t^*}^{-1} b_t \subseteq a_{t^*}^{-1} a_{t^*}^{1/2} b_t = a_{t^*}^{-1} b_t a_t^{1/2} = t a_t^{1/2}. \tag{13}$$

This implies that $\overline{a_{t^*}^{1/2} t}$ coincides with $ta_t^{1/2}$ on $\overline{\mathcal{D}(t^*t)} \subseteq \overline{\mathcal{D}(t)}$, since the latter operator is bounded on $E_t$.

With this representation of $z_t$ it is easily seen that $z_t$ maps $E_t$ into $F_{t^*}$: From [Lan95] Proposition 3.7 we get $\overline{\mathcal{R}(a_{t^*}^{1/2})} = \overline{\mathcal{R}(a_{t^*})} = \overline{\mathcal{D}(tt^*)} = F_{t^*}$. In particular the range of $z_t$ is contained in $\overline{\mathcal{R}(a_{t^*}^{1/2})}$. Hence $z_t$ can be considered as bounded operator on $E_t$ mapping into $F_{t^*}$. Analogously $z_{t^*} : F_{t^*} \to E_t$.

We prove $z_t^\star = z_{t^*}$. But first the adjoint $z_t^*$ of $z_t$ will be considered.

$$z_t^* = (\overline{a_{t^*}^{1/2} t} \restriction_{E_t})^* \supseteq (\overline{a_{t^*}^{1/2} t})^* \supseteq t^* a_{t^*}^{1/2} \supseteq z_{t^*}.$$

Hence $z_t^* \restriction_{F_{t^*}} = z_{t^*}$. Since $z_{t^*}$ already maps into $E_t$, this means in particular that $z_t^\star$ is indeed given by $z_{t^*}$. Hence $z_t \in \mathcal{L}(E_t, F_{t^*})$.

Next we prove the formulas for $a_t, a_{t^*}$, and $b_t$. First, with $\mathcal{R}(a_{t^*}^{1/2}) \subseteq F_{t^*}$ and for example equation (13) we note that

$$z_{t^*} z_t \restriction_{\mathcal{D}(t^*t)} = t^* a_{t^*}^{1/2} \restriction_{F_{t^*}} a_{t^*}^{1/2} t \restriction_{\mathcal{D}(t^*t)} = t^* a_{t^*} t \restriction_{\mathcal{D}(t^*t)} = 1 - a_t \restriction_{\mathcal{D}(t^*t)},$$

whence $a_t = (1 - z_{t^*} z_t)^{**}$ by Example 23. With $a_t = a_t^*$ the statement follows. Analogously the formula for $a_{t^*}$ is proven. Clearly $z_t a_t^{1/2} = ta_t^{1/2} \restriction_{E_t} a_t^{1/2} = ta_t = b_t$, since $\mathcal{R}(a_t^{1/2}) \subseteq E_t$.

The range of $z_t$ is contained in that one of $t$, as one can read off from the definition of $z_t$. We show that $\mathcal{N}(z_t) = \mathcal{N}(t)$. Let $x \in \mathcal{N}(t) \subseteq \mathcal{D}(t^*t) \subseteq \mathcal{D}(z_t)$. From equation (12) we deduce $z_t x = a_{t^*}^{1/2} tx = 0$, so $x \in \mathcal{N}(z_t)$. Assume $x \in \mathcal{N}(z_t)$ now. Then $(1 - a_t)x = z_{t^*} z_t x = 0$, and $x = a_t x \in \mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$. We obtain $(1 + t^*t)x = x$ and $t^*tx = 0$. From $\langle tx, tx\rangle = \langle t^*tx, x\rangle = 0$ we get $x \in \mathcal{N}(t)$. Together this proves $\mathcal{N}(z_t) = \mathcal{N}(t)$.

Using [Lan95] Proposition 3.7 again and the representation of $a_t$ that was shown above we get

$$\begin{aligned}\overline{\mathcal{R}(1 - z_{t^*} z_t)} &= \overline{\mathcal{R}(a_t \restriction_{\mathcal{D}(t^*t)})} = \overline{\mathcal{R}(a_t \restriction_{\overline{\mathcal{R}(a_t)}})} \supseteq \overline{\mathcal{R}(a_t \restriction_{\mathcal{R}(a_t)})} \\ &= \overline{\mathcal{R}(a_t^2)} = \overline{\mathcal{R}(a_t)} = \overline{\mathcal{D}(t^*t)} = E_t.\end{aligned}$$

That is $z_t \in \mathcal{Z}^d(E_t, F_{t^*})$. Hence $t_{z_t} : E_t \to F_{t^*}$ is a regular operator by [Lan95] Theorem 10.4.

We prove now that $(t_{z_t})^{**} = t$. First, $t_{z_t}$ is an restriction of $t$: Since $t_{z_t}$ is regular, $\mathcal{D}(t_{z_t}^\star t_{z_t}) = \mathcal{R}(1 - z_t^\star z_t) = \mathcal{R}(1 - z_{t^*} z_t) = \mathcal{R}(a_t \restriction_{E_t})$ is a core for $t_{z_t}$. Let $x \in \mathcal{R}(a_t \restriction_{E_t})$, that is $x = a_t y$ for some $y \in E_t$. In particular, $x \in \mathcal{R}(a_t) = \mathcal{D}(t^*t) \subseteq \mathcal{D}(t)$. Note that $a_t^{1/2} \restriction_{E_t}$ equals $(a_t \restriction_{E_t})^{1/2}$, since $a_t^{1/2} \restriction_{E_t}$ is a positive operator which squares to $a_t \restriction_{E_t}$. We compute

$$\begin{aligned}tx = ta_t y = ta_t^{1/2} \restriction_{E_t} a_t^{1/2} y = z_t a_t^{1/2} y = z_t (a_t \restriction_{E_t})^{-1/2} (a_t \restriction_{E_t})^{1/2} a_t^{1/2} y \\ = z_t (1 - z_t^\star z_t)^{-1/2} a_t^{1/2} \restriction_{E_t} a_t^{1/2} y = t_{z_t} a_t y = t_{z_t} x.\end{aligned}$$

That is, the restriction of $t_{z_t}$ to its core $\mathcal{D}(t_{z_t}^\star t_{z_t})$ is contained in $t$. Since $t : E \to F$ is closed (in the larger space), we get $t_{z_t} \subseteq t$.

Secondly, we prove that the range of $a_t^2$ is a core for $t$. From

$$\mathcal{D}(t_{z_t}) = \mathcal{R}((1 - z_t^\star z_t)^{1/2}) = \mathcal{R}(a_t^{1/2} \restriction_{E_t}) = \mathcal{R}(a_t^{1/2} \restriction_{\overline{\mathcal{R}(a_t)}}) \supseteq \mathcal{R}(a_t^2)$$

it will follow that $t = (t_{z_t})^{**}$, where we have used [Lan95] Proposition 3.7 again. Assume that $(x, tx) \perp \mathcal{G}(t \restriction_{\mathcal{R}(a_t^2)})$ for some $x \in \mathcal{D}(t)$. Then

$$\begin{aligned}\big\langle (x, tx), (a_t^2 y, ta_t^2 y)\big\rangle &= \big\langle x, a_t^2 y\big\rangle + \big\langle tx, ta_t^2 y\big\rangle = \langle a_t x, a_t y\rangle + \langle tx, b_t a_t y\rangle \\ &= \langle a_t x, a_t y\rangle + \langle b_t^* tx, a_t y\rangle\end{aligned}$$

vanishes for all $y \in E$. Since the range of $a_t$ is essential,

$$0 = (a_t + b_t^* t)x = (a_t + 1 - a_t)x = x.$$

So $\mathcal{G}(t) \cap \mathcal{G}(t \restriction_{\mathcal{R}(a_t^2)})^\perp = \{0\}$ is shown. With Proposition 13 we finally get

$$\mathcal{G}(t \restriction_{\mathcal{R}(a_t^2)})^\perp = \mathcal{G}(t \restriction_{\mathcal{R}(a_t^2)})^\perp \cap (\mathcal{G}(t) \oplus \mathcal{G}(t)^\perp) = \mathcal{G}(t)^\perp,$$

since $t$ is graph regular. □

DEFINITION. For $t \in \mathcal{R}_{gr}(E, F)$ we call $t_{z_t}$ the *regular operator associated to* $t$.

