Convergence of phase-field models and thresholding schemes via the gradient flow structure of multi-phase mean-curvature flow

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Abstract

This thesis is devoted to the rigorous study of approximations for (multi-phase) mean curvature flow and related equations. We establish convergence towards weak solutions of the according geometric evolution equations in the BV-setting of finite perimeter sets. Our proofs are of variational nature in the sense that we use the gradient flow structure of (multi-phase) mean curvature flow. We study two classes of schemes, namely phase-field models and thresholding schemes. The starting point of our investigation is the fact that both, the Allen-Cahn Equation and the thresholding scheme, preserve this gradient flow structure. The Allen-Cahn Equation is a gradient flow itself, while the thresholding scheme is a minimizing movements scheme for an energy that Γ-converges to the total interfacial energy. In both cases we can incorporate external forces or a volume-constraint. In the spirit of the work of Luckhaus and Sturzenhecker (Calc. Var. Partial Differential Equations 3(2):253–271, 1995), our results are *conditional* in the sense that we assume the time-integrated energies to converge to those of the limit. Although this assumption is natural, it is not guaranteed by the a priori estimates at hand.

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Contents

Introduction

Mean curvature flow is one of the most fundamental geometric evolution equations. It can be viewed as a system of degenerate parabolic equations or as the L^2-gradient flow of the area functional, and is thus a very natural object of study in geometry. However, it was not a mathematician who "discovered" the equation but W. W. Mullins, a physicist who devoted most of his career to the study of materials. As for many others after him, his motivation for studying mean curvature flow was grain growth, a coarsening process in polycrystals undergoing heat treatment. If we allow ourselves to call his paper [70] the hour of birth of mean curvature flow, then it is identical with the hour of birth of its *multi-phase* version, where each phase corresponds to a grain in Mullins' model. It seems that he was the first to write down the mean curvature flow equation

$$V = \sigma\mu H,$$

where V denotes the normal velocity and H the mean curvature of the interface. The free energy density σ and the mobility μ associated to a grain boundary depend on the mismatch of the two adjacent crystal lattices and on the orientation of the grain boundary w. r. t. these lattices. In other words, $\sigma = \sigma_{ij}(\nu)$ and $\mu = \mu_{ij}(\nu)$ are indexed by the two adjacent phases and are anisotropic, i. e. dependent on the normal ν of the interface. We restrict ourselves to the simplest choice of mobilities and neglect anisotropies. However, we consider a wide class of surface tensions σ_{ij}, including the most popular ansatz for small angle grain boundaries [74]. Especially because of the application to grain growth, the development and analysis of numerical schemes for simulating mean curvature flow and its multi-phase version have a long history. This work is devoted to the analysis of such algorithms. We prove rigorous convergence results towards weak solutions of multi-phase mean-curvature flow.

Mean curvature flow shares common features with other geometric evolution equations, such as Ricci flow and the harmonic map heat flow, but also with the semi-linear heat equation. The latter connection is not surprising since on the curvature level, mean curvature flow is a reaction-diffusion equation. Let us point out two compelling examples to illustrate these similarities. Only a few years after Giga and Kohn [39] derived a monotonicity formula for the semi-linear heat equation to study blow-ups of solutions, Struwe [84] developed a similar theory for the harmonic map heat flow, and so did Huisken [42] for mean curvature flow. The connection to Ricci flow is evident when comparing Hamilton's program for Ricci flow (and in particular Perelman's recent progress [72, 73]) to the work of Huisken and Sinestrari [43, 44, 45] on mean curvature flow with surgery. To understand the geometry of a surface, they alter it by smooth mean curvature flow (or its metric by smooth Ricci flow) interrupted by topological surgeries.

The mathematical analysis of mean curvature flow is challenging because generically solutions develop singularities in finite time. Only in a few examples, such as entire graphs [26], smooth solutions exist for all times. In the classical setting, the evolving surfaces are described by their parametrizations. The resulting equation is invariant under reparametrizations which causes its degeneracy. Starting from a smoothly embedded, compact surface, short-time existence follows from standard theory after an appropriate choice of a gauge, but the evolution develops singularities in finite time. Blow-ups at singularities are self-similar solutions which have already been studied by Mullins. He found examples of homothetically expanding and shrinking solutions and the translating solution which nowadays goes by the creative name of “grim reaper”. The characterization of these special solutions is important to understand singularities but is not yet settled.

In order to define solutions past singularities, several notions of weak solutions have been developed during the last decades. A very robust notion of weak solutions which goes by the somewhat misleading name of viscosity solution is based on the following comparison principle of two-phase mean-curvature flow. *Two disjoint surfaces stay disjoint during the evolution by mean curvature flow.* In fact, the distance of the surfaces is non-increasing. The viscosity solution $\Sigma(t)$ is the largest closed set enjoying this comparison property when tested with surfaces evolving smoothly by mean curvature. In the original papers of Chen, Giga and Goto [22], and Evans and Spruck [36], the (equivalent) definition of the viscosity solution is based on the level set formulation of Osher and Sethian [71]. Instead of evolving the surface Σ^0, they consider a generic function u^0 having Σ^0 as a level set, and evolve u^0 according to the degenerate parabolic equation

$$\partial_t u - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) |\nabla u| = 0. \tag{1}$$

Then they define the viscosity solution $\Sigma(t)$ as the corresponding level set of the solution u at time t. Equation (1) is designed such that – at least formally – every level set of u moves by mean curvature. Solutions to (1) are in general not smooth but may be defined by posing tangency properties for smooth functions [23]. Mean curvature flow is well-posed in this context, the viscosity solution is unique. But $\Sigma(t)$ may develop an interior, a phenomenon called “fattening”, which reflects the non-uniqueness of mean curvature flow in special situations. The level set function u gives a natural interpretation of *generic* flows and provides a useful framework to prove rigorous convergence results for the two-phase versions of the approximations we consider here. However, this notion cannot be generalized to multi-phase mean-curvature flow, where a comparison principle is clearly absent.

Our guiding principle in this work is instead the gradient flow structure of (multi-phase) mean curvature flow. In general, a gradient flow structure is given by an energy functional and a dissipation mechanism, given by the geometry of the space of configurations through a Riemannian metric. A simple computation reveals this structure for mean curvature flow. If the hypersurface $\Sigma = \Sigma(t)$ evolves smoothly by mean curvature (with $\sigma = \mu = 1$) the change of area is given by

$$\frac{d}{dt}|\Sigma| = -\int_\Sigma V\,H = -\int_\Sigma H^2, \tag{2}$$

where V denotes the normal velocity and H denotes the mean curvature of Σ. In view of (2), when fixing the energy to be the surface area, the metric tensor is given by the L^2-metric $\int_\Sigma V^2$ on the space of normal vector fields. However, some care needs to be taken when dealing with this metric as for example the geodesic distance vanishes identically [63, 64]. The implicit time discretization of Almgren, Taylor and Wang [3], and Luckhaus and Sturzenhecker [57] makes use of the gradient flow structure. In fact, it inspired De Giorgi to define a similar implicit time discretization for abstract gradient flows which he named "minimizing movements". His abstract scheme consists of a family of minimization problems which mimic the principle of a gradient flow moving in direction of the steepest descent. The configuration Σ^n at time step n is obtained from its predecessor Σ^{n-1} by minimizing $E(\Sigma) + \frac{1}{2h}\operatorname{dist}^2(\Sigma, \Sigma^{n-1})$, where dist denotes the geodesic distance induced by the Riemannian structure and $h > 0$ denotes the time-step size. In the Euclidean case, the scheme boils down to the implicit Euler scheme. It has been successfully utilized for applications in partial differential equations and for instance allowed Jordan, Kinderlehrer and Otto [49] to interpret diffusion equations as gradient flows for the entropy w. r. t. the Wasserstein distance. In view of the degeneracy in the case of mean curvature flow it is apparent that the scheme in [3, 57] uses a proxy for the geodesic distance. Their replacement for the distance of two boundaries $\Sigma = \partial\Omega$ and $\tilde{\Sigma} = \partial\tilde{\Omega}$ is the (non-symmetric) quantity $2\int_{\Omega\Delta\tilde{\Omega}} d_{\tilde{\Omega}}\,dx$, where $d_{\tilde{\Omega}}$ denotes the (unsigned) distance to $\partial\tilde{\Omega}$. This variational viewpoint of curvature-driven interface evolutions has proven to be flexible enough to study a tremendous amount of problems such as the Stefan Problem [57] and its anisotropic variant [38], the Mullins-Sekerka Flow [76] and its multi-phase variant [16], volume-preserving mean-curvature flow [69], the evolution of martensitic phase transitions [25], and many more. Furthermore, Chambolle [20] showed that the scheme [3, 57] which seems rather academic at a first glance can be implemented efficiently.

While a geometric comparison principle is absent, the gradient flow structure is still present in the multi-phase case. The energy is then a weighted sum of the interfacial energies, with a normal-dependent density in the anisotropic case. The metric tensor is the L^2-norm on the interfaces, possibly weighted by anisotropic mobilities. However, multi-phase mean-curvature flow is still poorly understood in comparison to its two-phase counterpart. The analytical study of the planar case started with the work of Mantegazza, Novaga and Tortorelli [59] who studied the evolution of a single triple junction. Recently Mantegazza, Novaga and Pluda [58] extended these results to the case of two triple junctions. Ilmanen, Neves and Schulze [47] proved short-time existence even when starting from certain non-regular networks, which should allow to continue the flow through all generic/stable singularities that form during the evolution of a planar network. Only recently, global weak solutions were constructed in the substantial work of Kim and Tonegawa [50]. They proved convergence of a variant of Brakke's original scheme towards a *non-trivial* Brakke flow in any space dimension. Uniqueness of the evolution is still unclear but is expected for a generic flow.

In retrospect, also Brakke's pioneering work [14] can be seen as a way of interpreting mean curvature flow as a gradient flow. His definition is similar to the one of an abstract gradient flow and characterizes solutions by the optimal dissipation of energy in the spirit of (2). Brakke's solutions are varifolds, a concept weak enough to obtain compactness under natural conditions and strong enough to give sense two either side of (2). In contrast to the abstract framework, Brakke measures the dissipation of energy only in terms of the gradient of the energy, here the mean curvature. Therefore he has to monitor *localized* versions of (2) and – as for an abstract gradient flow – only asks for an inequality instead of an equality. Since his definition does not involve the metric term, one loses control over the time derivative and thus weak solutions may be discontinuous in time and in particular mass can disappear instantly. One of Brakke's most important contributions is his regularity theory. He proved that for a k-dimensional Brakke flow with unit density, for all times t at which the mass does not drop, there exists an exceptional (closed) set with vanishing k-Hausdorff measure such that around all point outside this set the evolution is smooth (cf. Theorem 6.12 in [14]). Up to now, his regularity theory has only been improved in special situations such as for mean convex surfaces, see [86]. This is quite remarkable considering that the monotonicity formula had not yet been available at the time of his work [14].

In this work we use the gradient flow structure to prove rigorous convergence results for several schemes approximating (multi-phase) mean curvature flow. We consider two classes of schemes, namely phase-field models and thresholding schemes. Phase-field models are used to model various interfacial motions, replacing sharp interfaces by diffused transition layers. The Allen-Cahn Equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon) \tag{3}$$

started out as a physical model [2] for the evolution of antiphase boundaries but became a popular computational scheme. Variants of the equation can be used to model multi-phase systems with anisotropic surface energies incorporating external forces and even coupling with other equations. The equation is the (by the factor $\frac{1}{\varepsilon}$ accelerated) L^2-gradient flow of the Ginzburg-Landau Energy

$$E_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2}\left|\nabla u_\varepsilon\right|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,dx. \tag{4}$$

The derivation of motion by mean curvature as the singular limit of the Allen-Cahn Equation has a long history and is well-understood in the two-phase case. First formal asymptotic expansions were constructed by Rubinstein, Sternberg and Keller [78]. Convergence for a smooth evolution was proved independently by De Mottoni and Schatzman [24], and Chen [21]. Bronsard and Kohn [17] used the gradient flow structure of (3) to prove compactness, and, in the radially symmetric case, convergence to motion by mean curvature. For the long-time behavior past singularities the above mentioned notions of weak solutions have proven to be useful for understanding the singular limit of (3). Evans, Soner and Souganidis [35] rigorously proved the convergence towards the viscosity solution – at least if the viscosity solution does not fatten. Barles, Soner and Souganidis

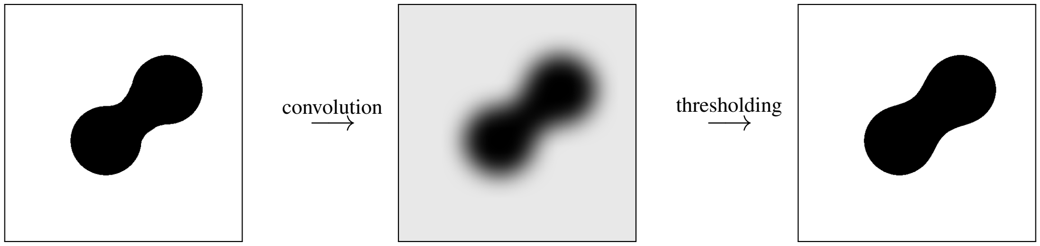

Figure 1: Thresholding for two phases.

[10] showed in particular that this holds true for mean convex or star shaped initial conditions. Ilmanen [46] proved convergence towards Brakke's formulation in the two-phase case by translating Huisken's monotonicity formula to the phase-field framework of (3). While the question of convergence of the Allen-Cahn Equation (3) seems to be almost settled in the two-phase case, little is known in the multi-phase case. Even the work of Ilmanen seems not to apply since he makes use of comparison techniques at a crucial point. In fact, he raised the question of how to deal with the multi-phase case in the same paper. Bronsard and Reitich [18] carried out a formal asymptotic expansion at a triple junction and proved short-time existence. However, to the best of our knowledge, rigorous long-time convergence results past singularities have not been available prior to the work [54] presented in Chapter 4.

The thresholding scheme is a time discretization for mean curvature flow. Its structural simplicity is intriguing to both, applied and theoretical scientists. Merriman, Bence and Osher [61] introduced the algorithm in 1992 to overcome the numerical difficulty of multiple scales in phase-field models like (3). Their idea is based on an operator splitting for the Allen-Cahn Equation, alternating between linear diffusion and thresholding. The latter replaces the fast reaction coming from the nonlinear right-hand side of (3). More precisely, given a time-step size $h > 0$ and the phase Ω^{n-1} at time step $n-1$, they convolve its characteristic function $\chi^{n-1} = \mathbf{1}_{\Omega^{n-1}}$ with a Gaussian kernel G_h of variance $\sqrt{h}$ and then define the evolved phase Ω^n at time step n as the super level set $\{G_h * \chi^{n-1} > \frac{1}{2}\}$, see Figure 1. The convolution can be implemented efficiently on a uniform grid using the discrete Fourier Transform and the thresholding step is a simple pointwise operation. Because of its simplicity and efficiency thresholding gained a lot of attention in the last decades. The popularity of the scheme comes from its natural extension to the multi-phase case [62]. First, one diffuses each phase independently, and the thresholding step is replaced by $\Omega_i = \{G_h * \chi_i^{n-1} > G_h * \chi_j^{n-1}$ for all $j \neq i\}$, see Figure 2. Large-scale simulations [28, 29, 30] demonstrate the efficiency of a slight modification of the scheme. For applications in materials science and image segmentation it is desirable to design algorithms that are efficient enough to handle large numbers of phases but flexible enough to incorporate external forces, variable surface tensions and even anisotropies. Surprisingly, it took more than twenty years to find a suitable generalization of the scheme to arbitrary surface tensions. The necessary impulse was an observation by Esedoğlu and Otto [33]. They realized that thresholding preserves the gradient flow structure of (multi-phase) mean curvature flow in the sense that it can be viewed as a minimizing movements

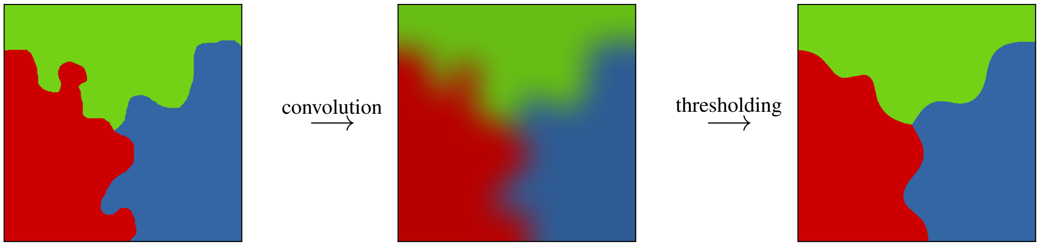

Figure 2: Thresholding for three phases

scheme for an energy that Γ-converges to the total interfacial area. This viewpoint allowed them to incorporate a wide class of surface tensions including the well-known Read-Shockley formulas [74]. The development of thresholding schemes for anisotropic motions started with the work [48] of Ishii, Pires and Souganidis. Efficient schemes were presented by Bonnetier, Bretin and Chambolle [13], where the convolution kernels are explicit and well-behaved in Fourier space but not necessarily in real space. The recent work [27] of Elsey and Esedoḡlu is inspired by the variational viewpoint [33] and shows that not all anisotropies can be obtained when structural features such as positivity of the kernel are needed. However, variants of the scheme developed by Esedoḡlu and Jacobs [32] share the same stability conditions even for more general kernels. Mascarenhas [60] incorporated external forces by changing the threshold value. In the same vein, Ruuth and Wetton [80] enforced a volume constraint by finding the threshold value that preserves the volume of the phase. The rigorous asymptotic analysis of thresholding schemes started with the independent convergence proofs of Evans [35], and Barles and Georgelin [10] in the isotropic two-phase case. Since the scheme preserves the above mentioned geometric comparison principle, they were able to prove convergence towards the viscosity solution of mean curvature flow. Recently, Swartz and Yip [85] proved convergence for a smooth evolution. In their proof they establish consistency and stability of the scheme, very much in the flavor of classical numerical analysis. They prove explicit bounds on the curvature and injectivity radius of the approximations and get a good understanding of the transition layer. However, also their result seems to be only applicable in the two-phase case.

Our convergence proofs for both, the Allen-Cahn Equation and thresholding schemes, are of variational nature. In particular, our analysis of the Allen-Cahn Equation uses some techniques known from the analytical study of its static analogue, initiated by the work of Modica and Mortola [67]. Modica [66] and Sternberg [83] provided the convergence of the Ginzburg-Landau Energy (4) towards a multiple of the perimeter functional in the sense of Γ-convergence. Kohn and Sternberg [51] were able to construct *local* minimizers of the Ginzburg-Landau Energy (4) based on the above Γ-convergence. Furthermore, it turns out that the convergence of the Ginzburg-Landau Energy towards the perimeter functional is even stronger: Luckhaus and Modica [56] proved that also the first variations of the energies converge towards the mean curvature – the first variation of the perimeter functional – by the clever use of a classical argument of Reshetnyak [75]. Extensions to the case of vector-valued order parameters u has been initiated by Sternberg [83], and Fonseca and Tartar [37] for two limiting phases, culminating in the work of Baldo [9] on the Γ-convergence in the multi-phase case.

The starting point for our analysis of thresholding schemes is the minimizing movements interpretation of Esedoğlu and Otto [33]. Let us explain this interpretation at the example of the two-phase scheme. They observed that the combination $\chi^n = \mathbf{1}_{\{G_h*\chi^{n-1}>\frac{1}{2}\}}$ of convolution and thresholding is equivalent to minimizing $E_h(\chi) + \frac{1}{2h}\mathrm{d}_h^2(\chi,\chi^{n-1})$, where

$$E_h(\chi) := \frac{1}{\sqrt{h}}\int (1-\chi)\, G_h * \chi \, dx \quad \text{and} \quad \mathrm{d}_h^2(\chi,\chi^{n-1}) := 2\sqrt{h}\int \left[G_{h/2} * \left(\chi - \chi^{n-1}\right)\right]^2 dx$$

are an approximation of the perimeter functional and the the square of a distance, respectively. The latter serves as a proxy for the induced distance, just like $2\int_{\Omega\Delta\Omega^{n-1}} d_{\Omega^{n-1}}dx$ in the minimizing movements scheme of Almgren, Taylor and Wang [3], and Luckhaus and Sturzenhecker [57]. The Γ-convergence of similar functionals has been developed some time ago by Alberti and Bellettini [1] and more recently by Ambrosio, De Philippis and Martinazzi [5], and was proven for the functionals E_h by Miranda, Pallara, Paronetti and Preunkert [65]. Esedoğlu and Otto found an independent, much simpler proof in the case of the energies E_h, which extends to the multi-phase case.

However, the Γ-convergence of the energies does not imply the convergence of the according gradient flows, or minimizing movements schemes. Since every gradient flow comes with a metric, it is evident that one needs conditions on both, the metric tensor and the energy, to verify the convergence. Sandier and Serfaty [81] provided sufficient conditions for this convergence. Serfaty [82] has already mentioned that in the case of the scalar Allen-Cahn Equation, these assumptions are guaranteed by the works of Röger and Schätzle [77] on the Willmore functional and Mugnai and Röger [68] on the action functional of the Allen-Cahn Equation. This result is restricted to two-phase mean-curvature flow in dimensions $d \leq 3$.

