Verallgemeinerungen und Interpretationen von Incipient-Infinite-Cluster-Maßen auf planaren Gittern und Platten

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

(John von Neumann)

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Abstract

Mathematics

Doctor of Philosophy

Generalizations and Interpretations of Incipient Infinite Cluster measure on Planar Lattices and Slabs

by Deepan Basu

This thesis generalizes and interprets Kesten's Incipient Infinite Cluster (IIC) measure in two ways. Firstly we generalize Járai's result which states that for planar lattices the local configurations around a typical point taken from crossing collection is described by IIC measure. We prove in Chapter 2 that for Backbone, Lowest crossing and set of Pivotals, the same hold true with multiple armed IIC measures. We develop certain tools, namely Russo Seymour Welsh Theorem and a strong variant of Quasi-multiplicativity for critical percolation on 2-dimensional slabs in Chapters 3 and 4 respectively. This enables us to first show existence of IIC in Kesten's sense on slabs in Chapter 4 and prove that this measure can be interpreted as the local picture around a point of crossing collection in Chapter 5.

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Symbols

$\mathbb{N}$	Set of natural numbers (including the origin)
$\mathbb{Z}^d$	d-dimensional square lattice
p	Probability parameter
$\theta(p)$	Percolation probability of origin
p_c	Critical probability
$\mathbb{S}_{k,d}$	d-dimensional slab $\mathbb{Z}^2 \times \{0, 1, \ldots, k\}^{d-2}$
∂	Interior boundary of a set
τ_u	Translation by a vertex u
$\mathbb{P}_p$	Bernoulli bond percolation measure with parameter p
Ω	Set of all possible configurations
$A \circ B$	Disjoint occurrence of two events A and B
$B(n)$	$[-n, n]^2$.
$L(n), R(n)$	Left and right boundary of $B(n)$
$T(n), D(n)$	Top and bottom boundary of $B(n)$
$x \leftrightarrow y$	x is connected to y by an open path
$x \xleftrightarrow{Z} y$	x is connected to y by an open path entirely inside Z
$SC(n)$	Crossing collection
$BB(n)$	Backbone of a horizontal open crossing of $B(n)$
$P(n)$	Set of pivotal edges for horizontal open crossing of $B(n)$
ν	IIC measure
ν_2, ν_3, ν_4	Specific multiple armed IIC measures
$\overline{A}$	For $A \subset \mathbb{Z}^2$, $A \times \{0, 1, \ldots, k\}^{d-2}$
$\alpha_\sigma(r)$	$\mathbb{P}\,[\text{Origin is connected to } \partial B(r) \text{ by multiple arms in the order } \sigma]$
$B(m, n)$	$\overline{[0, m) \times [0, n)}$
$L(m, n)$	$\overline{\{0\} \times [0, n)}$
$R(m, n)$	$\overline{\{m-1\} \times [0, n)}$
C_*, c_*	Constants depending on k, d and $p_c(\mathbb{S}_{k,d})$
$B'(n)$	$\overline{B(n)}$
$Q(v, n)$	Set of vertices having graph distance exactly n from v

Contents

Chapter 1

Introduction

Percolation has not only been a rich source of open problems which charms us with the beauty and apparent simplicity, but it also has a well-established origin in applied mathematics. The earliest treatment of percolation as a mathematical object date back to 1950s. Needless to say that much progress has since been made and the mathematics has developed; in the process it has built a reputation for being both difficult and important. For a person with some idea about elementary probability theory and real analysis, it is fairly easy to ask a number of questions about percolation. That these turn out to be surprisingly difficult to answer, gives the perception of how rich the study of percolation is.

1.1 Bernoulli Percolation

We begin by introducing Bernoulli bond percolation, one of the simplest yet content-wise rich models of percolation. For $x \in \mathbb{Z}^d$, we write x_i as the i-th co-ordinate of x. The graph theoretic distance between two points x and y is defined as

$$\delta(x,y) = \sum_{i=1}^{d} |x_i - y_i| \tag{1.1}$$

and two vertices x and y are called neighbors if $\delta(x,y) = 1$. Let us denote this graph as $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ where $\mathbb{E}^d$ is the set of edges between neighboring vertices. The bond percolation model on $\mathbb{Z}^d$ is defined as the following. Given any $p \in [0,1]$, any edge $e \in \mathbb{E}^d$ is open with probability p and closed with probability $1-p$, independent of all other edges. The sample space is chosen to be $\Omega = \Pi_{e \in \mathbb{E}^d}\{0,1\}$, elements of which are configurations indicated by $\omega = (\omega(e) : e \in \mathbb{E}^d)$. $\omega(e) = 0$ indicates the edge e is closed in configuration ω and $\omega(e) = 1$ indicates it being open. We take the σ-field $\mathcal{F}$ to be the one generated by the finite dimensional cylinders. Finally the probability measure is defined as the product measure:

$$\mathbb{P}_p = \Pi_{e \in \mathbb{E}^d} \mu_e$$

where $\mu_e(\omega(e) = 1) = p = 1 - \mu_e(\omega(e) = 0)$.

We say two vertices a and b are connected if there exist vertices $x_1, \dots x_n$ such that $a = x_1$, $b = x_n$, x_i is neighbor to x_{i+1} for all $i \in \{1, 2 \dots n-1\}$ and all of these edges between the neighbors are open and denote this by $a \leftrightarrow b$. The cluster $C(x)$ containing a vertex x is defined as the set of all vertices which are connected to x and let us write $C = C(O)$, the cluster of origin. The percolation probability is defined as $\theta(p) = \mathbb{P}_p(|C| = \infty)$. (Similarly we can define similar quantities for Bernoulli site percolation where each site $v \in \mathbb{Z}^d$ is open or closed instead and the connectivity $a \leftrightarrow b$ is defined by a series of neighboring open vertices instead of edges. For the sake of simplicity, we choose to deal with bond percolation, although the results we state hold true for site percolation as well.)

This quantity $\theta(p)$ is non-deceasing in p and it is well known (see [G99], for example), that for $d \geq 2$, there exists a critical value $p_c(d) \in (0, 1)$ such that $\theta(p) = 0$ if $p < p_c(d)$ and $\theta(p) > 0$ if $p > p_c(d)$. (For $d = 1$, it is not hard to figure out $p_c = 1$ and this makes most questions asked trivial for 1-dimension.) These two regimes are called sub-critical and super-critical, respectively.

We will narrow down our focus on critical Bernoulli percolation instead, although some of the results we prove would also hold true for other percolation models such as finite-dependent percolation at criticality. One of the primary justifications of focusing on criticality is that most of the questions posed in sub-critical and super-critical regime are better-understood (for example the asymptotics of cluster size or two point connections) but this is not true for all dimensions at criticality.

One of the immediate questions asked is whether $\theta(p_c) = 0$ or not. For critical bond percolation on $\mathbb{Z}^2$, it is known that there is almost surely no infinite cluster at criticality [H60, K80]. The proof vitally uses planarity, robbing the strategy of potentially being used in other dimensions. Also for $d \geq 11$, this has been shown to be true in [FH15] (which was improved from $d \geq 19$, proved in [HS90]) by lace expansion technique under the existence of a triangle condition which is not true for $d \leq 6$. The commonly shared belief is that $\theta(p_c) = 0$ holds true for any dimension and there would exist a general proof, but this currently eludes everyone.

We should highlight that critical planar percolation is better understood than its non-planar counterparts in general (although for sufficiently high dimension, critical percolation is understood well-enough using specific high-dimensional tools). Of course, two of the obvious aides are planarity and duality. For starters, we precisely know the critical threshold for some models. For example, for bond percolation on $\mathbb{Z}^2$ and site percolation on triangular lattice, $p_c = \frac{1}{2}$ and for bond percolation on triangular and hexagonal lattices, $p_c = 2\sin(\pi/18)$ and $p_c = 1 - 2\sin(\pi/18)$ respectively (see [G99]). (For critical thresholds of other planar lattices such as "Bow-tie" lattice, see [SZ06].) But the two other important tools present in the study of critical planar percolation are *Russo-Seymour-Welsh Theorem* and *Quasi-multiplicativity*.

Russo-Seymour-Welsh theorem states that in spite of absence of an infinite open cluster at criticality, there exists a non-vanishing probability for both the existence and absence of open clusters spanning arbitrarily large boxes [R78, SW78, R81, K82]. This is also known as the box-crossing theorem. Recently this theorem has been extended to some other planar models. To name a few, this has been proved for continuum percolation on $\mathbb{R}^2$ [R90], Voronoi percolation [BR06, T14] and most notably for FK-percolation [DCHN11, DCST17]. Such a result is not proved in other dimensions, and in fact, if the dimension is sufficiently high, it is proved in [A97] that these crossing probabilities tend to 1 as we take larger and larger boxes. *Quasi-multiplicativity* states that up to a universal multiplicative constant, the probability of an open crossing of an annulus can be decomposed into product of probabilities of open crossings of two sub-annuli that constitute it.

1.2 Incipient Infinite Cluster

For planar critical percolation, although there is almost surely no infinite cluster, there exist open clusters spanning arbitrarily large boxes [R78, SW78]. Aizenman [A97] posited that local patterns around vertices of large spanning clusters appear with frequencies given by a probability measure on occupancy configurations. This measure would inherit properties of critical percolation, but would be supported on configurations with an infinite open cluster at the origin. Informally, we can imagine this as the "birth" of the infinite cluster at criticality. One may call such a measure an incipient infinite cluster (IIC) measure.

Kesten [K86a] gave a first mathematically rigorous construction of such an IIC measure by conditioning on an open path from the origin to the boundary of a large box at critical percolation and increasing the size of the box to infinity. The resulting probability measure is supported on the configurations with an infinite open cluster at the origin.

He also described an alternative way of defining this measure by first conditioning on the event that the open cluster containing origin is infinite for $p > p_c$ and then looking at the limit as $p \searrow p_c$, and showed that the two limits are the same. These two interpretations already demonstrate the potential robustness of IIC measure.

Járai [J03] proved that Kesten's IIC measure indeed describes frequency of local patterns around a typical point in large crossing clusters. For example, if one chooses a vertex uniformly at random in a large crossing cluster, or for example k-th largest open cluster, then asymptotically, the occupancy configuration around this vertex has the law given by the IIC measure. Even if we change the conditional event as one particular vertex being in the crossing collection, as long as it is far away from the boundary, the limit as we take larger and larger boxes, will be the same IIC-measure as well. Thus, he unified several natural definitions of the IIC measure in the paper (see [J03, Theorems 1-4]).

It is quite natural to probe into how IIC "looks like". Kesten first showed that the size of IIC inside $B(n) = [-n, n]^2$ is comparable to $n^2\alpha(n)$, where $\alpha(n)$ is the one-arm connectivity, i.e. the probability of origin being connected to the boundary of $B(n)$. This implies that IIC is very thin, for example, compared with infinite clusters in supercritical phase. Later Kesten [K86b] also showed that although $\mathbb{Z}^2$ was recurrent, simple random walk on IIC is sub-diffusive (and later a stronger quenched version of this result was proved by Damron, Hanson and Sosoe in [DHS14]). This is characteristically different from simple random walk on supercritical clusters in $\mathbb{Z}^2$, which behaves "like" $\mathbb{Z}^2$ and expectedly, do not exhibit this property. (See [B04] for this result on $\mathbb{Z}^d$ in general.)

Van der Hofstad and Járai [HJ04, Corollary 4.2] first showed that for sufficiently high dimension, we can make sense of IIC by conditioning an open arm from the origin to a point and letting that point go to infinity. The crucial assumption is certain bounds on this connectivity probability (see [HJ04, (4.23),(4.24)]), which holds true for all $d \geq d_0$ for some $d_0 > 6$. (This version of IIC measure was also shown to exist for spread out percolation model in $d > 6$ for big enough parameter in [HJ04], and in [HHS02], Hofstad, Hollander and Slade constructed IIC for spread-out oriented percolation above 4+1 dimension as well.) They also conjecture in this paper that IIC measure exists in Kesten's sense and this matches with their established measure.

Later Heydenreich, Van der Hofstad and Hulshof showed in [HHH14a, Theorem 1.2] that under additional hypothesis, namely that the limit $\lim_{n\to\infty} n^2\alpha_d(n)$ exists (where $\alpha_d(n)$ is the one-arm connectivity at criticality in $\mathbb{Z}^d$), Kesten's IIC measure exists and is same with the one introduced in [HJ04]. Currently the best known result is $\alpha_d(n)$ is comparable to $1/n^2$ for high dimension (see [KN11, Theorem 1]), and given this, they showed existence of IIC in Kesten's sense, but only by taking the limit along an increasing subsequence, paying the cost for weaker asymptotics.

It was shown (for example in [B14]), as speculated by physicists, that IIC in high dimension is a 4-dimensional object. It is also well-understood, for example how simple random walk behaves on IIC in high dimension (see [HHH14b, Theorem 0, Theorem 1.6]). But this particular result is not true for $d \leq 6$ and more importantly the tools presented in aforementioned papers cannot work for this regime. Thus showing the existence of Kesten's IIC on $\mathbb{Z}^d$ for $3 \leq d \leq 6$ remains an open problem with very limited tools to attack.

As another direction of generalizing IIC measure, in [DS11], the so-called multiple-armed IIC measures on planar lattices were introduced, which are supported on configurations with several disjoint infinite open clusters meeting in a small neighborhood of the origin instead of one single open arm. In this paper, some of these measures were also explained as the local configurations around typical points from some sets significant to invasion percolation. For example, it was shown that the configuration around a typical point in the invasion cluster is explained by one-arm IIC measure, whereas that for the set of outlets is explained by a certain four-arm IIC measure. These measures have since come up in studying several objects, most notably, Chang-Long Yao proved CLT for multiple-armed IIC measure of winding angle in [Y13] and scaling limit of certain multiple armed IIC for site percolation on planar triangular lattice in [Y16].

The existence of some of the multiple armed IIC measure was theorized already before, and in fact [J03, Remark after Theorem 1] conjectured that such measures would describe local picture around typical points from sets significant to planar critical percolation as well. This will serve as the motivation of one of our results.

1.3 Our Contribution

Our contributions are twofold. Firstly we will prove what Járai conjectured, i.e. specific multiple armed IIC measures indeed describe the configuration around typical point from some sets significant for planar critical percolation. Secondly, we will prove the existence of IIC measure and extend its interpretation again as local limits of typical points in a giant crossing cluster on slabs in $\mathbb{Z}^d$, i.e., on graphs $\mathbb{S}_{k,d} = \mathbb{Z}^2 \times \{0, \dots k\}^{d-2}$ ($d \geq 2$, $k \geq 0$). We describe these results in detail in subsections 1.3.1 and 1.3.2.

1.3.1 Multiple-arm IIC as Local limits

Let us call $B(n) = [-n, n]^2$ and $\partial B(n) = B(n) \setminus B(n-1)$ as its interior boundary. For a vertex $v \in \mathbb{Z}^2$ and a set of vertices $X \subset \mathbb{Z}^2$, we denote $v \longleftrightarrow X$ as the event that there exists $x \in X$ such that $v \leftrightarrow x$. Kesten proved that, $\forall$ cylinder event E, the limit

$$\nu(E) = \lim_{n\to\infty} \mathbb{P}_{p_c}(E|O \leftrightarrow \partial B(n))$$

is well-defined, and by Kolmogorov's extension theorem, ν extends uniquely to a probability measure on configurations of edges, called *Kesten's incipient infinite cluster* measure. (Kolmogorov's extension theorem holds since the compatibility for events depending on finitely many edges hold true, and this can be checked immediately.) Járai [J03] proved that this measure is the specific measure around a typical point of crossing collection. More precisely, if one chooses a vertex uniformly at random in a large crossing collection or fix a vertex from it far from the boundary, then the occupancy configuration around this vertex has the law given by the IIC measure.

Let the left boundary of the square $B(n)$ be called $L(n) = \{-n\} \times [-n, n]$ and the right one be called $R(n) = \{n\} \times [-n, n]$. Járai made sense of IIC measure as the local configuration picture around a point of crossing collection

$$SC(n) = \{v \in B(n) : \mathrm{L}(n) \leftrightarrow v \leftrightarrow R(n) \text{ inside } B(n)\},$$

which is far away from the boundary. For $u \in \mathbb{Z}^2$, let us define translation τ_u acting on Ω by $\tau_u\omega(< x, y >) = \omega(< x - u, y - u >)$ where $< x, y > \in \mathbb{E}^2$, and on events by $\tau_u A = \{\tau_u\omega : \omega \in A\}$. Let us also call $\mathbb{P} := \mathbb{P}_{p_c}$. [J03, Theorem 2] states that, for any cylinder event E, any function $h(n)$ satisfying $h(n) \leq n$ but $\lim_{n\to\infty} h(n) = \infty$, and any sequence of vertices v_n,

$$\lim_{\substack{n\to\infty \\ |v_n|\leq n-h(n)}} \mathbb{P}\big[\tau_{v_n} E \mid v_n \in SC(n)\big] = \nu(E).$$

The "random" version of this theorem [J03, Theorem 1] states that, if I_n is a uniformly chosen point from the crossing collection $SC(n)$, for any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_n} E \mid SC(n) \neq \phi\big] = \nu(E).$$

(Although we slightly abuse the notation and still call this measure $\mathbb{P}$.)

Let us now define the "Special Sets" we are interested in. These sets carry certain significance in presence of an open horizontal crossing.

- **Backbone:** We define the backbone $BB(n)$ as the set of vertices in $B(n)$ which are connected to $L(n)$ and $R(n)$ by two disjoint paths, both being inside $B(n)$. In Figure 1.1, backbone vertices lie on red, blue, and green edges but not on black edges (although vertices on black edges are in crossing collection).

- **Lowest crossing:** If there exists an open path from $L(n)$ to $R(n)$ inside $B(n)$, there would exist a unique "lowest" such crossing (since we can define a partial relation in the set of left-right crossings characterized by the inclusion of the area 'under' the crossing inside $B(n)$). Let us call the set of edges on this path γ_{min} as $LC(n)$. In Figure 1.1, green and red edges constitute lowest crossing.

- **Set of pivotals:** An edge e is said to be pivotal (for the crossing event $LR(n) = \{$There exists an open path from $L(n)$ to $R(n)$ inside $B(n)\}$) in a given configuration ω if exactly one of ω or $\omega' := \{\omega'(f) = \omega(f) \text{ iff } f \neq e\}$ lies in $LR(n)$, i.e. switching that edge e in configuration ω from open to closed or vice versa affects the existence of an open horizontal crossing of $B(n)$. We denote by $P(n)$, the set of pivotal edges. In Figure 1.1, green edges are pivotal.

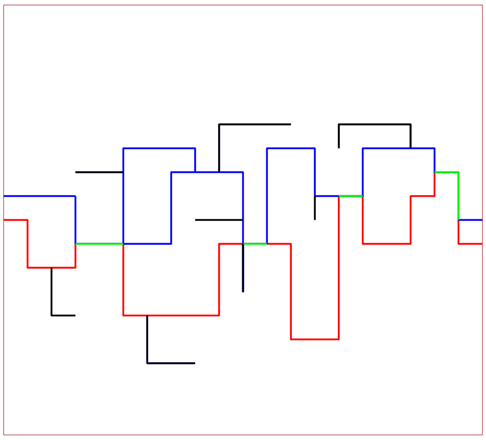

FIGURE 1.1: Edge sets for crossing

Note that the dual graph of $\mathbb{L}^2$ is given by $\mathbb{L}^{2^*} = \{(\frac{1}{2}, \frac{1}{2}) + x : x \in \mathbb{Z}^2\}$ which is isomorphic to $\mathbb{L}^2$ itself. Each edge e of the original graph intersects with a unique edge e^* in the dual graph and it is called as the dual edge of the edge e. We declare any edge in the dual graph open or closed corresponding to the status of its dual edge and call such open (or closed) paths in the dual graph as dual open (or dual closed, respectively) path. Also, for a set $X \subset \mathbb{Z}^2$, we call X^* as the set of edges dual to all the edges in the original graph with both vertices on X.

Let us call the top and bottom part of $\partial B(n)$ as $T(n) = [-n, n] \times \{n\}$ and $D(n) = [-n, n] \times \{-n\}$. We note two things. Firstly, for every edge e on the lowest crossing γ_{min}, e^* is connected to some edge of $D^*(n)$ by a dual closed path comprised of edges inside $B^*(n)$ (actually inside $U^*(\omega)$, where $U(\omega)$ is the area enclosed by $\gamma_{min}(\omega)$) and two disjoint open connections to $L(n)$ and $R(n)$ from its two vertices inside $B(n)$. Secondly, if e is a pivotal edge in horizontal crossing of $B(n)$, then one endpoint of e is connected to $L(n)$ and the other to $R(n)$ by open paths in $B(n)$ as well as two endpoints of e^*

being connected to some edge in $D^*(n)$ and $T^*(n)$ by two disjoint dual closed paths. Thus 2 open paths in $\mathbb{L}^2$ and 2 closed paths in $\mathbb{L}^{2^*}$ originate from e in alternate manner.

We will introduce some specific "multiple-arm" IIC measures now as the eligible candidates to describe the configurations around these sets. The existence of our 3 candidate measures we are about to define is already proved by virtue of [DS11, Theorem 1.6, Remark 7].

- Let us denote by $O \leftrightarrow_2 \partial B(n)$, the event that the origin is connected to $\partial B(n)$ by two disjoint open paths. For every cylinder event E, we define

$$\nu_2(E) = \lim_{n\to\infty} \mathbb{P}\big[E \mid O \leftrightarrow_2 \partial B(n)\big].$$

- Let us call $e_0 = ((0,0),(1,0))$ and denote by $e_0 \leftrightarrow_3 \partial B(n)$, the event that two endpoints of e_0 are connected to $\partial B(n)$ by two disjoint open paths, e_0^* is connected by a dual closed path inside $B(n)^*$ to some edge in $\partial B(n)^*$ and e_0 is open. For every cylinder event $E \subset \{\omega(e_0) = 1\}$, we define

$$\nu_3(E) = \lim_{n\to\infty} \mathbb{P}\big[E \mid e_0 \leftrightarrow_3 \partial B(n)\big].$$

- Let us denote by $e_0 \leftrightarrow_4 \partial B(n)$, the event that two endpoints of e_0 are connected to $\partial B(n)$ by two disjoint open paths and two endpoints of e_0^* are connected by two disjoint dual closed paths inside $B(n)^*$ to some edges in $\partial B(n)^*$. Notice that this event is independent of $\{\omega(e_0) = 1\}$. For every cylinder event E independent of $\{\omega(e_0) = 1\}$, we define

$$\nu_4(E) = \lim_{n\to\infty} \mathbb{P}[E \mid e_0 \leftrightarrow_4 \partial B(n)].$$

In $\mathbb{Z}^2$ any edge e is of the form $(v, v+e_x)$ or $(v, v+e_y)$ where e_x and e_y are unit vectors in positive direction of X or Y axis. We associate this vertex v with e and call as $v(e)$. For an edge e, also let ρ_e be the rotation that maps $e - v(e)$ to $e_0 := ((0,0),(1,0))$. Now we define τ_e as the operator on configurations such that for any edge f,

$$\tau_e(\omega)(f) = \omega(\rho_e(f - v(e))).$$

We state our theorems for each of the three sets– backbone, lowest crossing, and pivotals for both "local" and "random" versions.

Theorem 1.1. *Let there be sequences v_n of vertices and e_n of edges such that their distance from the boundary is at least $h(n)(\leq n)$ where $\lim_{n\to\infty} h(n) = \infty$.*
(a) For any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{v_n} E \mid v_n \in BB(n)\big] = \nu_2(E).$$

(b) For any cylinder event $E \subset \{\omega(e_0) = 1\}$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{e_n} E \mid e_n \in LC(n)\big] = \nu_3(E).$$

(c) For any cylinder event E independent of $\omega(e_0)$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{e_n} E \mid e_n \in P(n)\big] = \nu_4(E).$$

Theorem 1.2. *Let I_{n_2}, I_{n_3} and I_{n_4} be chosen uniformly from the sets $BB(n)$, $LC(n)$ and $P(n)$ respectively.*
(a) For any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,2}} E \mid BB(n) \neq \phi\big] = \nu_2(E).$$

(b) For any cylinder event $E \subset \{\omega(e_0) = 1\}$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,3}} E \mid LC(n) \neq \phi\big] = \nu_3(E).$$

(c) For any cylinder event E independent of $\omega(e_0)$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,4}} E \mid P(n) \neq \phi\big] = \nu_4(E).$$

With the present tools, Theorem 1.1(a) and Theorem 1.2(a) can be proved by replicating the proof of [J03, Theorem 1-2] with very little changes, as predicted by Járai himself. But to approach these results for the sets $LC(n)$ and $P(n)$, we need different tools. The proofs of Theorems 1.1 and 1.2 are similar in a sense that both rely on a certain decoupling argument. In Theorem 1.2 one first shows that the uniformly chosen point stays far away from the boundary of $B(n)$ with probability close to 1 (In Theorem 1.1 this is automatic by the assumption). Then, one shows that again with high probability, there exists an open circuit with 1 or 2 (for case(b) and case(c), respectively) defects. This circuit allows to decouple the local configuration around the given point (inside the circuit) from the rest of the configuration (outside the circuit, respectively). The conclusion then easily follows.

The heart of Járai's proof consists of the idea of decoupling a local neighborhood using an open circuit. Several technical issues, which arise since we are dealing with circuits with defects instead, thus need to be addressed. Perhaps the most conceptual difference between the proofs of Theorem 1.2 and Theorem 1.1 is that the former requires the tightness of the respective families (backbone, lowest crossing or pivotals) additionally.

1.3.2 IIC on Slabs

The slabs $\mathbb{S}_{k,d}$ garner attention primarily for two reasons. The first being that they serve as a natural extension of planar lattices, although proving results faces challenges such as lack of planarity and duality. The second is that the limiting results as k grows, mimic those of $\mathbb{Z}^d$. For example it is known that $p_c(\mathbb{S}_{k,d}) \to p_c(d)$ as $k \to \infty$. So, although their global behavior is quasi-planar, locally it represents $\mathbb{Z}^d$. It was a recent result where Duminil-Copin, Sidoravicius, and Tassion [DCST16] proved that $\theta(p_c(\mathbb{S}_{k,d})) = 0$. They used a certain technique of "glueing" open connections that made it amenable to planar treatment. This provided us with a much needed tool to approach the standard questions about critical percolation in slabs which are satisfactorily answered in planar models.

Our main objective, as we have highlighted, is to establish existence of IIC-measure on slabs and interpreting it as a local limit. $\theta(p_c(\mathbb{S}_{k,d})) = 0$ naturally makes the existence of IIC in slabs a non-trivial question. (Otherwise the conditional event would be an event with non-vanishing probability and it will yield the existence of IIC directly.) We will prove that both interpretations of Kesten's construction of the IIC work and coincide for slabs. The first measure defined by conditioning on an open path from the origin to the boundary of a large box at criticality and letting the size of the box run to infinity, exist and is equal to the second measure, where we condition on the open cluster of origin being infinite for supercritical p and let $p \searrow p_c(\mathbb{S}_{k,d})$. The resulting IIC measure, as before, will be supported on the configurations with an infinite open cluster at the origin.

For $A \subset \mathbb{Z}^2$, let us define $\overline{A} := A \times \{0, 1, \dots, k\}^{d-2}$. We define $B(n)$, $\partial B(n)$, $v \leftrightarrow x$, and $v \longleftrightarrow X$ as before. We prove existence of IIC in the following sense of Kesten:

Theorem 1.3. *Let $d \geq 2$ and $k \geq 0$ be integers. For any $v \in \mathbb{S}_{k,d}$ and any event E that depends on the state of finitely many edges of $\mathbb{S}_{k,d}$, the following two limits*

$$\lim_{n\to\infty} \mathbb{P}_{p_c}\left[E \mid v \longleftrightarrow \overline{\partial B(n)}\right] \quad \textit{and} \quad \lim_{p \searrow p_c} \mathbb{P}_p\left[E \mid |C(v)| = \infty\right], \tag{1.2}$$

exist and are equal.

The case $k = 0$ is naturally Kesten's result we have already mentioned (see [K86a, Theorem (3)]). By Kolmogorov's extension theorem, this measure (let us call this ν_v) extends uniquely to this *Kesten's IIC* measure on configurations of edges.

In fact, Theorem 1.3 will be a consequence of a more general result. We prove in Theorem 4.1 (in Chapter 4) that the two limits in (1.2) exist and are equal for any infinite connected bounded degrees graph satisfying the following two assumptions:

(A1) uniqueness of the infinite open cluster,

(A2) quasi-multiplicativity of crossing probabilities.

Assumption (A1) is true for a wide set of sufficiently regular amenable graphs which include $\mathbb{Z}^d$ and $\mathbb{S}_{k,d}$ (see e.g. [BS96]). Thus, we prove Theorem 1.3 by verifying (A2) for slabs. (See Section 4.2.2 for further discussions about validity of the assumption (A2) on other lattices.)

Let us recall that after Kesten's construction of IIC-measure, Járai showed that the measure could describe local occupancy configuration around a uniformly chosen point of some specific giant clusters, notably the crossing cluster (conditioned on the existence of having one) and the largest cluster ([J03, Theorems 1 and 3]) or around a point of the crossing cluster far away from the boundary ([J03, Theorem 2]). After establishing the existence of IIC-measure on slabs, it naturally begs the question whether this holds for slabs too. We will prove that we can indeed make sense of IIC measure as local limit of large crossing cluster in certain ways Járai did.

Let $L(n) = \{-n\} \times [-n, n]$ and $R(n) = \{n\} \times [-n, n]$ be left and right boundaries of $\partial B(n)$ and

$$SC(n) = \left\{ v \in \overline{B(n)} : \overline{R(n)} \overset{\overline{B(n)}}{\longleftrightarrow} v \overset{\overline{B(n)}}{\longleftrightarrow} \overline{L(n)} \right\}$$

be called the crossing collection. We say that a vertex $v \in \mathbb{S}_{k,d}$ has the 'level' $j \in \{0, 1, \ldots, k\}^{d-2}$ if last $d-2$ co-ordinates of v is given by j. For some vertex in the plane $u \in \mathbb{Z}^2$, and some level j, let us denote by u^j the vertex in $\mathbb{S}_{k,d}$ whose first 2 co-ordinates are given by u, and last $d-2$ of them by $j \in \{0, 1, \ldots, k\}^{d-2}$. For $u = (u_1, u_2) \in \mathbb{Z}^2$, let us define $u_S = (u_1, u_2, 0, \ldots, 0) \in \mathbb{S}_{k,d}$ and translation τ_u acting on Ω by $\tau_u \omega(< x, y >) = \omega(< x - u_S, y - u_S >)$, and on events by $\tau_u A = \{\tau_u \omega : \omega \in A\}$. Let us denote $\mathbb{P} = \mathbb{P}_{p_c(\mathbb{S}_{k,d})}$ from now on. We will prove that:

Theorem 1.4. *Let $h(n) \leq n$ be a function such that $\lim_{n \to \infty} h(n) = \infty$ and E be any event depending on the state of finitely many edges of $\mathbb{S}_{k,d}$. Then for any sequence of vertices $v_n \in Z^2$, and any fixed level $j \in \{0, 1, \ldots, k\}^{d-2}$,*

$$\lim_{\substack{n \to \infty \\ |v_n| \leq n - h(n)}} \mathbb{P}[\tau_{v_n} E | v_n^j \in SC(n)] = \nu_{(0,0)^j}(E).$$

The next natural question to ponder about is if we can make sense of the 'uniform' or 'global' variant of theorem 5.1. However, to prove this we need a certain tightness result of the crossing cluster $SC(n)$ (similar to [J03, Theorem 8(ii)]), which is currently missing. This result states that with high-probability, $|SC(n)|$ is at least bigger than some multiplicative factor times its expectation, whenever it is non-empty, i.e.

Conjecture 1.5.
$$\lim_{\epsilon\to 0}\inf_{n\geq 1}\mathbb{P}[\epsilon \leq \frac{|SC(n)|}{\mathbb{E}[|SC(n)|]}|SC(n)\neq\phi] = 1.$$

Let I_n indicate a vertex chosen uniformly at random from the crossing cluster $SC(n)$, when it is known to be non-empty. Here we abuse the notation and still call this measure as $\mathbb{P}$, and for $v = (v_1, v_2, \dots v_d) \in \mathbb{S}_{k,d}$, let us define $\tau_v = \tau_{(v_1,v_2)}$. The natural candidate for the limiting measure here is the average measure over every level j 'above' the origin. We show that this is indeed the case.

Theorem 1.6. *If Conjecture 1.5 holds, then*

$$\lim_{n\to\infty}\mathbb{P}[\tau_{I_n}E|SC(n)\neq\phi] = \frac{1}{(k+1)^{d-2}}\sum_{j\in\{0,1,\dots,k\}^{d-2}}\nu_{(0,0)^j}(E).$$

As we have mentioned, RSW theorem and quasi-multiplicativity are two important tools for critical planar percolation and the former specifically was also crucial component for Kesten's IIC construction. We will also prove these two results which will help us circumvent lack of planarity and other tools.

Let us recall that Russo-Seymour-Welsh theorem states that the probability that an open path connects the left and right sides of a rectangle is bounded away from 0 and 1 by constants that only depend on the aspect ratio of the rectangle. We will prove that the probability of crossing a "rectangular box" in $\mathbb{S}_{k,d}$ is bounded from below by a positive constant which only depends on the aspect ratio of the rectangle and the slab parameters k and d, but does not depend on the size of the rectangular box.

Let us define a rectangle and its left and right boundary regions by

$$B(m,n) = [0,m)\times[0,n), \qquad L(m,n) = \{0\}\times[0,n), \qquad R(m,n) = \{m-1\}\times[0,n).$$

Consider the crossing event

$$\mathrm{LR}(m,n) = \left\{\overline{L(m,n)} \text{ is connected to } \overline{R(m,n)} \text{ by an open path in } \overline{B(m,n)}\right\}.$$

Let us state the RSW theorem for slabs:

Theorem 1.7. *For any $\rho\in(0,\infty)$, there exists a constant $c_\rho = c_\rho(k,d) > 0$ such that*

$$\liminf_{n\to\infty}\mathbb{P}\left[\mathrm{LR}(\lfloor\rho n\rfloor, n)\right] \geq c_\rho. \tag{1.3}$$

We reiterate that to combat the obstacles (to connect open paths) created by lack of planarity we will adapt a certain technique for "glueing" open paths from [DCST16]. Although this proof can be extended for other models such as finite-range percolation, for simplicity we will be content working with slabs $\mathbb{S}_{k,d}$ alone. For the sake of completion, we also state, the high-probability variant of RSW theorem, which states that if the crossing probability in the easy direction of a rectangular box of fixed aspect ratio goes to 1 as the size increases, so must happen for the difficult direction of a rectangular box with arbitrarily large aspect ratios. Let us call $p(m,n) = \mathbb{P}[\mathrm{LR}(m,n)]$.

Corollary 1.8 (High-Probability version of RSW Theorem)**.**

$$\lim_{n\to\infty} p(\lfloor \rho n \rfloor, n) = 1 \text{ for some } \rho \in (0,1) \Rightarrow \lim_{n\to\infty} p(\lfloor \kappa n \rfloor, n) = 1 \text{ for all } \kappa > 0. \tag{1.4}$$

Another natural question to ask in the context of RSW theorem is whether for every $\rho > 0$, $\limsup_{n\to\infty} p(\lfloor \rho n \rfloor, n) < 1$. This was shown to be true very recently by Newman, Tassion and Wu in [NTW15, Theorem 3.1] for percolation on slabs. They also obtained independently and with different proofs the results of Theorem 1.7 and Corollary 1.8 (See [NTW15, Theorems 3.1 and 3.17]).

Let us introduce some notations before stating quasi-multiplicativity Lemma 1.9 for slabs. We define the annulus in slabs as $\mathrm{An}(m,n) = \overline{B(n) \setminus B(m-1)}$ for integers $m \leq n$. For $x, y \in \mathbb{S}_{k,d}$ and non-empty sets $X, Y, Z \subset \mathbb{S}_{k,d}$, we write

- $x \overset{Z}{\longleftrightarrow} y$ if there is a nearest neighbor path of open edges with all its vertices in Z.
- $x \overset{Z}{\longleftrightarrow} Y$ if there exists $y \in Y$ such that $x \overset{Z}{\longleftrightarrow} y$.
- $X \overset{Z}{\longleftrightarrow} Y$ in Z if there exists $x \in X$ such that $x \overset{Z}{\longleftrightarrow} Y$.

Lemma 1.9 (Quasi-multiplicativity)**.** *Fix $d \geq 2$, $k \geq 0$ and $\delta \in (0, 1 - p_c(\mathbb{S}_{k,d}))$. There exists $c > 0$ such that for any $p \in [p_c, p_c + \delta]$, integer $m > 0$, any finite connected $Z \subset \mathbb{S}_{k,d}$ such that $Z \supseteq \mathrm{An}(m, 3m)$, and any $X \subset Z \cap B'(m)$ and $Y \subset Z \setminus B'(3m)$,*

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y] \geq c \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \partial B'(2m)] \cdot \mathbb{P}_p[Y \overset{Z}{\longleftrightarrow} \partial B'(2m)]. \tag{1.5}$$

Notice that this is a stronger variant of the general quasi-multiplicativity lemma, whose planar version we are familiar with (albeit for one open arm). Apart from being uniformly true in $[p_c, p_c + \delta]$, its vital advantage lies in doing away with the 'shape' of the region and the only requirement being reasonable amount of space between the regions which are being connected, to split one long path into two.

As we discussed after the statement of Theorem 1.3 and will do so in details in Chapter 4, quasi-multiplicativity in the sense of (1.5) is one of the two prerequisite conditions for IIC to make sense of in Bernoulli percolation on a general graph. We expect that quasi-multiplicativity holds on $\mathbb{Z}^d$ if and only if $d < 6$. (We explain our intuition for believing this in details in Section 4.2.2.)

Let us sketch the outline of the proof of Theorem 1.3, which broadly follows the general scheme proposed by Kesten in [K86a] by attempting to decouple the configuration near v from infinity on multiple scales. Kesten's decoupling argument is based on the existence of an infinite collection of open circuits around v in disjoint annuli and utilizes two of their properties:
(a) Each path from v to infinity intersects every such circuit.
(b) By conditioning on the innermost open circuit in an annulus, the occupancy configuration in the region not surrounded by the circuit is still an independent Bernoulli percolation.

This approach explicitly uses planarity and thus cannot work in slabs. Instead, we take up the following strategy (assume for the ease of calculation that $v \in \overline{\{(0,0)\}}$). We define $S(m) := \overline{\partial B(m)}$.

- We identify a sufficiently fast growing sequence N_i such that given $v \longleftrightarrow S(n)$, the probability that there is a unique open cluster in $\mathrm{An}(N_i, N_{i+1})$ which connects $S(N_i)$ to $S(N_{i+1})$ (in percolation jargon, a unique crossing cluster) is asymptotically close to 1.

- Next, let an annulus $\mathrm{An}(N_i, N_{i+1})$ contain a unique crossing cluster. We explore all the open clusters in this annulus that intersect the interior boundary $S(N_i)$, call their union $\mathcal{C}_i$, and let $\mathcal{D}_i$ be the subset of $S(N_{i+1}+1)$ of vertices connected by an open edge to $\mathcal{C}_i$.

- Then, the configuration outside $\mathcal{C}_i$ is distributed as the original independent percolation and every vertex from $\mathcal{D}_i$ is connected by an edge to the same (crossing) cluster from $\mathcal{C}_i$. Thus, $v \longleftrightarrow S(n)$ if and only if

  (a) v is connected to $\mathcal{D}_i$ (this event only depends on the edges intersecting $\overline{B(N_i)} \cup \mathcal{C}_i$),

  (b) $\mathcal{D}_i$ is connected to $S(n)$ outside $\mathcal{C}_i$ (this only depends on the edges outside $\mathcal{C}_i$).

- This enables us to factorize $\mathbb{P}_p[E,\, v \longleftrightarrow S(n)]$ into sum over products of crossing probabilities $\mathbb{P}_p[E, v \overset{\overline{B(N_i)} \cup C_i}{\longleftrightarrow} D_i,\ \mathcal{D}_i = D_i,\ \mathcal{C}_i = C_i]$ and $\mathbb{P}_p[D_i \overset{\overline{B(n)} \setminus C_i}{\longleftrightarrow} S(n)]$. The rest of the proof is essentially the same as that of Kesten [K86a]. We repeat the described factorization on several scales, obtaining an approximation of $\mathbb{P}_p[E | v \longleftrightarrow S(n)]$ in terms of products of positive matrices M_i of such probabilities of annulus-crossing probabilities (where the rows and columns are over choices of 2-tuples $(\mathcal{C}_i, \mathcal{D}_i)$ and $(\mathcal{C}_{i+1}, \mathcal{D}_{i+1})$).