The next example shows, that if $E'$ and $F'$ are essential submodules of $E$ and $F$ respectively, not all regular operators $t : E' \to F'$ can be extended to a graph regular operator from $E$ into $F$. Again, the commutative case already inherits such a situation.

EXAMPLE 109. Let $E = C_0(\mathbb{R})$, $E' = C_0(\mathbb{R}^\times)$ and $m^\times : \mathbb{R}^\times \to \mathbb{C}$ be the sign function. Then $t_{m^\times} \in \mathcal{L}(E')$, since $\mathtt{reg}(m^\times) = \mathbb{R}^\times$. Assume there exists an operator $t \in \mathcal{R}_{gr}(E)$ with $t \supseteq t_{m^\times}$. Then there is a function $m : \mathbb{R} \to \mathbb{C}$ with $t = t_m \supseteq t_{m^\times}$. The intersection $\mathtt{reg}(m) \cap \mathbb{R}^\times$ is dense in $\mathbb{R}$. For $x \in \mathtt{reg}(m) \cap \mathbb{R}^\times$ there exists $f_x \in E'$ with $f_x(x) \neq 0$ and compact support (not intersecting 0); it is

$$m(x) f_x(x) = (t_m f_x)(x) = (t_{m^\times} f_x)(x) = m^\times(x) f_x(x), \quad \text{hence} \quad m(x) = m^\times(x).$$

This implies $\hat{m} = m^\times$ on $\mathbb{R}^\times$. Hence $t_m$ is not graph regular by Theorem 68, since $\texttt{sing-supp}_\texttt{r}(m) = \texttt{sing-supp}_\texttt{r}(\hat{m}) = \{0\}$.

Consider the bounded transform $z_t = t a_t^{1/2} \restriction_{E_t}$: Note that in general the range of $a_t^{1/2}$ is not contained in $\mathcal{D}(t)$. But as we have seen, the range of $a_t^{1/2} \restriction_{E_t}$ is contained in $\mathcal{D}(t)$.

Example 110. For the graph regular operator $t_m$ with symbol given by the function $m(x) := 1/x$ on $E = C_0(\mathbb{R})$, we compute (using Theorem 68)

$$a_{t_m} = t_{\frac{x^2}{1+x^2}}; \quad a_{t_m}^{1/2} = t_{\frac{|x|}{\sqrt{1+x^2}}},$$

since $a_{t_m}^{1/2}$ has to be positive. Let $f \in E = \mathcal{D}(a_{t_m}^{1/2})$ be such that $f \equiv 1$ on a neighbourhood $U$ of 0. For $x \in U \setminus \{0\}$ we have

$$g(x) := \frac{1}{x}(a_{t_m}^{1/2} f)(x) = \frac{1}{x}\frac{|x|}{\sqrt{1+x^2}} = \frac{\texttt{sgn}(x)}{\sqrt{1+x^2}},$$

hence $0 \notin \texttt{reg}(\hat{g})$ and $a_{t_m}^{1/2} f \notin \mathcal{D}(t_m)$.

Corollary 111. Let $z \in \mathcal{Z}(E,F)$. Then $(z_{t_z})^{**} = z$.

Proof. By Theorem 106 $t_z \in \mathcal{R}_{gr}(E,F)$ and $z = t_z a_{t_z}^{1/2} \in \mathcal{L}(E,F)$. By Theorem 108 it is $z_{t_z} = t_z a_{t_z}^{1/2} \restriction_{E_{t_z}}$. It follows $(z_{t_z})^{**} = z$ by Example 23, since $E_{t_z} = \overline{\mathcal{D}(t_z^* t_z)}$ is essential. □

Theorem 112. Let $t \in \mathcal{R}_{gr}(E,F)$ and assume $z := z_t^{**} \in \mathcal{L}(E,F)$. Then $z \in \mathcal{Z}(E,F)$ and $t = t_z$.

Proof. We compute, using Theorem 108 and $a_{t^*}^{1/2} t = t a_t^{1/2}$ on $\mathcal{R}(a_t) \subseteq \mathcal{D}(t)$ (see equation (13))

$$\begin{aligned} z^* z &= (t a_t^{1/2} \restriction_{E_t})^* (t a_t^{1/2} \restriction_{E_t})^{**} \supseteq (\overline{a_{t^*}^{1/2} t} \restriction_{E_t})^* t a_t^{1/2} \restriction_{E_t} \supseteq (a_{t^*}^{1/2} t)^* t a_t^{1/2} \restriction_{E_t} \\ &\supseteq t^* a_{t^*}^{1/2} t a_t^{1/2} \restriction_{E_t} \supseteq t^* a_{t^*}^{1/2} t a_t^{1/2} \restriction_{\mathcal{R}(a_t^{1/2})} = t^* t a_t \restriction_{\mathcal{R}(a_t^{1/2})} = (1 - a_t) \restriction_{\mathcal{R}(a_t^{1/2})}. \end{aligned}$$

Hence $z^* z = 1 - a_t$ by Example 23. That is $1 - z^* z = a_t$ is injective, hence $z \in \mathcal{Z}(E,F)$.

By Theorem 106 $t_z \in \mathcal{R}_{gr}(E,F)$ and $a_{t_z} = 1 - z^* z$. In particular $a_{t_z} = a_t$. We prove $b_{t_z} = b_t$ now and conclude then from Theorem 56 that $t = t_z$:

$$b_{t_z} = t_z a_{t_z} = z(1 - z^* z)^{-1/2} a_t = z a_t^{1/2} \supseteq z_t a_t^{1/2} = t a_t^{1/2} \restriction_{E_t} a_t^{1/2} = t a_t = b_t.$$

So $b_{t_z} = b_t$, since $b_t \in \mathcal{L}(E,F)$. □

Corollary 113. Let $t \in \mathcal{R}_{gr}(E,F)$. Equivalent are:

(1) $t = t_z$ for some $z \in \mathcal{Z}(E,F)$.
(2) $z_t^{**} \in \mathcal{L}(E,F)$.

In this case is $z = z_t^{**}$.

Proof. (1) ⇒ (2): Corollary 111. (2) ⇒ (1): Theorem 112. □

Definition. We say that $t$ has an *adjointable bounded transform* if one of the equivalent conditions of Corollary 113 is fulfilled for $t$.

By [Lan95] Theorem 10.4 regular operators have an adjointable bounded transform.

