The Emergence of Cosserat-type Structures in Metal Plasticity

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Contents

Notations

— We set $\mathbb{N} := \mathbb{Z} \cap [1, \infty)$, $\mathbb{N}^* := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := \mathbb{R} \cap [0, \infty)$;

— The letter C denotes constants whose values are allowed to vary from line to line and also within the same line. To emphasize that the constant C depends on the parameter δ, we will write $C = C_\delta$.

— With the notation $\alpha \gg C$ $(0 < \alpha \ll C)$ we mean $\alpha \geq C$ *$(0 < \alpha \leq C)$ is sufficiently large (sufficiently small)*;

— For $U \subset \mathbb{R}^n$, $B_r(U)$ is the (open) r-tubular neighborhood of U, i.e.

$$B_r(U) := \left\{ x \in \mathbb{R}^n \,\middle|\, \operatorname{dist}(x, U) < r \right\}.$$

$B(p, r)$ denotes the ball of radius r centered in p;

— $\mathcal{C}^k(\Omega)^m$ is the class of functions defined on $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$ which are continuous together with their derivatives of order up to k. We denote by $\mathcal{C}_c^k(\Omega)^m$ the subset of $\mathcal{C}_c^k(\Omega)^m$ whose elements have compact support. $L^p(\Omega)^m$ and $W^{k,p}(\Omega)^m$ denote the Lebesgue and Sobolev spaces of functions defined in Ω with values in $\mathbb{R}^m$;

— $\mathcal{M}_b(\Omega)^m$, with $\Omega \subset \mathbb{R}^n$, denotes the space of all measures on Ω with finite total variation;

— $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, while $\mathcal{L}^n$ (or $|\cdot|$) is the Lebesgue measure on $\mathbb{R}^n$;

— For a measure μ defined on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $\mu \llcorner A$ denotes the restriction of μ to A, i.e. $\mu \llcorner A(E) := \mu(A \cap E)$ for every subset E of $\mathbb{R}^n$;

— $|\mu|$ denotes the total variation of the vector valued measure μ defined on $\mathbb{R}^n$;

— $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$: $|v| := (v_1^2 + \cdots + v_n^2)^{\frac{1}{2}}$;

— If $v \in \mathbb{R}^n \setminus \{0\}$, we set $\widehat{v} := \frac{v}{|v|}$;

— For a matrix $A \in \mathbb{R}^{n \times n}$, A^{sym} denotes its symmetric part $A^{\mathrm{sym}} := \frac{1}{2}(A + A^T)$;

— Given $v, w \in \mathbb{R}^n$, $v \cdot w$ (also denoted as $\langle v, w \rangle$) is the scalar product of them, i.e. $v \cdot w := \sum_{i=1}^n v_i w_i$ and for two matrices $A, B \in \mathbb{R}^{n \times n}$, $A : B := \sum_{i,j=1}^n A^i_j B^i_j$;

— $\nabla^\perp$ is the "orthogonal" gradient: that is, for $f : \mathbb{R}^2 \to \mathbb{R}$, $\nabla^\perp f := \left(-\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\right)$.

Introduction

In this thesis we study the energy functional introduced in [1] by the author and S. Luckhaus–which is inspired by the ones introduced in [2] and [3]–able to describe low energy configurations of a two dimensional lattice with dislocations in the context of nonlinear elasticity. It consists of two terms: a nonlinear elastic energy outside the *core* of the dislocations and the core energy. Our main result says, roughly speaking, that low energy configurations consist of piecewise constant microrotations with small angle grain boundaries between them.

The plan of the the thesis is the following. In the first chapter, we begin recalling the physical background from which the problems arises, that is the one of metal plasticity, reassuming the basic notions of the theory of dislocations. We then give a review of the main results in the literature which motivated the model we study. In particular, we discuss the models introduced in [2], [3] and the Γ-convergence analysis given in [4] (see also [5] and [6]). The last section of the first chapter deals with Geometric Rigidity (especially with the results in [21] and [23]), which is a crucial ingredient in the proof of the main result. We also give a proof of a Geometric Rigidity estimate in dimension ≥ 3, which is scaling invariant for every exponent $p \in (\frac{n}{n-1}, 2]$. For the critical case $p^* = \frac{n}{n-1}$, we only obtain a weaker estimate which unfortunately misbehaves under rescaling, and thus cannot be applied to our analysis (cf. Theorem 1.4.9 and the Remark 1.4.1).

In Chapter 2, we study the functional introduced in [1]. For $\varepsilon > 0$, $L > 0$, $0 < \alpha \ll 1$ sufficiently small, $\tau > 0$ and $\lambda > 0$, we consider the family of *admissible strain fields* $\mathcal{A}(\varepsilon)$, whose elements $A : \Omega := [-L, L] \to \mathbb{R}^{2\times 2}$ satisfy the following conditions:

(i) *Regularity*: $A \in L^1_{\text{loc}}(\Omega)^{2\times 2}$ and $A \in L^2(\Omega \setminus B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl}(A)))$;

(ii) *Boundary Condition*: $A \equiv R_{\pm\alpha}$ near $x = \pm L$;

(iii) *(First) Quantization of the Burgers vector*: γ closed, Lipschitz, simple curve outside $B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl}(A))$:

$$\int_\gamma A \cdot t \mathrm{d}\mathcal{H}^1 \neq 0 \qquad \Rightarrow \left| \int_\gamma A \cdot t \mathrm{d}\mathcal{H}^1 \right| \geq \tau\varepsilon.$$

Then, for an admissible field A and a compact subset $S \Subset \Omega$ such that $S \supset B_{\lambda\varepsilon}(\operatorname{spt}(\operatorname{Curl}(A)))$

we define the energy functional

$$\mathcal{F}_\varepsilon(A,S) := \overbrace{\frac{1}{\tau}\int_{\Omega\setminus B_{\lambda\varepsilon}(S)} \mathrm{dist}^2(A,SO(2))\mathrm{d}x}^{\text{Elastic Energy}} + \underbrace{\frac{1}{\lambda}|B_{\lambda\varepsilon}(S)|}_{\text{Core Energy}}, \tag{0.0.1}$$

and we set $\mathcal{F}_\varepsilon(A,S) := \infty$ if S does not contain $B_{\lambda\varepsilon}(\mathrm{spt}(\mathrm{Curl}(A)))$.
An upper bound to the energy is given by the *Read-Shockley Formula*[18], which in our functional analytic setting reads as

Theorem 0.0.1. *There exists a constant $C_0 > 0$ such that*

$$\varlimsup_{\varepsilon\to 0} \inf_{(A,S)\ admissible} \frac{1}{\varepsilon}\mathcal{F}_\varepsilon(A,S) \leq C_0 L\alpha\left(|\log(\alpha)|+1\right). \tag{0.0.2}$$

The proof is carried out by the construction of a *small angle grain boundary*, that is of an array of edge dislocations, with adjusted boundary conditions. We call $E_{\mathrm{gb}}(\varepsilon) := C_0 L\varepsilon\alpha\left(|\log(\alpha)|+1\right)$ the *energy of a small angle grain boundary* at scale $\varepsilon > 0$ and with misorientation angle $\alpha > 0$, where C_0 is the constant in the right hand side of (0.0.2).
Our main result can then be stated as the *compactness in the class of microrotations*, in the following sense: every sequence (A_j, S_j) of admissible pairs whose energy does not exceed the one of a small angle grain boundary has a *competitor* (A'_j, S'_j) with "essentially the same energy", in the sense that $\mathcal{F}_\varepsilon(A'_j,S'_j) \leq C\mathcal{F}_\varepsilon(A_j,S_j)$ for a *universal* constant $C > 0$. The fields A'_j of such a competing sequence admit a subsequence which converges strongly in $L^2(\Omega)$ to a *piecewise constant* matrix field $A \in BV(\Omega, SO(2))$ (namely $A(x) \in SO(2)$ for a.e. $x \in \Omega$ and $DA = D^J A = (A^+ - A^-)\otimes \nu_A \mathcal{H}^1 \llcorner S_A$), i.e. to a *(generalized Cosserat) microrotation* (which can be seen as a generalization of the microrotations studied in [19]). This conclusion is achieved through a density estimate, which in turn is obtained coupling the existence of a harmonic competitor with the choice of an optimal foliation via a balls construction in the spirit of the one used for the Ginzburg-Landau functional (cf. [20] and the references therein). More precisely

Theorem 0.0.2. *There exists a constant $C > 0$ such that every sequence of admissible pair $\{(A_j,S_j)\} = \left\{(A_{\varepsilon_j},S_{\varepsilon_j})\right\}$, $\varepsilon_j \to 0$, with $\mathcal{F}_{\varepsilon_j}(A_j,S_j) \leq E_{gb}(\varepsilon_j)$, there exists another sequence (A'_j,S'_j) such that $\mathcal{F}_{\varepsilon_j}(A'_j,S'_j) \leq C\mathcal{F}_{\varepsilon_j}(A_j,S_j)$ which, up to subsequence, converges strongly in $L^2(\Omega)$ to a microrotation A and*

$$\left|A^+ - A^-\right| \sqrt{|\log\left(|A^+ - A^-|\right)|}\mathcal{H}^1 \llcorner S_A \leq C\mu,$$

where μ is the weak limit of the measures*

$$\mu_j := \frac{1}{\tau\varepsilon_j}\mathrm{dist}^2(A'_j,SO(2))\mathcal{L}^2 \llcorner \Omega + \frac{1}{\lambda\varepsilon_j}\mathcal{L}^2 \llcorner S'_j.$$

In particular

$$C^{-1}L\varepsilon_j\alpha\sqrt{|\log(\alpha)|} \leq \mathcal{F}_{\varepsilon_j}(A_j,S_j).$$

Theorem 0.0.2 is obtained directly from the following *density estimate*

Theorem 0.0.3. *Let $C > 0$ and $(A_j, S_j) = (A_{\varepsilon_j}, S_{\varepsilon_j})$ be a sequence of admissible pairs with $\mathcal{F}_{\varepsilon_j}(A_j, S_j) \leq CE_{gb}(\varepsilon_j)$ and such that $\Delta A_j = 0$ in $\Omega \setminus B_{\lambda\varepsilon_j}(S_j)$. Then, up to a subsequence, $A_j \to A \in BV(\Omega, SO(2))$ strongly in L^2. Moreover, there exist constants $C_0 > 0$, $\delta_1 \in (0,1)$ and $\omega_0 > 0$ such that for every $p \in \Omega$ and every $R > 0$ there exists an $\overline{R} \in [R, 2R]$ such that*

$$\left|\operatorname{Curl}(A)(B(p,\overline{R}))\right| \leq C\omega\left(\frac{\mu(B(p,3R))}{R}\right)\mu(B(p,3R)),$$

where $\omega : (0,\infty) \to (0,\infty)$ is the continuous function defined as

$$\omega(t) := \begin{cases} \omega_0 & \text{if } t \geq \delta_1, \\ (-\log t)^{-\frac{1}{2}} & \text{if } t \leq \delta_1. \end{cases}$$

The existence of a harmonic competitor is ensured by using the fact that the determinant is a null Lagrangian, and then replacing the "ground field" ∇u given by the Hodge decomposition of A ($A = \nabla u + F$, $\operatorname{div}(F) = 0$) by its harmonic extension, that is we have the

Proposition 0.0.4 (The Harmonic Competitor)**.** *There exists a constant $C > 0$, such that for every $\varepsilon > 0$ and $A \in \mathcal{A}(\varepsilon)$ there exists another field $A' \in \mathcal{A}(\varepsilon)$ which is harmonic outside $B_{\lambda\varepsilon}(A)$ and*

$$||A - A'||_{L^2(\Omega)} \leq C\,||\operatorname{dist}(A, SO(2))||_{L^2(\Omega)}.$$

Theorem 0.0.3 is then obtained via a balls construction, whose difficulty is due to the nonlinearity in the elastic energy. Indeed, one is not allowed to expand continuously the covering of the singular set as in the case of the Ginzburg-Landau functional, but is forced to use a *discrete* balls construction, as done in [6] in a different setting. Since in our context it is impossible to avoid the dislocations to collapse, we have to get rid of this obstruction via a particular *foliation*:

Lemma 0.0.5. *Let $\mathcal{A}_R := B(p,R) \setminus B(p,\frac{R}{2}) \subset \mathbb{R}^2$. There exist $\delta_0 > 0$ and $C > 0$ such that if $\{B(x_i,\varrho_i)\}_{i=1}^N$ are balls in $\mathbb{R}^2$ satisfying*

$$\mathcal{H}^1\left(\mathcal{A}_R \cap \bigcup_{i=1}^N \partial B(x_i,\varrho_i)\right) \leq \delta_0 R. \tag{0.0.3}$$

Then there exists a Lipschitz function $\varphi : \mathcal{A}_R \to [0,1]$ such that

1. *$||\nabla\varphi||_{L^\infty(\mathcal{A}_R)} \leq \frac{C}{R}$;*
2. *$\varphi \equiv 0$ on $\partial B(p,R)$ and $\varphi \equiv 1$ on $\partial B(p,\frac{R}{2})$;*
3. *If $U := \mathcal{A}_R \setminus \bigcup_{i=1}^N B(x_i,\varrho_i)$,*

$$\int_U \frac{|x|^2\,|\nabla\varphi|^2}{\operatorname{dist}^2(x,\partial U)}\mathrm{d}x \leq C(1+N). \tag{0.0.4}$$

The main obstruction in the construction of a foliation φ satisfying conditions $(i)-(iii)$ of Lemma 0.0.5 lies in the accumulation of balls at any "distance length scale" from U. In order to overcome this difficulty, we defined in [1] a *decision tree* which tells how and where to modify the natural radial foliation, through either a cut-off or setting the foliation to a constant (which involves an ad-hoc covering argument), possible because of (0.0.3). Using 0.0.5 and Proposition 0.0.4 one can start the balls construction and obtain Theorem 0.0.3. The idea is to start with a covering of $B_{\lambda\varepsilon_j}(S_j)$ with a family of disjoint balls $\mathcal{B}_0 := \{B(x_{i,0}, \varrho_{i,0})\}$ which are expanded until "there is only a small amount of singularities in the annuli", that is one considers the balls $B(x_{i_0}, \overline{\varrho}_{i,0})$, where

$$\overline{\varrho}_{i,0} := \sup \left\{ \varrho > 0 \,\middle|\, \left|\nabla\chi_{\bigcup\mathcal{B}_0}\right| \left(B(x_{i,0}, 2\varrho) \setminus B(x_{i,0}, \varrho)\right) > \delta_0 \varrho \right\}.$$

Then one needs another expansion (namely, an expansion of a factor of 30) in order to ensure the possibility of being able to extract a disjoint subfamily containing most of the mass of a given positive measure and a merging procedure in order to make the balls disjoint. That is, one can construct a family of coverings $\{\mathcal{B}_k\}_{k\geq 0}$ as follows:

$$\cdots \xrightarrow{\text{Merge}} \mathcal{B}_k = \left\{B\left(x_{k,i}, \varrho_{k,i}\right)\right\}_{i\in I_k} \xrightarrow{\text{Expand}} \left\{B\left(x_{k,i}, \overline{\varrho}_{k,i}\right)\right\}_{i\in I_k} \xrightarrow{\text{Vitali}} \left\{B\left(x_{k,i}, 6\overline{\varrho}_{k,i}\right)\right\}_{i\in I'_k} \xrightarrow{30\times}$$
$$\xrightarrow{30\times} \left\{B\left(x_{k,i}, 180\overline{\varrho}_{k,i}\right)\right\}_{i\in I'_k} \xrightarrow{\text{Merge}} \mathcal{B}_{k+1} = \left\{B\left(x_{k+1,i}, \varrho_{k+1,i}\right)\right\}_{i\in I_{k+1}} \xrightarrow{\text{Expand}} \cdots,$$

By construction, one can always find disjoint subfamilies where the measures

$$\tau_k^{(j)} := \sum_{j\in I_k} \left| \fint_{B_{k,j}} \operatorname{Curl} A_{\varepsilon_j} \mathrm{d}x \right| \mathcal{L}^2 \llcorner B_{k,j}$$

concentrate. We then combine this construction with the observation

$$\begin{aligned} \left|\int_{\varphi>0} \operatorname{Curl}(A_j)\mathrm{d}x\right| \leq & \sum_{B(q_i,r_i)\subset\{0<\varphi<1\}} \left|\int_{B(q_i,r_i)} \operatorname{Curl}(A_j)\mathrm{d}x\right| + \\ & \int_{0<\varphi<1} |x| \left|\nabla A_{\varepsilon_j,\mathrm{sym}}\right| |\nabla\varphi| \,\mathrm{d}x. \end{aligned}$$

Lemma 0.0.5, Proposition 0.0.4 and a bootstrap argument give then Theorem 2.7.1. It is then possible to give an estimate on the total variation of $\operatorname{Curl}(A)$ since the estimate in the "negative norm" holds for every point and every radius. Then one can finally estimate the total variation of the derivative DA in terms of $|\operatorname{Curl}(A)|$ since the geometric rigidity estimate in [23] gives that $A \in BV(\Omega, SO(2))$.

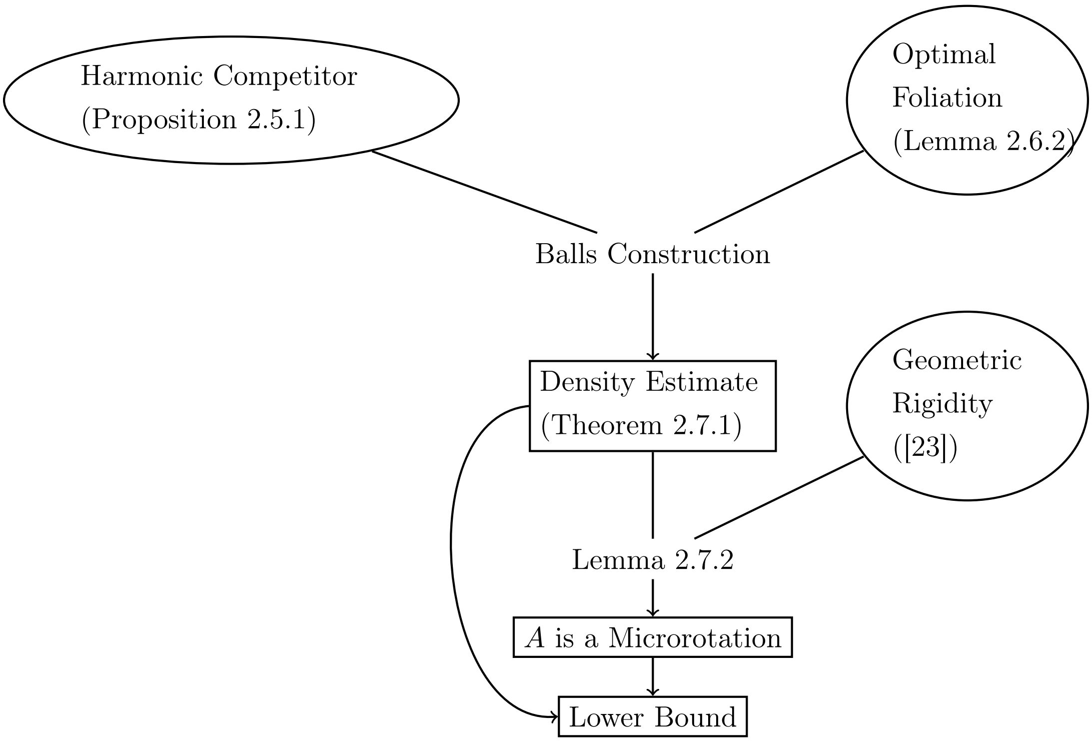

Figure 1: Compactness in the class of Microrotations and Lower Bound.

Chapter 1

Calculus of Variations and Defects in Solids

1.1 Defects in Crystals

It is well known that a wide class of materials, like metals, exhibit a *crystalline structure*, that is their atoms arrange themselves in patterns which are periodically repeated. Nevertheless, in real materials this structure is never perfect because of the presence of *defects*, which determinine important features of the material. We shall focus on *dislocations*, that are one dimensional defects responsible for the plastic behaviour of the material. We know want to examine the microscopic mechanism of plasticity, which can be explained through the (pattern) formation and motion of dislocations. We first start with a simple Gedankenexperiment. Consider a crystal $\mathcal{C}$ with a perfect cubic lattice structure, and suppose that a force is applied to the part of the crystal above the plane φ (which we call the *glide plane* of the dislocation). At the first stage, the bonds will break and reorganize as in Figure 1.1. If we let this force acting (assuming that it is strong enough in order to break the bonds but not to cause the fracture of the crystal), what we will see is the presence of an extra half plane q of atoms, which keeps moving on the glide plane. With this picture in mind, we consider the next element crucial for the determination of

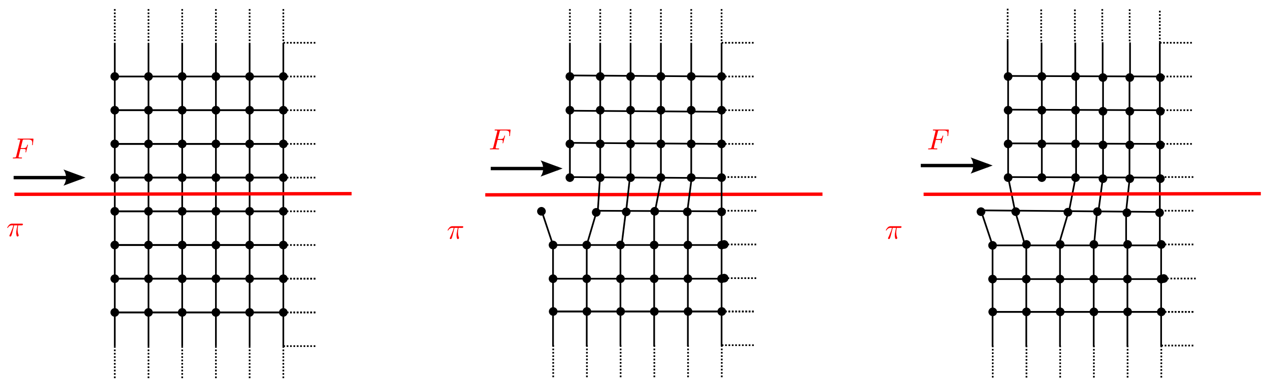

Figure 1.1: Motion of an edge dislocation along the glide plane π.

a dislocation, that is its *Burgers vector*. Consider a closed path in the undeformed lattice (like in figure (1a)) surrounding the last atom in the half line like in figure (1c). The image of this path in the deformed lattice will no longer be closed, and we call the (lattice) vector necessary in order to make it closed the *Burgers vector* of the dislocation, see figure 1.2. We can then define a dislocation as a pair $(\ell, \vec{b})$, where $\ell = \partial q \cap \mathcal{C}$ and $\bar{b}$ is the Burgers vector. We say that a pair $(\ell, \vec{b})$ an edge dislocation if $\vec{b}$ is orthogonal to ℓ and that it is a screw dislocation if it is parallel to ℓ. In general, dislocations need not to be edge or screw, but can be of mixed type.

Now that we have given a rough picture at the microscopic scale (i.e., the scale where one sees

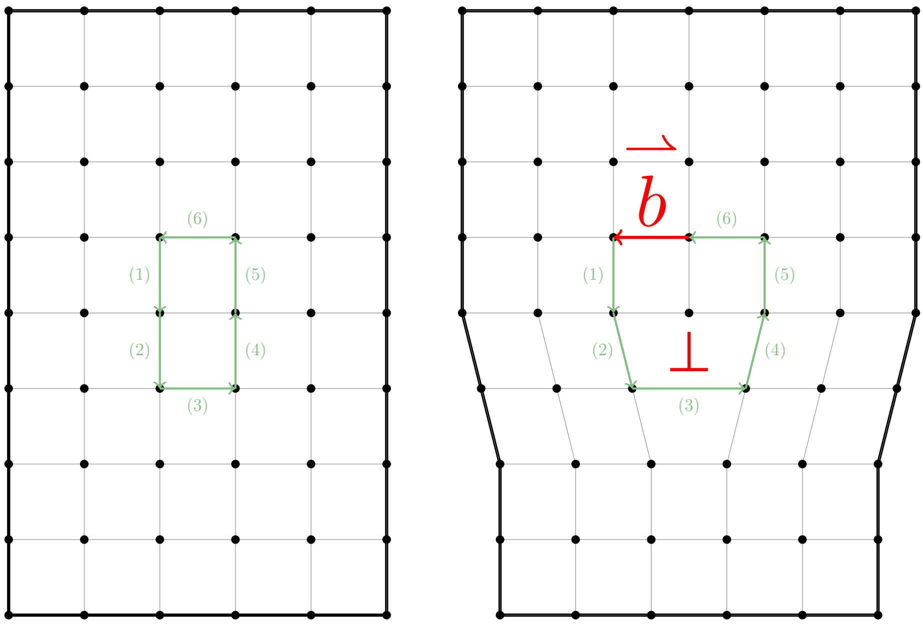

Figure 1.2: The Burgers' vector $\vec{b}$ of an edge dislocation.

atoms but cannot recognize the particular patterns they form), we can start looking at a higher scale, where one can recognize patterns of dislocations forming more complex structures. In the previous discussion we have seen the motion of a dislocation in a "cold" regime, namely confined to glide planes. We are actually more interested in the *annealed* context (Figure 1.3), where dislocations can move in any direction. We consider then another simple experiment. First, we take a thin bar of a metal, we bend it and then raise up the temperature, which will in particular make a "crystallization", that is it creates *small angle grain boundaries*. More precisely, a grain boundary is an interface where two crystals of different orientation join. Although in Chapter 2 we shall consider only one parameter in the description of the grain boundary, a complete analysis should take into account four different parameters (in three dimension even eight; cfr. Gottstein):

— α, the angle giving the orientation difference between two different crystals,

— the *grain boundary orientation angle*, which defines the spatial orientation of the grain boundary plane,

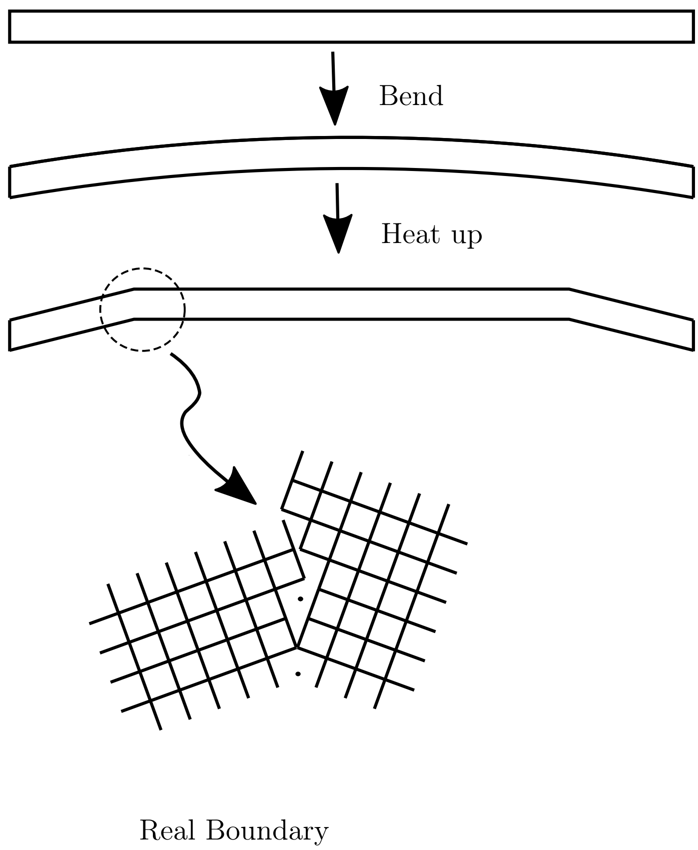

Figure 1.3: Formation of a grain boundary

— the two compontents of a two dimensional vector giving the relative translation of one grain with respect to each other.

We now discuss more in detail the reason for which we are interested in particular in grain boundaries. Considering that the distance between atoms in real crystals is extremely small (of the order of Angström), several authors (in particular Kondo, Kröner, Bilby et al., [11],[12] and [13]) discussed the possibility of a continuum description of the crystal which can still give a useful prediction of phenomena at (spatial) scales higher that the atomistic one. It turned out that a satisfactory approach is that of *non-Riemannian Geometry*, that is, to endow the region delimiting the crystal with the so callad *natural space structure*, i.e, with the metric $g(x)[v,w] := A(x)v \cdot A(x)w$ (where A is the *backstrain*) and with the connection $\Gamma^i_{jk}(x) := A^i_\beta(x)\partial_j A^\beta_k(x)$, which is curvature free but has torsion. Such a connection is also called a *Weitzenböck connection.* Being curvature free, it allows a distant parallelism, which is the feature which reminds of a underlying lattice structure, since then one is able to produce a global vector field displacing the primitive lattice vectors along themselves and each other. The torsion of this connection gives the dislocation density tensor.

In this construction, one makes a priori the assumption that a displaced vector in the continuized

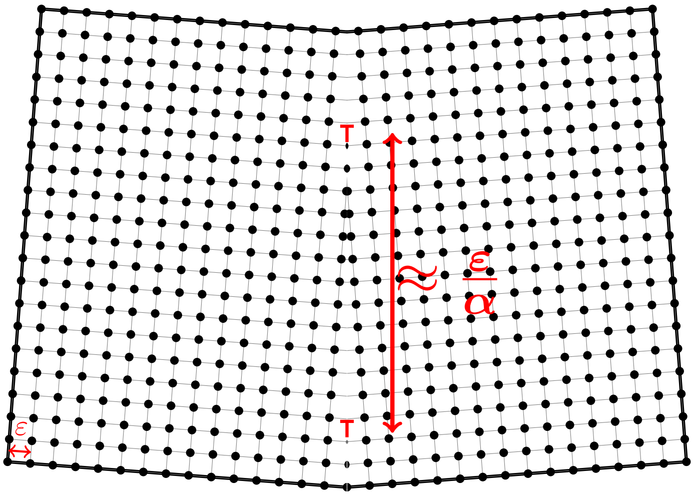

Figure 1.4: Approximation of a (symmetric tilt) small angle grain boundary with a row of dislocations.

crystal depends linearly on the starting vector and the displacement itself, making evindent the possibility of using a parallel displacement, that is an appropriate connection. As pointed out by Kondo, this assumption is reasonable in many applications, where the dislocation density is high but does not exceed a certain threshold depending on the desired accuracy of measurations of lengths (this is a consequence of what we call the *Kondo's uncertainty principle*). This condition is fulfilled in many applications but not, for example, in the case of grain boundaries.
This considerations contribute to make interesting the investigation of the continuum behavior of grain boundaries (more precisely, of *small angle symmetric tilt grain boundaries*), and we expect, at the mesoscopic scale, a picture different from the one predicted by the differential geometric approach.