- Finally, we use Lemma 1.9 to prove that the matrix operators are uniformly contracting, i.e. $\frac{(M_i)_{j,k}(M_i)_{j',k'}}{(M_i)_{j',k}(M_i)_{j,k'}}$ is bounded from both above and below uniformly in variables j, j', k, k', and i, where $(M_i)_{j,k}$ is the element of M_i situated at j-th row and k-th column. This is analogous to [K86a, Lemma (23)], thus the rest of the proof is same as in [K86a, pages 377-378], namely an application of Hopf's contraction property.

The outline of proving Theorems 1.4 and 1.6 is again a convenient adaptation of Járai's scheme, similar to how we approached Theorems 1.1 and 1.2 as well. One key change, is that instead of existence of open circuits, we exploit lack of percolation in slabs at criticality, and thus work with "shells" instead for decoupling events. Along the way we prove several useful properties of crossing collection, e.g moment bounds for crossing collection and bounds on one-arm connectivity apart from conjecturing the tightness result. (See Lemmas 5.4 and 5.5.)

It would be interesting to ponder over whether we can also make sense of IIC-measure in slabs as [J03, Theorem 3], i.e. by choosing a point randomly from the largest cluster in the box. For this we would require a result akin to [J03, Proposition 1] which states that the difference between the size of the largest and the second largest open cluster should diverge with probability 1 as we increase the size. This variant of Theorem 1.1, as well as Conjecture 1.5 seem hard to prove with the current set of tools we possess.

1.4 Organization of the thesis

We will prove the results related to multiple armed IIC in plane, e.g. Theorems 1.1, 1.2 and related tightness results in Chapter 2. Our aim in Chapter 3 will be to prove RSW Theorem 1.7 and its high-probability variant Corollary 1.8 alongwith some associated results. In subsequent Chapter 4 we prove Theorem 1.3 for general graphs whenever they satisfy uniqueness of the infinite open cluster and quasi-multiplicativity Lemma 1.9, followed by proving that the slabs satisfy the later condition (and the former one trivially). This proves the existence and coincidence of both the definitions of Kesten's IIC-measure in slabs. Finally in Chapter 5, we prove Theorem 1.4 first. Then we prove moment bound for crossing collection which, along with our tightness conjecture 1.5, constitute the proof of Theorem 1.6.

Before moving on to the subsequent chapters, let us recall some common definitions and tools for percolation in Section 1.5 that we will use quite frequently.

1.5 Definitions and Tools in Percolation

Let us establish a partial order $\prec$ on elements of Ω as follows. $\omega \prec \omega'$ if for every edge $e \in \mathbb{E}^d$, $\omega(e) \leq \omega'(e)$, i.e. ω' can be obtained from ω by opening a number of edges that were closed in ω. An event A in $\mathcal{F}$ is called *increasing* if for any two configurations $\omega \prec \omega'$, $\omega \in A$ implies $\omega' \in A$. Some common *increasing* events are existence of an open crossing in an rectangle, existence of an open circuit in an annuli or existence of k many disjoint open crossings in a rectangle. Intuitively, existence of one increasing event would typically imply an abundance of open edges, that might "encourage" any other increasing event, giving us the impression that they are positively correlated. The next inequality, commonly known as *FKG inequality*, is a formalization of this intuition.

Lemma 1.10 (FKG inequality). *Let A and B be two increasing events. Then*

$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\mathbb{P}_p[B].$$

Next, we are going to present *Reimer's inequality* which is a complementary correlation-type inequality for general events. Let A be an event that depends on finitely many edges $\{e_1, e_2, \ldots, e_m\}$. Let us define, for $K \subset \{1, 2, \ldots, m\}$, the cylinder event generated by ω on K as

$$C(\omega, K) = \{\omega' : \forall i \in K, \{\omega'(e_i) = \omega(e_i)\}.$$

For two events A, B depending on finitely many edges $\{e_1, e_2, \ldots, e_m\}$, let us denote their disjoint occurrence as $A \circ B$, which is defined as

$$A \circ B = \{\omega : \exists K \subset \{1, 2, \ldots, m\} \text{ such that } C(\omega, K) \subset A \text{ and } C(\omega, K^c) \subset B\},$$

where $K^c = \{1, 2, \ldots, m\} \setminus K$. For such "disjoint" occurrence, it is intuitive that conditioned on existence of one event, occurrence of the other requires more specific configurations in general. This is formalized as Reimer's inequality, which states

Lemma 1.11 (Reimer's inequality). *Let A and B be two events depending on finitely many edges. Then*

$$\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A]\mathbb{P}_p[B].$$

Precisely this inequality for A and B, both being increasing events in addition was proved by van den Berg and Kesten in [BK85] and commonly known as *BK inequality*. We can reformulate Lemma 1.11 in general where the probability of each edge being open is independent but not necessarily the same, but since we would not need this, we do not go into the details.

Chapter 2

Incipient Infinite clusters at Planar lattices

2.1 Introduction

For percolation on $\mathbb{Z}^2$ at criticality, it is known that there is almost surely no infinite cluster [H60, K80] and at the same time there exist open clusters spanning arbitrarily large boxes [R78, SW78]. Aizenman [A97] posited that local patterns around vertices of large spanning clusters appear with frequencies given by a probability measure on occupancy configurations. Although this measure would inherit properties of critical percolation, it would be supported on configurations with an infinite open cluster at the origin. One may call such a measure an incipient infinite cluster measure.

Kesten [K86a] gave a first mathematically rigorous construction of an incipient infinite cluster (IIC) by conditioning on an open path from the origin to the boundary of a large box at critical percolation and increasing the size of the box to infinity. The resulting probability measure is supported on the configurations with an infinite open cluster at the origin. Later, Járai [J03] proved that Kesten's IIC measure indeed describes frequency of local patterns in large crossing clusters. More precisely, if one chooses a vertex uniformly at random in a large crossing cluster, then the occupancy configuration around this vertex has the law given by the IIC measure. In the same paper, he verified that several other natural definitions of the IIC coincide with the one introduced by Kesten.

Later in [DS11, Theorem 1.6], a stronger variant of IIC measure was proved, where the conditioning event was more generalized. Instead of the origin, a small neighborhood around origin was connected to the boundary of a large box by several disjoint clusters.

To be more precise, this connection can be described as a finite series of arms, each of which might be either open or dual closed, described by a specific $\sigma \in \{open, closed\}^k$ for some $k \in N$. The resulting *multiple-arm IIC measure* is characterized solely by σ.

Our main objective in this chapter is to prove that some specific multiple-arm IIC measures, whose existence was validated by [DS11], also describe frequency of local patterns around some special sets. For example, we would prove that the local configuration around a point chosen uniformly from the "backbone" (the set of points which have disjoint paths to right and left side inside the large box) is described by the IIC measure where the conditioning event is that the origin is connected to the boundary of a big box by two disjoint open arms. The local configuration around an edge chosen randomly from the 'lowest' left-right crossing and set of pivotal edges for left-right crossing are shown to follow certain 3-arm and 4-arm IIC measures.

2.2 Notation and result

We recall the independent Bernoulli bond percolation measure on $\mathbb{Z}^2$. Given any $p \in [0,1]$, any edge $e \in \mathbb{E}^2$ is open with probability p and closed with probability $1-p$, independent of all other edges. The sample space is $\Omega = \Pi_{e\in\mathbb{E}^2}\{0,1\}$, elements of which are configurations indicated by $\omega = (\omega(e) : e \in \mathbb{E}^2)$. We define $\omega(e) = 1$ if the edge e is open in configuration ω and $\omega(e) = 0$ if e is closed. We take the σ-field $\mathcal{F}$ to be the one generated by the finite dimensional cylinders. Finally the probability measure is defined as the product measure:

$$\mathbb{P}_p = \prod_{e\in\mathbb{E}^d} \mu_e \ ,$$

where $\mu_e(\omega(e) = 1) = p = 1 - \mu_e(\omega(e) = 0)$. For $x, y \in \mathbb{Z}^2$ and $X, Y \subset \mathbb{Z}^2$, we write

- $x \leftrightarrow y$ if there exist vertices $x_1, x_2, \ldots x_n$ such that $a = x_1, b = x_n$, x_i is neighbour to x_{i+1} for all $i \in \{1, 2 \ldots n-1\}$ and all these edges between neighbours are open.
- $x \leftrightarrow Y$ if there exists $y \in Y$ such that $x \leftrightarrow y$.
- $X \leftrightarrow Y$ if there exists $x \in X$ such that $x \leftrightarrow Y$.

The cluster $C(x)$ containing a vertex x is defined as $C(x) = \{y : x \leftrightarrow y\}$, the set of all vertices which are connected to x. The critical threshold is defined as

$$p_c = \inf\{p : \mathbb{P}_p[C(O) \text{ is infinite}] > 0\},$$

where O is the origin. Let us also call $B(n) = [-n,n]^2$ and $\partial B(n)$ as the inner boundary of $B(n)$, i.e. $B(n) \setminus B(n-1)$.

Kesten [K86a, Theorem 3] proved that, for all cylinder event E, the limit

$$\nu(E) = \lim_{n\to\infty} \mathbb{P}_{p_c}(E|O \leftrightarrow \partial B(n))$$

is well-defined, and by Kolmogorov's extension theorem, ν extends uniquely to a probability measure on configurations of edges, called *Kesten's incipient infinite cluster* measure. (Kolmogorov's extension theorem holds since the compatibility for events depending on finitely many edges hold true, and this can be checked immediately.) In supercritical regime, i.e. for $p > p_c$, existence of the measure $\nu_p(E) = \mathbb{P}_p(E|O \leftrightarrow \infty)$ is trivial since $\mathbb{P}_p[0 \leftrightarrow \infty] > 0$. Kesten [K86a, Theorem 3] showed that $\lim_{p\searrow p_c} \nu_p = \nu$, providing another interpretation of IIC measure. Both of these definitions are quite intuitive, and the fact that they coincide makes this measure quite robust.

Let the left and right part of the interior boundary $\partial B(n)$ be called $L(n) = \{-n\}\times[-n, n]$ and $R(n) = \{n\} \times [-n, n]$ respectively. Járai [J03] made sense of IIC measure as the local configuration picture around a point of *crossing collection*, defined as

$$SC(n) = \{v \in B(n) : \mathrm{L}(n) \leftrightarrow v \leftrightarrow R(n) \text{ inside } B(n)\},$$

which is far away from the boundary. For $u \in \mathbb{Z}^2$ and $e =< x, y >\in \mathbb{E}^2$, let us define translation τ_u acting on Ω by $\tau_u\omega(< x, y >) = \omega(< x - u, y - u >)$, and on events by $\tau_u A = \{\tau_u\omega : \omega \in A\}$. Let us also denote $\mathbb{P}_{p_c}$ by $\mathbb{P}$, since we will only focus on the critical phase from now on.

[J03, Theorem 2] states that, for any cylinder event E, any function $h(n)$ satisfying $h(n) \le n$ but $\lim_{n\to\infty} h(n) = \infty$, and for any sequence of vertices $v_n \in B(n - h(n))$,

$$\lim_{\substack{n\to\infty \\ v_n\in B(n-h(n))}} \mathbb{P}[\tau_{v_n} E \;|v_n \in SC(n)] = \nu(E).$$

The "random" version of this theorem [J03, Theorem 1] states that, if I_n is a uniformly chosen point from the crossing collection $SC(n)$, for any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}[\tau_{I_n} E \;|SC(n) \ne \phi] = \nu(E).$$

(By calling this measure $\mathbb{P}$, we abuse the notation slightly here.) We define the crossing event $LR(n) = \{$There exists an open path from $L(n)$ to $R(n)$ inside $B(n)\}$. Let us now finally define the exceptional point sets whose typical local picture we are interested in understanding.

- **Backbone:** We define the backbone $BB(n)$ as the set of vertices in $B(n)$ which are connected to $L(n)$ and $R(n)$ by two disjoint paths, both being inside $B(n)$.
- **Lowest Crossing:** For every configuration ω in the event $LR(n)$, we can make sense of the unique 'lowest' crossing. We do this by defining a partial relation in

the set of open horizontal crossings in ω characterized by the inclusion of the area enclosed under it inside $B(n)$, and this will have a unique minimal element which we name $\gamma_{min}(\omega)$. Let us call the set of edges on this path γ_{min} as $LC(n)$.

- **Set of Pivotal edges:** An edge e is said to be pivotal for $LR(n)$ in a given configuration ω if exactly one of ω or $\omega' = \{\omega'(f) = \omega(f) \text{ iff } f \neq e\}$ lies in $LR(n)$, i.e. switching that edge e from open to closed or vice versa in ω alters the existence of an open horizontal crossing of $B(n)$. We denote by $P(n)$, the set of pivotal edges.

The dual graph of $\mathbb{L}^2$ is given by $\mathbb{L}^{2*} = \{(\frac{1}{2}, \frac{1}{2}) + x : x \in \mathbb{Z}^2\}$ which is isomorphic to $\mathbb{L}^2$ itself. Each edge e of the original graph intersects with a unique edge in the dual graph and it is called as the dual edge of the edge e. We declare any edge in the dual graph open or closed corresponding to the status of its dual edge and call such open (or closed) paths in the dual graph as dual open (or dual closed, respectively) path. Also, for a set $X \subset \mathbb{Z}^2$, we call X^* as the set of edges dual to all the edges in the original graph with both vertices on X.

Let us call the top and bottom part of $\partial B(n)$ as $T(n) = [-n, n] \times \{n\}$ and $D(n) = [-n, n] \times \{-n\}$. We note two things. Firstly, for every edge e on the lowest crossing γ_{min}, e^* is connected to some edge of $D^*(n)$ by a dual closed path comprised of edges inside $B^*(n)$ (actually inside $U^*(\omega)$, where $U(\omega)$ is the area enclosed by $\gamma_{min}(\omega)$) and two disjoint open connections to $L(n)$ and $R(n)$ from its two vertices inside $B(n)$. Secondly, if e is a pivotal edge in horizontal crossing of $B(n)$, then one endpoint of e is connected to $L(n)$ and the other to $R(n)$ by open paths in $B(n)$ as well as two endpoints of e^* being connected to some edge in $D^*(n)$ and $T^*(n)$ by two disjoint dual closed paths. Thus 2 open paths in $\mathbb{L}^2$ and 2 closed paths in $\mathbb{L}^{2*}$ originate from e in alternate manner.

We will introduce some specific "multiple-arm" IIC measures now as the eligible candidates to describe the configurations around these sets. The existence of our 3 candidate measures we are about to define is already proved (see [DS11, Theorem 1.6 and Remark 7]).

- Let us denote by $O \leftrightarrow_2 \partial B(n)$, the event that the origin is connected to $\partial B(n)$ by two disjoint open paths. For every cylinder event E, we define

$$\nu_2(E) = \lim_{n \to \infty} \mathbb{P}\big[E \mid O \leftrightarrow_2 \partial B(n)\big].$$

- Let us call $e_0 = ((0,0),(1,0))$ and denote by $e_0 \leftrightarrow_3 \partial B(n)$, the event that two endpoints of e_0 are connected to $\partial B(n)$ by two disjoint open paths, e_0^* is connected by a dual closed path inside $B(n)^*$ to some edge in $\partial B(n)^*$ and e_0 is open. For every cylinder event $E \subset \{\omega(e_0) = 1\}$, we define

$$\nu_3(E) = \lim_{n\to\infty} \mathbb{P}\big[E \mid e_0 \leftrightarrow_3 \partial B(n)\big].$$

- Let us denote by $e_0 \leftrightarrow_4 \partial B(n)$, the event that two endpoints of e_0 are connected to $\partial B(n)$ by two disjoint open paths and two endpoints of e_0^* are connected by two disjoint dual closed paths inside $B(n)^*$ to some edges in $\partial B(n)^*$. Notice that this event is independent of $\{\omega(e_0) = 1\}$. For every cylinder event E independent of $\{\omega(e_0) = 1\}$, we define

$$\nu_4(E) = \lim_{n\to\infty} \mathbb{P}\big[E \mid e_0 \leftrightarrow_4 \partial B(n)\big].$$

In $\mathbb{Z}^2$, any edge e is of the form $< x, y >$ where $y = x + (0, 1)$ or $y = x + (1, 0)$. For an edge e, also let ρ_e be the rotation that maps $< (0, 0), (1, 0) >$ to $e_0 =< (0, 0), (1, 0)) >$ for the former case and keeps the edge intact for the later one. We define τ_e as the shift operator on configurations such that for any edge f, $\tau_e(\omega)(f) = \omega(\rho_e(f - x))$. Now we state our main results.

Theorem 2.1. *Let there be sequences v_n of vertices and e_n of edges such that their distance from the boundary is at least $h(n)(\leq n)$ where $\lim_{n\to\infty} h(n) = \infty$.*
(a) For any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{v_n} E \mid v_n \in BB(n)\big] = \nu_2(E).$$

(b) For any cylinder event $E \subset \{\omega(e_0) = 1\}$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{e_n} E \mid e_n \in LC(n)\big] = \nu_3(E).$$

(c) For any cylinder event E independent of $\omega(e_0)$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{e_n} E \mid e_n \in P(n)\big] = \nu_4(E).$$

Theorem 2.2. *Let I_{n_2}, I_{n_3} and I_{n_4} be chosen uniformly from the sets $BB(n)$, $LC(n)$ and $P(n)$ respectively.*
(a) For any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,2}} E \mid BB(n) \neq \phi\big] = \nu_2(E).$$

(b) For any cylinder event $E \subset \{\omega(e_0) = 1\}$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,3}} E \mid LC(n) \neq \phi\big] = \nu_3(E).$$

(c) For any cylinder event E independent of $\omega(e_0)$,

$$\lim_{n\to\infty} \mathbb{P}\big[\tau_{I_{n,4}} E \mid P(n) \neq \phi\big] = \nu_4(E).$$

Remark 2.3. We restrict ourselves to critical bond percolation on $\mathbb{Z}^2$ here, but the results can be extended to site percolation and in fact can be generalized for large class $\mathfrak{C}$ of planar periodic graphs with invariance under reflections and rotation by some fixed angle $\frac{2\pi}{k}$ for some $k \geq 3$. This is due to the fact that the main ingredient of the proof is Russo-Seymour-Welsh theorem (see Theorem 2.4 below), which holds in such generality (see [R78],[SW78],[R81], and [K82], for example).

To prove these results, we will by and large follow the outline of strategy from [J03]. But first, we will require standard tools like Russo-Seymour-Welsh Theorem and quasi-multiplicativity along with several arm-estimates and tightness-results which we provide in Section 2.3. Most of the results come from existing literature, and we prove the other ones as required. One of our results worth highlighting for its novelty is the Lemma 2.10(c), the tightness of pivotals.

2.3 Tools and Arm estimates

It is interesting to notice that, at $p = p_c$ (for Bernoulli bond percolation on $\mathbb{Z}^2$, we actually know the precise value $p_c = \frac{1}{2}$ [K80, Theorem 1]), even if there is no infinite cluster, the probability that an open cluster spans from left to right side of a box of any fixed aspect ratio (but of any size) is uniformly bounded away from 0 and 1. This result, famously known as Russo-Seymour-Welsh theorem, describes that at critical percolation, there exists giant connected clusters at each scale with non-zero probability.

Let the left and right segment of the interior boundary of the rectangle $B(m,n) = [-m,m] \times [-n,n]$ be denoted by $L(m,n) = \{-m\} \times [-n,n]$ and $R(m,n) = \{m\} \times [-n,n]$ respectively. We define the horizontal crossing event

$$\mathrm{LR}(m,n) = \{L(m,n) \text{ is connected to } R(m,n) \text{ by an open path in } B(m,n)\},$$

and we denote the crossing probability as $p(m,n) = \mathbb{P}[\mathrm{LR}(m,n)]$.

Theorem 2.4 (RSW Theorem)**.** *For any $\rho \in (0,\infty)$, there exists $c_\rho > 0$ such that*

$$1 - c_\rho \geq \limsup_{n\to\infty} p(\lfloor \rho n \rfloor, n) \geq \liminf_{n\to\infty} p(\lfloor \rho n \rfloor, n) \geq c_\rho. \tag{2.1}$$

Remark 2.5. This theorem was first proved for critical Bernoulli percolation on planar lattices in [R78, SW78] and recently has been extended to some other planar models, notably to continuum percolation on $\mathbb{R}^2$ [R90], the FK-percolation [DCHN11, DCST17] and Voronoi percolation [BR06, T14]. We will in fact prove a part of this result for 2-dimensional slabs, in Chapter 3. The full result for slabs was proved independently as well by Newman, Tassion and Wu in [NTW15, Theorem 3.1].

Another important result, that we will repeatedly use throughout is quasi-multiplicativity lemma. It demonstrates that the probability of crossing a big annulus is comparable with the product of crossing probabilities for 2 smaller annuli that compose the bigger one. Thus at criticality, there exists constants $C_1 \leq C_2$ such that the fraction $\frac{\mathbb{P}[B(l)\leftrightarrow\partial B(n)]}{\mathbb{P}[B(l)\leftrightarrow\partial B(m)]\mathbb{P}[B(m)\leftrightarrow\partial B(n)]} \in [C_1, C_2]$ uniformly for any $l \leq m \leq n$. (By independence, one can take $C_2 = 1$.)

This does not only hold true for one open crossing but also for any general sequence σ consisting of a series of open and dual closed paths in a specific order. For $k \in \mathbb{N}$, and $\sigma \in \{open, closed\}^k$, let $k_1 \leq k$ be the number of 'open' entries and $k_2 = k - k_1$ are the number of closed entries of σ. For $k \leq l < n$, we denote $B(l) \leftrightarrow_\sigma \partial B(n)$ for the event that there exists k_1 open paths between $B(l)$ and $\partial B(n)$ inside $\mathrm{An}(l,n) = B(n) \setminus B(l-1)$ and k_2 dual closed paths connecting one edge of $B^*(l)$ to one edge of $\partial B(n)^*$ using only edges of $\mathrm{An}(l,n)^*$ such that the relative counterclockwise arrangement of them is given by σ. We also define

$$\alpha_\sigma(r,R) := \mathbb{P}\left[B(r) \leftrightarrow_\sigma \partial B(R)\right],$$

where $R \geq r \geq |\sigma|$. For simplicity we will call $\alpha_\sigma(|\sigma|, n)$ as $\alpha_\sigma(n)$.

Lemma 2.6 (Quasi-multiplicativity). *[N08, Proposition 16] There exists constant $Q_\sigma > 0$ only depending on $|\sigma|$ such that*

$$Q_\sigma \alpha_\sigma(R) > \alpha_\sigma(r,R)\alpha_\sigma(r) > \frac{\alpha_\sigma(R)}{Q_\sigma} \text{ whenever } |\sigma| < r < R. \tag{2.2}$$

Kesten [K87, Lemma 6] proved this result for $\sigma = \{open, closed, open, closed\}$ for a general set of periodic graphs which include $\mathfrak{C}$ (see Remark 2.3), and Nolin extended this result for any σ. (Although he proves it for site percolation on triangular lattices, his proof of quasi-multiplicativity does not rely crucially on specifics of that model.) We can define such arm-events in case of half and quarter plane by defining $\alpha_\sigma^+(r,R)$ as the same event but all $|\sigma|$ arms being restricted in $B(R)\setminus B(r) \cap (\mathbb{Z}\times\mathbb{Z}_+)$ and $\alpha_\sigma^{++}(r,R)$, the restricted region being $B(R)\setminus B(r)\cap(\mathbb{Z}_+\times\mathbb{Z}_+)$. Since they satisfy certain RSW theorem estimates in [K87, (2.15)] by virtue of having the same critical point as of percolation on $\mathbb{Z}^2$ ([GM90]), quasi-multiplicativity property holds true for them as well. (The argument is laid out in [K87, Page 112, Penultimate Paragraph].)

2.3.1 Arm Estimates

From the existing literature, here we gather some estimates of $\alpha_\sigma(r,R)$, for some specific sequences σ that we will later require. Before that let us define the following alternating sequences:

$$\sigma_{2j}^A \quad = \{open, closed, open, \ldots, closed\}, \quad |\sigma_{2j}^A| = 2j,$$
$$\sigma_{2j-1}^A \quad = \{open, closed, open, \ldots, open\}, \quad |\sigma_{2j-1}^A| = 2j-1,$$

for $j \in \mathbb{N}$, i.e. σ_{2j}^A (σ_{2j-1}^A, respectively) implies j open paths and j ($j-1$, respectively) dual closed paths alternately. We define $\alpha_j(r,R) := \alpha_{\sigma_j^A}(r,R)$ for $j \in \mathbb{N}$ and $\alpha_*(r,R) = \alpha_{\{closed\}}(r,R)$. Similarly, $\alpha_j^+(r,R)$, $\alpha_j^{++}(r,R)$, $\alpha_*^+(r,R)$ and $\alpha_*^{++}(r,R)$ are defined. Let us also specifically define $\sigma_3 = \{closed, open, closed\}$.

For two functions f and g defined on $\mathbb{N}^2$, we denote $f(r,R) = O(g(r,R))$ if there exists an universal constant $C > 0$ such that $f(r,R) < Cg(r,R)$ $\forall R \geq r \geq 1$. We also denote $f(r,R) \asymp g(r,R)$ when $f(r,R) = O(g(r,R))$ and $g(r,R) = O(f(r,R))$.

Lemma 2.7. [Arm Probabilities] *For any integers $R \geq 4r \geq 4$,*
(i) $(r/R) = O(\alpha_3(r,R))$, [DHS14, (16)]
(ii) $\alpha_5(r,R) \asymp (r/R)^2$, [KSZ98, Lemma 5]
(iii) ${\alpha_3}^+(r,R) \asymp (r/R)^2 \asymp {\alpha_{\sigma_3}}^+(r,R)$, [W07, First exercise sheet]
and there exists $\delta \in (0,1/2)$ such that
(iv) $max(\alpha_1(r,R), \alpha_*(r,R)) = O((r/R)^\delta)$, [G99, (11.90)]
(v) $(r/R)^{2-\delta} = O(\alpha_4(r,R))$, [GPS10, (2.6)]
(vi) ${\alpha_2}^{++}(r,R) = O((r/R)^{1+\frac{\delta}{2}})$. [GPS10, (4.13)]

Remark 2.8. For (i), the result in [DHS14] is slightly stronger, but we will not need the stronger form. For (iii), we cite an exercise sheet in [W07], which might feel improper, but we do so since the proof is similar to that of (ii) done by [KSZ98] and not central to our results. For (iv), [G99, 11.90] gives us the partial result that $\alpha_1(r,R) = O((r/R)^\delta)$ for some $\delta \in (0,1)$. The bound for α_* can be found in a similar way, utilizing two facts. Firstly, a dual closed path from $\partial B(r)$ to $\partial B(R)$ implies the absence of open circuits in concentric annuli in $B(R)\backslash B(r)$. Secondly, the probability of an open circuit in each annulus is bounded away from 1. This proves (iv), possibly for a smaller value of δ.

2.3.2 Expectation Estimates and Tightness Results

To prove results for crossing collection $SC(n)$, Járai needed results on how big $|SC(n)|$ typically is. [J03, Theorem 8(i)] and [J03, Theorem 8(ii)] provided him with expectation estimates and tightness results which were crucial for his proof. Similarly for us, two crucial ingredients to prove Theorems 2.1-2.3 would be estimates of expected sizes and tightness results for the sets $BB(n)$, $LC(n)$ and $P(n)$. Most of these results are already well-known in existing literature. The key novel result we want to emphasise on here is the tightness of pivotals, which is more challenging to prove.

The following Lemmas 2.9 and 2.10 have the equivalent statements posed in [J03, Theorem 8(i)] and [J03, Theorem 8(ii)], respectively. Let us call $\alpha_B(n) := \alpha_{\{open,open\}}(n)$.

Lemma 2.9. [Expectation Estimates]
(a) $\mathbb{E}[|BB(n)|] \asymp n^2\alpha_B(n)$.
(b) $\mathbb{E}[|LC(n)|] \asymp n^2\alpha_3(n)$, *[[MZ05],[DHS15, (12)]].*
(c) $\mathbb{E}[|P(n)|] \asymp n^2\alpha_4(n)$, *[[GPS10, Equation 7.3]].*

Proof. **(a):** We prove Lemma 2.9(a) by following the proof of [J03, Theorem 8(i)] for the sake of completeness, since this is short and would provide the readers with a quick glimpse of how such estimates are proved typically.

We define $v + B(n)$ as the square of sidelength $2n$ with centre at v. Similarly we define $\mathrm{An}(v, N, M) = \{v + B(M)\} \setminus \{v + B(N-1)\}$ for $N \le M$. For any $v = (v_1, v_2) \in B(n/2)$ we define the following events:

$$\begin{aligned} X(v) &= \{\text{There is an open horizontal crossing in } [-n, n] \times [v_2 - n/4, v_2 + n/4]\}, \\ A(v) &= \{\text{There is an open circuit in } \mathrm{An}(v, n/4, n/2)\}, \\ Y(v) &= \tau_v\{0 \leftrightarrow_{\{open,open\}} \partial B(n/2)\}. \end{aligned} \tag{2.3}$$

By RSW theorem, $\mathbb{P}[X(v)] > c_4$ and $\mathbb{P}[Y(v)] > c_3^4$, implying for each $v \in B(n/2)$,

$$\begin{aligned} \mathbb{P}[v \in BB(n)] \ge \mathbb{P}[X(v) \cap Y(v) \cap A(v)] &\ge \mathbb{P}[X(v)]\mathbb{P}[A(v)]\mathbb{P}[Y(v)] \\ &\ge c_4 {c_3}^4 \alpha_B(n/2) \ge c\, \alpha_B(n). \end{aligned}$$

We used FKG in the second step and quasi-multiplicativity in the last step. Summing over all such v gives us

$$\mathbb{E}[|BB(n)|] \ge \sum_{v \in B(n/2)} \mathbb{P}[v \in BB(n)] \ge cn^2\alpha_B(n).$$

For the other side, we classify each vertex by how close is it from either $L(n)$ or $R(n)$ and observe

$$\begin{aligned} \mathbb{E}[|BB(n)|] &= \sum_{v \in B(n)} \mathbb{P}[v \in BB(n)] \le 2n \sum_{r=0}^{n} \alpha_B(r)\alpha_1(r, n) \le Cn \sum_{r=0}^{n} \frac{\alpha_1(r, n)\alpha_B(n)}{\alpha_B(r, n)} \\ &\le C' n\alpha_B(n) \sum_{r=0}^{n} \frac{(r/n)^\delta}{r/n} \le C' n^{2-\delta}\alpha_B(n) \sum_{r=0}^{n} r^{\delta-1} \le C'' n^2\alpha_B(n). \end{aligned}$$

We used quasi-multiplicativity and Lemma 2.7(i),(iv) in third and fourth steps respectively. This completes the proof of Lemma 2.9(a). □

Before stating the tightness Lemma 2.10, we need to mention that both of these lemmas will be used to prove Theorem 2.2 but only Lemma 2.9 will be required for Theorem 2.1.

Lemma 2.10. [Tightness result]

(a) $\lim_{\epsilon\to 0}\inf_{n\geq 1}\mathbb{P}\left[\epsilon \leq \frac{|BB(n)|}{\mathbb{E}[|BB(n)|]} \leq \frac{1}{\epsilon}\ |BB(n)\neq\phi\right] = 1.$

(b) $\lim_{\epsilon\to 0}\liminf_{n\geq 1}\mathbb{P}\left[\epsilon \leq \frac{|LC(n)|}{\mathbb{E}[|LC(n)|]} \leq \frac{1}{\epsilon}\ |LC(n)\neq\phi\right] = 1$, *[DHS15, Lemma 24].*

(c) $\lim_{\epsilon\to 0}\inf_{n\geq 1}\mathbb{P}\left[\epsilon \leq \frac{|P(n)|}{\mathbb{E}[|P(n)|]} \leq \frac{1}{\epsilon}\ |P(n)\neq\phi\right] = 1.$

Remark 2.11. Notice that proving one side of the tightness result is immediate once we know expectation estimates. For example, for lowest crossing, by Markov inequality,

$$\begin{aligned}\mathbb{P}[\frac{|LC(n)|}{\mathbb{E}[|LC(n)|]} \geq \frac{1}{\epsilon}\ |LC(n)\neq\phi] &= \frac{1}{\mathbb{P}[LC(n)\neq\phi]}\mathbb{P}[\frac{|LC(n)|}{\mathbb{E}[|LC(n)|]} \geq \frac{1}{\epsilon}] \\ &\leq \frac{\epsilon}{\mathbb{P}[LC(n)\neq\phi]}. \end{aligned} \tag{2.4}$$

We know that $\mathbb{P}[BB(n)\neq\phi] = \mathbb{P}[LC(n)\neq\phi] = \mathbb{P}[\exists$ a horizontal crossing of $B(n)] > c$ for some constant $c > 0$. Thus RHS of (2.4) goes to 0 uniformly in n as $\epsilon \to 0$, and same holds for $BB(n)$. For $P(n)$, this will hold true similarly once we know $\mathbb{P}[P(n)\neq\phi] > C_1$ uniformly in n.

The other bound for backbone can be proved by practically following the proof of [J03, Theorem 8(ii)]. Since the proof is quite long and the only change required is re-defining one set, ($Y(m)$, to be precise, for having 2 disjoint connections instead of 1 to the lowest crossing path) we choose not to present it here.

As referred earlier, [DHS15, Lemma 24] in fact prove the difficult lower bound for lowest crossing, namely:

$$\lim_{\epsilon\to 0}\limsup_{n\geq 1}\mathbb{P}\left[0 \leq |LC(n)| \leq \epsilon\mathbb{E}[|LC(n)|]\right] = 0.$$

Therefore we only present the proof for pivotals here.

Tightness of Pivotals

As we described earlier, for the upper bound we need to prove that,

$$\mathbb{P}[P(n)\neq\phi] > C_1 > 0, \tag{2.5}$$

uniformly in n for some $C_1 > 0$. To prove (2.5), observe that the event $LR(n)\setminus\{LR(n)\circ LR(n)\}$ is exactly $\{P(n)\neq\phi, LR(n)\}$. (Recall that $\circ$ means disjoint occurrence.) This is because by Menger's theorem [M27], if it is possible to disconnect $L(n)$ from $R(n)$ by closing one edge, there cannot be more than one horizontal crossings that share some edge and vice versa. Let us call $p_n = \mathbb{P}[LR(n)]$. We have

$$\mathbb{P}[P(n)\neq\phi, LR(n)] = \mathbb{P}[LR(n)\setminus\{LR(n)\circ LR(n)\}] \geq p_n(1-p_n) \geq C_1, \tag{2.6}$$

where we use BK inequality in the second step, and RSW Theorem 2.4 that says p_n is bounded away from 0 and 1 uniformly in n in the last step. (Similarly we can find a lower bound for non-existence of left right crossing instead as well.) This completes the proof for the upper bound.

Proving the lower bound is more challenging, and in fact this will be the key ingredient to prove Theorem 2.2 (c). To explain the statement heuristically, we have to show that when a pivotal edge exists, it is very likely that many of them (i.e. asymptotic to their mean value) exists. Our strategy will be to show that in a square around any pivotal, there will be many pivotals with high probability. To have that space completely inside $B(n)$, it is convenient if we can ensure that pivotals are likely to be away from the boundary. This will be the first step of our proof. Let us call, for any set $S \subset \mathbb{Z}^2$ of vertices, the edges with at least one vertex inside S as $E(S)$.

Lemma 2.12 (**Boundary Lemma**). *Given $\epsilon > 0$, we can find $\alpha > 0$ small enough such that*

$$\mathbb{P}[E(\mathrm{An}(\lfloor(1-\alpha)n\rfloor, n)) \cap P(n) \neq \phi] < \epsilon.$$

This lemma will also find its use separately for the proof of Theorem 2.2(c). We prove this first before describing other components.

a) Proof of Boundary Lemma: Let us define k_0 as the integer such that $\frac{1}{2^{k_0}} \geq \lceil \alpha n \rceil > \frac{1}{2^{k_0+1}}$, and call $r = \lceil \frac{n}{2^{k_0}} \rceil \geq \lceil \alpha n \rceil$. We focus on dividing $\mathrm{An}(n - r(n), n)$ into two type of rectangles. First we take care of the four corner squares. Let us take one of them, say $C = [n-r, n] \times [n-r, n]$. If there exists any pivotal edge in $E(C)$ then there exists one open arm from C to $L(n)$ and one dual closed path to $D^*(n)$, both inside $B(n)$. So, by Lemma 2.7(vi),

$$\mathbb{P}[P(n) \cap E(C) \neq \phi] \leq {\alpha_2}^{++}(2r, 2n) \leq (r/n)^{1+\frac{\delta}{2}} < (2\alpha)^{1+\frac{\delta}{2}}. \tag{2.7}$$

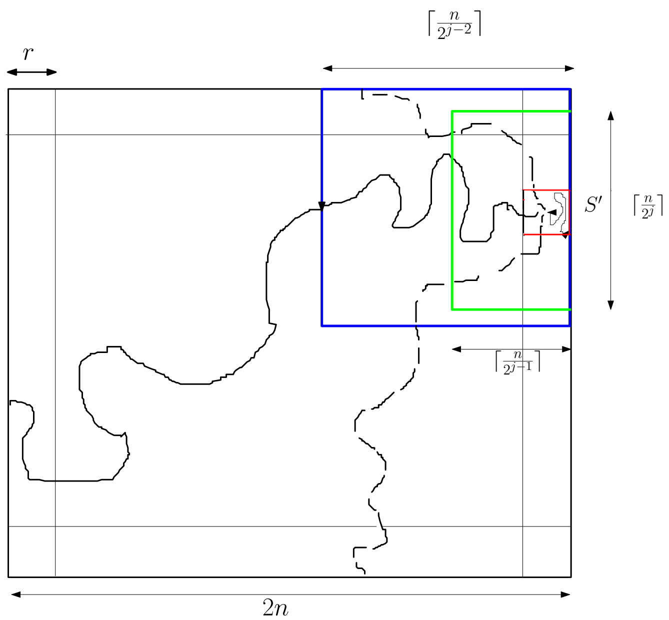

FIGURE 2.1: Pivotal edge close to boundary

We divide the rest of the boundary region in eight symmetric parts now, and choose one of them, say $S = [n-r, n] \times [0, n-r]$. We further divide S into sub-rectangles $S_j = [n-r, n] \times [n - \lceil n/2^{j-1} \rceil, n - \lceil n/2^j \rceil]$ for $j = 1, 2, \ldots, k_0$ and observe that each such sub-rectangle S_j can be covered by $2^{k_0 - j}$ distinct squares of dimension $r \times r$. If there exists a pivotal edge in such a component square S' centered at v' of dimension $r \times r$ in, say S_j, then there are three alternating arms to the boundary of the half-annulus $[v' + B(n/2^j) \cap B(n)] \setminus S'$,and then two alternating arms from $[n - \lceil n/2^{j-2} \rceil, n] \times [n - \lceil n/2^{j-2} \rceil, n]$ to left and bottom side of $B(n)$, both lying inside $B(n)$ (See Figure 2.1). This yields

$$\begin{aligned}
\mathbb{P}[E(S) \cap P(n) \neq \phi] &\leq \sum_{j=1}^{k_0} \mathbb{P}[E(S_k) \cap P(n) \neq \phi] \\
&\leq \sum_{j=1}^{k_0} 2^{k_0 - j} {\alpha_3}^{+}(r, \lceil n/2^j \rceil) {\alpha_2}^{++}(\lceil n/2^{j-2} \rceil, 2n) \qquad (2.8) \\
&\overset{[Lemma 2.7 (iii),(vi)]}{\leq} \sum_{j=1}^{k_0} C 2^{k_0 - j} (2^{j-k_0})^2 (1/2^{j-1})^{1+\delta/2} = \sum_{j=1}^{k_0} \frac{C}{2^{k_0-1}} 2^{-(j-1)\delta/2} \leq C'\alpha.
\end{aligned}$$

Thus by (2.7) and (2.8) and rotation invariance of the lattice, we get

$$\mathbb{P}[P(n) \cap E(\mathrm{An}(\lfloor (1-\alpha)n \rfloor, n)) \neq \phi] \leq 8C'\alpha + 4(2\alpha)^{1+\frac{\delta}{2}}.$$

Given $\epsilon > 0$, we can make the RHS of the above less than ϵ by choosing a suitable α (We can find a large constant C such that $\alpha = \frac{\epsilon}{C}$ works, for example.) and this completes our proof. □

b) Simplification: Let us recall that by $LR(n)$ we denote the existence of a horizontal open crossing of $B(n)$. Given $\epsilon > 0$, we will find $\theta > 0$ such that the following equations hold:

$$\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n) \; |P(n) \neq \phi, LR(n)] < \epsilon, \qquad (2.9)$$

$$\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n) \; |P(n) \neq \phi, LR(n)^c] < \epsilon, \qquad (2.10)$$

and this will complete the proof of Lemma 2.10(c), since Lemma 2.9(c) already tells us that $\mathbb{E}[|P(n)|] \asymp n^2 \alpha_4(n)$. Notice that by (2.6) and similar adaptation of it, the probability of both events $\{|P(n)| \neq \phi, LR(n)\}$ and $\{|P(n)| \neq \phi, LR(n)^c\}$ are bounded from below uniformly in n. Let us call the minimum of such bounds as $c_{\mathfrak{e}}$.