We will establish convergence towards a distributional formulation of (multi-phase) mean curvature flow in the setting of finite perimeter sets used by Luckhaus and Sturzenhecker [57]. In contrast to Brakke's concept of solution, here the normal velocity can be defined straightforwardly by the distributional equation $\partial_t\chi = V\,|\nabla\chi|\,dt$, where χ denotes the characteristic function of the phase Ω whose boundary $\Sigma = \partial\Omega$ evolves by mean curvature. We formulate the equation $V = H$ as

$$\int_0^T \int (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,|\nabla\chi|\,dt = \int_0^T \int V\,\xi\cdot\nu\,|\nabla\chi|\,dt. \tag{5}$$

In view of the integration by parts rule $\int_\Sigma (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu) = \int_\Sigma H\,\xi\cdot\nu$ for smooth surfaces Σ without boundary, the left-hand side is a natural way to encode the mean curvature. Furthermore, it can be naturally extended to the multi-phase case, automatically incorporating Herring's angle condition at triple junctions. However, this notion of solution is *not* stable under weak convergence in BV, which is the natural compactness coming from a priori estimates. Hence, as the result of Luckhaus and Sturzenhecker, also ours are only *conditional* convergence results. We assume the time-integrated energies of the approximations to converge to those of the limit. Under this strengthened convergence (sometimes called “strict” convergence in BV) and given the a priori estimates

$$\sup_t \int |\nabla\chi| \leq E_0, \quad \int_0^T \int V^2 \, |\nabla\chi| \, dt \leq E_0,$$

the notion of solution is stable. The main difficulty in the convergence proof of Luckhaus and Sturzenhecker for the minimizing movements scheme [3, 57], and for ours as well, is the right-hand side of (5). In fact, the stability of the left-hand side under this strengthened convergence is a classical result of Reshetnyak [75]. The structure of the right-hand side is more difficult, one has to pass to the limit in the product of the two weakly converging quantities V and ν. They overcome this difficulty by using regularity theory for (almost) minimal surfaces. Hence, their proof is restricted to dimensions $d \leq 7$. More precisely, their proof heavily relies on the fact that the metric term $\frac{1}{h}\int_{\Omega\Delta\Omega^{n-1}} d_{\Omega^{n-1}} dx$ is a compact perturbation on scales below $\sqrt{h}$. Therefore, they can control oscillations of the normal ν on these scales. Our proofs for the schemes considered here seem more robust in the sense that we only use mild regularity of the *limit*, namely the rectifiablity of the reduced boundary $\partial^*\Omega(t)$ for a.e. time slice, a consequence of De Giorgi’s Structure Theorem for sets of finite perimeter (cf. Theorem 4.4 in [41]). This regularity of the limit allows us to control the excess, a measure of the local flatness of $\partial^*\Omega(t)$.

This thesis is structured as follows. The first three chapters are devoted to the analysis of thresholding schemes. In Chapter 1 we present the work [53] with Felix Otto, which is the core of this thesis. We prove a conditional convergence result for the thresholding scheme in the multi-phase setting. We prove convergence towards a distributional solution of multi-phase mean-curvature flow similar to (5). Although this formulation is “stronger” than Brakke’s formulation in the sense that it requires more regularity, there is no direct way to infer Brakke’s inequality from this formulation. In Chapter 2 we derive Brakke’s inequality directly from the thresholding scheme, which will appear in the work [52] with Felix Otto. Chapter 3 provides generalizations of the work [53] presented in Chapter 1 and is based on the paper [55] with Drew Swartz. We treat external forces and a volume constraint. The reader will quickly realize that it is advantageous to familiarize him- or herself with Chapter 1 before turning to this chapter. In Chapter 4 we present the work [54] with Thilo Simon on the convergence of the Allen-Cahn Equation and can be read independently of the first three chapters (if one accepts the rather classical geometric property of BV-partitions proved in Section 5 of Chapter 1, which is a consequence of De Giorgi’s Structure Theorem). We prove the analogous results to the ones in Chapters 1 and 3. For the non-expert reader it is recommendable to start with this chapter to familiarize him- or herself with the application of the general strategy in a somewhat easier context.

Chapter 1

Multi-phase thresholding schemes

In this chapter we present the work [53] with Felix Otto. We prove a convergence result for the thresholding scheme in the multi-phase case for a wide class of surface tensions. Our result establishes convergence towards a weak formulation of mean curvature flow in the BV-framework of sets of finite perimeter. Like the result of Luckhaus and Sturzenhecker [57], ours is a *conditional* convergence result, which means that we assume the time-integrated energy of the approximation to converge to the time-integrated energy of the limit.

1 Introduction and main result

1.1 Idea of the proof

Let us start by giving a summary of the main steps and ideas of the convergence proof. In Section 2, we draw consequences from the basic estimate (10) in a minimizing movements scheme, like compactness, Proposition 2.1, coming from a uniform (integrated) modulus of continuity in space, Lemma 2.4, and in time, Lemma 2.5. We also draw the first consequence from the strengthened convergence (8) in Proposition 2.2. We strongly advise the reader to familiarize him- or herself with the argument for the modulus of continuity in time, Lemma 2.5, since it is there that the mesoscopic time scale $\sqrt{h}$ appears for the first time in a simple context before being used in Section 4 in a more complex context. In the same vein, the fudge factor α in the mesoscopic time scale $\alpha\sqrt{h}$, which will be crucial in Section 4, will first be introduced and used in the simple context when estimating the normal velocity V of the limit in Proposition 2.2.

Starting from Section 3, we also use the Euler-Lagrange equation (34) of the minimizing movement scheme. By Euler-Lagrange equation we understand the first variation w. r. t. the independent variables, as generated by a test vector field ξ. In Section 3, we pass to the limit in the energetic part of the first variation, recovering the mean curvature H via the term

$$\int_\Sigma H\,\xi\cdot\nu = \int_\Sigma \left(\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu\right).$$

This amounts to show that under our assumption of strengthened convergence (8), the Γ-convergence of the *functionals* can be upgraded to a distributional convergence of their *first variations*, cf. Proposition 3.1. It is a classical result credited to Reshetnyak [75] that under the strengthened convergence of sets of finite perimeter, the measure-theoretic normals and thus the distributional expression for mean curvature also converge. The fact that this convergence of the first variation may also hold when combined with a diffuse interface approximation is known for instance in case of the Ginzburg-Landau Energy, see [56]. In our case the convergence of the first variations relies on a localization of the ingredients for the Γ-convergence worked out in [33], like the consistency, i. e. pointwise convergence of these functionals.

Section 4 constitutes the central and, as we believe, most innovative piece of this chapter; we pass to the limit in the dissipation/metric part of the first variation, recovering the normal velocity V via the term $\int_\Sigma V\,\xi\cdot\nu$. In fact, we think of the test-field ξ as localizing this expression in time and space, and recover the desired limiting expression only up to an error that measures how well the limiting configuration can be approximated by a configuration with only two phases and a flat interface in the space-time patch under consideration; this is measured both in terms of area (leading to a multi-phase excess in the language of the regularity theory of minimal surfaces) and volume, see Proposition 4.1. The main difficulty of recovering the metric term $\int_\Sigma V\,\xi\cdot\nu$ in comparison to recovering the distributional form $\int_\Sigma (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)$ of the energetic term is that one has to recover both the normal velocity V, which is distributionally characterized by $\partial_t\chi = V|\nabla\chi|dt$ on the level of the characteristic function χ, and the (spatial) normal ν. In short: one has to pass to the limit in a *product*. More precisely, the main difficulty is that there is no good bound on the discrete normal velocity V at hand on the level of the *microscopic* time scale h; only on the level of the above-mentioned mesoscopic time scale $\sqrt{h}$, such an estimate is available. This comes from the fact that the basic estimate yields control of the time derivative of the characteristic function χ only when mollified on the spatial scale $\sqrt{h}$ in $u = G_h * \chi$. The main technical ingredient to overcome this lack of control in Proposition 4.1 is presented in Lemma 4.2 in the two-phase case and in Lemma 4.5 in the general setting: If one of the two (spatial) functions $u, \tilde u$ is not too far from being strictly monotone in a given direction (a consequence of the control of the tilt-excess, see Lemma 4.4), then the spatial L^1-difference between the level sets $\Omega = \{u > \frac12\}$ and $\tilde\Omega = \{\tilde u > \frac12\}$ is controlled by the squared L^2-difference between u and $\tilde u$.

In Section 5, we combine the results of the previous two sections yielding the weak formulation of $V = H$ on some space-time patch up to an error expressed in terms of the above mentioned (multi-phase) tilt-excess of the limit on that patch. Complete localization in time and partition of unity in space allows us to assemble this to obtain $V = H$ globally, up to an error expressed by the time integral of the sum of the tilt excess over the spatial patches of finite overlap. De Giorgi's Structure Theorem for sets of finite perimeter (cf. Theorem 4.4 in [41]), adapted to a multi-phase situation but just used for a fixed time slice, implies that the error expression can be made arbitrarily small by sending the length scale of the spatial patches to zero.

1.2 Notation

We denote by

$$G_h(z) := \frac{1}{(2\pi h)^{d/2}} \exp\left(-\frac{|z|^2}{2h}\right)$$

the Gaussian kernel of variance h. Note that $G_{2t}(z)$ is the fundamental solution to the heat equation and thus

$$\begin{aligned} \partial_h G - \tfrac{1}{2}\Delta G = 0 \quad &\text{in } (0,\infty)\times\mathbb{R}^d, \\ G = \delta_0 \quad &\text{for } h = 0. \end{aligned}$$

We recall some basic properties, such as the normalization, non-negativity, boundedness and the factorization property:

$$\int_{\mathbb{R}^d} G_h\,dz = 1, \quad 0 \le G_h \le Ch^{-d/2}, \quad \nabla G_h(z) = -\frac{z}{h} G_h(z), \quad G(z) = G^1(z_1)\,G^{d-1}(z'),$$

where G^1 denotes the 1-dimensional and G^{d-1} the $(d-1)$-dimensional Gaussian kernel; let us also mention the semi-group property

$$G_{s+t} = G_s * G_t.$$

Throughout this chapter, we will work with periodic boundary conditions, i. e. on the flat torus $[0,\Lambda)^d$. The thresholding scheme for multiple phases, introduced in [33], for arbitrary surface tensions σ_{ij} and mobilities $\mu_{ij} = 1/\sigma_{ij}$ is the following, cf. Figure 1.1 for an example.

Algorithm 1.1. *Given the partition $\Omega_1^{n-1}, \ldots, \Omega_P^{n-1}$ of $[0,\Lambda)^d$ at time $t = (n-1)h$, obtain the evolved partition $\Omega_1^n, \ldots, \Omega_P^n$ at time $t = nh$ by:*

1. *Convolution step:*

$$\phi_i := G_h * \left(\sum_{j=1}^{P} \sigma_{ij} \mathbf{1}_{\Omega_j^{n-1}}\right). \tag{1}$$

2. *Thresholding step:*

$$\Omega_i^n := \left\{x \in [0,\Lambda)^d \colon \phi_i(x) < \phi_j(x) \text{ for all } j \neq i\right\}. \tag{2}$$

We will denote the characteristic functions of the phases Ω_i^n at the n^{th} time step by χ_i^n and interpolate these functions piecewise constantly in time, i. e.

$$\chi_i^h(t) := \chi_i^n = \mathbf{1}_{\Omega_i^n} \quad \text{for } t \in [nh, (n+1)h).$$

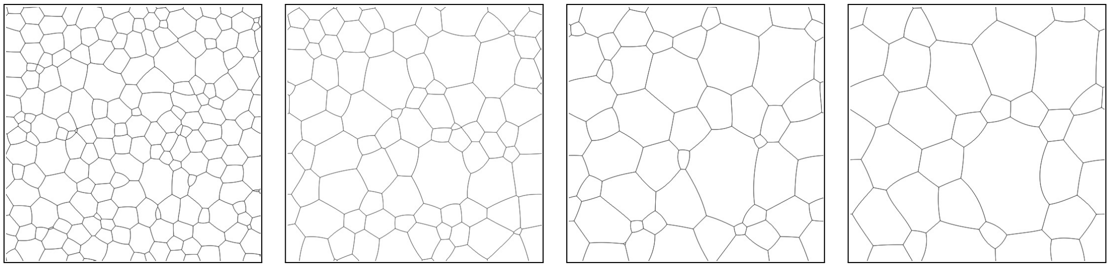

Figure 1.1: The evolution of a grain boundary network. Computation carried out with the code provided by Selim Esedoğlu [31].

As in [33], we define the *approximate energies*

$$E_h(\chi) := \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i \, G_h * \chi_j \, dx \tag{3}$$

for *admissible* measurable functions:

$$\chi = (\chi_1, \ldots, \chi_P)\colon [0,\Lambda)^d \to \{0,1\}^P \quad \text{s. t.} \quad \sum_{i=1}^P \chi_i = 1 \quad \text{a.e.} \tag{4}$$

Here and in the sequel $\int dx$ stands short for $\int_{[0,\Lambda)^d} dx$, whereas $\int dz$ stands short for $\int_{\mathbb{R}^d} dz$. The minimal assumption on the matrix of surface tensions $\{\sigma_{ij}\}$, next to the obvious

$$\sigma_{ij} = \sigma_{ji} \geq \sigma_{\min} > 0 \quad \text{if } i \neq j, \quad \sigma_{ii} = 0,$$

is the following triangle inequality

$$\sigma_{ij} \leq \sigma_{ik} + \sigma_{kj}.$$

It is known that (e. g. [33]), under the conditions above, these energies Γ-converge w. r. t. the L^1-topology to the *optimal partition energy* given by

$$E(\chi) := c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2}\Big(\int |\nabla \chi_i| + \int |\nabla \chi_j| - \int |\nabla(\chi_i + \chi_j)| \Big)$$

for *admissible* χ:

$$\chi = (\chi_1, \ldots, \chi_P)\colon [0,\Lambda)^d \to \{0,1\}^P \in BV \quad \text{s. t.} \quad \sum_{i=1}^P \chi_i = 1 \quad \text{a.e.}$$

The constant c_0 is given by

$$c_0 := \omega_{d-1} \int_0^\infty G(r) r^d dr = \frac{1}{\sqrt{2\pi}}.$$

For $\chi_i = \mathbf{1}_{\Omega_i}$, the interfaces $\Sigma_{ij} = \partial^*\Omega_i \cap \partial^*\Omega_j$ are defined as the intersections of the (reduced) boundaries of the respective phases and the energy $E(\chi)$ can be written as

$$E(\chi) = c_0 \sum_{i,j} \sigma_{ij} \left|\Sigma_{ij}\right| ,$$

cf. Figure 1.2. The proximity of E_h to E is intuitively clear. Indeed, roughly speaking, each summand in the definition of E_h measures the heat transfer from one phase to another after the short time h, see Figure 1.3. Parabolic rescaling by the length scale $\sqrt{h}$, i. e. the thickness of the red layer in the figure yields an approximation of the surface area of the interface between those two phases.

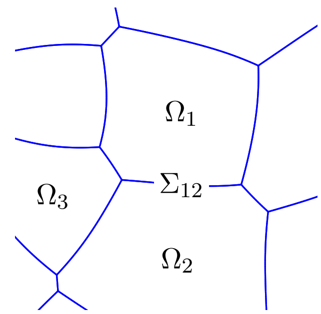

Figure 1.2: The phases Ω_i and the interfaces Σ_{ij} separating them.

For our purpose we ask the matrix of surface tensions σ to satisfy a *strict triangle inequality*:

$$\sigma_{ij} < \sigma_{ik} + \sigma_{kj} \quad \text{for pairwise different } i,\ j,\ k.$$

We recall the minimizing movements interpretation from [33] which is easy to check. The combination of convolution and thresholding step in Algorithm 1.1 is equivalent to solving the following minimization problem

$$\chi^n = \arg\min_{\chi} \left\{ E_h(\chi) - E_h(\chi - \chi^{n-1}) \right\} , \tag{5}$$

where χ runs over (4). The proof will mostly be based on the interpretation (5) and only once uses the original form (1) and (2) in Lemma 4.2 and Lemma 4.4, respectively. Following [33], we will additionally assume that σ is *conditionally negative-definite*, i. e.

$$\sigma \leq -\underline{\sigma} \quad \text{on } (1, \dots, 1)^\perp,$$

where $\underline{\sigma} > 0$ is a constant. That means, that σ is negative as a bilinear form on $(1, \dots, 1)^\perp$. This ensures that $-E_h(\chi - \chi^{n-1})$ in (5) is non-negative and penalizes the distance to the previous step. In the following we write $A \lesssim B$ to express that $A \leq CB$ for a generic constant $C < \infty$ that only depends on the dimension d, the total number of phases P and on the matrix of surface tensions σ through $\sigma_{\min} = \min_{i\neq j} \sigma_{ij}$, $\sigma_{\max} = \max \sigma_{ij}$, $\underline{\sigma}$ and $\min\{\sigma_{ik} + \sigma_{kj} - \sigma_{ij} \colon i,\ j,\ k \text{ pairwise different}\}$. Furthermore, we say a statement holds for $A \ll B$ if the statement holds for $A \leq \frac{1}{C} B$ for some generic constant $C < \infty$ as above.

1.3 Main result

The definition of our weak notion of mean-curvature flow is a distributional formulation which is suited to the framework of functions of bounded variation.

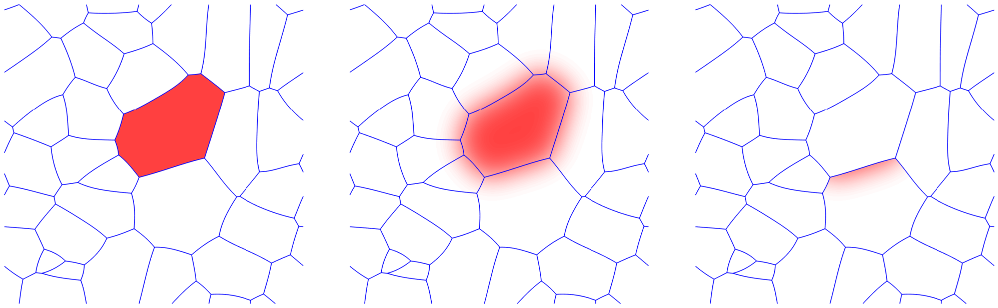

Figure 1.3: Each summand in the definition of the approximate energy E_h is constructed as follows. The phase's characteristic function χ_i (left) gets diffused, yielding $G_h * \chi_i$ (center), which is then tested against another characteristic function χ_j (right).

Definition 1.2 (Motion by mean curvature)**.** Fix some finite time horizon $T < \infty$, a matrix of surface tensions σ as above and initial data $\chi^0\colon [0,\Lambda)^d \to \{0,1\}^P$ with $E_0 := E(\chi^0) < \infty$. We say that the network

$$\chi = (\chi_1, \ldots, \chi_P) : (0,T) \times [0,\Lambda)^d \to \{0,1\}^P$$

with $\sum_i \chi_i = 1$ a. e. and

$$\sup_t E(\chi(t)) < \infty$$

moves by mean curvature if there exist functions $V_i\colon (0,T) \times [0,\Lambda)^d \to \mathbb{R}$ with

$$\int_0^T \int V_i^2 \, |\nabla \chi_i| \, dt < \infty$$

which satisfy

$$\sum_{i,j} \sigma_{ij} \int_0^T \int (\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i - 2\, \xi \cdot \nu_i \, V_i) \, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right) dt = 0 \qquad (6)$$

for all $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$ and which are normal velocities in the sense that for all $\zeta \in C^\infty([0,T] \times [0,\Lambda)^d)$ with $\zeta(T) = 0$ and all $i \in \{1, \ldots, P\}$

$$\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt + \int \zeta(0) \chi_i^0 \, dx = - \int_0^T \int \zeta \, V_i \, |\nabla \chi_i| \, dt. \qquad (7)$$

Note that (7) also encodes the initial conditions as well as (6) encodes the Herring angle condition. Indeed, for a smooth evolution, since for any interface Σ we have

$$\int_\Sigma (\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu) = \int_\Gamma b \cdot \xi + \int_\Sigma H \nu \cdot \xi,$$

where $\Gamma = \partial\Sigma$, b denotes the conormal and H the mean curvature of Σ, we do not only obtain the equation

$$H_{ij} = 2V_{ij} \quad \text{on } \Sigma_{ij} = \partial\Omega_i \cap \partial\Omega_j$$

along the smooth parts of the interfaces but also the Herring angle condition at triple junctions. If three phases Ω_1, Ω_2 and Ω_3 meet at a point x, then we have

$$\sigma_{12}\,\nu_{12}(x) + \sigma_{23}\,\nu_{23}(x) + \sigma_{31}\,\nu_{31}(x) = 0.$$

In terms of the opening angles θ_1, θ_2 and θ_3 at the junction, this condition reads

$$\frac{\sin\theta_1}{\sigma_{23}} = \frac{\sin\theta_2}{\sigma_{13}} = \frac{\sin\theta_3}{\sigma_{12}},$$

so that the opening angles at triple junctions are determined by the surface tensions.