1.2 Mesoscopic Theory

In the papers [2] and [3], the authors introduced and studied a mesoscopic reference-free model able to describe crystals with defects, starting from the assumption that the ground state lattice is a simple *Bravais* lattice, that is a lattice of the form $\mathcal{L}(\mathcal{G}) := \left\{ \mathcal{G}z \middle| z \in \mathbb{Z}^n \right\}$, where $\mathcal{G} \in GL^+(n, \mathbb{R})$ and $n = 2, 3$. Then, for any open set $\Omega \subset \mathbb{R}^n$ and particles configurations $\mathfrak{X} = \{x_i\}_{i \in I}$, $\#I < \infty$, they defined the Hamiltonian

$$H_\lambda(\mathfrak{X}, \Omega) := \int_\Omega \inf_{(A,\tau) \in A^+(n,\mathbb{R})} h_\lambda(x, (A, \tau), \mathfrak{X}) \mathrm{d}x, \tag{1.2.1}$$

where $A^+(n, \mathbb{R}) := GL(n, \mathbb{R})^+ \ltimes \mathbb{R}^n$ is the "positive" affine group, $\lambda > 0$ is a parameter giving the finite range interaction and h_λ is an energy density made up of three terms, which are taken as follows:

(i) The first term gives the nonlinear elastic energy associated to the matrix A, i.e. is of the form
$$F(A),$$
where $F : GL^+(n,\mathbb{R})$ is a function of class $\mathcal{C}^2$ satisfying the usual conditions of nonlinear elasticity:

— F is *frame indifferent*, i.e. $F(RA) = F(A)$ for every $R \in SO(n)$;

— F is invariant with respect to positive changes of the lattice basis of the Bravais lattice $\mathcal{L}(A)$, that is $F(A) = F(AB)$ for every $B \in SL(n,\mathbb{Z})$;

— F takes its minimum on the ground state lattice $\mathcal{L}_G$, i.e. $F(A) = 0$ if and only if $\mathcal{L}(A) = \mathcal{L}(G)$.

(ii) The second term measures locally how much the particle configuration differs from a perfect lattice. More precisely, to each particle $x_i \in \mathfrak{X}$ is assigned an excess energy through a periodic potential $W(\cdot, \mathcal{L}_x(A,\tau))$, where
$$\mathcal{L}_x(A,\tau) := \left\{ A(z-\tau) + x \middle| z \in \mathbb{Z}^n \right\},$$
whose periodicity is the one of the lattice $\mathcal{L}_x(A,\tau)$. It is of the form
$$\frac{1}{\lambda^n} \sum_{x_i \in I} \left(W(x_i, \mathcal{L}_x(A,\tau)) - \vartheta_0 \right) \varphi_{\lambda,x}(x_i),$$
where $\varphi_{\lambda,x} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a cut-off function whose support lies in the balls $B(x, 2\lambda)$ and $\vartheta_0 > 0$ is a constant. More precisely, besides other technical regularity conditions, the potential W satisfies the *growth condition*
$$C^{-1} \operatorname{dist}^2(x, \mathcal{L}(A,\tau)) \leq W(y,(A,\tau)) \leq C \operatorname{dist}^2(y, \mathcal{L}(A,\tau)), \qquad \forall (A,\tau) \in A^+(n,\mathbb{R}) \text{ and } y \in \mathbb{R}^n,$$
for some $C > 1$, and the *periodicity condition*
$$W(x,(A,\tau)) = W(x,(AB,\tau + Ab)) \quad \forall (B,b) \in A^+(n,\mathbb{Z}), \qquad W(x,(A,\tau)) = W(x-\tau,(A,0)).$$

(iii) The last term penalizes the presence in $\mathfrak{X} \cap B(x,\lambda)$ of *vacancies* with respect to $\mathcal{L}_x(A,\tau)$, by measuring the difference between $\det(A)^{-1}$ and the empirical density of $\mathfrak{X}$ in x:
$$\vartheta_1 \left(\det(A)^{-1} - \frac{1}{C_\varphi \lambda^n} \sum_{x_i \in \mathfrak{X}} \varphi_{\lambda,x}(x_i) \right),$$
where $\vartheta_1 > 0$ is a fixed constant and C_φ is a normalizing constant.

The energy density is then define as the sum of these three terms:

$$\begin{aligned} h_\lambda(x,(A,\tau),\mathfrak{X}) := F(A) + \frac{1}{\lambda^n}\sum_{x_i\in I}\left(W(x_i,\mathcal{L}_x(A,\tau)) - \vartheta_0\right)\varphi_{\lambda,x}(x_i) + \\ + \vartheta_1\left(\det(A)^{-1} - \frac{1}{C_\varphi\lambda^n}\sum_{x_i\in\mathfrak{X}}\varphi_{\lambda,x}(x_i)\right). \end{aligned} \tag{1.2.2}$$

The main results of [2] are of analytical-topological nature. The authors proved that low energy atomic configurations (satisfying an additional hard-core constraint) are characterized by a large set of low energy density, called *grains*. Moreover, each grain carries a natural fiber bundle structure, which we now describe more in details. First, given an admissible atoms configurations we look at a bounded subset $\widetilde{\Omega}$ of the trivial bundle $\Omega\times A^+(n,\mathbb{R})$, which can be chosen in such a way (see Theorem 4.4 of [2]) that for each $(x,g)\in\widetilde{\Omega}$ a large percentage of points $\mathfrak{X}\cap B(x,\lambda)$ are close to one and only one element of $\mathcal{L}_x(g)$, and there exists a unique (modulo $A^+(,\mathbb{Z})$) simple Bravais lattice $\mathcal{L}_x(g)$ "optimally fitted" with $\mathfrak{X}\cap B(x,\lambda)$. Now, consider the map ([2], Theorem 4.5)

$$\mathrm{Argmin}(x) := \{x\}\times[g]_x,$$

where g is a local minimum of $h_\lambda(x,\cdot,\mathfrak{X})$ (which exists and is unique modulo $A^+(n,\mathcal{Z})$, by the choice of $\widetilde{\Omega}$), and $[g]_x$ is the equivalence class of g modulo generation of the same simple Bravais lattice, that is

$$[(A,\tau)]_x := \left\{(AB, x - AB(B^{-1}\tau + b))\middle|(B,b)\in A^+(n,\mathbb{Z})\right\}.$$

Then it is possible to prove that for each $(x_0,g_0)\in\mathrm{Argmin}(x_0)$ and every $U\subset\pi_\Omega\widetilde{\Omega}$ open, simply connected neighborhood, there exits an open neighborhood $V\subset A^+(n,\mathbb{R})$ and a diffeomorphism $\Gamma=\Gamma_{g_0}=(A_{g_0},\tau_{g_0}):U\to V$[1] of class $\mathcal{C}^1$ such that

$$\Gamma(x_0)=g_0, \qquad (x,\Gamma(x))\in\mathrm{Argmin}(x) \qquad \forall x\in U. \tag{1.2.3}$$

Moreover, there exists a constant $C=C(g)$ (dependent also of other parameters which define $\widetilde{\Omega}$) such that

$$\lambda\,||\nabla A_{g_0}||_{L^\infty(U)} + \left|\left|\nabla\tau_{g_0} - A_{g_0}^{-1}\right|\right|_{L^\infty(U)} \le \frac{C}{\lambda}.$$

If we start from the representative g_1 instead of g_0, and we pick a different neighborhood U', we will end up with a different map $\Gamma'=\Gamma_{g_1}:U'\to V'$, but such that for every connected open subset $U''\subset U\cap U'$

$$\Gamma^{-1}(x)\circ\Gamma'(x)\equiv\Gamma_{g_0,g_1}\in A^+(n,\mathbb{Z}). \tag{1.2.4}$$

We now discuss how the authors used this results in order to give a notion of *generalized dislocations*, and how to detect them through a particular G-covering as follows (which gives in

[1] This is the starting point of the analysis in Chapter 2. Indeed, the matrix fields A in [1] should be interpreted locally as the $\nabla\tau$ in [2] and [3].

particular a fibre bundle structure, with a trivial connection though). Firstly, consider

$$P := \bigcup_{x \in \widetilde{\omega}} \operatorname{Argmin}(x),$$

where $\widetilde{\omega}$ is the projection of $\widetilde{\Omega}$ on Ω. It is possible to show that P is a $\mathcal{C}^1$ manifold embedded in $\Omega \times A^+(n, \mathbb{R})$. The discrete affine group $A^+(n, \mathbb{Z})$ (with product rule $(A, a)(B, b) = (AB, B^{-1}a + b)$) acts on P (freely, transitively and properly discontinuously) on the right via

$$(x, [(A, \tau)]_x) \cdot (B, b) := \left(x, \left[\left(AB, x = AB(B^1\tau + b)\right)\right]_x\right).$$

The projection π from P onto the orbit space $P/A^+(n, \mathbb{Z}) := \left\{p \cdot A^+(n, \mathbb{Z}) \middle| p \in P\right\}$ (where $p \cdot A^+(n, \mathbb{Z}) := \left\{p \cdot g \middle| g \in A^+(n, \mathbb{Z})\right\}$) gives a normal covering space

$$p : P \to P/A^+(n, \mathbb{Z}).$$

It is also possible to show that $P/A^+(n, \mathbb{Z})$ is homeomorphic to $\widetilde{\omega}$. Now, for every $x_0 \in \widetilde{\omega}$, we pick an element $[g]_{x_0} \in \operatorname{Argmin}(x_0)$. Let now $[\gamma] \in \pi_1(\widetilde{\omega}, x_0)$ be an element of the fundamental group of $\widetilde{\omega}$ based at x_0, and consider a representative $\gamma \in [\gamma]$ (here, $[\gamma]$ is as usual the equivalence modulo homotopy of loops). Since p defines a normal covering space (with deck transformations given simply by right multiplication), we can find an element $h = h([g]_{x_0}, \gamma) \in A^+(n, \mathbb{Z})$ such that

$$\widetilde{\gamma}_{[g]_{x_0}}(0) = \widetilde{\gamma}_{[g]_{x_0}} \cdot h,$$

where $\widetilde{\gamma}_{[g]_{x_0}}$ is the unique (horizontal) lift of γ such that $\widetilde{\gamma}_{[g]_{x_0}}(0) = [g]_{x_0}$. Moreover, using (1.2.4), one can show that

$$h\left([g]_{x_0} \cdot z, \gamma\right) = z^{-1} h\left([g]_{x_0}, \gamma\right) z, \qquad \forall z \in A^+(n, \mathbb{Z}),$$

and, for every curve $\gamma' \in [\gamma]$ there exists a $z \in A^+(n, \mathbb{Z})$ for which

$$h\left([g]_{x_0}, \gamma'\right) = z^{-1} h\left([g]_{x_0}, \gamma\right).$$

In particular, we end up with a well defined homomorphism

$$\begin{array}{rccc} \Psi : & \pi_1(\widetilde{\omega}, x_0) & \longrightarrow & A^+(n, \mathbb{Z})/\sim \\ & [\gamma] & \longmapsto & [h([g]_{x_0}, \gamma)]_\sim \end{array},$$

where $[g]_{x_0} \in \operatorname{Argmin}(x_0)$. The map Ψ is independent of the particular base point in the connected component C_{x_0} of $\widetilde{\omega}$ which contains x_0. As a corollary, one can see that the image of the fundamental group of C_{x_0} through Ψ is trivial, $\Psi\left(\pi_1(C_{x_0}, x_0)\right) = \{[(\mathrm{Id}, 0)]_\sim\}$ if and only if for every $U \Subset C_{x_0}$ and ever $g \in \operatorname{Argmin}(x_0)$ there exists a function $\Gamma : U \to A^+(n, \mathbb{R})$ as in the discussion before. We can rephrase this saying that the obstruction to the definition of global approximate Lagrangian coordinates on the whole connected component C_{x_0} is encoded by the *holonomy representation map* Ψ. We can then say that $[\gamma] \in \pi_1(\widetilde{\omega}, x_0)$ links a *generalized dislocation* if

$$\Psi([\gamma]) = [(\mathrm{Id}, b_\gamma)]_\sim ,$$

for some $b_\gamma \in \mathbb{Z}^n \setminus \{0\}$. The *Burgers vector* at the base point x_0 of $[\gamma]$ is then the projection on $\mathbb{R}^n$ of the difference between the map Γ obtained in the discussion before at the ending points of γ (which is well defined, that is independent of the particular representative of $[\gamma]$):

$$b_{\Omega,x_0}([\gamma]) := \pi_{\mathbb{R}^n} \left(\Gamma(\gamma(1)) - \Gamma(\gamma(0))\right).$$

Another approach to detect dislocations would be to consider the "meso-scale" system of local inverse deformations $\{(U_\alpha, \tau_\alpha)\}_{\alpha\in\mathcal{A}}$ (in the sense of [10]), where the τ_α are the translational components of the affine maps Γ_α obtained in (1.2.3), and then study the associated connection $\mathrm{d}\,(\nabla\tau)^{-1}\,\nabla\tau$ together with the holonomy group it generates. This approach is discussed more in detail in [3].

1.3 Γ-Convergence approach of Leoni, Garroni and Ponsiglione

In this section we discuss the Γ-convergence results proved in [4] in the context of linear elasticity (see also [5], [6] and [7], [8], [9] for the analysis of a related phase field model). The functional we shall study in Chapter 2 is similar to the one they study, although we are interest in different limit scales and we do not make any dislocation density assumption. The authors considered a two dimensional section of a crystal with dislocations, represented by an open and bounded subset Ω of $\mathbb{R}^2$ with Lipschitz boundary. They then considered the set of *Burgers vectors* $S := \{b_1, \cdots, b_s\} \subset \mathbb{R}^2$ and its $\mathbb{Z}$-linear combinations $\mathbb{S} := \mathrm{Span}_{\mathbb{Z}}(S)$, assuming $\mathbb{S} = \mathbb{R}^2$, and the three parameters

- ε, the lattice parameter;
- ϱ_ε, the radius of the core surrounding a dislocation;
- N_ε, the number of dislocations in Ω.

Such parameters should satisfy the conditions

- $\lim_{\varepsilon\to 0} \frac{\varrho_\varepsilon}{\varepsilon^s} = \infty$ for all $s \in (0,1)$ (i.e., the hard core region contains most of the self energy);
- $\lim_{\varepsilon\to 0} N_\varepsilon \varrho_\varepsilon^2 = 0$ (that is, the area of the hard core region tends to zero);

The class of *admissible dislocations* is then defined as the class of all vector valued Radon measures $\mu \in \mathcal{M}(\Omega)^2$ of the form

$$\mu = \sum_{i=1}^{M} \xi_i \delta_{x_i},$$

where δ_x denotes the Dirac δ concentrated at x, $x_i \in \mathbb{S}$, $M \in \mathbb{N}$, $B(x_i, \varrho_\varepsilon) \subset \Omega$ and $|x_i - x_j| \geq 2\varrho_\varepsilon$ for all $i \neq j$. For notational simplicity, $\Omega_r(\mu)$ denotes the "drilled" domain

$$\Omega_r(\mu) := \Omega \setminus \bigcup_{x_i \in \mathrm{spt}(\mu)} \overline{B(x_i, r)}.$$

Finally, the class of *admissible strains* $AS_\varepsilon(\mu)$ with respect to the admissible dislocation μ is given by all those matrix fields $\beta \in L^2(\Omega)^{2\times 2}$ which are identically 0 in $\Omega_\varepsilon(\mu)$, whose distributional Curl vanishes in $\Omega_\varepsilon(\mu)$ and

$$\int_{\partial B(x_i,\varepsilon)} \beta \cdot t \mathrm{d}\mathcal{H}^1 = \xi_i, \qquad \int_{\Omega_\varepsilon(\mu)} \left(\beta - \beta^T\right) \mathrm{d}x = 0 \qquad \forall i = 1, \cdots, M,$$

where $t(x)$ is the tangent vector to $B(x_i, \varepsilon)$ at the point x, and the integrand $\beta \cdot t$ is defined in the sense of traces (see also Section 1 in Chapter 2). The elastic energy of a pair (μ, β), with $\mu \in X_\varepsilon$ and $\beta \in AS_\varepsilon(\mu)$ is given by

$$E_\varepsilon(\mu, \beta) := \int_{\Omega_\varepsilon(\beta)} W(\beta)\mathrm{d}x = \int_\Omega W(\beta)\mathrm{d}x,$$

where W is of the form

$$W(A) := \frac{1}{2}\mathbb{C}A : A,$$

where $\mathbb{C}$ is the *linear elasticity* tensor, which is assumed to satisfy the growth bounds

$$C^{-1}\left|A^{\mathrm{sym}}\right|^2 \leq \mathbb{C}A : A \leq C\left|A^{\mathrm{sym}}\right|^2 \qquad \forall A \in \mathbb{R}^{2\times 2},$$

for some constant $C > 0$. They then consider three different energetic regimes, depending on the number of dislocations N_ε:

— The *subcritical (dilute) regime*, where $N_\varepsilon \ll |\log(\varepsilon)|$. In this case

$$\mathcal{F}_\varepsilon^{\mathrm{dilute}}(\mu, \beta) := \begin{cases} \frac{1}{N_\varepsilon|\log(\varepsilon)|} E_\varepsilon(\mu, \beta), & \text{if } \mu \in X_\varepsilon, \qquad \beta \in AS_\varepsilon(\mu), \\ \infty & \text{otherwise;} \end{cases}$$

— The *critical regime*, where $N_\varepsilon = |\log(\varepsilon)|$. Then the rescaling is of order $|\log(\varepsilon)|^2$:

$$\mathcal{F}_\varepsilon(\mu, \beta) := \begin{cases} \frac{1}{|\log(\varepsilon)|^2} E_\varepsilon(\mu, \beta), & \text{if } \mu \in X_\varepsilon, \qquad \beta \in AS_\varepsilon(\mu), \\ \infty & \text{otherwise;} \end{cases}$$

— The *supercritical regime*, where $N_\varepsilon \gg |\log(\varepsilon)|$. In this case

$$\mathcal{F}_\varepsilon^{\mathrm{super}}(\mu, \beta) := \begin{cases} \frac{1}{N_\varepsilon^2} E_\varepsilon(\mu, \beta), & \text{if } \mu \in X_\varepsilon, \qquad \beta \in AS_\varepsilon(\mu), \\ \infty & \text{otherwise;} \end{cases}$$

The definition of the Γ-limits of these sequences rely on the relaxation of a particular cell problem. More precisely, let $AS_{\varepsilon,1}(\xi)$ be the family of stress fields satisfying $\beta \in L^2(B(0,1) \setminus B(0,\varepsilon))^{2\times 2}$, $\operatorname{Curl}\beta = 0$ in $B(0,1) \setminus B(0,\varepsilon)$

$$\int_{\partial B(0,\varepsilon)} \beta \cdot t \mathrm{d}\mathcal{H}^1 = \xi,$$

and define, for $\xi \in \mathbb{R}^2$, the function

$$\psi_\varepsilon(\xi) := \frac{1}{|\log(\varepsilon)|} \min_{\beta \in AS_{\varepsilon,1}(\xi)} \int_{B(0,1)\setminus} .$$

In [4] the authors proved that the functions φ_ε converge pointwise to a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, and they also gave an explicit formula for it. They finally define the density $\varphi : \mathbb{R}^2 \to [0,\infty)$ as the relaxation

$$\varphi(\xi) := \inf \left\{ \sum_{i=1}^N \lambda_i \psi(\xi_i) \middle| \sum_{i=1}^N \lambda_i \xi_i = \xi, N \in \mathbb{N}, \lambda_i \geq 0, \xi_i \in \mathbb{S} \right\}.$$

We can then summarize the main results of [4] as

Theorem 1.3.1. *(a)* Critical Regime. *Define the functional* $\mathcal{F} : \mathcal{M}(\Omega, \mathbb{R}^2) \times L^2(\Omega)^{2\times 2} \to \mathbb{R}$ *as*

$$\mathcal{F}(\mu,\beta) := \begin{cases} \int_\Omega W(\beta)\mathrm{d}x + \int_\Omega \varphi\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right) \mathrm{d}\,|\mu|\,, & \textit{if } \mu \in H^{-1}(\Omega)^2,\ \operatorname{Curl}\beta = \mu, \\ \infty & \textit{otherwise.} \end{cases}$$

Then the following holds:

— *(Compactness) Let $\varepsilon_n \to 0$ and $\{(\mu_n, \beta_n)\}$ be a sequence in $\mathcal{M}(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ such that $\mathcal{F}_{\varepsilon_n}(\mu_n,\beta_n) \leq E$ for every n, for some positive constant $E > 0$. There, up to a subsequence, there exists a measure $\mu \in H^{-1}(\Omega)^2$ and $\beta \in L^2(\Omega)^{2\times 2}$ with* $\operatorname{Curl}\beta = \mu$ *and*

$$\frac{1}{|\log(\varepsilon_n)|}\mu_n \underset{*}{\rightharpoonup} \mu \textit{ in } \mathcal{M}(\Omega)^2, \tag{1.3.1}$$

$$\frac{1}{|\log(\varepsilon_n)|}\beta_n \rightharpoonup \beta \textit{ in } \mathcal{L}^2(\Omega)^{2\times 2}; \tag{1.3.2}$$

— *(Γ-convergence) The functionals $\mathcal{F}_\varepsilon$ Γ-converge to $\mathcal{F}$ as $\varepsilon \to 0$ with respect to the convergence of μ_n and β_n as above. More precisely one has the* $\Gamma - \liminf$ inequality*:*

For $(\mu,\beta) \in (\mathcal{M}(\Omega)^2 \cap H^{-1}(\Omega)^2) \times L^2(\Omega)^{2\times 2}$ with $\operatorname{Curl}\beta = \mu$ *and for every sequence (μ_n,β_n) satisfying* (1.3.1) *and* (1.3.2)*, we have*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon,\beta_\varepsilon) \geq \mathcal{F}(\mu,\beta),$$

and the $\Gamma - \limsup$ inequality*:*

For $(\mu,\beta) \in (\mathcal{M}(\Omega)^2 \cap H^{-1}(\Omega)^2) \times L^2(\Omega)^{2\times 2}$ with $\operatorname{Curl}\beta = \mu$ *there exists a recovery sequence (μ_n,β_n) satisfying* (1.3.1) *and* (1.3.2) *such that*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon,\beta_\varepsilon) \leq \mathcal{F}(\mu,\beta);$$

(b) Subcritical Regime. *Let $N_\varepsilon \to \infty$ in such a way that $\frac{N_\varepsilon}{|\log(\varepsilon)|} \to 0$, and define the functional*

$$\mathcal{F}^{dilute}(\mu,\beta) := \begin{cases} \int_\Omega W(\beta)\mathrm{d}x + \int_\Omega \varphi\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right)\mathrm{d}\,|\mu|\,, & \text{if } \mu \in H^{-1}(\Omega)^2,\ \operatorname{Curl}\beta = 0,\\ \infty & \text{otherwise.}\end{cases}$$

Then

— *(Compactness) Let $\varepsilon_n \to 0$ and $\{(\mu_n,\beta_n)\}$ be a sequence in $\mathcal{M}(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ such that $\mathcal{F}^{dilute}_{\varepsilon_n}(\mu_n,\beta_n) \le E$ for every n, for some positive constant $E > 0$. There exists a $\mu \in \mathcal{M}(\Omega)^2$ and $\beta \in L^2(\Omega)^{2\times 2}$ with $\operatorname{Curl}\beta = 0$ such that, up to a subsequence,*

$$\frac{1}{N_{\varepsilon_n}}\mu_n \underset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega)^2, \tag{1.3.3}$$

$$\frac{1}{(N_{\varepsilon_n}|\log(\varepsilon_n)|)^{\frac{1}{2}}}\beta_n \rightharpoonup \beta \text{ in } \mathcal{L}^2(\Omega)^{2\times 2}; \tag{1.3.4}$$

— *(Γ-convergence) The functionals $\mathcal{F}^{dilute}_\varepsilon$ Γ-converge to $\mathcal{F}$ as $\varepsilon \to 0$ with respect to the convergence of μ_n and β_n as above. More precisely one has the $\Gamma - \liminf$ inequality:*

For $(\mu,\beta) \in (\mathcal{M}(\Omega)^2 \cap H^{-1}(\Omega)^2) \times L^2(\Omega)^{2\times 2}$ with $\operatorname{Curl}\beta = 0$ and for every sequence (μ_n,β_n) satisfying (1.3.3) and (1.3.4), we have

$$\liminf_{\varepsilon\to 0} \mathcal{F}^{dilute}_\varepsilon(\mu_\varepsilon,\beta_\varepsilon) \ge \mathcal{F}^{dilute}(\mu,\beta),$$

and the $\Gamma - \limsup$ inequality:

For $(\mu,\beta) \in (\mathcal{M}(\Omega)^2 \cap H^{-1}(\Omega)^2) \times L^2(\Omega)^{2\times 2}$ with $\operatorname{Curl}\beta = 0$ there exists a recovery sequence (μ_n,β_n) satisfying (1.3.3) and (1.3.4) such that

$$\limsup_{\varepsilon\to 0} \mathcal{F}^{dilute}_\varepsilon(\mu_\varepsilon,\beta_\varepsilon) \le \mathcal{F}^{dilute}(\mu,\beta);$$

(c) Supercritical Regime. *Let $N_\varepsilon \to \infty$ in such a way that $\frac{N_\varepsilon}{|\log(\varepsilon)|} \to \infty$, and define the functional*

$$\mathcal{F}^{super}(\beta^{sym}) := \begin{cases} \int_\Omega W(\beta^{sym})\mathrm{d}x, & \text{if } \beta^{sym} \in L^2(\Omega)^{2\times 2},\\ \infty & \text{otherwise.}\end{cases}$$

Then

— *(Compactness) Let $\varepsilon_n \to 0$ and $\{(\mu_n,\beta_n)\}$ be a sequence in $\mathcal{M}(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ such that $\mathcal{F}^{super}_{\varepsilon_n}(\mu_n,\beta_n) \le E$ for every n, for some positive constant $E > 0$. There exists a $\beta^{sym} \in L^2(\Omega)^{2\times 2}$ such that, up to a subsequence,*

$$\frac{1}{N_{\varepsilon_n}}\beta^{sym}_n \rightharpoonup \beta^{sym} \text{ in } \mathcal{L}^2(\Omega)^{2\times 2}; \tag{1.3.5}$$

— *(Γ-convergence) The functionals $\mathcal{F}^{super}_\varepsilon$ Γ-converge to $\mathcal{F}^{dilute}$ as $\varepsilon \to 0$ with respect to the convergence of β^{sym}_n as above. More precisely one has the $\Gamma - \liminf$ inequality:*

For every symmetric matrix field $\beta^{sym} \times L^2(\Omega)^{2\times 2}$ and for every sequence (μ_n, β_n) satisfying (1.3.5)*, we have*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{super}(\mu_\varepsilon, \beta_\varepsilon) \geq \mathcal{F}^{super}(\mu, \beta),$$

and the $\Gamma - \limsup$ inequality*:*

Given a symmetric matrix field $\beta^{sym} \times L^2(\Omega)^{2\times 2}$, there exists a recovery sequence (μ_n, β_n) satisfying (1.3.5) *such that*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{super}(\mu_\varepsilon, \beta_\varepsilon) \leq \mathcal{F}^{super}(\mu, \beta).$$

1.4 Geometric Rigidity and its role in plasticity

In the study on linear elasticity models, Korn's inequality plays a crucial role. We can state it as follows (see [22] and the references therein):

Theorem 1.4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded connected Lipschitz domain. There exists a constant $C = C(\Omega) > 0$, depending only on the domain Ω, such that for every $u \in W^{1,2}(\Omega)^n$ there exists a skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$ (i.e. $S^T = -S$) such that*

$$||\nabla u - S||_{L^2(\Omega)^{n\times n}} \leq C \left|\left| \frac{\nabla u + (\nabla u)^T}{2} \right|\right|_{L^2(\Omega)^{n\times n}}.$$

That it, Korn's inequality says that the gradient of a Sobolev function can be estimated, after removing a constant antisymmetric matrix, by its symmetric part. The nonlinear counterpart of Korn's inequality is often called *geometric rigidity.* We give here a short review of the most important geometric rigidity estimates. The prototype of geometric rigidity estimate is Lioville's theorem, which we can state in modern terms ([21]) as follows:

$$u \in W^{1,2}(\Omega)^n, \quad \nabla u(x) \in SO(n) \text{ for a.e. } x \in \Omega \quad \Longrightarrow u \text{ is an affine map},$$

where $\Omega \subset \mathbb{R}^n$ is an open, connected set. Indeed, since $\nabla u \in SO(n)$ a.e., $\nabla u = \mathrm{cof}(\nabla u)$. But the *Piola's identity* gives $\mathrm{div}\,(\mathrm{cof}(\nabla u)) \equiv 0$, hence u is harmonic in Ω (hence, smooth). Then ∇u is harmonic itself, and hence constant, since it has constant norm. A natural question is whether or not Liouville's theorem is "stable", in the sense that if a gradient is close to the orthogonal group in average, is it close to a single rotation in average? A first answer in this direction was given by John ([14], [15]). He considered vector fields $u \in \mathcal{C}^1(Q)^n$ defined on a cube $Q = Q(x, L) \subset \mathbb{R}^n$, and the *strain matrix*

$$e_u(x) := \frac{1}{2}\left(\nabla u(x)\nabla u(x)^T - \mathrm{Id}\right),$$

and the *maximum strain*

$$\varepsilon(u) := \sup_{x\in Q} |u(x)|.$$

A vector field u is said to be *δ-quasiisometric* if $\varepsilon(u) \leq \delta$. In [14], the author proved the following

Theorem 1.4.2. *There exists a $\delta = \delta(n) > 0$ and a constant $C = C(n) > 0$ such that for any δ-quasiisometry $u \in \mathcal{C}^1(Q)$ there exists a rotation R and a vector $c \in \mathbb{R}^n$ such that*

— $\left|\frac{|u(x)-u(y)|}{|x-y|} - 1\right| \leq C\varepsilon(u)$ *for all* $x, y \in Q$;

— $|u(x) - Rx - c| \leq CL\varepsilon(u)$ *for all* $x \in Q$;

— $⨍_Q |\nabla u - R| \,\mathrm{d}x \leq C_n \varepsilon(u)$.

Then, in [15] he proved the L^p-version of the statement:

Theorem 1.4.3. *Let $p > 1$. There exists $\delta = \delta(n,p) > 0$ and $C = C(n,p) > 0$ such that for any δ-quasiisometry $u \in \mathcal{C}^1(Q)$ there exists a rotation R and a vector $c \in \mathbb{R}^n$ such that*

— $\left|\frac{|u(x)-u(y)|}{|x-y|} - 1\right| \leq C \,||e_u||_{L^p(Q)}$ *provided $p > n$ and $x, y \in Q$ are such that $|x-y| \geq \frac{L}{2}$;*

— $|u(x) - Rx - c| \leq CL \,||e_u||_{L^p(Q)}$ *provided $p > n$ and $x \in Q$;*

— $||\nabla f - R||_{L^p(Q)} \leq C \,||e_u||_{L^p(Q)}$.