# 9. Absolute value and polar decomposition

Before a polar decomposition can be stated the absolute value for graph regular operators has to be introduced. The absolute value of an graph regular operator is defined and shown to be graph regular again (Theorem 114). We proof the existence of the polar decomposition for an adjointable operator $t$ in the case that $\overline{\mathcal{R}(t)}$ and $\overline{\mathcal{R}(t^*)}$ are orthogonally closed (Theorem 118). The generalized concept of partial isometries is essentially used her. As a corollary we derive the polar decomposition for a graph regular operator $t_z$ having an adjointable bounded transform $z$ if $\overline{\mathcal{R}(t_z)}$ and $\overline{\mathcal{R}(t_z^*)}$ are orthogonally closed (Theorem 119).

Theorem 114. Suppose that $t \in \mathcal{R}_{gr}(E)$ and define

$$\mathcal{D}(|t|) := \mathcal{R}(a_t^{1/2}), \quad |t|(a_t^{1/2}x) := (1-a_t)^{1/2}x \quad (x \in E).$$

Then $|t| \in \mathcal{R}_{gr}(E)$ is self-adjoint and positive. It is $|t|^2 = t^*t$, $a_{|t|} = a_t$, and $b_{|t|} = |b_t|$. Moreover $\overline{\mathcal{R}(|t|)} = \overline{\mathcal{R}(t^*t)}$.

Proof. By Example 85 $|t| = |t|^* \in \mathcal{R}_{gr}(E)$. Further, $\mathcal{D}(|t|^2) \supseteq \mathcal{R}(a_t) = \mathcal{D}(t^*t)$. It is $|t|^2 a_t = 1 - a_t = t^*ta_t$, so $|t|^2 \supseteq t^*t$. Since $t^*t$ is self-adjoint and $|t|^2 = |t|^*|t|$ is symmetric, we obtain $|t|^2 = t^*t$. Since $a_t - a_t^2$ is positive,

$$\left\langle |t|(a_t^{1/2}x), a_t^{1/2}x \right\rangle = \left\langle (1-a_t)^{1/2}, a_t^{1/2}x \right\rangle = \left\langle (a_t - a_t^2)^{1/2}x, x \right\rangle \geq 0 \quad \text{for} \quad x \in E,$$

so $|t|$ is positive, too. Clearly, $a_t = a_{|t|}$. Further,

$$b_{|t|} = |t|a_t = |t|a_t^{1/2}a_t^{1/2} = (1-a_t)^{1/2}a_t^{1/2} = (a_t - a_t^2)^{1/2} = (b_t^* b_t)^{1/2} = |b_t|.$$

Using [Lan95] Proposition 3.7, we finally get

$$\overline{\mathcal{R}(|t|)} = \overline{\mathcal{R}((1-a_t)^{1/2})} = \overline{\mathcal{R}(1-a_t)} = \overline{\mathcal{R}(t^*t(1+t^*t)^{-1})} = \overline{\mathcal{R}(t^*t)}.$$

□

Let us consider the relation of the absolute value and the bounded transform in the case that the latter one is adjointable; we use this when transferring the polar decomposition from adjointable operators to graph regular ones.

Lemma 115. If $z \in \mathcal{Z}(E,F)$, then $|z| \in \mathcal{Z}(E)$ and $|t_z| = t_{|z|}$. Further, we have $\mathcal{N}(z) = \mathcal{N}(t_z) = \mathcal{N}(|z|) = \mathcal{N}(|t_z|)$.

Proof. Since $1 - z^*z = 1 - |z|^2$ and $\||z|\| = \|z\| \leq 1$, the first statement is clear. Further, $a_{t_z} = (1 + t_z^* t_z)^{-1} = 1 - z^*z$ by Theorem 106, so

$$\mathcal{D}(|t_z|) = \mathcal{R}(a_{t_z}^{1/2}) = \mathcal{R}((1-z^*z)^{1/2}) = \mathcal{R}((1-|z|^2)^{1/2}) = \mathcal{D}(t_{|z|}),$$

$$|t_z|a_{t_z}^{1/2} = (1-a_{t_z})^{1/2} = (z^*z)^{1/2} = |z| = t_{|z|}(1-|z|^2)^{1/2},$$

that is, $|t_z| = t_{|z|}$. Since kernels of orthogonally closed operators are orthogonally closed, we obtain $\mathcal{N}(z) = \mathcal{R}(z^*)^\perp = \mathcal{R}(t_{z^*})^\perp = \mathcal{R}(t_z^*)^\perp = \mathcal{N}(t_z)$ and analogously $\mathcal{N}(|z|) = \mathcal{N}(t_{|z|}) = \mathcal{N}(|t_z|)$. Since $\mathcal{N}(z) = \mathcal{N}(|z|)$, this completes the proof. □

Corollary 116. For $t \in \mathcal{R}_{gr}(E,F)$ is $z_{|t|} = (z_t^\star z_t)^{1/2} = |z_t|$ and $|t| = t_{z^*_{|t|}}$. In particular $|t|$ has an adjointable bounded transform.

Proof. Using Theorem 114 and Theorem 108 several times we conclude that

$$\begin{aligned} z_{|t|}a_t^{1/2} &= |t|a_t = b_{|t|} = |b_t| = (b_t^* b_t)^{1/2} = (a_t - a_t^2)^{1/2} \\ &= (1-a_t)^{1/2}a_t^{1/2} = (1-z_t^\star z_t)^{1/2}a_t^{1/2} = |z_t|a_t^{1/2}. \end{aligned}$$

With $\mathcal{D}(|z_t|) = \mathcal{D}(z_t) = \mathcal{D}(z_{|t|}) = E_t = \overline{\mathcal{R}(a_t^{1/2})}$ this proves that $|z_t| = z_{|t|} = (1-a_t)^{1/2} \lceil_{E_t}$. By Example 23 $z_{|t|}^* = (1-a_t)^{1/2}$, since the latter operator is self-adjoint. Hence $z_{|t|}^* \in \mathcal{Z}(E)$ is self-adjoint. Hence $t_{z_{|t|}^*} = |t|$ by Corollary 113. $\square$

In contrast to the Hilbert space case the domains $\mathcal{D}(t)$ and $\mathcal{D}(|t|)$ do not coincide in general, even more, neither $\mathcal{D}(t) \subseteq \mathcal{D}(|t|)$ nor $\mathcal{D}(t) \supseteq \mathcal{D}(|t|)$ holds.

Example 117. Let $E = C([0,1])$ and set $m(x) := x^{-1}e^{i/x}$ on $(0,1]$. Then, by Corollary 35, the operator $t_m$ is graph regular, since we have $\mathtt{reg}(m) = (0,1]$, $\mathtt{reg}_\infty(m) = \{0\}$ and $\mathtt{sing\text{-}supp_r}(m) = \emptyset$. Further, $|t_m| = t_{|m|}$. Let $f(x) := xe^{-i/x}$ for $x \in (0,1]$ and $f(0) = 0$. Then $f \in \mathcal{D}(t_m)$, but $f \notin \mathcal{D}(|t_m|)$. The function $g(x) = x$ on $[0,1]$ is in $\mathcal{D}(|t_m|)$, but not in $\mathcal{D}(t_m)$.

This observation produces a problem when considering the polar decomposition of a graph regular operator: We do not have the isometric mapping $|t|x \mapsto tx$ for $x \in \mathcal{D}(|t|) = \mathcal{D}(t)$. So in the first step we generalize the polar decomposition known from [Lan95] p. 29-30 by using the generalized concept of partial isometries. In the next step we transfer the result to those graph regular operator that admit an adjointable bounded transform. This extends some results of [FS10a] p.381-383.