Remark 1.3. To prove the convergence of the scheme, we will need the following convergence assumption:

$$\int_0^T E_h(\chi^h)\,dt \to \int_0^T E(\chi)\,dt. \tag{8}$$

This assumption makes sure that there is no loss of area in the limit $h \to 0$ as in Figure 1.4.

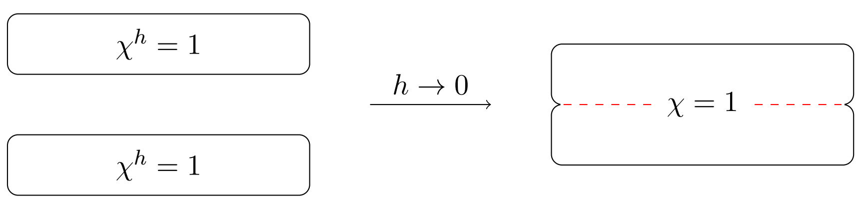

Figure 1.4: For fixed $t = t_0$ as $h \to 0$ there should be no loss of area. The ruled out case is illustrated here. The dashed line is sometimes called hidden boundary.

Theorem 1.4. *Let $P \in \mathbb{N}$, let the matrix of surface tensions σ satisfy the strict triangle inequality and be conditionally negative-definite, $T < \infty$ be a finite time horizon and let χ^0 be given with $E(\chi^0) < \infty$. Then for any sequence there exists a subsequence $h \downarrow 0$ and a partition $\chi\colon (0,T) \times [0,\Lambda)^d \to \{0,1\}^P$ with $E(\chi(t)) \le E_0$ such that the approximate solutions χ^h obtained by Algorithm 1.1 converge to χ. Given (8), χ moves by mean curvature in the sense of Definition 1.2 with initial data χ^0.*

Remark 1.5. In the following chapter, we will show that under the assumption (8) the limit χ solves a localized energy inequality and is thus a weak solution in the sense of Brakke.

Remark 1.6. Our proof uses the following three different time scales:

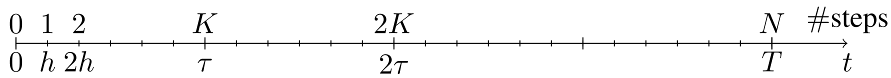

Figure 1.5: The micro-, meso-, and macroscopic time scales h, τ and T.

1. The *macroscopic time scale*, $T < \infty$, given by the finite time horizon,
2. the *mesoscopic time scale*, $\tau = \alpha\sqrt{h} \sim \sqrt{h} > 0$ and
3. the *microscopic time scale*, $h > 0$, coming from the time discretization.

The mesoscopic time scale arises naturally from the scheme: Due to the parabolic scaling, the microscopic time scale h corresponds to the length scale $\sqrt{h}$ as can be seen from the kernel G_h. Since for a smooth evolution, the normal velocity V is of order 1, this prompts the mesoscopic time scale $\sqrt{h}$.
The parameter α will be kept fixed most of the time until the very end, where we send $\alpha \to 0$. Therefore, it is natural to think of $\alpha \sim 1$, but small.
These three time scales go hand in hand with the following numbers, which we will for simplicity assume to be natural numbers throughout the proof:

1. N - the total number of microscopic time steps in a macroscopic time interval $(0, T)$,
2. K - the number of microscopic time steps in a mesoscopic time interval $(0, \tau)$ and
3. L - the number of mesoscopic time intervals in a macroscopic time interval.

The following simple identities linking these different parameters will be used frequently:

$$T = Nh = L\tau, \quad \tau = Kh, \quad L = \frac{N}{K} = \frac{T}{\tau}.$$

2 Compactness

In this section we prove the compactness of the approximate solutions, construct the normal velocities and derive bounds on these velocities. In the first subsection we present all results of this section; the proofs can be found in the subsequent subsection.

2.1 Results

The first main result of this section is the following compactness statement.

Proposition 2.1 (Compactness)**.** *There exists a sequence $h \downarrow 0$ and a limit $\chi\colon (0,T)\times[0,\Lambda)^d \to \{0,1\}^P$ such that*

$$\chi^h \longrightarrow \chi \quad \textit{a. e. in } (0,T)\times[0,\Lambda)^d \tag{9}$$

and the limit satisfies $E(\chi(t)) \le E_0$ and $\chi(t)$ is admissible in the sense of (4) for a. e. $t\in(0,T)$.

The second main result of this section is the following construction of the normal velocities and the square-integrability under the convergence assumption (8).

Proposition 2.2. *If the convergence assumption (8) holds, the limit $\chi = \lim_{h\to 0}\chi^h$ has the following properties.*

(i) $\partial_t\chi$ is a Radon measure with

$$\iint |\partial_t \chi_i| \lesssim (1+T)E_0$$

for each $i\in\{1,\dots,P\}$.

(ii) For each $i\in\{1,\dots,P\}$, $\partial_t\chi_i$ is absolutely continuous w. r. t. $|\nabla\chi_i|\,dt$. In particular, there exists a density $V_i \in L^1(|\nabla\chi_i|\,dt)$ such that

$$-\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt = \int_0^T\int \zeta\, V_i\,|\nabla\chi_i|\,dt$$

for all $\zeta\in C_0^\infty((0,T)\times[0,\Lambda)^d)$.

(iii) We have a strong L^2-bound: For each $i\in\{1,\dots,P\}$

$$\int_0^T\int V_i^2\,|\nabla\chi_i|\,dt \lesssim (1+T)\,E_0.$$

Both results essentially stem from the following basic estimate, a direct consequence of the minimizing movements interpretation (5).

Lemma 2.3 (Energy-dissipation estimate)**.** *The approximate solutions satisfy*

$$E_h(\chi^N) - \sum_{n=1}^N E_h(\chi^n - \chi^{n-1}) \le E_0. \tag{10}$$

$\sqrt{-E_h}$ defines a norm on the process space $\{\omega\colon [0,\Lambda)^d \to \mathbb{R}^P \,|\, \sum_i \omega_i = 0\}$*. In particular, the algorithm dissipates energy.*

In order to prove Proposition 2.1 we derive estimates on time- and space-variations of the approximations only using the basic estimate (10).

Estimate (10) bounds the (approximate) energies $E_h(\chi^h)$, which in turn control variations of $G_h * \chi^h$ in space via the term $\int \left|\nabla G_h * \chi^h\right| dx$. On length scales greater than $\sqrt{h}$, this estimate also survives for the approximations χ^h.

Lemma 2.4 (Almost BV in space)**.** *The approximate solutions satisfy*

$$\int_0^T \int \left|\chi^h(x+\delta e,t) - \chi^h(x,t)\right| dx\, dt \lesssim (1+T)E_0 \left(\delta + \sqrt{h}\right) \tag{11}$$

for any $\delta > 0$ *and* $e \in S^{d-1}$.

Variations in time are controlled by the following lemma coming from interpolating the (unbalanced) estimate (10) on time scales of order $\sqrt{h}$.

Lemma 2.5 (Almost BV in time)**.** *The approximate solutions satisfy*

$$\int_\tau^T \int \left|\chi^h(t) - \chi^h(t-\tau)\right| dx\, dt \lesssim (1+T)E_0 \left(\tau + \sqrt{h}\right) \tag{12}$$

for any $\tau > 0$.

Let us also mention that with the same methods we can prove $C^{1/2}$-Hölder-regularity of the volumes, i. e. $|\Omega(s)\Delta\Omega(t)| \lesssim |s-t|^{\frac{1}{2}}$. For the approximations this estimate of course only holds on time scales larger than the time-step size h.

Lemma 2.6 ($C^{1/2}$-Bounds)**.** *We have uniform Hölder-type bounds for the approximate solutions: I. e. for any pair* $s,t \in [0,T]$ *with* $|s-t| \geq h$ *we have*

$$\int \left|\chi^h(s) - \chi^h(t)\right| \, dx \lesssim E_0 \left|s-t\right|^{\frac{1}{2}}. \tag{13}$$

In particular, $\chi \in C^{1/2}([0,T], L^1([0,\Lambda)^d))$*: For almost every* $s,t \in (0,T)$*, we have*

$$\int \left|\chi(s) - \chi(t)\right| \, dx \lesssim E_0 \left|s-t\right|^{\frac{1}{2}}. \tag{14}$$

For the proof of the second main result of this section, Proposition 2.2, and also for later use in Section 4 it is useful to define certain measures which are induced by the metric term. These measures allow us to localize the result of Lemma 2.5. In the two-phase case this is enough to prove that the measure $\partial_t\chi$ is absolutely continuous w. r. t. the perimeter and the existence and integrability of the normal velocity, cf. (i) and (ii) of Proposition 2.2. The square-integrability follows then from a refinement of these estimates by localizing the fudge factor α (cf. Remark 1.6) after passage to the limit $h \to 0$.

Definition 2.7 (Dissipation measure)**.** For $h>0$, we define the *approximate dissipation measures* (associated to the approximate solution χ^h) μ_h on $[0,T]\times[0,\Lambda)^d$ by

$$\iint \zeta\, d\mu_h := \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \overline{\zeta}^n \left(\left| G_{h/2} * \left(\chi^n - \chi^{n-1}\right)\right|^2 + \left| G_h * \left(\chi^n - \chi^{n-1}\right)\right|^2 \right) dx, \tag{15}$$

where $\zeta \in C^\infty([0,T]\times[0,\Lambda)^d)$ and $\overline{\zeta}^n$ is the time average of ζ on the interval $[nh,(n+1)h)$. By the monotonicity of $h\mapsto \|G_h * u\|_{L^2}$ and the energy-dissipation estimate (10), we have

$$\mu_h([0,T]\times[0,\Lambda)^d) \lesssim E_0 \tag{16}$$

and $\mu_h \rightharpoonup \mu$ after passage to a further subsequence for some finite, non-negative measure μ on $[0,T]\times[0,\Lambda)^d$ with $\mu([0,T]\times[0,\Lambda)^d)\lesssim E_0$. We call μ the *dissipation measure*.

To prove Proposition 2.2 in the multi-phase case we have to ensure that the convergence assumption implies the convergence of the individual interfacial areas

$$\frac{1}{2}\int \left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right).$$

Lemma 2.8 (Implications of convergence assumption)**.** *The convergence assumption (8) ensures that for any pair $i\neq j$ and any $\zeta\in C^\infty([0,T]\times[0,\Lambda)^d)$,*

$$\begin{aligned}\int_0^T \frac{1}{\sqrt{h}}\int \zeta \left(\chi_i^h\, G_h * \chi_j^h + \chi_j^h\, G_h*\chi_i^h\right) dx\, dt \\ \to c_0 \int_0^T \int \zeta\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right) dt,\end{aligned} \tag{17}$$

as $h\to 0$.

The proof of Lemma 2.8 heavily relies on the fact that σ satisfies the *strict* triangle inequality so that we can preserve the triangle inequality after perturbing the energy functional. The following example shows that this is not a technical assumption but is a necessary condition for the lemma to hold and thus plays a crucial role in identifying the normal velocities V_i.

Example 2.9. To fix ideas let us consider three sets Ω_1, Ω_2 and Ω_3 in dimension $d=2$ with surface tensions $\sigma_{12}=\sigma_{23}=1$, $\sigma_{13}=2$ as illustrated in Figure 1.6. Then, the total energy is constant in h and due to the choice of the surface tensions the convergence assumption is fulfilled. Nevertheless, we clearly have

$$|\Sigma_{12}^h| = \text{const.} > 0 = |\Sigma_{12}| \quad \text{and} \quad |\Sigma_{13}^h| = 0 < \text{const.} = |\Sigma_{13}|.$$

This example also illustrates that although the energy functional E is lower semi-continuous, the individual interfacial energies $\frac{1}{2}\int\left(|\nabla\chi_i|+|\nabla\chi_j|-|\nabla(\chi_i+\chi_j)|\right)$ are not.

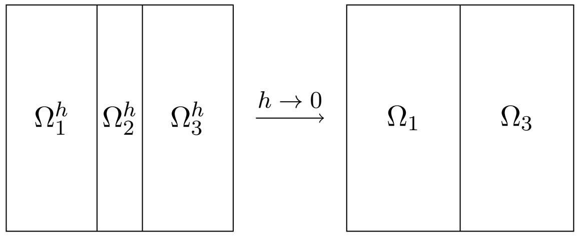

Figure 1.6: As $h \to 0$, the two interfaces Σ_{12}^h and Σ_{23}^h merge into one interface, Σ_{13}, between Phases 1 and 3. Therefore the measure of $|\Sigma_{13}^h|$ jumps up in the limit $h \to 0$ although the total interfacial energy converges due to the choice of surface tensions.

2.2 Proofs

Before proving the statements of this section we cite two results of [33] which will be used frequently in the proofs.

The following monotonicity statement is a key tool for the Γ-convergence in [33]. We will use it throughout our proofs but we seem not to rely heavily on it.

Lemma 2.10 (Approximate monotonicity)**.** *For all* $0 < h \leq h_0$ *and any admissible* χ*, we have*

$$E_h(\chi) \geq \left(\frac{\sqrt{h_0}}{\sqrt{h} + \sqrt{h_0}} \right)^{d+1} E_{h_0}(\chi). \tag{18}$$

Another important tool for the Γ-convergence in [33] is the following consistency, or pointwise convergence of the functionals E_h to E, which we will refine in Section 3.

Lemma 2.11 (Consistency)**.** *For any admissible* $\chi \in BV$*, we have*

$$\lim_{h \to 0} E_h(\chi) = E(\chi). \tag{19}$$

Taking the limit $h \to 0$ in (18) with $\chi = \chi^0$ and using (19), we see that that the interfacial energy E_0 of the initial data $\chi(0) \equiv \chi^0$ bounds the approximate energy of the initial data:

$$E_0 := E(\chi(0)) \geq E_h(\chi^0).$$

We first prove Proposition 2.1 which follows directly from the estimates in Lemmas 2.4 and 2.5. Then we give the proofs of the Lemmas used for Proposition 2.1. We present the proof of Proposition 2.2 at the end of this section since the proof heavily relies on the techniques developed in the proofs of the lemmas, especially in Lemma 2.5.

Proof of Proposition 2.1. The proof is an adaptation of the Riesz-Kolmogorov L^p-compactness theorem. By Lemma 2.4 and Lemma 2.5, we have

$$\int_0^T \int \left| \chi^h(x + \delta e, t + \tau) - \chi^h(t) \right| dx\, dt \lesssim (1 + T)\, E_0 \left(\delta + \tau + \sqrt{h} \right) \tag{20}$$

for any $\delta, \tau > 0$ and $e \in S^{d-1}$. For $\delta > 0$ consider the mollifier φ_δ given by the scaling $\varphi_\delta(x) := \frac{1}{\delta^{d+1}}\varphi(\frac{x}{\delta}, \frac{t}{\delta})$ and $\varphi \in C_0^\infty((-1,0) \times B_1)$ such that $0 \leq \varphi \leq 1$ and $\int_{-1}^0 \int_{B_1} \varphi = 1$. We have the estimates

$$\left|\varphi_\delta * \chi^h\right| \leq 1 \quad \text{and} \quad \left|\nabla(\varphi_\delta * \chi^h)\right| \lesssim \frac{1}{\delta}.$$

Hence, on the one hand, the mollified functions are equi-continuous and by Arzelà-Ascoli precompact in $C^0([0,T] \times [0,\Lambda)^d)$: For given $\epsilon, \delta > 0$ there exist functions $u_i \in C^0([0,T] \times [0,\Lambda)^d)$, $i = 1, \ldots, n(\epsilon, \delta)$ such that

$$\left\{\varphi_\delta * \chi^h \colon h > 0\right\} \subset \bigcup_{i=1}^{n(\epsilon,\delta)} B_\epsilon(u_i),$$

where the balls $B_\epsilon(u_i)$ are given w. r. t. the C^0-norm. On the other hand, for any function χ we have

$$\begin{aligned}\int_0^T \int |\varphi_\delta * \chi - \chi| \, dx \, dt &\leq \int \varphi_\delta(z,s) \int |\chi(x-z, t-s) - \chi(x,t)| \, d(x,t) \, d(z,s) \\ &\leq \sup_{(z,s) \in \operatorname{supp} \varphi_\delta} \int_0^T \int |\chi(x-z, t-s) - \chi(x,t)| \, dx \, dt.\end{aligned}$$

Using this for χ^h and plugging in (20) yields

$$\int_0^T \int \left|\varphi_\delta * \chi^h - \chi^h\right| dx \, dt \lesssim (1+T) \, E_0 \left(\delta + \sqrt{h}\right).$$

Given $\rho > 0$, fix $\delta, h_0 > 0$ such that

$$\int_0^T \int \left|\varphi_\delta * \chi^h - \chi^h\right| dx \, dt \leq \frac{\rho}{2} \quad \text{for all } h \in (0, h_0).$$

Then set $\epsilon := \frac{\rho}{T\Lambda^d}$ and find $u_1, \ldots, u_n$ from above. Note that only finitely many of the elements in the sequence $\{h\}$ are greater than h_0. Therefore,

$$\{\chi^h\}_h \subset \bigcup_{i=1}^n B_\rho(u_i) \cup \{\chi^h\}_{h > h_0} \subset \bigcup_{i=1}^n B_\rho(u_i) \cup \bigcup_{h > h_0} B_\rho(\chi^h)$$

is a finite covering of balls (w. r. t. L^1-norm) of given radius $\rho > 0$. Therefore, $\{\chi^h\}_h$ is precompact and hence relatively compact in L^1. Hence we can extract a converging subsequence. After passing to another subsequence, we can w. l. o. g. assume that we also have pointwise convergence almost everywhere in $(0,T) \times [0,\Lambda)^d$. $\square$

Proof of Lemma 2.3. By the minimality condition (5), we have in particular

$$E_h(\chi^n) - E_h(\chi^n - \chi^{n-1}) \leq E_h(\chi^{n-1})$$

for each $n = 1, \ldots, N$. Iterating this estimate yields (10) with $E_h(\chi^0)$ instead of $E_0 = E(\chi^0)$. Then (10) follows from the short argument after Lemma 2.11.

We claim that the pairing $-\frac{1}{\sqrt{h}} \int \omega \cdot \sigma \left(G_h * \tilde{\omega}\right) dx$ defines a scalar product on the process space. It is bilinear and symmetric thanks to the symmetry of σ and G_h. Since σ is conditionally negative-definite,

$$-\frac{1}{\sqrt{h}} \int \omega \cdot \sigma \left(G_h * \omega\right) dx = -\frac{1}{\sqrt{h}} \int \left(G_{h/2} * \omega\right) \cdot \sigma \left(G_{h/2} * \omega\right) dx \geq \frac{\underline{\sigma}}{\sqrt{h}} \|G_{h/2} * \omega\|_{L^2}^2 \geq 0.$$

Furthermore, we have equality only if $\omega \equiv 0$. Thus, $\sqrt{-E_h}$ is the induced norm on the process space. $\square$

Proof of Lemma 2.4. *Step 1:* We claim that

$$\int_0^T \int \left|\nabla G_h * \chi^h\right| dx\, dt \lesssim (1+T) E_0. \tag{21}$$

Indeed, for any characteristic function $\chi : [0,\Lambda)^d \to \{0,1\}$ we have

$$\nabla (G_h * \chi)(x) = -\int \nabla G_h(z) \left(\chi(x+z) - \chi(x)\right) dz.$$

Therefore, since $|\nabla G_h(z)| \lesssim \frac{1}{\sqrt{h}} |G_{2h}(z)|$,

$$\int |\nabla G_h * \chi| \; dx \lesssim \frac{1}{\sqrt{h}} \int G_{2h}(z) \int |\chi(x+z) - \chi(x)| \; dx\, dz.$$

By $\chi \in \{0,1\}$, we have $|\chi(x+z) - \chi(x)| = \chi(x) \left(1-\chi\right)(x+z) + \left(1-\chi\right)(x) \chi(x+z)$ and thus by symmetry of G_{2h}:

$$\int |\nabla G_h * \chi| \; dx \lesssim \frac{1}{\sqrt{h}} \int \left(1-\chi\right) \, G_{2h} * \chi \, dx.$$

Applying this on χ_i^h, summing over $i = 1, \ldots, P$, using $\chi_i^h = 1 - \sum_{j\neq i} \chi_j^h$ and $\sigma_{ij} \geq \sigma_{\min} > 0$ for $i \neq j$ we obtain

$$\int \left|\nabla G_h * \chi^h(t)\right| \; dx \lesssim E_{2h}(\chi^h) \lesssim E_h(\chi^h),$$

where we used the approximate monotonicity of E_h, cf. Lemma 2.10. Using the energy-dissipation estimate (10), we have

$$\int \left|\nabla G_h * \chi^h(t)\right| dx \lesssim E_0$$

and integration in time yields (21).