John's result was then improved by Kohn in [16], without assuming any a priori pointwise hypotheses on u:

Theorem 1.4.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and let $p \geq 1$, with $p \neq n$. There exist positive constants $C_1 = C_1(\Omega, p), C_2 = C_2(\Omega)$ such that for every bi-Lipschitzian map $u : \Omega \to \mathbb{R}^n$ there exist a rigid motion γ and $R \in SO(n)$ satisfying*

(i) if $1 \leq p < n$, then

$$||u - \gamma||_{L^q(\Omega)^n} + ||u - \gamma||_{L^p(\partial\Omega)^n} \leq C_1 \,||e(u)||_{L^p(\Omega)} \,,$$

with $q = \frac{np}{n-p}$, and "nonlinear elastic strain" $e(u)$ is defined as

$$e(u) := (\lambda_n - 1)_+ + (\lambda_2 \cdots \lambda_n - 1)_+ + |\det(G_u) - 1| \,,$$

where $G_u(x) := \sqrt{\nabla u(x)^T \nabla u}$ and $\lambda_1, \cdots, \lambda_n$ are its eigenvalues;

(ii) if $p > n$, then

$$||u - \gamma||_{L^\infty(\Omega)^n} \leq C_1 \,||e(u)||_{L^p(\Omega)} \,;$$

(iii) if $\widetilde{e}(u) := ||G_u^2 - Id||$, then

$$||\nabla u - R||^2_{L^2(\Omega)^{n\times n}} \leq C_2 \,||e(u) + \widetilde{e}(u)||_{L^1(\Omega)} \,.$$

The fundamental improvement was then achieved by Friesecke, James and Müller (see [21]). They indeed proved a geometric rigidity inequality without imposing any smallness of the elastic energy or invertibility assumptions on the fields:

Theorem 1.4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$, and let $1 < p < \infty$. There exists a constant $C = C(p, \Omega)$ such that for every $u \in W^{1,2}(\Omega)$ there exists a rotation $R \in SO(n)$ such that*

$$||\nabla u - R||_{L^p(\Omega)^{n\times n}} \leq C\, ||\mathrm{dist}(\nabla u, SO(n))||_{L^p(\Omega)^{n\times n}}\,.$$

We now need to recall the notion of weak-L^p spaces. A real-valued function f from a measure space (X, μ) is in $L^{p,\infty}(X, \mu)$ or $(L^p_w(X, \mu))$ if

$$||f||_{L^{p,\infty}(X,\mu)} := \sup_{t>0} t\mu\left(\left\{x \in X \middle| |f(x)| > t\right\}\right)^{\frac{1}{p}} < \infty.$$

Is easy to check that $||\cdot||_{L^{p,\infty}}\,(X, \mu)$ is only a quasi-norm, that is the triangle inequality holds just in the weak form

$$||f + g||_{L^{p,\infty}(X,\mu)} \leq C_p \left(||f||_{L^{p,\infty}(X,\mu)} + ||g||_{L^{p,\infty}(X,\mu)}\right).$$

We write $L^{p,\infty}(\Omega)$ for $L^{p,\infty}(\Omega, |\cdot|)$, when $\Omega \subset \mathbb{R}^n$ and $|\cdot|$ is the Lebesgue measure. Conti, Dolzmann and Müller proved in particular the following geometric rigidity estimate in weak-L^p spaces:

Theorem 1.4.6. *Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded connected domain. There exists a constant $C > 0$ depending only on p, n and Ω such that for every $u \in W^{1,1}(\Omega)^n$ such that $\mathrm{dist}(\nabla u, SO(n)) \in L^{p,\infty}(\Omega)^{n\times n}$ there exists a rotation $R \in SO(n)$ such that*

$$||\nabla u - R||_{L^{p,\infty}(\Omega)^{n\times n}} \leq C\, ||\mathrm{dist}(\nabla u, SO(n))||_{L^{p,\infty}(\Omega)^{n\times n}}\,. \tag{1.4.1}$$

We are going to apply Theorem 1.4.6 in order to obtain its generalization (on convex domains) in the case of incompatible fields, that is to tensor fields A which are not gradients. The idea is to correct the field A to a gradient via a averaged homotopy operator, and then to estimate the latter. In what follows, $\widehat{x} := \frac{x}{|x|}$, while $L^p(U, \Lambda^r)$ $(W^{m,p}(U, \Lambda^r))$ denotes the space of r-forms on U whose coefficients are L^p $(W^{m,p})$ functions. Moreover, recall that we can identify a tensor field $A \in L^1(\Omega)^{n\times n}$ with a vector of 1-forms of length n, that is with $\omega := (\omega^i)_{i=1}^n$, $\omega^i = A^i_j \mathrm{d}x^j$, and its Curl with $\mathrm{d}\omega$ (or, more precisely, with $(\star\mathrm{d}\omega)^\flat$), given by

$$\mathrm{d}\omega^i = \sum_{j<k} \left(\frac{\partial A^i_j}{\partial x^k} - \frac{\partial A^i_k}{\partial x^j}\right) \mathrm{d}x^j \wedge \mathrm{d}x^k.$$

We recall the following

Definition 1.4.1. Let $U \subset \mathbb{R}^n$ be a starshaped domain with respect to the point $y \in U$. The *linear homotopy operator* at the point y is the operator

$$k_y = k_{y,r} : \Omega^r(U) \to \Omega^{r-1}(U),$$

defined as

$$(k_y\omega)(x) := \int_0^1 s^{r-1}\omega(sx + (1-s)y) \llcorner (x - y)\mathrm{d}s,$$

where $(\omega(x) \llcorner v)[v_1, \cdots v_{n-1}] := \omega(x)[v, v_1, \cdots, v_{n-1}]$. It is well known that the linear homotopy operator satifies

$$\omega = k_{y,r+1}\mathrm{d}\omega + \mathrm{d}k_{y,r}\omega \quad \forall \omega \in \Omega^r(U). \tag{1.4.2}$$

In order to get more regularity, we consider the following *averaged* linear homotopy operator on $B := B(0,1)$, which coincides with the one introduced by Iwaniec and Lutoborski in [24], except for the choice of the weight function:

$$T = T_r : \Omega^r(B) \to \Omega^{r-1}(B),$$
$$T\omega(x) := \int_B \varphi(y)\,(k_y\omega)\,(x)\mathrm{d}y,$$

where $\varphi \in \mathcal{C}_c^\infty(B(0,2))$ is a positive cut-off function, with $\varphi \equiv 1$ in B and

$$\max\left\{||\varphi||_{L^\infty(\mathbb{R}^n)}, ||\nabla\varphi||_{L^\infty(\mathbb{R}^n)}\right\} \leq 3.$$

Clearly, (1.4.2) holds for T as well:

$$\omega = T\mathrm{d}\omega + \mathrm{d}T\omega. \tag{1.4.3}$$

We recall that, as proved in [24], T satisfies (for smooth forms ω) the pointwise bound

$$|T\omega(x)| \leq C_{n,r}\int_B \frac{|\omega(y)|}{|x-y|^{n-1}}\mathrm{d}y. \tag{1.4.4}$$

Indeed, for $\omega = \omega_\alpha \mathrm{d}x^\alpha \in \Omega^r(B)$ we have

$$T\omega(x) = \left(\int_B \mathrm{d}y\varphi(y)\int_0^1 t^{r-1}\langle x-y, e_i\rangle\,\omega_\alpha(tx+(1-t)y)\right)\mathrm{d}x^\alpha \llcorner e_i.$$

We then make the substitution $\Phi(y,t) := \left(tx+(1-t)y, \frac{t}{1-t}\right) \equiv (z(t,y), s(t))$, $\Phi : B(0,1)\times(0,1) \to B(0,1)\times(0,\infty)$ and we get

$$\begin{aligned}T\omega(x) &= \left(\int_B \mathrm{d}z\omega_\alpha(z)\langle x-z, e_i\rangle \int_0^\infty \mathrm{d}s\frac{s^{r-1}}{(1+s)^{r-1}}(1+s)(1+s)^{n-2}\varphi\left(z+s(z-x)\right)\right)\mathrm{d}x^\alpha \llcorner e_i = \\ &= \left(\int_B \mathrm{d}z\omega_\alpha(z)\frac{\langle x-z, e_i\rangle}{|x-z|^n}\int_0^2 s^{r-1}(1+s)^{n-r}\varphi(z+s\widehat{z-x})\right)\mathrm{d}x^\alpha \llcorner e_i \equiv \\ &\equiv \left(\int_B K_r^i(z, x-z)\omega_\alpha(z)\mathrm{d}z\right)\mathrm{d}x^\alpha \llcorner e_i,\end{aligned}$$

where

$$K_r^i(x,h) := \frac{\langle h, e_i\rangle}{|h|^n}\int_0^2 s^{r-1}(1+s)^{n-r}\varphi(x-s\widehat{h})\mathrm{d}s,$$

and we noticed that, since φ has compact support, the integral from 0 to ∞ actually reduces to an integral over a finite interval. That is, we get (1.4.4). It also follows easily from (1.4.4) that T is a compact operator from $L^p(B,\Lambda^r)$ to $L^p(B,\Lambda^{r-1})$. Moreover, by density, (1.4.3) extends to every differential form $\omega \in W^{1,p}(B,\Lambda^r)$, and to every differential form $\omega \in L^1(B,\Lambda^r)$ whose differential is a bounded Radon measure, $\mathrm{d}\omega \in \mathcal{M}_b(B,\Lambda^{r+1})$. In what follows, we will also need the *Hardy-Littlewood* inequality:

Theorem 1.4.7. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be two measurable functions vanishing at infinity, and let f^*, g^* be their symmetric decreasing rearrangements. Then*

$$\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \le \int_{\mathbb{R}^n} f^*(x)g^*(x)\mathrm{d}x.$$

Using the homotopy operator, we get the following weak-L^p geometric rigidity estimate for incompatible fields:

Theorem 1.4.8. *Let $p^* = p^*(n) := \frac{n}{n-1}$, and let $U \subset \mathbb{R}^n$ be an open, bounded and convex domain. There exists a constant $C = C(n, U) > 0$ such that for every $A \in L^{p^*}(U)$ whose* $\mathrm{Curl}(A)$ *is a vector measure on U with bounded total variation and whose support is contained in U, i.e.* $\operatorname{spt}\mathrm{Curl}(A) \Subset U$*, there exist a rotation $R \in SO(n)$ such that*

$$||A - R||_{L^{p^*,\infty}(U)} \le C\left(||\mathrm{dist}(A, SO(n))||_{L^{p^*,\infty}(U)} + |\mathrm{Curl}(A)|(U)\right).$$

Proof. Take any measurable subset $E \subset U$, and let $r > 0$ be such that $|B(0, r)| = |E|$. Then, using (1.4.4) and the Hardy-Littlewood inequality

$$\begin{aligned}
\int_E \mathrm{d}x\, |(T\omega)(x)| &\le C \int_E \mathrm{d}x \int_U \mathrm{d}y \frac{|\omega(y)|}{|x-y|^{n-1}} = \\
&= C \int_U \mathrm{d}y\, |\omega(y)| \int_E \frac{\mathrm{d}x}{|x-y|^{n-1}} \le \\
&\le C \int_U \mathrm{d}y\, |\omega(y)| \int_{\mathbb{R}^n} \chi_{E-x}(y) \frac{\mathrm{d}y}{|y|^{n-1}} \le \\
&\le C \int_U \mathrm{d}y\, |\omega(y)| \int_{\mathbb{R}^n} \chi_{B(0,r)} \frac{\mathrm{d}y}{|y|^{n-1}} \le \\
&= C \int_U \mathrm{d}y\, |\omega(y)| \int_0^r \mathrm{d}t \int_{\partial B(0,t)} \frac{\mathrm{d}y}{t^{n-1}} = \\
&= Cr\, ||\omega||_{L^1(U)} = C\,|E|^{\frac{1}{n}}\, ||\omega||_{L^1(U)}\,.
\end{aligned}$$

This gives immediately

$$||T\omega||_{L^1(U)} \le C(U)\, ||\omega||_{L^1(U)}\,,$$

and thus, using (1.4.3), $||A - T\mathrm{d}A||_{L^1(U)} \le C\,||\mathrm{d}A||_{L^1(U)}$, which extends immediately by density in the case when $\mathrm{d}A$ is a vector measure with bounded total variation. Choosing $E = \left\{x \in U \,\middle|\, |T\omega(x)| > t\right\}$, for $t > 0$

$$t\,|E| \le \int_E |T\omega(x)|\,\mathrm{d}x \le C\,|E|^{\frac{1}{n}}\,|\mathrm{d}A|\,(U).$$

Passing to the supremum over $t > 0$, we find

$$||T\mathrm{d}A||_{L^{p^*,\infty}(U)} \le C(U)\,|\mathrm{d}A|\,(U). \tag{1.4.5}$$

Since U is convex and $d(A-T\mathrm{d}A) = \mathrm{d}^2 TA = 0$, we can find a function g such that $\mathrm{d}g = A - T\mathrm{d}A$. From the estimates proven, is possible to apply Theorem 1.4.6 to g and find

$$||\mathrm{d}g - R||_{L^{p^*,\infty}(U)} \leq C\,||\mathrm{dist}(\mathrm{d}g, SO(n))||_{L^{p^*,\infty}(U)}\,.$$

But

$$||\mathrm{d}g - R||_{L^{p^*,\infty}(U)} \geq C\,||A - R||_{L^{p^*,\infty}(U)} - ||T\mathrm{d}A||_{L^{p^*,\infty}(U)}$$

and

$$||\mathrm{dist}(\mathrm{d}g, SO(n))||_{L^{p^*,\infty}(U)} \leq ||\mathrm{dist}(A, SO(n))||_{L^{p^*,\infty}(U)} + ||T\mathrm{d}A||_{L^{p^*,\infty}(U)}\,.$$

In particular,

$$||A - R||_{L^{p^*,\infty}(U)} \leq C\left(||\mathrm{dist}(A, SO(n))||_{L^{p^*,\infty}(U)} + |\mathrm{Curl}(A)|\,(U)\right). \qquad \square$$

We now give another estimate for L^p norms. It requires an L^∞-bound on the matrix field A, which is natural in the context of the theory of elasticity.

Theorem 1.4.9. *Let $n \geq 3$, $p^* := p^*(n) := \frac{n}{n-1}$, $p \in [p^*, 2]$ and fix $M > 0$. There exists a constant $C = C(n, M, p) > 0$, depending only on the dimension n, the exponent p and the constant M, such that for every $A \in L^\infty(B)$, with $||A||_\infty \leq M$ and $\mathrm{Curl}(A) \in \mathcal{M}_b(B, \Lambda^2)$, $B := B(0,1)$, there exists a corresponding rotation $R \in SO(n)$ for which, if $p > p^*$*

$$\int_B |A - R|^p\,\mathrm{d}x \leq C\left(\int_B \mathrm{dist}^p(A, SO(n))\mathrm{d}x + |\mathrm{Curl}(A)|^{p^*}(B)\right), \tag{1.4.6}$$

while, if $p = p^*$,

$$\begin{aligned}\int_B |A - R|^{p^*}\,\mathrm{d}x \leq & C\int_B \mathrm{dist}^{p^*}(A, SO(n))\mathrm{d}x + \\ & + C\,|\mathrm{Curl}(A)|^{p^*}(B)\left\{|\log\left(|\mathrm{Curl}(A)|\,(B)\right)| + 1\right\}.\end{aligned} \tag{1.4.7}$$

Remark 1.4.1. The constant C in (1.4.7) is *not* scaling invariant in the critical regime $p = p^*$. Thus, it cannot be used to extend the analysis in Chapter 2, Section 3, to any dimension.

Proof of Theorem 1.4.9. Without loss of generality, we can assume $T\mathrm{d}A$ not identically constant. Indeed, if $T\mathrm{d}A$ is identically constant, from the identity $T\mathrm{d}A = A + \mathrm{d}TA$, we see that $\mathrm{d}A = 0$, hence the result follows applying Theorem 1.4.5. As in the proof of Theorem 1.4.8, applying Theorem 1.4.5 (and using $|a-b|^p \geq 2^{1-p}\,|a|^p - |b|^p$) we find a rotation $R \in SO(n)$ for which the inequality

$$\int_B |A - R|^p\,\mathrm{d}x \leq C_n\left(\int_B |\mathrm{dist}(A, SO(n))|^p\,\mathrm{d}x + \int_B |T\mathrm{d}A(x)|^p\,\mathrm{d}x\right) \tag{1.4.8}$$

holds. We then just need to estimate the last term in the right hand side of (1.4.8). For, fix a $\Lambda > 1$ (to be chosen later), and define the integrals

$$I := \int_{|T\mathrm{d}A| > \Lambda} |T\mathrm{d}A|^p\,\mathrm{d}x, \qquad II := \int_{|T\mathrm{d}A| \leq \Lambda} |T\mathrm{d}A|^p\,\mathrm{d}x.$$

We now give an estimate for I. Firstly, we recall that T is a bounded operator from $L^p(B,\Lambda^r)$ into $W^{1,p}(B,\Lambda^{r+1})$, whenever $p\in(1,\infty)$ ([24], Proposition 4.1). Moreover, $T\mathrm{d}A=\mathrm{d}TA+A$, and $\nabla T=S_1+S_2$, where S_1 is a "weakly" singular operator which maps continuously L^∞ into itself, while S_2 is a Calderón-Zygmund operator (see the proof of Proposition 4.1 in [24]). In particular,

$$||T\mathrm{d}A||_{\mathrm{BMO}}\le C_n\,||A||_\infty\le C_nM,$$

where $C_n>0$ is a constant depending only on the dimension. Now, we can write

$$I=\Lambda^{p-p^*}\Lambda^{p^*}\left|\{|T\mathrm{d}A|>\Lambda\}\right|+\underbrace{\int_\Lambda^\infty\lambda^{p-1}\left|\{|T\mathrm{d}A|>\lambda\}\right|\mathrm{d}\lambda}_{=:I'}. \tag{1.4.9}$$

Clearly,

$$\Lambda^{p^*}\left|\{|T\mathrm{d}A|>\Lambda\}\right|\le||T\mathrm{d}A||^{p^*}_{L^{p^*,\infty}}\le C\,|\mathrm{d}A|\,(B)^{p^*}.$$

We now take a Calderón-Zygmund decomposition of $F(x):=|T\mathrm{d}A(x)|^p$: namely, we find a function $g\in L^\infty$, with $||g||_\infty\le2^{-n}\Lambda^p$ and disjoint cubes $\{Q_j\}_{j\ge1}$ such that, if $b:=\sum_{j\ge1}\chi_{Q_j}F$,

$$\begin{cases}F=g+b,\\ 2^{-n}\Lambda^p<\fint_{Q_j}F\mathrm{d}x\le\Lambda^p\quad\left(\text{Jensen }\Rightarrow\left|\fint_{Q_j}T\mathrm{d}A(x)\mathrm{d}x\right|\le\Lambda\right),\\ \left|\bigcup_{j\ge1}Q_j\right|<\frac{2^n}{\Lambda^p}\int|T\mathrm{d}A|^p\,\mathrm{d}x.\end{cases}$$

With such a decomposition, outside the cubes Q_j, $|T\mathrm{d}A|^p=|g(x)|\le2^{-n}\Lambda^p\le\Lambda^p$. Hence, using the John-Nirenberg inequality and the elementary estimate

$$\int_x^\infty\lambda^qe^{-\lambda}\mathrm{d}\lambda\le e^{-x}(1+x),\quad\forall q\le1\text{ and }x\ge1,$$

we find that (provided $p\le2$)

$$\begin{aligned}I'&=\int_\Lambda^\infty\lambda^{p-1}\sum_{j\ge1}\left|\left\{x\in Q_j\,\middle|\,|T\mathrm{d}A|>\lambda\right\}\right|\mathrm{d}\lambda\le\\
&\le\int_\Lambda^\infty\lambda^{p-1}\sum_{j\ge1}\left|\left\{x\in Q_j\,\middle|\,\left|T\mathrm{d}A(x)-\fint_{Q_j}T\mathrm{d}A\mathrm{d}x\right|>\lambda-\Lambda\right\}\right|\mathrm{d}\lambda\le\\
&\le C_1\int_\Lambda^\infty\lambda^{p-1}\left(\sum_{j\ge1}|Q_j|\right)\exp\left(-C_2\frac{\lambda-\Lambda}{||T\mathrm{d}A||_{\mathrm{BMO}}}\right)\mathrm{d}\lambda<\\
&<C_1\frac{2^n}{\Lambda^p}\left(\int|T\mathrm{d}A|^p\right)e^{C_2\frac{\Lambda}{||T\mathrm{d}A||_{\mathrm{BMO}}}}\left(\frac{||T\mathrm{d}A||_{\mathrm{BMO}}}{C_2}\right)^p\int_{\frac{C_2}{||T\mathrm{d}A||_{\mathrm{BMO}}}\Lambda}^\infty\lambda^{p-1}e^{-\lambda}\mathrm{d}\lambda\le\\
&\le C_1\frac{2^n}{\Lambda^p}\left(\int|T\mathrm{d}A|^p\right)\left(\frac{||T\mathrm{d}A||_{\mathrm{BMO}}}{C_2}\right)^p\left(1+\frac{C_2}{||T\mathrm{d}A||_{\mathrm{BMO}}}\Lambda\right)\le\\
&\le C_{n,M}\left(\int|T\mathrm{d}A|^p\right)\frac{1+\Lambda}{\Lambda^p}.\end{aligned} \tag{1.4.10}$$

Hence, if we choose Λ big enough (depending only on n and M) in (1.4.10),

$$I' \leq \frac{1}{2}\int |T\mathrm{d}A|^p \,. \tag{1.4.11}$$

Let us now estimate II. If $p > p^*$, we can write

$$\begin{aligned}
\int_{|T\mathrm{d}A|\leq\Lambda} |T\mathrm{d}A|^p \,\mathrm{d}x &= \int_{1<|T\mathrm{d}A|\leq\Lambda} |T\mathrm{d}A|^p \,\mathrm{d}x + \sum_{j\geq 0}\int_{2^{-j-1}<|T\mathrm{d}A|\leq 2^{-j}} \leq \\
&\leq C\left\{\Lambda^p \,|\mathrm{d}A|^{p^*}(B) + \sum_{j\geq 0} 2^{-(j+1)p}\left|\left\{|T\mathrm{d}A| > 2^{-(j+1)}\right\}\right|\right\} \leq \\
&\leq C\,|\mathrm{d}A|^{p^*}(B)\left(\Lambda^p + \sum_{j\geq 0} 2^{-j(p^*-p)}\right) \leq \\
&\leq C(n,p,M)\,|\mathrm{d}A|^{p^*}(B),
\end{aligned}$$

which gives (1.4.6). In the case $p = p^*$, we are going to make use of the increasing convex function Ψ, defined as the linear (convex) continuation of $t \mapsto t^p$ for $t \geq \Lambda$:

$$\Psi(t) := \begin{cases} t^{p^*} & \text{if } t \leq \Lambda, \\ p^*\Lambda^{p^*-1}t + (1-p^*)\Lambda^{p^*} & \text{if } t \geq \Lambda. \end{cases}$$

$$\begin{aligned}
II &\leq \int_B \Psi(|T\mathrm{d}A(x)|)\mathrm{d}x \leq \int_B \Psi\left(\fint_B \frac{C\,|\mathrm{d}A|\,(B)\mathrm{d}\,|\mathrm{d}A|\,(y)}{|x-y|^{n-1}}\right) \leq \\
&\leq \int_B \fint \Psi\left(\frac{C\,|\mathrm{d}A|\,(B)}{|x-y|^{n-1}}\right)\mathrm{d}\,|\mathrm{d}A|\,(y)\mathrm{d}x = \\
&= \fint_B \mathrm{d}\,|\mathrm{d}A|\,(y)\int_B \Psi\left(\frac{C\,|\mathrm{d}A|\,(B)}{|x-y|^{n-1}}\right)\mathrm{d}x \leq \\
&\leq \int_{B(0,2)} \Psi\left(\frac{C\,|\mathrm{d}A|\,(B)}{|z|^{n-1}}\right)\mathrm{d}z = C\int_0^2 \mathrm{d}\varrho\,\varrho^{n-1}\Psi\left(\frac{C\,|\mathrm{d}A|\,(B)}{\varrho^{n-1}}\right) = \\
&= \int_0^{C\left(|\mathrm{d}A|(B)\Lambda^{-1}\right)^{\frac{1}{n-1}}} \varrho^{n-1}\left(p^*\Lambda^{p^*-1}\frac{C\,|\mathrm{d}A|\,(B)}{\varrho^{n-1}} + (1-p^*)\Lambda^{p^*}\right)\mathrm{d}\varrho + \\
&\quad + C\int_{C\left(|\mathrm{d}A|(B)\Lambda^{-1}\right)^{\frac{1}{n-1}}}^2 \frac{|\mathrm{d}A|\,(B)^{p^*}}{\varrho}\mathrm{d}\varrho \leq \\
&\leq C\,|\mathrm{d}A|\,(B)^{p^*}\left(1 + |\log\left(|\mathrm{d}A|\,(B)\right)|\right).
\end{aligned} \tag{1.4.12}$$

Combining together (1.4.9), (1.4.11) and (1.4.12), we obtain (1.4.7). □

Remark 1.4.2. The same conclusions can be obtained considering the operator defined by an average on the sphere:

$$\widetilde{T}\omega(x) := \int_{\mathbb{S}^{n-1}} \mathrm{d}\mathcal{H}^{n-1}(y)k_y\omega(x).$$

The fundamental generalization of Theorem 1.4.5 proved by Müller, Scardia and Zeppieri in [23], that is a scaling-invariant geometric rigidity estimate for incompatible matrix fields. Unfortunately, such an estimate is valid only in dimension 2.

Theorem 1.4.10. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply connected Lipschitz subset. There exists a constant $C = C(\Omega) > 0$ (which is scaling and translation invariant) such that for every $A \in L^2(\Omega)^{2\times 2}$ whose Curl is a measure with bounded total variation there is a rotation $R \in SO(2)$ for which the following estimate holds:*

$$||A - R||_{L^2(\Omega)^{2\times 2}} \leq C \left(||\mathrm{dist}(A, SO(2)||_{L^2(\Omega)^{2\times 2}} + |\mathrm{Curl}\, A|\,(\Omega) \right).$$

Chapter 2

The First Mesoscopic Scale and Cosserat Microrotations

2.1 The Functional

In what follows

— $\Omega := [-L, L]^2$ represents a section of a crystal, $L > 0$;

— $\varepsilon > 0$ is the *lattice parameter*, i.e. the distance between atoms;

— $1 \gg \alpha > 0$ is the ("small") misorientation angle between two grains;

— $\ell > 0$ is a parameter (much smaller than L): in an ℓ-neighborhood of $x = \pm L$ we are going to impose the boundary conditions;

— $\lambda > 0$ is a parameter (independent of L, ε, α) so that $\lambda\varepsilon$ gives what physicists call the *core radius*;

— $\tau > 0$ is another parameter independent of all the others, which is defining the *minimal length of the Burgers' vector*, $\tau\varepsilon$.

We then restrict our attention to the following class of *admissible strain fields*, denoted by $\mathcal{A}(\varepsilon, \alpha, L, \tau, \lambda, \ell)$ (to which we shall simply refer to as $\mathcal{A}_\varepsilon$, in the case when the other parameters are clear from the context), which is defined as the family of matrix fields $A : \Omega \to \mathbb{R}^{2\times 2}$ satisfying the following conditions:

($\mathcal{A}_\varepsilon$, i) $A \in L^1_{\mathrm{loc}}(\Omega)^{2\times 2}$ and $A \in L^2\left(\Omega \setminus B_{\lambda\varepsilon}\left(\mathrm{spt}\,\mathrm{Curl}\,A\right)\right)^{2\times 2}$.

($\mathcal{A}_\varepsilon$, ii) (*Boundary Condition*) $A \equiv R_\alpha$ in $[-L, -L+\ell] \times [-L, L]$ and $A \equiv R_{-\alpha}$ in $[L-\ell, L] \times [-L, L]$, where R_α is the counter-clockwise rotation through the angle α, that is

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

($\mathcal{A}_\varepsilon$, iii) (*First Quantization of the Burgers' vector*) For every closed, Lipschitz simple curve $\gamma \subset \Omega \setminus B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl}(A))$, either

$$\int_\gamma A \cdot t \mathrm{d}\mathscr{H}^1 = 0$$

or

$$\left| \int_\gamma A \cdot t \mathrm{d}\mathscr{H}^1 \right| \geq \tau\varepsilon.$$

We call an *admissible core* any compact subset of $[-L+\ell, L-\ell] \times [-L, L]$, i.e. any element of $\mathcal{K}([-L+\ell, L-\ell]\times[-L,L])$. The *elastic energy* of a pair $(A, S) \in \mathcal{A}_\varepsilon \times \mathcal{K}([-L+\ell, L-\ell]\times[-L,L])$ is

$$\mathcal{E}_{\mathrm{el}}(A, S) := \frac{1}{\tau} \int_{\Omega \setminus B_{\lambda\varepsilon}(S)} \operatorname{dist}^2(A, \mathrm{SO}(2)) \mathrm{d}x,$$

while the *core energy* depends only on the core and is defined as

$$\mathcal{E}_{\mathrm{core}}(S) := \frac{1}{\lambda} \left| B_{\lambda\varepsilon}(S) \right|.$$

We define the set of admissible pairs

$$\mathcal{P}\left(\varepsilon, \alpha, L, \tau, \lambda, \ell\right) := \mathcal{A}\left(\varepsilon, \alpha, L, \tau, \lambda, \ell\right) \times \mathcal{K}([-L+\ell, L-\ell] \times [-L, L])$$

Whevener the constants $\alpha, L, \tau, \lambda, \ell$ are clear from the context, we shall simply write $\mathcal{P}_\varepsilon$ for $\mathcal{P}\left(\varepsilon, \alpha, L, \tau, \lambda, \ell\right)$. The (free) energy functional is defined on pairs $(A, S) \in \mathcal{P}_\varepsilon$ as

$$\mathcal{F}_\varepsilon(A, S) := \begin{cases} \mathcal{E}_{\mathrm{el}}(A, S) + \mathcal{E}_{\mathrm{core}}(S) & \text{if } \operatorname{spt}(\operatorname{Curl}(A)) \subset S, \\ +\infty & \text{otherwise.} \end{cases}$$

We also define the relaxed energy on admissible fields as

$$\mathcal{F}_\varepsilon(A) := \mathcal{F}_\varepsilon(A, \operatorname{spt}(\operatorname{Curl}(A))).$$

For notational simplicity, for a set S we let $\Omega_{\lambda\varepsilon}(S) := \Omega \setminus B_{\lambda\varepsilon}(S)$, while for a matrix field A $\Omega_{\lambda\varepsilon}(A) := \Omega \setminus B_{\lambda\varepsilon}(\operatorname{spt}(\operatorname{Curl}(A)))$.
We say that a matrix field A is a *microrotation* if $A \in BV(\Omega)^{n\times n}$ (see the Appendix), $A(x) \in SO(n)$ for every $x \in \Omega$ and $DA = D^J A$, i.e.

$$DA^{(i)} = \left(A^{(i),+} - A^{(i),-}\right) \otimes \nu_A \mathcal{H}^{n-1} \llcorner S_A, \qquad i = 1, \cdots, n.$$

Recall that is well defined a trace for matrix fields whose Curl is square-integrable, in the following sense. If U is a bounded Lipschitz domain in $\mathbb{R}^2$,

$$H(\operatorname{Curl}, U)^{2\times 2} := \left\{ A \in L^2(U)^{2\times 2} \middle| \operatorname{Curl}(A) \in L^2\left(U\right) \right\}.$$

Then, the operator

$$\gamma : A \in H(\operatorname{Curl}, U) \longmapsto A \cdot t \in H^{-\frac{1}{2}}(\partial U),$$

is well defined and continuous (where $t(x)$ is the tangent vector to ∂U at the point x), i.e. there exists a constant $C = C(U) > 0$ such that

$$\left| \int_{\partial U} A \cdot t \mathrm{d}\mathcal{H}^1 \right| \leq C \, ||A||_{H(\mathrm{Curl}, U)} \, .$$

Moreover, an approximation argument (see [27]) gives

$$\int_{\partial U} A \cdot t \mathrm{d}\mathcal{H}^1 = \int_U \mathrm{Curl}(A) \mathrm{d}x \qquad \forall A \in H \left(\mathrm{Curl}, \Omega\right)^{2\times 2} .$$

To every γ closed, Lipschitz, simple curve contained in $\Omega \setminus B_{\lambda\varepsilon}(\mathrm{spt}\,\mathrm{Curl}(A))$ we associate its *Burgers' vector* defined as

$$\vec{b}\,(\gamma) := \int_\gamma A \cdot t \mathrm{d}\mathcal{H}^1.$$

Remark 2.1.1. Although we chose $\mathrm{dist}^2(\cdot, SO(2))$ as the elastic energy density, all the results we prove remain valid if we consider instead a function $W : \mathbb{R}^{2\times 2} \to [0,\infty)$ which satisfies the usual assumptions of an elastic energy density in (two dimensional) nonlinear elasticity, that is

(i) W is continuous and of class $\mathcal{C}^2$ in a neighborhood of $SO(2)$;

(ii) $W(\mathrm{Id}) = 0$, i.e. the reference configuration is stress-free;

(iii) $W(RA) = W(A)$ for every matrix $A \in \mathbb{R}^{2\times 2}$, i.e. W is frame indifferent,

together with the growth assumption

(iv) There exists a constant $C > 1$ such that $C^{-1}\,\mathrm{dist}^2(A, SO(2)) \leq W(A) \leq C\,\mathrm{dist}^2(A, SO(2))$.