Theorem 118. Let $t \in \mathcal{L}(E,F)$. Then there is a partial isometry $v \in \mathcal{C}_o(E,F)$ with initial space $\overline{\mathcal{R}(t^*)}$ and final space $\overline{\mathcal{R}(t)}$ such that $t = v|t|$, $|t| = v^*t$ and $\mathcal{R}(v) = \overline{\mathcal{R}(t)}$, $\mathcal{R}(v^*) = \overline{\mathcal{R}(t^*)}$, $\mathcal{N}(v) = \mathcal{N}(t)$, and $\mathcal{N}(v^*) = \mathcal{N}(t^*)$ if and only if $\overline{\mathcal{R}(t)}$ and $\overline{\mathcal{R}(t^*)}$ are orthogonally closed.

Proof. The only if direction follows from the definition of partial isometries.

To prove the if part we assume that $\overline{\mathcal{R}(t)}$ and $\overline{\mathcal{R}(t^*)}$ are orthogonally closed. Using [Lan95] Proposition 3.7 twice (for $t$ and $|t|$) we get $\overline{\mathcal{R}(t^*)} = \overline{\mathcal{R}(t^*t)} = \overline{\mathcal{R}(|t|)}$. Define a mapping $w : \mathcal{R}(|t|) \to \mathcal{R}(t)$ by $w(|t|x) := tx$ for $x \in E$. Then $w$ is well-defined and isometric, since $\langle tx, tx\rangle = \langle |t|x, |t|x\rangle$ for $x \in E$. The continuous extension of $w$ to a mapping from $\overline{\mathcal{R}(|t|)}$ onto $\overline{\mathcal{R}(t)}$ is also an isometry which is denoted again by $w$. We define $v : E \to F$ by

$$v(x+y) := wx \quad (x \in \overline{\mathcal{R}(|t|)}, y \in \mathcal{N}(t)).$$

First, we want to show $v \in \mathcal{C}_o(E,F)$. Clearly,

$$\mathcal{N}(v) = \mathcal{N}(t), \quad \mathcal{N}(v)^\perp = \mathcal{R}(t^*)^{\perp\perp} = \overline{\mathcal{R}(t^*)} = \overline{\mathcal{R}(|t|)};$$

$v$ is in particular essentially defined. Further, $t = v|t|$ and $\mathcal{R}(v) = \overline{\mathcal{R}(t)}$. Let

$$v'(x+y) := w^{-1}x \quad (x \in \overline{\mathcal{R}(t)}, y \in \mathcal{N}(t^*)).$$

As above, $v'$ is essentially defined, $|t| = v't$, $\mathcal{N}(v') = \mathcal{N}(t^*)$ and $\mathcal{R}(v') = \overline{\mathcal{R}(t^*)}$. It is easily seen that $v' \subseteq v^*$ and $v \subseteq (v')^*$. Since $v'$ is essentially defined, so is $v^*$. Hence $v$ is orthogonally closable by Theorem 17.

We show that $v^* = v'$. Let $y \in \mathcal{D}(v^*)$. Then there is an element $z \in E$ such that $\left\langle v(x+x^\perp), y\right\rangle = \left\langle x + x^\perp, z\right\rangle$ for all $x \in \overline{\mathcal{R}(|t|)}$ and $x^\perp \in \mathcal{N}(t)$. Choosing $x = 0$, we conclude that $z \in \mathcal{N}(t)^\perp = \overline{\mathcal{R}(|t|)}$. Thus $\mathcal{R}(v^*) \subseteq \mathcal{R}(v')$. Putting now $x^\perp = 0$, we get $\langle tx', y\rangle = \langle v|t|x', y\rangle = \langle |t|x', z\rangle$ for all $x' \in E$. Hence $t^*y = |t|z = |t|v^*y$ and $\mathcal{N}(v^*) \subseteq \mathcal{N}(t^*) = \mathcal{N}(v)$. All in all we have $v' \subseteq v^*$, $\mathcal{R}(v^*) \subseteq \mathcal{R}(v')$ and $\mathcal{N}(v^*) \subseteq \mathcal{N}(v')$. This clearly implies that $v' = v^*$. Analogously, it is shown that

$v'^* = v$. Since $v^*v$ is the projection onto the orthogonally closed submodule $\overline{\mathcal{R}(t^*)}$ and $vv^*$ the projection onto $\overline{\mathcal{R}(t)}$, the assertion is proven. $\square$

For the polar decomposition for graph regular operators we have to restrict the statement to those operators having an adjointable bounded transform.

THEOREM 119. Assume $t \in \mathcal{R}_{gr}(E, F)$ has an adjointable bounded transform. There exists a partial isometry $v \in \mathcal{C}_o(E, F)$ with initial space $\overline{\mathcal{R}(t^*)}$ and final space $\overline{\mathcal{R}(t)}$, such that

$$t = v|t|, \quad |t| = v^*t,$$

$\mathcal{R}(v) = \overline{\mathcal{R}(t)}$, $\mathcal{R}(v^*) = \overline{\mathcal{R}(t^*)}$, $\mathcal{N}(v) = \mathcal{N}(t)$, and $\mathcal{N}(v^*) = \mathcal{N}(t^*)$ if and only if $\overline{\mathcal{R}(t)}$ and $\overline{\mathcal{R}(t^*)}$ are orthogonally closed.

PROOF. The "only if" part follows from the definition of partial isometries.

By Corollary 113 there exists $z \in \mathcal{Z}(E, F)$ with $t = t_z$. With Theorem 106, [Lan95] Proposition 3.7 and Lemma 115 it is

$$\overline{\mathcal{R}(t_z^*)} = \overline{\mathcal{R}(t_{z^*})} = \overline{\mathcal{R}(z^*)} = \overline{\mathcal{R}(|z|)} = \overline{\mathcal{R}(t_{|z|})} = \overline{\mathcal{R}(|t_z|)} \quad \text{and} \quad \overline{\mathcal{R}(t_z)} = \overline{\mathcal{R}(z)}.$$

Assume that $\overline{\mathcal{R}(t^*)} = \overline{\mathcal{R}(z^*)}$ and $\overline{\mathcal{R}(t)} = \overline{\mathcal{R}(z)}$ are orthogonally closed. By Theorem 118, there is a partial isometry $v \in \mathcal{C}_o(E, F)$ with $z = v|z|$, $|z| = v^*z$ and $\mathcal{R}(v) = \overline{\mathcal{R}(z)}$, $\mathcal{R}(v^*) = \overline{\mathcal{R}(z^*)}$, $\mathcal{N}(v) = \mathcal{N}(z)$ and $\mathcal{N}(v^*) = \mathcal{N}(z^*)$. We have $1 - |z|^2 = 1 - z^*z$ and $\mathcal{D}(t_{|z|}) = \mathcal{R}(1 - z^*z) = \mathcal{D}(t_z)$ by Theorem 106. With Lemma 115 we compute

$$v|t_z|(1 - |z|^2)^{1/2} = v|z| = z = t_z(1 - |z|^2)^{1/2},$$

so $v|t_z| = t_z$. One proves $v^*t_z = |t_z|$ in the same way. $\square$

Regular operators have adjointable bounded transforms, hence Theorem 119 applies in this case. That is, Theorem 119 generalizes [FS10a] Theorem 3.1 in two directions. First, it applies to a larger class of regular operators and secondly it applies to graph regular operators as well.