Step 2: By (21) and Hadamard's trick, we have on the one hand

$$\int_0^T \int \left| G_h * \chi^h(x+\delta e,t) - G_h * \chi^h(x,t) \right| dx\, dt \lesssim (1+T)E_0\delta.$$

Since $\chi \in \{0,1\}$, we have on the other hand

$$(\chi - G_h * \chi)_+ = \chi\, G_h * (1-\chi) \quad \text{and} \quad (\chi - G_h * \chi)_- = (1-\chi)\, G_h * \chi,$$

which yields

$$|\chi - G_h * \chi| = (1-\chi)\, G_h * \chi + \chi\, G_h * (1-\chi)\,. \tag{22}$$

Using the translation invariance and (22) for the components of χ^h, we have

$$\begin{aligned}\int_0^T \int \left| \chi^h(x+\delta e,t) - \chi^h(x,t) \right| dx\, dt \leq & 2\int_0^T \int \left| G_h * \chi^h - \chi^h \right| dx\, dt \\ & + \int_0^T \int \left| G_h * \chi^h(x+\delta e,t) - G_h * \chi^h(x,t) \right| dx\, dt \\ \lesssim & (1+T)\, E_0 \left(\sqrt{h} + \delta \right),\end{aligned}$$

which is precisely our claim. □

Proof of Lemma 2.5. In this proof, we make use of the mesoscopic time scale $\tau = \alpha\sqrt{h}$, see Remark 1.6 for the notation. First we argue that it is enough to prove

$$\int_\tau^T \int \left| \chi^h(t) - \chi^h(t-\tau) \right| dx\, dt \lesssim (1+T)E_0\tau \tag{23}$$

for $\alpha \in [1,2]$. If $\alpha \in (0,1)$, we can apply (23) twice, once for $\tau = \sqrt{h}$ and once for $\tau = (1+\alpha)\sqrt{h}$ and obtain (12). If $\alpha > 2$, we can iterate (23). Thus we may assume that $\alpha \in [1,2]$. We have

$$\begin{aligned}\int_\tau^T \int \left| \chi^h(t) - \chi^h(t-\tau) \right| dx\, dt = & h \sum_{k=0}^{K-1} \sum_{l=1}^{L} \int \left| \chi^{Kl+k} - \chi^{K(l-1)+k} \right| dx \\ = & \frac{1}{K} \sum_{k=0}^{K-1} \tau \sum_{l=1}^{L} \int \left| \chi^{Kl+k} - \chi^{K(l-1)+k} \right| dx.\end{aligned}$$

Thus, it is enough to prove

$$\sum_{l=1}^{L} \int \left| \chi^{Kl+k} - \chi^{K(l-1)+k} \right| dx \lesssim (1+T)E_0$$

for any $k = 0, \ldots, K-1$. By the energy-dissipation estimate (10), we have $E_h(\chi^k) \le E_0$ for all these k's. Hence we may assume w. l. o. g. that $k = 0$ and prove only

$$\sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim (1+T)E_0. \tag{24}$$

Note that for any two characteristic functions $\chi, \tilde{\chi}$ we have

$$\begin{aligned}
|\chi - \tilde{\chi}| =& (\chi - \tilde{\chi})\, G_h * (\chi - \tilde{\chi}) + (\chi - \tilde{\chi})(\chi - \tilde{\chi} - G_h * (\chi - \tilde{\chi})) \\
\le& (\chi - \tilde{\chi})\, G_h * (\chi - \tilde{\chi}) + |\chi - G_h * \chi| + |\tilde{\chi} - G_h * \tilde{\chi}|\,.
\end{aligned} \tag{25}$$

Now we post-process the energy-dissipation estimate (10). Using the triangle inequality for the norm $\sqrt{-E_h}$ on the process space and Jensen's inequality, we have

$$\begin{aligned}
-E_h\left(\chi^{Kl} - \chi^{K(l-1)}\right) \le& \left(\sum_{n=K(l-1)+1}^{Kl} \Big(- E_h\left(\chi^n - \chi^{n-1}\right) \Big)^{\frac{1}{2}} \right)^2 \\
\le& K \sum_{n=K(l-1)+1}^{Kl} -E_h\left(\chi^n - \chi^{n-1}\right).
\end{aligned} \tag{26}$$

Using (25) for χ_i^{Kl} and $\chi_i^{K(l-1)}$ with (22) for the second and the third right-hand side term and the conditional negativity of σ and the above inequality for the first right-hand side term we obtain

$$\sum_{l=1}^{L} \int |\chi_i^{Kl} - \chi_i^{K(l-1)}|\, dx \lesssim \sqrt{h} K \sum_{n=1}^{N} -E_h\left(\chi^n - \chi^{n-1}\right) + L \max_n \int \left(1 - \chi_i^n\right)\, G_h * \chi_i^n \, dx.$$

Since $(1 - \chi_i^n) = \sum_{j \ne i} \chi_j^n$ a. e. and $\sigma_{ij} \ge \sigma_{\min} > 0$ for all $i \ne j$, the energy-dissipation estimate (10) yields

$$\sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim \alpha E_0 + \frac{1}{\alpha} T E_0 \lesssim (1+T)E_0,$$

which establishes (24) and thus concludes the proof. □

Proof of Lemma 2.6. First note that (14) follows directly from (13) since we also have the convergence $\chi^h(t) \to \chi(t)$ in L^1 for almost every t. The argument for (13) comes in two steps. Let $s > t$, $\tau := s - t$ and $t \in [nh, (n+1)h)$.

Step 1: Let τ be a multiple of h. We may assume w. l. o. g. that $\tau = m^2 h$ for some $m \in \mathbb{N}$. As in the proof of Lemma 2.5, using (25) and (26) we derive

$$\int \left|\chi^{n+m} - \chi^n\right| \, dx \lesssim m\sqrt{h} \sum_{k=1}^{m} -E_h(\chi^{n+k} - \chi^{n+k-1}) + \sqrt{h} \max_t E_h(\chi^h(t)).$$

As before, we sum these estimates:

$$\begin{aligned}\int \left|\chi^{n+m^2} - \chi^n\right| dx \leq & \sum_{l=0}^{m-1} \int \left|\chi^{n+m(l+1)} - \chi^{n+ml}\right| dx \\ \lesssim & m\sqrt{h} \sum_{n'=n}^{n+m^2} -E_h(\chi^{n'} - \chi^{n'-1}) + m\sqrt{h} \max_t E_h(\chi^h(t)) \\ \lesssim & m\sqrt{h} E_0 = E_0\sqrt{\tau}.\end{aligned}$$

Step 2: Let $\tau \geq h$ be arbitrary. Take $m \in \mathbb{N}$ such that $s \in [(m+n)h, (m+n+1)h)$. From Step 2 we obtain the bound in terms of mh instead of τ. If $\tau \geq mh$, we are done. If $h \leq \tau < mh$, then $m \geq 2$ and thus $mh \leq \frac{m}{m-1}\tau \lesssim \tau$. □

Proof of Lemma 2.8. W. l. o. g. let $i = 1, j = 2$. We prove the statement in three steps. In the first step we reduce the statement to a time-independent one. In the second step, we show that due to the strict triangle inequality, the convergence of the energies implies the convergence of the individual perimeters. In the third step, we conclude by showing that this convergence still holds true if we localize with a test function ζ, which proves the time-independent statement formulated in the first step.

Step 1: Reduction to a time-independent problem. It is enough to prove that the convergence $\chi^h \to \chi$ in $L^1([0,\Lambda)^d, \mathbb{R}^P)$ and the convergence of the energies $E_h(\chi^h) \to E(\chi)$ imply

$$\frac{1}{\sqrt{h}} \int \zeta \left(\chi_1^h \, G_h * \chi_2^h + \chi_2^h \, G_h * \chi_1^h \right) dx \to c_0 \int \zeta \left(|\nabla \chi_1| + |\nabla \chi_2| - |\nabla(\chi_1 + \chi_2)| \right) \tag{27}$$

for any $\zeta \in C^\infty([0,\Lambda)^d)$.
Given $\chi^h \to \chi$ in $L^1((0,T) \times [0,\Lambda)^d)$, for a subsequence we clearly have $\chi^h(t) \to \chi(t)$ in $L^1([0,\Lambda)^d)$ for a. e. t. We further claim that for a subsequence

$$E_h(\chi^h) \to E(\chi) \quad \text{for a. e. } t. \tag{28}$$

Writing $\left|E_h(\chi^h) - E(\chi)\right| = 2\big(E(\chi) - E_h(\chi^h)\big)_+ + E_h(\chi^h) - E(\chi)$ and using the $\liminf$-inequality of the Γ-convergence of E_h to E, we have

$$\lim_{h\to 0} \Big(E(\chi) - E_h(\chi^h) \Big)_+ = 0 \quad \text{for a. e. } t.$$

Then Lebesgue's Dominated Convergence, cf. (10), and the convergence assumption (8) yield

$$\lim_{h\to 0} \int_0^T \Big| E_h(\chi^h) - E(\chi) \Big| \, dt = 0$$

and thus (28) after passage to a subsequence. Therefore, we can apply (27) for a. e. t and the time-dependent version follows from the time-independent one by Lebesgue's Dominated Convergence Theorem and (10).

Step 2: Convergence of perimeters. We claim that given $\chi^h \to \chi$ in $L^1([0,\Lambda)^d, \mathbb{R}^P)$ and the convergence of the total interfacial energy $E_h(\chi^h) \to E(\chi)$, the individual perimeters converge in the following sense: We have

$$F_h(\chi_1^h) \to F(\chi_1), \quad F_h(\chi_2^h) \to F(\chi_2) \quad \text{and} \quad F_h(\chi_1^h + \chi_2^h) \to F(\chi_1 + \chi_2).$$

where F_h and F are the two-phase analogues of the (approximate) energies:

$$F_h(\tilde{\chi}) := \frac{2}{\sqrt{h}} \int (1-\tilde{\chi})\, G_h * \tilde{\chi}\, dx \quad \text{and} \quad F(\tilde{\chi}) := 2c_0 \int |\nabla \tilde{\chi}| \,.$$

We will prove this claim by perturbing the functional E_h. We recall that the functionals F_h Γ-converge to F (see e. g. [65] or [33]). Since the argument for the three cases work in the same way, we restrict ourself to the first case, $F_h(\chi_1^h) \to F(\chi_1)$. Since the matrix of surface tensions σ satisfies the strict triangle inequality, we can perturb the functionals E_h in the following way: For sufficiently small $\epsilon > 0$, the associated surface tensions for the functional $\chi \mapsto E_h(\chi) - \epsilon F_h(\chi_1)$ satisfy the triangle inequality so that approximate monotonicity, Lemma 2.10, and consistency, Lemma 2.11, still apply. Therefore, by Lemma 2.10, we have for any $h_0 \geq h$

$$\begin{aligned} E_h(\chi^h) =& E_h(\chi^h) - \epsilon F_h(\chi_1^h) + \epsilon F_h(\chi_1^h) \\ \geq& \left(\frac{\sqrt{h_0}}{\sqrt{h} + \sqrt{h_0}}\right)^{d+1} \Big(E_{h_0}(\chi^h) - \epsilon F_{h_0}(\chi_1^h)\Big) + \epsilon F_h(\chi_1^h). \end{aligned}$$

By assumption, the left-hand side converges to $E(\chi)$. Since for fixed h_0, $\chi \mapsto E_{h_0}(\chi) - \epsilon F_{h_0}(\chi_1)$ is clearly a continuous functional on L^2, the first right-hand side term converges as $h \to 0$. Thus, for any $h_0 > 0$,

$$\limsup_{h\to 0} \epsilon F_h(\chi_1^h) \leq E(\chi) - \left(E_{h_0}(\chi) - \epsilon F_{h_0}(\chi_1)\right).$$

As $h_0 \to 0$, Lemma 2.11 yields

$$\limsup_{h\to 0} F_h(\chi_1^h) \leq F(\chi_1).$$

By the Γ-convergence we also have

$$\liminf_{h\to 0} F_h(\chi_1^h) \geq F(\chi_1)$$

and thus the convergence $F_h(\chi_1^h) \to F(\chi_1)$.

Step 3: Conclusion. We claim that given $\chi^h \to \chi$ in $L^1([0,\Lambda)^d, \mathbb{R}^P)$ and $E_h(\chi^h) \to E(\chi)$, for any $\zeta \in C^\infty([0,\Lambda)^d)$ we have (27).
We will not prove (27) directly but prove that for any $\zeta \in C^\infty([0,\Lambda)^d)$

$$F_h(\chi_1^h, \zeta) \to F(\chi_1, \zeta), \quad F_h(\chi_2^h, \zeta) \to F(\chi_2, \zeta) \quad \text{and} \quad F_h(\chi_1^h + \chi_2^h, \zeta) \to F(\chi_1 + \chi_2, \zeta) \tag{29}$$

for the localized functionals

$$F_h(\tilde\chi, \zeta) := \frac{1}{\sqrt{h}} \int \zeta \left[(1-\tilde\chi)\, G_h * \tilde\chi + \tilde\chi\, G_h * (1-\tilde\chi)\right] dx \quad \text{and} \quad F(\tilde\chi, \zeta) := 2c_0 \int \zeta \, |\nabla \tilde\chi| \tag{30}$$

instead. This is indeed sufficient since for any χ_1, χ_2, we clearly have

$$\begin{aligned} \chi_1\, G_h * \chi_2 + \chi_2\, G_h * \chi_1 = &\,(1-\chi_1)\, G_h * \chi_1 + (1-\chi_2)\, G_h * \chi_2 \\ &- (1-(\chi_1+\chi_2))\, G_h * (\chi_1 + \chi_2) \end{aligned}$$

and (29) therefore implies (27).

Now we give the argument for (29). As before, we only prove one of the statements, namely

$$F_h(\chi_1^h, \zeta) \to F(\chi_1, \zeta).$$

For this we use two lemmas that we will prove in Section 3. First, by applying Lemma 3.6, which is the localized version of Lemma 2.11, we have for the functional F_h instead of E_h we have $F_h(\chi_1) \to F(\chi_1)$. Then, by Lemma 3.7 we can estimate $\left|F_h(\chi_1) - F_h(\chi_1^h)\right| \to 0$ and thus conclude the proof. □

Let us mention that one can also follow a different line of proof for Lemma 2.8 by localizing the monotonicity statement of Lemma 2.10 with a test function ζ. Since Lemma 3.7 seems more robust, we only prove the statement in this fashion.

Proof of Proposition 2.2. We make use of the mesoscopic time scale τ, see Remark 1.6 for the notation.

Argument for (i): Let $\zeta \in C_0^\infty((0,T) \times [0,\Lambda)^d)$. We have to show that

$$-\int_0^T \int \partial_t \zeta \, \chi_i \, dx\, dt \lesssim (1+T) E_0 \, \|\zeta\|_\infty .$$

In this part we choose $\alpha = 1$. Using the notation $\partial^\tau \zeta = \frac{1}{\tau}\left(\zeta(t+\tau) - \zeta(t)\right)$ for the discrete time derivative, by the smoothness of ζ,

$$\partial^\tau \zeta \to \partial_t \zeta \quad \text{uniformly in } (0,T) \times [0,\Lambda)^d \text{ as } h \to 0.$$

Since $\chi^h \to \chi$ in $L^1((0,T)\times[0,\Lambda)^d)$, the product converges:

$$\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt = \lim_{h\to 0} \int_0^T \int \partial^\tau \zeta \, \chi_i^h \, dx \, dt.$$

Since $\operatorname{supp}\zeta$ is compact, by Lemma 2.5 we have

$$\begin{aligned}
-\int_0^T \int \partial^\tau \zeta \, \chi_i^h \, dx \, dt = &\int_0^T \int \zeta \, \partial^{-\tau} \chi_i^h \, dx \, dt \\
&\leq \|\zeta\|_\infty \int_\tau^T \int \left| \partial^{-\tau} \chi_i^h \right| dx \, dt \lesssim (1+T)\, E_0 \, \|\zeta\|_\infty
\end{aligned}$$

for sufficiently small h.

Argument for (ii): First we prove

$$-\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt \lesssim \frac{1}{\alpha} \int_0^T \int |\zeta| \, |\nabla \chi_i| \, dt + \alpha \iint |\zeta| \, d\mu \tag{31}$$

for any $\alpha > 0$ and any $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$. We fix ζ and by linearity we may assume that $\zeta \geq 0$ if we prove the inequality with absolute values on the left-hand side. We use the identity from above

$$-\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt = \lim_{h\to 0} \int_0^T \int \zeta \, \partial^{-\tau} \chi_i^h dx \, dt.$$

Setting

$$\zeta^n := \frac{1}{h} \int_{nh}^{(n+1)h} \zeta(t) \, dt$$

to be the time average over a microscopic time interval, we have

$$\left| \int_0^T \int \zeta \, \partial^{-\tau} \chi_i^h dx \, dt \right| \leq \frac{1}{K} \sum_{k=1}^K \sum_{l=1}^L \int \zeta^{Kl+k} \left| \chi_i^{Kl+k} - \chi_i^{K(l-1)+k} \right| dx.$$

Now fix $k \in \{1, \dots, K\}$. For simplicity, we will ignore k at first. We can argue as in the proof of Lemma 2.5, here with the localization ζ: By (22) we have for any $\chi \in \{0,1\}$

$$\frac{1}{\sqrt{h}} \int \zeta \, |G_h * \chi - \chi| \, dx = \frac{1}{\sqrt{h}} \int \zeta \left[(1-\chi) \, G_h * \chi + \chi \, G_h * (1-\chi) \right] dx = F_h(\chi, \zeta)$$

with F_h as in (30) and furthermore

$$\left| \int \left(\zeta^{K(l+1)} - \zeta^{Kl} \right) (1-\chi) \, G_h * \chi \, dx \right| \leq \|\partial_t \zeta\|_\infty \alpha \sqrt{h} \int (1-\chi) \, G_h * \chi \, dx.$$

Therefore, using (25) we obtain

$$\sum_{l=1}^{L}\int \zeta^{Kl}\left|\chi_i^{Kl}-\chi_i^{K(l-1)}\right|dx \lesssim \sum_{l=1}^{L}\int \zeta^{Kl}\big(\chi_i^{Kl}-\chi_i^{K(l-1)}\big)G_h*\big(\chi_i^{Kl}-\chi_i^{K(l-1)}\big)dx$$
$$+\frac{\tau}{\alpha}\sum_{l=1}^{L}F_h(\chi_i^{Kl},\zeta^{Kl})+\sqrt{h}\,\|\partial_t\zeta\|_\infty\,\tau\sum_{l=1}^{L}E_h(\chi^{Kl}),$$

where the last right-hand side term vanishes as $h\downarrow 0$ by (10). For the first right-hand side term we note that for any $\zeta\in C^\infty([0,\Lambda)^d)$ and any $\chi,\,\tilde\chi\in\{0,1\}$ we have

$$\begin{aligned}
&\left|\int \zeta\left[G_{h/2}*(\chi-\tilde\chi)\right]^2dx-\int\zeta\,(\chi-\tilde\chi)\,G_h*(\chi-\tilde\chi)\,dx\right|\\
&=\left|\int\big(\zeta\,G_{h/2}*(\chi-\tilde\chi)-G_{h/2}*[\zeta\,(\chi-\tilde\chi)]\big)\,G_{h/2}*(\chi-\tilde\chi)\,dx\right|\\
&\le\int G_{h/2}(z)\int|\zeta(x+z)-\zeta(x)|\,|\chi-\tilde\chi|\,(x+z)\,\big|G_{h/2}*(\chi-\tilde\chi)\big|\,(x)\,dx\,dz\\
&\lesssim\|\nabla\zeta\|_\infty\sqrt{h}\int\frac{|z|}{\sqrt{h}}G_{h/2}(z)\,dz\int|\chi-\tilde\chi|\,dx\\
&\lesssim\|\nabla\zeta\|_\infty\sqrt{h}\int|\chi-\tilde\chi|\,dx,
\end{aligned}$$

so that we can replace the first right-hand side term by

$$\sum_{l=1}^{L}\int\zeta^{Kl}\left(G_{h/2}*\big(\chi_i^{Kl}-\chi_i^{K(l-1)}\big)\right)^2dx,$$

up to an error that vanishes as $h\downarrow 0$, due to the above calculation and e. g. Lemma 2.6. As in (26) for $-E_h$, now for this localized version, we can use the triangle inequality and Jensen's inequality to bound this term by

$$\sum_{l=1}^{L}K\sum_{n=K(l-1)+1}^{Kl}\int\zeta^{Kl}\left(G_{h/2}*\big(\chi_i^{n}-\chi_i^{n-1}\big)\right)^2dx\le\alpha\iint\zeta\,d\mu_h+o(1),$$

as $h\downarrow 0$, where μ_h is the (approximate) dissipation measure defined in (15). Therefore we have

$$\sum_{l=1}^{L}\int\zeta^{Kl}\left|\chi_i^{Kl}-\chi_i^{K(l-1)}\right|dx\lesssim\frac{\tau}{\alpha}\sum_{l=1}^{L}F_h(\chi_i^{Kl},\zeta^{Kl})+\alpha\iint\zeta\,d\mu_h+o(1),$$

as $h\downarrow 0$. Taking the mean over the k's we obtain

$$\left|\int_0^T\int\zeta\,\partial^{-\tau}\chi_i^h dx\,dt\right|\lesssim\frac{1}{\alpha}\int_0^T F_h(\chi_i^h,\zeta)\,dt+\alpha\iint\zeta\,d\mu_h+o(1).$$

Passing to the limit $h \to 0$, (17), which is guaranteed by the convergence assumption (8), implies (31).
Now let $U \subset (0,T) \times [0,\Lambda)^d$ be open such that

$$\iint_U |\nabla \chi_i| \, dt = 0.$$

If we take $\zeta \in C_0^\infty(U)$, the first term on the right-hand side of (31) vanishes and therefore

$$-\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt \lesssim \alpha \iint |\zeta| \, d\mu.$$

Since the left-hand side does not depend on α, we have

$$-\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt \leq 0.$$

Taking the supremum over all $\zeta \in C_0^\infty(U)$ yields

$$\iint_U |\partial_t \chi_i| = 0.$$

Thus, $\partial_t \chi_i$ is absolutely continuous w. r. t. $|\nabla \chi_i| \, dt$ and the Radon-Nikodym theorem completes the proof.