Condition (iv) is rather restrictive, but it is essential in order to apply the Geometric Rigidity estimate of Müller, Scardia and Zeppieri (cf. [23]).

2.2 Upper Bound: the Read-Shockley Formula

Theorem 2.2.1. *There exists a constant $C_0 > 0$ such that*

$$\liminf_{\varepsilon \downarrow 0} \inf_{(A,S)\in\mathcal{P}_\varepsilon} \frac{1}{\varepsilon} \mathcal{F}_\varepsilon(A,S) \leq C_0 \alpha L \left(|\log(\alpha)| + 1\right). \tag{2.2.1}$$

Proof. Consider $\overline{n} \in \mathbb{N}$ such that $\frac{1}{\alpha} \in [2^{\overline{n}}, 2^{\overline{n}+1})$. Without loss of generality, we can assume $\varepsilon \in L2^{1-k}\frac{1}{2\mathbb{N}} := \left\{ \frac{L2^{1-k}}{2z} \middle| z \in \mathbb{N} \right\}$. Set $r_0 := \frac{\varepsilon}{2}$ and $N := \frac{L}{2^k r_0} \in 2\mathbb{N}$. Let $r_n := 2^n r_0$ and

$$\begin{aligned}
&p_n^1 := \left(-r_n, r_n\right), \, p_n^2 := \left(r_n, r_n\right), \, p_n^3 := \left(r_n, -r_n\right), \, p_n^4 := \left(-r_n, -r_n\right) \text{ for } n = 0, \cdots, \overline{n}, \\
&\Delta_n^1 := \Delta\left(p_n^1, p_{n-1}^1, p_{n-1}^4\right), \, \Delta_n^2 := \Delta\left(p_n^1, p_{n-1}^4, p_n^4\right), \\
&\Delta_n^3 := \Delta\left(p_n^2, p_{n-1}^3, p_n^3\right), \, \Delta_n^4 := \Delta\left(p_n^2, p_{n-1}^2, p_{n-1}^3\right) \text{ for } n = 1, \cdots, \overline{n},
\end{aligned}$$

where $\Delta(a, b, c)$ denotes the triangle whose vertices are a, b and c.

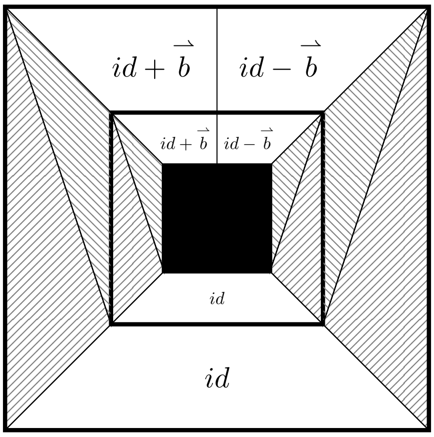

Figure 2.1: The map $v^{(1)}$ (the striped triangles are the ones where we are interpolating).

Let $Q_n := [-r_n, r_n]^2$ and $Q := [-r_{\overline{n}}, r_{\overline{n}}]^2$, $\vec{b} := (\varepsilon, 0)$ and define (see Figure 2.1)

$$v^{(1)} := \begin{cases} \text{id} & \text{in } \left(\left(Q \setminus \bigcup_{i,n} \Delta_n^i\right) \cap \{y < 0\}\right) \cup [-r_0, r_0]^2, \\ \text{id} + \vec{b} & \text{in } \left(Q \setminus \bigcup_{i,n} \Delta_n^i\right) \cap \{y > 0, x \leq 0\}, \\ \text{id} - \vec{b} & \text{in } \left(Q \setminus \bigcup_{i,n} \Delta_n^i\right) \cap \{y > 0, x > 0\}, \\ \text{linear interpolation} & \text{in } \bigcup_{i,n} \Delta_n^i. \end{cases}$$

It is readily seen that for $p \in \Delta_n^i$ we have

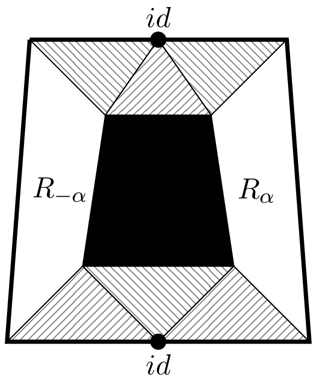

Figure 2.2: The map $v^{(2)}$ (as in Figure 2.1, the stripes denote the regions where we are interpolating).

$$\left|\nabla v^{(1)}(p) - \text{id}\right| \leq C \frac{1}{2^n}.$$

Now, we have to adjust the boundary condition. For, we consider the map $v^{(2)} : v^{(1)}(Q) \to \mathbb{R}^2$ defined as follows (see Figure 2.2). For $n = 1, \cdots, \overline{n}$, define the points

$$q_n^1 := (r_n - \varepsilon, r_n), \qquad q_n^2 := (r_n, -r_n), \qquad q_n^3 := (0, -r_n),$$
$$q_n^4 := (-r_n, -r_n), \qquad q_n^5 := (-r_n + \varepsilon, r_n), \qquad q_n^6 := (0, r_n),$$

and

$$
\begin{aligned}
&q_0^1 := (r_0, r_0), \qquad q_0^2 := (r_0, -r_0), \qquad q_0^3 := (0, -r_0),\\
&q_0^4 := (-r_0, -r_0), \qquad q_0^5 := (-r_0, r_0), \qquad q_0^6 := (0, r_0).
\end{aligned}
$$

Then, for $n = 0, \cdots, \overline{n}$, consider the triangles

$$
\begin{aligned}
&\widetilde{\Delta}_n^1 := \Delta\left(q_n^5, q_{n-1}^5, q_{n-1}^6\right), \qquad \widetilde{\Delta}_n^2 := \Delta\left(q_{n-1}^5, q_n^6, q_{n-1}^1\right), \qquad \widetilde{\Delta}_n^3 := \Delta\left(q_n^6, q_{n-1}^1, q_n^1\right),\\
&\widetilde{\Delta}_n^4 := \Delta\left(q_n^2, q_{n-1}^2, q_n^3\right), \qquad \widetilde{\Delta}_n^5 := \Delta\left(q_{n-1}^2, q_n^3, q_{n-1}^4\right), \qquad \widetilde{\Delta}_n^6 := \Delta\left(q_n^3, q_{n-1}^4, q_n^4\right).
\end{aligned}
$$

We then define $v^{(2)} : v^{(1)}(Q) \to \mathbb{R}^2$ as

$$
v^{(2)}(x) := \begin{cases} R_{-\alpha}x & \text{if } x \in \bigcup_{n=1}^{\overline{n}} \left\{ v^{(1)}\left(Q_n \setminus Q_{n-1}\right) \setminus \bigcup_{j=1}^{6} \widetilde{\Delta}_n^i \right\} \cap \{x < 0\},\\ R_{\alpha}x & \text{if } x \in \bigcup_{n=1}^{\overline{n}} \left\{ v^{(1)}\left(Q_n \setminus Q_{n-1}\right) \setminus \bigcup_{j=1}^{6} \widetilde{\Delta}_n^i \right\} \cap \{x > 0\},\\ x & \text{if } x = (0, \pm r_n),\\ \text{linear interpolation} & \text{otherwise.} \end{cases}
$$

It is easy to check that on each triangle we have

$$
\operatorname{dist}^2\left(\nabla v^{(2)}, \mathrm{SO}(2)\right) \le C\alpha^2.
$$

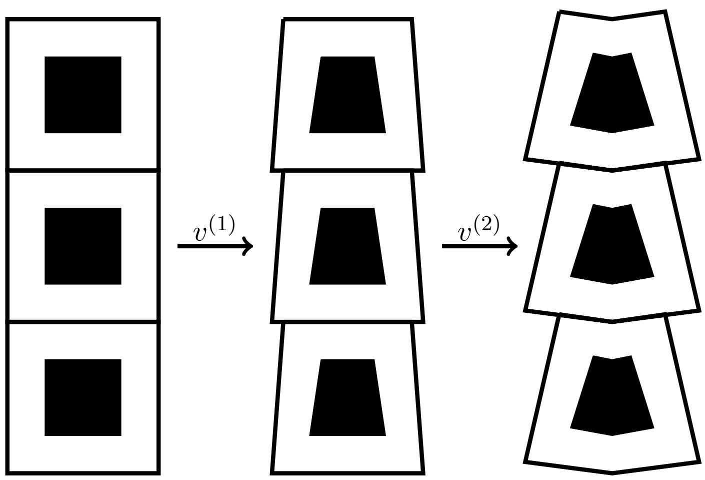

Figure 2.3: Schematic representation of the grain boundary constructed.

Thus, if $v := v^{(2)} \circ v^{(1)}$ (see Figure 2.3), on each triangle Δ_n^i,

$$
\operatorname{dist}^2\left(\nabla v, \mathrm{SO}(2)\right) \le C\frac{1}{4^n} + \alpha^2.
$$

This gives in particular

$$
\int_{Q\setminus[-r_0, r_0]^2} \operatorname{dist}^2\left(\nabla v, \mathrm{SO}(2)\right) \mathrm{d}x \le C\left(\varepsilon^2 \left|\log(\alpha)\right| + \varepsilon^2\right).
$$

The last step consists in gluing the maps constructed before. Namely, if $S := [-r_{\overline{n}}, r_{\overline{n}}] \times [-L, L]$ we define the map $u : \Omega \to \mathbb{R}^2$ as

$$u(x,y) := \begin{cases} R_{-\alpha} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } (x,y) \in (\Omega \setminus S) \cap \{x < 0\}\,, \\ R_{\alpha} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } (x,y) \in (\Omega \setminus S) \cap \{x > 0\}\,, \\ v\,(x, y + k r_{\overline{n}}) & \text{if } (x,y) \in Q + k\,(0, r_{\overline{n}})\,, \quad k \in \left\{-\frac{N}{2}, \frac{N}{2}\right\}. \end{cases}$$

Then, if $A_{\mathrm{gb}} := \nabla u$,

$$\mathcal{F}(A_{\mathrm{gb}}) \leq C\varepsilon\alpha h\,(|\log\alpha| + 1)\,. \qquad \square$$

We say that $E_{\mathrm{gb}}(\varepsilon) := C_0\varepsilon\alpha L\,(|\log(\alpha)| + 1)$ is the *energy of a grain boundary with misorientation angle α at the scale ε*, where $C_0 > 0$ is the constant from Theorem 2.2.1.

Remark 2.2.1. Although it not completely clear (at least, from the point of view of the Calculus of Variations) why the upper bound should also be (roughly) optimal, it is clear that the lower bound cannot be zero. Indeed, suppose it is. Then, because of the quantization of the core energy, minimizing fields need to be curl-free, meaning that we can look for minimizing sequence in the more restrictive class $W^{1,2}_{\alpha}(\Omega) := \left\{ u \in W^{1,2}(\Omega) \middle| \nabla u \equiv R_{\pm\alpha} \text{ near } x = \pm\ell \right\}$. But then

$$\inf_{u \in W^{1,2}_{\alpha}(\Omega)} \mathcal{F}(\nabla u) = \min_{u \in W^{1,2}_{\alpha}(\Omega)} \mathcal{F}(\nabla u) = C_{\alpha,h} > 0,$$

Indeed, the functional $u \in W^{1,2}(\Omega) \mapsto \int_{\Omega} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)(2))\mathrm{d}x$ is lower semicontinous with respect to the $W^{1,2}$-weak topology and is bounded from below. Then, the infimum is actually a minimum. But it cannot be zero, otherwise Liouville's Theorem would imply that the matrix field ∇u is constant, which is not compatible with the boundary conditions.

2.3 Surgery Lemmata

This section is dedicated to two technical lemmas. They allow us to find fields with energy comparable to a given one, but uniformly bounded and with |Curl| controlled by the core energy (when seen as a measure). These two technical requirements are essentials in the harmonic competitor lemma and in the balls construction.

Lemma 2.3.1. *There exists a constant $C > 0$ such that for every pair $(A, S) \in \mathcal{P}\,(\varepsilon, \alpha, L, \ell\lambda)$ whose energy satisfies $\mathcal{F}_{\varepsilon}(A, S) \leq E_{gb}(\varepsilon)$ there exists another pair $(\widetilde{A}, \widetilde{S}) \in \mathcal{P}\left(\varepsilon, \alpha, L, \frac{\ell}{2}, \lambda\right)$ s.t.*

(i) $\left|\left|\widetilde{A}\right|\right|_{L^{\infty}(\Omega)} \leq C$;

(ii) $\mathcal{F}_{\varepsilon}(\widetilde{A}, \widetilde{S}) \leq C\mathcal{F}_{\varepsilon}(A, S)$;

(iii) $\widetilde{A} \in \mathcal{C}^{\infty}\left(\overline{\Omega \setminus B_{\lambda\varepsilon}(\widetilde{S})}\right)$.

Proof. We will define the pair $(\widetilde{A}, \widetilde{S})$ by modifying it in several steps. Let $\omega := B_{\lambda\varepsilon}(S)$ and define

$$\widetilde{A}_1 := \begin{cases} \mathrm{id} & \text{in } \omega, \\ A & \text{in } \Omega \setminus \omega. \end{cases}$$

Clearly, $\operatorname{spt}\operatorname{Curl}\widetilde{A}_1 \subseteq \omega =: \widetilde{S}_1$, and by Vitali's Lemma we can find an at most countable collection of point $(x_j)_{j\in J} \in S$ such that the balls $B_{\lambda\varepsilon}(x_j)$ are mutually disjoint and

$$\omega = \bigcup_{x\in S} B_{\lambda\varepsilon}(x) \subset \bigcup_{j\in J} B_{5\lambda\varepsilon}(x_j).$$

Thus

$$\left|B_{\lambda\varepsilon}(\widetilde{S}_1)\right| \le |B_{\lambda\varepsilon}(\omega)| \le \left|\bigcup_{j\in J} B_{6\lambda\varepsilon}(x_j)\right| \le C_\lambda \sum_{j\in J} |B_{\lambda\varepsilon}(x_j)| \le C_{\lambda,n}\,|\omega| \le C_\lambda \mathcal{F}_\varepsilon(A,S).$$

Thus $\mathcal{F}_\varepsilon(\widetilde{A}_1, \widetilde{S}_1) \le C_\lambda \mathcal{F}_{\varepsilon,S}$ and $\left|\left|\widetilde{A}_1\right|\right|_{L^\infty(\omega)} \le M$. For notational simplicity, relabel $\widetilde{A}_1$ as A and $\widetilde{S}_1$ as S. Now we show that we can without loss of generality assume A to be smooth outside $B_{\lambda\varepsilon}(S)$. By the Hodge-Morrey decomposition, $A = \nabla u + F$, where $u \in W^{1,2}(\Omega)$ and $F \in L^2(\Omega)$ has zero divergence in the sense of distributions. Moreover, $\operatorname{Curl}(F) = 0$ in $\Omega_{\frac{\lambda\varepsilon}{2}}(S)$, and hence is harmonic (and, in particular, smooth) in $\Omega_{\frac{\lambda\varepsilon}{2}}(S)$. We then take a sequence $u_k \in \mathcal{C}^\infty(\Omega)\cap W^{1,2}(\Omega)$ converging to u in $W^{1,2}(\Omega)$. Set $A_k := \nabla u_k + F$. Clearly $\operatorname{Curl}(A_k) = \operatorname{Curl}(F) = \operatorname{Curl}(A)$ in Ω for every k and

$$\begin{aligned} \int_\Omega \operatorname{dist}^2(A_k, \mathrm{SO}(2))\mathrm{d}x &\le 2\left(\int_\Omega \operatorname{dist}^2(A, \mathrm{SO}(2))\mathrm{d}x + \int_\Omega |\nabla(u_k - u)|^2\,\mathrm{d}x\right) \\ &\le 3\int_\Omega \operatorname{dist}^2(A, \mathrm{SO}(2))\mathrm{d}x, \end{aligned}$$

provided k is chosen big enough. That is, we can without loss of generality assume A to be smooth in $\overline{\Omega_{\lambda\varepsilon}(S)}$. Now, fix $M > 1$ and consider the set of points

$$R := R_M := \left\{x \in \Omega \,\middle|\, \exists r > 0 : \fint_{B_r(x)} \operatorname{dist}^2(A, \mathrm{SO}(2)) \ge M\right\},$$

and define

$$r(x) := r_M(x) := \inf\left\{r > 0 \,\middle|\, \fint_{B_r(x)} \operatorname{dist}^2(A, \mathrm{SO}(2)) \ge M\right\}.$$

Clearly, $||A||_{L^\infty(\Omega\setminus R)} \le M + 2\sqrt{n}$. Let $R_1 := R \cap \{r(x) \ge \varepsilon\}$, and define the new field

$$\widetilde{A}_2 := \begin{cases} \mathrm{id} & \text{in } B_1, \\ A & \text{in } \Omega \setminus B_1, \end{cases}$$

where $B_1 := \bigcup_{x\in R_1} B_{r(x)}(x)$. Then $\operatorname{spt}\operatorname{Curl}\widetilde{A}_2 \subset B_1 \cup S$. Set $\widetilde{S}_2 := S \cup R_1$. Using Vitali's Lemma as before, we find a collection of (at most countable) mutually disjoint balls $B_j = B_{r(x_j)}(x_j)$ whose centers are in R_1 and

$$R_1 \subset \bigcup_{j\in J} B_{5r(x_j)}(x_j).$$

Thus, since $r(x_j) =: r_j \geq \varepsilon$ for every $j \in J$,

$$|B_{\lambda\varepsilon}(R_1)| \leq \sum_{j\in J} \left|B_{(5+\lambda)r_j}(x_j)\right| \leq C_\lambda \sum_j |B_j| \leq \frac{C_\lambda}{M} \sum_{j\in J} \int_{B_j} \mathrm{dist}^2(A, \mathrm{SO}(2)) \leq \frac{C_\lambda}{M} \mathcal{F}_\varepsilon(A, S).$$

As done before, relabel for simplicity $\widetilde{A}_2$ as A and $\widetilde{S}_2$ as S, and redefine the set R and the function r in function of this new pair (A, S). Then we reduced ourselves to the case when the potentially bad points, i.e. the ones in R, have $r(x) < \varepsilon$. Consider first those which lie in $B_{\lambda\varepsilon}(S)$, i.e. the points in $R_2 := R \cap B_{\lambda\varepsilon}(S)$. Consider the field

$$\widetilde{A}_3 := \begin{cases} \mathrm{id} & \text{in } B_2, \\ A & \text{in } \Omega \setminus B_2, \end{cases}$$

and the cores $\widetilde{S}_3 := S \cup R_2$, where $B_2 := \bigcup_{x\in R_2} B_{r(x)}(x)$. Using a covering argument as before, one can easily infer that $\left|B_{\lambda\varepsilon}(\widetilde{S}_3)\right| \leq C_{\lambda,n}\mathcal{F}_\varepsilon(A)$. Hence, relabeling $\widetilde{A}_3$ as A and $\widetilde{S}_3$ as S (and redefining R, r depending on the new field A) we are reduced to the case when R consists only points lying outside the $\lambda\varepsilon$-neighborhood of S and with $r(x) < \varepsilon$. In this case we are not allowed to merely cut off the fields, since we have no control of the singular set in terms of the covering $V := \bigcup_{x\in R} B_{r(x)}(x)$ of R (we can always assume V to be open, i.e. $r(x) > 0$ for every x). We then need to extend A in a Curl-free way. For, we first notice that using Vitali's Lemma again, we find

$$|V| \leq \frac{C}{M}\mathcal{F}_\varepsilon(A) \leq C\varepsilon L\alpha\,|\log(\alpha)| \leq \frac{\lambda}{2}\varepsilon.$$

In particular, this means that every ball of radius $\lambda\varepsilon$ must intersect the complement of V. We then cover $\Omega_{\lambda\varepsilon}(A)$ with (a finite number of) balls of such radius which overlap only finitely many times (depening only on the dimension):

$$\Omega_{\lambda\varepsilon}(A) \subset \bigcup_{j\geq 1} B_j, \qquad B_j := B\left(x_j, \lambda\varepsilon\right).$$

We only need to extend the field to those balls which are not intersecting the singular set (indeed, in those balls which do intersect the singular set we can simply set the field to be a constant). Following the proof of Whitney Lemma (see [25]), we define

$$\varrho(x) := \frac{1}{20}\min\left\{1, \mathrm{dist}(x, \mathcal{C})\right\}, \qquad \mathcal{C} := \Omega_{\lambda\varepsilon}(A) \setminus V.$$

By Vitali's Lemma, we find points $\{x_k\} \subset V$ such that

$$V = \bigcup_{k\geq 1} B\left(x_k, 5\varrho(x_k)\right),$$

and the balls $B\left(x_k, \varrho(x_k)\right)$ are disjoint. One can then prove that the sets

$$S_x := \left\{x_k \middle| B\left(x, 10\varrho(x)\right) \cap B\left(x_k, 10\varrho(x_k)\right) \neq \emptyset\right\},$$

have uniformly finite cardinality; more precisely, $\#S_x \leq (129)^2 =: C_2$ for all $x \in V$. Moreover, if $x_k \in S_x$, $\frac{1}{3}\varrho(x_k) \leq \varrho(x) \leq 3\varrho(x_k)$. One can then prove that is possible to contruct a partition of unity $\{\psi_k\}_{k\geq 1}$ such that

$$\begin{cases} \sum_{k\geq 1} \psi_k(x) \equiv 1, & \\ \sum_{k\geq 1} \nabla\psi_k(x) \equiv 0, & x \in U, \\ |\nabla\psi_k(x)| \leq \frac{C}{\varrho(x)}. & \end{cases}$$

For each k choose a point $s_k \in \mathcal{C}$ such that $|x_k - s_k| = \operatorname{dist}(x_k, \mathcal{C})$. Since the balls B_j are simply connected and A is Curl-free in $\Omega_{\lambda\varepsilon}(A)$, we can find a function $u \in \mathcal{C}^\infty(B_j)$ such that $A = \nabla u_j$ in B_j. We can then consider the extension in B_j

$$\overline{u}_j(x) := \begin{cases} u_j & \text{if } x \in B_j \setminus V, \\ \sum_{k\geq 1} \psi_k(x) \left(u_j(x) + A(s_k)\,(x - s_k)\right) & \text{if } x \in B_j \cap V. \end{cases}$$

It is then possible to show that $\overline{A}_j := \overline{u}_j \in \mathcal{C}^1(B_j)$ and $\nabla\overline{u}_j(x) = A(x)$ for all $x \in B_j \setminus V$. Moreover, if $x \in B_j \cap B_m$, then $\nabla\overline{u}_j(x) = \nabla\overline{u}_m(x)$. Indeed, since $\nabla u_j = \nabla u_m$ in $B_j \cap B_m$, there exists a constant $c_{jm} \in \mathbb{R}^2$ such that $u_j = c_{jm} + u_m$ in $B_j \cap B_m$, and hence $\nabla\overline{u}_j(x) = \nabla\overline{u}_m(x)$ since $\sum_{k\geq 1} \nabla\psi_k(x) = 0$ for every $x \in V$. In particular, the extension

$$\widetilde{A}(x) := \overline{A}_j(x), \text{if } x \in B_j$$

is well defined and Curl-free. It is also easy to verify that $|\nabla\overline{u}_j(x)| \leq C_2$, for some constant $C_2 > 0$ depending only on the dimension. Then, define $\widetilde{A}$ to be the identity in a $2\lambda\varepsilon$-neighborhood of S, which we call $\widetilde{S}$. This gives the desired field $\widetilde{A}$, since (arguing like in the discussion before)

$$|B_{\lambda\varepsilon}(S)| \leq C_2\, |B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl} A)|\,,$$

and

$$\int_V \operatorname{dist}^2(\widetilde{A}, \mathrm{SO}(2))\mathrm{d}x \leq C_2 \sum_j |B_j| \leq \frac{C}{M}\mathcal{F}_\varepsilon(A, S). \qquad \square$$

Lemma 2.3.2. *Let $(A, S) \in \mathcal{P}\,(\varepsilon, \alpha, L, \lambda, \ell)$. Then there exists another pair $(\widetilde{A}, \widetilde{S}) \in \mathcal{P}\left(\varepsilon, \alpha, L, \lambda, \frac{\ell}{2}\right)$ such that for a universal constant $C > 0$*

(i) $\mathcal{F}_\varepsilon\left(\widetilde{A}, \widetilde{S}\right) \leq C\mathcal{F}_\varepsilon(A, S)$;

(ii) $\operatorname{Curl}(\widetilde{A}) \in L^\infty(\Omega)$ *and* $\left|\operatorname{Curl}(\widetilde{A})\right| \leq C\mu_{2,\varepsilon}(\widetilde{S})$, *where*

$$\mu_{2,\varepsilon}(\widetilde{S}) := \frac{1}{\lambda\varepsilon}\mathcal{L}^2 \llcorner B_{\lambda\varepsilon}(\widetilde{S}).$$

Proof. By Lemma 2.3.1, we can assume A to be smooth in $\Omega_{\lambda\varepsilon}(A)$ and $||A||_{L^\infty(\Omega)} \leq C$. Consider a $\lambda\varepsilon$-mollifer $\varrho_{\lambda\varepsilon}$, that is $\varrho_{\lambda\varepsilon} \in \mathcal{C}_c^\infty\left(\mathbb{R}^n, [0,1]\right)$, $\operatorname{spt}(\varrho_{\lambda\varepsilon}) \subset \overline{B\left(0, \lambda\varepsilon\right)}$ and $\int \varrho_{\lambda\varepsilon} = 1$. Take a cut-off function ζ such that

$$\begin{cases} \zeta \in \mathcal{C}^\infty(\Omega), & \\ 0 \leq \zeta \leq 1, & \\ \zeta \equiv 1 & \text{in } B_{\lambda\varepsilon}(S), \\ \zeta \equiv 0 & \text{in } \Omega \setminus B_{2\lambda\varepsilon}(S), \\ ||\nabla\zeta||_{L^\infty(\Omega)} \leq \frac{C_0}{\lambda\varepsilon}. & \end{cases}$$

Define the new matrix field

$$\widetilde{A} := (1-\zeta)\, A + \zeta\, (A \star \varrho_{\lambda\varepsilon})\,.$$

Clearly, $\left|\left|\widetilde{A}\right|\right|_{L^\infty(\Omega)} \leq ||A||_{L^\infty(\Omega)} \leq C$ and

$$\operatorname{Curl}(\widetilde{A}) = (1-\zeta)\operatorname{Curl}(A) + (A \star \varrho_{\lambda\varepsilon} - A) \cdot \nabla^\perp \zeta + \zeta A \star \cdot \nabla^\perp \varrho_{\lambda\varepsilon},$$

where we used the notation $v \star \cdot w := \sum_{i=1}^n v_i \star w_i$ for $\mathbb{R}^n$-valued functions v, w. In particular,

— in $B_{\lambda\varepsilon}(S)$, $\zeta \equiv 1$, hence $\operatorname{Curl}(\widetilde{A}) = \zeta A \star \cdot \nabla^\perp \varrho_{\lambda\varepsilon}$, which in turn implies $\left|\operatorname{Curl}(\widetilde{A})\right| \leq \frac{C}{\lambda\varepsilon}$;

— in $B_{2\lambda\varepsilon}(S) \setminus B_{\lambda\varepsilon}(S)$, $\operatorname{Curl}(A) = 0$ and so $\operatorname{Curl}(\widetilde{A}) = (A \star \varrho_{\lambda\varepsilon} - A) \cdot \nabla^\perp \zeta + \zeta A \star \cdot \nabla^\perp \zeta$. This gives again $\left|\operatorname{Curl}(\widetilde{A})\right| \leq \frac{C}{\lambda\varepsilon}$;

— in $\Omega \setminus B_{2\lambda\varepsilon}(S)$, $\operatorname{Curl}(A) \equiv 0$ and $\zeta \equiv 0$, hence $\operatorname{Curl}(\widetilde{A}) = 0$.

From the discussion above, we have in particular that $\operatorname{spt}\operatorname{Curl}(\widetilde{A}) \subset B_{3\lambda\varepsilon}(S) =: \widetilde{S}$. Thus, for every $E \subset \Omega$

$$\begin{aligned} \left|\operatorname{Curl}(\widetilde{A})\right|(E) &= \int_E \left|\operatorname{Curl}\widetilde{A}\right| \mathrm{d}x = \int_{E \cap \operatorname{spt}\operatorname{Curl}(\widetilde{A})} \left|\operatorname{Curl}\widetilde{A}\right| \mathrm{d}x \leq \\ &\leq \left|\left|\operatorname{Curl}(\widetilde{A})\right|\right|_{L^\infty(\Omega)} \left|E \cap B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl}(\widetilde{A}))\right| \\ &\leq C\mu_{2,\varepsilon}(\widetilde{A})\,[E]\,. \end{aligned}$$

Moreover, a standard covering argument gives

$$\left|B_{\lambda\varepsilon}(\widetilde{S})\right| \leq C\left|B_{\lambda\varepsilon}(S)\right|,$$

which leads also to

$$\begin{aligned} \int_\Omega \operatorname{dist}^2(\widetilde{A}, \mathrm{SO}(2))\mathrm{d}x &= \int_{\Omega \setminus B_{2\lambda\varepsilon}} \operatorname{dist}^2(A, \mathrm{SO}(2))\mathrm{d}x + \int_{B_{2\lambda\varepsilon}(S)} \operatorname{dist}^2(\widetilde{A}, \mathrm{SO}(2))\mathrm{d}x \\ &\leq E_{\mathrm{el}}(A, S) + C\left|B_{\lambda\varepsilon}(S)\right| \leq C\mathcal{F}_\varepsilon(A, S). \end{aligned}$$

□

2.4 Structure of Limit Fields I: BV estimate

We can start the analysis of the structure of limits of energy-minimizing tensor fields. The following lemma gives a control on the BV norm of a function once a bound on its variance is known.

Lemma 2.4.1. *Let $u \in L^1(\Omega)^m$, $\Omega \subset \mathbb{R}^n$ open. There exists a constant $C = C(n) > 0$ depending only on the dimension n such that if for some positive measure μ on $\mathbb{R}^n$*

$$\operatorname{Var}_n(u; B(p,\varrho)) := \left(\int_{B(p,\varrho)} \left| u - \fint_{B(p,\varrho)} u \right|^{\frac{n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{n}} \leq \mu(B(p, 2\varrho)) \tag{2.4.1}$$

holds for every $p \in \Omega$ and $\varrho > 0$, then

$$|Du|(\Omega) \leq C\mu(\Omega).$$

In particular, $u \in BV(\Omega)$ provided $\mu(\Omega) < \infty$.