# 10. Functional calculus

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, where $\mathcal{A}$ is non-unital and $\mathcal{B}$ is unital. Clearly, each $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ extends uniquely to a $*$-homomorphism of the unitisation $\mathcal{A}^{\sim}$ of $\mathcal{A}$ via $\phi(a + \alpha 1_{\mathcal{A}}) := \phi(a) + \alpha 1_{\mathcal{B}}$ for $a \in \mathcal{A}$, $\alpha \in \mathbb{C}$.

Let $\zeta(z) := z$ for $z \in \mathbb{C}$. Then $t_\zeta : C_0(\mathbb{C}) \to C_0(\mathbb{C})$ is regular by Theorem 68, since $\mathtt{reg}(\zeta) = \mathbb{C}$.

The unitisation $\mathcal{C} := C_0(\mathbb{C})^{\sim}$ of $C_0(\mathbb{C})$ is isomorphic to $C(\overline{\mathbb{C}})$, where $\overline{\mathbb{C}}$ is the one-point compactification.[6]. The operator $t_\zeta^{\mathcal{C}}$ is no longer regular but still graph regular by Theorem 75, since $\mathtt{reg}(\zeta) = \mathbb{C}$, $\mathtt{reg}_\infty(\zeta) = \{\infty\}$ and $\mathtt{sing\text{-}supp_r}(\zeta) = \emptyset$ (where $\zeta$ also denotes any extension of $\zeta$ to $\overline{\mathbb{C}}$). We have

$$a_{t_\zeta^{\mathcal{C}}} = t^{\mathcal{C}}_{1/(1+|\zeta|^2)}, \quad b_{t_\zeta^{\mathcal{C}}} = t^{\mathcal{C}}_{\zeta/(1+|\zeta|^2)}.$$

Theorem 120. Let $E$ be a Hilbert $\mathcal{A}$-module and $t \in \mathcal{R}_{gr}(E)$ be normal and let $\zeta$ also denote the operator $t_\zeta^{\mathcal{C}}$. Then there exists an unique $*$-homomorphism $\phi_t : C_0(\mathbb{C})^{\sim} \to \mathcal{L}(E)$ with $\mathcal{N}(\phi_t(a_\zeta)) = \{0\}$ and $\phi_t(\zeta) = t$.

Proof. Let

$$D := \{z \in \mathbb{C} : |z| \le \frac{1}{2}\}, \quad F := \{(z_1, z_2) \in [0,1] \times D : |z_2|^2 = z_1 - z_1^2\} \subseteq [0,1] \times D.$$

Further let $F_\pi := \{(0,0)\}$ and $F_\iota := F \setminus F_\pi$. Since $\mathcal{C}$ is unital, we identify it with $\mathcal{L}(\mathcal{C})$. By Corollary 57 $a_\zeta$ is self-adjoint, $b_\zeta$ is normal and they commute. Further, their common spectrum $\sigma(a_\zeta, b_\zeta)$ is contained in $F$. Analogues statements hold for $a_t$ and $b_t$.

Uniqueness: Let $\phi : \mathcal{C} \cong \mathcal{L}(\mathcal{C}) \to \mathcal{L}(E)$ with $\mathcal{N}(\phi(a_\zeta)) = \{0\}$ and $\phi(\zeta) = t$. By Proposition 86 is

$$\phi(a_\zeta) = a_{\phi(\zeta)} = a_t, \quad \phi(b_\zeta) = b_{\phi(\zeta)} = b_t.$$

For $f \in C(F)$ is by functional calculus of bounded normal commuting operators

$$(*) \qquad \phi(f(a_\zeta, b_\zeta)) = f(\phi(a_\zeta), \phi(b_\zeta)) = f(a_t, b_t).$$

Every function $g + \beta \in C_0(\mathbb{C})^{\sim} = \mathcal{C}$ is of the form $f(a, b)$ for a function $f \in C(F)$ with $f \restriction_{F_\pi} \equiv \beta$, since

$$g(z) = g(a_\zeta(z)^{-1} b_\zeta(z)) = f(a_\zeta, b_\zeta)(z),$$

with

$$f(z_1, z_2) := \begin{cases} g(z_2/z_1) + \beta & , (z_1, z_2) \in F_\iota \\ \beta & , (z_1, z_2) \in F_\pi \end{cases}.$$

To show that $f$ is continuous on $F_\pi$, assume $(z_1, z_2) \to (0,0)$. From $|z_2|^2 = z_1 - z_1^2$ it follows $|z_2/z_1)| = \sqrt{1/z_1 - 1} \to \infty$ since $z_1 \to 0$. Therefore $g(z_2/z_1) \to 0$, since $g$ vanishes at infinity. This proves the uniqueness assertion.

Existence: On the other side equation $(*)$ gives a $*$-homomorphism from $C_0(\mathbb{C})^{\sim}$ into $\mathcal{L}(E)$. With $f(z_1, z_2) := z_1$ inserted in equation $(*)$ it follows $\mathcal{N}(\phi(a_\zeta)) = \mathcal{N}(a_t)$, and the latter is trivial. Analogously $\phi(b_\zeta) = b_t$. By Proposition 86 is $a_{\phi(\zeta)} = \phi(a_\zeta) = a_t$ and $b_{\phi(\zeta)} = \phi(b_\zeta) = b_t$. With Theorem 56 we finally conclude that $\phi(\zeta) = t$. □

[6] See [Kha09] Section 1.1.

# 11. Special matrices of $C^*$-algebras

## Counter examples

Several counter examples had already be given in the commutative case. Further phenomena depend on non-commutativity, which can be brought into the game when considering matrices with entries in (again commutative but also non-commutative) $C^*$-algebras.

We use the following notation: Let $\mathcal{A}$ be a $C^*$-algebra. For subsets $A_{ij} \subseteq \mathcal{A}$ for $i, j \in \{1, 2\}$ let

$$\begin{pmatrix} A_{11} & A_{22} \\ A_{21} & A_{22} \end{pmatrix} := \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} | a_{ij} \in A_{ij} \text{ for } i, j \in \{1, 2\} \right\}.$$

The following lemma describes a distinguished situation that we use below several times. For a (non-degenerated) $C^*$-algebra $\mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$ and a closed two-sided $\mathcal{I}$ of $\mathcal{A}$ we use the following sets of multipliers:

$$\begin{aligned} \mathtt{LM}(\mathcal{A}) &:= \{X \in \mathbf{B}(\mathcal{H}) | X\mathcal{A} \subseteq \mathcal{A}\}, \mathtt{RM}(\mathcal{A}) := \{X \in \mathbf{B}(\mathcal{H}) | \mathcal{A}X \subseteq \mathcal{A}\}, \\ \mathtt{M}(\mathcal{A}) &:= \mathtt{LM}(\mathcal{A}) \cap \mathtt{RM}(\mathcal{A}), \\ \mathtt{LM}(\mathcal{A}, \mathcal{I}) &:= \{X \in \mathbf{B}(\mathcal{H}) | X\mathcal{A} \subseteq \mathcal{I}\}, \mathtt{RM}(\mathcal{A}, \mathcal{I}) := \{X \in \mathbf{B}(\mathcal{H}) | \mathcal{A}X \subseteq \mathcal{I}\}. \end{aligned}$$

If $\mathcal{A}$ is unital, then obviously

$$\mathtt{LM}(\mathcal{A}) = \mathtt{RM}(\mathcal{A}) = \mathtt{M}(\mathcal{A}) = \mathcal{A}, \quad \mathtt{LM}(\mathcal{A}, \mathcal{I}) = \mathtt{RM}(\mathcal{A}, \mathcal{I}) = \mathcal{I}.$$