We will first prove (2.9) and it will be immediate how one little modification would prove (2.10). Let us call $A(n) = \{P(n) \neq \phi, LR(n)\}$. Let us first eliminate the pivotal edges close to the boundary using Lemma 2.12. Given $\epsilon > 0$, we choose α such that

$$\mathbb{P}[P(n) \subset E(B(\lfloor (1-\alpha)n \rfloor))] \geq 1 - \frac{\epsilon}{2c_{\mathfrak{e}}}, \qquad (2.11)$$

and call the event as G_n. For any configuration where both a horizontal crossing and pivotal edges for it exist, we notice two things. Firstly every pivotal edge must be open and secondly we can order the pivotal edges from first to last by the order in which any open horizontal crossing traverses through them from left to right. Let us denote the first pivotal edge by $\mathfrak{e}_f$. $\mathfrak{e}_f$ is naturally connected to $R(n)$ by an open path and to $L(n)$ by two open paths. These three paths are disjoint and inside $B(n)$. Also, $\mathfrak{e}_f{}^*$ is connected to $T^*(n)$ by a dual closed path, and let us call the leftmost path as Γ_t. Similarly let us call the leftmost path from $B^*(n)$ to the other end of $\mathfrak{e}_f{}^*$ as Γ_b. We will show later that the event H_n where the paths Γ_t and Γ_b are well-separated (which will be made rigorous later) has probability at least $1 - \frac{\epsilon}{4c_\mathfrak{e}}$. This would imply that

$$\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n) \; |A(n)] \leq \mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n), G_n, H_n \; |A(n)] + 3\epsilon/4. \tag{2.12}$$

We will now decompose the RHS of the above conditioning on $\mathfrak{e}_f$, Γ_t and Γ_b in the following way

$$\begin{aligned}
&\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n), G_n, H_n \; |A(n)] \\
&= \sum_e \sum_{\gamma_t, \gamma_b} \mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n), G_n, H_n \; |A(n), \mathfrak{e}_f = e, \Gamma_t = \gamma_t, \Gamma_b = \gamma_b] \\
&\qquad\qquad \cdot \mathbb{P}[\mathfrak{e}_f = e, \Gamma_t = \gamma_t, \Gamma_b = \gamma_b \; |A(n)].
\end{aligned}$$

Thus proving (2.9) reduces to proving that, uniformly in all permissible e, γ_t, and γ_b,

$$\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n), G_n, H_n \; |A(n), \mathfrak{e}_f = e, \Gamma_t = \gamma_t, \Gamma_b = \gamma_b] < \epsilon/4. \tag{2.13}$$

Notice that under the above condition, $B(n)$ is divided into two parts by $\gamma_t \cup \gamma_b$ which are connected through e. Let us call the part containing $R(n)$ as K. Let us denote γ_t and the part of $T^*(n)$ lying right of it together as $T^*(K)$. Similarly for γ_b and part of $D^*(n)$ right of it is denoted as $D^*(K)$. Conditioned on the event $\{\mathfrak{e}_f = e, \Gamma_t = \gamma_t, \Gamma_b = \gamma_b\}$, any other pivotal edge f must satisfy the following conditions:

- f must be open and in K.
- f is connected to e and $R(n)$ by two edge disjoint open paths, both inside K.
- f^* is connected to $T^*(K)$ and $D^*(K)$ by two edge disjoint dual closed paths, both inside K^*.

Let us call the set of such edges as $Y(n)$ and observe that once $\mathfrak{e}_f$, Γ_t and Γ_b are fixed, the event mentioned in (2.13) can be reinterpreted as

$$\begin{aligned}
&\mathbb{P}[0 < |P(n)| < \theta n^2 \alpha_4(n), G_n, H_n \; |A(n), \mathfrak{e}_f = e, \Gamma_t = \gamma_t, \Gamma_b = \gamma_b] \\
&\qquad\qquad = \mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n) - 1, G'_n \; |e \overset{K}{\leftrightarrow} R(n)],
\end{aligned}$$

where $G'(n)$ is the event that there exists no pivotal in $E(K \cap \mathrm{An}((1-\alpha)n, n))$ and $\{e \overset{K}{\leftrightarrow} R(n)\}$ indicates the existence of an open path from e to $R(n)$ inside K. The event H_n vanishes since the chosen γ_b and γ_t are deterministic paths which are "well-separated". Thus it would suffice for us to prove, uniformly over any $e \in E(B(\lfloor(1-\alpha)n\rfloor))$ and any such 'permissible' shape K,

$$\mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G'_n \mid e \overset{K}{\leftrightarrow} R(n)] < \epsilon/4. \tag{2.14}$$

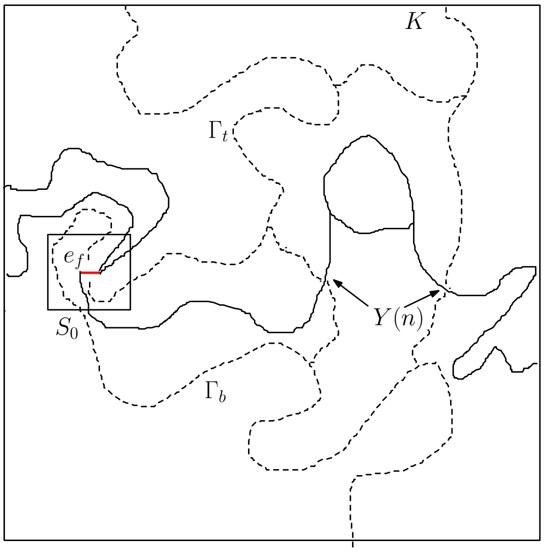

FIGURE 2.2: Conditioning on first pivotal

For any edge e, we define $v(e)$ as its left vertex if e is horizontal, or its bottom vertex if e is vertical. Since $e \in E(B(\lfloor(1-\alpha)n\rfloor))$, the square S_0 centered around $v(e)$ of side length $2\lfloor\alpha n\rfloor$ lies entirely inside $B(n)$. We intend to show that with high probability there exists at least $\theta n^2 \alpha_4(n)$ pivotals inside $S_0 \cap K$. We will split this square into disjoint annuli $A_i = \mathrm{An}(v(e), \lceil \frac{\alpha n}{2^i} \rceil, \lceil \frac{\alpha n}{2^{i-1}} \rceil)$ for $i = 1, 2, \ldots, k$. This k will be chosen later.

The heuristic argument which we will make rigorous later is that the probability of having so many pivotals in each annulus (from a certain fraction of the annuli) is bounded from below uniformly and such events are independent. This will bound the probability from above exponentially in k, and by choosing k large enough we can prove (2.14). There are two main challenges. Firstly we need some space between Γ_t and Γ_b to position certain "boxes" which will potentially contain pivotals, and that is why we needed them to be 'well-separated' in the first place. Secondly the existence of a long open connection $e \overset{K}{\leftrightarrow} R(n)$ robs the annuli of their independence. Let us first address the well separation issue and rigorously define the event H_n.

c) Well-separation of Boundaries: We say that K is well behaved, or equivalently Γ_t and Γ_b are β_0 *well separated*, if Γ_b and Γ_t have distance at least $\lceil \frac{\beta_i \alpha n}{2^i} \rceil$ inside $A_i \cap K$ for fractions $\{\beta_i = \frac{\beta_0}{2^{\frac{3i}{\delta}}}\}_{1 \le i \le k}$. As mentioned before, we call this event as $H_n = H_n(\beta_0, k)$ and will show $\mathbb{P}[H_n]$ can be made high enough by choosing β_0 small, irrespective of how big k is.

If Γ_t and Γ_b has distance $\le \lfloor \frac{\beta_i \alpha n}{2^i} \rfloor$ inside A_i, it implies that there exists a vertex $v \in A_i$ such that there exists six arms in the annulus $\mathrm{An}(v, \lfloor \frac{\beta_i \alpha n}{2^i} \rfloor, \lceil \frac{\alpha n}{2^i} \rceil)$ in the order $\sigma_6 = \{open, open, closed, open, open, closed\}$. Let us call this event az $Z_i(v)$. With Lemma 2.7 (ii),(iv) and Reimer's inequality, it is immediate that $\alpha_{\sigma_6}(m,n) = O((m/n)^{2+\delta})$ for some $\delta \in (0, 1/2]$. Thus if we cover the whole box $B(n)$ by squares of sidelength $\lceil \frac{\beta_i \alpha n}{2^{i+1}} \rceil$, such a six-arm event in annulus will be present with such a square completely inside the smaller square. This yields

$$\begin{aligned}\mathbb{P}[\bigcup_{v \in B(n)} Z_i(v) \text{ occurs}] &\le O\Big(\frac{n}{\lceil \frac{\beta_i \alpha n}{2^i} \rceil}\Big)^2 \alpha_{\sigma_6}\Big(\lceil \frac{3\beta_i \alpha n}{2^{i+1}} \rceil, \lceil \frac{\alpha n}{2^{i-1}} \rceil\Big) \\ &\le C \frac{2^{2i}}{\alpha^2 \beta_i^2} \Big(\frac{\beta_i}{2}\Big)^{2+\delta} \le C' \frac{2^{2i} {\beta_i}^{\delta}}{\alpha^2}.\end{aligned} \tag{2.15}$$

We will choose the fraction $\beta_0 < 1/2$ later. Using explicit form of β_i gives

$$\mathbb{P}[H_n^c] \le \mathbb{P}[\bigcup_{i \ge 1} \bigcup_{v \in B(n)} Z_i(v) \text{ occurs}] \le \sum_{1 \le i \le k} C' \frac{\beta_0^{\delta}}{\alpha^2 2^i} \le C' \frac{{\beta_0}^{\delta}}{\alpha^2}. \tag{2.16}$$

Given ϵ, we have chosen α first to satisfy (2.11) and then by choosing β_0 small, we make the RHS $< \frac{\epsilon}{4c_\epsilon}$ as promised before, and this does not depend on how big the value of k we pick.

d) Construction of 'good' annulli: To prove (2.14), we will decouple the long arm event $e \overset{K}{\leftrightarrow} R(n)$ into several independent events localized on disjoint annuli. For this we target to construct two circuits inside each $An_i = A_i \cap K$ slightly apart. By condition on innermost and outermost circuits as such, the conditioned event will be reduced to the existence of disjointed open connections between two such circuits in several annuli. Since existence of such circuits have probability uniformly bounded from below, it is intuitively clear that with high probability, we will get a certain fraction of 'good' annuli.

The obvious glitch is that we cannot get a complete circuit naturally inside An_i which we will later address by replacing them with suitable open arcs. Another subtle nuance is that had the distance of Γ_b and Γ_t been $\ge \lceil \frac{\beta \alpha n}{2^i} \rceil$ in An_i uniformly for some β, we could have positioned a box of length $\asymp \lceil \frac{\beta \alpha n}{2^i} \rceil$ comfortably inside these circuits and attempted to show that such a box has many pivotals. (This could not be proved since the RHS in (2.16) would have blown up in such a case.) But since β_i is also changing, such a strategy is not universal. Thus, we will categorize the annuli into two groups.

We call an annulus A_i of 'Type-A' if the distance between γ_b and γ_t inside An_i is $\geq \lceil \frac{\beta_0 \alpha n}{2^i} \rceil$. Otherwise the distance lies between $\lfloor \frac{\beta_0 \alpha n}{2^i} \rfloor$ and $\lceil \frac{\beta_i \alpha n}{2^i} \rceil$ and we call the annulus of 'Type-B'. Also, for some space constraints, we will work only with every third annuli $D_i = A_{3i-2}$. Now we describe our definition of 'good' annuli for each of the two types.

Type-A For the Type-A annuli, we do the following. We sample independently a configuration ω' on edges of D_i and then superimpose these two configuration in ω'' as $\omega''(e) = \omega(e)$ if $e \in E(K)$ and $\omega''(e) = \omega'(e)$ if $e \notin E(K)$. Let us call the corresponding probability measures as $\mathbb{P}'$ and $\mathbb{P}''$. If there are open circuits in ω' in each sub annuli $D_i^{in} = \mathrm{An}(v(e), \lfloor \frac{\alpha n}{2^{3i-3}} \rfloor, \lfloor \frac{5\alpha n}{2^{3i-1}} \rfloor)$ and $D_i^{ex} = \mathrm{An}(v(e), \lfloor \frac{7\alpha n}{2^{3i-1}} \rfloor, \lfloor \frac{\alpha n}{2^{3i-2}} \rfloor)$, inside configuration ω'' it would create open paths O_i^{in} and O_i^{ou} in both of the aforementioned two sub-annuli D_i^{in}, D_i^{ex} connecting γ_t with γ_b, along with (possibly) a series of open paths joining γ_t (or γ_b) with itself strictly inside those two sub-annuli.

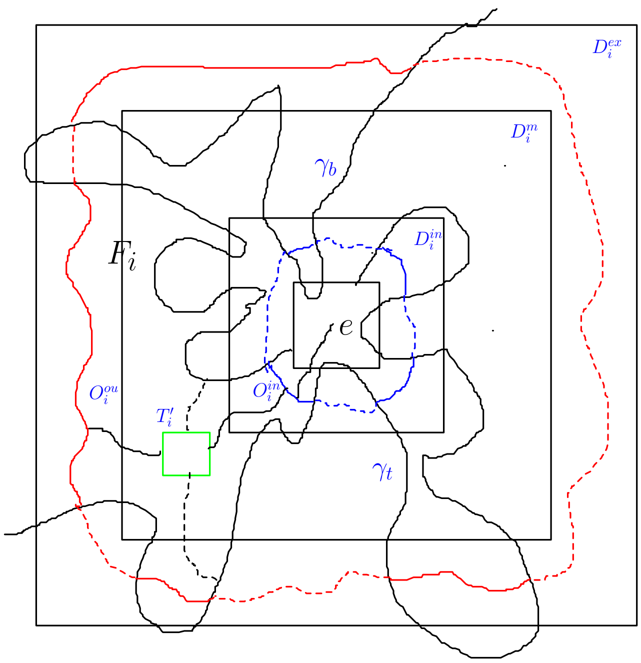

FIGURE 2.3: Good annuli of Type A

Any two such circuits create, in ω'', a closed area F_i inside $K \cap D_i$ such that

– its boundary constitutes of O_i^{in}, segments of γ_t inside $K \cap D_i$ joined possibly by open paths inside D_i^{ex} (remnants of edges of the open circuit), O_i^{ou} and then again segments of γ_b inside $K \cap D_i$ joined possibly by open paths inside D_i^{ex} (again, remnants of the circuit),

– for any open connection from e to $R(n)$ inside K, the path must first cross O_i^{in}, then through F_i to O_i^{ou}.

Let us call this Type-A annuli 'good' if there exists such a region F_i. We have, for any Type A annuli D_i,

$$\mathbb{P}[D_i \text{ is good}] = \mathbb{P}''[D_i \text{ is good}] \geq \mathbb{P}'[\exists \text{ open circuits in both } D_i^{in} \text{ and } D_i^{ex}] \geq c_c, \tag{2.17}$$

for some universal constant $c_c > 0$. The first equality comes from the fact that the 'goodness' of D_i is good only depends on the edges inside $K \cap D_i$. Given a deterministic K, we can find a box T_i inside the middle sub-annulus $D_i^m = \mathrm{An}(v(e), \lceil \frac{5\alpha n}{2^{3i-1}} \rceil, \lceil \frac{7\alpha n}{2^{3i-1}} \rceil)$ and K, of sidelength $\eta_i = \lfloor \frac{\beta_0 \alpha n}{2^{3i-2}} \rfloor$ such that it does not intersect γ_t or γ_b. (We can deterministically decide on such a box given K, such that it would surely be inside F_i irrespective of the positions of open circuits.) We will try to show, for 'good' annuli of Type-A, that there are possibly many pivotals in the square T_i' concentric with T_i but with sidelength $\lfloor \eta_i/2 \rfloor$ (See Figure 2.3).

Type-B For a Type-B annuli, let the distance between γ_t and γ_b inside $D_i \cap K$ be $L_i \in [\lceil \frac{\beta_i \alpha n}{2^{3i-2}} \rceil, \lceil \frac{\beta_0 \alpha n}{2^i} \rceil]$. Let us assume for the sake of simplicity, L_i is divisible by 4. We take a $v_i \in K \cap D_i$ for which the boundary of $v_i + B(L_i/2)$ *just* touches both γ_b and γ_t (In presence of multiple such boxes, we follow some pre determined order). We resample here the edges of the annulus $\mathrm{An}(v_i, L_i/2, L_i)$ independently again in a configuration ω' (with measure $\mathbb{P}'$) and stitch ω and ω' into ω'' together as before (with measure $\mathbb{P}''$), depending on whether they are from K or not.

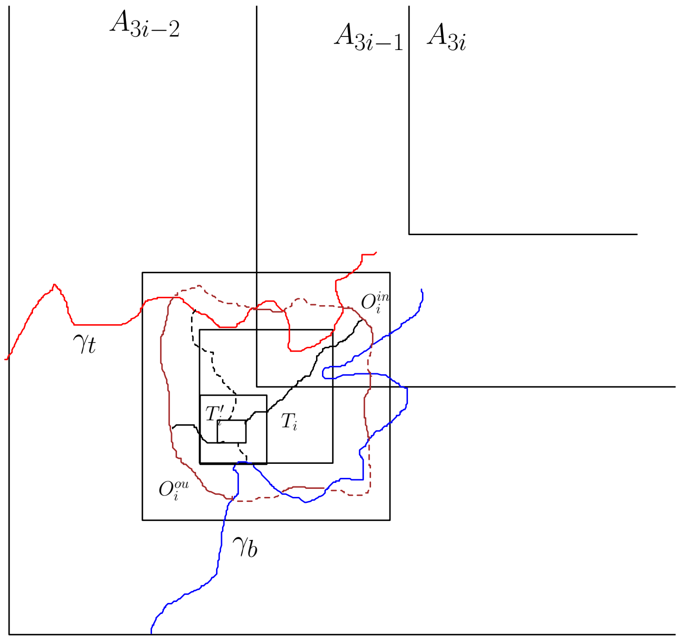

FIGURE 2.4: Good annuli of Type B

If there is an open circuit in $\mathrm{An}(v_i, L_i/2, L_i)$ in ω', it would similarly create a closed area F_i inside this annulus whose boundaries would consist of parts of the open circuit, and segments of γ_t and γ_b inside $\mathrm{An}(v_i, L_i/2, L_i)$ including two open segments O_i^{in} and O_i^{ou} of the circuit, such that any path from e to $R(n)$ would have to first pass through O_i^{in}, then through F_i to O_i^{ou}. We call D_i 'good' if there exists such an area inside $v_i + B(L_i)$ (which have $v_i + B(L_i/2)$ completely inside). For any Type-B annuli D_i,

$$\mathbb{P}[D_i \text{ is good}] = \mathbb{P}''[D_i \text{ is good}] \geq \mathbb{P}'[\exists \text{ open circuit in } \mathrm{An}(v_i, L_i/2, L_i)] \geq c_c', \tag{2.18}$$

for a universal constant c_c'. Notice that $v_i + B(L_i)$ may not be completely inside D_i, but since $\beta_0 < 1/2$, this box $v_i + B(L_i)$ is disjoint from previous and successive annuli D_{i-1} and D_{i+1} (this was the precise reason for considering every third annulus instead of all). Also at least one of the four quarters of this box (let us call it T_i) must be inside D_i, and let us take T_i' as the square concentric with T_i but with sidelength $L_i/4$. We will try to show, in this case, that there are possibly many pivotals inside T_i' (see Figure 2.4).

This shows that each annuli is 'good', irrespective of its type, with probability at least $c_0 = c_c \wedge c_c'$ (by (2.17) and (2.18)). Let us take annuli D_i for $i = 1, 2, \ldots, k$ for some k we will choose later. By independence, we can choose k_0 large enough such that

$$\mathbb{P}[\text{At least } \frac{c_0}{2} \text{ fraction of annuli are 'good'}] \geq 1 - \frac{\epsilon}{8}, \tag{2.19}$$

for any $k \geq k_0$. Since this event (let us call it I_n) is increasing, and so is $e \overset{K}{\leftrightarrow} R(n)$, by FKG inequality we have

$$\begin{aligned} \mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G_n' \mid e \overset{K}{\leftrightarrow} R(n)] \quad &- \mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G_n', I_n \mid e \overset{K}{\leftrightarrow} R(n)] \\ &\leq \mathbb{P}[I_n^c \mid e \overset{K}{\leftrightarrow} R(n)] \leq \mathbb{P}[I_n^c] < \tfrac{\epsilon}{8}. \end{aligned} \tag{2.20}$$

Thus with (2.14) and (2.20) at our disposal, the problem is reduced to prove the following

$$\mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G_n', I_n \mid e \overset{K}{\leftrightarrow} R(n)] < \frac{\epsilon}{8}. \tag{2.21}$$

e) Decoupling: For Type-A 'good' annuli D_i, let us take the innermost such open path O_i^{in} in $D_i^{in} \cap K$ and a collection of outermost open paths in $D_i^{in} \cap K$, including O_i^{ou} and possibly other open paths joining γ_b or γ_t with itself and call this ordered collection as $\mathfrak{C}_i$. For Type-B 'good' annuli D_i let us take the outermost of such open paths in $\mathrm{An}(v_i, L_i/2, L_i) \cap K$ including O_i^{in}, O_i^{ou} and other open paths joining γ_b or γ_t with itself and call this $\mathfrak{C}_i$. Notice that for a deterministic collection of paths C_i, the event $\mathfrak{C}_i = C_i$ does not depend on what happens inside F_i (which is determined by C_i). Let us denote the random set $\mathfrak{S} \subset \{1, 2, \ldots k\}$ such that D_i is 'good' iff $i \in \mathfrak{S}$. (We know that under

I_n, $|\mathfrak{S}| \geq \frac{c_0 k}{2}$.)

$$\begin{aligned}
&\mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G'_n, I_n \,| e \overset{K}{\leftrightarrow} R(n)] \\
&= \sum_S \sum_{i \in S} \sum_{C_i} \mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G'_n, I_n \mid e \overset{K}{\leftrightarrow} R(n),\, \mathfrak{S} = S,\, \mathfrak{C}_i = C_i \;\forall i \in S] \\
&\qquad\qquad\qquad\qquad \cdot \mathbb{P}[\mathfrak{S} = S,\; \mathfrak{C}_i = C_i \;\forall i \in S \mid e \overset{K}{\leftrightarrow} R(n)].
\end{aligned}$$

Notice that we need to prove now, uniformly over any choice of S and collection C_i for $i \in S$,

$$\mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G'_n, I_n \mid e \overset{K}{\leftrightarrow} R(n), \mathfrak{S} = S,\; \mathfrak{C}_i = C_i \;\forall i \in S] < \frac{\epsilon}{8}. \tag{2.22}$$

If there is an edge from $Y(n)$ inside T'_i (in either case), it is open and must have two disjoint open arms to O_i^{in} and O_i^{ou} inside F_i, and two disjoint dual closed paths to γ_t and γ_b inside F_i^*. Let us call such edges as Y'_i and observe that this event $\{e \in Y'_i\}$ depends only on edges inside F_i. Let us denote the event that O_i^{in} is connected by an open path in F_i to O_i^{ou} as $O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}$. We break down the long connection $e \overset{K}{\leftrightarrow} R(n)$ as

$$\begin{aligned}
&\mathbb{P}[|Y(n)| < \theta n^2 \alpha_4(n), G'_n, I_n \mid e \overset{K}{\leftrightarrow} R(n),\, \mathfrak{S} = S,\, \mathfrak{C}_i = C_i \;\forall i \in S] \\
&\leq \mathbb{P}[|Y'(i)| < \theta n^2 \alpha_4(n) \;\forall i \in S \mid e \overset{K}{\leftrightarrow} R(n),\, \mathfrak{S} = S,\, \mathfrak{C}_i = C_i \;\forall i \in S] \\
&= \mathbb{P}[|Y'(i)| < \theta n^2 \alpha_4(n) \;\forall i \in S \mid O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou} \;\forall i \in S] \\
&= \prod_{i \in S} \mathbb{P}[|Y'(i)| < \theta n^2 \alpha_4(n) \mid O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}].
\end{aligned} \tag{2.23}$$

In the second step we reduce the conditioned event to only the relevant part for our event $\{|Y'(i)| < \theta n^2 \alpha_4(n) \;\forall i \in S\}$ and in the third step we break the event into several conditioned events on different annuli. We need to prove that for all i and for any permissible shape F_i,

$$\mathbb{P}[|Y'(i)| \geq \theta n^2 \alpha_4(n) \,|O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] > c, \tag{2.24}$$

for some universal constant c, possibly dependent on β_0. Then we will have

$$\prod_{i \in S} \mathbb{P}[|Y'(i)| < \theta n^2 \alpha_4(n) \,|O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] \leq (1-c)^{|S|} \leq (1-c)^{\frac{c_0 k}{2}}. \tag{2.25}$$

By choosing large enough k in the final step, we can make the RHS less than $\frac{\epsilon}{8}$. By (2.22) and (2.23), it suffices to prove only (2.24) now, which says that uniformly over any 'permissible' shape of F_i, the number of pivotals inside the box T'_i is large with probability uniformly bounded from below.

f) Many Pivotals for 'good' Boxes: We will prove this in two steps. First we will prove that

$$\mathbb{P}[|Y'(i)| \geq \mathbb{E}[Y'(i)]/2 \,|O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] > c, \tag{2.26}$$

and then we will prove that we can choose a θ small enough such that $\min_{1\le i\le k} \mathbb{E}[|Y'(i)|] \ge 2\theta n^2\alpha_4(n)$. To prove (2.26), we will use Paley-Zygmund inequality which states

$$\mathbb{P}[|Y'(i)| \ge 1/2\mathbb{E}[Y'(i)] \; |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] \ge \frac{(\mathbb{E}[|Y'(i)| \; |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}])^2}{4\mathbb{E}[|Y'(i)|^2 \; |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}]}.$$

As the second moment method requires, we will need to find suitable lower bound for $\mathbb{E}[|Y'(i)| \; |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}]$ and upper bound for $\mathbb{E}[|Y'(i)|^2 \; |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}]$. We will do this separately depending on the type of the 'good' annuli.

Type A : For any edge $e \in E(T_i')$, it needs to have four alternating arms inside F_i, two disjoint open arms to O_i^{in} and O_i^{ou}, and two disjoint dual closed ones to segments of γ_t and γ_b that make up the boundary of F_i, avoiding other open segments from C_i. Notice that from any edge in T_i', the distance to either of γ_t, γ_b, O_i^{in} or O_i^{ou} lies in $\left[\lfloor\frac{\beta_0\alpha n}{2^{3i-1}}\rfloor, \lceil\frac{\alpha n}{2^{3i-3}}\rceil\right]$. Also the lengths of the segments O_i^{in} and O_i^{ou} are bounded from below by $\lfloor\frac{\beta_0\alpha n}{2^{3i-1}}\rfloor$ by separation of γ_t and γ_b. Similarly the size of 'permissible' segments of γ_t and γ_b inside D_i is at least the width of D_i^m, i.e. $\lfloor\frac{\alpha n}{2^{3i-2}}\rfloor$. By delicate use of arm separation techniques akin to [DS11, Proof of Lemma 2], these restrictions enforce the existence of a constant c_u (possibly depending on β_0) uniformly over the shape of F_i such that for every $e \in E(T_i')$,

$$\mathbb{P}[e \in Y'(i)] \ge c_u\alpha_4(\lceil\frac{\alpha n}{2^{3i-3}}\rceil).$$

Summing over all edges of $E(T_i')$, we get $\mathbb{E}[|Y'(i)|] \ge C_u(\lfloor\frac{\beta_0\alpha n}{2^{3i-1}}\rfloor)^2\alpha_4(\lceil\frac{\alpha n}{2^{3i-3}}\rceil)$.

Let us define $m_i = \lfloor\frac{\beta_0\alpha n}{2^{3i-1}}\rfloor$. If two edges e_x, e_y are both from $Y'(i)$, there must be four arms in $v(e_x) + B(|e_x - e_y|/2)$, $v(e_y) + B(|e_x - e_y|/2)$ and $\mathrm{An}(v(e_x) + v(e_y)/2, |e_x - e_y|, m_i)$. Using this we get

$$\begin{aligned}
\mathbb{E}[|Y'(i)|^2] &\le \sum_{e_x,e_y\in E(T_i')} \mathbb{P}[e_x, e_y \in Y'(i)] \\
&\le \sum_{e_x,e_y\in E(T_i')} (\alpha_4(|e_x - e_y|/2))^2\alpha_4(|e_x - e_y|, m_i) \\
&\le \sum_{e_x\in E(T_i')} \sum_{l=1}^{m_i} \sum_{e_y:|e_y-e_x|=l} (\alpha_4(l/2))^2\alpha_4(l, m_i) \\
&\le Cm_i^3\alpha_4(m_i)^2 \sum_{l=1}^{m_i} 1/\alpha_4(l, m_i) \\
&\le Cm_i^3\alpha_4(m_i)^2 \sum_{l=1}^{m_i} (m_i/l)^\delta \le C'm_i^4\alpha_4(m_i)^2. \qquad (2.27)
\end{aligned}$$

In the fourth step we have used the bound $|\{e_y : |e_y - e_x| = l\}| \le Cm_i$ and quasi-multiplicativity Lemma 2.6. Finally in the fifth step we have used Lemma 2.7(v).

Thus using Paley-Zygmund inequality we get

$$\begin{aligned}\mathbb{P}[|Y'(i)| \geq \mathbb{E}[Y'(i)]/2 \ |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] \ &\geq \ \frac{C''\alpha_4(\lceil \frac{\alpha n}{2^{3i-3}} \rceil)^2}{\mathbb{P}[O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}]\alpha_4(\lfloor \frac{\beta_0 \alpha n}{2^{3i-1}} \rfloor)^2} \\ &\geq \ O(1)\alpha_4(\lfloor \frac{\beta_0 \alpha n}{2^{3i-1}} \rfloor, \lceil \frac{\alpha n}{2^{3i-3}} \rceil)^2 \geq c_{\beta_0}.\end{aligned} \tag{2.28}$$

The last step comes from the fact that for any fraction κ, we can find an universal constant c_κ such that $\alpha_4(\kappa n, n) > c_\kappa$. (This is immediate by constructing four disjoint tunnels in the annuli and having one arm through each of them.)

Type B : The argument is similar in a certain sense. Distance of T_i' from either of γ_t, γ_b, O_i^{in} or O_i^{ou} lies between $L_i/4$ and $3L_i$. Thus here by repeating the same argument, we have $\mathbb{E}[|Y'(i)|] \geq C_u' L_i^2 \alpha_4(3L_i)$ and $\mathbb{E}[|Y'(i)|^2] \leq C' L_i^4 \alpha_4(L_i/4)^2$. Similarly by using Paley-Zygmund inequality we get

$$\mathbb{P}[|Y'(i)| \geq \mathbb{E}[Y'(i)]/2 \ |O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}] \geq \frac{O(1)}{\mathbb{P}[O_i^{in} \overset{F_i}{\leftrightarrow} O_i^{ou}]} \geq c_o.$$

We now choose $c = c_{\beta_0} \wedge c_o$ in (2.26). Now we are left to choose a θ small enough such that $\min_{i \leq k} \mathbb{E}[|Y'(i)|] \geq 2\theta n^2 \alpha_4(n)$. Notice that

$$\min_{i \leq k} \mathbb{E}[|Y'(i)|] \geq C \min_{i \leq k} [(L_i)^2 \alpha_4(3L_i)] \wedge [(m_i)^2 \alpha_4(\lceil \frac{\alpha n}{2^{3i-3}} \rceil)] \geq C(\frac{\lfloor \beta_k \alpha n}{2^{3k-2}} \rfloor)^2 \alpha_4(n). \tag{2.29}$$

We recall how we choose variables step by step. Given ϵ we first choose α to satisfy (2.11). Then we set the relation $\beta_j = \frac{\beta_0}{2^{\frac{3j}{\delta}}}$ for $1 \leq j \leq k$ and later choose small β_0 satisfying RHS of (2.16) $< \frac{\epsilon}{4c_{\mathfrak{e}}}$. Then we choose k large enough to satisfy both (2.19) and (2.25) (recall that c in (2.25) will depend on β_0). Finally we choose $\theta < C/2(\frac{\beta_k \alpha}{2^{2k-1}})^2$ and by (2.29) this completes our proof of (2.9).

For proving (2.10), all pivotal edges will be closed and we have to condition on the first closed pivotal edge from top to bottom and then the 'topmost' open crossings from this edge to $R(n)$ and $L(n)$. We will change the circuits from open to dual closed and FKG will work in the analogous equation to (2.20) because both conditioned event and presence of dual closed circuits are decreasing. Since the rest of the proof is identical, we do not repeat it.

Remark 2.13. In fact, to prove Lemma 2.7(ii), it is shown in [KSZ98, Lemma] that the probability of having some vertex in $B(n)$ being σ_5^A connected to the boundary $\partial B(n)$ is uniformly positive, and it is possible to prove a stronger result than $\mathbb{P}[P(n) \neq \phi, LR(n)] > c_{\mathfrak{e}}$- in the form that $\mathbb{P}[|P(n)| \geq c \log n] \geq c$ for some constant $c > 0$.

2.4 Proof of Theorems

With the current set of tools present, Theorem 2.1(a) and 2.2(a) can be proved by replicating the proof of [J03, Theorem 1-2] mutatis mutandis. (As a side-note, even to prove the existence of 2-arm IIC, Kesten's strategy in [K86a] is good enough, and we do not require the delicate treatment meted out in [DS11].) The only concern is that, for the analogous statement of [J03, (2.20)], it is required that $\mathbb{E}[|BB(n)|]/n \to \infty$ as $n \to \infty$. This holds true, since Lemma 2.7(i),(iv) and Reimer's inequality imply

$$c/n \leq \alpha_3(n) \leq \alpha_B(n)\alpha_*(n) \leq C\alpha_B(n)n^{-\delta},$$

thus making $\mathbb{E}[|BB(n)|]/n \asymp n\alpha_B(n) \geq C'n^{\delta}$, which serves our purpose.

But proofs for lowest crossing and pivotals indeed require new tools. We will prove Theorem 2.1 in the next Section 2.4.1 followed by the proof of Theorem 2.2 in Section 2.4.2.2.

2.4.1 Local variant

Notice that unlike the conditioning event in Theorem 2.1(a), the events in 2.1(b),(c) has the existence of a closed path in dual graph. Thus most of the events arising are not increasing and it creates inconvenience since FKG inequality will not suffice alone. We will circumnavigate this problem with Reimer's inequality and a generalized version of quasi-multiplicativity. We will first prove Theorem 2.1(c) in subsection 2.4.1.1 and subsequently highlight the key alterations required for Theorem 2.1(b) in subsection 2.4.1.2, which is comparatively easier.

2.4.1.1 Local limit for Pivotals

For any vertex in crossing collection, it is likely that there will be an open circuit in an annulus around it if it is made thick enough. This was used crucially by Járai as a key component of his proof. Naturally for pivotals this cannot be true, since there are two dual closed paths around every pivotal that would prohibit existence of open circuits around it. We will settle for open circuits with defects instead. Let a circuit with k defects mean a circuit open at all but k many edges. We consider integers $1 << N << M << h(n)/8$, and the choices of variables will be clearer later in the proof. We define the event $F_k = F_k(M, N) = \{$ there is a circuit with k defects in $\mathrm{An}(N, M)\}$.

On the event $\{e \in P(n)\}$, there cannot exist a circuit with k defects when $k < 2$. So $\mathbb{1}\{e \in P(n), \tau_e F_2^c\} = 1$ implies every circuit with k defects around e must have $k \geq 3$. By Menger's Theorem [M27], this implies that there must be at least 3 edge disjoint closed paths in $\mathrm{An}(N, M)$ in the dual graph. Let $\sigma_5 = \{open, closed, open, closed, closed\}$. Let $Z(e, M, n)$ indicate the event that there are two open paths from $v(e) + B(M)$ to left and right boundary of $B(n)$ and there exists two dual closed paths from $v(e) + B(M)$ to top and bottom boundary of $B(n)$, all the arms being inside $B(n) \setminus v(e) + B(M)$.

$$\begin{aligned}
& \mathbb{E}[\mathbb{1}\{[e \in P(n), \tau_e F_2^c]\}] \\
\leq\ & \mathbb{P}[\tau_e\{e_0 \leftrightarrow_{\sigma_4^A} \partial B(N)\}]\mathbb{P}[\tau_e\{B(N) \leftrightarrow_{\sigma_5} \partial B(M)\}]\mathbb{P}[Z(e, M, n)] \\
\leq\ & C\mathbb{P}(F_0^c)\mathbb{P}[\tau_e\{e_0 \leftrightarrow_{\sigma_4^A} \partial B(N)\}\mathbb{P}[\tau_e\{B(N) \leftrightarrow_{\sigma_4^A} \partial B(M)\}\mathbb{P}[Z(e, M, n)] \\
\leq\ & C'\mathbb{P}(F_0^c)\mathbb{P}[\tau_e\{e_0 \leftrightarrow_{\sigma_4^A} \partial B(M)\}\mathbb{P}[Z(e, M, n)] \\
\leq\ & C''\mathbb{P}(F_0^c)\mathbb{P}[e \in P(n)] \leq O((N/M)^{\delta})\mathbb{P}[e \in P(n)], \qquad (2.30)
\end{aligned}$$

where we used Reimer's inequality, a stronger form of quasi-multiplicativity, and Lemma 2.7(iv) to glue connections. We choose $M/N > N_1(\epsilon)$ large enough so that we can make RHS of (2.30) $< \epsilon\mathbb{P}[e \in P(n)]$.

Let us define, when e is pivotal, $F_e(D_{f_1,f_2}) = \{\text{In } \mathrm{An}(v(e), N, M),\ D_{f_1,f_2} \text{ is the outermost circuit with defects at } f_1 \text{ and } f_2\}$ and so $\tau_e F_2$ becomes a disjoint union over $F_e(D_{f_1,f_2})$ over all permissible circuits D_{f_1,f_2} with two defects. Hence we can write

$$\begin{aligned}
\mathbb{P}(\tau_e E | e \in P(n)) & \overset{(2.30)}{\leq} \epsilon + \mathbb{P}(\tau_e E \cap \tau_e F_2 | e \in P(n)) \\
& \leq \epsilon + \sum_{D_{f_1,f_2}} \mathbb{P}(\tau_e E \cap F_e(D_{f_1,f_2})\ | e \in P(n)) \\
& \leq \epsilon + \mathbb{P}(\tau_e E | e \in P(n)). \qquad (2.31)
\end{aligned}$$

We will need the following result from [DS11] involving the measure ν_4. Let D_{f_1,f_2} indicate a circuit D with all edges but f_1 and f_2 open. Let us indicate, by $e \leftrightarrow_4 D_{f_1,f_2}$, that an edge e 'inside' D_{f_1,f_2} is connected to the two open arcs of D_{f_1,f_2} and there are closed paths in a dual lattice from endpoints of e^* to ${f_1}^*$ and ${f_2}^*$.

Lemma 2.14. *[DS11, (7.15)]* $\lim_{N\to\infty} \mathbb{P}(E|e_0 \leftrightarrow_4 D^N_{f_1,f_2}) = \nu_4(E)\ \forall D^N$ *surrounding* $B(N)$, *for any two distinct edges* $f_1, f_2 \in D^N$ *and for any cylinder event* E *independent of* $\omega(e_0)$.