Proof. We can assume $\mu(\Omega) < \infty$, otherwise there is nothing to prove. Extend u to 0 outside Ω and consider a tessellation of $\mathbb{R}^n$ with closed cubes of side $\delta > 0$, $\left\{Q_i^{(\delta)}\right\}_{i\geq 1} \equiv \{Q(x_i, \delta)\}_{i \geq 1}$, whose side length is δ and whose interiors are pairwise disjoint. Define

$$u_\delta := \sum_{i\geq 1} \chi_{Q_i^{(\delta)}} u_i^{(\delta)}, \qquad u_i^{(\delta)} := \fint_{Q_i^{(\delta)}} u \mathrm{d}x.$$

Clearly, $u_\delta \to u$ in $L^1(\Omega)$. The divergence theorem gives for any test function $\varphi \in \mathcal{C}_c^1(\Omega)$ such that $||\varphi||_\infty \leq 1$

$$\int_\Omega u_\delta \operatorname{div}(\varphi)\mathrm{d}x = \sum_{i\geq 1} u_i^\delta \int_{Q_i^{(\delta)}} \operatorname{div}(\varphi)\mathrm{d}x = \sum \sigma_{ij} \left(u_i^{(\delta)} - u_j^{(\delta)} \right) \int_{\partial Q_i^{(\delta)} \cap \partial Q_j^{(\delta)}} \varphi \cdot \nu \mathrm{d}\mathcal{H}^{n-1},$$

where the last sum is extended over all those $i < j$ such that $\mathcal{H}^{n-1}\left(\partial Q_i^{(\delta)} \cap \partial Q_j^{(\delta)}\right) > 0$, $\sigma_{ij} \in \{1, -1\}$ are constants giving the correct sign and ν denotes the outer unit normal. Taking the absolute values, we find

$$\left| \int_\Omega u_\delta \operatorname{div}(\varphi)\mathrm{d}x \right| \leq \sum \left| u_i^\delta - u_j^\delta \right| \mathcal{H}^{n-1}\left(\partial Q_i^{(\delta)}\right).$$

Now, consider the balls $B_i^{(\delta)} := B(x_i, 2\delta)$ which cover all the squares with an edge in common with $Q_i^{(\delta)}$ and use Hölder's inequality in order to find

$$\begin{aligned} \left| u_i^\delta - u_j^\delta \right| &\leq \frac{1}{\left|Q_i^{(\delta)}\right|} \int_{B_i^{(\delta)}} \left| u - \fint_{B_i^{(\delta)}} u \mathrm{d}x \right| \mathrm{d}x \frac{1}{\left|Q_i^{(\delta)}\right|} \operatorname{Var}_n\left(u; B_i^{(\delta)}\right) \left|B_i^{(\delta)}\right|^{\frac{1}{n}} \leq \\ &\leq C_n \delta^{1-n} \operatorname{Var}_n\left(u; B_i^{(\delta)}\right). \end{aligned}$$

Thus, since the balls B_i^δ overlap only finitely many times (depending on the dimension n), we have from (2.4.1)

$$\left| \int_\Omega u_\delta \operatorname{div}(\varphi)\mathrm{d}x \right| \leq C_n \sum \delta^{n-1}\delta^{1-n} \operatorname{Var}_n\left(u; B_i^{(\delta)}\right) \leq C_n \sum_{i\geq 1\,:\, B_i^{(\delta)} \cap \Omega \neq \emptyset} \mu(2B_i^{(\delta)}) \leq C_n \mu(\Omega).$$

Since the estimate is independent of $\delta > 0$ and the test function φ, we can first take the limit as $\delta \downarrow 0$ and then the supremum on φ in order to infer $|Du|(\Omega) \leq C_n \mu(\Omega)$. □

Then, in order to conclude that a limit field is in BV, we just need to bound its variance. This is an application of the geometric rigidity estimate for incompatible fields due to Müller, Scardia and Zeppieri, which we recalled in Theorem 1.4.10.

Proposition 2.4.2. *There exists a constant $C > 0$ such that if $\varepsilon_j \to 0$ and $(A_j, S_j) \in \mathcal{P}_{\varepsilon_j,\alpha,L}$ be such that $\mathcal{F}_j(A_j, S_j) \le E_{gb}(\varepsilon_j)$. Then, there exists another sequence, still denoted by (A_j, S_j), such that $\mathcal{F}_j(A_j, S_j) \le CE_{gb}(\varepsilon_j)$, $A_j \to A \in BV(\Omega)$ in $L^2(\Omega)$ and $A(x) \in \mathrm{SO}(2)$ for every $x \in \Omega \setminus \mathcal{M}$, where $\mathcal{M} \subset \Omega$ is a set of Hausdorff dimension at most 1.*

Proof. By Lemma 2.3.2, we can always assume $|\mathrm{Curl}(A_j)| \le C\mu_{2,j}$, where $\mu_{2,j} := \mu_{2,\varepsilon_j}(A_j)$. Since $\int_\Omega |A_j|^2 \mathrm{d}x \le C$, there exists (up to passing to a subsequence) a matrix field $A \in L^2(\Omega)$ such that $A_j \rightharpoonup A$ in $L^2(\Omega)$. Pick then $x \in \Omega$ and $\varrho > 0$. By Theorem 1.4.10, there exists a rotation $R^j_{\varrho,x}$ such that

$$\fint_{B(x,\varrho)} \left|A_j - R^j_{\varrho,x}\right|^2 \mathrm{d}y \le \frac{C(B(0,1))}{\varrho^2}\left(E_{\mathrm{gb}}(\varepsilon_j) + |T_j|^2 (B(x,\varrho))\right),$$

where $T_j := |\mathrm{Curl}(A_j)| \overset{*}{\rightharpoonup} T$. Thus, taking the lim sup and passing to a subsequence, we find

$$\lim_{j\to\infty} \sqrt{\fint \left|A_j - R^j_{\varrho,x}\right|^2 \mathrm{d}y} \le C \limsup_{j\to\infty} \frac{|T_j|(B(x,\varrho))}{\varrho^2} \le C\frac{T(B(x,2\varrho))}{\varrho^2}.$$

Now, up to another subsequence, $R^j_{\varrho,x} \to R_{\varrho,x} \in \mathrm{SO}(2)$, by the lower semicontinuity of the L^2 norm,

$$\int_{B(x,\varrho)} |A - R_{\varrho,x}|^2 \mathrm{d}y \le CT(B(x,2\varrho))^2. \tag{2.4.2}$$

In particular, we see from (2.4.2) that we can apply Lemma 2.4.1 and conclude that $A \in BV(\Omega)$, with $|DA|(\Omega) \le CT(\Omega) \le C \liminf_{j\to\infty} T_j(\Omega) < \infty$. Consider now the following sets, for $\delta, c > 0$:

$$\mathcal{M}_{\delta,c} := \left\{x \in \Omega \middle| \exists \varrho < \delta \text{ s.t. } \int_{B(x,\varrho)} |A - R_{\varrho,x}|^2 \mathrm{d}y > c\varrho^2\right\}.$$

We can find points $x_j \in \mathcal{M}_{\delta,c}$ such that, if $B_j := B(x_j, \varrho(x_j))$, then

$$\mathcal{M}_{\delta,c} \subset \bigcup_{j\ge1} B_j.$$

In particular, $c\varrho_j^2 \le C\mu(B_j)^2$. We define

$$\mathcal{M}_c := \left\{x \in \Omega \middle| \exists \varrho_i \downarrow 0 \text{ s.t. } \int_{B_{\varrho_i}(x)} |A - Q_{\varrho_i,x}|^2 \mathrm{d}y > c\varrho_i^2\right\}.$$

By the definition of Hausdorff measure, we find

$$\mathcal{H}^1(\mathcal{M}_c) \le \sqrt{c}\sum_j T(B_j) + 1 \le \sqrt{c}T(\mathcal{M}_c) + 1 \le C\,(T(\Omega) + 1) < \infty.$$

Clearly, for every $x \in \Omega \setminus \mathcal{M}_c$,

$$\limsup_{\varrho \downarrow 0} \fint_{B(x,\varrho)} |A - Q_{\varrho,x}|^2 \, \mathrm{d}y \le c.$$

By the arbitrariness of $c > 0$, $A(x) \in \mathrm{SO}(2)$ for every $x \in \Omega \setminus \bigcap_{c>0} \mathcal{M}_c$, that is $A(x)$ belongs to SO(2) for all points in Ω, except for a set of Hausdorff dimension at most 1.
Now we have, in particular, $A_j \rightharpoonup A$ and $A(x) \in \mathrm{SO}(2)$ for almost every $x \in \Omega$. Denote as $R_j(x)$ the projection of $A_j(x)$ on SO(2). Then $A_j = R_j + (A_j - R_j)$. We know that $A_j - R_j \to 0$ in $L^2(\Omega)$ while, up to a subsequence, $R_j \rightharpoonup A$. But then $R_j \to A$ (because the L^2 norms converge to the norm of A), and thus $A_j \to A$ in $L^2(\Omega)$. □

Using a slicing argument and Proposition 2.4.2, we obtain the estimate $\mu_2(\Omega) \ge |DA|(\Omega) \ge C\alpha L$, that is a (weak) lower bound to the energy. We are going to improve this result in a first qualitative, and then quantitative way. By qualitative we mean that the limit field is actually a microrotation, while the quantitative improvement is an estimate involving a power of the logarithm of α^{-1}. These facts rely essentially on two basic tools: the existence of a harmonic competitor and an "optimal foliation" lemma. We give here the proof of the first one.

2.5 The Harmonic Competitor

Proposition 2.5.1. *Let $\Omega \subset \mathbb{R}^n$ be open, and $A \in L^\infty(\Omega)^{n\times n}$ be a matrix field such that $||A||_\infty \le M$, and let $O \subset \Omega \setminus B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl} A)$ be an open, connected subset with Lipschitz boundary. Then there exists a matrix field $\widetilde{A} \in L^2(\Omega)^{n\times n}$ which is harmonic in O and a constant $C_{n,M} > 0$ (depending only on the dimension n and M) such that*

$$\left|\left|A - \widetilde{A}\right|\right|_{L^2(O)} \le C_{n,M} \left|\left|\operatorname{dist}(A, SO(n))\right|\right|_{L^2(O)}.$$

Proof. Let $E := ||\operatorname{dist}(A, SO(n))||^2_{L^2(O)}$. The Hodge decomposition of A gives a vector field $u \in W_0^{1,2}(\Omega)^n$ and a divergence-free (in the sense of distributions in Ω) matrix field $F \in L^2(\Omega)^{n\times n}$ such that

$$A = \nabla u + F.$$

As in the proof of Lemma 2.3.1, we can assume A to be smooth in $\overline{B_{\lambda\varepsilon}(\operatorname{spt}\operatorname{Curl} A)}$. Consider the function $u_h^1 \in W^{1,2}(O)$ defined as the harmonic extension of u in O:

$$\begin{cases} \Delta u_h^1 = 0 & \text{in } O, \\ u_h^1 = u^1 & \text{on } \partial O, \end{cases}$$

and let then $A_h := \nabla(u_h^1, u^2, \cdots, u^n) + F$. Define $G := O \cap \left\{\det(A) > \frac{1}{2}\right\}$, and $U(x) := \sqrt{AA^T}\chi_G + (1 - \chi_G)\mathrm{id}$, together with the vector fields $R_i(x) := \left[U(x)^{-1}(\nabla u + F)\right]^i$ and $R_{1h} := \left[U(x)^{-1}(\nabla u_h + F)\right]^1$. In what follows, we identify vector fields with their associated differential 1-forms. We first notice that

$$\int_O \det(A)\mathrm{d}x = \int_O \det(A_h)\mathrm{d}x. \tag{2.5.1}$$

Indeed, since the determinant is a null Lagrangian, (2.5.1) is equivalent to

$$\sum_{i=2}^{n} \int_O \mathrm{d}\left(u_h^1 - u^1\right) \wedge \bigwedge_{j=2}^{n} \left((1-\delta_{ij})\mathrm{d}u^j + F^j\right) = 0,$$

which holds because of the Leibniz formula for forms, the fact that $\operatorname{Curl} F^i = 0$ in O and Stokes' theorem (together with $u_h^1 = u^1$ on ∂O). Hence, we can write (notice that, since $R_1, \cdots, R_n$ are orthonormal, for any vector field $\mathcal{A}$ we have $\mathcal{A} \wedge R_2 \wedge \cdots \wedge R_n = \sum_{k=1}^{n} \langle \mathcal{A}, R_k \rangle R_k \wedge R_2 \wedge \cdots \wedge R_n = \langle \mathcal{A}, R_1 \rangle R_1 \wedge \cdots \wedge R_n = \langle \mathcal{A}, R_1 \rangle \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$)

$$\begin{aligned}
\int_O \det(A)\mathrm{d}x = \int_O \det(A_h)\mathrm{d}x &= \int_O \left(\mathrm{d}u_h^1 + F^1\right) \wedge \left(\mathrm{d}u^2 + F^2\right) \wedge \cdots \wedge (\mathrm{d}u^n + F^n) = \\
&= \int_O (R_{1h} \wedge R_2 \wedge \cdots \wedge R_n) \det(U) = \\
&= \int_G \langle R_{1h}, R_1 \rangle \det(U)\mathrm{d}x + \int_{O\setminus G} R_{1h} \wedge R_2 \wedge \cdots \wedge R_n \det(U).
\end{aligned} \tag{2.5.2}$$

On the other hand,

$$\int_O \det(A)\mathrm{d}x = \int_G \langle R_1, R_1 \rangle \det(U) + \int_{O\setminus G} R_1 \wedge \cdots \wedge R_n \det(U). \tag{2.5.3}$$

Subtracting (2.5.2) from (2.5.3), we obtain

$$0 = \int_G \langle R_1 - R_{1h}, R_1 \rangle \det(U)\mathrm{d}x + \int_{O\setminus G} (R_1 - R_{1h}) \wedge R_2 \wedge \cdots \wedge R_n \det(U). \tag{2.5.4}$$

Rewrite (2.5.4) as

$$\begin{aligned}
\int_G \left\langle \nabla u^1 - \nabla u_h^1, \nabla u^1 + F^1 \right\rangle = &- \int_G \left\langle \nabla u^1 - \nabla u_h^1, \left(\det(U)U^{-2} - \mathrm{id}\right)\left(\nabla u^1 + F^1\right) \right\rangle \mathrm{d}x + \\
&+ \int_{O\setminus G} (R_1 - R_{1h}) \wedge R_2 \wedge \cdots \wedge R_n \det(U),
\end{aligned}$$

and then add $\int_{O\setminus G} \langle \nabla u^1 - \nabla u_h^1, \nabla u^1 + F^1 \rangle$ on both sides. Since u_h^1 is the harmonic extension of u^1 and $\operatorname{div}(F^1) = 0$, we have

$$\int_O \left|\nabla u^1 - \nabla u_h^1\right|^2 \mathrm{d}x = \int_O \left\langle \nabla u^1 - \nabla u_h^1, \nabla u^1 + F^1 \right\rangle \mathrm{d}x = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &:= \int_{O\setminus G} \left\langle \nabla u^1 - \nabla u_h^1, \nabla u^1 + F^1 \right\rangle, \\
I_2 &:= - \int_G \left\langle \nabla u^1 - \nabla u_h^1, \left(\det(U)U^{-2} - \mathrm{id}\right)\left(\nabla u^1 + F^1\right) \right\rangle, \\
I_3 &:= \int_{O\setminus G} (R_1 - R_{1h}) \wedge R_2 \wedge \cdots \wedge R_n.
\end{aligned}$$

Now, because of the continuity of the determinant, there exists a dimensional constant $c_n > 0$ such that $\left\{\det(A) \leq \frac{1}{2}\right\} \subset \{\mathrm{dist}(A, SO(n)) \geq C_n\}$. Thus, since $||A||_\infty \leq M$,

$$\left|\int_{O\setminus G} \left\langle \nabla u^1 - \nabla u_h^1, \nabla u^1 + F^1 \right\rangle \mathrm{d}x\right| \leq C_n \int_{\{\mathrm{dist}(A,SO(n))\geq C_n\}} \left|\nabla u^1 - \nabla u_h^1\right| \mathrm{d}x \leq$$
$$\leq C_n \sqrt{E} \sqrt{\int_O |\nabla u^1 - \nabla u_h^1|^2 \,\mathrm{d}x}.$$

Let us now estimate I_2. Since the function $f(U) := U^{-2}\det(U)$ is smooth on G, $||A||_\infty \leq M$ and $f(\mathrm{id}) = \mathrm{id}$, there exists a constant, depending only on n and M,$C_n = C_n(M) > 0$ such that

$$|f(U) - \mathrm{id}| \leq C_n |U - \mathrm{id}| = C_n \,\mathrm{dist}(A, SO(n)).$$

Then

$$|I_2| \leq C_n \int_G \left|\nabla u^1 - \nabla u_h^1\right| \left|U^{-2}\det(U) - \mathrm{id}\right| \left|\nabla u^1 + F^1\right| \mathrm{d}x \leq$$
$$\leq C_n \sqrt{E} \sqrt{\int_O |\nabla u^1 - \nabla u_h^1|^2 \,\mathrm{d}x}.$$

Finally, let us estimate I_3. Again because of the boundedness of A,

$$|I_3| \leq C_n \int_{O\setminus G} |R_1 - R_{1h}| \frac{c_n}{c_n} \mathrm{d}x \leq C_n \sqrt{E} \sqrt{\int_O |\nabla u^1 - \nabla u_h^1|^2 \,\mathrm{d}x}.$$

Combining these estimates together, we find

$$\int_O \left|\nabla u^1 - \nabla u_h^1\right|^2 \mathrm{d}x \leq C_n \sqrt{E} \sqrt{\int_O |\nabla u^1 - \nabla u_h^1|^2 \,\mathrm{d}x},$$

i.e.

$$\int_O \left|\nabla u^1 - \nabla u_h^1\right|^2 \mathrm{d}x \leq C_n E.$$

Applying the same procedure to each component, we find

$$\int_O |\nabla u - \nabla u_h|^2 \,\mathrm{d}x \leq C_n E,$$

where $u_h = (u_h^1, \cdots, u_h^n)$. Now we can define

$$\widetilde{u} := u_h \chi_O + u \chi_{\Omega\setminus O},$$

and set $\widetilde{A} := \nabla\widetilde{u} + F$. Since $\mathrm{Div}(\widetilde{A}) = \Delta\widetilde{u} = 0$ and $\mathrm{Curl}(\widetilde{A}) = 0$ in O, from the identity

$$-\Delta L_j^i + \partial_j \,\mathrm{Div}\, L^i = -\sum_{k=1}^n \partial_k \left(\partial_k L_j^i - \partial_j L_k^i\right),$$

valid for any matrix field $L \in L^1(\Omega)^{n\times n}$, we infer that $\Delta\widetilde{A} = 0$ in O. □

Remark 2.5.1. Combining together the lemmata 2.3.1, 2.3.2 and Proposition 2.5.1, we have that for every $(A,S) \in \mathcal{P}_\varepsilon$ such that $\mathcal{F}_\varepsilon(A,S) \leq E_{\text{gb}}(\varepsilon)$, we can find a competitor $(\widetilde{A},\widetilde{S}) \in \mathcal{A}_\varepsilon$ whose energy can be estimated in terms of the original one, i.e. $\mathcal{F}_\varepsilon(\widetilde{A},\widetilde{S}) \leq C\mathcal{F}_\varepsilon(A,S)$, where $C > 0$ is a universal constant, satisfying the following properties:

(a) $\left|\left|\widetilde{A}\right|\right|_\infty \leq C$,

(b) $\left|\operatorname{Curl}(\widetilde{A})\right| \leq C\mu_{2,A}$,

(c) $\Delta\widetilde{A} = 0$ in $\Omega_{\lambda\varepsilon}(\widetilde{A})$.

That is, since we are interested in a lower bound to the energy, we can restrict our attention to those pairs in $\mathcal{P}_\varepsilon$ satisfying (a), (b) and (c).

Remark 2.5.2. If $A \in \mathcal{A}_\varepsilon \cap \{G : ||G||_\infty \leq M\}$ and $\widetilde{A}$ is the matrix field given by Lemma 2.5.1, then the Burgers' vectors relative to A still define a bounded functional from 1-cycles into $\mathbb{R}^2$, and it can also be proved without employing the maximum principle. Indeed, if we identify $\widetilde{A}$ with a vector of 1-forms, the Burgers' vector

$$\begin{array}{rccc} \vec{b}_{\widetilde{A}}: & Z_1(\Omega_{\lambda\varepsilon}(\widetilde{A});\mathbb{R}) & \longrightarrow & \mathbb{R}^2 \\ & T & \longmapsto & \left\langle T,\widetilde{A}\right\rangle \end{array}$$

defines a bounded operator (where the space of 1-cycles is endowed with the mass norm). Indeed, in $\Omega \setminus B_{\lambda\varepsilon}(\widetilde{A})$, we can write $\widetilde{A} = \mathrm{d}u_h + F$, where $A = \mathrm{d}u + F$. Then, since T is a closed current,

$$\left\langle T,\widetilde{A}\right\rangle = \langle T, \mathrm{d}u_h + F\rangle = \langle T, F\rangle = \langle T, \mathrm{d}u + F\rangle = \langle T, A\rangle\,.$$

But $|\langle T, A\rangle| \leq ||A||_\infty \mathbf{M}(T) \leq M\mathbf{M}(T)$, hence the claim.

We shall need the following Lemma, which gives an expression for the Burgers' vector in terms of the gradient of the fields and the position of the points on the curve.

Lemma 2.5.2. *Suppose $\gamma \subset \mathbb{R}^2$ is a closed, simple Lipschitz curve, and V is a $\mathcal{C}^1$ vector field defined in a neighborhood of γ. Then*

$$\int_\gamma V(x)\cdot t(x)\mathrm{d}\mathcal{H}^1 = -\int_\gamma \nabla V(x)x\cdot t(x)\mathrm{d}\mathcal{H}^1,$$

where $t(x)$ is the tangent vector of γ at x.

Proof. Let $\gamma = \left\{f(t)\middle| t \in [0,1)\right\}$, where f is a Lipschitz parametrization of γ, and set $x_0 :=$

$f(0) = f(1)$. Then

$$\begin{aligned}
\int_\gamma V \cdot t \mathrm{d}\mathcal{H}^1 &= \int_\gamma (V(x) - V(x_0)) \cdot t \mathrm{d}\mathcal{H}^1 = \int_0^1 (V(f(t)) - V(f(0))) \cdot \dot{f}(t) \mathrm{d}t = \\
&= \int_0^1 \left(\int_0^t \nabla V(f(s)) \dot{f}(s) \mathrm{d}s \right) \cdot \dot{f}(t) \mathrm{d}t = \int_0^1 \nabla V(f(s)) \dot{f}(s) \cdot \int_s^1 \dot{f}(t) \mathrm{d}t \mathrm{d}s = \\
&= -\int_0^1 \nabla V(f(s)) \dot{f}(s) \cdot f(s) \mathrm{d}s = -\int_0^1 \nabla V(f(s)) f(s) \cdot \frac{\dot{f}(s)}{\left|\dot{f}(s)\right|} \left|\dot{f}(s)\right| \mathrm{d}s = \\
&= -\int_\gamma \nabla V(x) x \cdot t(x) \mathrm{d}\mathcal{H}^1. \qquad \square
\end{aligned}$$

As an immediate application of Lemma 2.5.2, we see that if γ lies in a region where A is both Curl and divergence free, then

$$\begin{aligned}
\overrightarrow{b}(\gamma) &= -\left(\int_\gamma \left(x \cdot \nabla A_1^1, x^\perp \cdot \nabla A_1^1 \right) \cdot t(x) \mathrm{d}\mathcal{H}^1, \int_\gamma \left(-x^\perp \cdot \nabla A_2^2, x \cdot \nabla A_2^2 \right) \cdot t(x) \mathrm{d}\mathcal{H}^1 \right) \equiv \\
&\equiv -\int_\gamma \left(\begin{pmatrix} x & x^\perp \\ -x^\perp & x \end{pmatrix} \cdot \begin{pmatrix} \nabla A_1^1 & \nabla A_2^2 \end{pmatrix} \right) t(x) \mathrm{d}\mathcal{H}^1.
\end{aligned} \tag{2.5.5}$$

2.6 The Foliation Lemma

We are left with the second fundamental tool, that is the foliation Lemma. In the proof, we will need the following technical covering lemma:

Lemma 2.6.1. *Let $R > 0$, $\delta \in (0,1)$, $M > 10$ and consider a family $I = \{x_i\}_{i=1}^N$ of points in $\mathbb{R}^n$ whose subfamily $J \subset I$ has the property that for each $j \in J$ there exists a $k \in \mathbb{N}$ such that $\left(B\left(x_i, \left(\frac{M}{2}\right) - 2\right) r_k \setminus B\left(x_i, r_k\right)\right) \cap I = \emptyset$, where $r_k := \delta^k R$. Set*

$$r_j := \max \left\{ \delta^k R \middle| k \geq 0 \text{ and } \left(B\left(x_i, \left(\frac{M}{2} - 2\right) r_k\right) \setminus B\left(x_i, r_k\right) \right) \cap I = \emptyset \right\}.$$

Then there exists a subfamily $\widetilde{J} \subset J$ such that the balls $\left\{B\left(x_i, \left(x_i, \left(\frac{M}{4} - 1\right) r_i\right)\right)\right\}_{i \in \widetilde{J}}$ are disjoint and

$$\bigcup_{j \in J} B\left(x_j, \left(\frac{M}{8} - \frac{5}{4}\right) r_j\right) \subset \bigcup_{i \in \widetilde{J}} B\left(x_i, \left(\frac{M}{8} - \frac{1}{4}\right) r_i\right).$$

Proof. Let $\beta := \frac{M}{2} - 2$. Define inductively the family $\widetilde{J}$ as follows. Select a maximal family of points J_0 from $\left\{ j \in J \middle| r_j = \delta^k R \right\}$ such that

$$|x_i - x_j| \geq \frac{\beta}{2} (r_i + r_j) \qquad \forall i, j \in J_0,$$

and set $\widetilde{J}_0 := J_0$. Suppose then that the family $\widetilde{J}_k$ has been defined, $k \geq 0$, and select a maximal family of points J_{k+1} from $\left\{ j \in J \middle| r_j = \delta^{k+1} R \right\}$ such that

$$|x_i - x_j| \geq \frac{\beta}{2} (r_i + r_j) \qquad \forall i, j \in \widetilde{J}_k \cup J_{k+1},$$

and then set $\widetilde{J}_{k+1} = \widetilde{J}_k \cup J_{k+1}$. The set $\widetilde{J}$ is given by

$$\widetilde{J} := \bigcup_{k \geq 0} \widetilde{J}_k.$$

Clearly, the balls $\left\{B\left(x_i, \frac{\beta}{2} r_i\right)\right\}_{i \in \widetilde{J}}$ are disjoint. Moreover, for every $x_j \in J$ we can find an $x_i \in \widetilde{J}$ such that $r_i = r_j$ and

$$|x_i - x_j| < \frac{\beta}{2}(r_i + r_j) \leq \beta r_i,$$

which means, by the definition of r_i, that $|x_i - x_j| \leq r_i$. Hence, if $x \in B\left(x_j, \left(\frac{M}{8} - \frac{5}{4}\right) r_j\right)$, $j \in J$, then there exists an $i \in \widetilde{J}$ such that

$$|x - x_i| \leq r_i + \left(\frac{M}{8} - \frac{5}{4}\right) r_i = \left(\frac{M}{8} - \frac{1}{4}\right) r_i. \qquad \square$$

We are now in position to prove a key step, that is the lemma which gives the optimal foliation.

Lemma 2.6.2. *There exist $\delta_0 \in (0,1)$ and $C > 0$ such that if $\{B(x_i, \varrho_i)\}_{i=1}^N$ are balls in $\mathbb{R}^2$ satisfying*

$$\mathcal{H}^1\left(\mathcal{A} \cap \bigcup_{i=1}^N \partial B(x_i, \varrho_i)\right) \leq \delta_0, \quad \mathcal{A} := B(0,1) \setminus B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2, \tag{2.6.1}$$

then there exists a Lipschitz function $\varphi : \mathcal{A} \to [0,1]$ such that

(i) $||\nabla\varphi||_{L^\infty(\mathcal{A})} \leq C$;

(ii) $\varphi \equiv 0$ *on* $\partial B(0,1)$ *and* $\varphi \equiv 1$ *on* $\partial B\left(0, \frac{1}{2}\right)$;

(iii) If $U := \mathcal{A} \setminus \bigcup_{i=1}^N B(x_i, \varrho_i)$,

$$\int_U \frac{|\nabla\varphi(x)|^2}{\mathrm{dist}^2(x, \partial U)} \mathrm{d}x \leq C(1 + N). \tag{2.6.2}$$

Proof. We shall modify in an appropriate way the natural radial foliation. First of all, define

$$\delta_1 := \inf\left\{r \geq \delta_0 \middle| \partial B\left(0, \frac{1}{2} + r\right) \cap \bigcup_{i=1}^N B(x_i, \varrho_i) = \emptyset\right\},$$

and

$$\delta_2 := \inf\left\{r \geq \delta_0 \middle| \partial B(0, 1 - r) \cap \bigcup_{i=1}^N B(x_i, \varrho_i) = \emptyset\right\}.$$

By a simple geometric argument, one can see that $\delta_0 \leq \min\{\delta_1, \delta_2\} \leq \max\{\delta_1, \delta_2\} \leq \frac{3}{2}\delta_0$. Define then the function

$$\varphi_0(x) := \begin{cases} C(\delta_1, \delta_2)(1 - \delta_2 - |x|) & \text{if } |x| \in B(0, 1 - \delta_2) \setminus B\left(0, \frac{1}{2} + \delta_1\right), \\ 0 & \text{if } |x| \geq 1 - \delta_2, \\ 1 & \text{if } |x| \leq \frac{1}{2} + \delta_1, \end{cases}$$

where $C(\delta_1, \delta_2) := \frac{1}{\frac{1}{2}-\delta_1-\delta_2}$ (clearly φ_0 is Lipschitz, with Lipschitz constant $C(\delta_1, \delta_2) \le \frac{1}{\frac{1}{2}-3\delta_0}$ and satisfies (ii)). We will then split the integral I in the left hand side of (2.6.2) in three terms: one where, roughly speaking, we see enough space in order to interpolate the function with a constant, another one where the balls accumulate (where we will use a covering argument) and a last one where we are very close to the balls $B(x_i, \varrho_i)$ (of which we will get rid of simply by using a "cutting-out" function, possible because of (2.6.1)).