Argument for (iii): We refine the estimate in the argument for (ii). Instead of estimating the right-hand side of (31) and optimizing afterwards, which leads to a weak L^2-bounds, we localize. Starting from (31), we notice that we can localize with the test function ζ. Thus, we can post-process the estimate and obtain

$$\left| \int_0^T \int V_i \, \zeta \, |\nabla \chi_i| \, dt \right| \leq C \int_0^T \int \frac{1}{\alpha} \, |\zeta| \, |\nabla \chi_i| \, dt + C \iint \alpha \, |\zeta| \, d\mu$$

for any integrable $\zeta : (0,T) \times [0,\Lambda)^d \to \mathbb{R}$, any measurable $\alpha \colon (0,T) \times [0,\Lambda)^d \to (0,\infty)$ and some constant $C < \infty$ which depends only on the dimension d, the number of phases P and the matrix of surface tensions σ. Now choose

$$\zeta = V_i \quad \text{and} \quad \alpha = \frac{2C}{|V_i|},$$

where we set $\alpha := 1$ if $V_i = 0$, in which case all other integrands vanish. Then, the first term on the right-hand side can be absorbed in the left-hand side and we obtain

$$\int_0^T \int V_i^2 \, |\nabla \chi_i| \, dt \lesssim \mu([0,T] \times [0,\Lambda)^d) \lesssim E_0. \qquad \square$$

3 Energy functional and curvature

It is a classical result by Reshetnyak [75] that the convergence $\chi^h \to \chi$ in L^1 and

$$\int |\nabla \chi^h| \to \int |\nabla \chi| =: E(\chi)$$

imply convergence of the first variation

$$\delta E(\chi, \xi) = \int (\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu) \, |\nabla \chi| \, .$$

A result by Luckhaus and Modica [56] shows that this may extend to a Γ-convergence situation, namely in case of the Ginzburg-Landau functional

$$E_h(u) := \int h \, |\nabla u|^2 + \frac{1}{h} \left(1 - u^2\right)^2 dx.$$

We show that this also extends to our Γ-converging functionals E_h. Let us first address why the first variation of the approximate energies is of interest in view of our minimizing movements scheme. We recall (5): the approximate solution χ^n at time nh minimizes $E_h(\chi) - E_h(\chi - \chi^{n-1})$ among all χ. The natural variations of such a minimization problem are *inner variations*, i. e. variations of the independent variable. Given a vector field $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$ and an admissible χ, we define the deformation χ_s of χ along ξ by the distributional equation

$$\frac{\partial}{\partial s} \chi_{i,s} + \nabla \chi_{i,s} \cdot \xi = 0, \quad \chi_{i,s}\big|_{s=0} = \chi_i,$$

which means that the phases are deformed by the flow generated through ξ. The inner variation δE_h of the energy E_h at χ along the vector field ξ is then given by

$$\delta E_h(\chi, \xi) := \frac{d}{ds} E_h(\chi_s)\big|_{s=0} = \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i \, G_h * (-\nabla \chi_j \cdot \xi) \, dx. \tag{32}$$

For an admissible $\tilde{\chi}$ the inner variation of the metric term $-E_h(\chi - \tilde{\chi})$ is given by

$$-\delta E_h(\,\cdot\, - \tilde{\chi})(\chi, \xi) := \frac{d}{ds} \big(- E_h(\chi_s - \tilde{\chi}) \big)\big|_{s=0} = \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int (\chi_i - \tilde{\chi}_i) \, G_h * (\nabla \chi_j \cdot \xi) \, dx. \tag{33}$$

The (chosen and not necessarily unique) minimizer χ^n in Algorithm 1.1 therefore satisfies the Euler-Lagrange equation

$$\delta E_h(\chi^n, \xi) - \delta E_h(\,\cdot\, - \chi^{n-1})(\chi^n, \xi) = 0 \tag{34}$$

for any vector field $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$.

3.1 Results

The goal of this section is to prove the following statement about the convergence of the first term in the Euler-Lagrange equation.

Proposition 3.1. *Let $\chi^h,\, \chi\colon (0,T)\times[0,\Lambda)^d \to \{0,1\}^P$ be such that $\chi^h(t),\, \chi(t)$ are admissible in the sense of (4) and $E(\chi(t))<\infty$ for a. e. t. Let*

$$\chi^h \longrightarrow \chi \quad \text{a. e. in } (0,T)\times[0,\Lambda)^d, \tag{35}$$

and furthermore assume that

$$\int_0^T E_h(\chi^h)\,dt \longrightarrow \int_0^T E(\chi)\,dt. \tag{36}$$

Then, for any $\xi \in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^d)$, we have

$$\begin{aligned}\lim_{h\to 0}\int_0^T &\delta E_h(\chi^h,\xi)\,dt\\ &= c_0 \sum_{i,j}\sigma_{ij}\int_0^T\int \left(\nabla\cdot\xi - \nu_i\cdot\nabla\xi\,\nu_i\right)\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right)dt.\end{aligned}$$

It is easy to reduce the statement to the following time-independent statement.

Proposition 3.2. *Let $\chi^h,\, \chi\colon [0,\Lambda)^d \to \{0,1\}^P$ be admissible in the sense of (4) with $E(\chi)<\infty$ such that*

$$\chi^h \longrightarrow \chi \quad \text{a. e.}, \tag{37}$$

and furthermore assume that

$$E_h(\chi^h) \longrightarrow E(\chi). \tag{38}$$

Then, for any $\xi\in C^\infty([0,\Lambda)^d,\mathbb{R}^d)$, we have

$$\lim_{h\to 0}\delta E_h(\chi^h,\xi) = c_0\sum_{i,j}\sigma_{ij}\int \left(\nabla\cdot\xi - \nu_i\cdot\nabla\xi\,\nu_i\right)\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right).$$

Remark 3.3. Proposition 3.2 and all other statements in this section hold also in a more general context. We do not need the approximations χ^h to be characteristic functions. In fact the statements hold for any sequence $u^h\colon [0,\Lambda)^d \to [0,1]^P$ with $\sum_i u_i^h = 1$ a. e. converging to some $\chi\colon [0,\Lambda)^d\to\{0,1\}^P$ with $E(\chi)<\infty$ in the sense of (37)–(38).

The following first lemma brings the first variation δE_h of E_h into a more convenient form, up to an error vanishing as $h \to 0$ because of the smoothness of ξ. Already at this stage one can see the structure

$$\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu = \nabla \xi \colon (Id - \nu \otimes \nu)$$

in the first variation of E in the form of $\nabla \xi \colon (G_h Id - h\nabla^2 G_h)$ on the level of the approximation.

Lemma 3.4. *Let χ be admissible and $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$ then*

$$\delta E_h(\chi, \xi) = \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i \, \nabla \xi : \big(G_h Id - h\nabla^2 G_h\big) * \chi_j \, dx + O\left(\|\nabla^2 \xi\|_\infty E_h(\chi) \sqrt{h}\right). \tag{39}$$

We have already seen in Lemma 2.8 that we can pass to the limit in the term involving only the kernel $G_h Id$:

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta \, \chi_i^h \, G_h * \chi_j^h \, dx = c_0 \sum_{i,j} \sigma_{ij} \int \zeta \, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right),$$

where now $\zeta = \nabla \cdot \xi$. The next proposition shows that we can also pass to the limit in the term involving the second derivatives $h\nabla^2 G_h$ of the kernel, which yields the projection $\nu \otimes \nu$ onto the normal direction in the limit.

Proposition 3.5. *Let χ^h, χ satisfy the convergence assumptions (37) and (38). Then for any $A \in C^\infty([0,\Lambda)^d, \mathbb{R}^{d\times d})$*

$$\begin{aligned} \lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i^h \, A : h\nabla^2 G_h * \chi_j^h \, dx & \\ = c_0 \sum_{i,j} \sigma_{ij} \int \nu \cdot A \, \nu \, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right). \end{aligned}$$

The following two statements are used to prove Proposition 3.5. The following lemma yields in particular the construction part in the Γ-convergence result of E_h to E. We need it in a localized form; the proof closely follows the proof of Lemma 4 in Section 7.2 of [33].

Lemma 3.6 (Consistency)**.** *Let $\chi \in BV([0,\Lambda)^d, \{0,1\}^P)$ be admissible in the sense of (4). Then for any $\zeta \in C^\infty([0,\Lambda)^d)$*

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta \, \chi_i \, G_h * \chi_j \, dx = c_0 \sum_{i,j} \sigma_{ij} \int \zeta \, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right)$$

and for any $A \in C^\infty([0,\Lambda)^d, \mathbb{R}^{d\times d})$

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i \, A\colon h\nabla^2 G_h * \chi_j \, dx$$
$$= c_0 \sum_{i,j} \sigma_{ij} \int \nu \cdot A\, \nu \, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right).$$

The next lemma shows that under our convergence assumption of χ^h to χ, the corresponding spatial covariance functions f_h and f are very close and allows us to pass from Lemma 3.4 and Lemma 3.6 to Proposition 3.2.

Lemma 3.7 (Error estimate)**.** *Let χ^h, χ satisfy the convergence assumptions (37) and (38) and let k be a non-negative kernel such that*

$$k(z) \leq p(|z|) G(z)$$

for some polynomial p. Then

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int k_h(z) |f_h(z) - f(z)| \, dz = 0, \tag{40}$$

where

$$f_h(z) := \sum_{i,j} \sigma_{ij} \int \chi_i^h(x) \chi_j^h(x+z)\, dx \quad \textit{and} \quad f(z) := \sum_{i,j} \sigma_{ij} \int \chi_i(x)\chi_j(x+z)\, dx.$$

3.2 Proofs

Proof of Proposition 3.1. The proposition is an immediate consequence of the time-independent analogue, Proposition 3.2. Indeed, according to Step 1 in the proof of Lemma 2.8 we have $E_h(\chi^h) \to E(\chi)$ for a. e. t. Thus all conditions of Proposition 3.2 are fulfilled. Proposition 3.1 follows then from Lebesgue's Dominated Convergence Theorem. □

Proof of Proposition 3.2. We may apply Lemma 3.4 for χ^h and obtain by the energy-dissipation estimate (10) that

$$\delta E_h(\chi, \xi^h) = \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i^h \, \nabla \xi : \left(Id\, G_h - h\nabla^2 G_h \right) * \chi_j^h \, dx + O\left(\|\nabla^2 \xi\|_\infty E_0 \sqrt{h} \right).$$

Applying Proposition 3.5 for the kernel $\nabla^2 G$ with $\nabla\xi$ playing the role of the matrix field A and Lemma 2.8 for the kernel G with $\zeta = \nabla \cdot \xi$, we can conclude the proof. □

Proof of Lemma 3.4. Recall the definition of δE_h in (32). Since $-\nabla\tilde\chi\cdot\xi = -\nabla\cdot(\tilde\chi\,\xi)+\tilde\chi\,(\nabla\cdot\xi)$ for any function $\tilde\chi\colon [0,\Lambda)^d\to\mathbb{R}$, we can rewrite the integral on the right-hand side of (32):

$$\begin{aligned}\int \chi_i\, G_h*(-\nabla\chi_j\cdot\xi)\,dx &= \int -\chi_i\, G_h*(\nabla\cdot(\chi_j\,\xi)) + \chi_i\, G_h*(\chi_j\,\nabla\cdot\xi)\,dx\\ &= \int -\chi_i\,\nabla G_h*(\chi_j\,\xi) + \chi_j\,(\nabla\cdot\xi)\,G_h*\chi_i\,dx.\end{aligned}$$

Let us first turn to the first right-hand side term. For fixed (i,j), we can collect the two terms in the sum that belong to the interface between phases i and j and obtain by the antisymmetry of the kernel ∇G_h that the resulting term with the prefactor $\frac{2\sigma_{ij}}{\sqrt{h}}$ is

$$\begin{aligned}&\int -\chi_i\,\nabla G_h*(\chi_j\,\xi) - \chi_j\,\nabla G_h*(\chi_i\,\xi)\,dx\\ &\qquad = \int \chi_i(x)\int (\xi(x)-\xi(x-z))\cdot\nabla G_h(z)\,\chi_j(x-z)\,dz\,dx.\end{aligned}$$

A Taylor expansion of ξ around x gives the first-order term

$$\frac{2\sigma_{ij}}{\sqrt{h}}\int \chi_i(x)\int (\nabla\xi(x)\,z)\cdot\nabla G_h(z)\,\chi_j(x-z)\,dz\,dx.$$

Now we argue that the second-order term is controlled by $\|\nabla^2\xi\|_\infty E_h(\chi)\sqrt{h}$. Indeed, since $|z|^3G(z)\lesssim G_2(z)$, the contribution of the second-order term is controlled by

$$\begin{aligned}&\|\nabla^2\xi\|_\infty\frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int |z|^2\frac{|z|}{h}G_h(z)\int \chi_i(x)\,\chi_j(x+z)\,dx\,dz\\ &\lesssim \|\nabla^2\xi\|_\infty\sum_{i,j}\sigma_{ij}\int G_{2h}(z)\int\chi_i(x)\,\chi_j(x+z)\,dx\,dz\\ &\sim \|\nabla^2\xi\|_\infty\sqrt{h}\,E_{2h}(\chi).\end{aligned}$$

Using the approximate monotonicity (18) of E_h, we have suitable control over this term. After distributing the first-order term on both summand (i,j) and (j,i) we therefore have

$$\begin{aligned}\delta E_h(\chi,\xi) =& \frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int \chi_i(x)\nabla\xi(x) : \int (2G_h(z)Id + z\otimes\nabla G_h(z))\,\chi_j(x+z)\,dz\,dx\\ &+ O\left(\|\nabla^2\xi\|_\infty E_h(\chi)\sqrt{h}\right)\end{aligned}$$

and since $\nabla^2 G(z) = -Id\,G - z\otimes\nabla G(z)$, we conclude the proof. □

Proof of Proposition 3.5. By Lemma 3.6 we know that the term converges if we take χ instead of the approximation χ^h on the left-hand side of the statement. Lemma 3.7 in turn controls the error by substituting χ^h by χ on the left-hand side. □

Proof of Lemma 3.6. Our main focus in this proof lies on the anisotropic kernel $\nabla^2 G$. The statement for G is – up to the localization – already contained in the proof of Lemma 4 in Section 7.2 of [33].

Step 1: Reduction of the statement to a simpler kernel. Since $\nabla^2 G(z)$ is a symmetric matrix, the inner product

$$A : \nabla^2 G(z) = A^{\text{sym}} : \nabla^2 G(z).$$

depends only the symmetric part A^{sym} of A; hence w. l. o. g. let A be a symmetric matrix field. But then there exist functions $\zeta_{ij} \in C^\infty([0,\Lambda)^d)$, such that

$$A(x) = \sum_{i,j} \frac{1}{2} \zeta_{ij}(x) \left(e_i \otimes e_j + e_j \otimes e_i\right).$$

We also note

$$e_i \otimes e_j + e_j \otimes e_i = (e_i + e_j) \otimes (e_i + e_j) - (e_i \otimes e_i + e_j \otimes e_j).$$

Hence by linearity it is enough to prove the statement for A of the form

$$A(x) = \zeta(x)\, \xi \otimes \xi$$

for some $\xi \in S^{d-1}$. By rotational invariance we may assume

$$A(x) = \zeta(x)\, e_1 \otimes e_1.$$

Hence the statement can be reduced to

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta\, \chi_i\, h \partial_1^2 G_h * \chi_j \,dx = c_0 \sum_{i,j} \sigma_{ij} \int \zeta\, \nu_1^2\, \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) \tag{41}$$

for any $\zeta \in C^\infty([0,\Lambda)^d)$. In the following we will show that for any such test function ζ and any pair of characteristic functions $\chi,\, \tilde\chi \in BV([0,\Lambda)^d, \{0,1\})$ such that

$$\chi\, \tilde\chi = 0 \quad \text{a.e.} \tag{42}$$

and for the anisotropic kernel $k(z) = z_1^2 G(z)$ we have

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta\, \tilde\chi\, k_h * \chi \,dx = c_0 \int \zeta \left(\nu_1^2 + 1\right) \frac{1}{2} \left(|\nabla \chi| + |\nabla \tilde\chi| - |\nabla(\chi + \tilde\chi)|\right). \tag{43}$$

The analogous statement for the Gaussian kernel G instead of the anisotropic kernel k is – up to the localization with ζ – contained in [33]. In that case the right-hand side of (43) turns into the localized energy, i. e. replacing the anisotropic term $(\nu_1^2 + 1)$ by 1. Since $\partial_1^2 G(z) = \left(z_1^2 - 1\right) G(z)$

it is indeed sufficient to prove (43). We will prove this in five steps. Before starting, we introduce spherical coordinates $z = r\xi$ on the left-hand side:

$$\begin{aligned}\frac{1}{\sqrt{h}}\int \zeta\,\tilde{\chi}\,k_h * \chi\,dx =& \frac{1}{\sqrt{h}}\int k(z)\int \zeta(x)\tilde{\chi}(x)\chi(x+\sqrt{h}z)\,dx\,dz\\ =& \int_0^\infty G(r)r^{d+2}\frac{1}{\sqrt{h}r}\int_{S^{d-1}}\xi_1^2\int \zeta(x)\tilde{\chi}(x)\chi(x+\sqrt{h}r\xi)\,dx\,d\xi\,dr. \quad (44)\end{aligned}$$

In the following two steps of the proof, we simplify the problem by disintegrating in r (Step 2) and ξ (Step 3). Then we explicitly calculate an integral that arises in the second reduction and which translates the anisotropy of the kernel k into a geometric information about the normal (Step 4). We simplify further by disintegration in the vertical component (Step 5) and conclude by solving the one-dimensional problem (Step 6).

Step 2: Disintegration in r. We claim that it is sufficient to show

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{\sqrt{h}}\int_{S^{d-1}}\xi_1^2\int \zeta(x)\tilde{\chi}(x)\chi(x+\sqrt{h}\xi)\,dx\,d\xi\\ &\quad = \frac{|B^{d-1}|}{d+1}\int \zeta\left(\nu_1^2+1\right)\frac{1}{2}\left(|\nabla\chi|+|\nabla\tilde{\chi}|-|\nabla(\chi+\tilde{\chi})|\right). \quad (45)\end{aligned}$$

Indeed, note that since $G(z) = G(|z|)$ and $\frac{d}{dr}G(r) = -rG(r)$ we have, using integration by parts,

$$\int_0^\infty G(r)r^{d+2}\,dr = -\int_0^\infty \frac{d}{dr}(G(r))r^{d+1}\,dr = (d+1)\int_0^\infty G(r)r^d\,dr.$$

Replacing $\sqrt{h}$ by $\sqrt{h}\,r$ on the left-hand side of (45) and integrating w. r. t. the non-negative measure $G(r)r^d dr$ and using the equality from above shows that (45), in view of (44), formally implies (43). To make this step rigorous, we use Lebesgue's Dominated Convergence Theorem. A dominating function can be obtained as follows:

$$\begin{aligned}&\left|\frac{1}{\sqrt{h}r}\int_{S^{d-1}}\xi_1^2\int \zeta(x)\tilde{\chi}(x)\chi(x+\sqrt{h}r\xi)\,dx\,d\xi\right|\\ &\quad\overset{(42)}{=} \left|\frac{1}{\sqrt{h}r}\int_{S^{d-1}}\xi_1^2\int \zeta(x)\tilde{\chi}(x)\left(\chi(x+\sqrt{h}r\xi)-\chi(x)\right)\,dx\,d\xi\right|\\ &\quad\leq \|\zeta\|_\infty\frac{1}{\sqrt{h}r}\int_{S^{d-1}}\int\left|\chi(x+\sqrt{h}r\xi)-\chi(x)\right|\,dx\,d\xi\\ &\quad\leq \|\zeta\|_\infty\,|S^{d-1}|\int|\nabla\chi|,\end{aligned}$$

which is finite and independent of r. Hence, it is integrable w. r. t. the finite measure $G(r)r^{d+2}dr$.