Let us abuse the notation to write $D_{f_1,f_2} = D^N_{f_1,f_2}$. By Lemma 2.14 we know if $N > N_3(\epsilon, E)$ then

$$\frac{1}{1+\epsilon}\mathbb{P}(E, e_0 \leftrightarrow_4 D_{f_1,f_2}) \leq \nu_4(E)\mathbb{P}(e_0 \leftrightarrow_4 D_{f_1,f_2}) \leq (1+\epsilon)\mathbb{P}(E, e_0 \leftrightarrow_4 D_{f_1,f_2}). \qquad (2.32)$$

We will choose N large enough so that $B(N)$ contains all edges on which the cylinder event E depends. On the event $\{e \in P(n)\}$, all circuits around e must have at least two defects and so it makes sense to define the 'outermost' *circuit with two defects* among all others in the annulus $\mathrm{An}(v(e), N, M)$. On the event $F_e(D_{f_1,f_2})$, $\{e \in P(n)\}$ if and only if $\{e \leftrightarrow_4 D_{f_1,f_2}\}$ and each arc of D_{f_1,f_2} is connected to exactly one of $L(n)$ or $R(n)$, and each of ${f_1}^*$ and ${f_2}^*$ is connected by closed path in dual lattice to $T^*(n)$ and $D^*(n)$. We denote this later event by $D_{f_1,f_2} \leftrightarrow_4 B(n)$. We write $\mathbb{1}\{\tau_e E, e \in P(n), F_e(D_{f_1,f_2})\}$ as the product of $\mathbb{1}\{\tau_e E, e \leftrightarrow_4 D_{f_1,f_2}\}$ and $\mathbb{1}\{F_e(D_{f_1,f_2}), D_{f_1,f_2} \leftrightarrow_4 B(n)\}$, so that the first part only depends on the interior of D_{f_1,f_2} except e whereas the second part depends on the edges on or in the exterior of D_{f_1,f_2}, making them independent. We use this fact to decouple them and observe

$$\begin{aligned}
& \mathbb{P}(\tau_e E | e \in P(n)) \\
\overset{(2.31)}{\leq} \quad & \epsilon + \frac{1}{\mathbb{P}(e \in P(n))} \sum_{D_{f_1,f_2}} \mathbb{P}(\tau_e E, F_e(D_{f_1,f_2}), e \in P(n)) \\
\leq \quad & \epsilon + \frac{1}{\mathbb{P}(e \in P(n))} \sum_{D_{f_1,f_2}} \mathbb{P}(\tau_e E, e \leftrightarrow_4 D_{f_1,f_2}) \mathbb{P}(F_e(D_{f_1,f_2}), D_{f_1,f_2} \leftrightarrow_4 B(n)) \\
\overset{(2.32)}{\leq} \quad & \epsilon + \frac{(1+\epsilon)\nu_4(E)}{\mathbb{P}(e \in P(n))} \sum_{D_{f_1,f_2}} \mathbb{P}(\tau_e \Omega, e \leftrightarrow_4 D_{f_1,f_2}) \mathbb{P}(F_e(D_{f_1,f_2}), D_{f_1,f_2} \leftrightarrow_4 B(n)) \\
\leq \quad & \epsilon + \frac{(1+\epsilon)\nu_4(E)}{\mathbb{P}(e \in P(n))} \sum_{D_{f_1,f_2}} \mathbb{P}(\tau_e \Omega, F_e(D_{f_1,f_2}), e \in P(n)) \\
\leq \quad & \epsilon + (1+\epsilon)\nu_4(E). \qquad (2.33)
\end{aligned}$$

Now given $\epsilon > 0$, we have defined the variables in the following manner. First we have chosen $N > max(N_3(\epsilon, E), N_3(\epsilon, \Omega))$ (recall (2.32)), then we have made $M/N > N_1(\epsilon)$ such that RHS of (2.30) is $< \epsilon \mathbb{P}[e \in P(n)]$. Then we have taken n large enough such that $h(n) > 8M$. Similarly we can prove the lower bound

$$\mathbb{P}(\tau_e E | e \in P(n)) \geq -\epsilon + \frac{1}{(1+\epsilon)} \nu_4(E). \qquad (2.34)$$

Since these two inequalities hold true for any arbitrary $\epsilon > 0$, this completes the proof. □

2.4.1.2 Local limit for Lowest Crossing

Due to existence of 3 arms instead of 4, we will look at circuits with one defect instead. Firstly we prove, analogous to (2.30), $\mathbb{P}[\tau_e F_1^c \; | e \in LC(n)] < \epsilon$. Then using the result analogous to Lemma 2.14, we state, for any cylinder event $E \subset \{\omega(e_0) = 1\}$,

$$\frac{1}{1+\epsilon} \mathbb{P}(E, e_0 \leftrightarrow_3 D_f) \leq \nu_3(E) \mathbb{P}(e_0 \leftrightarrow_3 D_f) \leq (1+\epsilon) \mathbb{P}(E, e_0 \leftrightarrow_3 D_f), \qquad (2.35)$$

for $N > N_2(\epsilon, E)$ and any circuit $D = D^N$ with one defect f surrounding $B(N)$. (Note that This is analogous to (2.32).) We make sense of $\{e_0 \leftrightarrow_3 D_f\}$ by having two disjoint open paths from both ends of e_0 (which itself is open) to some vertex in D and a dual closed path from e_0^* to f^*. Similarly $D_f \leftrightarrow_3 B(n)$ is made sense of similarly by two disjoint open connections from D_f to $R(n)$ and $L(n)$, and one closed connection in the dual graph from f^* to $D^*(n)$. The key decoupling strategy (2.33) remains exactly the same. $\square$

2.4.2 Uniform variant

As before, we first deal with pivotals in Section 2.4.2.1 and then come to the lowest crossing in Section 2.4.2.2.

2.4.2.1 Uniform limit for Pivotals

By Lemma 2.10(c), for fixed $\epsilon > 0$ and conditioned on A_n, we can find $x > 0$ such that the event $H_n := \{|P(n)| \geq x\mathbb{E}(|P(n)|)\}$ has probability at least $1 - C_1\epsilon$. Recall that $I_{n,4}$ is chosen uniformly from $P(n)$. Let us also define $G_n = G_n(\alpha) = \{I_{n,4} \in E((1-\alpha)n)\}$, whose probability is proved to be close to 1 by Lemma 2.12, for small enough α. We prove that all these events G_n, H_n and $\tau_{I_{n,4}} F_2$ together are very likely when conditioned by $A_n := \{P(n) \neq \phi\}$, i.e.

Lemma 2.15. *Given $\epsilon > 0$, the quotient M/N can be chosen large enough and α small enough such that*

$$\mathbb{P}(\tau_{I_{n,4}} E | A_n) \leq 3\epsilon + \mathbb{P}(\tau_{I_{n,4}} E \cap \tau_{I_{n,4}} F_2 \cap H_n \cap G_n(\alpha) | A_n) \leq 3\epsilon + \mathbb{P}(\tau_{I_{n,4}} E | A_n). \quad (2.36)$$

Proof. To integrate the boundary condition, for α small enough, we can make

$$\mathbb{P}(G_n^c(\alpha)|A_n) \leq \frac{\mathbb{P}[G_n^c(\alpha)]}{\mathbb{P}[A_n]} \leq \frac{\mathbb{P}[G_n^c(\alpha)]}{C_1} \overset{Lemma\ 2.12}{\leq} \epsilon. \quad (2.37)$$

We will now bound $\mathbb{P}(\tau_{I_{n,4}} F_2^c \cap H_n \cap G_n | A_n)$.

$$\begin{aligned}
\mathbb{P}(\tau_{I_{n,4}} F_2^c \cap H_n \cap G_n | A_n) &\leq \sum_{e \in E(n-r(n))} \mathbb{E}\Big(\frac{\mathbb{1}\{e \in P(n), \tau_e F_2^c, H_n\}}{|P(n)|} \Big| A_n\Big) \\
&\leq \sum_{e \in E(n-r(n))} \mathbb{E}\Big(\frac{\mathbb{1}\{e \in P(n), \tau_e F_2^c, H_n\}}{x\mathbb{E}_{p_c}(|P(n)|)} \Big| A_n\Big) \\
&\leq \frac{1}{x\mathbb{E}_{p_c}(|P(n)|)} \sum_{e \in E(n-r(n))} \mathbb{E}_{p_c}(\mathbb{1}\{e \in P(n), \tau_e F_2^c\} | A_n).
\end{aligned} \quad (2.38)$$

From (2.38) and (2.30), we have

$$\mathbb{P}(\tau_{I_{n,4}} F_2^c \cap H_n \cap G_n | A_n) \leq \frac{\mathbb{P}(F_0^c)}{x\mathbb{P}(A_n)} \overset{Lemma\ 2.7(iv)}{\leq} O((N/M)^\delta) < \epsilon. \quad (2.39)$$

We choose $M/N > N_2(\epsilon)$ large enough so that we can make RHS of (2.39) $< \epsilon$. Taking all components together, we get

$$\begin{aligned}
&\mathbb{P}(\tau_{I_{n,4}}E|A_n)\\
\leq\ &\mathbb{P}(H_n^c|A_n)+\mathbb{P}(G_n^c(\alpha)|A_n)+\mathbb{P}(\tau_{I_{n,4}}E\cap H_n\cap G_n(\alpha)|A_n)\\
\overset{(2.37)}{\leq}\ &2\epsilon+\mathbb{P}(\tau_{I_{n,4}}F_2^c\cap H_n\cap G_n(\alpha)|A_n)+\mathbb{P}(\tau_{I_{n,4}}E\cap\tau_{I_{n,4}}F_2\cap H_n\cap G_n(\alpha)|A_n)\\
\leq\ &3\epsilon+\mathbb{P}(\tau_{I_{n,4}}E\cap\tau_{I_{n,4}}F_2\cap H_n\cap G_n(\alpha)|A_n)\leq 3\epsilon+\mathbb{P}(\tau_{I_{n,4}}E|A_n),
\end{aligned}\tag{2.40}$$

and our proof of Lemma 2.15 is complete. □

Let us decompose $\tau_e F_2$ over disjoint unions of $F_e(D_{f_1,f_2})$, as we did in (2.31).

$$\begin{aligned}
\mathbb{P}(\tau_{I_{n,4}}E|A_n)&\overset{(2.40)}{\leq} 3\epsilon+\mathbb{P}(\tau_{I_{n,4}}E\cap\tau_{I_{n,4}}F_2\cap H_n\cap G_n|A_n)\\
&\leq 3\epsilon+\sum_{e\in E(n-r(n))}\sum_{D_{f_1,f_2}}\mathbb{E}(\frac{\mathbb{1}\{\tau_e E, e\in P(n), F_e(D_{f_1,f_2}), H_n\}}{|P(n)|}|A_n)\\
&\leq 3\epsilon+\mathbb{P}(\tau_{I_{n,4}}E|A_n).
\end{aligned}\tag{2.41}$$

One difference with the proof of Theorem 2.1(c) is that we need to deal with the denominator $|P(n)|$ which depends on both of the sets of edges as well as on e. We will define some set whose cardinality is close enough to $|P(n)|$, but only depends on $ext(D_{f_1,f_2})$ and $\omega(e)$. We define $P^1(D_{f_1,f_2},n)$ as the set of edges f from $ext(D_{f_1,f_2})$ which satisfy:

a) There are dual closed paths C_1 and C_2 from f^* to edges of $T^*(n)$ and $D^*(n)$.

b) There is an open path O_1 from f to $L(n)$ or $R(n)$.

c) There is an open path O_2, disjoint from O_1 from f to one arc of D_{f_1,f_2}.

d) The paths C_1, C_2, O_1 and O_2 are disjoint and lie completely 'outside' D_{f_1,f_2}.

This set is defined such that under $\{\omega(e)=1, e\in P(n), F_e(D_{f_1,f_2})\}$, $ext(D_{f_1,f_2})\cap P(n)=P^1(D_{f_1,f_2},n)$. With the same intention for $\{\omega(e)=0\}$ instead, we define $P^0(D_{f_1,f_2},n)$ as the set of edges f outside $ext(D_{f_1,f_2})$ which satisfy:

a) There are open paths O_1 and O_2 from f to $L(n)$ and $R(n)$.

b) There is a dual closed path C_1 from f^* to edge of $T^*(n)$ or $D^*(n)$.

c) There is a dual closed path C_2, disjoint from C_1, from f^* to either $f_1{}^*$ or $f_2{}^*$.

d) The paths C_1, C_2, O_1 and O_2 are disjoint and lie completely outside D_{f_1,f_2}.

Now we define random variables:

$$X_{D,e,E}=\mathbb{1}\{\tau_e E, e\leftrightarrow_4 D_{f_1,f_2}\},$$
$$Y^i_{D_{f_1,f_2},n}=\frac{\mathbb{1}\{F_e(D_{f_1,f_2}), D_{f_1,f_2}\leftrightarrow_4 B(n), \omega(e)=i, P^i(D_{f_1,f_2},n)\neq\emptyset\}}{|P^i(D_{f_1,f_2},n)|}\qquad \text{for } i\in\{0,1\}.$$

Notice that, by virtue of the definition, $X_{D,e,E}$ is independent of either of the events $Y^i_{D_{f_1,f_2},n}$ (recall E is a cylinder event independent of $\omega(e)$). For fixed ϵ, M we can define $N_1(M,\epsilon)$ such that

$$|E(B(M))| = O(1)M^2 < (\epsilon/2)\cdot Cn^\delta \leq (\epsilon/2)\mathbb{E}[|P(n)|], \tag{2.42}$$

$\forall n > N_1(M,\epsilon)$ from Lemma 2.9(c).

Now given $\epsilon > 0$, we define the variables in the following manner. First we choose $N > max(N_3(\epsilon, E), N_3(\epsilon,\Omega))$ (recall (2.32)), then we take M large enough such that (2.39) holds by making $M/N > N_2(\epsilon)$. We choose α small enough such that it satisfies (2.37). Then we fix $n > max(\lceil M/\alpha\rceil, N_1(M,\epsilon))$.

$$\begin{aligned}
& \mathbb{P}(\tau_{I_{n,4}}E|A_n) \\
\overset{(2.41)}{\leq} \quad & 3\epsilon + \frac{1}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \mathbb{E}\left(\frac{\mathbb{1}\{\tau_e E, e\in P(n), F_e(D_{f_1,f_2}), H_n\}}{|P(n)|}\right) \\
\leq \quad & 3\epsilon + \frac{1}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \sum_{i=0,1} \mathbb{E}\left(X_{D,e,E} Y^i_{D_{f_1,f_2},n}\right) \\
\overset{(2.32)}{\leq} \quad & 3\epsilon + \frac{(1+\epsilon)\nu_4(E)}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \sum_{i=0,1} \mathbb{E}\left(X_{D,e,\Omega} Y^i_{D_{f_1,f_2},n}\right) \\
\leq \quad & 4\epsilon + \frac{(1+\epsilon)\nu_4(E)}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \sum_{i=0,1} \mathbb{E}\left(X_{D,e,\Omega} Y^i_{D_{f_1,f_2},n}, \mathbb{1}\{H_n\}\right) \\
\overset{(2.42)}{\leq} \quad & 4\epsilon + \frac{(1+\epsilon)\nu_4(E)}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \sum_{i=0,1} \mathbb{E}\left(\frac{\mathbb{1}\{e\in P(n), F_e(D_{f_1,f_2}), \omega(e)=i\}}{(1-\epsilon/2)|P(n)|}\right) \\
\leq \quad & 4\epsilon + \frac{(1+\epsilon)^2\nu_4(E)}{\mathbb{P}(A_n)} \sum_{e\in E(n-r(n))} \sum_{D_{f_1,f_2}} \mathbb{E}\left(\frac{\mathbb{1}\{e\in P(n), F_e(D_{f_1,f_2})\}}{|P(n)|}\right) \\
\leq \quad & 4\epsilon + (1+\epsilon)^2\nu_4(E).
\end{aligned} \tag{2.43}$$

Similarly we can prove the lower bound

$$\mathbb{P}(\tau_{I_{n,4}}E|A_n) \geq -\epsilon + \frac{1}{(1+\epsilon)^2}\nu_4(E). \tag{2.44}$$

We have proved thus, for any choice of $\epsilon > 0$,

$$-\epsilon + \frac{1}{(1+\epsilon)^2}\nu_4(E) \leq \liminf_{n\to\infty} \mathbb{P}(\tau_{I_{n,4}}E|A_n) \leq \limsup_{n\to\infty} \mathbb{P}(\tau_{I_{n,4}}E|A_n) \leq 4\epsilon + (1+\epsilon)^2\nu_4(E).$$

This completes the proof. $\square$

2.4.2.2 Uniform limit for Lowest Crossing

We again highlight below the key changes, in addition to what we did in Section 2.4.1.2.

- Since Lemma 2.10(b) is slightly weaker than Lemma 2.10(c), given $\epsilon > 0$, we would first find x and then integer N_0 such that the event $H_n = \{|LC(n)| \geq x\mathbb{E}[|LC(n)|]\}$ has probability $> 1 - \epsilon/2$ for all $n \geq N_0$. We will choose $N \geq max(N_0(\epsilon), N_1(E), N_3(\epsilon, E), N_3(\epsilon, \Omega))$ where N_1 is the smallest integer such that the cylinder event E depends only on the edges inside $B(N_1)$.

- We will show that $\mathbb{P}[G_n^c \,|LC(n) \neq \phi]$ can be made $< \epsilon$, where $G_n := \{I_{n,3} \notin E_f\}$, E_f indicating edges with at least one vertex in $\mathrm{An}(n - r(n), n)$. This boudary $r(n)$ will be chosen suitably later. We do not have any lemma akin to Lemma 2.12. But, in fact, we will not need such sophisticated bound, and Lemma 2.7 will suffice to prove something similar.

$$\begin{aligned}
\mathbb{P}[G_n^c \,|LC(n) \neq \phi] &\leq \epsilon/2 + \mathbb{P}[H_n \cap G_n^c \,|LC(n) \neq \phi] \\
&\leq \epsilon/2 + \frac{1}{C_1} \sum_{e \in E_f} \mathbb{E}[\frac{\mathbb{1}[e \in LC(n)]}{x\mathbb{E}[|LC(n)|]}] \\
&\leq \epsilon/2 + \frac{1}{C_1} \sum_{e \in E_f} \mathbb{E}[\frac{\mathbb{1}[\tau_e\{0 \leftrightarrow \partial B(n)\}]}{x\mathbb{E}[|LC(n)|]}] \\
&= \epsilon/2 + \frac{1}{C_1} \sum_{e \in E_f} \frac{\alpha_1(n)}{x\mathbb{E}[|LC(n)|]} \leq \epsilon/2 + \frac{Cr(n)}{xn^\delta}.
\end{aligned}$$

  In the last step we use Lemma 2.7(i), (iv) and Lemma 2.9(b). We thus impose $r(n) = n^{\delta/2}$.

- We would then prove the following result analogous to Lemma 2.15 which states, given any ϵ, we can choose n and M/N to be large enough such that

$$\begin{aligned}
\mathbb{P}(\tau_{I_{n,4}} E \,|LC(n) \neq \phi) &\leq 3\epsilon + \mathbb{P}(\tau_{I_{n,4}} E \cap \tau_{I_{n,4}} F_1 \cap H_n \cap G_n \,|LC(n) \neq \phi) \\
&\leq 3\epsilon + \mathbb{P}(\tau_{I_{n,4}} E \,|LC(n) \neq \phi).
\end{aligned}$$

- Instead of $P^1(D_{f_1,f_2}, n)$ and $P^0(D_{f_1,f_2}, n)$, we define the set $LC(D_f, n)$ for the circuit D with sole defect on f as the set of edges e' from $ext(D_f)$ which satisfies:
  a) There is an open path O_1 from e' to $L(n)$ or $R(n)$.

  b) There is an open path O_2 from e' to some edge of D other than f.

  c) There is a dual closed path C_1 from e'^* to some edge of $D(n)^*$.

  d) O_1, O_2, and C_1 are disjoint.

- We define, naturally $F_e(D_f) = \{D_f$ is the outermost circuit in $\mathrm{An}(e, N, M)\}$ and $D_f \leftrightarrow_3 B(n)$ as the event that f^* is connected by a dual closed path to $D(n)^*$ and D_f is connected by two open disjoint paths to $L(n)$ and $R(n)$. Finally we define

$$X_{D_f,e,E} = \mathbb{1}\{\tau_e E, e \leftrightarrow_3 D_f\}, \qquad Y_{D_f,n} = \frac{\mathbb{1}\{F_e(D_f), D_f \leftrightarrow_3 B(n)\}}{|LC(D_f, n)|}.$$

  Notice that they are independent since they depend on disjoint set of edges.

- We choose again $N_1(M, \epsilon)$ as done in (2.42) (since $|P(n)| \leq |LC(n)|$, this choice of N works) and ensure that $8M < r(n)$. The central argument in (2.43) remains the same after we substitute all the pieces mentioned. This completes the proof. □

Remark 2.16. Since we were very crude with this estimate, the bound on $r(n)$ turned out a bit stringent – we needed to impose that $r(n) = o(n^\delta)$. But since this is enough for our case, we do not strive to obtain a more sophisticated bound which can be obtained with careful calculations.

Chapter 3

Russo-Seymour-Welsh-Theorem in Slabs

3.1 Introduction

Russo-Seymour-Welsh theorem is one of the main tools in the study of planar percolation models at criticality, which states that the probability that an open path connects the left and right sides of a rectangle is bounded away from 0 and 1 by constants that only depend on the aspect ratio of the rectangle. This theorem was first proved for critical Bernoulli percolation on planar lattices in [R78, SW78, R81, K82] and recently has been extended to some other planar models, perhaps most notably to the FK-percolation [DCHN11, DCST17] and Voronoi percolation [BR06, T14].

In this chapter our main aim is to establish Russo-Seymour-Welsh theorem or commonly known as box-crossing theorem for critical Bernoulli percolation on two dimensional slabs in $\mathbb{Z}^d$, i.e. $\mathbb{S}_{k,d} := \mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ for $d \geq 3, k \geq 0$. We prove that the probability of crossing a "rectangular box" is bounded from below by a positive constant which only depends on the aspect ratio of the rectangle and the slab parameters k, d, but does not depend on the size of the rectangular box. This is the main result of our paper [BS15].

As one can imagine, lack of planarity creates some obstacle to connecting paths which are obvious and straightforward in plane. For this, we will introduce a certain technique for "glueing" open paths which is inspired by a recent paper of Duminil-Copin, Sidoravicius, and Tassion [DCST16] in which they use this crucially to prove $\theta(p_c(\mathbb{S}_{k,d})) = 0$ for any $k \geq 0$ and $d \geq 3$. This proof can be extended for other models such as finite-range percolation,but as we have highlighted, we work with solely slabs $\mathbb{S}_{k,d}$ for the sake of simplicity.

3.2 Notation and result

For integers $k \geq 0$, $d \geq 3$, we consider Bernoulli bond percolation on $\mathbb{S}_{k,d}$ with parameter $p \in [0,1]$, and denote the corresponding measure by $\mathbb{P}_p$. Let p_c be the critical threshold for percolation, i.e.,

$$p_c = \inf \{p : \mathbb{P}_p[\text{open connected component of } 0 \text{ in } \mathbb{S}_{k,d} \text{ is infinite}] > 0\},$$

and define the measure $\mathbb{P} = \mathbb{P}_{p_c}$.

For a subset A of vertices of $\mathbb{Z}^2$, let

$$\overline{A} = A \times \{0, \ldots, k\}^{d-2}.$$

Define a rectangle and its left and right boundary regions by

$$B(m,n) = \overline{[0,m) \times [0,n)}, \qquad L(m,n) = \overline{\{0\} \times [0,n)}, \qquad R(m,n) = \overline{\{m-1\} \times [0,n)}.$$

Consider the crossing event

$$\mathrm{LR}(m,n) = \{L(m,n) \text{ is connected to } R(m,n) \text{ by an open path in } B(m,n)\}$$

and the crossing probability $p(m,n) = \mathbb{P}[\mathrm{LR}(m,n)]$.

The main result of this chapter is the RSW theorem :

Theorem 3.1. *For any $\rho \in (0, \infty)$,*

$$\liminf_{n \to \infty} p(\lfloor \rho n \rfloor, n) > 0. \tag{3.1}$$

Next, we will state the high-probability variant of RSW theorem, which states that if the crossing probability in the easy direction of a rectangular box of fixed aspect ratio goes to 1 as the size increases, so must happen for the difficult direction of a rectangular box with arbitrarily large aspect ratios, i.e :

Corollary 3.2 (High-Probability version of RSW Theorem)**.**

$$\lim_{n \to \infty} p(\lfloor \rho n \rfloor, n) = 1 \textit{ for some } \rho \in (0,1) \Rightarrow \lim_{n \to \infty} p(\lfloor \kappa n \rfloor, n) = 1 \textit{ for all } \kappa > 0. \tag{3.2}$$

Remark 3.3. For $\rho < 1$, the result of Theorem 3.1 holds in any dimension $d \geq 2$. We believe that it also holds for $\rho \geq 1$, but no such proof is currently known. If dimension is sufficiently high, it is proved in [A97] that the crossing probabilities tend to 1 as $n \to \infty$. Unfortunately our method relies crucially on quasi-planarity of slabs. So it sheds no insight about existence (or lack of it) of Theorem 3.1 for general values of d.

Another question which arises in this context is whether $\limsup_{n\to\infty} p(\lfloor \rho n \rfloor, n) < 1$ holds for every $\rho > 0$. This was shown to be true very recently by Newman, Tassion and Wu in [NTW15, Theorem 3.1] for percolation on slabs. They also obtained independently and with different proofs the results of Theorem 3.1 and Corollary 3.2 (see [NTW15, Theorems 3.1 and 3.17]).

We will prove Theorem 3.1 in Section 3.4 and Corollary 3.2 in Section 3.5. Finally in Section 3.6 we provide some related results. But first in Section 3.3, we introduce the new technique for glueing paths via local modifications from [DCST16]. This sort of "surgery" would be used repeatedly throughout this chapter.

3.3 Surgery for Glueing paths

We describe one technique for glueing paths, inspired by [DCST16], which will be used to adapt some arguments from planar percolation to slabs. We begin with a classical combinatorial lemma about local modifications, see, e.g., [DCST16, Lemma 7].

Lemma 3.4. *Let $n \geq 1$ and $p \in (0,1)$. Let $A, B \subseteq \{0,1\}^n$ and $\mathbf{P}_p$ a product measure on $\{0,1\}^n$ with parameter p, i.e.,*

$$\mathbf{P}_p[\omega] = \prod_{i=1}^{n} p^{\omega_i}(1-p)^{1-\omega_i}, \quad \omega \in \{0,1\}^n.$$

If there exists a relation $\mathfrak{R} \subset A \times B$ such that

(a) if $(\omega, \omega') \in \mathfrak{R}$, there exists a set $S \subseteq \{1, \ldots, n\}$ such that $|S| \leq s$ and

$$\omega_i = \omega_i', \quad \textit{for all } i \notin S,$$

(b) for every $\omega \in A$, the set $R_\omega = \{\omega' : (\omega, \omega') \in \mathfrak{R}\}$ has at least $t \in \mathbb{N}$ many elements,

then

$$\mathbf{P}_p[A] \leq \frac{\left(\frac{2}{\min(p,1-p)}\right)^s \cdot \mathbf{P}_p[B]}{t}.$$

If we get a function $f : A \to B$ instead which satisfies the condition (a), needless to say that it would satisfy the inequality with $t = 1$. We will often apply Lemma 3.4 in case s is not bigger than the number of edges in $[-3,3]^2 \times \{0, 1, \ldots k\}^{d-2}$ and $p = p_c(\mathbb{S}_{k,d})$. Therefore, we define

$$C_* = \left(\frac{2}{\min(p_c(\mathbb{S}_{k,d}), 1 - p_c(\mathbb{S}_{k,d}))}\right)^{d \cdot 7^2 \cdot k^{d-2}}, \qquad c_* = \frac{1}{1 + 3C_*}.$$

Earlier we defined $\overline{A}$ as a subset of $\mathbb{S}_{k,d}$ for each $A \subset \mathbb{Z}^2$. In the proofs we will often use the same notation $\overline{A}$ for $A \subset \mathbb{S}_{k,d}$ meaning

$$\overline{A} = \{z = (z_1, \dots, z_d) \in \mathbb{S}_{k,d} \ : \ (z_1, z_2, x_3, \dots, x_d) \in A \text{ for some } x_3, \dots, x_d \ \}.$$

This way, for each $A \subset \mathbb{Z}^2$, $\overline{A}$ defined earlier is the same as $\overline{A \times \{0\}^{d-2}}$ defined just above.

For $x, y \in \mathbb{S}_{k,d}$ and $X, Y, Z \subset \mathbb{S}_{k,d}$, we write

- $x \overset{Z}{\longleftrightarrow} y$ if there is a nearest neighbor path of open edges from x to y with all its vertices in Z.
- $x \overset{Z}{\longleftrightarrow} Y$ if there exists $y \in Y$ such that $x \overset{Z}{\longleftrightarrow} y$.
- $X \overset{Z}{\longleftrightarrow} Y$ in Z if there exists $x \in X$ such that $x \overset{Z}{\longleftrightarrow} Y$.

If we do not mention Z, it is understood that $Z = \mathbb{S}_{k,d}$ and for $X, Y, Z \subset \mathbb{Z}^2$, we define $X \overset{Z}{\longleftrightarrow} Y := \overline{X} \overset{\overline{Z}}{\longleftrightarrow} \overline{Y}$. Let us also use $B(m,n) = [0,m) \times [0,n)$ and finally call $\mathbb{P} := \mathbb{P}_{p_c(\mathbb{S}_{k,d})}$, the usual Bernoulli product measure at criticality.

The following lemma is essentially proven in [DCST16, Lemma 6].

Lemma 3.5. *Let X_1, X_2, Y_1, and Y_2 be disjoint connected subsets of the interior vertex boundary of $[0,m) \times [0,n)$ arranged in a counter-clockwise order. Then*

$$\mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} X_2\right] \geq c_* \cdot \mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} Y_1, X_2 \overset{B(m,n)}{\longleftrightarrow} Y_2\right].$$

Proof. Let

$$X = \{X_1 \overset{B(m,n)}{\longleftrightarrow} X_2\}, \qquad E_i = \{X_i \overset{B(m,n)}{\longleftrightarrow} Y_i\}, \qquad \text{for } i = 1, 2.$$

It suffices to prove that $\mathbb{P}[E_1 \cap E_2 \cap X^c] \leq 3C_* \cdot \mathbb{P}[X]$. For $i \in \{1, 2\}$, consider events

$$F_i = \bigcup_{z \in X_{3-i}} \{\overline{X}_i \text{ is connected to } \overline{z + [-3,3]^2} \text{ in } B(m,n)\}.$$

We will prove first that

$$\mathbb{P}[E_1 \cap E_2 \cap F_1^c \cap F_2^c] \leq C_* \cdot \mathbb{P}[X], \tag{3.3}$$

and later that $\mathbb{P}[F_i \cap X^c] \leq C_* \cdot \mathbb{P}[X]$, for $i = 1, 2$. These together will be sufficient, since $X \subset F_1 \cap F_2$. To prove (3.3), we intend to use Lemma 3.4. Thus we will construct a suitable function $f : E_1 \cap E_2 \cap F_1^c \cap F_2^c \to X$. This tricky construction is already elaborated in the proof of [DCST16, Lemma 6, Fact 2]. Nevertheless, we show it again in this context as this is a key tool and some variant of it will be used repeatedly throughout this chapter.

We fix an order $\prec$ on edges $\{e : |e| = 1\}$ in $\mathbb{Z}^d$ and enumerate all the vertices of $\mathbb{S}_{k,d}$ arbitrarily. Define an order $<$ on self-avoiding paths from $\overline{X_1}$ to $\overline{Y_1}$ in $B(m, n)$ as follows. If $\gamma = (\gamma_0, \dots, \gamma_n)$ and $\gamma' = (\gamma'_0, \dots, \gamma'_{n'})$ are two such paths, then $\gamma < \gamma'$ if either of the following holds :

- γ_0 has a smaller number than γ'_0.
- $n < n'$ and $\gamma = (\gamma'_0, \dots, \gamma'_n)$.
- There exists $k < \min(n, n')$ such that $(\gamma_0, \dots, \gamma_k) = (\gamma'_0, \dots, \gamma'_k)$, and the edge $\{0, \gamma_{k+1} - \gamma_k\} \prec \{0, \gamma'_{k+1} - \gamma'_k\}$.

Take $\omega \in E_1 \cap E_2 \cap F_1^c \cap F_2^c$. Let $\gamma_{\min}(\omega)$ be the minimal open self-avoiding path from $\overline{X_1}$ to $\overline{Y_1}$ for the above defined order. We look at the set

$$U(\omega) = \{z : z \in \gamma_{min}(\omega), \exists y \in \overline{\{z\}} \text{ such that } y \overset{B(m,n)}{\longleftrightarrow} X_2\}$$

Since $\omega \in E_2$, $U(\omega)$ is non-empty. Also, since $\omega \in F_1^c \cap F_2^c \subset X^c$, for a vertex $z \in U(\omega)$, $\overline{\{z\}}$ is connected to $\overline{X_2}$ by an open path not using any edges of $\overline{\gamma_{\min}(\omega)}$ and the set $\overline{z + [-3, 3]^2 \times \{0\}^{d-2}}$ is disjoint from $\overline{X_1} \cup \overline{X_2}$.

We will choose any such $z \in U(\omega)$, and will locally modify the occupancy configuration in its neighborhood $B_z = \overline{z + [-3, 3]^2 \times \{0\}^{d-2}}$ so that we get a function $f : E_1 \cap E_2 \cap F_1^c \cap F_2^c \to E_1 \cap X$ with the properties:

- (i) $z \in \gamma_{\min}(f(\omega))$,
- (ii) z is a unique vertex on $\gamma_{\min}(f(\omega))$ connected to $\overline{X_2}$ by an open path that does not use edges of $\gamma_{\min}(f(\omega))$,
- (iii) $\omega_e = f(\omega)_e$ for all $e \notin B_z$.

Given the altered configuration, we will first track the minimal path γ_{min} and then spot z as the unique vertex satisfying property (ii). The function f would thus satisfy the conditions of Lemma 3.4 with s being the number of edges in $\overline{[-3, 3]^2}$ (by properties (ii) and (iii)), and $\mathbb{P}[E_1 \cap E_2 \cap F_1^c \cap F_2^c] \leq C_* \cdot \mathbb{P}[E_1 \cap X] \leq C_* \cdot \mathbb{P}[X]$ will be an immediate consequence, completing the proof.

Coming back to the local modification, we do the following:

- Mark the vertices by which γ_{min} enters $B_z^i := \overline{z + [-2,2]^2 \times \{0\}^{d-2}}$, the "inner" neighborhood, for the first time by v_i. Let us call the segment of γ_{min} from beginning till v_i as γ_{min}^i.
- Mark the vertices by which γ_{min} leaves B_z^i for the last time by v_o. Let us call the segment of γ_{min} from v_o till end as γ_{min}^o.
- Mark a path from $\overline{z}$ to X_2 by β and mark the vertex by which β leaves B_z^i for the last time by v_β.
- Find three neighboring vertices of z (say z_i, z_o and z_β) such that:
  - There exist three self avoiding paths γ_i, γ_o and γ_β, not using z or three aforementioned neighbors of it, which connects v_i to z_i, v_o to z_o and v_β to z_β inside B_z^i,
  - $(z, z_o) \prec (z, z_\beta)$.
- Close all edges of B_z except the edges with both vertices in $B_z \setminus B_z'$ which are in γ_{min}^i, γ_{min}^o or β.
- Open all the edges in paths $\gamma_i, \gamma_o, \gamma_\beta$ and three edges $(z, z_i), (z, z_o), (z, z_\beta)$ (while keeping every other edges with both vertices in B_z^i closed).

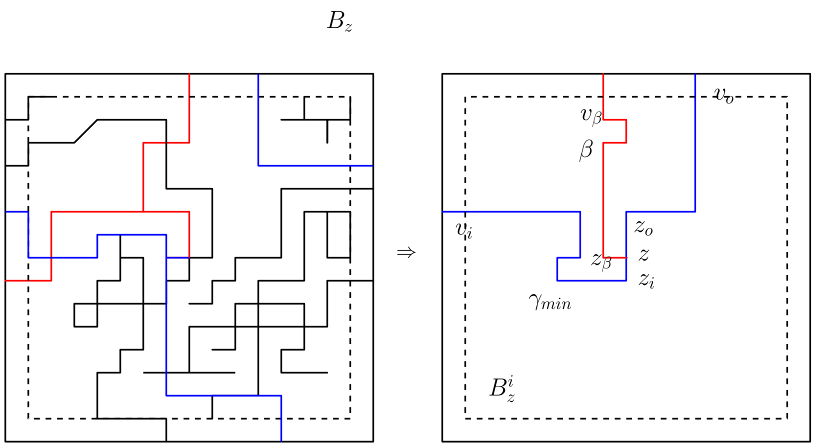

FIGURE 3.1: Local Modification

In the new configuration z is connected to $\overline{X_1}, \overline{X_2}$ and $\overline{Y_1}$ (making the altered configuration $\omega' \in E_1 \cap X$) and $\gamma_{min}(\omega')$ matches with $\gamma_{min}^i(\omega)$ from the starting point to v_i, then due to lack of choice leads to z, chooses (z, z_o) over (z, z_β) and then again leads to v_o and finally matches with $\gamma_{min}^o(\omega)$ again after v_o. This takes care of property (i) and existence of a connection from z to $\overline{X_2}$ without using edges of γ_{min}, which is a part of property (ii). The uniqueness of z satisfying property (ii) stems from the fact that had there been another contender $z' \in \gamma_{min} \cap B_z^c$ (inside B_z, by our descriptions there cannot be another contender), at least one of the occupancy configurations in B_z or $B_{z'}$ would have been preserved in the initial configuration ω as well, contradicting $\omega \in X^c$. Property (iii) is obvious from our construction and this proves (3.3).

To prove $\mathbb{P}[F_i \cap X^c] \leq C_* \cdot \mathbb{P}[X]$, notice that for any $\omega \in F_i \cap X^c$, one can choose $z \in X_{3-i}$ satisfying the requirement of F_i so that after modifying the occupancy configuration in $\overline{z + [-3,3]^2}$, one obtains a configuration in which $z \times \{0\}^{d-2}$ is the unique vertex of $\overline{X_{3-i}}$ which is connected to $\overline{X_i}$ in $B(m,n)$ and by Lemma 3.4, we are done. We do not give more details about the surgery since it is similar to the one we described and, in fact, simpler. This completes our proof. □

Remark 3.6. The choice of the size of neighborhood around z is chosen to be three in the case of slabs so that for any choice of three points on the interior boundary of B_z^i, we can find three disjoint paths from these three points to three neighbors of z inside B_z^i. For other models like finite-range percolation, the same approach would work but with a bigger neighborhood.

Lemma 3.5 and the FKG inequality imply the following corollary:

Corollary 3.7. *Let X_1, X_2, Y_1, Y_2 be as in Lemma 3.5. Then*

$$\mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} X_2\right] \geq c_* \cdot \mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} Y_1\right] \mathbb{P}\left[X_2 \overset{B(m,n)}{\longleftrightarrow} Y_2\right].$$

Remark 3.8. We are primarily using rectangular blocks as the glueing areas, but this can be generalized to quite general shapes. In fact, this can be generalized in the following way :

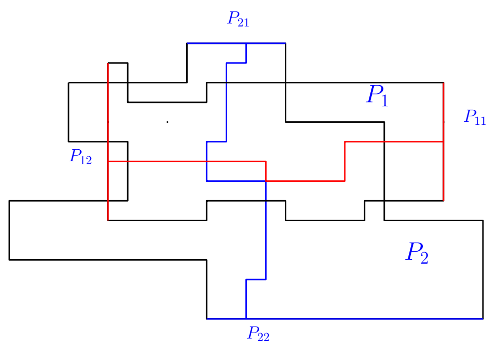

FIGURE 3.2: Glueing for polygon-boxes

Let two simple polygons P_1, P_2 with vertices from $\mathbb{Z}^2$ have regions P_{ij} (for $i,j \in \{1,2\}$), which are disjoint connected subset of the interior vertex boundary of P_i. For an event A, the polygons P_1, P_2 are called "glueing-friendly" *under the event* A for regions P_{ij} if for any $\omega \in A$, any two open paths γ_i connecting P_{i1} to P_{i2} in P_i (for $i = 1,2$) necessarily have an intersection point z such that $z + [-4,4]^2 \subset P_1 \cap P_2$, then

$$\mathbb{P}[P_{11} \xleftrightarrow{P_1 \cup P_2} P_{22}] \geq c_* \cdot \mathbb{P}[P_{11} \xleftrightarrow{P_1} P_{12}, P_{21} \xleftrightarrow{P_2} P_{22}, A] \tag{3.4}$$

If the boundary of the polygon is regular enough, i.e. the polygons can be represented by union of finitely many rectangles with both dimensions bigger than 6 (Let us call them *glueing-regular*), the result mentioned above holds true with $A = \Omega$ (see Figure 3.2). The core of the proofs is the surgery exactly similar to what we did while proving Lemma 3.5.

Another version of "glueing" is a tool to glue events with probability close to 1 to yield a glued event of probability close to 1. This was used in [DCST16] and we would revisit them in Section 3.5.