In order to detect the regions where we have to modify the foliation, it is convenient to introduce particular coverings and organize them in a graph. For, define the sets

$$U_k := \left\{ x \in U \middle| r_{k-1} < \operatorname{dist}(x, \{x_i\}_{i=1}^N) \le r_k \right\},$$

where $r_k := M^k r_0$ and $r_0 := \frac{c_0}{n}$, for some constants $c_0 > 0$ and $M > 2$ to be chosen later, and $k \in \{0, \cdots, K\}$, $K := \left[\frac{1}{2}\log(N)\right]$. Let $I := \{x_1, \cdots, x_N\}$ and for each $k \in \{0, \cdots, K\}$ choose a maximal family I_k of points in I whose reciprocal distances are $\ge r_k$. Notice that for each k the balls $\{B(x_i, 2r_k)\}_{i \in I_k}$ are a cover of U_k. We then define the *edge maps*

$$E_k : I_k \longrightarrow I_{k+1},$$

which have the property that, for $x_i \in I_k$,

$$|E_k(x_i) - x_i| = \min \left\{ |x_j - x_i| \middle| x_j \in I_{k+1} \right\}.$$

Clearly, $|E_k(x_i) - x_i| < r_{k+1}$; indeed, either $x_i \in I_{k+1}$ (and in such a case $E_k(x_i) = x_i$) or $x_i \notin I_{k+1}$. But then $|x_j - x_i| < r_{k+1}$ for some $j \in I_{k+1}$ in order to not contradict the maximality of I_{k+1}.

We can now define the directed graph (actually, the forest) $G = (V, E)$ whose vertices are given by

$$V := \left\{ (x_i, k) \middle| x_i \in I_k, \quad k \in \{0, \cdots, K\} \right\},$$

and whose edges are

$$E := \left\{ ((x_i, k), (E_k(x_i), k+1)) \middle| i \in I_k, \quad k \in \{0, \cdots, K-1\} \right\}.$$

We write $v \sim w$ if either $(v, w) \in E$ or $(w, v) \in E$. Notice that G is the disjoint union of (directed) trees whose roots are the points (x_i, K), $x_i \in I_K$. Given an edge $e = (x_i, k)$, we denote by T_e the subtree rooted at e. We also define the "pruned" tree at the vertex e as

$$T_e^{\mathrm{pr}} := T_e \setminus \bigcup_{\substack{e' \in V \\ T_{e'} \subset T_e, \quad T_{e'} \neq T_e \\ \deg(e') = 2}} T_{e'} \setminus \{e'\}.$$

We then have the pruned forest

$$G^{\mathrm{pr}} := \bigcup_{x_i \in I_K} T_{(x_i, K)}^{\mathrm{pr}} =: (V^{\mathrm{pr}}, E^{\mathrm{pr}}).$$

The vertices of degree 2 where we prune the tree are the ones which we will see to correspond to empty annuli. To see this, notice that if an edge $e = (x_i, k) \in V$, $k \le K-1$, has degree 2 and $e' = (x_j, k') \in T_e$ with $k' \le k-1$, then

(a) $I_{k-1} \cap B\left(x_i, \frac{r_k}{2}\right) = \{x_{i_0}\}$ is a singleton. Indeed,

$$2 = \deg(x_i) = \# E_{k-1}^{-1}(x_i) + 1.$$

But $E_{k-1}^{-1}(x_i) \supset I_{k-1} \cap B\left(x_i, \frac{r_k}{2}\right)$ (indeed, if $x_j \in B\left(x_i, \frac{r_k}{2}\right) \cap I_{k-1}$, then for every $x_{i'} \in I_k \setminus \{x_i\}$ we have $|x_j - x_{i'}| \ge |x_i - x_{i'}| - |x_i - x_j| > r_k - \frac{r_k}{2} = \frac{r_k}{2} > |x_j - x_i|$, that is $E_{k-1}(x_j) = x_i$) which is always not empty. Otherwise, $x_i \notin I_{k-1}$ and for every $x_j \in I_{k-1}$ we have $|x_j - x_i| > \frac{r_k}{2} > r_{k-1}$, i.e. $\{x_i\} \cup I_{k-1}$ would be a family whose points have reciprocal distance is $\ge r_{k-1}$ and which strictly contains I_{k-1}, which was assumed to be a maximal family. Hence,

$$1 = \# E_{k-1}^{-1}(x_i) \ge \# I_{k-1} \cap B\left(x_i, \frac{r_k}{2}\right) \ge 1,$$

i.e. $\# I_{k-1} \cap B\left(x_i, \frac{r_k}{2}\right) = 1$, say $I_{k-1} \cap B\left(x_i, \frac{r_k}{2}\right) = \{x_{i_0}\}$;

(b) $|x_i - x_{i_0}| < r_{k-1}$. This is clear, because of what we said at the point (a);

(c) $\left(B\left(x_i, \frac{r_k}{2} - r_{k-1}\right) \setminus B\left(x_{i_0}, r_{k-1}\right)\right) \cap I = \emptyset$. This is also a direct consequence of the previous two points. In particular,

$$\left(B\left(x_{i_0}, \frac{r_k}{2} - 2r_{k-1}\right) \setminus B\left(x_{i_0}, r_{k-1}\right)\right) \cap I = \emptyset;$$

(d) $|x_j - x_{i_0}| < \frac{M}{M-1} r_{k-1} = \frac{r_k}{M-1}$. Indeed, since e has degree 2, the only vertex at level $k-1$ is precisely $(x_{i_0}, k-1)$. Hence, if we set $y_0 := x_j$ and define inductively $y_{i+1} := E_{k'+i}(y_i)$, $i = 0, \cdots, k-k'-2$, we have (since $E_{k-2}(y_{k-k'-2}) = x_{i_0}$)

$$|x_j - x_{i_0}| \le \sum_{i=0}^{k-k'-2} |y_{i+1} - y_i| \le r_{k-1} \sum_{i=0}^{k-k'-2} M^{-i} \le r_{k-1} \frac{M}{M-1} < \frac{r_k}{M-1}.$$

Define then the family of points J as

$$J := \left\{ x_j \in I \,\middle|\, \exists x_i \in I, \quad k \in \{1, \cdots, K-1\} \text{ such that } (x_j, k-1) \sim (x_i, k) \text{ and } \deg((x_i, k)) = 2 \right\}.$$

Lemma 2.6.1 gives a subfamily $\widetilde{J}$ such that

— $\left\{B\left(x_i, \left(\frac{M}{4} - 1\right) r_i\right)\right\}_{i \in \widetilde{J}}$ are disjoint;

— $\left[B\left(x_i, \left(\frac{M}{2} - 2\right) r_i\right) \setminus B\left(x_i, r_i\right)\right] \cap I = \emptyset$;

— Provided $\frac{M}{8} - \frac{5}{4} > 3$,

$$\bigcup_{j \in J} B\left(x_j, 3r_j\right) \subset \bigcup_{j \in \widetilde{J}} B\left(x_j, \left(\frac{M}{8} - \frac{1}{4}\right) r_j\right).$$

Let now $c_1 := \frac{M}{8} + \frac{3}{4}$ and $c_2 := \frac{M}{4} - 2$, and consider the Lipschitz function $\eta : \mathbb{R}^+ \to [0,1]$

$$\eta(t) := \begin{cases} 1 & \text{if } t \in [0, c_1], \\ \frac{1}{c_1 - c_2} t - \frac{c_2}{c_1 - c_2} & \text{if } t \in [c_1, c_2], \\ 0 & \text{if } t \geq c_2, \end{cases}$$

whose Lipschitz constant is $\frac{1}{c_1 - c_2}$. Define

$$\varphi_1(x) := \sum_{j \in \widetilde{J}} \left(\eta\left(\frac{|x - x_j|}{r_j}\right) \overline{\varphi}_{0,j} + \left(1 - \eta\left(\frac{|x - x_j|}{r_j}\right)\right) \varphi_0(x) \right),$$

where $\overline{\varphi}_{0,j} = \fint_{B(x_j, c_2 r_j)} \varphi_0(y) \mathrm{d}y$. Since the balls defined by the family $\widetilde{J}$ are disjoint, we easily infer

$$\begin{aligned} ||\nabla \varphi_1||_\infty &\leq ||\nabla \varphi_0||_\infty + \max_{j \in \widetilde{J}} \left(|\eta'|_\infty \frac{1}{r_j} \left|\left|\varphi_0 - \overline{\varphi}_{0,j}\right|\right|_{L^\infty(B(x_j, c_2 r_j))} \right) \leq \\ &\leq C(\delta_0, M). \end{aligned}$$

Finally, consider the set $\mathcal{I} := \varphi_1 \left(\bigcup_{i=1}^N B\left(x_i, 2\varrho_i + r_0\right) \right)$. Then

$$\mathcal{L}^1\left(\mathcal{I}\right) \leq \mathrm{Lip}(\varphi_1) \sum_{i=1}^N \left(4\varrho_i + 2r_0\right) \leq \frac{1}{2},$$

provided we take $\delta_0 \leq \frac{1}{16\,\mathrm{Lip}(\varphi_1)}$ and $c_0 = 2\delta_0$. Consider then the Lipschitz function $\varphi : [0,1] \to [0,1]$ defined by

$$\psi' := \frac{\chi_{[0,1] \setminus \mathcal{I}}}{1 - |\mathcal{I}|}, \qquad \psi(0) = 0, \qquad \psi(1) = 1.$$

Define $\varphi := \psi \circ \varphi_1$. Clearly φ satisfies (i) and (ii). Let us prove it satisfies (iii). For, notice first that if $x \notin \bigcup_{i=1}^N B\left(x_i, 2\varrho_i\right)$, then $d(x) := \mathrm{dist}\left(x, \{x_i\}_{i=1}^N\right) \geq \frac{1}{2} \mathrm{dist}\left(x, \partial U\right)$. Set $U' := \left(B(0,1) \setminus B\left(0, \frac{1}{2}\right)\right) \setminus \bigcup_{i=1}^N B\left(x_i, 2\varrho_i + r_0\right)$ and $U'' := \bigcup_{i=1}^N B\left(x_i, 2\varrho_i + r_0\right) \setminus \bigcup_{i=1}^N B\left(x_i, \varrho_i\right)$. Since φ is constant on $\bigcup_{i=1}^N B\left(x_i, 2\varrho_i + r_0\right)$

$$\begin{aligned} \int_U \frac{|\nabla \varphi|^2}{\mathrm{dist}^2(x, \partial U)} \mathrm{d}x &\leq C \left(\int_{U'} \frac{|\nabla \varphi|^2}{d^2(x)} \mathrm{d}x + \int_{U''} \frac{|\nabla \varphi|^2}{\mathrm{dist}^2(x, \partial U)} \mathrm{d}x \right) = \\ &= C \int_{U'} \frac{|\nabla \varphi|^2}{d^2(x)} \mathrm{d}x \leq C \int_{U' \cap \left\{d < \frac{1}{\sqrt{N}}\right\}} \frac{|\nabla \varphi|^2}{d^2(x)} \mathrm{d}x + N. \end{aligned}$$

Write

$$U' \cap \left\{ d < \frac{1}{\sqrt{N}} \right\} = U_1' \cup U_2',$$

$$U_1' := \left(U' \cap \left\{ d < \frac{1}{\sqrt{N}} \right\} \right) \setminus \bigcup_{j \in \widetilde{J}} B\left(x_j, \left(\frac{M}{4} - 2\right) r_j\right),$$

$$U_2' = U \setminus U_1'.$$

Notice that $U_k \cap U_1' \subset \bigcup_{i\,:\,(x_i,k)\in V^{\mathrm{pr}}} B(x_i, 2r_k)$. Since any non-trivial tree $T = (V, E)$ satisfies

$$\#V \le 2\#\left\{v \in V \middle| \deg(v) = 1\right\} + \#\left\{v \in V \middle| \deg(v) = 2\right\}$$

and the total number of leaves in the forest is always $\le N$, we have

$$\begin{aligned}\int_{U_1'} \frac{|\nabla\varphi|^2}{d^2(x)}\mathrm{d}x &\le \sum_{k=0}^{K} \int_{U_k\cap U_1'} \frac{|\nabla\varphi|^2}{d^2(x)}\mathrm{d}x \le C(\delta_0, M) \sum_{k=0}^{K} \sum_{i\,:\,(x_i,k)\in V^{\mathrm{pr}}} \frac{1}{r_{k-1}^2} \int_{B(x_i,r_k)} \mathrm{d}x \le \\ &\le C(\delta_0, M)\,\#V^{\mathrm{pr}} \le C(\delta_0, M)\, N.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{U_2'} \frac{|\nabla\varphi|^2}{d^2(x)}\mathrm{d}x &\le \sum_{j\in\widetilde{J}} \int_{B\left(x_j,\left(\frac{M}{4}-3\right)r_j\right)} \frac{|\nabla\varphi|^2}{d^2(x)}\mathrm{d}x \le \\ &\le C(\delta_0, M)\,\#\widetilde{J} \le C(\delta_0, M)\, N.\end{aligned}$$ □

By a scaling argument we get the following

Corollary 2.6.3. *Let $\mathcal{A} := B(p, 2R) \setminus B(p, R) \subset \mathbb{R}^2$. There exist $\delta_0 \in (0, 1)$ and $C > 0$ such that if $\{B(x_i, \varrho_i)\}_{i=1}^N$ are balls satisfying*

$$\mathcal{H}^1\left(\mathcal{A} \cap \bigcup_{i=1}^{N} \partial B(x_i, \varrho_i)\right) \le \delta_0 R, \tag{2.6.3}$$

then there exists a Lipschitz function $\varphi : \mathcal{A} \to [0, 1]$ such that

(i) $||\nabla\varphi||_{L^\infty(\mathcal{A})} \le \frac{C}{2R}$*;*

(ii) $\varphi \equiv 0$ *on* $\partial B(p, 2R)$ *and* $\varphi \equiv 1$ *on* $\partial B(p, R)$*;*

(iii) If $U := \mathcal{A} \setminus \bigcup_{i=1}^N B(x_i, \varrho_i)$*,*

$$\int_U \frac{|x|^2\,|\nabla\varphi(x)|^2}{\mathrm{dist}^2(x, \partial U)}\mathrm{d}x \le C(1 + N). \tag{2.6.4}$$

Remark 2.6.1. The proof shows that the foliation φ constructed in Lemma 2.6.2 is constant on (a neighborhood of) each ball (and in a neighborhood of the boundary of the annulus). Moreover, due to the choice of δ_1 and δ_2, the superlevel sets $\{\varphi \ge 1\} = \{\varphi = 1\}$ and $\{\varphi > 0\}$ contain all the balls $B(x_i, \varrho_i)$ they intersect.

In the balls construction we shall need to choose, from a family of balls covering the support of a measure μ, a well disjoint subfamily containing a relevant fraction of the total mass. This is exactly the content of the following Lemma.

Lemma 2.6.4. *Suppose $\bigcup_{i\in I} B(x_i, 30\varrho_i) \subset B(0, R) \subset \mathbb{R}^n$ and μ is a measure on $\mathbb{R}^n$ whose support is contained in $\bigcup_{i\in I} B(x_i, \varrho_i)$. Then there exists a subfamily of indices $\widetilde{I} \subset I$ and radii $R_i > 3\varrho_i$ such that the balls $\{B(x_i, 2R_i)\}_{i\in\widetilde{I}}$ are mutually disjoint, contained in $B(0, R)$ and*

$$\sum_{i\in\widetilde{I}} \mu(B(x_i, R_i)) \ge \frac{1}{2\cdot 13^n}\mu(B(0, R)).$$

Proof. Let $U_k := B_{R\left(1-2^{k+1}\right)} \backslash B_{R\left(1-2^{k}\right)}$. If $x_i \in U_k$, then $3\varrho_i < \frac{1}{10}2^{-k}R =: r_k$ (since $R\left(1-2^{-(k+1)}\right) + 30\varrho_i \leq |x_i| + 30\varrho_i < R$) and if $|k-k'| \geq 2$, $x_i \in U_k$ and $x_j \in U_{k'}$, then $B\left(x_i, r_k\right) \cap B\left(x_j, r_{k'}\right) = \emptyset$. Choose then a maximal family of indices $I_k \subset U_k \cap I$ such that $|x_i - x_j| \geq \frac{1}{3}r_k$. Then

$$\bigcup_{i \in I} B\left(x_i, \varrho_i\right) \subset \bigcup_{k \geq 0} \bigcup_{i \in I_k} B\left(x_i, 2r_k\right).$$

Indeed,

$$\bigcup_{i \in I} B\left(x_i, \varrho_i\right) = \bigcup_{k \geq 0} \bigcup_{\substack{i \in I \\ x_i \in U_k}} B\left(x_i, \varrho_i\right).$$

But

$$V_k := \bigcup_{\substack{i \in I \\ x_i \in U_k}} B\left(x_i, \varrho_i\right) \subset \bigcup_{i \in I_k} B\left(x_i, r_k\right).$$

For, if $x \in B\left(x_i, \varrho_i\right)$ for some $x_i \in U_k$ then either $x_i \in I_k$ (and in such a case there is nothing to show) or $x_i \notin I_k$. But in the latter case $|x_i - x_\ell| < \frac{1}{3}r_k$ in order to not contradict the maximality of I_k. Hence $x_i \in B\left(x_\ell, \frac{2}{3}r_k\right)$. Clearly, either

$$\sum_{k \geq 0} \mu\left(\bigcup_{i \in I_{2k}} B\left(x_i, r_{2k}\right)\right) \geq \frac{1}{2}\mu(B_R) \qquad \text{or} \qquad \sum_{k \geq 0} \mu\left(\bigcup_{i \in I_{2k+1}} B\left(x_i, r_{2k+1}\right)\right) \geq \frac{1}{2}\mu(B_R). \tag{2.6.5}$$

If $i, j \in I_k$ and $|x_i - x_j| \geq 4r_k$, then $B\left(x_i, \frac{1}{3}r_k\right) \subset B\left(x_j, \frac{13}{3}r_k\right)$, which in turn implies that the balls $\left\{B\left(x_i, 2r_k\right)\right\}_{i \in I_k}$ can intersect at most $13^n - 1$ times. Therefore I_k can be split in $N := 13^n$ subsets $I_{k,j}$ such that the balls $B\left(x_i, 2r_k\right)$ are disjoint. Suppose that (2.6.5) holds for even indices (the other case is completely analogous). For every $k \geq 0$, choose a $j(k) \in \{1, \cdots, N\}$ in guise that

$$\mu\left(\bigcup_{i \in I_{2k, j(2k)}} B\left(x_i, r_{2k}\right)\right) \geq \frac{1}{N}\mu\left(\bigcup_{i \in I_{2k}} B\left(x_i, r_{2k}\right)\right).$$

Then, since the families $I_{2k,j}$ are disjoint,

$$\frac{1}{2 \cdot 13^n}\mu\left(B_R(0)\right) \leq \sum_{k \geq 0} \sum_{i \in I_{2k, j(2k)}} \mu\left(B\left(x_i, r_{2k}\right)\right).$$

Define then the family of indices

$$\widetilde{I} := \left\{ i \in I \middle| \exists k \geq 0 \text{ such that } i \in I_{2k, j(2k)} \right\},$$

and the corresponding radii

$$R_i := \max\left\{ r_{2k} \middle| i \in I_{2k, j(2k)} \right\},$$

which are > 0 for $i \in \widetilde{I}$. Then

$$\frac{1}{2 \cdot 13^n}\mu\left(B_R(0)\right) \leq \sum_{k \geq 0} \sum_{i \in I_{2k, j(2k)}} \mu\left(B\left(x_i, r_{2k}\right)\right) = \sum_{i \in \widetilde{I}} \mu\left(B\left(x_i, R_i\right)\right).$$

□

2.7 Stucture of Limit Fields II: The Microrotations

We recall here an elementary geometric construction which is standard in the Vortex-Balls argument (for more details, we refer to [20] and the references therein and also to [6], where the authors applied the balls construction in order to obtain a Γ-convergence result in the study of systems of edge dislocations). First, recall that any two non-disjoint closed balls $B_i = B(p_i, r_i)$, $i = 1, 2$, can be *merged* in another ball, i.e. there exists a ball B whose radius is $\le r_1 + r_2$ and which contains $B_1 \cup B_2$ (for, take $B = B\left(\frac{r_1p_1+r_2p_2}{r_1+r_2}, r_1 + r_2\right)$). Also, recall that any finite family of closed balls $\{B(p_i, r_i)\}_{i\in I}$ can be covered by another family of closed balls $\{B(p'_i, r'_i)\}_{i\in I'}$, $\#I' \le \#I$, which are disjoint and $\sum_{i\in I'} r'_i \le \sum_{i\in I} r_i$. For, let $m = 0$ (which should be thought as a counter) and define $I_0 := I$ and $\mathcal{B}_0 := \{B^0_i\}$, where $B^0_i := B(p_i, r_i)$. Let

$$\mathcal{D}_m := \left\{ i \in I_m \middle| \exists i' \in I_m \setminus \{i\} \ : \ B^m_i \cap B^m_{i'} \neq \emptyset \right\}.$$

Pick an $i \in \mathcal{D}_m$, and set $J_i := \left\{ i' \in I_m \setminus \{i\} \middle| B^m_i \cap B^m_{i'} \neq \emptyset \right\}$. Let B^{m+1}_1 be the balls obtained by merging the ones in $J_i \cup \{i\}$, and let $B^{m+1}_2, \cdots, B^{m+1}_{\#\mathcal{B}_m-\#I_i-1}$ be the remaining ones. Define then

$$\mathcal{B}_{m+1} := \left\{B^{m+1}_i\right\}_{i=1}^{\#\mathcal{B}_m-\#I_i-1} \equiv \left\{B^{m+1}_i\right\}_{i\in I_{m+1}}.$$

Define then $\mathcal{D}_{m+1}$ and relabel m as $m+1$. As long as $\mathcal{D}_m \neq \emptyset$, repeat the procedure. After finitely many steps we end up with a family of closed balls which are disjoint and whose sum of the radii does not exceed the sum of the original ones.

Henceforth, we deal with competitors of minimizing sequences, that is for every $\varepsilon_j \downarrow 0$ and every pair $(A_j, S_j) \in \mathcal{P}(\varepsilon, \alpha, L, \lambda, \tau, \ell)$, we can find a competing sequence $(A'_j, S'_j) \in \mathcal{P}\left(\varepsilon, \alpha, L, \lambda, \tau, \frac{\ell}{2}\right)$, which we denote again (with an abuse of notation) by (A_j, S_j), which has the properties reassumed in Remark 2.5.1. In particular, each field A_j of such a competing sequence is harmonic outside the singular set $B_{\lambda\varepsilon}(S_j)$, and, up to a subsequence, Proposition 2.4.2 ensures $A_j \to A \in BV(\Omega)$ strongly in $L^2(\Omega)$. Associated to this sequence, we define the measures

$$\mu_{1,j} := \frac{1}{\tau\varepsilon_j}\operatorname{dist}^2(A_j, \mathrm{SO}(2))\mathcal{L}^2 \llcorner \Omega, \qquad \mu_{2,j} := \frac{1}{\lambda\varepsilon_j}\mathcal{L}^2 \llcorner B_{\lambda\varepsilon}(S_j), \qquad \mu_j := \mu_{1,j} + \mu_{2,j}.$$

which, up to subsequences, converge weakly in the sense of measures to μ_1, μ_2 and μ respectively. We combine together this property and the foliation Lemma 2.6.3 through a balls construction, in order to obtain the following density estimate.

Theorem 2.7.1 (Pseudolinear 1-density estimate)**.** *Let $(A_j, S_j) \in (\varepsilon_j, \alpha, L, \tau, \lambda, \ell)$ be a sequence of admissible pairs such that $\mathcal{F}_{\varepsilon_j}(A_j, S_j) \le E_{gb}(\varepsilon_j)$, and consider the competing sequence (A'_j, S'_j) as in Remark 2.5.1, which (up to a subsequece) converges stronly in $L^2(\Omega)$ to $A \in BV(\Omega)$. There exist constants $C_0 > 0$, $\delta_1 \in (0,1)$ and $\omega_0 > 0$ such that for every $p \in \Omega$ and every $R > 0$ there exists an $\overline{R} \in [R, 2R]$ such that*

$$\left|\mathrm{Curl}(A)(B(p, \overline{R})\right| \le C\omega\left(\frac{\mu(B(p, 3R))}{R}\right)\mu(B(p, R)), \tag{2.7.1}$$

where $\omega : (0,\infty) \to (0,\infty)$ *is the continuous increasing function defined as*

$$\omega(t) := \begin{cases} \omega_0 & \textit{if } t \geq \delta_1, \\ (-\log(t))^{-\frac{1}{2}} & \textit{if } t < \delta_1. \end{cases} \tag{2.7.2}$$

Proof. We can assume $\mu(B(p,R)) > 0$, otherwise there is nothing to prove. We relabel the competing sequence (A'_j, S'_j) as (A_j, S_j). Let $\delta_1 > 0$ to be chosen later. If $\mu_2\left(B(p,3R)\right) \geq \delta_1 R$, then by Remark 2.5.1 we have

$$\mu(B(p,3R)) \geq C_2\delta_1 \left|\mathrm{Curl}(B(p,R))\right|.$$

If $\mu_2(B(p,3R)) < \delta_1 R$,

$$\limsup_{m\to\infty} \mu_{2,\varepsilon_m}\left(B\left(p,2R\right)\right) \leq \mu_2\left(B\left(p,3R\right)\right) \leq \delta_1 R.$$

Hence, up to a subsequence, which we denote again by ε_m, we have that

$$\left|B_{\lambda\varepsilon_m}\left(S_m\right) \cap B\left(p,2R\right)\right| \leq \lambda\varepsilon_m\delta_1 R, \qquad S_m := S_{\varepsilon_m}.$$

Write $B_{\lambda\varepsilon_m}(S_m) = \bigcup_{j\in J_{H,m}} H_{j,m}$, where $H_{j,m}$ are the (closed) connected components of $B_{\lambda\varepsilon_m}(S_m)$, and consider only those ones which intersect $B(p,\frac{R}{2})$, that is

$$J'_{H,m} := \left\{ j \in J_{H,m} \middle| H_{j,m} \cap B\left(p, \frac{R}{2}\right) \neq \emptyset \right\}.$$

Next, cover these components by disjoint balls $\mathcal{B}_0 := \{B\left(x_{0,i},\varrho_{0,i}\right)\}_{i\in I_0} \equiv \{B_{0,i}\}_{i\in I_0}$ such that $\sum_{i\in I_0} \varrho_{0,i} \leq \sum_{j\in J'_{H,m}} \mathrm{diam}\left(()\, H_{j,m}\right) \leq \frac{\mu}{\lambda\varepsilon_m}\sum_{j\in J'_{H,m}} |H_{j,m}| \leq \mu\left(B(p,3R)\right)$. Now, we let these balls grow. Namely, for any positive measure μ define

$$\overline{\varrho}_\mu(x) := \sup\left\{ \varrho > 0 \middle| \mu\left(B\left(x,2\varrho\right) \setminus B\left(x,\varrho\right)\right) > \delta_0\varrho \right\}.$$

Set $\overline{\varrho}_0 := \overline{\varrho}_{\left|\nabla\chi_{\bigcup\mathcal{B}_0}\right|}$ and $\overline{\varrho}_{i,0} := \overline{\varrho}_0(x_{i,0})$. We can then use Vitali in order to obtain a cover $\left\{B\left(x_{i,0},6\overline{\varrho}_{i,0}\right)\right\}_{i\in I'_0}$ such that the balls $B\left(x_{i,0},2\overline{\varrho}_{0,i}\right)$ are disjoint. Then

$$6\delta_0\sum_{i\in I'_0}\overline{\varrho}_{i,0} = 6\sum_{i\in I'_0}\int_{B\left(x,2\overline{\varrho}_{i,0}\right)\setminus B\left(x,\overline{\varrho}_{i,0}\right)}\left|\nabla\chi_{\bigcup\mathcal{B}_0}\right| \leq 6\int\left|\nabla\chi_{\bigcup\mathcal{B}_0}\right| \leq 6\sum\varrho_{i,0}.$$

Then, we expand again these balls by a factor of 30: that is, we consider $\left\{B\left(x_{i,0},180\overline{\varrho}_{i,0}\right)\right\}_{i\in I'_0}$. By a merging, we get a new family of balls (whose closures are pairwise disjoint) $\mathcal{B}_1 := \{B\left(x_{i,1},\varrho_{i,1}\right)\}_{i\in I_1}$ such that $\sum_{i\in I_1}\varrho_{1,i} \leq C_0\sum_{i\in I'_0}\overline{\varrho}_{0,i}$, where $C_0 := \frac{180}{\delta_0}$, which is in turn smaller than $\frac{1}{2}R$, provided δ_1 was chosen small enough. We can then iterate this procedure in order to construct a family of coverings $\{\mathcal{B}_k\}_{k\geq 0}$, which we can schematize as follows:

$$\begin{aligned} \cdots \xrightarrow{\text{Merge}} \mathcal{B}_k = \{B\left(x_{k,i},\varrho_{k,i}\right)\}_{i\in I_k} \xrightarrow{\text{Expand}} \left\{B\left(x_{k,i},\overline{\varrho}_{k,i}\right)\right\}_{i\in I_k} \xrightarrow{\text{Vitali}} \left\{B\left(x_{k,i},6\overline{\varrho}_{k,i}\right)\right\}_{i\in I'_k} \xrightarrow{30\times} \\ \xrightarrow{30\times} \left\{B\left(x_{k,i},180\overline{\varrho}_{k,i}\right)\right\}_{i\in I'_k} \xrightarrow{\text{Merge}} \mathcal{B}_{k+1} = \{B\left(x_{k+1,i},\varrho_{k+1,i}\right)\}_{i\in I_{k+1}} \xrightarrow{\text{Expand}} \cdots, \end{aligned}$$

where $\overline{\varrho}_{k,i} := \overline{\varrho}_{\left|\nabla \chi_{\bigcup \mathcal{B}_k}\right|}(x_{k,i})$. Notice that $\sum_{i\in I_{k+1}} \varrho_{k+1,i} \leq C_0 \sum_{i\in I_k} \varrho_{k,i}$, i.e. $\sum_{i\in I_k} \varrho_{k,i} \leq C_0^k \sum_{i\in I_0} \varrho_{0,i}$. Moreover, by construction, each of the balls $B\left(x_{k,i}, 180\overline{\varrho}_{k,i}\right)$ is contained in precisely one of the $B\left(x_{k+1,i}, \varrho_{k+1,i}\right)$. That is, we have the inclusions

$$B\left(x_{k+1,i}, \varrho_{k+1,i}\right) \supset \bigcup_{j\in I'_{k,i}} B\left(x_{k,j}, 180\overline{\varrho}_{k,j}\right) \supset \bigcup_{j\in I'_{k,i}} B\left(x_{k,j}, 6\overline{\varrho}_{k,j}\right) \supset \operatorname{spt}\left(\tau_k \llcorner B\left(x_{k+1,i}, \varrho_{k+1,i}\right)\right), \tag{2.7.3}$$

where τ_k is the measure defined by

$$\tau_k := \sum_{i\in I_k} a_{k,j} \mathcal{L}^2 \llcorner B_{k,j}, \text{where } a_{k,j} := \left| \fint_{B_{k,j}} \operatorname{Curl} A_{\varepsilon_m} \mathrm{d}x \right|.$$

By Lemma 2.6.4, for each $i \in I_{k+1}$ we find a subfamily $I''_{k,i} \subset I'_{k,i}$ and radii $R_{k,\nu} > 18\overline{\varrho}_{k,\nu}$ such that

$$\begin{cases} B\left(x_{k,\nu}, 2R_{k,\nu}\right) \subset B\left(x_{k+1,i}, \varrho_{k+1,i}\right) & \forall \nu \in I''_{k,i}, \\ B\left(x_{k,\nu}, 2R_{k,\nu}\right) \cap B\left(x_{k,\nu'}, 2R_{k,\nu'}\right) = \emptyset & \forall \nu \neq \nu', \\ \sum_{\nu\in I''_{k,i}} \tau_k\left(B\left(x_{k,\nu}, R_{k,\nu}\right)\right) \geq \frac{1}{C_2}\tau_k\left(B\left(x_{k+1,i}, \varrho_{k+1,i}\right)\right), & C_2 := 2\cdot(13)^2. \end{cases} \tag{2.7.4}$$