Step 3: Disintegration in ξ. We claim that it is sufficient to show that for each $\xi \in S^{d-1}$,

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta(x)\tilde{\chi}(x) \left(\chi(x+\sqrt{h}\xi) + \chi(x-\sqrt{h}\xi)\right)\, dx$$
$$= \int \zeta \, |\xi\cdot\nu| \, \frac{1}{2} \left(|\nabla\chi| + |\nabla\tilde{\chi}| - |\nabla(\chi+\tilde{\chi})|\right). \tag{46}$$

Indeed, if we integrate w. r. t. the non-negative measure $\frac{1}{2}\xi_1^2 d\xi$ we obtain the left-hand side of (45) from the left-hand side of (46). At least formally, this is obvious because of the symmetry under $\xi \mapsto -\xi$. The dominating function to interchange limit and integration is obtained as in Step 1:

$$\left|\frac{1}{\sqrt{h}} \int \zeta(x)\tilde{\chi}(x) \left(\chi(x+\sqrt{h}\xi) + \chi(x-\sqrt{h}\xi)\right)\, dx\right|$$
$$\overset{(42)}{\leq} \frac{1}{\sqrt{h}} \sup|\zeta| \int \left|\chi(x+\sqrt{h}\xi) - \chi(x)\right| + \left|\chi(x-\sqrt{h}\xi) - \chi(x)\right|\, dx \leq 2\|\zeta\|_\infty \int |\nabla\chi|.$$

For the passage from the right-hand side of (46) to the right-hand side of (45) we note that since

$$\int_{S^{d-1}} \xi_1^2 \int \zeta \, |\xi\cdot\nu| \, |\nabla\chi| \frac{1}{2} d\xi = \frac{1}{2} \int \int_{S^{d-1}} \xi_1^2 \, |\xi\cdot\nu| \, d\xi \, \zeta \, |\nabla\chi|$$

and $|\nu| = 1$ $|\nabla\chi|$- a. e. it is enough to prove

$$\frac{1}{2} \int_{S^{d-1}} \xi_1^2 |\xi\cdot\nu| \, d\xi = \frac{|B^{d-1}|}{d+1} \left(\nu_1^2 + 1\right) \quad \text{for all } \nu \in S^{d-1} \tag{47}$$

to obtain the equality for the right-hand side.

Step 4: Argument for (47). By symmetry of $\int_{S^{d-1}} d\xi$ under the reflection that maps e_1 into ν, we have

$$\int_{S^{d-1}} \xi_1^2 |\xi\cdot\nu| \, d\xi = \int_{S^{d-1}} (\xi\cdot\nu)^2 |\xi_1| \, d\xi.$$

Applying the divergence theorem to the vector field $|\xi_1| \, (\xi\cdot\nu) \, \nu$, we have

$$\int_{S^{d-1}} (\xi\cdot\nu)^2 |\xi_1| \, d\xi = \int_B \nabla\cdot\left(|\xi_1| \, (\xi\cdot\nu) \, \nu\right) d\xi.$$

Since $\nabla\cdot\left(|\xi_1| \, (\xi\cdot\nu) \, \nu\right) = \operatorname{sign}\xi_1 \, (\xi\cdot\nu) \, \nu_1 + |\xi_1|$, the right-hand side is equal to

$$\left(\int_B \operatorname{sign}\xi_1 \, \xi \, d\xi\right) \cdot \nu \, \nu_1 + \int_B |\xi_1| \, d\xi.$$

By symmetry of $d\xi$ under rotations that leave e_1 invariant, we see that $\int_B \operatorname{sign}\xi_1 \, \xi \, d\xi$ points in direction e_1, so that the above reduces to

$$\left(\nu_1^2 + 1\right) \int_B |\xi_1| \, d\xi.$$

We conclude by observing

$$\int_B |\xi_1|\, d\xi = \int_{-1}^{1} |\xi_1|\, |B^{d-1}| \left(1-\xi_1^2\right)^{\frac{d-1}{2}} d\xi_1$$
$$= 2|B^{d-1}| \int_0^1 \frac{d}{d\xi_1}\left[-\frac{1}{d+1}\left(1-\xi_1^2\right)^{\frac{d-1}{2}}\right] d\xi_1 = 2\frac{|B^{d-1}|}{d+1}.$$

Step 5: One-dimensional reduction. The problem reduces to its one-dimensional analogue, namely: For all $\chi,\, \tilde\chi \in BV([0,\Lambda),\{0,1\})$ such that

$$\chi\,\tilde\chi = 0 \quad \text{a.e.} \tag{48}$$

and every $\zeta \in C^\infty([0,\Lambda))$ we have

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int_0^\Lambda \zeta(s)\tilde\chi(s)\left(\chi(s+\sqrt{h}) + \chi(s-\sqrt{h})\right) ds = \int_0^\Lambda \zeta\, \frac{1}{2}\Big(|\frac{d\chi}{ds}| + |\frac{d\tilde\chi}{ds}| - |\frac{d(\chi+\tilde\chi)}{ds}|\Big). \tag{49}$$

Indeed, by symmetry, it suffices to prove (46) for $\xi = e_1$. Using the decomposition $x = se_1 + x'$ we see that (46) follows from (49) using the functions $\chi_{x'}(s) := \chi(se_1 + x')$, $\tilde\chi_{x'}$, $\zeta_{x'}$ in (49) and integrating w. r. t. dx'. For the left-hand side, this is formally clear. For the right-hand side, one uses BV-theory: If $\chi \in BV([0,\Lambda)^d)$, we have $\chi_{x'} \in BV([0,\Lambda))$ for a. e. $x' \in [0,\Lambda)^{d-1}$ and

$$\int_{[0,\Lambda)^{d-1}} \int_0^\Lambda \zeta_{x'}(s)\, |\frac{d\chi_{x'}}{ds}|\, dx' = \int_{[0,\Lambda)^d} \zeta\, |e_1\cdot\nu|\, |\nabla\chi|$$

for any $\zeta \in C^\infty([0,\Lambda)^d)$. To make the argument rigorous, we use again Lebesgue's Dominated Convergence. As before, using (48), we obtain

$$\left|\frac{1}{\sqrt{h}} \int_0^\Lambda \zeta_{x'}(s)\tilde\chi_{x'}(s)\left(\chi_{x'}(s+\sqrt{h}) + \chi_{x'}(s-\sqrt{h})\right)\, ds\right|$$
$$\le \|\zeta\|_\infty \frac{1}{\sqrt{h}} \int_0^\Lambda \left|\chi_{x'}(s+\sqrt{h}) - \chi_{x'}(s)\right| + \left|\chi_{x'}(s-\sqrt{h}) - \chi_{x'}(s)\right|\, ds$$
$$\le 2\|\zeta\|_\infty \int_0^\Lambda |\frac{d\chi_{x'}}{ds}|.$$

Since

$$\int_{[0,\Lambda)^{d-1}} \int_0^\Lambda |\frac{d\chi_{x'}}{ds}| dx' = \int_{[0,\Lambda)^d} |e_1\cdot\nu|\, |\nabla\chi| \le \int_{[0,\Lambda)^d} |\nabla\chi|,$$

this is indeed an integrable dominating function.

Step 6: Argument for (49). Since $\chi,\ \tilde\chi$ are $\{0,1\}$-valued, every jump has height 1 and since $\chi,\ \tilde\chi \in BV([0,\Lambda))$, the total number of jumps is finite. Let $J,\ \tilde J \subset [0,\Lambda)$ denote the jump sets of χ and $\tilde\chi$, respectively. Now, if $\sqrt{h}$ is smaller than the minimal distance between two different points in $J\cup\tilde J$, then in view of (48), the only contribution to the left-hand side of (49) comes from neighborhoods of points where both, χ and $\tilde\chi$, jump:

$$\begin{aligned}\frac{1}{\sqrt{h}}\int_0^\Lambda \zeta(s)\,\tilde\chi(s)\left(\chi(s+\sqrt{h})+\chi(s-\sqrt{h})\right)\,ds \\ = \sum_{s\in J\cap\tilde J}\frac{1}{\sqrt{h}}\int_{s-\sqrt{h}}^{s+\sqrt{h}}\zeta(\sigma)\,\tilde\chi(\sigma)\left(\chi(\sigma+\sqrt{h})+\chi(\sigma-\sqrt{h})\right)\,d\sigma.\end{aligned}$$

Note that $\chi(\sigma+\sqrt{h})+\chi(\sigma-\sqrt{h})\equiv 1$ on each of these intervals and that

$$\tilde\chi = \mathbf{1}_{I_s^h} \quad \text{on } (s-\sqrt{h}, s+\sqrt{h})$$

for intervals of the form

$$I_s^h = (s-\sqrt{h}, s) \quad \text{or} \quad I_s^h = (s, s+\sqrt{h}).$$

Since $|I_s^h| = \sqrt{h}$, we have

$$\frac{1}{\sqrt{h}}\int_0^\Lambda \zeta(s)\tilde\chi(s)\left(\chi(s+\sqrt{h})+\chi(s-\sqrt{h})\right)\,ds = \sum_{s\in J\cap\tilde J}\frac{1}{\sqrt{h}}\int_{I_s^h}\zeta(\sigma)\,d\sigma \longrightarrow \sum_{s\in J\cap\tilde J}\zeta(s).$$

Note that by (48), $\chi+\tilde\chi$ jumps precisely where either χ or $\tilde\chi$ jumps. Thus

$$\int_0^\Lambda \zeta\,\frac{1}{2}\Big(|\frac{d\chi}{ds}|+|\frac{d\tilde\chi}{ds}|-|\frac{d(\chi+\tilde\chi)}{ds}|\Big) = \frac{1}{2}\Big(\sum_{s\in J}\zeta(s)+\sum_{s\in\tilde J}\zeta(s)-\sum_{s\in J\Delta\tilde J}\zeta(s)\Big) = \sum_{s\in J\cap\tilde J}\zeta(s).$$

Therefore, (49) holds, which concludes the proof. □

Proof of Lemma 3.7. The proof is divided into two steps. First, we prove the claim for $k = G$, to generalize this result for arbitrary kernels k in the second step.

Step 1: $k = G$. By Lemma 3.6 and the convergence assumption (38), we already know

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int G_h(z)\left(f_h(z)-f(z)\right)\,dz = 0.$$

Hence, it is sufficient to show that

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int G_h(z)\left(f(z)-f_h(z)\right)_+\,dz = 0.$$

Fix $h_0 > 0$ and $N \in \mathbb{N}$ and set $h := \frac{1}{N^2} h_0$. We will make use of the following triangle inequality for $f_{(h)} = f,\ f_h$:

$$f_{(h)}(z+w) \leq f_{(h)}(z) + f_{(h)}(w) \quad \text{for all } z,\ w \in \mathbb{R}^d. \tag{50}$$

This inequality has been proven in the proof of Lemma 3 in Section 7.1 of [33]. For the convenience of the reader we reproduce the argument here: Using the admissibility of χ in the form of $\sum_k \chi_k = 1$, we obtain the following identity for any pair $1 \leq i, j \leq P$ of phases and any points $x, x', x'' \in [0, \Lambda)^d$:

$$\begin{aligned}
&\chi_i(x)\chi_j(x'') - \chi_i(x)\chi_j(x') - \chi_i(x')\chi_j(x'') \\
&\quad = \chi_i(x)\sum_k \chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\sum_k \chi_k(x'') - \sum_k \chi_k(x)\chi_i(x')\chi_j(x'') \\
&\quad = \sum_k \left[\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right].
\end{aligned}$$

Note that the contribution of $k \in \{i, j\}$ to the sum has a sign:

$$\begin{aligned}
&\sum_{k\in\{i,j\}} \left[\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right] \\
&\qquad = \chi_i(x)\chi_i(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_i(x'') - \chi_i(x)\chi_i(x')\chi_j(x'') \\
&\qquad\quad + \chi_i(x)\chi_j(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_j(x'') - \chi_j(x)\chi_i(x')\chi_j(x'') \\
&\qquad = -\left[\chi_i(x)\chi_j(x')\chi_i(x'') + \chi_j(x)\chi_i(x')\chi_j(x'')\right] \leq 0.
\end{aligned}$$

We now fix $z,\ w \in \mathbb{R}^d$ and use the above inequality for $x' = x + z$, $x'' = x + z + w$ so that after multiplication with σ_{ij}, summation over $1 \leq i, j \leq P$ and integration over x, we obtain $f(z+w) - f(z) - f(w)$ on the left-hand side. Indeed, using the translation invariance for the term appearing in $f_\zeta(w)$, we have

$$\begin{aligned}
&f(z+w) - f(z) - f(w) \\
&= \int \sum_{i\neq j} \sigma_{ij}\left[\chi_i(x)\chi_j(x+z+w) - \chi_i(x)\chi_j(x+z) - \chi_i(x+z)\chi_j(x+z+w)\right] dx \\
&\leq \int \sum_{i\neq j, k\neq i,j} \sigma_{ij}\big[\chi_i(x)\chi_k(x+z)\chi_j(x+z+w) - \chi_i(x)\chi_j(x+z)\chi_k(x+z+w) \\
&\qquad\qquad\qquad - \chi_k(x)\chi_i(x+z)\chi_j(x+z+w)\big] dx.
\end{aligned}$$

Using the triangle inequality for the surface tensions, we see that the first right-hand side integral is non-positive:

$$\begin{aligned}
&\sum_{i\neq j,k\neq i,j} \sigma_{ij}\left(\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right)\\
&\quad \leq \sum_{i\neq j,k\neq i,j} \sigma_{ik}\chi_i(x)\chi_k(x')\chi_j(x'') + \sum_{i\neq j,k\neq i,j} \sigma_{kj}\chi_i(x)\chi_k(x')\chi_j(x'')\\
&\qquad - \sum_{i\neq j,k\neq i,j} \sigma_{ij}\chi_i(x)\chi_j(x')\chi_k(x'') - \sum_{i\neq j,k\neq i,j} \sigma_{ij}\chi_k(x)\chi_i(x')\chi_j(x'') = 0.
\end{aligned}$$

Indeed, the first and the third term, and the second and the last term cancel since the domain of indices in the sums is symmetric and thus we have (50).

By iterating the triangle inequality (50) for $f_{(h)} = f,\ f_h$ we have

$$f_{(h)}(Nz) \leq N f_{(h)}(z) \quad \text{for all } z \in \mathbb{R}^d.$$

Hence, by the definition of h,

$$\frac{1}{\sqrt{h_0}} f_{(h)}(\sqrt{h_0}z) \leq \frac{1}{\sqrt{h}} f_{(h)}(\sqrt{h}z) \quad \text{for all } z \in \mathbb{R}^d. \tag{51}$$

Therefore, using (51) for f_h, the subadditivity of $u \mapsto u_+$ and finally (51) for f, we obtain

$$\begin{aligned}
&\Big(\frac{1}{\sqrt{h}} f(\sqrt{h}z) - \frac{1}{\sqrt{h}} f_h(\sqrt{h}z)\Big)_+\\
&\qquad \leq \Big(\frac{1}{\sqrt{h}} f(\sqrt{h}z) - \frac{1}{\sqrt{h_0}} f_h(\sqrt{h_0}z)\Big)_+\\
&\qquad \leq \Big(\frac{1}{\sqrt{h}} f(\sqrt{h}z) - \frac{1}{\sqrt{h_0}} f(\sqrt{h_0}z)\Big)_+ + \Big(\frac{1}{\sqrt{h_0}} f(\sqrt{h_0}z) - \frac{1}{\sqrt{h_0}} f_h(\sqrt{h_0}z)\Big)_+\\
&\qquad \leq \frac{1}{\sqrt{h}} f(\sqrt{h}z) - \frac{1}{\sqrt{h_0}} f(\sqrt{h_0}z) + \frac{1}{\sqrt{h_0}} \Big| f(\sqrt{h_0}z) - f_h(\sqrt{h_0}z)\Big|.
\end{aligned}$$

Integrating w. r. t. the positive measure $G(z)\,dz$ yields

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int G_h(z)\left(f(z) - f_h(z)\right)_+ dz \leq& \frac{1}{\sqrt{h}} \int G(z) f(\sqrt{h}z)\,dz - \frac{1}{\sqrt{h_0}} \int G(z) f(\sqrt{h_0}z)\,dz\\
&+ \frac{1}{\sqrt{h_0}} \int G(z) \Big| f(\sqrt{h_0}z) - f_h(\sqrt{h_0}z)\Big|\,dz\\
=& E_h(\chi) - E_{h_0}(\chi) + \frac{1}{\sqrt{h_0}} \int G_{h_0}(z)\,|f(z) - f_h(z)|\,dz.
\end{aligned} \tag{52}$$

Given $\delta > 0$, by Lemma 3.6 we may first choose $h_0 > 0$ such that for all $0 < h < h_0$:

$$|E_h(\chi) - E_{h_0}(\chi)| < \frac{\delta}{2}.$$

We note that we may now choose $N \in \mathbb{N}$ so large that for all $0 < h < \frac{1}{N^2} h_0$:

$$\left| f(\sqrt{h_0} z) - f_h(\sqrt{h_0} z) \right| \leq \frac{\delta}{2} \sqrt{h_0} \quad \text{for all } z \in \mathbb{R}^d.$$

Indeed, using the triangle inequality and translation invariance we have

$$\begin{aligned}
&\left| f(\sqrt{h_0} z) - f_h(\sqrt{h_0} z) \right| \\
&\leq \sum_{i,j} \sigma_{ij} \int \left| \chi_i(x)\chi_j(x+z) - \chi_i(x)\chi_j^h(x+z) \right| + \left| \chi_i(x)\chi_j^h(x+z) - \chi_i^h(x)\chi_j^h(x+z) \right| dx \\
&\lesssim \sum_{i=1}^{P} \int \left| \chi_i(x) - \chi_i^h(x) \right| dx,
\end{aligned}$$

which tends to zero as $h \to 0$ because by Lebesgue's Dominated Convergence and (37). Hence also the second term on the right-hand side of (52) is small:

$$\frac{1}{\sqrt{h_0}} \int G_{h_0}(z) \left| f(z) - f_h(z) \right| dz \leq \frac{\delta}{2}.$$

Step 2: $k = p\,G$. Fix $\epsilon > 0$. Since G is exponentially decaying, we can find a number $M = M(\epsilon) < \infty$ such that

$$k(z) \leq \epsilon\, G\big(\frac{z}{\sqrt{2}}\big) = \epsilon\, G_2(z) \quad \text{for all } |z| > M. \tag{53}$$

Hence we can split the integral into two parts. On the one hand, using (40) for $k = G$,

$$\begin{aligned}
&\frac{1}{\sqrt{h}} \int_{\{|z| \leq M\}} k(z) \left| f_h(\sqrt{h} z) - f(\sqrt{h} z) \right| dz \\
&\qquad \leq \left(\sup_{[0,M]} p \right) \frac{1}{\sqrt{h}} \int G(z) |f_h(\sqrt{h} z) - f(\sqrt{h} z)|\, dz \to 0
\end{aligned}$$

as $h \to 0$, and on the other hand, using (53) and the approximate monotonicity in Lemma 2.10,

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int_{\{|z| > M\}} k(z) \left| f_h(\sqrt{h} z) - f(\sqrt{h} z) \right| dz \leq& \epsilon \frac{1}{\sqrt{h}} \int G_2(z) \left(f_h(\sqrt{h} z) + f(\sqrt{h} z) \right) dz \\
\lesssim& \epsilon \left(E_h(\chi^h) + E_h(\chi) \right).
\end{aligned}$$

By the convergence assumption (38) and the consistency, cf. Lemma 2.11, we can take the limit $h \to 0$ on the right-hand side and obtain

$$\limsup_{h \to 0} \frac{1}{\sqrt{h}} \int k_h(z) |f_h(z) - f(z)|\, dz \lesssim \epsilon \sum_{i,j} \sigma_{ij} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right).$$

Since the left-hand side does not depend on $\epsilon > 0$, this implies (40). $\square$

4 Dissipation functional and velocity

As for any minimizing movements scheme, the time derivative of the solution should arise from the metric term in the minimization scheme. For the minimizing movements scheme of our interfacial motion, the time derivative is the normal velocity. The goal of this section, which is the core of the chapter, is to compare the first variation of the dissipation functional to the normal velocity.

4.1 Idea of the proof

Let us first give an idea of the proof in a simplified setting with only two phases, a constant test vector field ξ and no localization. Then the first variation (33) of the metric term reads

$$\frac{2}{\sqrt{h}} \int \left(\chi^n - \chi^{n-1}\right) G_h * \left(-\nabla \chi^n \cdot \xi\right) dx.$$

Using the distributional equation $\nabla \chi \cdot \xi = \nabla \cdot (\chi \xi) - (\nabla \cdot \xi)\, \chi$, this is equal to

$$\frac{2}{\sqrt{h}} \int \left(\chi^n - \chi^{n-1}\right) \left(-\nabla G_h * (\chi^n \xi) + G_h * (\chi^n \xi)\right) dx \approx -2 \int \frac{\chi^n - \chi^{n-1}}{h}\, \xi \cdot \sqrt{h} \nabla G_h * \chi^n dx$$

as $h \to 0$. We will prove this in Lemma 4.7. Since $\partial_t^{-h} \chi^h = \frac{\chi^n - \chi^{n-1}}{h} \rightharpoonup V \left|\nabla \chi\right| dt$ and $\sqrt{h} \nabla G_h * \chi^n \approx c_0 \nu$ only in a weak sense, we cannot pass to the limit a priori. Our strategy is to freeze the normal and to control

$$\int_0^T \int \partial_t^{-h} \chi^h\, \xi \cdot \sqrt{h} \nabla G_h * \chi^h \, dx\, dt - \int_0^T \int \partial_t^{-h} \chi^h\, c_0\, \xi \cdot \nu^* \, dx\, dt \tag{54}$$

by the excess

$$\varepsilon^2 := \int_0^T \left(E_h(\chi^h) - E_h(\chi^*) \right) dt,$$

where $\chi^* = \mathbf{1}_{\{x \cdot \nu^* > \lambda\}}$ is a half space in direction of ν^*. By the convergence assumption ε^2 converges to

$$\mathscr{E}^2 := c_0 \int_0^T \left(\int \left|\nabla \chi\right| - \int \left|\nabla \chi^*\right| \right) dt,$$

as $h \to 0$, which is small by De Giorgi's Structure Theorem – at least after localization in space and time; i. e. sets of finite perimeter have (approximate) tangent planes almost everywhere. To be self-consistent we will prove this application of De Giorgi's result in Section 5.