3.4 Proof of Theorem 3.1

Since the case $k = 0$ is classical (see e.g. [R81, SW78, K82].) and as some of the "glueing" ideas used in the proof are unnecessary for $k = 0$ and easier for $k \geq 1$, we *assume from now on without further mentioning that* $k \geq 1$. The theorem is proved in 3 steps:

- The result holds for all $\rho \in (0,1)$. This is well known. We give a proof in Proposition 3.9.
- If the result holds for some $\rho > 1$, then it holds for all $\rho > 1$. This is a well known fact in planar percolation. We prove the slab version in Proposition 3.10 using the planar approach together with a novel technique for glueing paths from [DCST16] (see Lemma 3.5).
- There exist $c > 0$ and $C < \infty$ such that for all $n \geq 1$, $p(44n, 43n) \geq c \cdot p(43n, 44n)^C$. This inequality is the crucial component and we prove it in Proposition 3.11 using various "paths glueing" procedures. □

3.4.1 Crossings of narrow rectangles

The following proposition is an adaptation to slabs of a well known fact about the probabilities of crossing hypercubes of fixed aspect ratio in the easy direction. Its proof is standard and does not require the "glueing" Lemma 3.5.

Proposition 3.9. *For any* $\rho \in (0,1)$, (3.1) *holds.*

Proof. Let $0 < a < b$ be integers. It suffices to prove that $\liminf_{n\to\infty} p(an, bn) > 0$. We will prove a standard recursive inequality which states that for $C = 2\lceil \frac{b}{b-a} \rceil + 1$, every $p \in [0,1]$ and $n \geq 1$,

$$\mathbb{P}_p\left[\mathrm{LR}(2an, 2bn)\right] \leq \left(C\, \mathbb{P}_p\left[\mathrm{LR}(an, bn)\right]\right)^2. \tag{3.5}$$

Let us denote, for $v = (v_1, v_2) \in \mathbb{Z}^2, v + B(m,n) := [v_1, v_1 + m) \times [v_2, v_2 + n)$, i.e the rectangle of dimension $m \times n$ with left bottom corner v. Any open left-right crossing of $\overline{B(2an, 2bn)}$ produces open left-right crossings of $\overline{B(an, 2bn)}$ and $\overline{(an, 0) + B(an, 2bn)}$ giving $\mathbb{P}_p\left[\mathrm{LR}(2an, 2bn)\right] \leq \mathbb{P}_p\left[\mathrm{LR}(an, 2bn)\right]^2$. We will suitably cover $\overline{B(an, 2bn)}$ by C many copies of $B(an, bn)$ or its rotated version $B(bn, an)$ (see Figure 3.3) such that the existence of an open left-right crossing of $\overline{B(an, 2bn)}$ indicates that at least one of the copies is crossed in the easy direction. Let us define the set $W = \{0, b-a, 2(b-a), \ldots, \lceil \frac{b}{b-a} \rceil (b-a)\}$. Indeed, we see that any open left-right crossing of $\overline{B(an, 2bn)}$ either crosses horizontally one of the rectangles $\overline{(0, kn) + B(an, bn)}$, $k \in W$, or crosses vertically one of the rectangles $\overline{(0, kn) + B(bn, an)}$, $k \in W \setminus \{0\}$, and by union bound we have: $\mathbb{P}_p\left[\mathrm{LR}(an, 2bn)\right] \leq C \cdot \mathbb{P}_p\left[\mathrm{LR}(an, bn)\right]$.

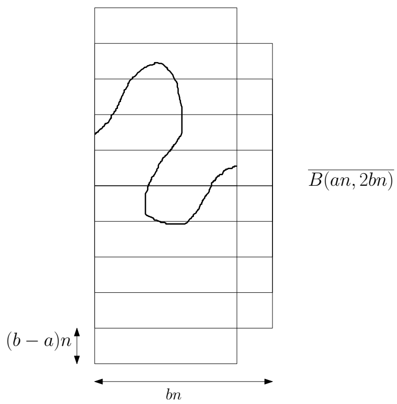

FIGURE 3.3: Covering Boxes

By (3.5), for all $p \in [0,1]$, $n \geq 1$, and $s \geq 0$,

$$\mathbb{P}_p\left[\mathrm{LR}(2^s an, 2^s bn)\right] \leq \left(C^2\, \mathbb{P}_p\left[\mathrm{LR}(an, bn)\right]\right)^{2^s}.$$

If $\liminf_{n\to\infty} p(an, bn) \leq \frac{1}{2C^2}$, then there exists $n \in \mathbb{N}$ such that $C^2 p(an, bn) < 2/3$. Since the crossing probability $\mathbb{P}_p\left[\mathrm{LR}(an, bn)\right]$ is continuous in p, there also exists $p > p_c$ such that $C^2\, \mathbb{P}_p\left[\mathrm{LR}(an, bn)\right] < 2/3$. For this choice of parameters, $\lim\limits_{s\to\infty} \mathbb{P}_p\left[\mathrm{LR}(2^s an, 2^s bn)\right]$ equals 0, which is impossible, since for every $p > p_c$, this limit equals to 1 (see e.g. [G99, Theorem 8.97]). Thus,

$$\liminf_{n\to\infty} p(an, bn) > \frac{1}{2(2\lceil \frac{b}{b-a} \rceil + 1)^2} > 0. \tag{3.6}$$

□

3.4.2 Crossings of wide rectangles

Proposition 3.10. *If* (3.1) *holds for some* $\rho > 1$, *then it holds for all* $\rho > 1$.

Proof. This is immediate from the following inequality, which relates the crossing probability of a long rectangle with that of a shorter one. For all $m > n$,

$$p(2m-n,n) \geq \frac{1}{4} \cdot {c_*}^3 \cdot p(m,n)^4. \tag{3.7}$$

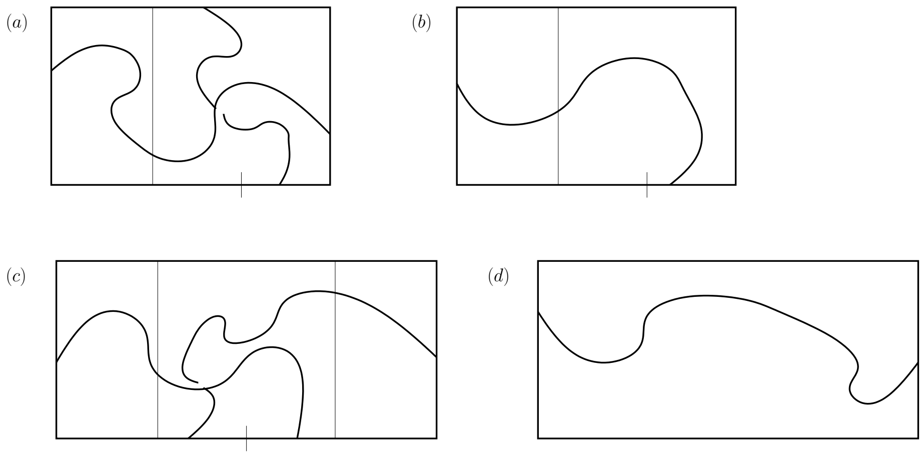

FIGURE 3.4: (a) left-right crossing of $B(m,n)$ and top-bottom crossing of a $\overline{[m-n,m)\times[0,n)}$ landing on the right half of the bottom, (b) path from $L(m,n)$ to $\overline{[m-\frac{n}{2},m)\times\{0\}}$ in $B(m,n)$, (c) paths from $L(2m-n,n)$ to $\overline{[m-\frac{n}{2},m)\times\{0\}}$, and from $\overline{[m-n,m-\frac{n}{2})\times\{0\}}$ to $R(2m-n,n)$ in $B(2m-n,n)$, (d) left-right crossing of the wide rectangle $B(2m-n,n)$.

The inequality (3.7) follows from two applications of Corollary 3.7 illustrated on Figure 3.4. ☐

3.4.3 Crossings of rectangles: short and long directions

The main contribution of this section is the following proposition, which relates the crossing probability of a rectangle in the long direction with the one in the short. The exact values of the aspect ratios do not matter as long as one of them is smalle than 1 and the other one is greater than 1. We thus choose them to be 43/44 and 44/43 for the sake of ease in calculations.

Proposition 3.11. *For all* $n \in \mathbb{N}$,

$$p(44n,43n) \geq \frac{{c_*}^{21} \cdot p(43n,44n)^{198}}{10^{154}}. \tag{3.8}$$

Proof. Fix $n \in \mathbb{N}$. We write

$$B = B(43n,44n), \quad L = L(43n,44n), \quad R = R(43n,44n),$$

and define

$$c = p(43n, 44n), \qquad c' = \frac{{c_*}^{21} \cdot c^{198}}{10^{154}}.$$

We prove the proposition by considering several cases. The first two steps are inspired by the ideas of Bollobás and Riordan from [BR06], and aimed at restricting possible shapes of left-right crossings. Steps 3 and 4 contain preliminary estimates needed to implement the main idea in Step 5.

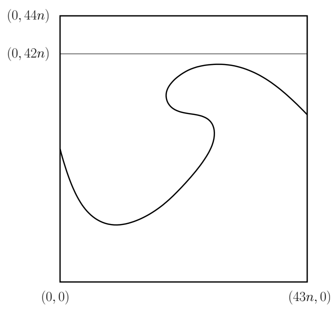

FIGURE 3.5: Left-right crossing staying at least $2n$ away from the top of $B(43n, 44n)$.

Step 1. We first consider the case when there is a considerable probability that a left-right crossing of B stays away from the top or bottom boundary of B, see Figure 3.5. Assume that $p(43n, 42n) \geq \frac{c}{100}$. Then by (3.7),

$$\begin{aligned} p(44n, 43n) &\geq p(44n, 42n) \\ &\geq \frac{1}{4}{c_*}^3 p(43n, 42n)^4 \geq c', \end{aligned}$$

which implies (3.8). Thus, we may assume that

$$p(43n, 42n) < \frac{c}{100}. \tag{3.9}$$

Step 2. Next, we consider the case when there is a considerable probability that a left-right crossing of B starts sufficiently far away from the middle of L. Let

$$S = \overline{\{0\} \times [20n, 24n)} \tag{3.10}$$

be the middle of L. Assume that

$$\mathbb{P}\left[L \setminus S \overset{B}{\longleftrightarrow} R\right] \geq \frac{c}{10}.$$

Then, by reflectional symmetry,

$$\mathbb{P}\left[\{0\} \times [24n, 44n) \overset{B}{\longleftrightarrow} R\right] \geq \frac{c}{20}.$$

By assumption (3.9),

$$\mathbb{P}\left[\{0\} \times [24n, 44n) \overset{B}{\longleftrightarrow} [0, 43n) \times \{2n\}\right] \geq \frac{c}{20} - \frac{c}{100} \geq \frac{c}{100}.$$

By rotational symmetry, the above display states that

$$\mathbb{P}\left[\{0\} \times [0, 43n) \overset{B(42n,43n)}{\longleftrightarrow} [22n, 42n) \times \{0\}\right] \geq \frac{c}{100}.$$

Similarly to the second application of Corollary 3.7 in the proof of (3.7), see Figure 3.6, one gets

$$p(44n, 43n) \geq c_* \cdot \left(\frac{c}{100}\right)^2 \geq c',$$

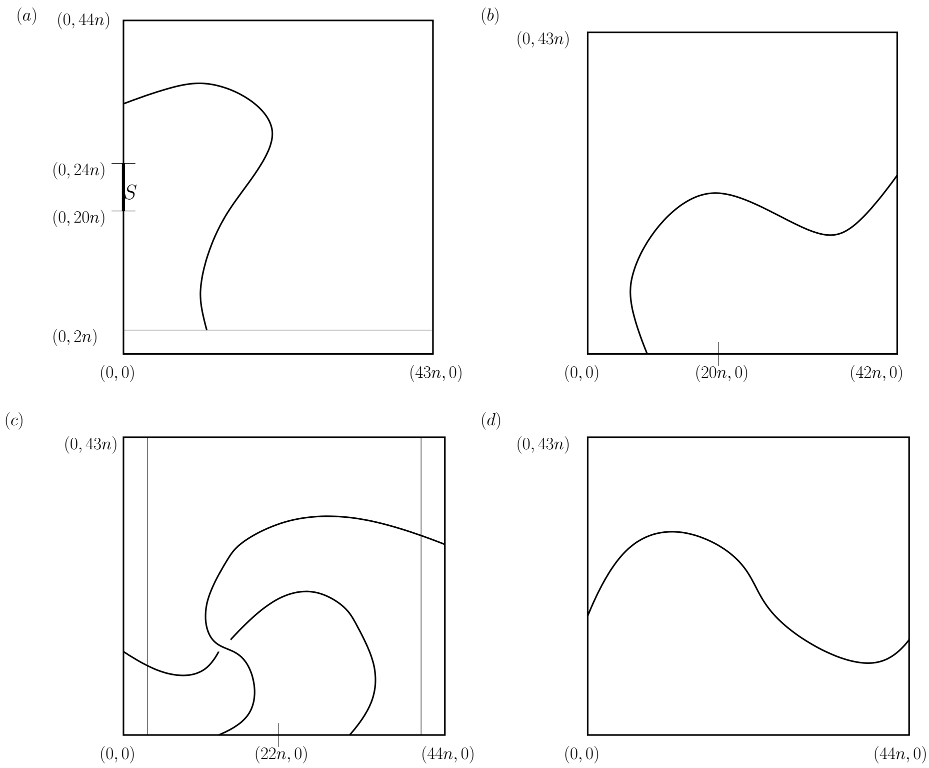

FIGURE 3.6: (a) part of L above S is connected to $\overline{[0,43n)\times\{2n\}}$ in B, (b) rotation of (a) by $\frac{\pi}{2}$, (c) $L(44n,43n)$ is connected to $\overline{[22n,42n)\times\{0\}}$ and $\overline{[2n,22n)\times\{0\}}$ is connected to $R(44n,43n)$, (d) left-right crossing of $B(44n,43n)$.

which is precisely (3.8). Thus, we may assume, in addition to (3.9), that

$$\mathbb{P}\left[L\setminus S \overset{B}{\longleftrightarrow} R\right] < \frac{c}{10}. \tag{3.11}$$

Step 3. Here we consider the case when there is a considerable probability that two well-separated subsegments of L are connected. For integers $a<b$, let

$$T_{ab} = \overline{[0,43n)\times[a,b)} \quad \text{and} \quad T = \overline{[0,43n)\times\mathbb{Z}}.$$

Assume that for some $a<b$,

$$\mathbb{P}\left[\{0\}\times[0,4n) \overset{T_{ab}}{\longleftrightarrow} \{0\}\times[8n,12n)\right] \geq \frac{c_*\cdot c^{18}}{10^{14}}.$$

Then, by repetitive use of Corollary 3.7, see Figure 3.7, for each $m\geq 1$,

$$\mathbb{P}\left[\{0\}\times[0,4n) \overset{T}{\longleftrightarrow} \{0\}\times[4n(m+1),4n(m+2))\right] \geq \frac{{c_*}^{2m-1}\cdot c^{18m}}{10^{14m}}.$$

Note that if $m=11$, then the event on the left hand side implies that there is a vertical crossing of $\overline{[0,43n)\times[4n,48n)}$. Thus,

$$p(44n,43n) \geq \frac{{c_*}^{21}\cdot c^{198}}{10^{154}} = c',$$

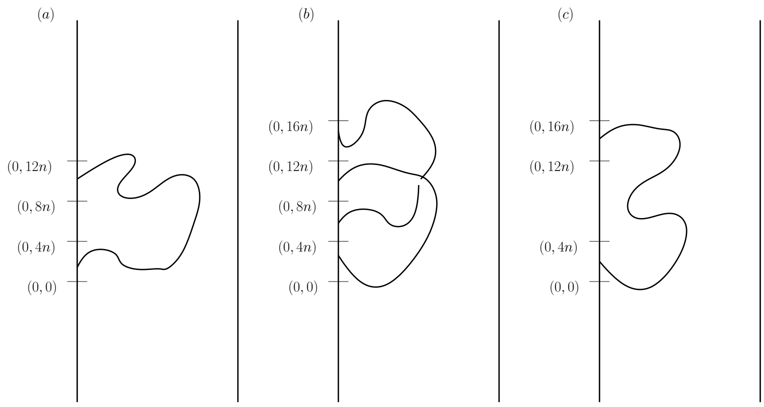

FIGURE 3.7: Vertical extension of open paths.

which gives (3.8). Therefore, we may assume, in addition to (3.9) and (3.11), that

$$\mathbb{P}\left[\{0\}\times[0,4n)\xleftrightarrow{T_{ab}}\{0\}\times[8n,12n)\right]<\frac{c_*\cdot c^{18}}{10^{14}},\qquad \text{for all } a<b. \tag{3.12}$$

Next, we derive several corollaries of assumption (3.12).

Corollary 3.12. *Under the assumption* (3.12)*, for all* $a<b$*,*

$$\mathbb{P}\left[\{0\}\times[8n,12n)\xleftrightarrow{T_{ab}}\{43n-1\}\times[0,4n)\right]<\frac{c^9}{10^7}. \tag{3.13}$$

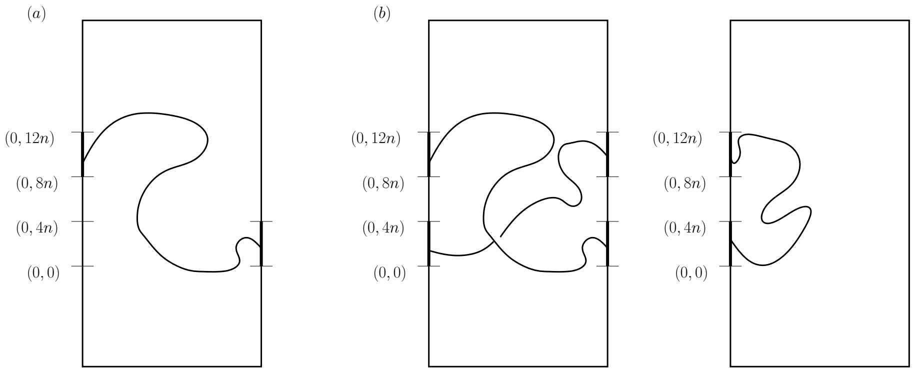

FIGURE 3.8: (a) illustration of the event in (3.13), (b) proof of Corollary 3.12.

Proof of Corollary 3.12. Using reflectional symmetry and Corollary 3.7,

$$\begin{aligned}\mathbb{P}&\left[\{0\}\times[8n,12n)\xleftrightarrow{T_{ab}}\{43n-1\}\times[0,4n)\right]^2\\ &=\mathbb{P}\left[\{0\}\times[8n,12n)\xleftrightarrow{T_{ab}}\{43n-1\}\times[0,4n)\right]\\ &\qquad\cdot\mathbb{P}\left[\{0\}\times[0,4n)\xleftrightarrow{T_{ab}}\{43n-1\}\times[8n,12n)\right]\\ &\le {c_*}^{-1}\cdot\mathbb{P}\left[\{0\}\times[0,4n)\xleftrightarrow{T_{ab}}\{43n-1\}\times[8n,12n)\right]\overset{(3.12)}{<}\frac{c^{18}}{10^{14}}.\end{aligned}$$

□

Corollary 3.13. *Under the assumption* (3.12), *for all $a < b$,*

$$\mathbb{P}\left[\begin{array}{c}\textit{there exist a simple path } \gamma \textit{ from } \overline{\{0\}\times[0,4n)} \textit{ to } \overline{\{43n-1\}\times[0,4n)} \\ \textit{and a path } \gamma' \textit{ from } \overline{\{0\}\times[8n,12n)}, \textit{ both in } T_{ab}, \textit{ such that} \\ \textit{the distance between } \overline{\gamma} \textit{ and } \overline{\gamma'} \textit{ is } \leq 2\end{array}\right] < \frac{3\cdot c^9}{10^7}. \tag{3.14}$$

In particular,

$$\mathbb{P}\left[\begin{array}{c}\textit{there exist a simple path } \gamma \textit{ from } \overline{\{0\}\times[0,4n)} \textit{ to } \overline{\{43n-1\}\times[0,4n)} \\ \textit{and a path } \gamma' \textit{ from } \overline{\{0\}\times[8n,12n)}, \textit{ both in } T, \textit{ such that} \\ \textit{the distance between } \overline{\gamma} \textit{ and } \overline{\gamma'} \textit{ is } \leq 2\end{array}\right] \leq \frac{3\cdot c^9}{10^7}. \tag{3.15}$$

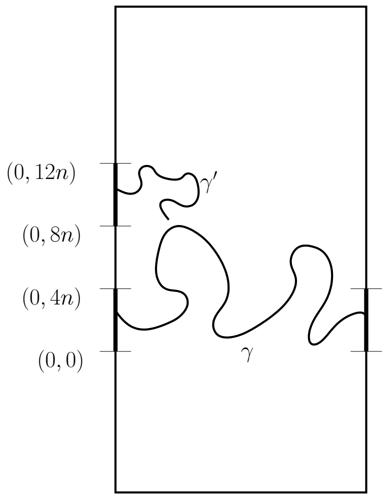

FIGURE 3.9: An illustration of the event in (3.14).

Proof of Corollary 3.13. It suffices to prove (3.14), as (3.15) follows from (3.14) when $a \to -\infty$ and $b \to +\infty$.

Denote the event in (3.14) by A. By the total probability formula,

$$\begin{aligned}\mathbb{P}[A] \leq \mathbb{P}&\left[\{0\}\times[8n,12n) \xleftrightarrow{T_{ab}} \{0\}\times[0,4n)\right] \\ &+ \mathbb{P}\left[\{0\}\times[8n,12n) \xleftrightarrow{T_{ab}} \{43n-1\}\times[0,4n)\right] \\ &\quad + \mathbb{P}\left[A, \{43n-1\}\times[0,4n) \overset{T_{ab}}{\not\leftrightarrow} \{0\}\times[8n,12n) \overset{T_{ab}}{\not\leftrightarrow} \{0\}\times[0,4n)\right] \\ \overset{(3.12)}{\underset{(3.13)}{\leq}} &\frac{c_*\cdot c^{18}}{10^{14}} + \frac{c^9}{10^7} + \mathbb{P}\left[A, \{43n-1\}\times[0,4n) \overset{T_{ab}}{\not\leftrightarrow} \{0\}\times[8n,12n) \overset{T_{ab}}{\not\leftrightarrow} \{0\}\times[0,4n)\right]\end{aligned}$$

Denote by A' the event in the RHS. For a configuration ω, let $P(\omega)$ be the set of vertices, which belong to at least one self-avoiding path from $\overline{\{0\}\times[0,4n)}$ to $\overline{\{43n-1\}\times[0,4n)}$ in T_{ab}, one may call it a backbone. For $\omega \in A'$, backbone is non-empty and, contains at least one point $z(\omega)$ such that $\overline{z + [-2,2]^2 \times \{0\}^{d-2}}$ is connected to $\overline{\{0\} \times [8n, 12n)}$ although $\overline{\{z\}}$ is not. Expectedly, we now consider a local modification map f from A' to the event

$$A'' = \left\{ \omega'' \ : \ \begin{array}{c} \text{there exists a unique } z(\omega'') \in P(\omega'') \text{ connected to } \overline{\{0\} \times [8n, 12n)} \\ \text{by an open path contained in } T_{ab} \setminus P(\omega'') \text{ except for the vertex } z(\omega'') \end{array} \right\}$$

such that for all $\omega' \in A'$ and all $e \notin \overline{z(f(\omega')) + [-3,3]^2 \times \{0\}^{d-2}}$, $f(\omega')_e = \omega'_e$. By Lemma 3.4, $\mathbb{P}[A'] \leq C_* \cdot \mathbb{P}[A''] \leq {c_*}^{-1} \cdot \mathbb{P}[A'']$.

$$A'' \subseteq \left\{ \{0\} \times [0, 4n) \xleftrightarrow{T_{ab}} \{0\} \times [8n, 12n) \right\},$$

implies that

$$\mathbb{P}[A'] \leq {c_*}^{-1} \cdot \mathbb{P}\left[\{0\} \times [0, 4n) \xleftrightarrow{T_{ab}} \{0\} \times [8n, 12n) \right] < \frac{c^{18}}{10^{14}},$$

where the last inequality follows from the assumption (3.12). Putting the bounds together,

$$\mathbb{P}[A] < \frac{c_* \cdot c^{18}}{10^{14}} + \frac{c^9}{10^7} + \frac{c^{18}}{10^{14}} \leq \frac{3 \cdot c^9}{10^7}.$$

□

Corollary 3.14. *Under the assumptions* (3.11) *and* (3.12),

$$\mathbb{P}\left[\begin{array}{c} \textit{there exist a path } \gamma' \textit{ from } \overline{\{0\} \times [0, 4n)} \textit{ in } T \\ \textit{and a path } \gamma'' \textit{ from } \overline{\{0\} \times [16n, 20n)} \textit{ in } T, \textit{ such that} \\ \textit{the distance between } \overline{\gamma'} \textit{ and } \overline{\gamma''} \textit{ is} \leq 4 \end{array} \right] \leq \frac{12 \cdot c^8}{10^7}. \tag{3.16}$$

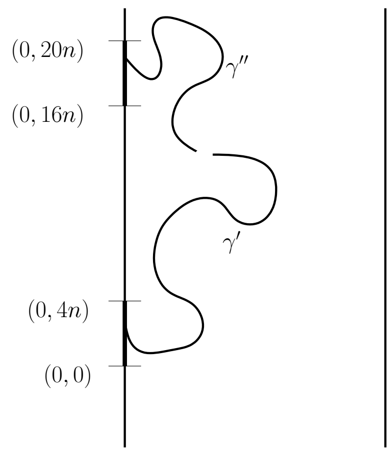

FIGURE 3.10: An illustration of the event in (3.16).

Proof of Corollary 3.14. Denote the event in (3.16) by A. By assumption (3.11),

$$\mathbb{P}\left[\{0\}\times[8n,12n)\overset{T}{\longleftrightarrow}\{43n-1\}\times[8n,12n)\right]\geq c-2\,\frac{c}{10}\geq\frac{c}{2}.$$

The event above and the event A are increasing, thus :

$$\mathbb{P}[A]\overset{FKG}{\leq}\frac{2}{c}\cdot\mathbb{P}\left[A,\{0\}\times[8n,12n)\overset{T}{\longleftrightarrow}\{43n-1\}\times[8n,12n)\right]\overset{(3.15)}{\leq}\frac{2}{c}\cdot 2\,\frac{3\cdot c^9}{10^7}=\frac{12\cdot c^8}{10^7}.$$

The last inequality is due to the fact that intersection of the two events implies that for any path γ from $\overline{\{0\}\times[8n,12n)}$ to $\overline{\{43n-1\}\times[8n,12n)}$ in T, the distance from $\overline{\gamma}$ to $\overline{\gamma'}\cup\overline{\gamma''}$ is ≤ 2. $\square$

Step 4. The aim of this step is to introduce a certain event of positive probability, see Proposition 3.15. Our choice of this event will be clarified in Step 5.

Recall the definition of S from (3.10). For a configuration ω, let $C_S = C_S(\omega)$ be the set of all $z\in T$ connected to S by an open path in T. Let

$$f(\omega)=\mathbb{P}\left[\{0\}\times[4n,8n)\overset{T\setminus\overline{C_S}}{\longleftrightarrow}\{43n-1\}\times\mathbb{Z}\;\middle|\;C_S\right](\omega),$$

$$g(\omega)=\mathbb{P}\left[\begin{array}{c}\text{there exists a path } \gamma' \text{ from } \overline{\{0\}\times[4n,8n)} \text{ in } T\text{, such that}\\ \text{the distance between } \overline{\gamma'} \text{ and } \overline{C_S} \text{ is } \leq 4\end{array}\;\middle|\;C_S\right](\omega).$$

We consider the following events:

$$A_1=\left\{S\overset{T}{\longleftrightarrow}[0,43n)\times\{2n\}\right\},\quad A_2=\left\{w: f(\omega)\geq\frac{c^2}{10}\right\},\quad A_3=\left\{w: g(\omega)\leq\frac{c^4}{1000}\right\}.$$

Proposition 3.15. *Under the assumptions* (3.9), (3.11), *and* (3.12),

$$\mathbb{P}[A_1\cap A_2\cap A_3]\geq\frac{c^4}{10^3}.$$

Proof of Proposition 3.15. By assumptions (3.9) and (3.11),

$$\mathbb{P}[A_1]\geq c-\frac{c}{10}-\frac{c}{100}\geq\frac{c}{2}.$$

By the Markov inequality and (3.16),

$$\mathbb{P}[A_3^c]\leq\frac{1000}{c^4}\cdot\mathbb{E}[g]<\frac{1000}{c^4}\cdot\frac{12\cdot c^8}{10^7}=\frac{12\cdot c^4}{10^4}. \tag{3.17}$$

To bound $\mathbb{P}[A_1\cap A_2]$ from below we use the Paley-Zygmund inequality, which states that for non-negative random variable X states that $\mathbf{P}[X\geq\frac{1}{2}\mathbf{E}[X]]\geq\frac{1}{4}\frac{(\mathbf{E}[X])^2}{\mathbf{E}[X^2]}$. We intend to apply it to the measure $\mathbf{P}[\cdot] = \mathbb{E}\left[\mathbb{1}.\frac{\mathbb{1}_{A_1}}{\mathbb{P}[A_1]}\right]$ so that we get

$$\mathbb{E}\left[\mathbb{1}_{f(\omega)\geq\frac{1}{2}\cdot\mathbb{E}[f(\omega)\cdot\frac{\mathbb{1}_{A_1}}{\mathbb{P}[A_1]}]}\cdot\frac{\mathbb{1}_{A_1}}{\mathbb{P}[A_1]}\right] \geq \frac{1}{4}\cdot\left(\mathbb{E}[f(\omega)\cdot\frac{\mathbb{1}_{A_1}}{\mathbb{P}[A_1]}]\right)^2. \tag{3.18}$$

We have already defined the event A_2 retroactively such that if we prove a suitable lower bound of $\mathbb{E}[f(\omega)\cdot\mathbb{1}_{A_1}]$, it will simultaneously bound the LHS of (3.18) from above and the RHS from below (by $\frac{\mathbb{P}[A_1\cap A_2]}{\mathbb{P}[A_1]}$ and $\frac{c^4}{100\mathbb{P}[A_1]^2}$ respectively, if we prove the lower bound to be $c^2/5$, which we prove now).

$$\begin{aligned}
&\mathbb{E}[f(\omega)\cdot\mathbb{1}_{A_1}]\\
= \quad & \mathbb{P}\left[S \overset{T}{\longleftrightarrow} [0,43n)\times\{2n\}, \{0\}\times[4n,8n) \overset{T\setminus\overline{C_S}}{\longleftrightarrow} \{43n-1\}\times\mathbb{Z}\right]\\
\overset{(3.16)}{\geq} \quad & \mathbb{P}\left[S \overset{T}{\longleftrightarrow} [0,43n)\times\{2n\}, \{0\}\times[4n,8n) \overset{T}{\longleftrightarrow} \{43n-1\}\times\mathbb{Z}\right] - \frac{12\cdot c^8}{10^7}\\
\overset{(FKG)}{\geq} \quad & \mathbb{P}[A_1]\cdot\mathbb{P}\left[\{0\}\times[4n,8n) \overset{T}{\longleftrightarrow} \{43n-1\}\times\mathbb{Z}\right] - \frac{12\cdot c^8}{10^7}\\
\geq \quad & \frac{c}{2}\left(c-\frac{c}{10}\right) - \frac{12\cdot c^8}{10^7} \geq \frac{c^2}{5}.
\end{aligned}$$

Now simplification yields

$$\mathbb{P}[A_1\cap A_2] \geq \frac{c^4}{100} \Rightarrow \mathbb{P}[A_1\cap A_2\cap A_3] \overset{(3.17)}{\geq} \frac{c^4}{100} - \frac{12\cdot c^4}{10^4} \geq \frac{c^4}{10^3}. \qquad \square$$

Step 5. We are ready to conclude. For a configuration ω, let $Q(\omega)$ be the set of vertices from T, which are connected to S by an open path in $\overline{[0,43n)\times[2n,\infty)}$.

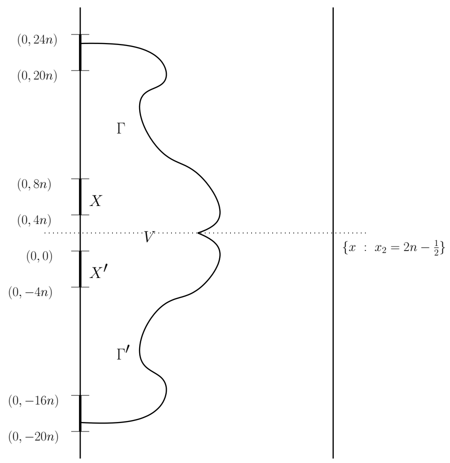

FIGURE 3.11: An illustration of Γ, Γ', and V for a configuration from the event $A_1\cap A_2\cap A_3$. Γ is the outer vertex boundary of the cluster of S in $\overline{[0,43n)\times[2n,+\infty)}$, Γ' is its mirror reflection with respect to the hyperplane $\{x : x_2 = 2n-\frac{1}{2}\}$, and V is the connected component of $T\setminus(\Gamma\cup\Gamma')$ containing the origin.

Let $\Gamma(\omega)$ be the outer vertex boundary of $\overline{Q(\omega)}$, and $\Gamma'(\omega)$ the mirror reflection of Γ with respect to the hyperplane $\{x \;:\; x_2 = 2n - \frac{1}{2}\}$.

We denote the connected component of $T \setminus (\Gamma \cup \Gamma')$ which contains 0 by V, which is finite for any $\omega \in A_1$. Let $X = \overline{\{0\} \times [4n, 8n)}$, and $X' = \overline{\{0\} \times [-4n, 0)}$. Note that X' is the mirror reflection of X with respect to the hyperplane $\{x \;:\; x_2 = 2n - \frac{1}{2}\}$. Moreover, if $\omega \in A_2 \cap A_3$, then both X and X' are contained in V. We consider an auxiliary probability space Ω' with configurations ω' and the same probability measure $\mathbb{P}$ on it, and compute :

$$\mathbb{P}\left[X\text{is connected to } X' \text{ in } T \text{ by an open path in } \omega'\right]$$

$$\geq \mathbb{P} \otimes \mathbb{P}\left[(\omega, \omega') \;:\; \begin{array}{c} \omega \in A_1 \cap A_2 \cap A_3, \\ X\text{is connected to } X' \text{ in } V(\omega) \text{ by an open path in } \omega' \end{array}\right]$$

$$\overset{(*)}{\geq} C_*^{-1} \cdot \mathbb{P} \otimes \mathbb{P}\left[(\omega, \omega') \;:\; \begin{array}{c} \omega \in A_1 \cap A_2 \cap A_3, \\ X \text{ is } \textit{not} \text{ connected to } X' \text{ in } V(\omega) \text{ by an open path in } \omega' \\ X\text{is connected to } \Gamma'(\omega) \text{ in } V(\omega) \text{ by an open path in } \omega', \\ X'\text{is connected to } \Gamma(\omega) \text{ in } V(\omega) \text{ by an open path in } \omega', \\ \text{there is no open path } \pi \text{ in } \omega' \text{ from } X \text{ in } V(\omega) \\ \text{so that the distance between } \overline{\pi} \text{ and } \Gamma(\omega) \text{ is} \leq 4, \\ \text{there is no open path } \pi' \text{ in } \omega' \text{ from } X' \text{ in } V(\omega) \\ \text{so that the distance between } \overline{\pi'} \text{ and } \Gamma'(\omega) \text{ is} \leq 4 \end{array}\right]$$

$$\geq C_*^{-1} \cdot \mathbb{E}_\omega\left[\mathbb{1}_{A_1 \cap A_2 \cap A_3}(\omega) \cdot \mathbb{P}_{\omega'}\left[\begin{array}{c} \text{Both } X \text{ and } X' \text{ are connected to } \Gamma'(\omega) \text{ in } V(\omega), \\ \text{each by an open path in } \omega' \end{array}\right]\right]$$

$$- C_*^{-1} \cdot \mathbb{E}_\omega\left[\mathbb{1}_{A_1 \cap A_2 \cap A_3}(\omega) \cdot \mathbb{P}_{\omega'}\left[\begin{array}{c} \text{there is an open path } \pi \text{ in } \omega' \text{ from } X \text{ in } V(\omega) \\ \text{so that the distance between } \overline{\pi} \text{ and } \Gamma(\omega) \text{ is} \leq 4, \\ \text{or} \\ \text{there is an open path } \pi' \text{ in } \omega' \text{ from } X' \text{ in } V(\omega) \\ \text{so that the distance between } \overline{\pi'} \text{ and } \Gamma'(\omega) \text{ is} \leq 4 \end{array}\right]\right]$$

$$- C_*^{-1} \cdot \mathbb{P}\left[X\text{is connected to } X' \text{ in } T \text{ by an open path in } \omega'\right]$$

$$\geq C_*^{-1}\left[\mathbb{E}_\omega\left[\mathbb{1}_{A_1 \cap A_2 \cap A_3}(\omega) \cdot \left[f(\omega)^2 - 2g(\omega)\right]\right] - \mathbb{P}\left[X\text{is connected to } X' \text{ in } T \text{ in } \omega'\right]\right].$$

Every path from X to Γ' in V and every path from X' to Γ have intersecting projections, and all the contender 'intersection points' to locally modify upon are sufficiently far away from the possibly 'rough-boundary' $\Gamma \cup \Gamma'$ to allow for a local modification successfully. This makes the shape V to be "glueing-friendly" under the event, and thus the inequality

($*$) follows from Lemma 3.4 and Remark 3.8. The last inequality comes from the FKG inequality and the definitions of event A_1 and functions f and g.

By the definition of events A_2 and A_3 and Proposition 3.15,

$$\mathbb{P}\left[X \overset{T}{\longleftrightarrow} X'\right] \geq c_* \cdot \left(\frac{c^4}{100} - 2 \cdot \frac{c^4}{1000}\right) \cdot \frac{c^4}{10^3}.$$

In particular, there exist $a < b$ such that

$$\mathbb{P}\left[X \overset{T_{ab}}{\longleftrightarrow} X'\right] \geq \frac{c_* \cdot c^8}{10^6}.$$

From this we conclude, as in the argument of Step 3, that $p(44n, 43n) \geq c'$ (or simply observe that the above inequality contradicts the assumption (3.12)). This completes the proof of Proposition 3.11. $\square$

As we already mentioned, there was a recent improvement of this result. Newman, Tassion and Wu [NTW15, Theorem 3.1] were successful in proving that $p(\lfloor \rho n \rfloor, n)$ is bounded away from both 0 and 1. They used a form of surgery that is slightly different from us. They also showed that the existence of an open circuit in an annulus has probability bounded away from 0 and 1 in all scales and were successful in glueing an open path to an open circuit. Variants of this type of glueing will be described by us later while proving the existence of IIC on slabs in Chapter 4.

3.5 RSW: High Probability Version

To prove Corollary 3.2, we would need to revisit our "glueing" techniques. We would need to prove that glueing two crossing events of probability close to 1 yields a new crossing event with probability close to 1 as well.

3.5.1 Glueing Revisited

We would state and prove, a result inspired from [DCST16, Fact 1, Fact 2] and similar in spirit with [NTW15, Theorem 3.7] in the form as below:

Lemma 3.16. *Let X_1, X_2, Y_1, and Y_2 be as in Lemma 3.5. For every $\epsilon > 0$, $\exists\ \delta > 0$ such that if $\mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} Y_1\right] \wedge \mathbb{P}\left[X_2 \overset{B(m,n)}{\longleftrightarrow} Y_2\right] > 1 - \delta$, then $\mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} X_2\right] > 1 - \epsilon$.*

Proof of Lemma 3.16. As before we define an order $<$ on self-avoiding paths from $\overline{X_1}$ to $\overline{Y_1}$ in $\overline{B(m,n)}$ and let γ_{min} be the minimal open path among them. We define E_1, E_2, X, and the neighborhoods B_z, B_z^i for any point $z \in \mathbb{S}_{k,d}$ as done in the proof of Lemma 3.5. Let us also define, for two vertices $u = (u_1, u_2, \ldots, u_d)$ and $v =$

$(v_1, v_2, \ldots, v_d)$ in $\mathbb{S}_{k,d}$, the distance of their projection as $dist_2(u,v) = |u_1 - v_1| \vee |u_2 - v_2|$ and the distance of projection of two sets $U, V \subset \mathbb{S}_{k,d}$ as $dist_2(U,V) = \min_{u \in U, v \in V} dist_2(u,v)$.

Finally we define $B_1(z) = \overline{z + [-1,1]^2 \times \{0\}^{d-2}}$ and the set $U(\omega)$ (slightly different from before):

$$U(\omega) = \left\{ z \in \gamma_{min} : \begin{array}{c} B_1(z) \text{ is connected to } X_2 \text{ in } \overline{B(m,n)} \text{ by an open path} \\ \beta \text{ such that } dist_2(\beta, \overline{\gamma_{min}}) = 1 \end{array} \right\},$$

for $\omega \in E_1 \cap E_2 \cap X^c$ (this path β is allowed to be singleton). We will split the event $J = E_1 \cap E_2 \cap X^c$ into $J_> = J \cap \{|U(\omega)| > t\}$ and $J_< = J \cap \{|U(\omega)| \leq t\}$ for some large integer t we will choose later. We will do two separate surgeries on these two sets.