Let

$$K := \max\left\{ k \geq 1 \,\middle|\, \sum_{i\in I_k} \varrho_{k,i} < R \right\} + 1.$$

From the discussion above, we have that for a universal constant $c_0 > 0$ (namely, $c_0 = \log\left(C_0\right)^{-1}$)

$$K \geq c_0 \log\left(\frac{R}{\sum_{i\in I_0} \varrho_{0,i}}\right) \geq c_0 \log\left(\frac{R}{\mu(B(p,3R))}\right).$$

Now that we constructed the family of coverings $\{\mathcal{B}_k\}_{k=0}^K$, we shall discuss how to combine it with Lemma 2.6.2 and Proposition 2.5.1. Firstly, consider a ball $B\left(q,r\right)$ and balls $\{B\left(q_i, r_i\right)\}_{i=1}^N$ which satisfy the conditions of Corollary 2.6.3. Notice that since φ is constant on each ball $B\left(q_i, r_i\right)$, we have $\varphi\left(\bigcup_{i=1}^N B\left(q_i, r_i\right)\right) = \{\varphi_i\}_{i=1}^{L-1}$. Define $\varphi_0 := 0$ and $\varphi_L := 1$, and re-label, if necessary, the φ_i in such a way that $0 = \varphi_0 \leq \varphi_1 < \varphi_2 < \cdots < \varphi_{L-1} \leq \varphi_L = 1$. Using the fact that each connected component of $\partial\{\varphi > h\}$ is a closed, simple Lipschitz curve, and that clearly $\{\varphi_i < \varphi < \varphi_{i+1}\} \cap B_{\lambda\varepsilon}(S_{\varepsilon_m}) = \emptyset$, we have that for each $h \in (\varphi_i, \varphi_{i+1})$

$$\begin{aligned} \int_{\partial\{\varphi>h\}} A_m \cdot t \mathrm{d}\mathcal{H}^1 = \int_{\{\varphi>h\}} \operatorname{Curl}(A_m)\mathrm{d}x = \int_{\{h<\varphi<\varphi_{i+1}\}} \operatorname{Curl}(A_m)\mathrm{d}x + \int_{\{\varphi\geq\varphi_{i+1}\}} \operatorname{Curl}(A_m)\mathrm{d}x = \\ = \int_{\{\varphi\geq\varphi_{i+1}\}} \operatorname{Curl}(A_m)\mathrm{d}x. \end{aligned}$$

Thus, setting $b_i := \int_{\{\varphi=\varphi_i\}} \operatorname{Curl}(A_m)\mathrm{d}x$, we have

$$\int_{\{\varphi\geq\varphi_i\}} \operatorname{Curl}(A_m)\mathrm{d}x = \sum_{j=i}^{L} b_i.$$

Integrate then for $h \in (0,1)$ in order to get

$$\int_0^1 \mathrm{d}h \int_{\partial\{\varphi>h\}} A_m \cdot t \mathrm{d}\mathcal{H}^1 = \sum_{i=0}^{L} \int_{\varphi_i}^{\varphi_{i+1}} \int_{\{\varphi>h\}} \mathrm{Curl}(A_m)\mathrm{d}x = \\ = \sum_{i=0}^{L} (\varphi_{i+1}-\varphi_i) \sum_{j=i}^{L} b_i = \sum_{i=1}^{L} \varphi_i b_i .$$

On the other hand, as a consequence of Lemma (2.5.2) we have that, for $h \notin \{\varphi_i\}_{i=1}^L$,

$$\int_{\partial\{\varphi>h\}} A_m \cdot t \mathrm{d}\mathcal{H}^1 = -\int_{\partial\{\varphi>h\}} \left(\begin{pmatrix} x & x^\perp \\ -x^\perp & x \end{pmatrix} \cdot \begin{pmatrix} \nabla(A_m)_1^1 & \nabla(A_m)_2^2 \end{pmatrix} \right) t(x)\mathrm{d}\mathcal{H}^1 .$$

In particular we see that

$$b_L = -\sum_{i=1}^{L-1} \varphi_i b_i - \int_0^1 \mathrm{d}h \int_{\partial\{\varphi>h\}} \left(\begin{pmatrix} x & x^\perp \\ -x^\perp & x \end{pmatrix} \cdot \begin{pmatrix} \nabla(A_m)_1^1 & \nabla(A_m)_2^2 \end{pmatrix} \right) t(x)\mathrm{d}\mathcal{H}^1 .$$

Then, adding $\sum_{i=1}^{L-1} b_i$ on both sides, we get

$$\sum_{i=1}^{L} b_i = \sum_{i=1}^{L-1} (1-\varphi_i)\, b_i - \int_0^1 \mathrm{d}h \int_{\partial\{\varphi>h\}} \left(\begin{pmatrix} x & x^\perp \\ -x^\perp & x \end{pmatrix} \cdot \begin{pmatrix} \nabla(A_m)_1^1 & \nabla(A_m)_2^2 \end{pmatrix} \right) t(x)\mathrm{d}\mathcal{H}^1 .$$

Passing to the absolute values and using the Fleming-Rishel formula, we obtain

$$\left| \int_{\{\varphi>0\}} \mathrm{Curl}(A_m)\mathrm{d}x \right| \le \sum_{B(q_i,r_i)\subset\{0<\varphi<1\}} \left| \int_{B(q_i,r_i)} \mathrm{Curl}(A_m)\mathrm{d}x \right| + \int_{\{0<\varphi<1\}} |x|\, |\nabla A_{m,\mathrm{sym}}|\, |\nabla\varphi| \,\mathrm{d}x. \tag{2.7.5}$$

We can now apply (2.7.5) to the balls $\widetilde{B}_{k,\nu} := B(x_{k,\nu}, R_{k,\nu})$ obtained in (2.7.4) and the foliation $\varphi_{\nu,i}^{(k)}$ given by Lemma 2.6.3, for every $k \ge 0$ and $\nu \in I''_{k,i}$, $i \in I_{k+1}$ as in the discussion before. This gives

$$\sum_{\nu\in I''_{k,i}} \left| \int_{\left\{\varphi_{\nu,i}^{(k)}>0\right\}} \mathrm{Curl}(A_m)\mathrm{d}x \right| \le \sum_{\nu\in I''_{k,i}} \sum_{B_{k,j}\subset\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}} \left| \int_{B_{k,j}} \mathrm{Curl}(A_m)\mathrm{d}x \right| + \\ + \sum_{\nu\in I''_{k,i}} \int_{\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}} |x|\, |\nabla A_{m,\mathrm{sym}}| \left|\nabla\varphi_{\nu,i}^{(k)}\right| \mathrm{d}x. \tag{2.7.6}$$

to both sides of (2.7.6). Now, for $i \in I_{k+1}$, define the quantities

$$\mathcal{I}_i^{(k)} := \bigcup_{\nu\in I''_{k,i}} \left\{\varphi_{\nu,i}^{(k)} = 1\right\} \text{ (inner balls)},$$

$$\mathcal{A}_i^{(k)} := \bigcup_{\nu\in I''_{k,i}} \left\{0 < \varphi_{\nu,i}^{(k)} < 1\right\} \text{ (annuli)},$$

$$\mathcal{R}_i^{(k)} := \bigcup_{B_{k,j}\subset B_{k+1,i}\setminus\bigcup_{\nu\in I''_{k,i}}\left\{\varphi_{\nu,i}^{(k)}>0\right\}} B_{k,j} \text{ (remaining balls)},$$

$$\mathcal{J}_i^{(k)} := \sum_{\nu\in I''_{k,i}} \int_{\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}} |x|\, |\nabla A_{\varepsilon,\mathrm{sym}}| \left|\nabla\varphi_{\nu,i}^{(k)}\right| \mathrm{d}x.$$

Since the balls $\left\{\widetilde{B}_{k,\nu}\right\}_{\nu\in I''_{k,i}}$ were given by Lemma 2.6.4 we have

$$\tau_k\left(\mathcal{A}_i^{(k)}\right)=\tau_k\left(B_{k+1,i}\right)-\tau_k\left(\mathcal{I}_i^{(k)}\right)-\tau_k\left(\mathcal{R}_i^{(k)}\right)\leq\left(1-\varepsilon_0\right)\tau_k\left(B_{k+1,i}\right)-\tau_k\left(\mathcal{R}_i^{(k)}\right),\tag{2.7.7}$$

where $\varepsilon_0:=C_2^{-1}=\left(2\,(13)^2\right)^{-1}<1$. We then add the term

$$P_{k,i}:=\left|\sum_{B_{k,j}\subset B_{k+1,i}\setminus\bigcup_{\nu\in I''_{k,i}}\left\{\varphi_{\nu,i}^{(k)}>0\right\}}\int_{B_{k,j}}\mathrm{Curl}(A_{\varepsilon_m})\mathrm{d}x\right|$$

to both sides of (2.7.6), which gives, using (2.7.7),

$$\begin{aligned}\left|\int_{B_{k+1,i}}\mathrm{Curl}(A_{\varepsilon_m})\mathrm{d}x\right|&\leq P_{k,i}+\sum_{\nu\in I''_{k,i}}\left|\int_{\left\{\varphi_{\nu,i}^{(k)}>0\right\}}\mathrm{Curl}(A_{\varepsilon_m})\mathrm{d}x\right|\leq\\&\leq\sum_{\nu\in I''_{k,i}}\sum_{B_{k,j}\subset\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}}\left|\int_{B_{k,j}}\mathrm{Curl}(A_{\varepsilon_m})\mathrm{d}x\right|+P_{k,i}+\mathcal{J}_i^{(k)}\leq\\&\leq\tau_k\left(\mathcal{A}_i^{(k)}\right)+P_{k,i}+\mathcal{J}_i^{(k)}\leq\left(1-\varepsilon_0\right)\tau_k\left(B_{k+1,i}\right)+\mathcal{J}_i^{(k)}.\end{aligned}\tag{2.7.8}$$

We then just need to sum up (2.7.8) for $i\in I_{k+1}$ in order to get

$$\tau_{k+1}\left(B\left(p,2R\right)\right)\leq\left(1-\varepsilon_0\right)\tau_k\left(B\left(p,2R\right)\right)+\mathcal{J}^{(k)},\qquad\mathcal{J}^{(k)}:=\sum_{i\in I_{k+1}}\mathcal{J}_i^{(k)}.\tag{2.7.9}$$

Moreover, we notice that if we set $\varphi^{(k)}:=\sum_{i\in I_{k+1}}\sum_{\nu\in I''_{k,i}}\varphi_{\nu,i}^{(k)}\chi_{\left\{\varphi_{\nu,i}^{(k)}>0\right\}}$ and $\mathcal{A}^{(k)}:=\bigcup_{i\in I_{k+1}}\mathcal{A}_i^{(k)}$, then (since $\varphi^{(k)}$ is constant on $\bigcup\mathcal{B}_k$)

$$\begin{aligned}\mathcal{J}^{(k)}&=\int_{\mathcal{A}^{(k)}}|x|\left|\nabla\left(A_{\varepsilon_m,\mathrm{sym}}-\mathrm{id}\right)\right|\left|\nabla\varphi^{(k)}\right|=\int_{\mathcal{A}^{(k)}\setminus\bigcup\mathcal{B}_k}|x|\left|\nabla\left(A_{\varepsilon_m,\mathrm{sym}}-\mathrm{id}\right)\right|\left|\nabla\varphi^{(k)}\right|=\\&=\int_{\bigcup\mathcal{B}_{k+1}\setminus\bigcup\mathcal{B}_k}|x|\left|\nabla\left(A_{\varepsilon_m,\mathrm{sym}}-\mathrm{id}\right)\right|\left|\nabla\varphi^{(k)}\right|.\end{aligned}$$

Recall that, using a Whitney covering, one can prove the existence of a constant $c=c_n>0$ such that for every harmonic function u in an open set $U\subset\mathbb{R}^n$

$$\int_U|\nabla u|^2\,\mathrm{dist}^2(x,\partial U)\mathrm{d}x\leq c_n\int_U|u|^2\,\mathrm{d}x.\tag{2.7.10}$$

Now, we sum (2.7.9) for $k\in\left\{\left[\frac{K}{2}\right]-1,\cdots,K-1\right\}$. Using (2.7.10) and the fact that $\tau_k\left(B\left(p,2R\right)\right)$

is decreasing, we find

$$\begin{aligned}\frac{K}{2}\tau_K\left(B\left(p,2R\right)\right) &\leq \sum_{k=\left[\frac{K}{2}\right]-1}^{K-1}\tau_{k+1}\left(B\left(p,2R\right)\right) \leq \sum_{k=\left[\frac{K}{2}\right]-1}^{K-1}\left(\left(1-\varepsilon_0\right)^k\tau_0+\mathcal{J}^{(k)}\right) \leq \\ &\leq \frac{1}{\varepsilon_0}\left(1-\varepsilon_0\right)^{\frac{K}{3}}\tau_0\left(B\left(p,2R\right)\right)+\sum_{k=1}^{K}\mathcal{J}^{(k)} \leq \\ &\leq \frac{1}{\varepsilon_0}\left(\frac{\mu_{2,\varepsilon_m}\left(B\left(p,2R\right)\right)}{R}\right)^{\beta}\mu_{2,\varepsilon_m}\left(B\left(p,2R\right)\right)+ \\ &\quad+\frac{c_0}{\varepsilon_0}\sqrt{\int_{B(p,2R)}\left|A_{\varepsilon_m,\mathrm{sym}}-\mathrm{id}\right|^2}\sqrt{\sum_{k=0}^{K-1}n_k},\end{aligned} \tag{2.7.11}$$

where $n_k := \sum_{i\in I_{k+1}}\sum_{\nu\in I''_{k,i}}\left(\left(\sum_{B_j^{(k)}\subset\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}}1\right)+1\right)$ is the total number of balls contained in the union of the annuli $\left\{0<\varphi_{\nu,i}^{(k)}<1\right\}$, which is decreasing by construction, i.e. $n_k\leq N_0$, N_0 being the number of connected components of $B_{\lambda\varepsilon_m}\left(S_{\varepsilon_m}\right)$ inside $B\left(p,3R\right)$. Notice that $n_k\leq N_0$ and

$$N_0\leq C\frac{\left|B_{\lambda\varepsilon_m}(S_{\varepsilon_m})\cap B\left(p,2R\right)\right|}{\left(\lambda\varepsilon_m\right)^2}=C\frac{\mu_{2,\varepsilon_m}\left(B(p,2R)\right)}{\lambda\varepsilon_m}.$$

In particular, if we divide (2.7.11) by $\sqrt{K}$, we obtain

$$\begin{aligned}\tau_K\left(B\left(p,2R\right)\right)\sqrt{\left|\log\left(\frac{\mu_{\varepsilon_m}(B\left(p,2R\right))}{R}\right)\right|} &\leq C_\lambda\frac{\mu_{\varepsilon_m}\left(B\left(p,2R\right)\right)\left(\frac{\mu_{\varepsilon_m}(B(p,2R))}{R}\right)^{\beta}}{\sqrt{\log\left(\frac{\mu_{\varepsilon_m}(B(p,2R))}{R}\right)}}+ \\ &\quad+\mu_{\varepsilon_m}\left(B\left(p,2R\right)\right)\leq \\ &\leq C_\lambda\mu_m\left(B\left(p,2R\right)\right),\end{aligned} \tag{2.7.12}$$

where $\mu_m := \mu_{1,\varepsilon_m}+\mu_{2,\varepsilon_m}$. Now, since $\left|B_{\lambda\varepsilon_m}(S_m)\cap B\left(p,3R\right)\right|\leq\lambda\varepsilon_m\delta_1 R$, we can find an $R(m)\in[R,2R]$ such that

$$\tau_K\left(B(p,2R)\right)\geq\left|\int_{B(p,R(m))}\mathrm{Curl}\,A_m\mathrm{d}x\right|.$$

Up to a subsequence, we can always assume that $R(m)\to\overline{R}\in[R,2R]$. Moreover, since $\{\mathrm{Curl}\,A_{\varepsilon_m}\}$ quasi-converges to $(\mathrm{Curl}\,A,\xi)$, with $\xi(\Omega)<\infty$, we can also assume $\xi\left(\partial B(p,\overline{R})\right)=0$ (up to increasing the constant C_λ in the right hand side of (2.7.12) by a factor of 2). In particular, we have

$$\limsup_{m\to\infty}\left|\int_{B(p,R(m))}\mathrm{Curl}\,A_m\mathrm{d}x\right|\geq\left|\int_{B(p,\overline{R})}\mathrm{d}\,\mathrm{Curl}\,A\right|.$$

Taking the limit superior as $m\to\infty$ in (2.7.12), we find

$$\left|\int_{B(p,\overline{R})}\mathrm{d}\,\mathrm{Curl}(A)\right|\sqrt{\left|\log\left(\frac{\mu(B\left(p,3R\right))}{R}\right)\right|}\leq C_0\mu\left(B(p,3R)\right). \tag{2.7.13}$$

In particular, we can choose ω as in (2.7.2) and obtain (2.7.1). $\square$

Theorem 2.7.1 is giving an estimate of the norm of Curl(A) on balls, while in order to obtain informations about the derivative DA we would need (by virtue of Proposition A.3.1) an upper bound on the total variation of Curl(A). The key observation in order to prove such an estimate is that, by the definition of supremum limit, we are allowed to take a covering with balls of the same radii.

Lemma 2.7.2. *Let T be a vector valued Radon measure and μ be a positive finite Radon measure, both defined on $\mathbb{R}^n$. Suppose that there exists a constant $C_0 > 1$ such that for every $x \in \Omega$ and every $R > 0$*

$$|T\left(B(x,R)\right)| \leq \omega\left(\frac{\mu\left(B(x,C_0R)\right)}{R^\beta}\right)\mu\left(B(x,C_0R)\right), \tag{2.7.14}$$

where $\beta \in \{1, \cdots, n-1\}$ and $\omega : (0,\infty) \to (0,\infty)$ is an increasing function such that $\omega(\delta) \to 0$ as $\delta \to 0$. Then

(a) $|T|\left(\Omega \setminus S\right) = 0$, where

$$S := \left\{x \in \Omega \middle| \Theta^*(x) > 0\right\}, \qquad \Theta^*(x) := \Theta^*_\beta(\mu, x) := \limsup_{R\downarrow 0} \frac{\mu\left(B(x,R)\right)}{R^\beta}.$$

(b) $\mathcal{H}^\beta \llcorner S$ is σ-finite.

(c) $|T| \leq C_n\left(\omega \circ \Theta^*\right)\mu \llcorner S$, where $C_n > 0$ is a constant depending only on the dimension.

In particular, if $T = DA$ for some $A \in BV(\Omega)^n$, then $DA = D^JA = |A^+ - A^-| \otimes \nu_A\mathcal{H}^{n-1}\llcorner S_A$ and

$$g^{-1}\left(\left|A^+ - A^-\right|\right)\mathcal{H}^{n-1}\llcorner S_A \leq C\mu, \tag{2.7.15}$$

where $g(t) := t\omega(t)$.

Proof. From the definition of limit superior,

$$G_s := \left\{x \in \Omega \middle| \Theta^*(x) \leq s\right\} \subset \bigcap_{s>0}\bigcup_{R>0} G_{s,R,\delta},$$

where

$$G_{s,R,\delta} := \left\{x \in \Omega \middle| \frac{\mu\left(B(x,C_0\varrho)\right)}{\varrho^\beta} < s + \delta \quad \forall \varrho < R\right\}.$$

For any μ-measurable set E, consider the r-tubular neighborhood $U_r = B_r\left(E \cap G_{s,R,\delta}\right)$. If $\varrho < \min\left\{R, \frac{r}{C_0}\right\}$, then we can find $K = K(n)$ (depending only on the dimension n) disjoint families of balls balls $\mathcal{B}_k := \left\{B_i^{(k)}\right\}_{i\in I_k} \equiv \left\{B\left(x_i^{(k)}, \varrho\right)\right\}_{i\in I_k}$, $k = 1, \cdots, K$ whose union covers $E_{s,R,\delta} := E \cap G_{s,R,\delta}$, that is

$$E_{s,R,\delta} \subset \bigcup_{k=1}^K \bigcup_{i\in I_k} B_i^{(k)}, \qquad B_i^{(k)} \cap B_j^{(k)} = \emptyset \forall i \neq j.$$

Moreover, the choice of ϱ ensures

$$\frac{\mu\left(B(x_i^{(k)}, C_0\varrho)\right)}{\varrho^\beta} < s+\delta, \quad \forall i \in I_k, \quad k \in \{1, \cdots K\}$$

and

$$C_0 B_i^{(k)} \subset U_r.$$

Let $f := \frac{\mathrm{d}T}{\mathrm{d}|T|}$, $|f| = 1$ $|T|$-a.e., and $\varphi \in \mathcal{C}_c(\Omega)$. Then, using (2.7.14),

$$\begin{aligned}
|T|\left(E_{s,R,\delta}\right) &\leq \sum_{k=1}^{K}\sum_{i\in I_k}\left|\int_{B_i^{(k)}}\langle f, \mathrm{d}T\rangle\right| \leq \\
&\leq \sum_{k=1}^{K}\sum_{i\in I_k}\left\{\int_{B_i^{(k)}}|f-\varphi|\,\mathrm{d}\,|T| + \int_{B_i^{(k)}}\left|\varphi(x)-\varphi\left(x_i^{(k)}\right)\right|\mathrm{d}\,|T|\,(x) + \right. \\
&\qquad\qquad \left. + \left|\left\langle\varphi\left(x_i^{(k)}\right), T\left(B_i^{(k)}\right)\right\rangle\right|\right\} \leq \\
&\leq K\left\{\int_{U_r}|f-\varphi|\,\mathrm{d}\,|T| + \left(\sup_{|x-y|<\varrho}|\varphi(x)-\varphi(y)|\right)|T|\,(U_r)\right\} + \\
&\quad + C_n\,||\varphi||_\infty\,\omega\,(\delta+s)\,\mu(U_r).
\end{aligned}$$

Define

$$(I) := \int_{U_r}|f-\varphi|\,\mathrm{d}\,|T|\,, \qquad (II) := \left(\sup_{|x-y|<\varrho}|\varphi(x)-\varphi(y)|\right)|T|\,(U_r)\,.$$

As $\varrho \to 0$, we see that $(II) \to 0$, while if we consider a sequence of functions φ converging to f, also $(I) \to 0$. As $G_{s,R,\delta}$ is increasing in R, taking $R \to \infty$ we can replace $E_{s,R,\delta}$ on the left hand side with the union $E_{s,\delta} := \bigcup_{R>0} E_{s,R,\delta}$. Since this holds for every $\delta > 0$, we can let $\delta \to 0$ and recover $E_s = E \cap \{\Theta^* > s\}$ on the left hand side. Finally, taking $r \to 0$ and using the fact that μ is a Radon measure, we find that for every μ-measurable set E

$$|T|\,(E \cap G_s) \leq \omega(s)\mu\,(E \cap G_s)\,. \tag{2.7.16}$$

Since $\omega(s) \to 0$ as $s \to 0$, we have that

$$|T|\,(\Omega \setminus S) = 0,$$

i.e. (a). Set $S_\delta := \left\{x \middle| \Theta^*(x) > \delta\right\}$. Then clearly $\mathcal{H}^\beta(S_\delta) \leq C_n\frac{1}{\delta}\mu(\mathbb{R}^n) < \infty$. In particular, $\mathcal{H}^\beta \llcorner S$ is σ-finite, thus (b) is proven.

Now, for every $\zeta > 0$, we can find a compact set $H = H(\zeta)$ such that $\Theta^*|_H$ is continuous and $|T|\,(\Omega \setminus H) < \zeta$. For $\eta > 0$, let $\Phi^*(\eta) > 0$ be such that

$$x, y \in H, |x-y| \leq \Phi^*(\eta) \Longrightarrow |\Theta^*(x) - \Theta^*(y)| \leq \eta.$$

Consider a sequence $\{a_i\}_{i\geq 1}$ such that $(0,\infty) = \bigcup_{i\geq 1}(a_i, a_{i+1}]$ and $|a_{i+1} - a_i| < \Phi^*(\eta)$. For any Borel set F, let

$$F_i := F \cap \left\{x \middle| \Theta^*(x) \in (a_i, a_{i+1}]\right\}.$$

Let $\mu_0 := \mu \llcorner S$. Using (2.7.16) with $E = F \cap \left\{ x \middle| \Theta^*(x) > a_i \right\}$,

$$
\begin{aligned}
|T|(F) &\le \zeta + |T|(F \cap H) \le \zeta + \sum_{i \ge 1} |T|(F_i \cap H) \le \\
&\le \zeta + \sum_{i\ge 1} |T| \left(F \cap H \cap \left\{ x \middle| \Theta^*(x) > a_i \right\} \cap G_{a_{i+1}} \right) \le \\
&\le \zeta + C_n \sum_{i \ge 1} \omega\left(a_{i+1}\right) \mu_0(F_i \cap H) = \zeta + C_n \sum_{i\ge 1} \int_{F_i \cap H} \omega\left(a_{i+1}\right) \mathrm{d}\mu_0 \le \\
&\le \zeta + C_n \sum_{i\ge 1} \int_{F_i \cap H} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu_0 + C_n \sum_{i \ge 1} \int_{F_i \cap H} \left|\omega\left(\Theta^*(x)\right) - \omega\left(a_{i+1}\right)\right| \mathrm{d}\mu_0 \le \\
&\le \zeta + C_n \sum_{i\ge 1} \int_{F_i \cap H} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu_0 + C_n \eta \mu_0(F \cap H) = \\
&= \zeta + C_n \int_{F \cap H} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu_0 + C_n \eta \mu_0(F \cap H).
\end{aligned}
$$

By the arbitrariness of ζ, η and the Borel set F, we infer that

$$|T| \le C_n \left(\omega \circ \Theta^*\right) \mu \llcorner S,$$

i.e. (c). Now, suppose $T = DA$ for some $A \in BV(\Omega)^m$. Then from (c), we see that $DA = |A^+(x) - A^-(x)| \otimes \nu_A \mathcal{H}^\beta \llcorner (S \cap S_A)$, where $\beta := n-1$. Our first claim is that

$$\left|A^+(x) - A^-(x)\right| \le C \Theta^*(x) \omega\left(\Theta^*(x)\right) \qquad \text{for } \mathcal{H}^\beta - \text{a.e. } x \in S \cap S_A.$$

Let $E \subset \mathbb{R}^n$ be a Borel set. For any $\zeta > 0$, we can find $H = H(\zeta)$ compact such that $\Theta^*|_H$ is continuous and $\mu\left(\mathbb{R}^n \setminus H\right) \le \zeta$. Since S is rectifiable, we can assume without loss of generality that the β-density of each $x \in S \cap H \cap E$ is 1, namely

$$\lim_{\varrho \downarrow 0} \frac{\mathcal{H}^\beta\left(S \cap H \cap E \cap B\left(x, \varrho\right)\right)}{c_\beta \varrho^\beta} = 1,$$

where $c_\beta > 0$ is a constant dependent only on $\beta > 0$. From this and the definition of limit superior, for every $\eta > 0$, $k \in \mathbb{N}$ and $x \in E \cap S_\xi \cap H =: G_\xi$, $\xi > 0$, we can find a radius $\varrho_k(x) \le k^{-1}$ such that, for a constant $C = C(\beta) > 0$,

$$
\begin{cases}
C\left(1-\eta\right) \varrho_k(x)^\beta \le \mathcal{H}^\beta\left(G_\xi \cap \overline{B\left(x, \varrho_k(x)\right)}\right) \le C(1+\eta) \varrho_k(x)^\beta, \\
\Theta^*(x) \ge \frac{\mu(B(x, \varrho_k(x)))}{\varrho_k(x)^\beta} - \eta.
\end{cases}
\tag{2.7.17}
$$

We then consider, for $N > 1$, the fine cover of G_ξ

$$\mathcal{F}_N := \left\{ \overline{B\left(x, \varrho_k(x)\right)} \middle| x \in G_\xi, \quad k \ge N \right\}.$$

from which, by Vitali-Besicovitch Theorem, we can extract a disjoint family $\mathcal{F}'_N = \left\{B(x_i, \varrho_i)\right\}_{i \ge 1}$ such that

$$\mu\left(G_\xi \setminus \bigcup \mathcal{F}'_N\right) = 0.$$

Then

$$
\begin{aligned}
\int_{E\cap S_\xi} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu(x) &\le C\zeta + \int_G \omega\left(\Theta^*(x)\right) \mathrm{d}\mu(x) = \\
&= C\zeta + \sum_i \int_{\overline{B(x_i,\varrho_i)}\cap G_\xi} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu(x) \le \\
&\le C\zeta + \sum_i \omega\left(\Theta^*(x_i)\right) \mu\left(\overline{B\left(x_i,\varrho_i\right)} \cap G\right) + \\
&\quad + \left(\sup_{\substack{x,y\in G \\ |x-y|\le N^{-1}}} \left|\omega\left(\Theta^*(x)\right) - \omega\left(\Theta^*(y)\right)\right| \right) \mu(G) \le \\
&\le C\zeta + \sum_i \omega\left(\Theta^*(x_i)\right) \varrho_i^\beta \left(\eta + \Theta^*(x_i)\right) + o_N(1) \le \\
&\le C\zeta + \sum_i \Theta^*(x_i)\omega\left(\Theta^*(x_i)\right) \varrho_i^\beta + \eta \sum_i \omega\left(\Theta^*(x_i)\right) \varrho_i^\beta + o_N(1).
\end{aligned}
$$

Using (2.7.17), we find (setting $g(s) := s\omega(s)$ and $\widetilde{g} := g \circ \Theta^*$),

$$
\begin{aligned}
\sum_i \widetilde{g}(x_i)\varrho_i^\beta &\le \frac{C}{1-\eta} \sum_i \widetilde{g}(x_i)\mathcal{H}^\beta(G_\xi \cap \overline{B\left(x_i,\varrho_i\right)}) \le \\
&\le \int_{S\cap E} \widetilde{g}(y)\mathrm{d}\mathcal{H}^\beta(y) + \left(\sup_{\substack{x,y\in G, \\ |x-y|\le N^{-1}}} \left|\widetilde{g}(x) - \widetilde{g}(y)\right| \right) \mathcal{H}^\beta\left(S_\xi\right) \le \\
&\le \int_{S\cap E} \widetilde{g}(y)\mathrm{d}\mathcal{H}^\beta(y) + o_N(1)\frac{\mu(\mathbb{R}^n)}{\xi}.
\end{aligned}
$$

and, since ω is bounded,

$$
\eta \sum_i \omega\left(\Theta^*(x_i)\right) \varrho^\beta \le C\eta \left|\left|\omega\right|\right|_\infty \frac{\mathcal{H}^\beta(S_\xi)}{1-\eta}.
$$

That is,

$$
\int_{E\cap S_\xi} \omega\left(\Theta^*(x)\right) \mathrm{d}\mu(x) \le C\zeta + o_N(1)\frac{1}{\xi} + C\eta\frac{||\omega_\infty||}{\xi(1-\eta)} + \int_{S\cap E} \widetilde{g}(x)\mathrm{d}\mathcal{H}^\beta(x). \tag{2.7.18}
$$