The main difficulty in controlling (54) lies in finding good bounds on

$$\int_0^T \int \left| \partial_t^h \chi^h \right| dx\, dt.$$

For the sake of simplicity we set $E_0 = T = \Lambda = 1$ and write χ instead of χ^h in the following. In Section 2 we have seen the bound

$$\iint |\partial_t^\tau \chi| \, dx \, dt = O(1) \quad \text{for } \tau \sim \sqrt{h}. \tag{55}$$

For this, we used the energy-dissipation estimate (10) to bound the dissipation

$$\sqrt{h} \iint \left(G_{h/2} * \partial_t^h \chi \right)^2 dx \, dt = \sum_n \frac{1}{\sqrt{h}} \int \left(\chi^n - \chi^{n-1} \right) G_h * \left(\chi^n - \chi^{n-1} \right) dx \lesssim 1$$

and Jensen's inequality gave us control over the function

$$\alpha^2(t) := \frac{1}{\sqrt{h}} \int \left(G_{h/2} * (\chi(t+\tau) - \chi(t)) \right)^2 dx = \alpha^2 \sqrt{h} \int \left(G_{h/2} * \partial_t^\tau \chi \right)^2 dx \tag{56}$$

by the fudge factor α appearing in the definition of the mesoscopic time scale $\tau = \alpha\sqrt{h}$:

$$\int_0^T \alpha^2(t) \, dt \lesssim \alpha^2. \tag{57}$$

This estimate is the reason for the slight abuse of notation: We call the function in (56) $\alpha^2(t)$ in order to keep the relation (57) between the two quantities in mind. In the following we will always carry along the argument t of the function $\alpha^2(t)$ to make the difference clear. Writing χ^τ short for $\chi(\,\cdot\, + \tau)$ we have shown in the proof of Lemma 2.5 that (55) holds in the more precise form of

$$\iint |\chi^\tau - \chi| \, dx \, dt \lesssim \sqrt{h} \int \alpha^2(t) \, dt + \sqrt{h} \int E_h(\chi) \, dt \lesssim \sqrt{h} \left(\varepsilon^2 + 1 \right) + \frac{\tau^2}{\sqrt{h}}. \tag{58}$$

In this section we will derive the following more subtle bound:

$$\iint |\partial_t^\tau \chi| \, dx \, dt = O(1) \quad \text{for } \tau = o(\sqrt{h}). \tag{59}$$

While the argument for (55) was based on

$$\chi^\tau - \chi = G_h * (\chi^\tau - \chi) + (\chi - G_h * \chi) + (\chi^\tau - G_h * \chi^\tau)$$

we now start from the thresholding scheme:

$$\chi^\tau - \chi = \mathbf{1}_{\{u^\tau > \frac{1}{2}\}} - \mathbf{1}_{\{u > \frac{1}{2}\}} \quad \text{with} \quad u^\tau := G_h * \chi^{\tau - h} \quad \text{and} \quad u := G_h * \chi^{-h}.$$

We will use an elementary one-dimensional estimate, Lemma 4.2 (cf. Corollary 79 for this rescaled version), in direction $\nu^* = e_1$ (w. l. o. g.) and integrate transversally to obtain

$$\frac{1}{\sqrt{h}} \int |\chi^\tau - \chi| \, dx \lesssim \frac{1}{\sqrt{h}} \int_{\frac{1}{3} \le u \le \frac{2}{3}} \left(\sqrt{h} \partial_1 u - \overline{c} \right)_-^2 dx + s + \frac{1}{s^2} \frac{1}{\sqrt{h}} \int (u^\tau - u)^2 dx. \tag{60}$$

The first right-hand side term measures the monotonicity of the phase function u in normal direction in the transition zone $\{\frac{1}{3} \leq u \leq \frac{2}{3}\}$. It is clear that this term vanishes for $\chi^{-h} = \chi^*$, provided the universal constant $\overline{c} > 0$ is sufficiently small. In Lemma 4.4 we will indeed bound this term by the excess

$$\varepsilon^2(-h) := E_h(\chi^{-h}) - E_h(\chi^*)$$

at the previous time step. Compared to the first approach which yielded (58), where the limiting factor is that the first right-hand side term is only $O(\sqrt{h})$, the result of the latter approach yields the improvement

$$\iint |\chi^\tau - \chi| \, dx \, dt \lesssim \sqrt{h} \left(\varepsilon^2 + s\right) + \frac{1}{s^2} \frac{\tau^2}{\sqrt{h}} \tag{61}$$

for an arbitrary (small) parameter $s > 0$. Now we show how to use the bound (61) in order to estimate (54). First, in Lemma 4.7 by freezing time for χ on the mesoscopic time scale $\tau = \alpha\sqrt{h}$ and using a telescoping sum for the first term $\partial_t^h \chi$ we will show that

$$\begin{aligned} \iint \partial_t^h \chi \, \xi \cdot \sqrt{h} \nabla G_h * \chi \, dx \, dt = & \iint \partial_t^\tau \chi \, \xi \cdot \sqrt{h} \nabla G_h * \frac{\chi + \chi^\tau}{2} \, dx \, dt \\ & + O\left(\left(\frac{\tau}{\sqrt{h}} \iint |\partial_t^\tau \chi| \, dx \, dt \right)^{\frac{1}{2}} \right). \end{aligned} \tag{62}$$

By (61) the error term is controlled by

$$\left(\varepsilon^2 + s + \frac{1}{s^2} \alpha^2 \right)^{\frac{1}{2}} \lesssim \frac{1}{\alpha} \varepsilon^2 + \alpha^{\frac{1}{3}} \tag{63}$$

by choosing $s \sim \alpha^{\frac{2}{3}}$. Second, in Lemma 4.8 we will show how to use the algebraic relation $(\chi^\tau - \chi)(\chi^\tau + \chi) = \chi^\tau - \chi$ for the product $(\chi^\tau - \chi)\sqrt{h}\nabla G_h * (\chi^\tau + \chi)$ so that we can rewrite the right-hand side of (62) as

$$\iint \partial_t^\tau \chi \, c_0 \, \xi \cdot e_1 \, dx \, dt + O\left(\iint |\partial_t^\tau \chi| \ k_h * |\chi^\tau - \chi| \, dx \, dt \right) + O\left(\varepsilon^2\right) \tag{64}$$

for some kernel k. Third, in Lemma 4.9 we will control the first error term by using its quadratic structure and the estimate (61) before the transversal integration in x':

$$\begin{aligned} \int |\partial_t^\tau \chi| \ k_h * |\chi^\tau - \chi| \, dx & \lesssim \frac{1}{\tau} \int \left(\int |\chi^\tau - \chi| \, dx_1 \right) k_h' *' \left[1 \wedge \left(\frac{1}{\sqrt{h}} \int |\chi^\tau - \chi| \, dx_1 \right) \right] dx' \\ & \lesssim \frac{1}{\alpha} \left[\varepsilon^2 + \frac{1}{s^2} \alpha^2 + s \left(\sqrt{h} \left(\tilde{s} + \varepsilon^2 \right) + \frac{1}{\tilde{s}^2} \alpha^2 \right) \right] \\ & \lesssim \frac{1}{\alpha} \varepsilon^2 + \frac{1}{\alpha} s \tilde{s} + \left(\frac{s}{\tilde{s}^2} + \frac{1}{s^2} \right) \alpha \sim \frac{1}{\alpha} \varepsilon^2 + \alpha^{\frac{1}{9}}, \end{aligned} \tag{65}$$

by choosing $\tilde{s} \sim \alpha^{\frac{2}{3}}$ and $s \sim \alpha^{\frac{4}{9}}$. We note that the values of the exponents of α in (63) and (65) do not play any role and can be easily improved. We only need the extra terms, here $\alpha^{\frac{1}{3}}$ and $\alpha^{\frac{1}{9}}$, to be $o(1)$ as $\alpha \to 0$; the prefactor of the excess ε^2, here $\frac{1}{\alpha}$, can be large. Indeed, after sending $h \to 0$ we will obtain the error $\frac{1}{\alpha}\mathscr{E}^2 + \alpha^{\frac{1}{9}}$. We will handle this term in Section 5 by first sending the fineness of the localization to zero so that $\mathscr{E}^2$ vanishes, and then sending the parameter $\alpha \to 0$.

In the following we will make the above steps rigorous and give a full proof in the multi-phase case. First we state the main result, Proposition 4.1, then we explain the tools we will be using more carefully in the subsequent lemmas. We turn first to the two-phase case to present the one-dimensional estimate (60) in Lemma 4.2, its rescaled and localized version Corollary 4.3 and the estimate for the error term Lemma 4.4. Subsequently we state the same results in Lemma 4.5 and Corollary 4.6 for the multi-phase case. These estimates are the core of the proof of Proposition 4.1 and use the explicit structure of the scheme. Let us note that in these estimates we are using the two steps of the scheme, the convolution step (1) and the thresholding step (2), in a well-separated way. Indeed, the one-dimensional estimate, Lemma 4.5, analyzes the thresholding step (2); and Corollary 4.6 brings the (transversally integrated) error term in the form of the excess ε^2 at the previous time step by analyzing the convolution step (1).

4.2 Results

The main result of this section is the following proposition which will be used for small time intervals in Section 5 where we will control the limiting error terms which appear here with soft arguments from Geometric Measure Theory. In view of the definition of $\mathscr{E}^2$ below, the proposition assumes that $\chi_3, \ldots, \chi_P$ are the *minority phases* in the space-time cylinder $(0,T) \times B_r$; likewise it assumes that the normal between χ_1 and χ_2 is close to the first unit vector e_1. This can be assumed since on the one hand we can relabel the phases in case we want to treat another pair of phases as the majority phases. On the other hand, due to the rotational invariance, it is no restriction to assume that e_1 is the approximate normal.

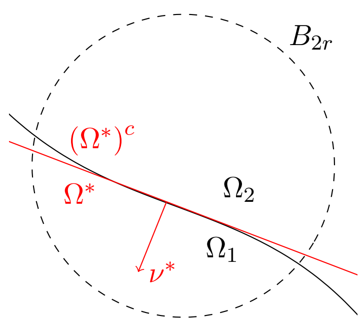

Figure 1.7: The majority phases Ω_1 and Ω_2 and the half space $\Omega^* = \{x \cdot \nu^* > \lambda\}$ approximating Ω_1 inside the ball B_{2r}. Its complement $(\Omega^*)^c$ approximates Ω_2 inside B_{2r}.

Proposition 4.1. *For any $\alpha \ll 1$, $T>0$, $\xi \in C_0^\infty((0,T)\times B_r,\mathbb{R}^d)$ and any $\eta \in C_0^\infty(B_{2r})$ radially symmetric and radially non-increasing cut-off for B_r in B_{2r} with $|\nabla \eta| \lesssim \frac{1}{r}$ and $\left|\nabla^2\eta\right| \lesssim \frac{1}{r^2}$, we have*

$$\limsup_{h\to 0}\left|\int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t),\xi(t)) + 2c_0\sigma_{12}\left(\int \xi_1\, V_1\, |\nabla \chi_1| - \int \xi_1\, V_2\, |\nabla\chi_2|\right)dt\right|$$
$$\lesssim \|\xi\|_\infty \left[\int_0^T \left(\frac{1}{\alpha^2}\mathscr{E}^2(t) + \alpha^{\frac19} r^{d-1}\right)dt + \alpha^{\frac19}\iint \eta\, d\mu\right]. \qquad (66)$$

Here we use the notation

$$\mathscr{E}^2(t) := \sum_{i=3}^P \int \eta\, |\nabla \chi_i(t)| + \inf_{\chi^*}\Bigg\{\left|\int \eta\,(|\nabla\chi_1(t)| - |\nabla\chi^*|)\right| + \frac{1}{r}\int_{B_{2r}} |\chi_1(t)-\chi^*|\,dx$$
$$+\left|\int \eta\,(|\nabla\chi_2(t)| - |\nabla\chi^*|)\right| + \frac{1}{r}\int_{B_{2r}} |\chi_2(t)-(1-\chi^*)|\,dx\Bigg\},$$

where the infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x_1>\lambda\}}$ in direction e_1.

The exponents of α in this statement are of no importance and can be easily improved. It is only relevant that the two extra error terms, i. e. $r^{d-1}T$ and $\iint \eta\, d\mu$, are equipped with prefactors which vanish as $\alpha \to 0$. In Section 5 we will show that – even after summation – the excess will vanish as the fineness of the localization, i. e. the radius r of the ball in the statement of Proposition 4.1 tends to zero. There we will take first the limit $r\to 0$ and then $\alpha \to 0$ to prove Theorem 1.4. The prefactor of the excess, here $\frac{1}{\alpha^2}$ differs from the one in the two-phase case since the one-dimensional estimate is slightly different in the multi-phase case.

Let us comment on the structure of $\mathscr{E}^2$. The first term, describing the surface area of Phases $3,\dots,P$ inside the ball B_{2r}, will be small in the application when $\chi_3,\dots,\chi_P$ are indeed the minority phases. The second term, sometimes called the *excess energy* describes how far χ_1 and χ_2 are away from being half spaces in direction e_1 or $-e_1$, respectively. The terms comparing the surface energy inside B_{2r} do not see the orientation of the normal, whereas the bulk terms measuring the L^1-distance inside the ball B_{2r} do see the orientation of the normal.

The estimates in Section 2 are not sufficient to understand the link between the first variation of the metric term and the normal velocities. For this, we need refined estimates which we will first present for the two-phase case, where only one interface evolves. The main tool of the proof is the following one-dimensional lemma. For two functions u, $\tilde u$, it estimates the L^1-distance between the characteristic functions $\chi = \mathbf{1}_{\{u\geq \frac12\}}$ and $\tilde\chi = \mathbf{1}_{\{\tilde u\geq\frac12\}}$ in terms of the L^2-distance between the u's - at the expense of a term that measures the strict monotonicity of one of the functions u. We will apply it in a rescaled version for x_1 being the normal direction.

Lemma 4.2. *Let $I\subset\mathbb{R}$ be an interval, Let $u,\ \tilde u \in C^{0,1}(I)$, $\chi := \mathbf{1}_{\{u\geq\frac12\}}$ and $\tilde\chi := \mathbf{1}_{\{\tilde u\geq\frac12\}}$. Then*

$$\int_I |\chi-\tilde\chi|\,dx_1 \lesssim \int_{\{|u-\frac12|<s\}} (\partial_1 u - 1)_-^2\,dx_1 + s + \frac{1}{s^2}\int_I (u-\tilde u)^2\,dx_1 \qquad (67)$$

for every $s>0$.

The following modified version of Lemma 4.2 is the estimate one would use in the two-phase case.

Corollary 4.3. *Let u, $\tilde u \in C^{0,1}(I)$, $\chi := \mathbf{1}_{\{u\geq \frac12\}}$, $\tilde\chi := \mathbf{1}_{\{\tilde u\geq \frac12\}}$ and $\eta\in C_0^\infty(\mathbb{R})$, $0\leq\eta\leq 1$ radially non-increasing. Then*

$$\frac{1}{\sqrt{h}}\int \eta\,|\chi-\tilde\chi|\,dx_1 \lesssim \frac{1}{\sqrt{h}}\int_{\{|u-\frac12|<s\}}\eta\left(\sqrt{h}\,\partial_1 u-1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int\eta\,(u-\tilde u)^2\,dx_1$$

for any $s>0$.

In the previous corollary, it was crucial to control strict monotonicity of one of the two functions via the term

$$\frac{1}{\sqrt{h}}\int_{\{|u-\frac12|<s\}}\eta\left(\sqrt{h}\,\partial_1 u-1\right)_-^2 dx_1.$$

In the following lemma, we consider the d-dimensional version, i. e. dx_1 replaced by dx, of this term in case of $u=G_h*\chi$. We show that this term can be controlled in terms of the excess, measuring the energy difference to a half space χ^* in direction e_1.

Lemma 4.4. *Let $\chi\colon[0,\Lambda)^d\to\{0,1\}$, $\chi^*=\mathbf{1}_{\{x_1>\lambda\}}$ a half space in direction e_1 and $\eta\in C_0^\infty(B_{2r})$ a cut-off of B_r in B_{2r} with $|\nabla\eta|\lesssim\frac1r$ and $\left|\nabla^2\eta\right|\lesssim\frac{1}{r^2}$. Then there exists a universal constant $\overline{c}>0$ such that*

$$\frac{1}{\sqrt{h}}\int_{\{z_1\lessgtr 0\}} G_h(z)\int\eta(x)\,(\chi(x+z)-\chi(x))_\pm\,dx\,dz \lesssim \varepsilon^2+\sqrt{h}\frac{1}{r^2}, \tag{68}$$

$$\frac{1}{\sqrt{h}}\int_{\{\frac13\leq G_h*\chi\leq\frac23\}}\eta\left(\sqrt{h}\,\partial_1(G_h*\chi)-\overline{c}\right)_-^2 dx \lesssim \varepsilon^2+\sqrt{h}\frac{1}{r^2}+\sqrt{h}\frac{1}{r}E_h(\chi), \tag{69}$$

where ε^2 is defined via

$$\begin{aligned}\varepsilon^2 :=& \frac{1}{\sqrt{h}}\int\eta\left[(1-\chi)\,G_h*\chi+\chi\,G_h*(1-\chi)\right]dx\\ &-\frac{1}{\sqrt{h}}\int\eta\left[(1-\chi^*)\,G_h*\chi^*+\chi^*\,G_h*(1-\chi^*)\right]dx+\frac1r\int_{B_{2r}}|\chi-\chi^*|\,dx\end{aligned}$$

and the integral on the left-hand side of (68) with the two cases $<,+$ and $>,-$, respectively is a short notation for the sum of the two integrals.

In our application, we use the following lemma which is valid for any number of phases with arbitrary surface tensions instead of Lemma 4.2 or Corollary 4.3. Nevertheless, the core of the proof is already contained in the respective estimates in the two-phase case above. As in Proposition 4.1, we assume that χ_1 and χ_2 are the majority phases and that e_1 is the approximate normal to $\Omega_1=\{\chi_1=1\}$.

Lemma 4.5. *Let $I \subset \mathbb{R}$ be an interval, $h > 0$, $\eta \in C_0^\infty(\mathbb{R})$, $0 \le \eta \le 1$ radially non-increasing and $u,\, \tilde u\colon I \to \mathbb{R}^P$ be two smooth maps into the standard simplex $\{U_i \ge 0,\, \sum_i U_i = 1\} \subset \mathbb{R}^P$. We define $\phi_i := \sum_j \sigma_{ij} u_j$, $\tilde\phi_i := \sum_j \sigma_{ij}\tilde u_j$, $\chi_i := \mathbf{1}_{\{\phi_i < \phi_j\, \forall j \neq i\}}$ and $\tilde\chi_i := \mathbf{1}_{\{\tilde\phi_i < \tilde\phi_j\, \forall j \neq i\}}$. Then*

$$\begin{aligned}\frac{1}{\sqrt h}\int \eta\,|\chi-\tilde\chi|\,dx_1 \lesssim & \frac{1}{\sqrt h}\int_{\{\frac13 \le u_1 \le \frac23\}} \left(\sqrt h \partial_1 u_1 - \overline{c}\right)_-^2 dx_1 + \frac1s \frac{1}{\sqrt h}\sum_{j\ge 3}\int \eta\,[u_j \wedge (1-u_j)]\,dx_1 \\ & + s + \frac{1}{s^2}\frac{1}{\sqrt h}\int \eta\,|u-\tilde u|^2\,dx_1 \end{aligned} \tag{70}$$

for any $s \ll 1$.

As Lemma 4.4 can be used to estimate the integrated version of the error in Corollary 4.3 against the excess, the following corollary shows that the integrated version of the corresponding error term in the multi-phase version, Lemma 4.5, can be estimated against a multi-phase version of the excess ε^2.