Lemma 121. Let $\mathcal{I}$ be a closed two-sided ideal of a $C^*$-algebra $\mathcal{A}$ and set

$$\mathcal{B} := \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & \mathcal{A} \end{pmatrix}.$$

Then $\mathcal{B}$ is itself a $C^*$-algebra (with obvious addition, multiplication, adjoint and norm). If $\mathcal{I}$ is non-degenerated on $\mathcal{H}$, then $\mathcal{B}$ is non-degenerated as well and

$$\mathtt{LM}(\mathcal{B}) = \begin{pmatrix} \mathtt{LM}(\mathcal{I}) & \mathtt{LM}(\mathcal{A}, \mathcal{I}) \\ \mathtt{LM}(\mathcal{I}) & \mathtt{LM}(\mathcal{A}) \end{pmatrix}, \mathtt{M}(\mathcal{B}) = \begin{pmatrix} \mathtt{M}(\mathcal{I}) & \mathtt{LM}(\mathcal{A}, \mathcal{I}) \cap \mathtt{RM}(I) \\ \mathtt{LM}(\mathcal{I}) \cap \mathtt{RM}(\mathcal{A}, \mathcal{I}) & \mathtt{M}(\mathcal{A}) \end{pmatrix}.$$

If in particular $\mathcal{A}$ is unital, then

$$\mathtt{LM}(\mathcal{B}) = \begin{pmatrix} \mathtt{LM}(\mathcal{I}) & \mathcal{I} \\ \mathtt{LM}(\mathcal{I}) & \mathcal{A} \end{pmatrix}, \qquad \mathtt{M}(\mathcal{B}) = \begin{pmatrix} \mathtt{M}(\mathcal{I}) & \mathcal{I} \\ \mathcal{I} & \mathcal{A} \end{pmatrix}.$$

Proof. Clearly, $\mathcal{B}$ is a $C^*$-algebra; we omit the details. If $\mathcal{I}$ is non-degenerated, the same is clearly true for $\mathcal{B} \subseteq \mathbf{B}(\mathcal{H} \oplus \mathcal{H})$. Now, we try to calculate the left-multiplier of $\mathcal{B}$. Let $a \in \mathbf{B}(\mathcal{H})$, that is, $A = (a_{ij})_{ij}$ with $a_{ij} \in \mathbf{B}(\mathcal{H})$ for $i, j \in \{1, 2\}$. For each $b = (b_{ij})_{ij} \in \mathcal{B}$ we compute

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \in \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & \mathcal{A} \end{pmatrix}$$

we deduce: $a_{11}\mathcal{I}, a_{21} \subseteq \mathcal{I}$, (set $b_{21} = 0$), $a_{12}\mathcal{A} \subseteq \mathcal{I}$, $a_{22}\mathcal{A} \subseteq \mathcal{A}$ (set $b_{12} = 0$). That is $a_{11} \in \mathtt{LM}(\mathcal{I})$, $a_{12} \in \mathtt{LM}(\mathcal{A}, \mathcal{I})$, $a_{21} \in \mathtt{LM}(\mathcal{I})$, $a_{22} \in \mathtt{LM}(\mathcal{A})$. Hence one inclusion for $\mathtt{LM}(\mathcal{B})$ is proven. The inverse inclusion is easily checked. The other statement follow now directly. $\square$

We will use this result for two cases. Let $\mathcal{A} = C_0(X)^\sim$ and $\mathcal{I} = C_0(X)$ for some locally compact Hausdorff space $X$. Then $\mathcal{A}$ is unital. We let

$$\mathcal{B}_0 = \begin{pmatrix} C_0(X) & C_0(X) \\ C_0(X) & C_0(X) \end{pmatrix}.$$

It is $\mathtt{LM}(C_0(X)) = \mathtt{M}(C_0(X)) = C_b(X)$; with Lemma 121 it is

$$\mathcal{B} = \begin{pmatrix} C_0(X) & C_0(X) \\ C_0(X) & C_0(X)^\sim \end{pmatrix},$$

$$\mathtt{LM}(\mathcal{B}) = \begin{pmatrix} C_b(X) & C_0(X) \\ C_b(X) & C_0(X)^\sim \end{pmatrix}, \qquad \mathtt{M}(\mathcal{B}) = \begin{pmatrix} C_b(X) & C_0(X) \\ C_0(X) & C_0(X)^\sim \end{pmatrix}.$$

From this we can read off that all elements of the form

$$\begin{pmatrix} * & * \\ f & * \end{pmatrix} \in \mathtt{LM}(\mathcal{B}) \quad \text{with} \quad f \in C_b(X) \setminus C_0(X)$$

act as operators on $\mathcal{B}$ such that the adjoint is only defined on $\mathcal{B}_0$.

In the next example we construct an operator $t$ on $\mathcal{B}$ with $a_t$, $b_t$ adjointable but $a_{t^*}$ is not.

Example 122. Let $f, g \in C(\mathbb{R})$ be functions given by $f(x) := x\sqrt{1+\sin^2(x)}$ and $g(x) := x\sqrt{1+\cos^2(x)}$. Then $|f(x)|^2 + |g(x)|^2 = 3x^2$. Define $t : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{D}(t) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B} |\, fc, fd, gc \in C_0(\mathbb{R}), gd \in C_0(\mathbb{R})^\sim \right\}, \quad t = \begin{pmatrix} 0 & f \\ 0 & g \end{pmatrix}.$$

For the adjoint $t^*$ we obtain

$$\mathcal{D}(t^*) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B} |\, \overline{f}a + \overline{g}c \in C_0(\mathbb{R}), \overline{f}b + \overline{g}d \in C_0(\mathbb{R})^\sim \right\}, \quad t = \begin{pmatrix} 0 & 0 \\ \overline{f} & \overline{g} \end{pmatrix}.$$

It is now easily verified that $1 + t^*t$ is surjective and

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+|f|^2+|g|^2} \end{pmatrix} \in \mathcal{B}, \quad b_t = \begin{pmatrix} 0 & \frac{f}{1+|f|^2+|g|^2} \\ 0 & \frac{g}{1+|f|^2+|g|^2} \end{pmatrix} \in \mathcal{B}.$$

The operator $a_{t^*}$ is computed as

$$\mathcal{D}(a_{t^*}) = \mathcal{B}_0, \quad a_{t^*} = \frac{1}{1+|f|^2+|g|^2} \begin{pmatrix} 1+|g|^2 & -\overline{f}g \\ -\overline{g}f & 1+|f|^2 \end{pmatrix}.$$

That is, $a_t \in \mathcal{A}$ and $b_t \in \mathcal{A}$ are adjointable but $a_{t^*} \notin \mathtt{M}(\mathcal{A})$ is not adjointable, since

$$\frac{\overline{f(x)}g(x)}{1+|f(x)|^2+|g(x)|^2} = \frac{x^2}{1+3x^2}\sqrt{(1+\sin^2(x))(1+\cos^2(x))}$$

does not vanish at infinity.

Now, we sate a result for a (non-degenerated) $C^*$-algebra $\mathcal{B} \subseteq \mathbf{B}(\mathcal{H})$. The possibility of fulfilling the assumptions will be demonstrated thereafter in Example 124. Compare this example with Proposition 91.

Lemma 123. Let $x \in \mathbf{B}(\mathcal{H})$ with trivial kernel and dense range in $\mathcal{H}$. Assume further, $x \in \mathtt{M}(\mathcal{B}) \subseteq \mathbf{B}(\mathcal{H})$ and $x\mathcal{B}$ is dense in $\mathcal{B}$ but $x^*\mathcal{B}$ is not dense in $\mathcal{B}$.