For $\omega \in J_<$, we choose all such points $z \in U(\omega)$ and close every edge not in γ_{min} with at least one vertex in $B_1(z)$. This makes it impossible for X_2 to be connected to Y_2 anymore, giving us an"anti-gluing" map $f : J_< \to E_1 \cap {E_2}^c$. Both minimality of γ_{min} and the set $U(\omega)$ is preserved, making us able to identify at most t many neighborhoods where the surgery has been done. This gives :

$$\mathbb{P}[E_1 \cap E_2 \cap X^c \cap \{|U(\omega)| \leq t\}] \leq C_*^t \mathbb{P}[E_1 \cap {E_2}^c] \leq C_*^t \delta. \tag{3.19}$$

For the sub-event $J_>$, we will define a relation $\mathfrak{R} \subset J_> \times X$. The surgery is quite similar to the one used in Lemma 3.5. The key difference is that instead of picking one point of $U(\omega)$ and glueing immediately, we exploit the fact that all points of $U(\omega)$ are eligible for the new connection to $\overline{X_2}$. We need to be cautious so that it is possibly to identify correctly the neighborhood where the surgery has been done, and snce the relation would satisfy $|\{\omega' : (\omega, \omega') \in \mathfrak{R}\}| > t$ for every $\omega \in J_>$, Lemma 3.4 would yield:

$$\mathbb{P}[J \cap \{|U(\omega)| > t\}] \leq \frac{C_* \mathbb{P}[X]}{t} \leq \frac{C_*}{t}. \tag{3.20}$$

Coming back to the surgery, for any $z \in U(\omega)$,

- Mark the vertex by which γ_{min} enters "inner" neighborhood B_z^i for the first time by v_i and call the segment of γ_{min} upto v_i as γ_{min}^i. Similarly we mark the vertex by which γ_{min} leaves it for the last time as v_i and the segment of γ_{min} from v_o onwards as γ_{min}^o.
- Mark the vertex by which an open self-avoiding path β from X_2 first enters B_z^i as v_β.
- Find three neighboring vertices of z (say z_i, z_o and z_β) with $(z, z_o) \prec (z, z_\beta)$, following the guidelines:

(a) If $z = v_i$ or z is a neighbor of v_i, we take $z_i = v_i$.

(b) If $z = v_o$ or z is a neighbor of v_o, we take $z_o = v_0$.

(c) Otherwise we take z_i, z_o, z_β distinct from v_i, v_o, v_β.

We note three self avoiding paths γ_i, γ_o and γ_β, entirely in B^i_z, not using z or its aforementioned three neighbors which connects v_i to z_i, v_o to z_o and v_β to z_β respectively.

- Close all edges of B_z except the edges in $B_z \setminus B'_z$ which are in $\gamma^i_{min}, \gamma^o_{min}$ or β and open all the edges in paths $\gamma_i, \gamma_o, \gamma_\beta$ and three edges $(z, z_i), (z, z_o), (z, z_\beta)$.

The altered configuration ω' is a configuration from $E_1 \cap X$ and given an ω' in the range of $\mathfrak{R}$, by the same argument, we are able to identify γ_{min} and the unique vertex of it which is connected to X_2 without using the edges of γ_{min}.

Given ϵ we first choose integer t large enough to make RHS of (3.20) $< \epsilon/3$ and then choose small $\delta < \epsilon/6$ such that, having known ϵ and t, we can make RHS of (3.19) $< \epsilon/3$. This gives us $\mathbb{P}[X] \geq \mathbb{P}[E_1 \cap E_2] - \mathbb{P}[J] \geq 1 - 2\delta - 2\epsilon/3 \geq 1 - \epsilon$. □

This argument also holds true for non-regular shape (Let us recall Remark 3.8). Although the statement can be stated in a more generalized way, we choose to state it in a way that would suffice for us:

Corollary 3.17. *Let P_1, P_2 be two simple "glueing-regular" polygons with vertices from $\mathbb{Z}^2$ having regions P_{ij} (for $i, j \in \{1, 2\}$), which are disjoint connected subset of the interior vertex boundary of P_i. If any open path γ_1 connecting P_{11} to P_{12} in P_1 must intersect with any open path γ_2 connecting P_{21} to P_{22} in P_2, then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mathbb{P}[P_{11} \overset{P_1}{\longleftrightarrow} P_{12}] \wedge \mathbb{P}[P_{21} \overset{P_2}{\longleftrightarrow} P_{22}] \geq 1 - \delta \Rightarrow \mathbb{P}[P_{11} \overset{P_1 \cup P_2}{\longleftrightarrow} P_{22}] \geq 1 - \epsilon.$$

3.5.2 Proof of Corollary 3.2

One final requirement is the square root trick whose proof is elementary from FKG inequality.

Lemma 3.18. *(Square-root trick) Let $\mathfrak{E}_1, \ldots, \mathfrak{E}_k$ be increasing events, and $\mathfrak{E} := \bigcup_{i=1}^{k} \mathfrak{E}_i$. Then*

$$\max_{1 \leq i \leq k} \mathbb{P}[\mathfrak{E}_i] \geq 1 - (1 - \mathbb{P}[\mathfrak{E}])^{1/k}.$$

We would first prove that:

$$\lim_{n\to\infty} p(\lfloor \theta n\rfloor, n) = 1 \text{ for some } \theta > 1 \Rightarrow \lim_{n\to\infty} p(\lfloor \kappa n\rfloor, n) = 1 \text{ for all } \kappa > 0. \tag{3.21}$$

By monotonicity $p(\lfloor \kappa n\rfloor, n) \to 1$ for all $\kappa \leq \theta$. For $\kappa > \theta$, we would use the high-probability glueing introduced in Lemma 3.16. The strategy would be exactly similar to that in Proposition 3.10 (see Picture 3.4 again).

By Lemma 3.18, if the top-bottom crossing of the square has probability $1-\delta$, then with probability $1-\sqrt{\delta}$ the top is connected to the right half of bottom (making this event of high-probability whenever the probability of top-bottom crossing is also high). Now by using Corollary 3.17 twice, we can obtain $p(\lfloor(2\theta-1)n\rfloor, n) \to 1$. By using the same trick as many times as required and by virtue of monotonicity, one can prove $p(\lfloor \kappa n\rfloor, n) \to 1$ for any κ.

Thus our proof reduces to proving $\lim_{n\to\infty} p(\lfloor \kappa n\rfloor, n) = 1$ for some aspect ratio $\kappa > 1$ from $\lim_{n\to\infty} p(\lfloor \rho n\rfloor, n) = 1$ for some aspect ratio $\rho \in (0,1)$. We will do this by glueing translated copies of rectangles by 'circuit-like' structures (which exists with high probability as well).

Let us define $K_{m,n} := B(n+1,n)\setminus[(\frac{n-m}{2}, \frac{n-m}{2}) + B(m, m+1)\cup[n/2-1, n/2]\times[0, n/2]]$.

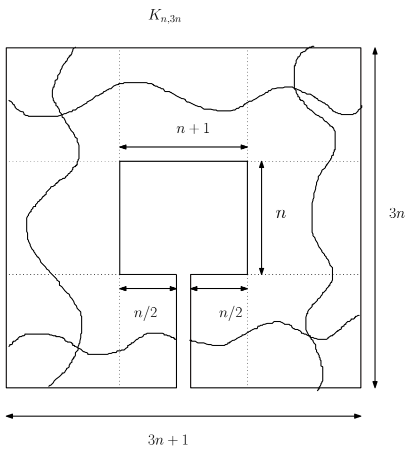

FIGURE 3.12: Key-hole $K_{n,3n}$

This "keyhole" shape is obtained by deleting edges across $[n/2-1, n/2]\times[0, n/2]$ from the annulus (see Figure 3.12). By Theorem 3.1 and (3.4), we can glue the open paths in five constituent rectangular boxes making up $\overline{K_{n,3n}}$ to get :

$$\mathbb{P}\left[\{3n/2\}\times[0,n] \overset{K_{n,3n}}{\longleftrightarrow} \{1+3n/2\}\times[0,n]\right] > c_K, \tag{3.22}$$

for some $c_K \in (0,1)$ and for all n. (This can also be proved by using [NTW15, Corollary 3.2.1], by the uniform lower bound of the probability of having a circuit in an annulus.)

Now we cover the right side of $B(\lceil \rho n\rceil, n)$ in $\lceil 1/\phi\rceil$ segments of length ϕn. For the simplicity of calculation, let us assume $\rho n/4$ and $\phi n/2$ are integers. By Lemma 3.18, $\exists y_\rho \in [0,n]$ such that

$$\mathbb{P}\left[L(\rho n, n) \overset{B(\rho n, n)}{\longleftrightarrow} \{n\}\times[y_\rho - \phi n/2, y_\rho + \phi n/2]\right] \geq 1 - (1-p(\rho n, n))^{1/\phi}).$$

Let us denote concentric keyholes by $K_k := (n - 3^k\phi n/2, y_\rho - 3^k\phi n/2) + K_{3^{k-1}\phi n, 3^k\phi n}$ for $k \in \{0, 1, \ldots, N = \lfloor \log_3 \frac{\rho}{4\phi} \rfloor\}$ and $K = (n - \phi n/2, y_\rho - \phi n/2) + K_{\phi n, \rho n/4}$. We can choose ϕ small enough such that

$$\mathbb{P}\left[\{n\} \times [y_\rho - n/4, y_\rho - \phi n] \overset{K}{\longleftrightarrow} \{n+1\} \times [y_\rho - n/4, y_\rho - \phi n]\right] > 1 - \epsilon. \tag{3.23}$$

(Since presence of 'broken circuit' in each concentric keyhole K_k is independent of each other, we choose N such that $(1 - c_K)^N < \epsilon$ and choose $\phi := \frac{\rho}{3^{N+2}}$.) Again, by glueing the paths in the event $\{L(\rho n, n) \overset{B(\rho n, n)}{\longleftrightarrow} \{n\} \times [y_\rho - \phi n/2, y_\rho + \phi n/2]\}$ and loop event mentioned in (3.23) (both of which has probability close to 1) we get the event

$$E_\rho = \{\{L(\rho n, n) \overset{B(\rho n, n) \cup A_{\phi, n}}{\longleftrightarrow} \{n+1\} \times [y_\rho - n/4, y_\rho - \phi n]\}\}$$

with probability going to 1 by virtue of Corollary 3.17 (see Figure 3.13).

Finally inside $\overline{(n+1, 0) + B(\lfloor \rho n \rfloor, n)}$ we have a path from right side to $\overline{\{n+1\} \times [y_\rho - \phi n/2, y_\rho + \phi n/2]}$ which by symmetry has probability going to 1 as well. We finally glue this with the open connection in E_ρ, and obtain, with probability going to 1, a left right crossing of the rectanglular box $\overline{(0, -\rho n/4) + B(2\rho n + 1, (1 + \rho/2)n)}$, giving us $\lim_{n \to \infty} p(n, \frac{2\rho}{1+\rho/2}) = 1$.

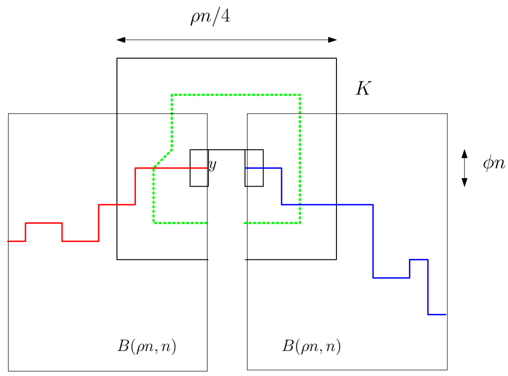

FIGURE 3.13: Glueing via fat annulus

We proved that the crossing of the rectangular box of aspect ratio $\frac{2\rho}{1+\rho/2}$ has high probability if that is true also for the rectangular box of aspect ratio ρ. To prove this for a rectangular box with aspect ratio $\kappa > 1$, we simply keep on repeating this procedure, (this works since the sequence $f(x), f(f(x)), \ldots, f^n(x)$ eventually crosses the value 1 irrespective of initial value $x \in (0, 1)$ for $f(x) = \frac{2x}{1+x/2}$) and by (3.21), that suffices.

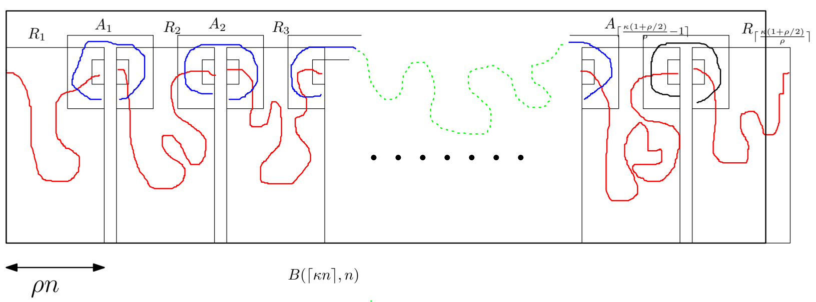

FIGURE 3.14: Glueing a series of boxes

Another alternate method would be to put $\lceil \frac{\kappa(1+\rho/2)}{\rho} \rceil$ many copies of $\overline{B(\lfloor \rho n \rfloor, n)}$ in series, consecutive ones being 1 distance apart from each other and connected via "keyhole" and then to glue them step by step. Instead of going into too much details, let us present the idea by Figure 3.14. $\square$

3.6 Associated Results

In this section, we will describe two other corollaries which stems out from Theorem 3.1. These two results are interesting in their own right. As we have stated, glueing paths in slabs are faced with some challenges. Although we now know that RSW theorem holds true, there are still quite a lot of questions that are not addressed, although the answers to them in plane are quite straightforward. For example, for planar lattices, when we know there is a top-bottom crossing and a left-right crossing in a rectangle, it instantly gives us a cluster spanning each of the 4 sides of the rectangle. But existence of such a cluster is not obvious in slabs. We will prove the existence of such a cluster as well with probability bounded uniformly from below.

Similar to $L(m,n)$ and $R(m,n)$ let us also define $T(m,n)$, $D(m,n)$ as top and bottom surfaces of $\overline{B(m,n)}$. We introduce the event $A_4(m,n)$ of having an open cluster inside $\overline{B(m,n)}$ connected to each of the four surfaces $L(m,n), R(m,n), T(m,n)$ and $D(m,n)$.

Corollary 3.19. *For every $\rho > 0$, there exists $x_\rho > 0$ such that $\mathbb{P}[A_4(\lfloor \rho n \rfloor, n)] \geq c_\rho$ $\forall n \in \mathbb{N}$.*

Before we begin with the proof, let us highlight the key challenge of glueing a top-bottom crossing with a left-right crossing into a cluster spanning all four sides. Since the proof of this result relies on simple yet careful circumnavigation of the specific challenge, let us describe at first the naive attempt of glueing which does not work.

Let us define $TD(m,n) := \{T(m,n) \overset{B(m,n)}{\longleftrightarrow} D(m,n)\}$ and we take any configuration ω from the event $X(n,\rho) = LR(\lfloor \rho n \rfloor, n) \cap TD(\lfloor \rho n \rfloor, n) \cap A_4(\lfloor \rho n \rfloor, n)^c$. Let us look at the minimal left-right path $\gamma_l(\omega)$ and the minimal top-bottom path (defined in a similar way) $\gamma_t(\omega)$. If it happens that, for example, $\gamma_l(\omega) \overset{B(\lfloor \rho n \rfloor, n)}{\longleftrightarrow} T(\lfloor \rho n \rfloor, n)$, it will be challenging to glue these two paths (at one of the points where their projections intersect). This is due to the fact that the conventional "glueing" might alter γ_t significantly because the path by which γ_l is connected to the top side might become the part of the new $\gamma_t(\omega')$. Same thing will happen with γ_l if γ_t is connected to left side. This will create problems because given a changed configuration, we would not be able to precisely point out at which region the surgery has been done (see Figure 3.15).

Although our simplistic attempt fails, it provides an important fact– namely, for any configuration in $\{\gamma_l(\omega) \overset{B(\lfloor \rho n \rfloor, n)}{\longleftrightarrow} T(\lfloor \rho n \rfloor, n)\} \cap \{\gamma_t(\omega) \overset{B(\lfloor \rho n \rfloor, n)}{\not\leftrightarrow} L(\lfloor \rho n \rfloor, n)\}$, we can do the

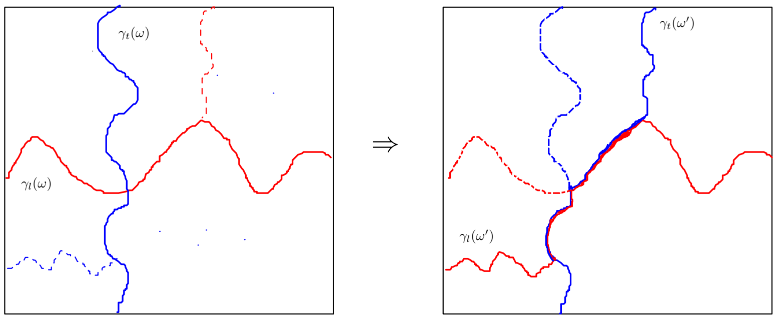

FIGURE 3.15: Issues with direct glueing

glueing on γ_l. Although the γ_t might change, but γ_l would not (outside the small box naturally, where the surgery has been done). Moreover, we can find the precise point at which the surgery has been done by identifying the unique vertex from which there is a path to $D(\lfloor \rho n \rfloor, n)$ $\big($since γ_l could not be connected to the bottom as well initially, otherwise $\omega \in A_4(\lfloor \rho n \rfloor, n)\big)$. We will capitalize on this fact in the proof repeatedly.

Proof. By FKG inequality and RSW theorem, we have

$$\mathbb{P}[LR(\lfloor \rho n \rfloor, n) \cap TD(\lfloor \rho n \rfloor, n)] \geq c_\rho c_{1/\rho}$$

As done previously, we will take a configuration from $X(n, \rho)$ and after surgery the changed configuration ω' would be in $A_4 = A_4(\lfloor \rho n \rfloor, n)$. We need some notations for splitting up the event. We have already defined γ_l and γ_t as minimal left-right and top-bottom path respectively. Similarly we would have γ_r and γ_b as the minimal right-left and bottom-top paths. Our definition does not dictate that, for example, γ_t and γ_b need be same or even intersect. Now both γ_t and γ_b can be connected to either with $L(\lfloor \rho n \rfloor, n)$ or $R(\lfloor \rho n \rfloor, n)$ or neither of them. Similarly each of γ_l, γ_r might be connected to either top side or bottom or none of them. We will split $X(n, \rho)$ over 81 disjoint sub-events X^{ij}_{kl} when $i, j \in \{T, D, N\}$ and $k, l \in \{L, R, N\}$. Here i, j, k, l indicates which side is connected with $\gamma_l, \gamma_r, \gamma_t$ and γ_b respectively other than the two sides they are already connecting, N indicating with neither of the rest two sides. For example, the sub-event X^{NT}_{LR} indicates that γ_r, γ_t and γ_b are connected with top, left and right sides respectively whereas γ_l is connected to neither top or bottom side. We will now group these events suitably and will do surgeries accordingly.

We divide the cases in three groups depending on the value of, say i. The first case will be where $i = N$. The cases $i = T$ and $i = D$ can be treated similarly by switching l and k and hence without loss of generality we will only describe about $i = T$ in Case 2.

Case 1: We divide this into two sub-cases depending on the value of k. One will be when $k \neq L$ and the other being $k = L$.

Subcase (1a): If $k \neq L$, we would look at the set $U(\omega)$ where the projections of γ_l and γ_t meet and pick a point z from the set. Let $B_z := \overline{z + [-3,3]^2 \times \{0\}^{d-2}}$, $B_z^i := \overline{z + [-2,2]^2 \times \{0\}^{d-2}}$ and $\prec$ be the order on edges $\{e : |e| = 1\}$ in $\mathbb{Z}^d$. We will do the following:

- Mark the first entries to and last exits from B_z^i for γ_l and γ_t by v_l^i, v_l^o, v_t^i and v_t^o respectively. Call the segments of γ_l from the beginning upto v_l^i as γ_l^i and from v_l^o to the end as γ_l^o and similarly define γ_t^i and γ_t^o.
- If one of v_l^i or v_l^o lies in $\overline{\{z\}}$, set $v_l = v_l^i$ or $v_l = v_l^o$ accordingly. (Notice that at most one of these can occur).
- If one of v_t^i or v_t^o lies in $\overline{\{z\}}$, set $v_t = v_t^i$ or $v_t = v_t^o$ accordingly. (Notice that at most one of these can occur).
- Otherwise fix two vertices v_l and v_t in $\overline{\{z\}}$, and one of the shortest paths ρ entirely inside $\overline{\{z\}}$ connecting them (for $d = 3$, this is an unique line segment).
- Find two non intersecting open paths π_l from v_l^i to v_l^o via v_l and π_t from v_t^i to v_t^o via v_t inside $\overline{B_z}$, neither of them sharing any vertex with ρ (other than v_l and v_t, respectively) such that:
  - The edge e_l emerging from v_l in π_l satisfies $e_l \prec \rho_l$ where ρ_l is the first edge emerging from v_l in ρ.
  - The edge e_t emerging from v_t in π_t satisfies $e_t \prec \rho_t$ where ρ_t is the first edge emerging from v_t in ρ.
- Close all the edges in $\overline{B_z}$ except those in $B_z \setminus B_z^i$ which are in γ_l^i, γ_l^o, γ_t^i or γ_t^o and open all the edges in π_l, π_t and ρ.

The resultant configuration ω' is in A_4 and by the nature of the surgery $\gamma_t(\omega')$ matches with $\gamma_t(\omega)$ and enters it through v_t^i, then follows π_t upto v_t, chooses the smaller edge e_t over (v_t, v_l), continues in π_t to v_t^o and again matches with $\gamma_t(\omega)$ after the last exit from $\overline{B_z}$. Same holds true for γ_l as well. Since the only possible new connection to the left side has to occur after the intersection of these two, minimality of γ_l is not challenged and we can easily spot the "surgery-box" as the point where $\gamma_t'(\omega')$ and $\gamma_l'(\omega')$ meet.

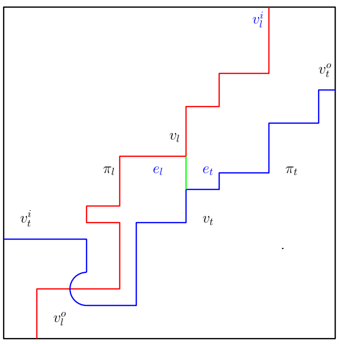

FIGURE 3.16: Glueing four arms

Subcase (1b): This case is similar, but there is a subtle difference in the surgery done. Since $k = L$, if we follow through as the previous case, the minimality of γ_l in the changed configuration would not go unchallenged as before. But γ_t would not face such a problem, and after the same surgery (although the restrictions in the previous surgery can be relaxed slightly, we do not state them for the sake of simplicity), we will be able to detect the "surgery-box" as the unique vertex on γ_t which is connected to the left side without using any of the edges of γ_t. The uniqueness is again implicit from the fact that $\gamma_t(\omega)$ is not connected to the right side and the only connection comes through the intersection point with the old γ_l, which we might not identify completely now.

Case 2: Case 1 can be generalized for any one of the four variables instead of i, and hence without loss of generality, we will assume that none of the variable takes the value N here along with the existing restriction $i = T$. Depending on whether $l = R$ or $l = L$, we again divide this into two sub-cases- each being akin to the respective sub-cases of the previous case with γ_l and γ_b (instead of γ_t). The arguments remain exactly the same.

After taking into account all the cases, we have

$$\mathbb{P}[LR(\lfloor \rho n \rfloor, n) \cap TD(\lfloor \rho n \rfloor, n) \cap A_4(\lfloor \rho n \rfloor, n)^c] \leq 81C_* \mathbb{P}[A_4(\lfloor \rho n \rfloor, n)]$$

which finally yields $\mathbb{P}[A_4(\lfloor \rho n \rfloor, n)] \geq \frac{c_\rho c_{1/\rho}}{(1+81C_*)} := x_\rho$. □

The result above can be again generalized for some boundary segments instead of four sides specifically, i.e. in the form :

Corollary 3.20. *Let X_1, X_2, Y_1, and Y_2 be disjoint connected subsets on the four boundary surfaces $T(m,n), R(m,n), D(m,n)$ and $L(m,n)$ of $\overline{B(m,n)}$ respectively. Then*

$$\begin{aligned} \mathbb{P} & \left[\exists \text{ an open cluster inside } \overline{B(m,n)} \text{ connected to each } X_1, X_2, Y_1 \text{ and } Y_2\right] \\ \geq & \ \frac{1}{1+81C_*} \cdot \mathbb{P}\left[X_1 \overset{B(m,n)}{\longleftrightarrow} Y_1, X_2 \overset{B(m,n)}{\longleftrightarrow} Y_2\right] \end{aligned} \tag{3.24}$$

The following result provides a lower bound for a certain crossing probability of annulus. For positive integers $m \leq n$, let $B'(n) = [-n,n]^2 \times \{0,\ldots,k\}^{d-2}$ be the box of side length $2n$ in $\mathbb{S}_{k,d}$ centered at 0, and $S(n)$ be $B'(n) \setminus B'(n-1)$, the inner boundary of $B'(n)$. Also let $\mathrm{An}(m,n) = B'(n) \setminus B'(m-1)$ be the annulus of side lengths $2m$ and $2n$ and $Z(m,n) = \{S(m) \longleftrightarrow S(n)\}$.

Corollary 3.21. $\limsup_{n\to\infty} \mathbb{P}[Z_{n,\rho n}] \geq 1/\sqrt{\rho}$ *for any $\rho > 1$.*

Proof. We will prove the result for $\rho = 2$ since the proof for any other ρ will be identical. Let us denote $\limsup_{n\to\infty} \mathbb{P}[Z_{n,2n}]$ by c_0. For any $m < n$, we have

$$\mathbb{P}[\bigcup_{k=0}^{\frac{n}{m}-1}\{\overline{z'_k + B'(m)} \text{ is connected by two disjoint paths to } \overline{z'_k + B'(n)}\}] \geq p(2n,2n) \geq c,$$

where $z'_k = (n, 2km+m)$, since we know, by Theorem 3.1, $p(n,n) \geq c$ for some $c > 0$. The left-hand side here can be bounded from above by $\lceil (n/m) \rceil \mathbb{P}[S(m) \longleftrightarrow S(n)]^2$, using union bound first and then by virtue of BK inequality and translation invariance of $\mathbb{P}$, which culminates to:

$$\mathbb{P}[S(m) \longleftrightarrow S(n)] \geq \sqrt{cm/n}. \tag{3.25}$$

Now we will bound this quantity from above by existence of open paths in each annulus $A(i) = B'(2^i m) \setminus B'(2^{i-1}m)$ for $i \in \{1, 2, \ldots \lfloor \log_2(n/m) \rfloor\}$. Fix any $\epsilon > 0$, for m large enough we have:

$$\sqrt{cm/n} \leq \mathbb{P}[S(m) \longleftrightarrow S(n)] \leq \prod_{i=1}^{\lfloor \log_2(n/m) \rfloor} \mathbb{P}[Z_{2^{i-1},2^i}] \leq (c_0+\epsilon)^{\lfloor \log_2(n/m) \rfloor} \leq (2m/n)^{\log_{1/2}(c_0+\epsilon)}.$$

Since this holds true for arbitrarily large n, we must have $\log_{1/2}(c_0+\epsilon) \leq 1/2$, otherwise the reverse would hold true for large enough n. So $c_0 + \epsilon \geq 1/\sqrt{2}$ for any choice of ϵ and that completes our proof for $\rho = 2$. □

Remark 3.22. Only two prerequisites yield us the Corollary 3.21. First one is that $\limsup_{n\to\infty} \mathbb{P}[Z(n, \rho n)] < 1$ holds for some $\rho > 1$ and second one is $\liminf_{n\to\infty} p(n,n) > 0$ holds. The dimension or quasi-planarity is not used in the following proof, and as a result, when these results hold true (for any dimension), so does this corollary.

Moreover, we can do away with the assumption that $\liminf_{n\to\infty} p(n,n) > 0$ since we know $\liminf_{n\to\infty} p(\lfloor \alpha n \rfloor, n) > 0$ fo $\alpha < 1$. If we substitute squares in the proof with rectangles of aspect ratio α, we can prove the same result, only altering the other assumption to $\limsup_{n\to\infty} \mathbb{P}[Z(n, \frac{\rho}{\alpha} n)] < 1$ for some $\alpha < 1$ instead.

Chapter 4

Incipient Infinite Cluster and Quasi-multiplicativity of connections

4.1 Introduction

We have discussed in Chapter 2 that Kesten [K86a] gave a first mathematically rigorous construction of an incipient infinite cluster (IIC) for Bernoulli critical percolation on $\mathbb{Z}^2$. This was obtained in two ways. The first way was to condition on an open path from the origin to the boundary of a large box at criticality and increasing the size of the box to infinity. The second way was to condition on the origin being in an infinite open cluster for supercritical percolation (say with parameter p) and letting $p \searrow p_c$, where p_c is the critical threshold for $\mathbb{Z}^2$. Both of these probability measures were shown to exist and coincide. This measure (known as IIC measure) is supported on the configurations with an infinite open cluster at the origin.

Later, versions of the incipient infinite cluster were shown to exist on $\mathbb{Z}^d$ with sufficiently high dimension [HJ04, HHH14a], but the tools used are completely different. In fact, it is still an open problem to show the existence of either of the definitions of Kesten's IIC measure on $\mathbb{Z}^d$ for $d \geq 3$ (for a partial progress in high-dimensions see [HHH14a, Theorem 1.2]).

In this chapter, we will first prove the existence of IIC measure on a general class of infinite connected bounded degrees graphs whenever they satisfy two prerequisite criteria. One of them is known to be true for $\mathbb{Z}^d$ and $\mathbb{S}_{k,d}$, and the other one is expected to be true for low dimension $d < 6$. More importantly we will then prove that the slabs

$\mathbb{S}_{k,d} = \mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ indeed satisfy the second criteria and thus Kesten's IIC is well-defined for slabs.

4.2 Notation and Results

Let G be an infinite connected bounded degrees graph with a vertex set V. Let ρ be the graph metric on V, and define for $v \in V$ and positive integers $m \leq n$,

$$Q(v,n) = \{x \in V \ : \ \rho(v,x) \leq n\}, \quad S(v,n) = \{x \in V \ : \ \rho(v,x) = n\},$$

$$A(v,m,n) = Q(v,n) \setminus Q(v,m-1).$$

Consider Bernoulli bond percolation on G with parameter $p \in [0,1]$ and denote the corresponding probability measure by $\mathbb{P}_p$. The open cluster of $v \in V$ is denoted by $C(v)$. Let p_c be the critical threshold for percolation, i.e., for $v \in V$,

$$p_c = \inf \{p : \mathbb{P}_p[|C(v)| = \infty] > 0\}.$$

For $x, y \in V$ and $X, Y, Z \subset V$, we write $x \stackrel{Z}{\longleftrightarrow} y$ if there is a nearest neighbour path of open edges from x to y such that all its vertices are in Z, $x \stackrel{Z}{\longleftrightarrow} Y$ if there exist $y \in Y$ such that $x \stackrel{Z}{\longleftrightarrow} y$ and $X \stackrel{Z}{\longleftrightarrow} Y$, if there exist $x \in X$ such that $x \stackrel{Z}{\longleftrightarrow} Y$. If $Z = V$, we omit Z from the notation. We use $\not\leftrightarrow$ instead of $\longleftrightarrow$ to denote complements of the respective events.

We are interested in the existence and equality of the limits

$$\lim_{n \to \infty} \mathbb{P}_{p_c}[E \mid w \longleftrightarrow S(w,n)] \quad \text{and} \quad \lim_{p \searrow p_c} \mathbb{P}_p[E \mid |C(w)| = \infty], \tag{4.1}$$

where E is a cylinder event. The question is highly non-trivial if $\mathbb{P}_{p_c}[|C(w)| = \infty] = 0$. The seminal result of Kesten [K86a, Theorem (3)] states that if G is from a class of two dimensional graphs, such as $\mathbb{Z}^2$, then the above two limits exist and have the same value $\nu_{G,w}(E)$. By Kolmogorov's extension theorem, $\nu_{G,w}$ extends uniquely to a probability measure on configurations of edges, which is often called *Kesten's incipient infinite cluster* measure. It is immediate that $\nu_{G,w}[|C(w)| = \infty] = 1$. Kesten's argument is based on the existence of an infinite collection of open circuits around w in disjoint annuli and the properties that

(a) each path from w to infinity intersects every such circuit, and

(b) by conditioning on the innermost open circuit in an annulus, the occupancy configuration in the region not surrounded by the circuit is still an independent Bernoulli percolation.

These properties are no longer valid when one considers higher dimensional lattices. A partial progress, as we have mentioned earlier, has been recently made in sufficiently high dimensions by Heydenreich, van der Hofstad and Hulshof [HHH14a, Theorem 1.2], who showed using lace expansions the existence of the first limit in (4.1) under the assumption that $n^2\,\mathbb{P}_{p_c}[0 \longleftrightarrow S(0,n)]$ converges. Concerning low dimensional lattices, almost nothing is known there about critical and near critical percolation, and the existence of Kesten's IIC seems particularly hard to show. Several other constructions of incipient infinite clusters are obtained by Járai [J03] for planar lattices and van der Hofstad and Járai [HJ04] for high dimensional lattices.

The main result of this chapter is the existence and the equality of the two limits in (4.1) for graphs satisfying two assumptions: (A1) uniqueness of the infinite open cluster and (A2) quasi-multiplicativity of crossing probabilities. While (A1) is satisfied by many amenable graphs, most notably $\mathbb{Z}^d$, (A2) can be expected only in low dimensional graphs. For instance, we argue below that (A2) holds for $\mathbb{Z}^d$ if and only if $d < 6$. In our second result, we prove that (A2) is satisfied by slabs $\mathbb{S}_{k,d}$ ($d \geq 2$, $k \geq 0$), thus showing for these graphs the existence and equality of the limits in (4.1). We now state the assumptions and the main result, and then comment more on the assumptions.

(A1) (Uniqueness of the infinite open cluster) For any $p \in [0,1]$ there exists almost surely at most one infinite open cluster.

(A2) (Quasi-multiplicativity of crossing probabilities) Let $v \in V$ and $\delta > 0$. There exists $c_* > 0$ such that for any $p \in [p_c, p_c+\delta]$, integer $m > 0$, a finite connected set $Z \subset V$ such that $Z \supseteq A(v,m,4m)$, and sets $X \subset Z \cap Q(v,m)$ and $Y \subset Z \setminus Q(v,4m)$,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y] \geq c_* \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} S(v,2m)] \cdot \mathbb{P}_p[Y \overset{Z}{\longleftrightarrow} S(v,2m)]. \tag{4.2}$$

Theorem 4.1. *Assume that the graph G satisfies the assumptions (A1) and (A2) for some choice of $v \in V$ and $\delta > 0$. Then, for any cylinder event E, the two limits in (4.1) exist and have the same value.*

If the assumptions (A1) and (A2) are satisfied at $p = p_c$, then the first limit in (4.1) exists.

Before we discuss the strategy of the proof, let us comment on the assumptions.

4.2.1 Comments on (A1):

1. (A1) is satisfied by many sufficiently regular (e.g., vertex transitive) amenable graphs, most notably lattices $\mathbb{Z}^d$ and slabs $\mathbb{Z}^2 \times \{0,\dots,k\}^{d-2}$ ($d \geq 2$, $k \geq 0$), see, e.g., [BS96].

2. (A1) is equivalent to the assumption that for some $\delta > 0$ there exists at most one infinite open cluster for any fixed $p \in [p_c, p_c + \delta]$. Indeed, if for a given p the infinite open cluster is unique almost surely, then the same holds for any $p' > p$, see, e.g., [HP99, S99].

3. For $v \in V$ and $m \le n$, let $E_1(v,m,n) = \{S(v,m) \longleftrightarrow S(v,n)\}$ and $E_2(v,m,n)$ the event that in the annulus $A(v,m,n)$ there are at least two disjoint open crossing clusters.

   Assumption (A1) is equivalent to the following one, which will be used in the proof of Theorem 4.1: For any $v \in V$, $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $n > 4m$ such that

$$\sup_{p \in [0,1]} \mathbb{P}_p \left[E_2(v,m,n) \right] < \varepsilon \tag{4.3}$$

   or, equivalently,

$$\sup_{p \in [0,1]} \mathbb{P}_p \left[E_2(v,m,n) \mid E_1(v,m,n) \right] < \varepsilon. \tag{4.4}$$

   The equivalence of the claims (4.3) and (4.4) follows from the inequalities

$$\mathbb{P}_p \left[E_2(v,m,n) \right] \le \mathbb{P}_p \left[E_2(v,m,n) \mid E_1(v,m,n) \right] \le \mathbb{P}_p \left[E_2(v,m,n) \right]^{\frac{1}{2}},$$

   where the second one is a consequence of the BK inequality.

   It is elementary to see that (4.3) implies (A1). On the other hand, if (4.3) does not hold, then there exist $v_0 \in V$, $\varepsilon_0 > 0$ and $m_0 \in \mathbb{N}$ such that for all $n > 4m_0$, $\sup_{p \in [0,1]} \mathbb{P}_p \left[E_2(v_0, m_0, n) \right] \ge \varepsilon_0$. The function $\mathbb{P}_p \left[E_2(v_0, m_0, n) \right]$ is continuous in $p \in [0,1]$ and monotone decreasing in n. Thus, there exists $p_0 \in [0,1]$ such that $\mathbb{P}_{p_0} \left[E_2(v_0, m_0, n) \right] \ge \varepsilon_0$ for all $n > 4m_0$. By passing to the limit as $n \to \infty$, we conclude that for $p = p_0$, with positive probability there exist at least two infinite open clusters and (A1) does not hold.

4.2.2 Comments on (A2):

4. It follows from the Russo-Seymour-Welsh Theorem [R78, SW78] that (A2) holds for two dimensional graphs, such as $\mathbb{Z}^2$, considered by Kesten in [K86a]. Russo-Seymour-Welsh ideas have been recently extended to slabs in [NTW15, BS15], after the absence of percolation at criticality in slabs was proved by Duminil-Copin, Sidoravicius and Tassion [DCST16]. In Lemma 4.4 of the present paper we prove that (A2) is fulfilled by slabs $\mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}$ ($d \ge 2$, $k \ge 0$), thus verifying the existence and equality of the limits (4.1) for slabs.

5. We believe that assumption (A2) holds for lattices $\mathbb{Z}^d$ if $d < 6$, but does not hold if $d > 6$. Dimension $d_c = 6$ is called the *upper critical dimension* above which the

percolation phase transition should be described by mean-field theory, see, e.g., [CC87]. This was rigorously confirmed in sufficiently high dimensions by Hara and Slade [HS90, H08].

It is easy to see that the mean-field behavior excludes (A2). Indeed, it is believed that above d_c, the two point function decays as

$$\mathbb{P}_{p_c}[x \longleftrightarrow y] \asymp (1+\rho(x,y))^{2-d}.$$

(Here $f(z) \asymp g(z)$ if for some c, $cf(z) \leq g(z) \leq c^{-1}f(z)$ for all z.) Hara [H08] proved it rigorously in sufficiently high dimensions. Given this asymptotics, Aizenman showed in [A97, Theorem 4(2)] that for all $m(n) \leq n$ such that $\frac{m(n)}{n^{2/(d-4)}} \to \infty$,

$$\mathbb{P}_{p_c}\left[S(0,m(n)) \longleftrightarrow S(0,n)\right] \to 1, \quad \text{as } n \to \infty,$$

and Kozma and Nachmias [KN11] that $\mathbb{P}_{p_c}\left[0 \longleftrightarrow S(0,n)\right] \asymp n^{-2}$. Thus, the inequality

$$\mathbb{P}_{p_c}[0 \longleftrightarrow S(0,n)] \geq c\,\mathbb{P}_{p_c}[0 \longleftrightarrow S(0,m(n))]\,\mathbb{P}_{p_c}[S(0,m(n)) \longleftrightarrow S(0,n)]$$

cannot hold for large n.

The situation below d_c is much more subtle. With the exception of $d=2$, where planarity helps enormously, the (near-)critical behavior below d_c is widely unknown. Let us nevertheless give a few words about why we think (A2) should hold below d_c. It is believed that the number of clusters crossing any annulus $A(0,m,2m)$ is bounded uniformly in m if $d<d_c$ and grows at $p=p_c$ like m^{d-6} above d_c, with log-correction for $d=d_c$, and this dichotomy is intimately linked to the transition at d_c from the hyperscaling to the mean-field; see [C85, BCKS99]. Thus, it would be not unreasonable to expect that below d_c,

$$\mathbb{P}_p[\exists!\,\text{crossing cluster of } A(0,m,2m) \mid X \overset{Z}{\longleftrightarrow} S(0,2m),\, Y \overset{Z}{\longleftrightarrow} S(0,m)] \geq c > 0,$$

which is enough to establish (A2). We are not able to prove it yet or give a simpler sufficient condition for it. It would already be very nice if, for instance, (A2) was derived from the assumption that $\mathbb{P}_p[\exists!\,\text{crossing cluster of } A(0,m,2m)] \geq c$, or from the assumptions of [BCKS99].