Then, in (2.7.18) we first let $N\to\infty$, then $\zeta\to 0$ and $\eta\to 0$. By the arbitrariness of $\xi > 0$ and the set E, we finally get

$$
\left(\omega\circ\Theta^*\right)\mu\llcorner \left\{x\in S \middle| \Theta_1(S,x) = 1\right\} \le \widetilde{g}\mathcal{H}^\beta\llcorner \left\{x\in S \middle| \Theta_1(S,x) = 1\right\}.
$$

That is, since S is rectifiable,

$$
\left|A^+(x) - A^-(x)\right| \le C\Theta^*(x)\omega\left(\Theta^*(x)\right), \qquad \text{for } \mathcal{H}^\beta - \text{a.e. } x\in S. \tag{2.7.19}
$$

We rewrite (2.7.19) as

$$
f\left(\left|A^+(x) - A^-(x)\right|\right) \le C\Theta^*(x), \qquad \text{for } \mathcal{H}^\beta - \text{a.e. } x\in S. \tag{2.7.20}
$$

where $f := g^{-1}$. We proceed now with the proof of the second step. Let $E \subset \mathbb{R}^n$ Borel and $\xi > 0$. We re-define G_ξ as

$$G_\xi := E \cap \left\{ x \in S \middle| \Theta_1(S,x) = 1 \text{ and } \Theta^*(x) > \xi \right\}.$$

For every $\eta > 0$ and $k \in \mathbb{N}$, we can find a $\varrho_k(x) \le k^{-1}$ such that

$$\begin{cases} C\,(1-\eta)\,\varrho_k(x)^\beta \le \mathcal{H}^\beta\left(G_\xi \cap \overline{B\,(x,\varrho_k(x))}\right) \le C(1+\eta)\varrho_k(x)^\beta, \\ \Theta^*(x) \le \frac{\mu(B(x,\varrho_k(x)))}{\varrho_k(x)^\beta} + \eta, \\ A^+(y) - A^-(y) = A^+(x) - A^-(x), \qquad \forall y \in \overline{B\,(x,\varrho_k(x))} \cap G_\xi, \\ f\left(|A^+(x) - A^-(x)|\right) \le C\Theta^*(x) \qquad \forall x \in G_\xi. \end{cases} \tag{2.7.21}$$

As before, for $N > 1$, we define the fine cover

$$\mathcal{F}_N := \left\{ \overline{B\,(x,\varrho_k(x))} \middle| x \in G_\xi, \quad k \ge N \right\},$$

from which we extract a disjoint family $\mathcal{F}'_N = \{B\,(x_i,\varrho_i)\}_{i\ge1}$ such that

$$\mathcal{H}^\beta\left(G_\xi \setminus \bigcup \mathcal{F}'_N\right) = 0.$$

We have

$$\begin{aligned} \int_{E\cap S_\xi} f\left(\left|A^+(x) - A^-(x)\right|\right) \mathrm{d}\mathcal{H}^\beta &= \int_{G_\xi} f\left(\left|A^+(x) - A^-(x)\right|\right) \mathrm{d}\mathcal{H}^\beta = \\ &= \sum_i \int_{G_\xi\cap\overline{B(x_i,\varrho_i)}} f\left(\left|A^+(x) - A^-(x)\right|\right) \mathrm{d}\mathcal{H}^\beta = \\ &= \sum_i f\left(\left|A^+(x_i) - A^-(x_i)\right|\right) \mathcal{H}^\beta(G_\xi \cap \overline{B\,(x_i,\varrho_i)}) \le \\ &\le C\,(1+\eta) \sum_i \Theta^*(x_i)\varrho_i^\beta \le \\ &\le C\,(1+\eta) \sum_i \left(\frac{\mu\,(B(x,\varrho_i))}{\varrho_i^\beta} + \eta \right) \varrho_i^\beta \le \\ &\le C\,(1+\eta)\,\mu\left(B_{\frac{1}{N}}\,(G_\xi)\right) + C\eta\frac{1+\eta}{1-\eta}\mathcal{H}^\beta(S_\xi). \end{aligned}$$

As $N \to \infty$ and $\eta \to 0$, by the arbitrariness of $\xi > 0$ and E we have

$$f\left(\left|A^+ - A^-\right|\right) \mathcal{H}^\beta \llcorner S \le C\mu. \qquad \square$$

From Theorem 2.7.1, Lemma 2.7.2 and Proposition A.3.1 we immediately infer the following

Corollary 2.7.3. *There exists a constant $C > 0$ such that for every sequence of pairs $(A_j, S_j) \in \mathcal{P}\,(\varepsilon_j, \alpha, L, \tau, \lambda, \ell)$, $\varepsilon_j \to 0$ with $\mathcal{F}_{\varepsilon_j}(A_j, S_j) \le E_{gb}(\varepsilon_j)$, there exists another sequence $(A'_j, S'_j) \in$*

$\mathcal{P}\left(\varepsilon_j, \alpha, L, \tau, \lambda, \frac{\ell}{2}\right)$ *such that* $\mathcal{F}_{\varepsilon_j}(A'_j, S'_j) \leq C\mathcal{F}_{\varepsilon_j}(A'_j, S'_j)$ *which, up to a subsequence, converges strongly in* $L^2(\Omega)$ *to a microrotation* A *and*

$$\left|A^+ - A^-\right| \sqrt{|\log(|A^+ - A^-|)|}\mathcal{H}^1 \llcorner S_A \leq C\mu,$$

where μ *is the weak** *limit of the measures*

$$\mu_j := \frac{1}{\tau\varepsilon_j} \operatorname{dist}^2(A'_j, \mathrm{SO}(2))\mathcal{L}^2 \llcorner \Omega + \frac{1}{\lambda\varepsilon_j}\mathcal{L}^2 \llcorner S'_j.$$

In particular,

$$C\alpha L\varepsilon_j\sqrt{|\log(\alpha)|} \leq \mathcal{F}_{\varepsilon_j}(A_j, S_j).$$

Appendix A

Appendix

A.1 A short review of Calderón-Zygmund Operators

We recall that a *standard kernel* is a function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{D}_n \to \mathbb{R}$, where

$$\mathcal{D}_n := \left\{ (x, x) \in \mathbb{R}^n \times \mathbb{R}^n \middle| x \in \mathbb{R}^n \right\}$$

is the diagonal, satisfying, for some constant $C > 0$, the following:

(i) $|K(x, y)| \leq C \left|x - y\right|^{-n}$, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{D}_n$;

(ii) $|K(x, y_1) - K(x, y_2)| \leq C \frac{|y_1 - y_2|^\alpha}{|x - y_1|^{|n+\alpha|}}$ for every x, y_1, y_2 satisfying $|y_1 - y_2| \leq \frac{1}{2} \left|x - y_1\right|$;

(iii) $|K(x_1, y) - K(x_2, y)| \leq C \frac{|x_1 - x_2|^\alpha}{|x_1 - y|^{n+\alpha}}$ for every x_1, x_2, y satisfying $\left| x_) 1 - x_2 \right| \leq \frac{1}{2} \left|x_1 - y\right|$,

for some $\alpha \in (0, 1]$. It can be shown that conditions (ii) and (iii) are implied by the easier (but weaker) conditions

(ii') $|\nabla_x K(x, y)| \leq C \left|x - y\right|^{-n-1}$;

(iii') $|\nabla_y K(x, y)| \leq C \left|x - y\right|^{-n-1}$.

A *Calderón-Zygmund* operator (of the "second generation") is a bounded linear operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, i.e. for some $C = C(T)$

$$||Tf||_{L^2} \leq C \left|\left|f\right|\right|_{L^2},$$

and such that there exists a singular kernel K satisfying

$$Tf(y) = \int_{\mathbb{R}^n} K(x, y) f(x) \mathrm{d}x,$$

for every $f \in \mathbb{L}^2(\mathbb{R}^n)$ with compact support, and for every $y \in \mathbb{R}^n \setminus \operatorname{spt}(f)$. We give now an example of such an operator particularly relevant for the analysis in section 1.4.

Example A.1.1. Let $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{D}_n \to \mathbb{R}$ be a function satisfying, for some $C > 0$,

1. $|K(x, y)| \leq C\,|y|^{-n}$;
2. $|\nabla_x K(x, y)| \leq C\,|y|^{-n-1}$ and $|\nabla_y K(x, y)| \leq C\,|y|^{-n-1}$;
3. $K(x, \lambda y) = \lambda^{-n} K(x, y)$ for every x, y and $\lambda > 0$;
4. $$\int_{\mathbb{S}^{n-1}} K(x, y) \mathrm{d}\mathcal{H}^{n-1} = 0, \qquad \forall x \in \mathbb{R}^n.$$

Then it is possible to show that the operator

$$T : f \in L^2(\mathbb{R}^n) \longmapsto \int K(y, x - y) f(y) \mathrm{d}y \in L^2(\mathbb{R}^n),$$

is a Calderón-Zygmund operator.

Calderón-Zygmund operator behave particularly well in L^p spaces, $1 < p < \infty$, since in such a case they are bounded operators (with operator norm depending on p). For $p = 1$, they are bounded only weakly, that is the exists a constant such that

$$||Tf||_{L^{1,\infty}(\mathbb{R}^n)} \leq C\,||f||_{L^1(\mathbb{R}^n)}.$$

Such operators also "almost" bounded in L^∞, in the sense that they map L^∞ into BMO, the space of functions of *bounded mean oscillation*, which can often be used as a replacement for L^∞ (as we did in the proof of Theorem 1.4.9):

Definition A.1.1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ has *bounded mean oscillation* ($f \in BMO(\mathbb{R}^n)$) provided

$$||f||_{BMO} := \sup_Q \fint_Q \left| f - \fint_Q f \mathrm{d}x \right| \mathrm{d}x < \infty,$$

where the supremum is taken over all the cubes Q in $\mathbb{R}^n$.

Notice that $||\cdot||_{BMO}$ is a norm only on functions defined modulo constants. We also recall the fundamental *John-Nirenberg* inequality ([**?**]), which shows that any BMO function is exponentially integrable:

Theorem A.1.1. *There exist constant $C_1, C_2 > 0$, depending only on the dimension n, such that for every $f \in BMO(\mathbb{R}^n)$, any cube $Q \subset \mathbb{R}^n$ and $\lambda > 0$ one has*

$$\left| \left\{ x \in Q \middle| \left| f(x) - \fint_Q f(y) \mathrm{d}y \right| > \lambda \right\} \right| \leq C_1\,|Q| \exp\left(-C_2 \frac{\lambda}{||f||_{BMO}} \right).$$

It is possible to show that Calderón-Zygmund operators are bounded from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, i.e. there exists a constant $C = C(T)$ such that for every $f \in L^\infty(\mathbb{R}^n)$

$$||Tf||_{BMO} \leq C\,||f||_\infty.$$

A.2 A short review of BV functions

We start with the definition of *countably rectifiable set.* We say that a subset $M \subset \mathbb{R}^N$ is *countably n-rectifiable* if there exist (at most) countably many n-dimensional embedded submanifolds $\{M_i\}_{i\geq 1}$ of class $\mathcal{C}^1$ and $M_0 \subset \mathbb{R}^N$ with $\mathcal{H}^n(M_0) = 0$ such that

$$M \subset M_0 \cup \bigcup_{i\geq 1} M_i,$$

i.e. if M is contained, up to a $\mathcal{H}^n$-negligible set, in the countable union of submanifolds. If, moreover, the set is $\mathcal{H}^n$-measurable and $\mathcal{H}^n(M) < \infty$, we say that M is a *rectifible set.* When M is n-countably rectifiable and also $\mathcal{H}^n$-measurable, then it is possible to assume that at each point $x \in M_i$

$$\Theta^n(M_i, x) := \lim_{r\to 0} \frac{\mathcal{H}^n(M_i \cap B(x,r))}{\omega_n r^n} = 1,$$

where ω_n is the Lebesgue measure of the unit ball of $\mathbb{R}^n$, and

$$\Theta^n(M \setminus M_i, x) = 0.$$

A locally integrable function defined on an open set $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, is said to belong to $BV(\Omega)$ if its distributional gradient is a vector valued Radon measure, that is if there exist measures $D_i u$ such that

$$\int_\Omega \varphi \cdot \mathrm{d}Du = -\int_\Omega u \mathrm{div}\varphi \mathrm{d}x,$$

for all $\varphi \in \mathcal{C}^1_c(\Omega)^n$, where $Du = (D_1 u, \cdots, D_n u)$. We recall that the following structure theorem holds:

Theorem A.2.1. *Let $u \in BV(\Omega)$. Then the gradient Du can be splitted as*

$$Du = \nabla u \mathrm{d}x + (u_+ - u_-)\, n(x, J_u)\mathcal{H}^{n-1} \llcorner J_u + D^C u \equiv \widetilde{D}u + D^J u,$$

where

$$\nabla u \textit{ is the approximate differential of } u, i.e. \nabla u(x) = \frac{\mathrm{d}Du}{\mathrm{d}\mathcal{L}^n},$$

$$u_-(x) := \mathrm{apliminf}_{y\to x}\, u(y) := \sup\left\{t \middle| \Theta^n(E_t(u), x) = 1\right\}, \quad E_t(u) := \left\{x \in \Omega \middle| u(x) > t\right\},$$

$$u_+(x) := \mathrm{aplimsup}_{y\to x}\, u(y) := \inf\left\{t \middle| \Theta^n(E_t(u), x) = 0\right\},$$

$$S_u := \textit{singular set of } u := \left\{x \in \Omega \middle| u_-(x) < u_+(x)\right\},$$

$$n(x, J_u) := \textit{inner normal to } S_u := \lim_{r\to 0} \frac{D\chi_{S_u}(B(x,r))}{|D\chi_{S_u}|(B(x,r))}.$$

$D^C u$ is called the Cantor part *of the derivative, $D^J u$ is the* jump part*, $\widetilde{D}u := \nabla u \mathcal{L}^n + D^C u$ is the* diffuse part *and $D^a u := \nabla u \mathcal{L}^n$ is the* absolutely continuous part*. $D^a u$, $D^J u$ and $D^C u$ are mutually orthogonal. Moreover, the set S_u is $\mathcal{H}^{n-1}$-measurable and countably $n-1$-rectifiable.*

For BV functions, the following chain rule holds:

Theorem A.2.2. *Let $u \in BV(\Omega)^m$ and $f \in \mathcal{C}^1(\mathbb{R}^m)^p$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. Then $v := f \circ u$ belongs to $BV(\Omega)^p$*

$$\begin{cases} \widetilde{D}v & = \nabla f(u)\nabla u\mathcal{L}^n + \nabla f(\widetilde{u})D^c u = \nabla f(\widetilde{u})\widetilde{D}u, \\ D^J v & = (f(u^+) - f(u^-)) \otimes n(u, S_u) \llcorner S_u, \end{cases}$$

where $\widetilde{u}(x)$ is the approximate limit of u at x, i.e. the unique vector $z \in \mathbb{R}^m$ (which exists for any $x \in \Omega \setminus S_u$) such that

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - z| \,\mathrm{d}y = 0.$$

A.3 Curl Bounds Gradient on $SO(n)$

Proposition A.3.1. *There exists a dimensional constant $C = C(n) > 0$ such that for every $\Omega \subset \mathbb{R}^n$ open*

$$|\mathrm{Curl}(A)| \geq C\,|DA|\,, \qquad \forall A \in BV\left(\Omega, \mathrm{SO}(n)\right),$$

where

$$BV\left(\Omega, \mathrm{SO}(n)\right) := \left\{ A \in BV\left(\Omega, \mathbb{R}^{n\times n}\right) \middle| A(x) \in \mathrm{SO}(n) \text{ for } \mathcal{L}^n - a.e.\ x \in \Omega \right\}.$$

Here, $\mathrm{Curl}(A)$ is the vector $\mathrm{Curl}(A) = \left(\mathrm{Curl}(A^1), \cdots, \mathrm{Curl}(A^n)\right)$ whose components are the measure-valued 2-forms

$$\mathrm{Curl}(A^i) = \left(D_j A^i_k - D_k A^i_j\right) \mathrm{d}x^j \wedge \mathrm{d}x^k \equiv \mathrm{Curl}(A^i)_{jk} \mathrm{d}x^j \wedge \mathrm{d}x^k \equiv \Gamma^i_{jk} \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

and

$$|\mathrm{Curl}(A)| := \frac{1}{2} \sum_{i,j,k=1}^{n} \left| D_j A^i_k - D_k A^i_j \right|.$$

Proof. We can assume $A(x) \in \mathrm{SO}(n)$ for every $x \in \Omega$. We discuss separately the diffuse part $\widetilde{D}A = \nabla A\mathcal{L}^n + D^c A$ and the jump part $D^J A = (A^+ - A^-) \otimes \nu_A \mathcal{H}^{n-1} \llcorner S_A$. Since $A \in \mathrm{SO}(n)$, $A^\nu_i A^\nu_j = \delta_{ij}$. Thus, by Theorem A.2.2,

$$0 = \widetilde{D}_k\left(\delta_{ij}\right) = \widetilde{D}_k\left(A^\nu_i A^\nu_j\right) = A^\nu_i \widetilde{D}_k A^\nu_j + A^\nu_j \widetilde{D}_k A^\nu_i,$$

where, with an abuse of notation, we denoted by A its precise representative $\widetilde{A}$. That is, the 3-"tensor" $L := A^T \nabla A$, whose components are $L^i_{jk} = A^\nu_i \widetilde{D}_k A^\nu_j$, satisfies $L^i_{jk} = -L^j_{ik}$. A straightforward computation then gives

$$A^\nu_\mu \widetilde{D}_k A^\nu_j = L^\mu_{jk} = \frac{1}{2}\left(A^\nu_j \widetilde{\Gamma}^\nu_{\mu k} + A^\nu_k \widetilde{\Gamma}^\nu_{\mu j} - A^\nu_\mu \widetilde{\Gamma}^\nu_{jk}\right).$$

Multiplying by A^i_μ an summing over μ the previous identity, we get

$$\widetilde{D}_k A^i_j = \frac{1}{2} A^i_\mu \left(A^\nu_j \widetilde{\Gamma}^\nu_{\mu k} + A^\nu_k \widetilde{\Gamma}^\nu_{\mu j}\right) - \frac{1}{2} \widetilde{\Gamma}^i_{jk}.$$

In particular,

$$\left|\widetilde{D}A\right| \leq C\left|\widetilde{\Gamma}\right|.$$

For the jump part, we notice that

$$\mathrm{Curl}(A^i)^J = \left(A^{i+} - A^{i-}\right) \cdot J_{jk}\nu_A \mathcal{H}^{n-1} \llcorner S_A \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

where J_{jk} is the linear operator defined by

$$(J_{jk}v)^\ell := \begin{cases} v_j & \text{if } \ell = k, \\ -v_k & \text{if } \ell = j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $J_{jk}v \cdot v = 0$ for every $j < k$ and the $\{J_{jk}v\}_{1 \leq j < k \leq n}$ are linearly independent, for every vector $v \in \mathbb{R}^n$. Thus, $\{J_{jk}\nu_A(x)\}_{1 \leq j < k \leq n}$ is a basis for the tangent space $T(x) := \nu_A(x)^\perp$. In particular, there exists a dimensional constant $C(n)$ such that for every open subset $U \subset \Omega$

$$\sum_{i,j,k=1}^n \left|\mathrm{Curl}(A^i)^J_{jk}\right|(U) \geq C(n) \sum_{i=1}^n \int_{S_A \cap U} \left|\pi_T\left(A^{i+} - A^{i-}\right)\right| \mathrm{d}\mathcal{H}^{n-1}.$$

But, since $T(x)$ is an $(n-1)$-dimensional subspace and $A \in \mathrm{SO}(n)$, we have another dimensional constant (denoted again by $C(n)$) such that $\left|\pi_T\left(A^{i+} - A^{i-}\right)\right| \geq C(n)\left|A^{i+} - A^{i-}\right|$. That is

$$\sum_{i,j,k=1}^n \left|\mathrm{Curl}(A^i)^J_{jk}\right|(U) \geq C(n) \sum_{i=1}^n \int_{S_A \cap U} \left|A^{i+} - A^{i-}\right| \mathrm{d}\mathcal{H}^{n-1} = C(n)\left|D^J A\right|(U). \qquad \square$$

A.4 Some technical Lemmas

A.4.1 Bound vertices in a tree by its vertices of degree 1 and 2

Lemma A.4.1. *A non-trivial tree $T = (V, E)$ satisfies*

$$\#T \leq 2\#\left\{v \in V \,\middle|\, \deg(v) = 1\right\} + \#\left\{v \in V \,\middle|\, \deg(v) = 2\right\}. \tag{A.4.1}$$

Proof. We prove it by induction on $n = \#V$. If $n = 1$ there is nothing to prove. Suppose now that (A.4.1) holds when $\#V = n$, and let us prove it when $\#V = n + 1$. So, let $T = (V, E)$ be a tree with $n + 1$ vertices. Take a leaf $\ell \in V$ (which exists since T is a non-trivial tree) and consider then the tree $T^* := (V^*, E^*)$, whose vertices are $V^* := V \setminus \{\ell\}$ and whose edges are given by $E^* := E \setminus \{\{\ell, \ell'\}\}$, where ℓ' is the only neighbor of ℓ. Consider then

$$N_i := \#\left\{v \in V \,\middle|\, \deg(v) = i\right\}, \qquad N_i^* := \#\left\{v \in V^* \,\middle|\, \deg^*(v) = i\right\},$$

where $\deg^*(v)$ denotes the degree of v seen as a vertex of T^*. Then we have the following

$$N_1^* = \begin{cases} N_1 & \text{if } \deg(\ell') = 2, \\ N_1 - 1 & \text{if } \deg(\ell') > 2, \end{cases} \qquad N_2^* = \begin{cases} N_2 - 1 & \text{if } \deg(\ell') = 2, \\ N_2 + 1 & \text{if } \deg(\ell') = 3, \\ N_2 & \text{if } \deg(\ell') > 3. \end{cases}$$

Then one can easily check, using the induction hyphothesis, that (A.4.1) holds also when $\#V = n+1$. $\square$

A.4.2 Whitney Covering and an estimate for harmonic functions

We recall now the *Whitney covering Lemma* (cf. [29] and [31])

Theorem A.4.2 (Whitney covering Lemma). *Let $\Omega \subset \mathbb{R}^n$ be an open and proper subset. There exists a countable collection $\{Q_j\}_{j\geq 1}$, $Q_j = Q(x_j, \ell_j)$ of closed cubes such that*

(i) $\Omega = \bigcup_{j\geq 1} Q_j$ *and the Q_j's have disjoint interiors;*

(ii) $C^1 \ell_j \leq \operatorname{dist}(Q_j, \partial\Omega) \leq C\ell_j$, *where $C = C(n) > 1$ is a dimensional constant;*

(iii) $\sum_{j\geq 1} \chi_{\frac{9}{8}Q_j} \leq 12^n$.

Lemma A.4.3. *There exists a dimensional constant $C = C(n) > 0$ such that for every $\Omega \subset \mathbb{R}^n$ open and proper and every $u : \Omega \to \mathbb{R}$ harmonic,*

$$\int_\Omega \operatorname{dist}^2(x, \partial\Omega)\, |\nabla u|^2 \,\mathrm{d}x \leq C \int_\Omega |u|^2 \,\mathrm{d}x.$$

Proof. Let $\{Q_j\}_{j\geq 1}$ be a covering of Ω as in Theorem A.4.2. Then

$$\begin{aligned}\int_\Omega \operatorname{dist}^2(x, \partial\Omega)\, |\nabla u|^2 \,\mathrm{d}x &= \sum_{j\geq 1} \int_{Q_j} \operatorname{dist}^2(x, \partial\Omega)\, |\nabla u|^2 \,\mathrm{d}x \leq C \sum_{j\geq 1} \ell_j^2 \int_{Q_j} |\nabla u|^2 \,\mathrm{d}x \leq \\ &\leq C \sum_{j\geq 1} \int_{\frac{9}{8}Q_j} |u|^2 \,\mathrm{d}x \leq C \int_\Omega |u|^2 \,\mathrm{d}x.\end{aligned}$$

$\square$

A.4.3 A lemma for vector valued measures

We recall that a sequence of vector valued measure (defined on a locally compact separable metric space X; in our case, we can just take $X = \mathbb{R}^n$ with the usual Euclidean metric) $\{\mu_j\}_{j\geq 1}$ *quasi-converges* to (μ, ξ), $\mu_j \xrightarrow{q} (\mu, \xi)$, where μ is a vector valued measure and ξ is a positive measure, if $\mu_j \xrightarrow{*} \mu$ and $|\mu_j| \xrightarrow{*} \xi$. Quasi-convergence is equivalent to the fact that $\mu_j(B) \to \mu(B)$ and $|\mu_j|(B) \to \xi(B)$ for all relatively compact subsets B with $\xi(\partial B) = 0$. It is then easy to prove the following

Lemma A.4.4. *Suppose that a sequence $\{\mu_j\}_{j\geq 1}$ of vector valued measures on X quasi-converges to (μ, ξ), where ξ is a positive Radon measure. Then, if $\xi(\partial B(p, R)) = 0$, for every sequence $R_j \to R$,*

$$\limsup_{j\to\infty} |\mu_j(B(p, R_j))| \geq |\mu(B(p, R))|.$$

Proof. Since $\mu_j \xrightarrow{q} (\mu, \xi)$ and $\xi(\partial B(p, R)) = \emptyset$,

$$|\mu(B(p, R))| = \lim_{j\to\infty} |\mu_j(B(p, R))|.$$

Without loss of generality, we can assume $R_j \downarrow R$. Then

$$\limsup_{j\to\infty} |\mu_j(B(p,R))| \leq \limsup_{j\to\infty} |\mu_j(B(p,R_j) \setminus B(p,R))| + \limsup_{j\to\infty} |\mu_j(B(p,R_j))| \,.$$

We just have to prove $\limsup_j |\mu_j(B(p,R_j) \setminus B(p,R))| = 0$. Since $R_j \to R$, for every $\varepsilon > 0$, $B(p,R_j) \setminus B(p,R) \subset A_{\varepsilon,R} := \overline{B(p,(1+\varepsilon)R)} \setminus B(p,(1-\varepsilon)R)$, provided j is sufficiently large. In particular, since $|\mu_j| \xrightarrow{*} \xi$,

$$\limsup_j |\mu_j(B(p,R_j) \setminus B(p,R))| \leq \limsup_j |\mu_j| \,(A_{\varepsilon,R}) \leq \xi(A_{\varepsilon,R}).$$

But ξ is a Radon measure, thus by the arbitrariness of ε we obtain

$$\limsup_j |\mu_j(B(p,R_j) \setminus B(p,R))| \leq \xi(\partial B(p,R)) = 0. \qquad \square$$

Bibliography

[1] G. Lauteri and S. Luckhaus. *An Energy Estimate for Dislocation Configurations and the Emergence of Cosserat-Type Structures in Metal Plasticity.* arXiv.org (August 2016).

[2] S. Luckhaus and L. Mugnai. *On a mesoscopic many-body Hamiltonian describing elastic shears and dislocations.* Continuum Mechanics and Thermodynamics **22**, 251–290 (2010).

[3] S. Luckhaus and J. Wohlgemuth. *Study of a model for reference-free plasticity.* arXiv.org (August 2014).

[4] A. Garroni, G. Leoni and M. Ponsiglione. *Gradient theory for plasticity via homogenization of discrete dislocations.* Journal of the European Mathematical Society (JEMS) **12**, 1231–1266 (2010).

[5] M. Ponsiglione. *Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuous.* SIAM Journal on Mathematical Analysis **39**, 449–469 (2007).

[6] L. De Luca, A. Garroni and M. Ponsiglione. Γ-*convergence analysis of systems of edge dislocations: the self energy regime.* Archive for Rational Mechanics and Analysis **206**, 885–910 (2012).

[7] A. Garroni and S. Müller. Γ-*limit of a phase-field model of dislocations.* SIAM Journal on Mathematical Analysis **36**, 1943–1964 (2005).

[8] A. Garroni and S. Müller. *A variational model for dislocations in the line tension limit.* Archive for Rational Mechanics and Analysis **181**, 535–578 (2006).

[9] S. Conti, A. Garroni and S. Müller. *Singular kernels, multiscale decomposition of microstructure, and dislocation models.* Archive for Rational Mechanics and Analysis **199**, 779–819 (2011).

[10] P. Cermelli. *Material symmetry and singularities in solids.* The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences **455**, 299–322 (1999).

[11] K. Kondo. *On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua.* International Journal of Engineering Science **2**, 219–251 (1964).

[12] E. Kröner. *Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen.* Archive for Rational Mechanics and Analysis **4**, 273–334 (1959).

[13] B.A. Bilby, R. Bullough and E. Smith. *Continuous Distributions of Dislocations: A New Application of the Methods of Non-Riemannian Geometry.* Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences **231**, 263–273 (1955).

[14] F. John. *Rotation and Strain.* Communications on Pure and Applied Mathematics **14**, 391–413 (1961).

[15] F. John. *Bounds for deformations in terms of average strains.* Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), 129–144 (1972).

[16] R.V. Kohn. *New integral estimates for deformations in terms of their nonlinear strains.* Archive for Rational Mechanics and Analysis **78**, 131–172 (1982).

[17] G. Gottstein. *Physical foundations of materials science.* Springer Science & Business Media (2013).

[18] W.T. Read and W. Shockley. *Dislocation models of crystal grain boundaries.* Physical Review **78**, 275 (1950).

[19] E. Cosserat and F. Cosserat. *Théorie des corps déformables.* Hermann et fils, Paris (1909).

[20] E. Sandier and S. Serfaty. *Vortices in the magnetic Ginzburg-Landau model.* Birkhäuser Boston, Inc., Boston, MA (2007).

[21] G. Friesecke, R. James and S. Müller. *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity.* Communications on Pure and Applied Mathematics **55**, 1461–1506 (2002).

[22] S. Conti, G. Dolzmann and S. Müller. *Korn's second inequality and geometric rigidity with mixed growth conditions.* Calculus of Variations and Partial Differential Equations **50**, 437–454 (2014).

[23] S. Müller, L. Scardia and C.I. Zeppieri. *Geometric rigidity for incompatible fields, and an application to strain-gradient plasticity.* Indiana University Mathematics Journal **63**, 1365–1396 (2014).

[24] T. Iwaniec and A. Lutoborski. *Integral Estimates for null Lagrangians.* Archive for Rational Mechanics and Analysis **1**, 25–79 (1993).

[25] L.C. Evans and R. Gariepy. *Measure theory and fine properties of functions.* CRC Press, Boca Raton, FL, (2015).

[26] L. Ambrosio, N. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems.* The Clarendon Press, Oxford University Press, New York, (2000).

[27] R. Dautray and J-L. Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques. Tome 3.* Masson, Paris (1985).

[28] A. P. Calderón and A Zygmund. *On singular integrals.* American Journal of Mathematics **78**, 289–309 (1956).

[29] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series **30**, xiv+290 pages (1970).

[30] C. Muscalu and W. Schlag. *Classical and multilinear harmonic analysis. Vol. I.* Cambridge Studies in Advanced Mathematics **137**, xviii+370 pages (2013).

[31] L. Grafakos. *Classical Fourier Analysis.* Graduate Texts in Mathematics **249**, Springer, New York, xviii+638 pages (2014).

[32] M. Giaquinta, G. Modica and J. Souček. *Cartesian currents in the calculus of variations. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] **37**, Springer-Verlag, Berlin, xxiv+711 pages (1998).

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