Then $T\mu\mathcal{B}$, but $T$ is not affiliated with $\mathcal{B}$; it is

(R) $\qquad T^{-1}\mathcal{B}$ is dense $\mathcal{B}$, but $(T^*)^{-1}\mathcal{B}$ is not dense $\mathcal{B}$,

(A) $\qquad a_T\mathcal{B}$ is dense $\mathcal{B}$, but $a_{T^*}\mathcal{B}$ is not dense $\mathcal{B}$.

PROOF. Clearly, $T := x^{-1} \in \mathcal{C}(\mathcal{H})$ and $T^{-1} \in \mathtt{M}(\mathcal{B})$, hence $T\mu\mathcal{B}$ by Proposition 94. The statement (R) follows directly from the assumptions. Since $I+(T^*T)^{-1}$ is a bijective element of $\mathtt{M}(\mathcal{B})$, we compute

$$(I + T^*T)^{-1}\mathcal{B} = (T^*T)^{-1}(I + (T^*T)^{-1})^{-1}\mathcal{B} = (T^*T)^{-1}\mathcal{B} = T^{-1}(T^*)^{-1}\mathcal{B} = xx\mathcal{B}.$$

Analogously

$$(I + TT^*)^{-1}\mathcal{B} = T^{-1}(T^*)^{-1}\mathcal{B} = x^*x\mathcal{B}.$$

Hence $a_T\mathcal{B} = xx^*\mathcal{B}$ and $a_{T^*}\mathcal{B} = x^*x\mathcal{B}$, so (A) follows from Lemma 90. □

Now we construct such a $x \in \mathbf{B}(\mathcal{H})$, where we chose $\mathcal{H}$ to be the Hilbert space $\ell^2(\mathbb{N}^2)$. Further let $\mathcal{A} = \mathbf{B}(\mathcal{H})$ and $\mathcal{I} = \mathcal{K}(\mathcal{H})$. With Lemma 121 it is

$$\mathcal{B} = \begin{pmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{K}(\mathcal{H}) \\ \mathcal{K}(\mathcal{H}) & \mathbf{B}(\mathcal{H}) \end{pmatrix}, \quad \mathtt{M}(\mathcal{B}) = \begin{pmatrix} \mathbf{B}(\mathcal{H}) & \mathcal{K}(\mathcal{H}) \\ \mathcal{K}(\mathcal{H}) & \mathbf{B}(\mathcal{H}) \end{pmatrix}.$$

EXAMPLE 124. Let $\{e_{kl}\}_{k,l\in\mathbb{N}}$ be the standard orthonormal basis of $\mathcal{H}$. Let $s \in \mathbf{B}(\mathcal{H})$ be the shift operator given by $se_{k,l} = e_{k+1,l}$ for $k, l \in \mathbb{N}$; the adjoint $s^*$ acts as $s^*e_{k+1,l} = e_{k,l}$ for $k, l \in \mathbb{N}$ and $s^*e_{1,l} = 0$ for $l \in \mathbb{N}$. Let $P$ be the orthogonal projection onto $\mathcal{N}(s^*)$. Clearly, $\{e_{1,l}\}_{l\in\mathbb{N}}$ is an orthonormal basis of $P\mathcal{H}$. Further, let $\{\lambda_{k,l}\}_{k,l\in\mathbb{N}}$ be a double sequence of positive numbers such that $\lim_{k,l\to\infty} \lambda_{kl} = 0$. Define a self-adjoint compact operator on $\mathcal{H}$ by $re_{k,l} := \lambda_{k,l}e_{k,l}$, $k, l \in \mathbb{N}$.

Let $x \in \mathbf{B}(\mathcal{H} \oplus \mathcal{H})$ be defined by the operator matrix

$$x := \begin{pmatrix} s & r \\ 0 & s^* \end{pmatrix} \in \mathtt{M}(\mathcal{B}). \tag{14}$$

Since $\lambda_{kl} > 0$ for all $k, l \in \mathbb{N}$, the compression $Pr\lceil_{P\mathcal{H}}$ of $r$ to $P\mathcal{H}$ has trivial kernel and dense range. Using this fact it is easily seen that $\mathcal{N}(x) = \{0\}$ and $\mathcal{R}(x)$ is dense in $\mathcal{H} \oplus \mathcal{H}$.

$x\mathcal{A}$ is dense in $\mathcal{A}$: Let $y$ be an element of $\mathcal{A}$. Then $y$ is given by

$$y := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$a, b, c \in \mathcal{K}(\mathcal{H})$ and $d \in \mathbf{B}(\mathcal{H})$, and we have

$$xy = \begin{pmatrix} sa + rc & sb + rd \\ s^*c & s^*d \end{pmatrix}. \tag{15}$$

Since $\mathcal{K}(\mathcal{H}) = s^*s\mathcal{K}(\mathcal{H}) \subseteq s^*\mathcal{K}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, we have $s^*\mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$. Similarly, $s^*\mathbf{B}(\mathcal{H}) = \mathbf{B}(\mathcal{H})$. Let $e := e_{k,l}\langle e_{n,m}, .\rangle$ be a rank one operator on $\mathcal{H}$. If $k > 1$, then $se_{k-1,l}\langle e_{n,m}, .\rangle = e$. If $k = 1$, then $e \in \mathcal{N}(s^*) \cap \mathcal{K}(\mathcal{H})$ and $re/\lambda_{1,l} = e$. Hence $s\mathcal{K}(\mathcal{H}) + r(\mathcal{N}(s^*) \cap \mathcal{K}(\mathcal{H}))$ is dense in $\mathcal{K}(\mathcal{H})$. Therefore, by (15), $x\mathcal{B}$ is dense in $\mathcal{B}$.

$x^*\mathcal{A}$ is not dense in $\mathcal{A}$: First we note that $Ps = 0$ and $P$ is not compact, $P$ is not in the closure of $r\mathcal{K}(\mathcal{H}) + s\mathbf{B}(\mathcal{H})$. Therefore, since

$$x^*y = \begin{pmatrix} s^*a & s^*b \\ ra + sc & rb + sd \end{pmatrix},$$

it follows that the set $x^*\mathcal{A}$ is not dense in $\mathcal{A}$.

# Abstract and open Questions

Let $E$ and $F$ be Hilbert $C^*$-modules over a $C^*$-algebra $\mathcal{A}$. New classes of (possibly unbounded) operators $t : E \to F$ are introduced and investigated - first of all *graph regular operators*. Instead of the density of the domain $\mathcal{D}(t)$ we only assume that $t$ is essentially defined, that is, $\mathcal{D}(t)^\perp = \{0\}$. Then $t$ has a well-defined adjoint. We call an essentially defined operator $t$ graph regular if its graph $\mathcal{G}(t)$ is orthogonally complemented in $E \oplus F$ and orthogonally closed if $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t)$. A theory of these operators and related concepts is developed: polar decomposition, functional calculus. Various characterizations of graph regular operators are given: $(a, a_*, b)$-transform and bounded transform. A number of examples of graph regular operators are presented ($E = C_0(X)$, a fraction algebra related to the Weyl algebra, Toeplitz algebra, Heisenberg group). A new characterization of affiliated operators with a $C^*$-algebra in terms of resolvents is given as well as a Kato-Rellich theorem for affiliated operators. The association relation is introduced and studied as a counter part of graph regularity for concrete $C^*$-algebras.