4.2.3 Sketch of proof for Theorem 4.1

We finish the introduction with a brief description of the proof of Theorem 4.1. Our proof follows the general scheme proposed by Kesten in [K86a] by attempting to decouple the configuration near w from infinity on multiple scales. The implementations are however rather different.

- Using (4.4) we identify a sufficiently fast growing sequence N_i such that given $w \longleftrightarrow S(w,n)$, the probability that the annulus $A(w,N_i,N_{i+1}) \subset Q(w,n)$ contains a unique crossing cluster is asymptotically close to 1; see (4.6).
- Next, let an annulus $A(w,N_i,N_{i+1})$ contain a unique crossing cluster. We explore all the open clusters in this annulus that intersect the interior boundary $S(w,N_i)$, call their union $\mathcal{C}_i$, and let $\mathcal{D}_i$ be the subset of $S(w,N_{i+1}+1)$ of vertices connected by an open edge to $\mathcal{C}_i$; see (4.7).
- Then, the configuration outside $\mathcal{C}_i$ is distributed as the original independent percolation and every vertex from $\mathcal{D}_i$ is connected by an edge to the same (crossing) cluster from $\mathcal{C}_i$. Thus, $w \longleftrightarrow S(w,n)$ if and only if
  (a) w is connected to $\mathcal{D}_i$ (this event only depends on the edges intersecting $S(w,N_i) \cup \mathcal{C}_i$),
  (b) $\mathcal{D}_i$ is connected to $S(w,n)$ outside $\mathcal{C}_i$ (this only depends on the edges outside $\mathcal{C}_i$).
- This allows to factorize $\mathbb{P}_p[E,\, w \longleftrightarrow S(w,n)]$; see (4.8). The rest of the proof is essentially the same as that of Kesten [K86a]. We repeat the described factorization on several scales, obtaining in (4.10) an approximation of $\mathbb{P}_p[E|w \longleftrightarrow S(w,n)]$ in terms of products of positive matrices.
- Finally, we use (A2) to prove that the matrix operators are uniformly contracting, which is enough to conclude the proof; see (4.11) and the text below.

4.3 Proof of Theorem 4.1

We will prove the first claim of the theorem. The proof of the second one follows from the proof below by replacing everywhere p by p_c. The general outline of the proof is the same as the original one of Kesten [K86a, Theorem (3)], but the choice of scales and the decoupling are done differently.

First of all, it suffices to prove that for any $w \in V$ and a cylinder event E,

$$\mathbb{P}_p[E|w \longleftrightarrow S(w,n)] \text{ converges to some } \nu_p(E) \textit{ uniformly} \text{ on } [p_c, p_c+\delta] \text{ for some } \delta > 0. \tag{4.5}$$

Indeed, (4.5) implies the existence of the first limit in (4.1) and that $\nu_p(E)$ is continuous. Since for any $p > p_c$, $\nu_p(E) = \mathbb{P}_p[E \mid |C(w)| = \infty]$, the existence of the second limit in (4.1) and its equality to the first one follows from the continuity of $\nu_p(E)$.

Actually, by the inclusion-exclusion formula, it suffices to prove (4.5) for all events E of the form {edges $e_1, \ldots, e_k$ are open}. Although our proof could be implemented for any cylinder event E, calculations are neater for increasing events.

Fix $w \in V$ and an increasing event E. Also fix $v \in V$ and $\delta > 0$ for which the assumption (A2) is satisfied. Consider a sequence of scales N_i such that $N_{i+1} > 4N_i$ for all i, $Q(v, N_0)$ contains w and the states of its edges determine E. We will write $B_i = Q(v, N_i)$, $S_i = S(v, N_i)$ and $A_i = A(v, N_i, N_{i+1})$. Let F_i be the event that there exists a unique open crossing cluster in A_i. Define

$$\varepsilon_i = \sup_{p\in[p_c,p_c+\delta]} \mathbb{P}_p\left[F_i^c \mid S_i \longleftrightarrow S_{i+1}\right].$$

By (4.4), we can choose the scales N_i so that $\varepsilon_i \to 0$ as $i \to \infty$.

We first note that for $n > N_{i+1} + N_0$,

$$\mathbb{P}_p[w \longleftrightarrow S(w,n), F_i^c] \le c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p[w \longleftrightarrow S(w,n)], \tag{4.6}$$

where c_* is the constant in the assumption (A2). Indeed, by independence,

$$\begin{aligned}
\mathbb{P}_p[w \longleftrightarrow S(w,n), F_i^c] &\le \mathbb{P}_p[w \longleftrightarrow S_i] \cdot \mathbb{P}_p[S_i \longleftrightarrow S_{i+1}, F_i^c] \cdot \mathbb{P}_p[S_{i+1} \longleftrightarrow S(w,n)] \\
&\le \varepsilon_i \cdot \mathbb{P}_p[w \longleftrightarrow S_i] \cdot \mathbb{P}_p[S_i \longleftrightarrow S_{i+1}] \cdot \mathbb{P}_p[S_{i+1} \longleftrightarrow S(w,n)] \\
&\le c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p\left[w \longleftrightarrow S(w,n)\right],
\end{aligned}$$

where the last inequality follows from the assumption (A2).

We begin to describe the main decomposition step. Consider the random sets

$$\begin{aligned}
\mathcal{C}_i &= \left\{x \in Q(v, N_{i+1}) \;:\; x \overset{Q(v,N_{i+1})}{\longleftrightarrow} Q(v, N_i)\right\}, \\
\mathcal{D}_i &= \{x \in S(v, N_{i+1}+1) \;:\; \exists\, y \in \mathcal{C}_i, \text{ a neighbour of } x, \text{ such that edge } \langle x, y\rangle \text{ is open}\}.
\end{aligned} \tag{4.7}$$

Note that $\mathcal{C}_i$ contains $Q(v, N_i)$, the event $\{\mathcal{C}_i = U\}$ depends only on the states of edges in $Q(v, N_{i+1})$ with at least one end-vertex in U, and either $\{\mathcal{C}_i = U\} \subset F_i$ or $\{\mathcal{C}_i = U\} \cap F_i = \emptyset$. Also note that the event $\{\mathcal{C}_i = U, \mathcal{D}_i = R\}$ depends only on the states of edges in $Q(v, N_{i+1}+1)$ with at least one end-vertex in U.

For any $U \subset Q(v, N_{i+1})$ and $R \subset S(v, N_{i+1}+1)$, consider the event

$$F_i(U, R) = \{\mathcal{C}_i = U,\, \mathcal{D}_i = R\},$$

and let Π_i be the collection of all such pairs (U,R) that $\{\mathcal{C}_i = U\} \subset F_i$ and $F_i(U,R) \neq \emptyset$.

Then $F_i = \cup_{(U,R)\in\Pi_i} F_i(U,R)$, and for all $n > N_{i+1} + N_0$,

$$\begin{aligned}\mathbb{P}_p\left[E, w \longleftrightarrow S(w,n), F_i\right] &= \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \longleftrightarrow S(w,n), F_i(U,R)\right] \\ &= \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \longleftrightarrow S_{i+1}, F_i(U,R)\right] \cdot \mathbb{P}_p\left[R \overset{Q(w,n)\setminus U}{\longleftrightarrow} S(w,n)\right].\end{aligned}$$

Together with (4.6), this gives the inequality

$$\begin{aligned}&\Big|\mathbb{P}_p\left[E, w \longleftrightarrow S(w,n)\right] - \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \longleftrightarrow S_{i+1}, F_i(U,R)\right] \cdot \mathbb{P}_p\left[R \overset{Q(w,n)\setminus U}{\longleftrightarrow} S(w,n)\right]\Big| \\ &\qquad \leq c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p[w \longleftrightarrow S(w,n)] \leq \frac{c_*^{-2}\varepsilon_i}{\mathbb{P}_{p_c}[E]} \cdot \mathbb{P}_p[E, w \longleftrightarrow S(w,n)], \quad (4.8)\end{aligned}$$

where the last step follows from the FKG inequality, since E is increasing. Define the constant $C_* = (c_*^2\,\mathbb{P}_{p_c}[E])^{-1}$ and for $(U,R) \in \Pi_i$, let

$$\begin{aligned}u'_p(U,R) &= \mathbb{P}_p\left[E, w \longleftrightarrow S_{i+1}, F_i(U,R)\right], \\ u''_p(U,R) &= \mathbb{P}_p\left[w \longleftrightarrow S_{i+1}, F_i(U,R)\right], \\ \gamma_p(U,R,n) &= \mathbb{P}_p\left[R \overset{Q(w,n)\setminus U}{\longleftrightarrow} S(w,n)\right].\end{aligned}$$

In this notation, (4.8) becomes

$$(1 - C_*\varepsilon_i) \leq \frac{\sum_{(U,R)\in\Pi_i} u'_p(U,R)\,\gamma_p(U,R,n)}{\mathbb{P}_p\left[E, w \longleftrightarrow S(w,n)\right]} \leq (1 + C_*\varepsilon_i),$$

and by replacing E above with the sure event, we also get

$$(1 - C_*\varepsilon_i) \leq \frac{\sum_{(U,R)\in\Pi_i} u''_p(U,R)\,\gamma_p(U,R,n)}{\mathbb{P}_p\left[w \longleftrightarrow S(w,n)\right]} \leq (1 + C_*\varepsilon_i).$$

Now we iterate. Let $(U,R) \in \Pi_i$. We can apply a similar reasoning as in (4.6) and (4.8) to $\gamma_p(U,R,n)$ and obtain that for any $j > i+2$ and $n > N_{j+1} + N_0$,

$$\begin{aligned}&\Big|\gamma_p(U,R,n) - \sum_{(U',R')\in\Pi_j} \mathbb{P}_p\left[R \overset{B_{j+1}\setminus U}{\longleftrightarrow} S_{j+1}, F_{j-1}, F_j(U',R')\right] \cdot \gamma_p(U',R',n)\Big| \\ &\qquad \leq c_*^{-2}(\varepsilon_{j-1} + \varepsilon_j) \cdot \gamma_p(U,R,n). \quad (4.9)\end{aligned}$$

For $j > i+2$, $(U,R) \in \Pi_i$ and $(U',R') \in \Pi_j$, define

$$M_p(U,R;\,U',R') = \mathbb{P}_p\left[R \overset{B_{j+1}\setminus U}{\longleftrightarrow} S_{j+1}, F_{j-1}, F_j(U',R')\right].$$

Then (4.9) becomes

$$(1 - c_*^{-2}\,(\varepsilon_{j-1}+\varepsilon_j))\,\gamma_p(U,R,n) \le \sum_{(U',R')\in\Pi_j} M_p(U,R;\,U',R')\,\gamma_p(U',R',n)$$
$$\le (1 + c_*^{-2}\,(\varepsilon_{j-1}+\varepsilon_j))\,\gamma_p(U,R,n).$$

Iterating further gives that for any $\varepsilon > 0$ and $s \in \mathbb{N}$, there exist indices $i_1,\dots,i_s$ such that $i_{k+1} > i_k + 2$ and for all $n > N_{i_s+1} + N_0$,

$$e^{-\varepsilon}\,\mathbb{P}_p\left[E \,|\, w \longleftrightarrow S(w,n)\right] \le \frac{\sum u'_p(U_1,R_1)\,M_p(U_1,R_1;\,U_2,R_2)\dots M_p(U_{s-1},R_{s-1};,U_s,R_s)\,\gamma_p(U_s,R_s,n)}{\sum u''_p(U_1,R_1)\,M_p(U_1,R_1;\,U_2,R_2)\dots M_p(U_{s-1},R_{s-1};,U_s,R_s)\,\gamma_p(U_s,R_s,n)}$$
$$\le e^{\varepsilon}\,\mathbb{P}_p\left[E \,|\, w \longleftrightarrow S(w,n)\right], \quad (4.10)$$

where the two sums are over $(U_1,R_1) \in \Pi_{i_1},\dots,(U_s,R_s) \in \Pi_{i_s}$.

We will prove that (A2) implies that there exists κ such that for all i, $j > i+2$, all pairs $(U_1,R_1),(U_2,R_2) \in \Pi_i$, $(U_1',R_1'),(U_2',R_2') \in \Pi_j$, and all $p \in [p_c, p_c+\delta]$,

$$\frac{M_p(U_1,R_1;\,U_1',R_1')\,M_p(U_2,R_2;\,U_2',R_2')}{M_p(U_1,R_1;\,U_2',R_2')\,M_p(U_2,R_2;\,U_1',R_1')} \le \kappa^2. \quad (4.11)$$

(This is an analogue of [K86a, Lemma (23)].) If so, then we can use Hopf's contraction property of multiplication by positive matrices as in [K86a, pages 377-378][1] to conclude from (4.10) that there exists $\xi \le 1$, which depends on E, p, and the scales $i_1,\dots,i_s$, such that for all $n > N_{i_s+1} + N_0$,

$$e^{-\varepsilon}\left(\xi - \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right) \le \mathbb{P}_p\left[E \,|\, w \longleftrightarrow S(w,n)\right] \le e^{\varepsilon}\left(\xi + \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right). \quad (4.12)$$

It follows from (4.12) and the fact that $\xi \le 1$ that for any $m,n > N_{i_s+1} + N_0$ and $p \in [p_c, p_c+\delta]$,

$$\Big|\mathbb{P}_p\left[E \,|\, w \longleftrightarrow S(w,m)\right] - \mathbb{P}_p\left[E \,|\, w \longleftrightarrow S(w,n)\right]\Big| \le \left(e^{\varepsilon} - e^{-\varepsilon}\right) + \left(e^{\varepsilon} + e^{-\varepsilon}\right)\left(\frac{\kappa-1}{\kappa+1}\right)^{s-1},$$

[1] There is a mathematical typo in the first inequality on [K86a, page 378] – $\mathrm{osc}(u',u'')$ is missing. However, one can show using RSW techniques that the missing term there is bounded from above by a constant independent of j_1, and the remaining argument goes through. In our case, the situation is simpler, since for our choice of u' and u'', $\mathrm{osc}(u',u'') \le 1$.

which implies (4.5).

It remains to prove (4.11). Let $j > i+2$. Consider the random sets

$$\mathcal{X}_j = \left\{ x \in A_{j-1} \ : \ x \overset{A_{j-1}}{\longleftrightarrow} S_j \right\},$$
$$\mathcal{Y}_j = \left\{ y \in S(v, N_{j-1}-1) : \exists\, x \in \mathcal{X}_j, \text{ a neighbour of } y, \text{ such that } \langle x, y\rangle \text{ is open} \right\}.$$

Note that $\mathcal{X}_j$ contains S_j, the event $\{\mathcal{X}_j = X\}$ depends only on the states of edges in A_{j-1} with at least one end-vertex in X, and either $\{\mathcal{X}_j = X\} \subset F_{j-1}$ or $\{\mathcal{X}_j = X\} \cap F_{j-1} = \emptyset$. Also note that the event $\{\mathcal{X}_j = X,\, \mathcal{Y}_j = Y\}$ depends only on the states of edges in B_j with at least one end-vertex in X. For any $X \subset A_{j-1}$ and $Y \subset S(v, N_{j-1}-1)$, consider the event

$$G_j(X,Y) = \{\mathcal{X}_j = X,\, \mathcal{Y}_j = Y\},$$

and let Γ_j be the collection of all such pairs (X,Y) that $\{\mathcal{X}_j = X\} \subset F_{j-1}$ and $G_j(X,Y) \neq \emptyset$. Then $F_{j-1} = \cup_{(X,Y)\in\Gamma_j} G_j(X,Y)$ and for any $(U,R) \in \Pi_i$, $(U',R') \in \Pi_j$,

$$M_p(U,R;U',R') = \sum_{(X,Y)\in\Gamma_j} \mathbb{P}_p\left[R \overset{B_j\setminus(X\cup U)}{\longleftrightarrow} Y\right] \cdot \mathbb{P}_p\left[G_j(X,Y), F_j(U',R'), Y \longleftrightarrow R'\right].$$

By the assumption (A2),

$$c_* \le \frac{\mathbb{P}_p\left[R \overset{B_j\setminus(X\cup U)}{\longleftrightarrow} Y\right]}{\mathbb{P}_p\left[R \overset{Q(v,2N_{i+1})\setminus U}{\longleftrightarrow} S(v,2N_{i+1})\right] \cdot \mathbb{P}_p\left[S(v,2N_{i+1}) \overset{B_j\setminus X}{\longleftrightarrow} Y\right]} \le 1.$$

This easily implies (4.11) with $\kappa = c_*^{-1}$. The proof of Theorem 4.1 is complete. □

Remark 4.2. Instead of conditioning on the events $\{w \longleftrightarrow S(w,n)\}$, one could instead condition on the generalized event ensuring long connections and obtain the same limit in the following way. Let Z_n be any finite connected set containing $Q(w,n)$ and f_n be one of the interior boundary edges of Z_n. We will have, for any cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}[E | w \overset{Z_n}{\longleftrightarrow} f_n] = \nu_w[E], \tag{4.13}$$

the same limits as in (4.1). We can also generalize this to a set of edges Y_n instead of one, and the set (or the edge) need not even be in the boundary. We will obtain the same limits as in (4.1) as long as $Y_n \subset Z_n \setminus Q(w,n)$. This is immediate after observing that $\mathbb{P}_p[E | w \overset{Z_n}{\longleftrightarrow} Y_n]$ satisfies inequalities (4.12).

4.4 Quasi-multiplicativity for slabs

In this section we prove that the assumption (A2) is fulfilled by slabs $\mathbb{S}_{k,d}$ for any $d \geq 2$ and $k \geq 0$ and for any $\delta > 0$ such that $p_c + \delta < 1$, thus proving

Theorem 4.3. *The two limits in* (4.1) *exist and coincide for* $\mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ *(for any* $d \geq 2$, $k \geq 0$*).*

Fix $d \geq 2$ and $k \geq 0$. For positive integers $m \leq n$, let $B'(n) = [-n, n]^2 \times \{0, \ldots, k\}^{d-2}$ be the box of side length $2n$ in $\mathbb{S}_{k,d}$ centered at 0, $\partial B'(n) = B'(n) \setminus B'(n-1)$ the inner boundary of $B'(n)$, and $\mathrm{An}(m, n) = B'(n) \setminus B'(m-1)$ the annulus of side lengths $2m$ and $2n$. We will prove the following lemma.

Lemma 4.4. *Let* $d \geq 2$ *and* $k \geq 0$. *Let* $\delta > 0$ *such that* $p_c + \delta < 1$. *There exists* $c > 0$ *such that for any* $p \in [p_c, p_c + \delta]$, *integer* $m > 0$, *any finite connected* $Z \subset \mathbb{S}_{k,d}$ *such that* $Z \supseteq \mathrm{An}(m, 3m)$, *and any* $X \subset Z \cap B'(m)$ *and* $Y \subset Z \setminus B'(3m)$,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y] \geq c \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \partial B'(2m)] \cdot \mathbb{P}_p[Y \overset{Z}{\longleftrightarrow} \partial B'(2m)]. \tag{4.14}$$

To see that Lemma 4.4 implies (A2), note that it suffices to prove (4.2) for $m \geq m_0$ and sufficiently large m_0. One can choose $m_0 = m_0(d, k)$ large enough so that $A(0, m, 4m) \supset \mathrm{An}(m, 3m)$. Thus, Lemma 4.4 implies (A2).

Before proving Lemma 4.4, we will state and prove an easier variant of it at the critical regime $p = p_c$. We denote $\mathbb{P}_{p_c}$ by $\mathbb{P}$.

Lemma 4.5. *There exists* $c > 0$ *such that for any integers* $n/2 \geq k \geq 2m \geq 4$

$$\mathbb{P}[\partial B'(m) \longleftrightarrow \partial B'(n)] \geq c \cdot \mathbb{P}[\partial B'(m) \longleftrightarrow \partial B'(k)] \cdot \mathbb{P}[\partial B'(k) \longleftrightarrow \partial B'(n)]. \tag{4.15}$$

Let us recall Remark 3.8. For two "smooth" polygons P_1, P_2 with vertices from $\mathbb{Z}^2$ having regions P_{ij} (for $i, j \in \{1, 2\}$), which are disjoint connected subset of the interior vertex boundary of P_i, if any two open paths γ_i connecting P_{i1} to P_{i2} in P_i (for $i = 1, 2$) necessarily have an intersection point z, then

$$\mathbb{P}[P_{11} \overset{P_1 \cup P_2}{\longleftrightarrow} P_{22}] \geq c_* \cdot \mathbb{P}[P_{11} \overset{P_1}{\longleftrightarrow} P_{12}]\mathbb{P}[P_{21} \overset{P_2}{\longleftrightarrow} P_{22}]. \tag{4.16}$$

By "smooth" polygon, we mean simple polygon that can be represented as a finite union of rectangles with both dimensions ≥ 6. Since Theorem 3.1 tells us that probability of left-right crossing of a rectangular box is uniformly bounded from below by a non-negative constant depending only on the aspect ratio of it, we can create long connections in polygons with uniformly positive probability as well.

Proof of lemma 4.5: Let us recall, for $A \subset \mathbb{Z}^2$, we defined $\overline{A} = A \times \{0,1,\ldots,k\}^{d-2}$ and call the segment $\overline{\{k\} \times [0,k]}$ of $\partial B'(k)$ by $Z(k)$. Symmetry dictates that $\mathbb{P}[\partial B'(u) \overset{\mathrm{An}(u,v)}{\longleftrightarrow} Z(v)] \geq \mathbb{P}[\partial B'(u) \longleftrightarrow \partial B'(v)]/8$ for any integers $u < v$. Our aim is to glue two paths in $\{\partial B'(m) \overset{\mathrm{An}(m,k)}{\longleftrightarrow} Z(k)\}$ and $\{Z(k) \overset{\mathrm{An}(k,n)}{\longleftrightarrow} \partial B'(n)\}$ with the help of a open path in a "tunnel" around $Z(k)$ as we did in Figure 3.13. The shape of this "tunnel" $T(k)$ is given by the union of following five rectangular boxes (See Figure 4.1):

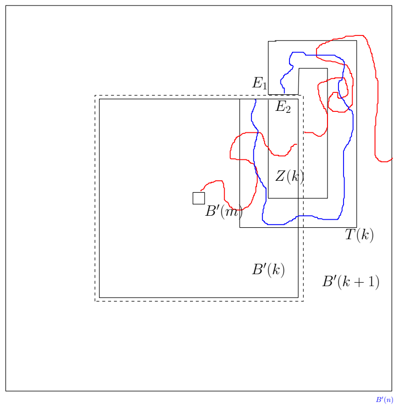

FIGURE 4.1: Quasimultiplicativity

- $\overline{[k/2, 3k/4] \times [-k/4, k]}$
- $\overline{[k/2, 3k/2] \times [-k/4, 0]}$
- $\overline{[5k/4, 3k/2] \times [-k/4, 3k/2]}$
- $\overline{[3k/4, 3k/2] \times [5k/4, 3k/2]}$
- $\overline{[3k/4, k] \times [k+1, 3k/2]}$

We mark two "ends" of this tunnel

$$E_1 = \overline{[k/2, 3k/4] \times \{k\}}, \qquad \text{and} \qquad E_2 = \overline{[3k/4, k] \times \{k+1\}}.$$

By repeated use of RSW theorem 3.1 and (4.16), we glue open paths in the aforementioned five constituent rectangular boxes, and obtain, for some $c_T > 0$, $\mathbb{P}[E_1 \overset{T(k)}{\longleftrightarrow} E_2] \geq c_T$. Glueing this event with $\{\partial B'(m) \overset{\mathrm{An}(m,k)}{\longleftrightarrow} Z(k)\}$ yields:

$$\mathbb{P}[\partial B'(m) \overset{\mathrm{An}(m,k) \cup T(k)}{\longleftrightarrow} E_2] \geq \frac{c_* c_T \mathbb{P}[\partial B'(m) \longleftrightarrow \partial B'(k)]}{8}.$$

The next step is glueing this modified event with the event $\{Z(k+1) \overset{\mathrm{An}(k+1,n)}{\longleftrightarrow} \partial B'(n)\}$ to obtain

$$\mathbb{P}[\partial B'(m) \overset{\mathrm{An}(m,n)}{\longleftrightarrow} \partial B'(n)] \geq \frac{{c_*}^2 c_T \mathbb{P}[\partial B'(m) \longleftrightarrow \partial B'(k)] \mathbb{P}[\partial B'(k+1) \longleftrightarrow \partial B'(n)]}{64}.$$

This yields (4.15) with $c = \frac{{c_*}^2 c_T}{64}$. $\square$

Remark 4.6. Given any $\delta < 1 - p_c$, this result will also hold true uniformly for $p \in [p_c, p_c + \delta]$. This happens since by monotonicity $\mathbb{P}_p[E_1 \overset{T(k)}{\longleftrightarrow} E_2] \geq \mathbb{P}_{p_c}[E_1 \overset{T(k)}{\longleftrightarrow} E_2] \geq c_T$ and the rest follows through with a slightly different constant $c'_* = \frac{1}{1+3C'_*}$ (Recall Section 3.3) where

$$C'_* = \left(\frac{2}{\min(p_c(\mathbb{S}_{k,d}), 1 - p_c(\mathbb{S}_{k,d}) - \delta)} \right)^{d \cdot 7^2 \cdot (k+1)^{d-2}}.$$

We will furnish the proof of Lemma 4.4 now as promised. Its major improvement lies in doing away with the 'shape' of the region (the regions need not be rectangular, or even "glueing-friendly" as mentioned in Remark 3.8) and the only requirement being reasonable amount of space between the regions which are being connected, to split one long path into two (albeit at the cost of a universal constant).

Proof of Lemma 4.4. Instead of (4.14), it suffices to prove that there exists $c > 0$ such that for any $m > 0$, any finite connected $Z \subset \mathbb{S}_{k,d}$ such that $Z \supseteq \mathrm{An}(m, 2m)$, and any $X \subset Z \cap Q(m)$ and $Y \subset Z \setminus Q(2m)$,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y] \geq c \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(3m)] \cdot \mathbb{P}_p[B'(2m) \overset{Z}{\longleftrightarrow} Y]. \tag{4.17}$$

Indeed, for Z as in the statement of the lemma, by (4.17),

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(3m)] \geq c \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(2m)] \cdot \mathbb{P}_p[\partial B'(\frac{4}{3}m) \overset{Z}{\longleftrightarrow} \partial B'(3m)],$$

and $\mathbb{P}_p[\partial B'(\frac{4}{3}m) \overset{Z}{\longleftrightarrow} \partial B'(3m)] \geq \mathbb{P}_{p_c}[\partial B'(\frac{4}{3}m) \longleftrightarrow \partial B'(3m)] \geq c > 0$, an immediate corollary of RSW Theorem 3.1.

We proceed to prove (4.17). Let E_m be the event that there exists an open circuit (nearest neighbour path with the same start and end points) around $B'(2m)$ contained in $\mathrm{An}(2m, 3m)$. It is shown in [NTW15, Corollary 3.2.1] that $\mathbb{P}_p[E_m] \geq \mathbb{P}_{p_c}[E_m] > c > 0$ for some $c > 0$ independent of m. Thus, by the FKG inequality,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(3m),\ Y \overset{Z}{\longleftrightarrow} B'(2m),\ E_m] \geq c \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(3m)] \cdot \mathbb{P}_p[Y \overset{Z}{\longleftrightarrow} B'(2m)].$$

Consider an arbitrary deterministic ordering of all circuits in $\mathbb{S}_{k,d}$. We describe one of the ordering following [NTW15, Definition before theorem 3.8] for the sake of completeness. We first order the vertices and then classify the circuits on the basis of the minimal vertex they contain. We interpret the circuit as a self avoiding path starting from that minimal vertex, having the "smaller" edge out of two edges emanating from the minimal vertex chosen as the first edge of the path, and we already know a way to order these self-avoiding paths from Chapter 3. For a configuration in E_m, let Γ be the minimal (with respect to this ordering) open circuit around $B'(2m)$ contained in $\mathrm{An}(2m, 3m)$.

Recall that for $A \subset \mathbb{S}_{k,d}$, we defined $\overline{A} := \overline{\{z \in \mathbb{Z}^2 : \overline{\{z\}} \cap A \neq \phi\}}$. Note that

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} B'(3m),\ Y \overset{Z}{\longleftrightarrow} B'(2m),\ E_m] \leq \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ Y \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ E_m].$$

Thus, to prove (4.17), it suffices to show that for some $C < \infty$,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ Y \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ E] \leq C \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y].$$

This will be achieved using local modification arguments similar to those in [NTW15]. In fact, for the above inequality to hold, it suffices to show that for some $C < \infty$,

$$\mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ Y \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ E, X \overset{Z}{\not\leftrightarrow} Y] \leq C \cdot \mathbb{P}_p[\overset{Z}{\longleftrightarrow} Y]. \tag{4.18}$$

We write the event in the left hand side of (4.18) as the union of three subevents satisfying additionally

$$(a) X \overset{Z}{\not\leftrightarrow} \Gamma\ , Y \overset{Z}{\not\leftrightarrow} \Gamma, \qquad (b) X \overset{Z}{\not\leftrightarrow} \Gamma\ , Y \overset{Z}{\longleftrightarrow} \Gamma, \qquad (c) X \overset{Z}{\longleftrightarrow} \Gamma\ , Y \overset{Z}{\not\leftrightarrow} \Gamma.$$

It suffices to prove that the probability of each of the three sub-events can be bounded from above by $C \cdot \mathbb{P}_p[X \longleftrightarrow Y \text{ in } Z]$.

Case (a): We prove that for some $C < \infty$,

$$\mathbb{P}_p\left[\begin{array}{c} X \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ Y \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ E_m, X \overset{Z}{\not\leftrightarrow} Y, \\ X \overset{Z}{\not\leftrightarrow} \Gamma, Y \overset{Z}{\not\leftrightarrow} \Gamma \end{array}\right] \leq C \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \Gamma \overset{Z}{\longleftrightarrow} Y] \leq C \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y]. \tag{4.19}$$

Denote by G_a the event on the left hand side.

We will again construct a map $f : G_a \to \{X \overset{Z}{\longleftrightarrow} \Gamma \overset{Z}{\longleftrightarrow} Y\}$ to invoke Lemma 3.4. Let us call a map $f : G_a \to \{X \longleftrightarrow Y \text{ in } Z\}$ *D-good* if

(1) For each $\omega \in G_a$, ω and $f(\omega)$ differ in at most D edges,

(2) At most 2^D many configurations ω can be mapped to the same configuration, i.e., for each $\omega \in G_a$, $|\{\omega' \in G_a : f(\omega') = f(\omega)\}| \leq 2^D$.

By Lemma 3.4, if we can construct a *D-good* map, the desired inequality is satisfied with $C = \frac{2^D}{\min(p_c, 1-p_c-\delta))^D}$ for $p \in [p_c, p_c + \delta]$

Take a configuration $\omega \in G_a$. Let $U(\omega)$ be the set of all points $u \in \overline{\Gamma}$ such that u is connected to X in Z by an open self-avoiding path π_u that from the first step on does not visit $\overline{\{u\}}$. Similarly, let $V(\omega)$ be the set of all points $v \in \overline{\Gamma}$ such that v is connected to Y in Z by an open self-avoiding path π_v that does not visit $\overline{\{v\}}$ after the first step.

Subcase (a1): Assume first that we can choose $u \in U(\omega)$ and $v \in V(\omega)$ such that $\overline{\{u\}} = \overline{\{v\}}$. For such ω's, the configuration $f(\omega)$ is defined as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the (unique) edge of π_u, the (unique) edge of π_v, and the edges belonging to Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths ρ (unique line segment if $d = 3$) between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths between v and $\Gamma \cup \rho$ in $\overline{\{u\}}$.

Notice that ω and $f(\omega)$ differ in at most $2d\,(k+1)^{d-2}$ edges. Moreover, since u, v, and Γ are all in different open clusters in ω, after connecting them by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique set of the form $\overline{\{z\}}$ where X (and Y) is connected to Γ.

Subcase (a2): Assume next that $\overline{U}(\omega) \cap \overline{V}(\omega) = \emptyset$. Choose $u \in U(\omega)$ and $v \in V(\omega)$. Note that $\overline{\{u\}}$ is not connected to Y in Z and $\overline{\{v\}}$ is not connected to X in Z. The configuration $f(\omega)$ is defined as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}} \cup \overline{\{v\}}$ except for the edges of π_u, π_v, and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{v\}}$ that belong to one of the shortest paths between v and Γ in $\overline{\{v\}}$.

Notice that ω and $f(\omega)$ differ in at most $4d\,(k+1)^{d-2}$ edges. Step (a) of the construction does not alter the paths π_u and π_v. Finally, since u, v, and Γ are all in different open clusters in ω, after connecting u, v, and Γ by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}} \cup \overline{\{v\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique such set where X and Y are connected to Γ.

The constructed function f thus satisfies the condition stated with $D = 4d\,(k+1)^{d-2}$, and the proof of (4.19) is complete.

Case (b): We prove that for some $C < \infty$,

$$\mathbb{P}_p\left[\begin{array}{c} X \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ Y \overset{Z}{\longleftrightarrow} \overline{\Gamma},\ E, X \overset{Z}{\not\leftrightarrow} Y, \\ X \overset{Z}{\not\leftrightarrow} \Gamma,\ Y \overset{Z}{\longleftrightarrow} \Gamma \end{array}\right] \leq C \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} \Gamma \overset{Z}{\longleftrightarrow} Y] \leq C \cdot \mathbb{P}_p[X \overset{Z}{\longleftrightarrow} Y]. \tag{4.20}$$

Denote by G_b the event on the left hand side. As in Case (a), (4.20) will follow if we construct a suitable *D-good* map $f : G_b \to \{X \overset{Z}{\longleftrightarrow} Y\}$.

Take a configuration $\omega \in G_b$. Let $U(\omega)$ be the set of all points $u \in \overline{\Gamma}$ such that u is connected to X in Z by an open self-avoiding path π_u that does not visit $\overline{\{u\}}$ after the first step.

Subcase (b1): We first assume that there exists $u \in U(\omega)$ such that Y is connected to Γ in $Z \setminus \overline{\{u\}}$. For such ω's, we define $f(\omega)$ as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the edges of π_u and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths between u and Γ in $\overline{\{u\}}$.

Notice that ω and $f(\omega)$ differ in at most $2d\,(k+1)^{d-2}$ edges. Y is connected to Γ in $Z \setminus \overline{\{u\}}$ in the configuration $f(\omega)$. Finally, since u and Γ are in different open clusters in ω, after connecting u and Γ by a simple open path as in (b), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique such set where X is connected to Γ.

Subcase (b2): Assume next that for any $u \in U(\omega)$, Y is not connected to Γ in $Z \setminus \overline{\{u\}}$. Take $u \in U(\omega)$. There exists $v \in \overline{\{u\}}$ such that v is connected to Y in Z by an open self-avoiding path π_v that from the first step on does not visit $\overline{\{v\}}$. For such ω's, we define $f(\omega)$ exactly as in Subcase (a1). We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the edges of π_u, π_v, and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths ρ between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{u\}}$ that belong to one of the shortest paths between v and $\Gamma \cup \rho$ in $\overline{\{u\}}$.

Notice that unlike in Subcase(a1), it is allowed here that $v \in \Gamma$, but this makes no difference in the construction. Indeed, after closing edges as in (a1), Y remains connected to Γ only if $v \in \Gamma$. Thus, after modifying ω according to (a1), either u, v, and Γ are all in different open clusters or $v \in \Gamma$ and the clusters of u and Γ are different. In both cases, after connecting u, v, and Γ by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique set of the form $\overline{\{z\}}$ where X (and Y) is connected to Γ.

The proof of (4.20) is complete, since the constructed function f satisfies the condition stated with $D = 2d\,(k+1)^{d-2}$.

Since the proof of Case (c) is essentially the same as the proof of Case (b), we omit it. Cases (a)-(c) imply (4.18). The proof of Lemma 4.4 is complete. $\square$

By independence, the complementary inequality also holds:

$$\mathbb{P}[X \overset{Z}{\longleftrightarrow} Y] \leq \mathbb{P}[X \overset{Z}{\longleftrightarrow} \partial B'(m)] \cdot \mathbb{P}[\partial B'(m) \overset{Z}{\longleftrightarrow} Y].$$

Obviously Lemma 4.5, the quasi-multiplicativity of square-boxes, is a special case of Lemma 4.4.

Remark 4.7. (1) As a result of (4.13) and Theorem 4.3, we are able to apply a decoupling argument similar to the one used in the proof of Theorem 4.1 to extend various results of Járai [J03] to slabs. For instance, we demonstrate in Chapter 5 that the local limit of the occupancy configurations around vertices in the bulk of a crossing cluster of large box are given by the IIC measures.

(2) Using Lemma 4.4, one can show that the expected number of vertices of the IIC in $B'(n)$ is comparable to $n^2\mathbb{P}[0 \longleftrightarrow \partial B'(n)]$.

(3) In [DS11], the so-called multiple-armed IIC measures were introduced for planar lattices, which are supported on configurations with several disjoint infinite open clusters meeting in a neighbourhood of the origin. We have seen in Chapter 2 that these measures describe the local occupancy configurations around outlets of the invasion percolation [DS11] and pivotals for open crossings of large boxes. It would be interesting to construct multiple-armed IIC measures on slabs, but at the moment it seems quite difficult with current set of tools present.

Chapter 5

IIC as local limits in Slabs

5.1 Introduction

We have described in Chapter 2 that after Kesten constructed IIC-measure for planar percolation in [K86a], Járai showed that the measure could describe local occupancy configuration around a point chosen uniformly from some specific giant clusters, notably the crossing cluster (conditioned on the existence of having one) and the largest cluster ([J03, Theorems 1 and 3]) or around a point of the crossing cluster far away from the boundary ([J03, Theorem 2]). We have already established the existence of IIC-measure on slabs $\mathbb{S}_{k,d} = \mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ (for integers $d \geq 2$ and $k \geq 0$) in Kesten's sense, thus it naturally begs whether these results are true for slabs as well. In this chapter, we prove that we can indeed make sense of IIC measure as local limit of a vertex, away from the boundary, from the crossing collection. We also show that under a certain assumption, occupancy configuration around a point chosen uniformly from the crossing collection is also described by IIC measure.

5.2 Notation and Results

Let us call, as before, $B'(n) = [-n, n]^2 \times \{0, 1, \ldots, k\}^{d-2}$, $S(n) = B'(n) \setminus B'(n-1)$ and $p_c(\mathbb{S}_{k,d})$ is the critical threshold for percolation, i.e.,

$$p_c(\mathbb{S}_{k,d}) = \inf \{p : \mathbb{P}_p[\text{open connected component of } 0 \text{ in } \mathbb{S}_{k,d} \text{ is infinite}] > 0\} .$$

We denote the independent Bernoulli bond percolation measure on $\mathbb{S}_{k,d}$ with parameter $p_c(\mathbb{S}_{k,d})$ as $\mathbb{P} = \mathbb{P}_{p_c(\mathbb{S}_{k,d})}$. For $x, y \in \mathbb{S}_{k,d}$ and $X, Y, Z \subset \mathbb{S}_{k,d}$, we write

- $x \overset{Z}{\longleftrightarrow} y$ if there is a nearest neighbour path of open edges from x to y with all its vertices in Z.

- $x \xleftrightarrow{Z} Y$ if there exists $y \in Y$ such that $x \xleftrightarrow{Z} y$.
- $X \xleftrightarrow{Z} Y$ in Z if there exists $x \in X$ such that $x \xleftrightarrow{Z} Y$.

If we do not mention Z, it is understood that $Z = \mathbb{S}_{k,d}$. We showed in Theorem 4.3 that for any $v \in \mathbb{S}_{k,d}$ and any event E that depends on the state of finitely many edges of $\mathbb{S}_{k,d}$, there exists the limit

$$\nu_v[E] = \lim_{n\to\infty} \mathbb{P}[E \mid v \longleftrightarrow S(n)],$$

called *Kesten's incipient infinite cluster* (IIC) measure.