A further development of the theory of orthogonally closed and graph regular operators is possible e.g. concerning the spectrum of operators and the Cayley-transform (theory of symmetric and self-adjoint operators). We have seen that more examples of graph regular operators can easy be obtained via inverses and quotients of adjointable operators.

Considering the bounded transform a question at hand is: Let $t \in \mathcal{R}(E, F)$ (or $\mathcal{R}(E, F)$) and $E'$, $F'$ are Hilbert $C^*$-modules such that $E \subseteq E'$ and $F \subseteq F'$ are essential. Is there a $t' \in \mathcal{R}_{gr}(E', F')$ such that $t$ is the regular operator that is associated with $t'$?

Another point is the following. For $t \in \mathcal{C}_o(E, F)$ it was shown (Lemma 42) that

$$\mathcal{R}(1 + t^*t) \oplus \mathcal{R}(1 + tt^*) \subseteq \mathcal{G}(t) \oplus v\mathcal{G}(t^*) \subseteq E \oplus F.$$

For weakly regular operators the left hand side is essential. But for each operator $\mathcal{G}(t) \oplus v\mathcal{G}(t^*)$ is essential in any case (Corollary 16). Is every orthogonally closed operator already weakly regular? It can be shown that this is true for finite linear combinations of pairwise orthogonal projections. For operators on commutative $C^*$-algebras we have seen that the answer is affirmative (Theorem 68).

Another open question is whether Corollary 87 (2) remains valid for arbitrary $C^*$-algebras or not? This would connect graph regularity and association even more. For example the results for graph regular operators could be transferred to associated operators. E.g. it is unknown if $(T + A)\mu\mathcal{A}$ if $T\mu\mathcal{A}$ and $A \in \mathtt{M}(\mathcal{A})$. This would also improve the resolvent criterion for this relation.

As it was already noted, it is unknown whether or not $t^*$ is bounded provided $t \in \mathcal{C}_o(E, F)$ is bounded.

# Notations

**Numbers:**

| | |
|---|---|
| $\mathbb{N}$ | natural numbers; $0 \in \mathbb{N}$ |
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}_+$ | nonnegative real numbers |
| $\mathbb{R}^\times$ | nonzero real numbers |
| $\mathbb{C}$ | complex numbers: $\overline{\lambda}$ denotes the complex conjugate of $\lambda \in \mathbb{C}$ |
| $\overline{\mathbb{C}}$ | one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ |

**Function spaces:**

| | |
|---|---|
| $C(X)$ | continuous $\mathbb{C}$-valued functions on $X$ |
| $C_b(X)$ | continuous $\mathbb{C}$-valued bounded functions on $X$ |
| $C_0(X)$ | continuous $\mathbb{C}$-valued functions on $X$ vanishing at infinity |

**Vector spaces:**

If $E$ is a *vector space* over $\mathbb{C}$ and $(F_i)_{i \in I}$ is a family of subsets of $E$, then $\mathtt{span}\{F_i | i \in I\}$ denotes the $\mathbb{C}$-*linear span* of this family, that is the set all elements of the form $\sum_{i \in I} \lambda_i x_i$, where $x_i \in F_i$, $\lambda_i \in \mathbb{C}$ for all $i \in I$ and $\lambda_i = 0$ for almost all indices $i \in I$.

**Topology:**

If $X$ is a topological space and $A \subseteq X$ is a subset, then $\overline{A}$ denotes the *closure*, $\partial A$ the *boundary* and $A^\circ$ the *interior* of $A$.

**$*$-algebras:**

If $\mathcal{A}$ is a $*$-algebra, $\mathcal{A}_h := \{a \in \mathcal{A} | a = a^*\}$ denotes the set of *hermitian* elements of $\mathcal{A}$.

**$C^*$-algebras:**

If $\mathcal{A}$ is a $C^*$-algebra, $\mathcal{A}_+ := \{a \in \mathcal{A} | a \geq 0\}$ denotes the set of *positive* elements of $\mathcal{A}$. If additionally $\mathcal{A} \subseteq \mathbf{B}(\mathcal{H})$ is non-degenerately represented on a Hilbert space $\mathcal{H}$, $\mathtt{M}(\mathcal{A}) := \{x \in \mathbf{B}(\mathcal{H}) | x\mathcal{A}, \mathcal{A}x \subseteq \mathcal{A}\} = \{x \in \mathbf{B}(\mathcal{H}) | x\mathcal{A}, x^*\mathcal{A} \subseteq \mathcal{A}\}$ denotes the *multiplier algebra* of $\mathcal{A}$. The *unitisation* of the non-unital $C^*$-algebra $\mathcal{A}$ is denoted by $\mathcal{A}^\sim$.

# Bibliography

Together with Konrad Schüdgen the author already published parts of this thesis in a paper. A preprint appeared on arXiv in September 2014. A second and corrected version of this can be found under http://arxiv.org/abs/1409.8523v2 (July 2015).

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*Literatur that was used in this thesis*

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*Further literature that is concerned with (unbounded operators on) Hilbert $C^*$-modules*

The paper [FS10b] of M. Frank and K. Sharifi inspired me, especially Theorem 2.1 therein; trying to understand why density in a biorthogonal complement is necessary brought the idea to me that adjointability should be connected to essential domains. In the same light the concept of generalized projections appeared.

Further inspiration was given to me by [Izu89], [Lan95], [MM70], [MT05] and [Wor91].


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# Dank

> Dankbarkeit, man spürt sie ja so selten bei den Menschen, und gerade die Dankbarsten finden nicht den Ausdruck dafür, sie schweigen verwirrt, sie schämen sich und tun manchmal stockig, um ihr Gefühl zu verbergen.
>
> *Stefan Zweig*, 1881-1942

Prof. Dr. Konrad Schmüdgen danke ich für die mir immer in Erinnerung bleibenden Vorlesungen zu unbeschränkten Operatoren und Operatoralgebren. Beeindruckt hat mich immer die Fähigkeit zur Einordnung, welche Aussagen von besonderem Nutzen sind, und die gelungene Balance zwischen Theorie und Beispielen. Auch für die privaten Unterrichtseinheiten zu den $*$-Algebren zu Beginn meiner Dissertation bedanke ich mich. Durch seine Lehre im Allgemeinen und durch seine Kritik zum Inhalt des hier vorliegenden Textes ist er zwischen allen Zeilen anwesend. Schließlich bin ich Herrn Schmüdgen für die Findungen des Themas und die jahrelange Betreuung der Arbeit verbunden; insbesondere für die schwierigen Zeiten zu Beginn und am Ende.

Der International Max Planck Research School IMPRS des Max-Planck-Insituts für Mathematik in den Naturwissenschaften danke ich für die Unterstützung der Dissertation durch ein (verlängertes) Stipendium und die Nutzung der Ressourcen vor Ort, insbesondere der wunderbaren Bibliothek. Dem Max-Planck-Istituts verdanken wir die Anwesenheit der vorliegenden Zeilen.

Florian Fischer danke ich besonders freundschaftlich: für die vielen Diskusionsstunden über alles was uns einfiel, wann und wo immer es uns einfiel, für das mathematische Interesse an meiner Arbeit und nochmals für die Freundschaft, die von alltäglichen Abenteuern bis ins Berufliche Feld reicht.

*Meiner Frau und meiner Familie*
*widme ich diese Arbeit*
*als Dank für die vielseitige Unterstützung.*

# Erklärung

Hiermit erkläre ich, die vorliegende Dissertation selbstständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellenten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

Wernigerode, den 10. September 2015

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René Gebhardt