Recall that for $A \subset \mathbb{Z}^2$, we defined $\overline{A} := A \times \{0, 1, \ldots, k\}^{d-2}$. Let $L(n) = \overline{\{-n\} \times [-n, n]}$ and $R(n) = \overline{\{n\} \times [-n, n]}$ be left and right boundaries of $S(n)$ and

$$SC(n) = \{v \in B'(n) : R(n) \xleftrightarrow{B'(n)} v \xleftrightarrow{B'(n)} L(n)\}$$

be called the crossing collection. We say that a vertex $v \in \mathbb{S}_{k,d}$ has the 'level' $j \in \{0, 1, \ldots, k\}^{d-2}$ if last $d-2$ co-ordinates of v is given by j. It is quite immediate that the local pictures must be different when we look from 2 different levels $j \neq j'$. Thus the only translations which make sense in slabs $\mathbb{S}_{k,d}$ are the translations in $\mathbb{Z}^2$. For some vertex in the plane $u \in \mathbb{Z}^2$, and some level j, let us denote by u^j the vertex in $\mathbb{S}_{k,d}$ whose first 2 co-ordinates are given by u, and last $d-2$ of them by $j \in \{0, 1, \ldots, k\}^{d-2}$.

For $u = (u_1, u_2) \in \mathbb{Z}^2$, let us define $u_S = (u_1, u_2, 0, \ldots, 0) \in \mathbb{S}_{k,d}$ and translation τ_u acting on Ω by $\tau_u\omega(< x, y >) = \omega(< x - u_S, y - u_S >)$, and on events by $\tau_u A = \{\tau_u\omega : \omega \in A\}$. We will prove that:

Theorem 5.1. *Let $h(n) \leq n$ be a function such that $\lim_{n\to\infty} h(n) = \infty$ and E be any event depending on the state of finitely many edges of $\mathbb{S}_{k,d}$. Then for any sequence of vertices $v_n \in Z^2$, and any fixed level $j \in \{0, 1, \ldots, k\}^{d-2}$,*

$$\lim_{\substack{n\to\infty \\ |v_n| \leq n - h(n)}} \mathbb{P}[\tau_{v_n} E | v_n^j \in SC(n)] = \nu_{(0,0)^j}(E).$$

The next natural question to ponder about is if we can make sense of the 'uniform' or 'global' variant of theorem 5.1. To prove this, we need the tightness result of the crossing collection (similar to [J03, Theorem 8(ii)]). This states that with high-probability, $|SC(n)|$ is at least bigger than some multiplicative factor times its expectation, whenever it is non-empty, i.e.

Conjecture 5.2.

$$\lim_{\epsilon\to 0} \inf_{n\geq 1} \mathbb{P}[\epsilon \leq \frac{|SC(n)|}{\mathbb{E}[|SC(n)|]} | SC(n) \neq \phi] = 1.$$

This seems difficult to prove, since both planarity and duality, which were vital ingredients in proving [J03, Theorem 8(ii)], are absent here. Additionally, glueing tools apparently are not adequate enough to prove this. Nevertheless, assuming this conjecture holds true, we can extend the 'global' variant as follows.

Let I_n indicate a vertex chosen uniformly at random from the crossing cluster $SC(n)$, when it is known to be non-empty. Here we abuse the notation and still call this measure as $\mathbb{P}$, and for $v = (v_1, v_2, \dots v_d) \in \mathbb{S}_{k,d}$, let us define $\tau_v = \tau_{(v_1,v_2)}$. The natural candidate for the limiting measure here is the average measure over every level j 'above' the origin. We show that this is indeed the case.

Theorem 5.3. *If Conjecture 5.2 holds, then*

$$\lim_{n\to\infty} \mathbb{P}[\tau_{I_n} E | SC(n) \neq \phi] = \frac{1}{(k+1)^{d-2}} \sum_{j\in\{0,1,\dots,k\}^{d-2}} \nu_{(0,0)^j}(E).$$

We will prove Theorems 5.1 and 5.3 in Section 5.4. But before that let us first replicate some tools for this setting in Section 5.3 which are known to be true for planar critical percolation. Although the main purpose of the following results is helping to prove the above-mentioned theorems, they highlight the similarity with planar critical percolation as well.

5.3 Auxiliary properties of crossing collection in slabs

In this section, we will prove moment bounds on crossing collection and a slightly stronger variant of quasi-multiplicativity lemma 4.4. But before presenting these results, we give an elementary bound on one-arm connectivity, i.e $\alpha(n) = \mathbb{P}[0 \longleftrightarrow S(n)]$.

5.3.1 One-arm connectivity bound

It is known that for site percolation on planar triangular lattices, $\alpha(n) \asymp n^{-5/48+o(1)}$ in [LSW02, Theorem 1.1]. The exponent is expected to be same for other lattices as well but not yet proved. The general result that can be proved is that for some small $\eta \in (0, 1/2)$ such that, $n^{-\eta} \geq \alpha(n) \geq n^{-1/2}$ (see Lemma 2.7(i)). We will prove a similar bound for slabs as well. We define $\alpha(m,n) = \mathbb{P}[S(m) \longleftrightarrow S(n)]$ for integers $m \leq n$.

Lemma 5.4 (One-arm connectivity)**.** *There exists $\eta \in (0, 1/2]$ and constant C_1 such that for all integers $m \leq n$,*

$$(m/n)^\eta \geq \alpha(m,n) \geq C_1\sqrt{m/n}.$$

Proof. The upper bound of $\alpha(m,n)$ has already proved in [NTW15, Corollary 3.2.3], which is a direct consequence of RSW Theorem [NTW15, Theorem 3.1] in slabs. The lower bound is given in (3.25), which states that $\mathbb{P}[S(m) \longleftrightarrow S(n)] \geq \sqrt{cm/n}$ for integers $m < n$. □

5.3.2 Moments of crossing collection

The quasi-multiplicativity of crossings in boxes (recall (4.15)) stated that there exists $c > 0$ such that for any integers $n/2 \geq l \geq 2m \geq 4$,

$$\mathbb{P}[B'(m) \longleftrightarrow S(n)] \geq c \cdot \mathbb{P}[B'(m) \longleftrightarrow S(l)] \cdot \mathbb{P}[B'(l) \longleftrightarrow S(n)]. \tag{5.1}$$

Using (5.1), we will next state and prove that $\mathbb{E}[|SC(n)|^k] \asymp [n^2\alpha(n)]^k$ holds for any $k \in \mathbb{N}$, giving us bounds for every finite moments of crossing collection. This result is well-known and fairly easy to obtain in the plane. (see e.g [J03, Theorem 8(i)] and [K86a, Theorem 8].)

Lemma 5.5. *$\mathbb{E}[|SC(n)|^t] \asymp (n^2\alpha(n))^t$ for all $t \in \mathbb{N}$.*

Proof. We will first prove for $t = 1$. This part closely follows that of [J03, Theorem 8(i)]. Nevertheless, we include this for the sake of completeness. Let us call, for $u = (u_1, u_2, \ldots, u_d) \in \mathbb{S}_{k,d}$, $u + B'(N) = [u_1 - N, u_1 + N] \times [u_2 - N, u_2 + N] \times \{0, 1, \ldots, k\}^{d-2}$, the translated box centred around u. We extend the definitions naturally to $u + \mathrm{An}(N, M) := \{u + B'(M)\} \setminus \{u + B'(N)\}$ and $u + S(N) := u + \mathrm{An}(N - 1, N)$. One side of the moment bound is quite immediate, if we notice:

$$\begin{aligned}
\mathbb{E}[|SC(n)|] &= \sum_{v \in B'(n)} \mathbb{P}[R(n) \overset{B'(n)}{\longleftrightarrow} v \overset{B'(n)}{\longleftrightarrow} L(n)] \\
&\leq \sum_{v \in B'(n)} \mathbb{P}[v \longleftrightarrow v + S(n)] = (2n+1)^2(k+1)^{d-2}\alpha(n) \leq C_{k,d}\, n^2\alpha(n).
\end{aligned}$$

For $v = (v_1, v_2, \ldots v_d) \in B'(n/2)$, let $H(v) := \overline{[-n, n] \times [v_2, v_2 + n]}$. The top side of square $[-n, n]^2$ is denoted by $T(n) = [-n, n] \times \{n\}$. By invoking (3.24), we are again able to glue paths in $Y(v) = \{\exists \text{ an open horizontal crossing in } H(v)\}$ and $X(v) := \tau_v\{0 \overset{B'(n/2)}{\longleftrightarrow} \overline{T(n/2)}\}$ to obtain, for such $v \in B'(n/2)$,

$$\mathbb{P}[v \in SC(n)] \geq \frac{c_*}{27}\mathbb{P}[X(v)]\mathbb{P}[Y(v)] \overset{RSW}{\geq} \frac{c_* c_4 \mathbb{P}[v \longleftrightarrow v + S(n/2)]}{108} \geq \frac{c_* c_4 \alpha(n/2)}{108}.$$

If we sum over all such $v \in B'(n/2)$, we get:

$$\mathbb{E}[|SC(n)|] \geq \sum_{v \in B'(n/2)} \mathbb{P}[v \in SC(n)] \geq C'_{k,d}\, n^2\alpha(n/2) \geq C'_{k,d}\, n^2\alpha(n).$$

By Jensen's inequality, we have, for any $t > 1$,

$$\mathbb{E}[|SC(n)|^t] \geq [\mathbb{E}[|SC(n)|]]^t \geq C(t)(n^2\alpha(n))^t.$$

So for any $t \in \mathbb{N}$, we have the lower bound. For the upper bound, let us work with $t = 2$ for the sake of simplicity (although this method works for any integer t).

For a vertex $v \in \mathbb{S}_{k,d}$, let us define the event $A(v,l,m) := \{v + S(l) \overset{v+\mathrm{An}(l,m)}{\longleftrightarrow} v + S(m)\}$ and $|v|_2 := |v_1| \vee |v_2|$. We also call $A(v,m) := A(v,0,m)$.

$$
\begin{aligned}
& \mathbb{E}[|SC(n)|^2] \\
= & \sum_{v,w \in B'(n)} \mathbb{P}[v \in SC(n), w \in SC(n)] \\
\leq & \sum_{v,w \in B'(n)} \mathbb{P}[A(v, \lfloor \frac{|v-w|_2}{3} \rfloor) \cap A(w, \lfloor \frac{|v-w|_2}{3} \rfloor) \cap A(\frac{v+w}{2}, |v-w|_2, n)] \\
= & \sum_{v,w \in B'(n)} \mathbb{P}[A(0, \lfloor \frac{|v-w|_2}{3} \rfloor)]^2 \mathbb{P}[A(0, |v-w|_2, n)]. \qquad (5.2)
\end{aligned}
$$

We use translation invariance of the model in the last step. Let us denote $\mathbb{P}[A(0,m,n)]$ by $\alpha(m,n)$. By quasi-multiplicativity (5.1), we have $\alpha(m)\alpha(m,n) \asymp \alpha(n)$. Since by RSW Theorem 3.1, $\alpha(\lfloor m/2 \rfloor, m) \asymp \alpha(\lfloor m/3 \rfloor, m) \asymp 1$ for any m, we have $\alpha(m) \asymp \alpha(m/2) \asymp \alpha(m/3)$. Using these facts and repeatedly using (5.1), we get

$$
\begin{aligned}
\mathbb{E}[|SC(n)|^2] & \leq \sum_{v,w \in B'(n)} \alpha(\frac{|v-w|_2}{3})^2 \alpha(|v-w|_2, n) \\
& \leq C[\alpha(n)]^2 \sum_{v,w \in B'(n)} \frac{1}{\alpha(|v-w|_2, n)} \\
& = C[\alpha(n)]^2 \sum_{v \in B'(n)} \sum_{k=1}^{n} \sum_{w:|v-w|_2 = k} \frac{1}{\alpha(k,n)} \\
& \leq C' n^2 [\alpha(n)]^2 \sum_{k=1}^{n} \frac{k}{\alpha(k,n)}
\end{aligned}
$$

In the last step we use the fact that number of vertices w which are exactly k away from v is $O(k)$, and then sum over all v. Now using Lemma 5.4, we get

$$
\mathbb{E}[|SC(n)|^2] \leq C'' n^{5/2} [\alpha(n)]^2 \sum_{k=1}^{n} \sqrt{k} \leq C''' [n^2 \alpha(n)]^2,
$$

which completes the proof. □

5.3.3 Quasi-multiplicativity revisited

We will prove the following stronger form of quasi-multiplicativity Lemma 4.4, which will help us to decouple configurations with a little more restriction than one-arm connectivity:

Lemma 5.6. *For any $v \in B'(n - 2M)$ with $n > 2M$ integers,*

$$
\mathbb{P}[v \in SC(n)] \asymp \mathbb{P}[v \longleftrightarrow v + S(M)] \cdot \mathbb{P}[R(n) \longleftrightarrow v + S(M) \longleftrightarrow L(n)].
$$

Proof. One side of the proof is immediate by independence. The proof of the other side is similar to that of Lemma 4.4. We would, thus, present a brief sketch, highlighting the necessary alterations, while heavily referring to the aforementioned lemma. We call $E_M(v) := \{\text{There is an open circuit in } v + \mathrm{An}(M, 2M)\}$, and we know already by [NTW15, Corollary 3.2.1] that $\mathbb{P}[E_M(v)] = \mathbb{P}[E_M(0)] \geq c$ for a constant $c > 0$ independent of M. Thus,

$$\begin{aligned}
&\mathbb{P}[v \longleftrightarrow v + S(M)] \cdot \mathbb{P}[R(n) \longleftrightarrow v + S(M) \longleftrightarrow L(n)] \\
\overset{(Lemma\ 4.5)}{\leq}\ & \frac{C}{\mathbb{P}[E_M(v)]} P[v \longleftrightarrow v + S(2M)] \cdot \mathbb{P}[R(n) \longleftrightarrow v + S(M) \longleftrightarrow L(n)] \cdot \mathbb{P}[E_M(v)] \\
\overset{(FKG)}{\leq}\ & \frac{C}{c} \mathbb{P}[v \longleftrightarrow v + S(2M), E_M(v), R(n) \longleftrightarrow v + S(M) \longleftrightarrow L(n)]. \qquad (5.3)
\end{aligned}$$

Let us denote the event on the right side as X. It suffices to prove that for some constant $C > 0$, $\mathbb{P}[X] \leq C\mathbb{P}[v \in SC(n)]$.

For configurations in $E_M(v)$ (and thus also for X) we can make sense of minimal open circuit in $v + \mathrm{An}(M, 2M)$ as done in [NTW15, Definition before theorem 3.8] (or as we described in the proof of Lemma 4.4). We call it Γ as before, and define $C_{\Gamma,n}$ as the vertices of $B'(n)$ which are on or outside of $\overline{\Gamma}$. (Recall that for $A \subset \mathbb{S}_{k,d}$, we defined $\overline{A} := \overline{\{z : \overline{\{z\}} \cap A \neq \phi\}}$.) Let us call :

$$X'' := X \cap [\{\overline{\Gamma} \overset{C_{\Gamma,n}}{\longleftrightarrow} R(n)\} \circ \{\overline{\Gamma} \overset{C_{\Gamma,n}}{\longleftrightarrow} L(n)\}],$$

where $\circ$ denoted the disjoint occurrence of those two specific events. We will separately treat X'' and $X' = X \cap X''^c$ and show that $\mathbb{P}[X'] \vee \mathbb{P}[X''] \leq C\mathbb{P}[v \in SC(n)]$.

Case 1: The key strategy is again, finding a *D-good* map $f : X' \cap \{v \notin SC(n)\} \to \{v \in SC(n)\}$ to invoke Lemma 3.4. Let us define $Y := X' \cap \{v \notin SC(n)\}$, and recall that a map is *D-good* if

(1) For each $\omega \in Y$, ω and $f(\omega)$ differ in at most D edges,

(2) At most 2^D many configurations ω can be mapped to the same configuration, i.e., for each $\omega \in Y$, $|\{\omega' \in Y : f(\omega') = f(\omega)\}| \leq 2^D$.

The desired inequality is satisfied with $C = \frac{2^D}{\min(p_c, 1-p_c))^D} + 1$ if we can construct a *D-good* map, and this would complete the proof.

Let us call the set of vertices in $B'(n)$ which have two disjoint paths to $R(n)$ and $L(n)$ as the "Backbone" and denote it by $BB(n)$. If we take a configuration $\omega \in X'$, there would be a vertex $u(\omega) \in C_{\Gamma,n} \setminus \overline{\Gamma}$ that is connected to $\overline{\Gamma}$ without using any other vertex of $BB(n)$, and this would be unique, otherwise $\overline{\Gamma}$ will be connected to $R(n)$ and $L(n)$ by two edge disjoint paths resulting in $\omega \in X''$. We can now construct the *D-good* map exactly as done in the proof of Lemma 4.4 by subdividing in three parts and constructing a map for each of them, the three parts being:

$$(a)\{v \overset{B'(n)}{\not\leftrightarrow} \Gamma\}\cap\{u \overset{B'(n)}{\not\leftrightarrow} \Gamma\},\quad (b)\{v \overset{B'(n)}{\not\leftrightarrow} \Gamma\}\cap\{u \overset{B'(n)}{\longleftrightarrow} \Gamma\},\quad (c)\{v \overset{B'(n)}{\longleftrightarrow} \Gamma\}\cap\{u \overset{B'(n)}{\not\leftrightarrow} \Gamma\}.$$

Then for each case the proofs are similar to that of Lemma 4.4. We locally modify to glue v, Γ, and u together. The uniqueness of u is preserved under the map, which enables us to identify the location of the surgery and hence, helps us to use Lemma 3.4.

Case 2: In this case we need two successive surgeries. Let $N(v)$ denote the set of neighbours of a vertex $v \in \mathbb{S}_{k,d}$. We will construct two *D-good* maps

- $f : X'' \cap \{v \in SC(n)\}^c \to \{v \overset{B'(n)}{\longleftrightarrow} \Gamma \overset{B'(n)}{\longleftrightarrow} L(n), S_\Gamma\}$
- $f' : \{v \overset{B'(n)}{\longleftrightarrow} \Gamma \overset{B'(n)}{\longleftrightarrow} L(n), S_\Gamma\} \cap \{v \in SC(n)\}^c \to \{v \in SC(n)\}$

where S_Γ is defined as:

$$S_\Gamma := \{\exists v \in \overline{\Gamma} \text{ such that for some } v' \in \{v\} \cup N(v),\ v' \overset{C_{\Gamma,n}}{\longleftrightarrow} R(n)\}$$

These two together imply $\mathbb{P}[X''] \leq [\frac{2^D}{\min(p_c, 1-p_c)^D} + 1]^2 \mathbb{P}[v \in SC(n)]$ and that completes the proof. The construction of the first map is done exactly how we glue a path from v to $\overline{\gamma}$ with a path from $\overline{\gamma}$ to $L(n)$ and invoke (4.19) and (4.20). Since in X'' we have two disjoint paths from $\overline{\gamma}$ to $L(n)$ and $R(n)$, and the surgery can only alter edges with at least one vertex on $\overline{\gamma}$, the range of the map must be inside S_Γ.

If we take a configuration in $Y = \{v \overset{B'(n)}{\longleftrightarrow} \Gamma \overset{B'(n)}{\longleftrightarrow} R(n), S_\Gamma\} \cap \{v \in SC(n)\}^c$, it is enforced that $\Gamma \overset{B'(n)}{\not\leftrightarrow} R(n)$ but there exists a vertex z connected to $R(n)$ inside $B'(n)$ such that $\exists z' \in \overline{\gamma} \cap \{\{z\} \cup N(z)\}$. If $\overline{\Gamma} \overset{B'(n)}{\not\leftrightarrow} R(n)$, we open the edge $e_z = (z, z')$. (The edge had to be closed before, otherwise it violates the assumption.) If it happens that $z' \in \Gamma$, then the surgery is immediate, since this is the unique vertex by which Γ (which is preserved since the open clusters of $R(n)$ are different from the cluster of Γ and v) will be connected to $R(n)$ in the image as well. So without loss of generality, we can assume $z \in \overline{\Gamma}$ and $z' \in \overline{\Gamma} \setminus \Gamma$.

Let us call $U(\omega) = \{z \in \overline{\Gamma} : z \overset{C_{\Gamma,n}}{\longleftrightarrow} R(n)\}$. We separate the event Y into further sub-cases and perform local modifications as required :

(a) $\exists u \in U(\omega)$ such that both $v \overset{B'(n)\setminus\overline{\{u\}}}{\longleftrightarrow} \Gamma$ and $\Gamma \overset{B'(n)\setminus\overline{\{u\}}}{\longleftrightarrow} L(n)$ hold.

(b) $\exists u \in U(\omega)$ such that $v \overset{B'(n)\setminus\overline{\{u\}}}{\longleftrightarrow} \Gamma$ hold but $\Gamma \overset{B'(n)\setminus\overline{\{u\}}}{\not\leftrightarrow} L(n)$ does not.

(c) $\exists u \in U(\omega)$ such that $\Gamma \overset{B'(n)\setminus\overline{\{u\}}}{\longleftrightarrow} L(n)$ hold but $v \overset{B'(n)\setminus\overline{\{u\}}}{\nleftrightarrow} \Gamma$ does not.

(d) $\nexists u \in U(\omega)$ for which either $\Gamma \overset{B'(n)\setminus\overline{\{u\}}}{\longleftrightarrow} L(n)$ or $v \overset{B'(n)\setminus\overline{\{u\}}}{\nleftrightarrow} \Gamma$ hold.

While these cases are not mutually exclusive (case(b) and case(c) might hold simultaneously) they exhaust Y. Thus doing surgery on each of the cases suffices by the union bound. The sub-case (a) can be dealt exactly like sub-case (b1) and all other three cases can be dealt like sub-case (b2) in the proof of Lemma 4.4, the only notable difference being that for case (d), we open up three paths one by one in $\overline{\{u\}}$ instead of two. This completes the proof. □

5.4 IIC as local limits

We will follow the scheme of Járai broadly, only suitably substituting circuits with certain structures called 'shells' as it benefits us. Before starting with proofs, we recall the following alternate definition of IIC-measure from Remark 4.2. Let Z_n be any finite connected set containing $B'(n)$ and f_n be one of its boundary edges, i.e. connecting some $x_n \in Z_n$ with some $y_n \in Z_n^c$. We will have, for fixed level $j \in \{0,1,\ldots,k\}^{d-2}$ and some cylinder event E,

$$\lim_{n\to\infty} \mathbb{P}[E|(0,0)^j \overset{Z_n}{\longleftrightarrow} f_n] = \nu_{(0,0)^j}[E]. \tag{5.4}$$

5.4.1 Proof of Theorem 5.1

We refer $v^j + B'(N)$ as $v + B(N)$ for any vertex $v \in \mathbb{Z}^2$, (since the box does not alter if the level j changes) and similarly denote $v^j + \mathrm{An}(N,M)$ by $v + \mathrm{An}(N,M)$. We denote by $F(N,M,v_n)$ the event that there exists an open path from $v_n + B'(N)$ to $v_n + S(M)$ in $v_n + B'(M)$ and any two such open paths have at least one edge in common (to establish non-existence of two disjoint such paths and have some sort of 'uniqueness' of the path). By suitably choosing M/N large enough, we would aim to make $\mathbb{P}[F(N,M,v_n)\ |v_n^j \in SC(n)] \geq 1-\epsilon$ for any level j, $v_n \in B(n-h(n))$ and any $\epsilon > 0$. We will choose these parameters $1 << N << M < h(n) < n$ suitably later.

For a connected set $V \subset \mathrm{An}(N,M)$ containing $S(M)$ and g as one of its boundary edge, let us redefine $G_{(N,M)}(V,g)$ as the event that

(a) g is open and connected to $S(M)$ by two edge-disjoint open paths in V (or has an end-vertex in $S(M)$),

(b) every other boundary edge of V is closed and connected to $S(M)$ by an open path in V.

We call this shape V as 'shell' and g as its 'orifice'. For any configuration in $G_{(N,M)}(V,g)$, any path, by which $S(M)$ is connected to $S(N)$, must exit V by the orifice g. Let us denote $A_n^j = \{v_n^j \in SC(n)\}$, $F'(N,M,v_n) := \{v_n + S(N) \overset{v_n+\mathrm{An}(N,M)}{\longleftrightarrow} v_n + S(M)\}$, and $O(v_n, M) = B(n) \setminus \{v_n + S(M)\}$. Notice that $\mathbb{P}[F'(N,M,v_n)] = \mathbb{P}[F'(N,M,(0,0))] = \alpha(N,M)$ holds true by translation invariance, and conditioned on A_n^j, $F(N,M,v_n)^c$ implies two disjoint connections from $v_n + S(N)$ to $v_n + S(M)$. Thus, we obtain

$$\begin{aligned}
&\mathbb{P}[F(N,M,v_n)^c|A_n^j] \\
\leq\ & \frac{1}{\mathbb{P}[A_n^j]}\mathbb{P}\left[\begin{array}{c} v_n \longleftrightarrow v_n + S(N),\ \text{there exist two disjoint open paths from} \\ v_n + S(N) \text{ to } v_n + S(M), R(n) \overset{O(v_n,M)}{\longleftrightarrow} v_n + S(M) \overset{O(v_n,M)}{\longleftrightarrow} L(n) \end{array}\right] \\
\overset{(BK)}{\leq}\ & \frac{\mathbb{P}[v_n \longleftrightarrow v_n + S(N)]\mathbb{P}[F'(N,M,v_n)]^2\mathbb{P}[R(n) \overset{O(v_n,M)}{\longleftrightarrow} v_n + S(M) \overset{O(v_n,M)}{\longleftrightarrow} L(n)]}{\mathbb{P}[A_n^j]} \\
\overset{(Lemma\ 4.5)}{\leq}\ & \frac{C\mathbb{P}[v_n \longleftrightarrow v_n + S(M)]\mathbb{P}[R(n) \overset{O(v_n,M)}{\longleftrightarrow} v_n + S(M) \overset{O(v_n,M)}{\longleftrightarrow} L(n)]\alpha(N,M)}{\mathbb{P}[A_n^j]} \\
\overset{(Lemma\ 5.6)}{\leq}\ & C'\alpha(N,M) \overset{(Lemma\ 5.4)}{\leq} C''(N/M)^{\eta}.
\end{aligned} \tag{5.5}$$

Now we choose M/N to be large enough so as to make RHS of (5.5) less than ϵ.

If the event $F = F(N,M,v_n)$ occurs, the non-existence of two edge-disjoint paths imply there exists a 'cut-edge' by Menger's theorem [M27]. We take the first cut-edge g which any path travelling from $v_n + S(M)$ to $v_n + S(N)$ encounters. Thus under occurrence of F, there exists unique (V,g) for which $\tau_{v_n}G(V,g) = \tau_{v_n}G_{(N,M)}(V,g)$ occurs , which is measurable "within" $v_n + V$ (i.e with respect to the state of edges with at least one end-vertex in $\tau_{v_n}V$). Let us denote by $\mathfrak{V}$ the all possible 2-tuples of $(V,g)(\omega)$ over all $\omega \in F(N,M,(0,0))$ and define $V'(n) := B'(n) \cap \tau_{v_n}\{[B'(M)]^c \cup V\}$, the region "outside" the inner boundary of V. We pick N to be large enough such that E depends only on the edges of $B(N)$ and for any $v \in \mathbb{Z}^2$, define the following variables

- $X(v^j, E) := \mathbb{1}[\tau_v E, v^j \overset{\tau_v\{B'(M)\setminus V\}}{\longleftrightarrow} \tau_v g]$,
- $Y(V,n) := \mathbb{1}[\tau_{v_n}G(V,g), L(n) \overset{V'(n)}{\longleftrightarrow} \tau_{v_n}g \overset{V'(n)}{\longleftrightarrow} R(n)]$.

Observe that $X(v_n^j, E)$ and $Y(V,n)$ are independent since latter depends on the edges of $V'(n)$ and the former on the edges of $\tau_{v_n}\{B'(M) \setminus V\}$. We thus obtain

$$\begin{aligned}
\mathbb{P}[\tau_{v_n}E|A_n^j] &\leq \mathbb{P}[F^c|A_n^j] + \frac{1}{\mathbb{P}[A_n^j]}\sum_{(V,g)\in\mathfrak{V}} \mathbb{P}[\tau_{v_n}E, \tau_{v_n}G(V,g), v_n^j \in SC(n)] \\
&\leq \epsilon + \frac{1}{\mathbb{P}[A_n^j]}\sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X(v_n^j, E)]\mathbb{E}[Y(V,n)] \\
&= \epsilon + \frac{1}{\mathbb{P}[A_n^j]}\sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X(v_n^j, E)]\mathbb{E}[Y(V,n)] \\
&= \epsilon + \frac{1}{\mathbb{P}[A_n^j]}\sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X((0,0)^j, E)]\mathbb{E}[Y(V,n)]
\end{aligned}$$

Given $\epsilon > 0$, we choose N to be large enough such that

$$\mathbb{P}[E'|(0,0)^j \overset{B'(M)\setminus V}{\longleftrightarrow} g] \in (\frac{1}{1+\epsilon}\nu_{(0,0)^j}(E'), (1+\epsilon)\nu_{(0,0)^j}(E')), \tag{5.6}$$

for any $(V, g) \in \mathfrak{V}$ and for E' being E or Ω. This holds by (5.4), where we choose $Z_N = B'(M) \setminus V \supset B'(N)$. This yields

$$\begin{aligned}
\mathbb{P}[\tau_{v_n} E|A_n^j] &\leq \epsilon + \frac{(1+\epsilon)\nu_{(0,0)^j}(E)}{\mathbb{P}[A_n^j]} \sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X((0,0)^j, \Omega)]\mathbb{E}[Y(V,n)] \\
&= \epsilon + \frac{(1+\epsilon)\nu_{(0,0)^j}(E)}{\mathbb{P}[A_n^j]} \sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X(v_n^j, \Omega)]\mathbb{E}[Y(V,n)] \\
&= \epsilon + \frac{(1+\epsilon)\nu_{(0,0)^j}(E)}{\mathbb{P}[A_n^j]} \sum_{(V,g)\in\mathfrak{V}} \mathbb{P}[\tau_{v_n}\Omega, G(V,g), v \in SC(n)] \\
&\leq \epsilon + (1+\epsilon)\nu_{(0,0)^j}(E).
\end{aligned} \tag{5.7}$$

Similarly by working the other way we have :

$$\mathbb{P}[\tau_{v_n} E|A_n^j] \geq -\epsilon + \frac{1}{1+\epsilon}\nu_{(0,0)^j}(E). \tag{5.8}$$

So given $\epsilon > 0$, first we choose N large enough to "include" E and satisfy (5.6), then we choose M/N to be large enough to control RHS of (5.5), and finally choose n large enough such that $h(n) > 2M$ holds. (We need to have $v_n + B'(2M)$ lying entirely inside $B'(n)$, and this is the only reason we need to take vertices 'away' from boundary.) This completes the proof of Theorem 5.1, since (5.7) and (5.8) holds for arbitrary $\epsilon > 0$. $\square$

5.4.2 Proof of Theorem 5.3

Let us recall the tightness conjecture 5.2 which insists that given ϵ, we can find small $x(\epsilon)$ such that for the event $H_n = H_n(x) = \{|SC(n)| > x\mathbb{E}[|SC(n)|]\}$,

$$\mathbb{P}[H_n^c|SC(n) \neq \phi] > 1 - \epsilon. \tag{5.9}$$

For $u = (u_1, u_2, \ldots, u_d) \in \mathbb{S}_{k,d}$, we denote $F(N, M, (u_1, u_2))$ and $F'(N, M, (u_1, u_2))$ by $F(N, M, u)$ and $F'(N, M, u)$ respectively. We write:

$$\begin{aligned}
\mathbb{P}[\tau_{I_n} F^c, H_n|SC(n) \neq \phi] &= \frac{1}{\mathbb{P}[SC(n) \neq \phi]} \sum_{u\in B'(n)} \mathbb{E}\left[\frac{\mathbb{1}[u \in SC(n), \tau_u F^c], H_n}{|SC(n)|}\right] \\
&\overset{(5.5)}{\leq} \frac{1}{\mathbb{P}[SC(n) \neq \phi]x\mathbb{E}|SC(n)|} \sum_{u\in B'(n)} \mathbb{P}[u \in SC(n)]\mathbb{P}[F'] \\
&= \frac{\mathbb{P}[F']}{x\mathbb{P}[SC(n) \neq \phi]} \leq \frac{\mathbb{P}[F']}{xc_1} \leq \epsilon.
\end{aligned} \tag{5.10}$$

In the last step we choose M/N to be large enough to make $\mathbb{P}[F'] < xc_1\epsilon$. We will show that the vertices close to the boundary contribute negligibly, and this will give us necessary space to make use of the strategy used in the proof of Theorem 5.1. For some suitable function $f(n)$ (that we will choose later) let us define $G_n = G_n^f = \{I_n \in B'(n - f(n))\}$.

$$\begin{aligned}
\mathbb{P}[G_n^c, H_n|SC(n) \neq \phi] &\leq \sum_{u \in \mathrm{An}(n-f(n),n)} \mathbb{E}\left[\frac{\mathbb{1}[u \in SC(n), H_n]}{|SC(n)|}|SC(n) \neq \phi\right] \\
&\leq \sum_{u \in \mathrm{An}(n-f(n),n)} \mathbb{E}\left[\frac{\mathbb{1}[u \in SC(n), H_n]}{x\mathbb{E}[|SC(n)|]}|SC(n) \neq \phi\right] \\
&\leq \sum_{u \in \mathrm{An}(n-f(n),n)} \frac{C\alpha(n/2)}{x\mathbb{E}[|SC(n)|]} \leq \frac{C'nf(n)\alpha(n)}{x\mathbb{E}[|SC(n)|]} \leq \epsilon. \quad (5.11)
\end{aligned}$$

In the last step, we choose n large enough and $f(n) = o(n)$ to make $\frac{C'nf(n)\alpha(n)}{x\mathbb{E}[|SC(n)|]} < \epsilon$. If we combine this boundary adjustment, the existence of unique connections in thick annuli and the tightness result, we get, for a cylinder event E:

$$\begin{aligned}
&\mathbb{P}[\tau_{I_n} E|SC(n) \neq \phi] \\
&\leq 3\epsilon + \frac{1}{\mathbb{P}[SC(n) \neq \phi]} \sum_{u \in B'(n-f(n))} \sum_{(V,g) \in \mathfrak{V}} \mathbb{E}\left[\frac{\mathbb{1}[\tau_u E, u \in SC(n), \tau_u G(V,g)]}{|SC(n)|} : H_n\right] \\
&\leq 3\epsilon + \frac{1}{\mathbb{P}[SC(n) \neq \phi]} \sum_{u \in B'(n-f(n))} \sum_{(V,g) \in \mathfrak{V}} \mathbb{E}\left[\frac{\mathbb{1}[\tau_u E, u \in SC(n), \tau_u G(V,g)]}{|SC(n)|}\right] \\
&\leq 3\epsilon + \mathbb{P}[\tau_{I_n} E|SC(n) \neq \phi]. \quad (5.12)
\end{aligned}$$

As before, we would decouple the event $\mathbb{1}[\tau_u E, u \in SC(n), \tau_u G(V,g)]$ in two parts as

$$\mathbb{1}[\tau_u E, u \stackrel{\tau_u\{B'(M)\setminus V\}}{\longleftrightarrow} \tau_u g] \cdot \mathbb{1}[\tau_u G(V,g), L(n) \stackrel{V_u(n)}{\longleftrightarrow} \tau_u g \stackrel{V_u(n)}{\longleftrightarrow} R(n)],$$

where $V_u(n) = B'(n) \cap \tau_u\{[B'(M)]^c \cup V\}$. To deal with the denominator $|SC(n)|$, we define (following [J03, (2.24)])

$$W_n(V) = \{w \in V_u(n) : w \in SC(n)\}.$$

For this quantity we naturally have

$$|W_n(V)| \leq |SC(n)| \leq |W_n(V)| + C'M^2 \leq (1+\epsilon)|W_n(V)|, \quad (5.13)$$

by choosing n large enough so that $\frac{C'M^2}{n} \leq \epsilon/2$, since $|SC(n)| \geq n$. Using $W_n(V)$ makes it easy to split since $W_n(V)$ depends on edges of $V_u(n)$. We define :

$$\begin{aligned} X(u,E) &= \mathbb{1}[\tau_u E, u \overset{\tau_u\{B'(M)\setminus V\}}{\longleftrightarrow} \tau_u g], \\ Y(V,n) &= \frac{\mathbb{1}[\tau_u G(V,g), L(n) \overset{V_u(n)}{\longleftrightarrow} \tau_u g \overset{V_u(n)}{\longleftrightarrow} R(n)]}{|W_n(V)|}, \end{aligned}$$

with the understanding of $Y = 0$ when it is of the form $0/0$. Let us call the summand under expectation in third line of (5.12) as $E(u,E,V,n)$, we have

$$E(u,E,V,n) \le \mathbb{E}[X(u,E)]\mathbb{E}[Y(V,n)] \le (1+\epsilon)E(u,E,V,n) \tag{5.14}$$

for any event E depending only on the edges of $B(N)$. We choose N again large enough to 'contain' E as well as following (5.6), M/N to be large enough for (5.10) to hold, $f(n) > 2M$ to make use of Lemma 5.6 and finally n to be large enough for (5.11) and (5.13) to hold. Let us denote square $B_2(m) = [-m,m]^2$. We obtain

$$\begin{aligned} &\mathbb{P}[\tau_{I_n} E | SC(n) \neq \phi] \\ \le\ & 3\epsilon + \frac{1}{\mathbb{P}[SC(n)\neq\phi]} \sum_{u\in B'(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} E(u,E,V,n) \\ \overset{(5.14)}{\le}\ & 3\epsilon + \frac{1}{\mathbb{P}[SC(n)\neq\phi]} \sum_{u\in B'(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} \mathbb{E}[X(u,E)]\mathbb{E}[Y(V,n)] \\ =\ & 3\epsilon + \frac{1}{\mathbb{P}[SC(n)\neq\phi]} \sum_{v\in B_2(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} \sum_{j\in\{0,1,\dots k\}^{d-2}} \mathbb{E}[X(v^j,E)]\mathbb{E}[Y(V,n)] \\ \overset{(5.6)}{\le}\ & 3\epsilon + \frac{(1+\epsilon)}{\mathbb{P}[SC(n)\neq\phi]} \sum_{v\in B_2(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} \sum_{j\in\{0,1,\dots k\}^{d-2}} \nu_{(0,0)^j}(E)\mathbb{E}[X(v^j,\Omega)]\mathbb{E}[Y(V,n)] \\ \overset{(5.14)}{\le}\ & 3\epsilon + \frac{(1+\epsilon)^2}{\mathbb{P}[SC(n)\neq\phi]} \sum_{v\in B_2(n-f(n))} \sum_{j\in\{0,1,\dots k\}^{d-2}} \nu_{(0,0)^j}(E) \sum_{(V,g)\in\mathfrak{V}} E(v^j,\Omega,V,n) \\ \le\ & 3\epsilon + (1+\epsilon)^2 \frac{1}{(k+1)^{d-2}} \sum_{j\in\{0,1,\dots k\}^{d-2}} \nu_{(0,0)^j}(E) \end{aligned}$$

In the last step, we use the fact that by symmetry, for any fixed level $j \in \{0,1,\dots k\}^{d-2}$,

$$(k+1)^d \sum_{v\in B_2(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} E(v^j,\Omega,V,n) = \sum_{u\in B'(n-f(n))} \sum_{(V,g)\in\mathfrak{V}} E(u,\Omega,V,n).$$

Similarly, for the other side, we obtain

$$\mathbb{P}[\tau_{I_n} E | SC(n)\neq\phi] \ge -3\epsilon + \frac{1}{(1+\epsilon)^2(k+1)^{d-2}} \sum_{j\in\{0,1,\dots k\}^{d-2}} \nu_{(0,0)^j}(E).$$

This completes the proof. □

Remark 5.7. The other side of tightness result, i.e. the size of the crossing collection cannot be too big compared to its expected size, follows from the moment bounds and the Markov inequality. We emphasised on the difficult bound since that is required for our result.

It is not hard to observe that by simple manipulations of (5.10) and (5.11), we do not need the tightness conjecture in the strong form we posed earlier (and believe to be true). Instead proving something weaker akin to $\mathbb{P}[0 < |SC(n)| < [\mathbb{E}[|SC(n)|]]^{1-\delta}] \to 0$ as $n \to \infty$ for some small $\delta < \eta^2/2$ (recall η from Lemma 5.4) would also suffice.

Also, it is possible to prove the tightness result for other clusters, e.g. the largest cluster using certain glueing tricks we used. However, for crossing collection, the key strategy lies in retaining the long path while applying the glueing trick cleverly, and due to lack of any immediate alternative this looks quite challenging.

5.5 Discussions

Apart from the tightness conjecture, it would be also interesting to see whether we can also make sense of IIC-measure in slabs as [J03, Theorem 3], i.e by choosing a point randomly from the largest cluster in the box. For this we will require a result akin to [J03, Proposition 1] which states that the difference between the size of the largest and the second largest open cluster should diverge with probability 1 as we increase the size. Both of these seem hard to prove with the current tools we have.

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