Corollary 4.6. *Let χ be admissible, $\chi^* = \mathbf{1}_{\{x_1 > \lambda\}}$ a half space in direction e_1 and $\eta \in C_0^\infty(B_{2r})$ a cut-off of B_r in B_{2r} with $|\nabla \eta| \lesssim \frac1r$ and $\left|\nabla^2\eta\right| \lesssim \frac{1}{r^2}$. Then there exists a universal constant $\overline{c} > 0$ such that for $u = G_h * \chi$*

$$\begin{aligned}\frac{1}{\sqrt h}\int_{\{\frac13 \le u_1 \le \frac23\}} \eta \left(\sqrt h\, \partial_1 u_1 - \overline{c}\right)_-^2 dx + \frac{1}{\sqrt h}\sum_{j\ge3}\int \eta\,[u_j\wedge(1-u_j)]\,dx & \\ \lesssim \varepsilon^2(\chi) + \frac{\sqrt h}{r^2} + \frac{\sqrt h}{r}E_h(\chi), &\end{aligned}$$

where the functional $\varepsilon^2(\chi)$ is defined via

$$\begin{aligned}\varepsilon^2(\chi) := \sum_{i\ge 3} F_h(\chi_j,\eta) + & F_h(\chi_1,\eta) - F_h(\chi^*,\eta) + \frac1r\int_{B_{2r}} |\chi_1 - \chi^*|\,dx \\ & + F_h(\chi_2,\eta) - F_h(\chi^*,\eta) + \frac1r \int_{B_{2r}} |\chi_2 - (1-\chi^*)|\,dx\end{aligned}$$

and the functional F_h is the following localized version of the approximate energy in the two-phase case

$$F_h(\tilde\chi,\eta) := \frac{1}{\sqrt h}\int \eta\,[(1-\tilde\chi)\,G_h * \tilde\chi + \tilde\chi\, G_h * (1-\tilde\chi)]\,dx, \quad \tilde\chi \in \{0,1\}.$$

With these tools we can now turn to the rigorous proof of (62)–(65) in the following lemmas. In the next two lemmas, we approximate the first variation of the metric term by an expression that makes the normal velocity appear. The main idea is to work, as for Lemma 2.5, on a mesoscopic time scale $\tau \sim \sqrt h$, introducing a fudge factor α, cf. Remark 1.6. The first lemma shows that we may coarsen the first variation from the microscopic time scale h to the mesoscopic time scale $\alpha\sqrt h$ and is therefore the rigorous analogue of (62). It also shows that we may pull the test vector field ξ out of the convolution.

Lemma 4.7. *Let $\xi \in C_0^\infty((0,T)\times B_r,\mathbb{R}^d)$. Then*

$$\int_0^T -\delta E_h(\,\cdot - \chi^h(t-h))(\chi^h(t),\xi(t))\,dt$$
$$\approx \sum_{i,j}\sigma_{ij}\tau\sum_{l=1}^L\int \frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx$$

in the sense that the error is controlled by

$$\|\xi\|_\infty\left(\frac{1}{\alpha}\tau\sum_{l=1}^L\varepsilon^2(\chi^{Kl-1})+\alpha^{\frac13}r^{d-1}T+\alpha^{\frac13}\iint\eta\,d\mu_h\right)+o(1),\quad \text{as } h\to 0,$$

where $\eta\in C_0^\infty(B_{2r})$ is a radially symmetric, radially non-increasing cut-off for B_r in B_{2r} with $|\nabla\eta|\lesssim\frac1r$ and the functional $\varepsilon^2(\chi)$ is defined in Corollary 4.6.

While the first lemma made the mesoscopic time derivative $\frac1\tau\big(\chi_i^{Kl}-\chi_i^{K(l-1)}\big)$ appear, the upcoming second lemma makes the approximate normal, here e_1, appear. This is the analogue of (64).

Lemma 4.8. *Given ξ and η as in Lemma 4.7 we have*

$$\sum_{i,j}\sigma_{ij}\,\tau\sum_{l=1}^L\int\frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx$$
$$\approx -2c_0\,\sigma_{12}\,\tau\sum_{l=1}^L\left(\int\xi_1(l\tau)\,\frac{\chi_1^{Kl}-\chi_1^{K(l-1)}}{\tau}dx-\int\xi_1(l\tau)\,\frac{\chi_2^{Kl}-\chi_2^{K(l-1)}}{\tau}dx\right),$$

in the sense that the error is controlled by $\|\xi\|_\infty$ times

$$\frac1\alpha\tau\sum_{l=1}^L\varepsilon^2(\chi^{Kl})+\alpha\iint\eta\,d\mu_h$$
$$+\tau\sum_{l=1}^L\frac1\tau\int\eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|k_h*\left(\eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|\right)dx+o(1),$$

as $h\to0$, where $0\le k(z)\le|z|G(z)$ and the functional $\varepsilon^2(\chi)$ is defined in Corollary 4.6.

Let us comment on the error term: The first part of the error term arises because e_1 is only the approximate normal. The last part arises in the passage from a diffuse to a sharp interface and formally is of quadratic nature.

The following lemma deals with the error term in the foregoing lemma and brings it into the standard form. The only difference to the two-phase case in (65) is the prefactor in front of the excess ε^2 which comes from the slight difference in the two one-dimensional estimates.

Lemma 4.9. *With η as in Lemma 4.7 we have*

$$\tau \sum_{l=1}^{L} \frac{1}{\tau} \int \eta \left|\chi^{Kl} - \chi^{K(l-1)}\right| k_h * \left(\eta \left|\chi^{Kl} - \chi^{K(l-1)}\right|\right) dx \lesssim \frac{1}{\alpha^2} \tau \sum_{l=1}^{L} \varepsilon^2(\chi^{Kl-1}) + \alpha^{\frac{1}{9}} r^{d-1} T + \alpha^{\frac{1}{9}} \iint \eta \, d\mu_h,$$

where the functional $\varepsilon^2(\chi)$ is defined in Corollary 4.6.

With the above lemma we can conclude the proof of Proposition 4.1. Since one of the error terms includes the factor r^{d-1} we will only use the proposition in case there the behavior in the ball B_r is non-trivial. In the trivial case – meaning that the measure of the boundary inside B is much smaller than r^{d-1} – we can use the following easy estimate.

Lemma 4.10. *In the situation as in Proposition 4.1, we have*

$$\left| \sum_{i,j} \sigma_{ij} \int_0^T \int \left(\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i - 2 \, \xi \cdot \nu_i \, V_i\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt \right| \lesssim \|\xi\|_\infty \left[\sum_{i=1}^{P} \int_0^T \int \eta \left(\frac{1}{\alpha} + \alpha \, V_i^2\right) |\nabla \chi_i| \, dt + \alpha \iint \eta \, d\mu \right].$$

4.3 Proofs

Proof of Proposition 4.1. Step 1: The discrete analogue of (66). The statement follows easily from

$$\left| \int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t), \xi(t)) \, dt + 2 c_0 \sigma_{12} \int_0^T \left(\int \xi_1 \, V_1 \, |\nabla \chi_1| - \int \xi_1 \, V_2 \, |\nabla \chi_2| \right) dt \right| \lesssim \|\xi\|_\infty \left[\frac{1}{\alpha^2} \int_0^T \varepsilon^2(t) \, dt + \alpha^{\frac{1}{9}} r^{d-1} T + \alpha^{\frac{1}{9}} \iint \eta \, d\mu_h \right] + o(1), \quad \text{as } h \to 0. \tag{71}$$

Here we use the notation $\varepsilon^2(t) := \varepsilon^2(\chi^h(t))$, where the functional $\varepsilon^2(\chi)$ is defined in Corollary 4.6. The infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x_1 > \lambda\}}$ in direction e_1. All terms appearing in ε^2 correspond to terms in $\mathscr{E}^2$. The first term is the sum of the localized approximate energies of $\chi_3, \ldots, \chi_P$, the second term describes the approximate energy excess of Phases 1 and 2. The convergence of these terms as $h \to 0$ for a fixed half space χ^* follows as in the proof of Lemma 2.8. Taking the infimum over the half spaces yields (66).

Step 2: Choice of appropriately shifted mesoscopic time slices. In order to prove (71), we use the machinery that we develop later on in this section. There we work on the mesoscopic time scale $\tau = \alpha\sqrt{h}$ instead of the microscopic time scale h, see Remark 1.6 for the notation. To apply these results, we have to adjust the time shift of time slices of mesoscopic distance. At the end, we will choose a microscopic time shift $k_0 \in \{1, \ldots, K\}$ such that the average over time slices of mesoscopic distance is controlled by the average over all time slices:

$$\tau \sum_{l=1}^{L} \left[\varepsilon^2(\chi^{Kl+k_0}) + \varepsilon^2(\chi^{Kl+k_0-1}) \right] \lesssim h \sum_{n=1}^{N} \varepsilon^2(\chi^n) = \int_0^T \varepsilon^2(t)\, dt. \tag{72}$$

This follows from the simple fact that $\varepsilon^2(k_0) \leq \frac{1}{K}\sum_{k=1}^{K} \varepsilon^2(k)$ for some k_0. For notational simplicity, we shall assume that $k_0 = 0$ in (72).

Step 3: Argument for (71). Using Lemmas 4.7, 4.8 and 4.9, we obtain

$$\begin{aligned} &\int_0^T -\delta E_h(\,\cdot\,, \chi^h(t-h))(\chi^h(t), \xi(t))\, dt \\ &\qquad \approx -2c_0\sigma_{12}\, \tau \sum_{l=1}^{L} \left(\int \xi_1(l\tau) \frac{\chi_1^{Kl} - \chi_1^{K(l-1)}}{\tau}\, dx - \int \xi_1(l\tau) \frac{\chi_2^{Kl} - \chi_2^{K(l-1)}}{\tau}\, dx \right) \end{aligned} \tag{73}$$

up to an error

$$\|\xi\|_\infty \left(\frac{1}{\alpha^2} \int_0^T \varepsilon^2(t)\, dt + \alpha^{\frac{1}{9}} r^{d-1} T + \alpha^{\frac{1}{9}} \iint \eta\, d\mu_h \right) + o(1), \quad \text{as } h \to 0,$$

where we used the choice of time slices (72). Since ξ has compact support in $(0, T)$, a discrete integration by parts yields

$$\tau \sum_{l=1}^{L} \int \xi_1(l\tau) \frac{1}{\tau} \left(\chi_i^{Kl} - \chi_i^{K(l-1)} \right) dx = -\tau \sum_{l=0}^{L-1} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl}\, dx.$$

By the Hölder-type bounds in Lemma 2.6 we can replace the mesoscopic scale on the right-hand side by the microscopic scale for χ:

$$\begin{aligned} &\tau \sum_{l=0}^{L-1} \left| \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl}\, dx - \frac{1}{K} \sum_{k=1}^{K} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl+k}\, dx \right| \\ &\qquad \leq \|\partial_t \xi\|_\infty h \sum_{l=0}^{L-1} \sum_{k=1}^{K} \int \left| \chi^{Kl} - \chi^{Kl+k} \right| dx \lesssim \|\partial_t \xi\|_\infty E_0 T \sqrt{\tau}. \end{aligned}$$

By the smoothness of ξ, we can easily do the same for ξ to obtain by (iii) in Proposition 2.2 that for $h \to 0$

$$\tau \sum_{l=0}^{L-1} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl}\, dx \to \int_0^T \int \partial_t \xi_1\, \chi_i\, dx\, dt = -\int_0^T \int \xi_1 V_i\, |\nabla \chi_i|\, dt.$$

Using this for the right-hand side of (73) establishes (71) and thus concludes the proof. $\square$

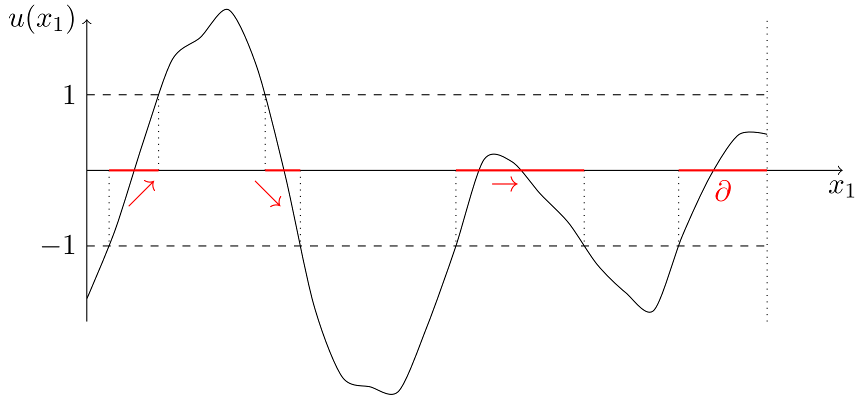

Figure 1.8: The four cases (i)–(iv) for an interval $J \subset I$ (from left to right).

Proof of Lemma 4.2. Step 1: An easier inequality. We claim that for any function $u \in C^{0,1}(I)$, we have

$$|\{|u| \le 1\}| \lesssim \int_{\{|u|\le 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1. \tag{74}$$

In order to prove (74), we decompose the set that we want to measure on the left-hand side

$$\{|u| \le 1\} = \bigcup_{J \in \mathscr{J}} J$$

into countably many pairwise disjoint intervals. As illustrated in Figure 1.8, we distinguish the following four different cases for an interval $J = [a,b] \in \mathscr{J}$:

(i) $J \in \mathscr{J}_{\nearrow}$: $u(a) = -1$ and $u(b) = 1$

(ii) $J \in \mathscr{J}_{\searrow}$: $u(a) = 1$ and $u(b) = -1$

(iii) $J \in \mathscr{J}_{\rightarrow}$: $u(a) = u(b)$,

(iv) $J \in \mathscr{J}_{\partial}$: J contains a boundary point of I.

By Jensen's inequality for the convex function $z \mapsto z_-^2$, we have

$$\begin{aligned}
\frac{1}{|J|}\int_J (\partial_1 u - 1)_-^2 \, dx_1 &\ge \left(\frac{1}{|J|}\int_J (\partial_1 u - 1)\, dx_1\right)_-^2 = \left(1 - \frac{u(b)-u(a)}{|J|}\right)_+^2 \\
&= \begin{cases} \left(1 - \frac{2}{|J|}\right)_+^2, & \text{if } J \in \mathscr{J}_{\nearrow}, \\ \left(1 + \frac{2}{|J|}\right)^2, & \text{if } J \in \mathscr{J}_{\searrow}, \\ 1, & \text{if } J \in \mathscr{J}_{\rightarrow}. \end{cases}
\end{aligned}$$

If $|J| \geq 4$, then $-1 \leq 2(u(b) - u(a))/|J| \leq 1$ and so

$$\frac{1}{|J|} \int_J (\partial_1 u - 1)_-^2 \, dx_1 \geq \frac{1}{4}. \tag{75}$$

Thus, we have

$$|J| \lesssim 1 \vee \int_J (\partial_1 u - 1)_-^2 \, dx_1$$

for any interval $J \in \mathscr{J}$. Since $\# \mathscr{J}_\partial \leq 2$, we have

$$\sum_{J \in \mathscr{J}_\partial} |J| \lesssim 1 + \int_J (\partial_1 u - 1)_-^2 \, dx_1,$$

which is enough in case (iv). In case (iii), we immediately have

$$|J| \lesssim \int_J (\partial_1 u - 1)_-^2 \, dx_1, \tag{76}$$

while in case (ii) we even have the stronger estimate

$$\int_J (\partial_1 u - 1)_-^2 \, dx_1 \gtrsim |J| \left(1 + \frac{2}{|J|}\right)^2 \gtrsim 1 \vee |J|$$

since $1 + s^2 \geq 1$ and $1 + s^2 \geq 2s$ for all $s \in \mathbb{R}$. Thus on the one hand we can estimate the measure of such an interval $J \in \mathscr{J}_\searrow$ as in (76). On the other hand, we can bound the total number of these intervals:

$$\# \mathscr{J}_\searrow \lesssim \sum_{J \in \mathscr{J}_\searrow} \int_J (\partial_1 u - 1)_-^2 \, dx_1 \leq \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1, \tag{77}$$

which clearly yields

$$\# \mathscr{J}_\nearrow \leq \# \mathscr{J}_\searrow + 1 \lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1.$$

Hence, using (75) for those $J \in \mathscr{J}_\nearrow$ with $|J| \geq 4$, we have

$$\begin{aligned} \sum_{J \in \mathscr{J}_\nearrow} |J| = \sum_{\substack{J \in \mathscr{J}_\nearrow \\ |J| \geq 4}} |J| + \sum_{\substack{J \in \mathscr{J}_\nearrow \\ |J| < 4}} |J| &\lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + \# \mathscr{J}_\nearrow \\ &\lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1. \end{aligned}$$

Using these estimates, we derive

$$|\{|u| \le 1\}| = \sum_{J \in \mathscr{J}} |J| \lesssim \int_{\{|u|\le 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1.$$

Step 2: Rescaling (74). Let $s > 0$. We use Step 1 for $\hat{u}$ and set $u := s\hat{u}$, $x_1 = s\hat{x}_1$. Then $\partial_1 u = \hat{\partial}_1 \hat{u}$ and

$$|\{|u| \le s\}| = s\,|\{\hat{u} \le 1\}| \overset{(74)}{\lesssim} s \int_{\{|\hat{u}|\le 1\}} \left(\hat{\partial}_1 \hat{u} - 1\right)_-^2 d\hat{x}_1 + s = \int_{\{|u|\le s\}} (\partial_1 u - 1)_-^2 \, dx_1 + s.$$

Therefore, using this for $u - \frac{1}{2}$ instead of u, we have

$$|\{|u - \tfrac{1}{2}| \le s\}| \lesssim \int_{\{|u-\frac{1}{2}|\le s\}} (\partial_1 u - 1)_-^2 \, dx_1 + s. \tag{78}$$

Step 3: Introducing $\tilde{u}$. By Chebyshev's inequality, we have

$$|\{|u - \tilde{u}| \ge s\}| \le \frac{1}{s^2} \int_I (u - \tilde{u})^2 dx_1$$

for all $s > 0$. Set

$$E := \{|u - \tfrac{1}{2}| \le s\} \cup \{|u - \tilde{u}| \ge s\} \subset I.$$

Then, since e. g. $u \ge \frac{1}{2} > \tilde{u}$ and $|u - \frac{1}{2}| > s$ imply $|\tilde{u} - u| > s$,

$$\{\chi \ne \tilde{\chi}\} = \{u \ge \tfrac{1}{2}\} \Delta \{\tilde{u} \ge \tfrac{1}{2}\} \subset E.$$

Hence,

$$\int_I |\chi - \tilde{\chi}| \, dx_1 \le |E| \lesssim \int_{\{|u-\frac{1}{2}|<s\}} (\partial_1 u - 1)_-^2 \, dx_1 + s + \frac{1}{s^2} \int_I (u - \tilde{u})^2 \, dx_1,$$

which concludes the proof. □

Proof of Corollary 4.3. By rescaling $x_1 = \sqrt{h}\,\hat{x}_1$, $\hat{u}(\hat{x}_1) = u(\sqrt{h}\,\hat{x}_1)$, and analogously for $\tilde{u}$ and using Lemma 4.2 for the transformed functions we obtain:

$$\frac{1}{\sqrt{h}} \int_I |\chi - \tilde{\chi}| \, dx_1 \lesssim \frac{1}{\sqrt{h}} \int_{\{|u-\frac{1}{2}|<s\}} \left(\sqrt{h}\,\partial_1 u - 1\right)_-^2 dx_1 + s + \frac{1}{s^2} \frac{1}{\sqrt{h}} \int_I (u - \tilde{u})^2 \, dx_1. \tag{79}$$

Now we approximate η by simple functions: Let

$$\tilde{\eta} := \frac{[N\eta]}{N} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{J_n}, \quad \text{where} \quad J_n := \left\{ x \in I \colon \eta(x) > \frac{n}{N} \right\}.$$

Then $0 \le \tilde\eta \le \eta$, $|\eta - \tilde\eta| \le \frac{1}{N}$ and since η is radially non-increasing, each J_n is an open interval. We can apply (79) with J_n playing the role of I. By linearity we have

$$\begin{aligned}\frac{1}{\sqrt{h}}\int \tilde\eta\,|\chi-\tilde\chi|\,dx_1 &\lesssim \frac{1}{\sqrt{h}}\int_{\{|u-\frac12|<s\}} \tilde\eta\left(\sqrt{h}\,\partial_1 u - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int \tilde\eta\,(u-\tilde u)^2\,dx_1\\ &\le \frac{1}{\sqrt{h}}\int_{\{|u-\frac12|<s\}} \eta\left(\sqrt{h}\,\partial_1 u - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int \eta\,(u-\tilde u)^2\,dx_1.\end{aligned}$$

Passing to the limit $N\to\infty$, the left-hand side converges to $\frac{1}{\sqrt{h}}\int \eta\,|\chi-\tilde\chi|\,dx_1$ and we obtain the claim. □

Proof of Lemma 4.4. Argument for (68). As in Step 1 of the proof of Lemma 2.4, by (22) we have

$$\frac{1}{\sqrt{h}}\int \eta\,[(1-\tilde\chi)\,G_h*\tilde\chi + \tilde\chi\,G_h*(1-\tilde\chi)]\,dx = \frac{1}{\sqrt{h}}\int G_h(z)\int \eta(x)\,|\tilde\chi(x+z)-\tilde\chi(x)|\,dx\,dz. \tag{80}$$

Using $|\chi^*(x+z)-\chi^*(x)| = \operatorname{sign}(z_1)\,(\chi^*(x+z)-\chi^*(x))$, and $2u_+ = |u|+u$ on the set $\{z_1>0\}$ and $2u_- = |u|-u$ on $\{z_1<0\}$, we thus obtain

$$\begin{aligned}&\frac{2}{\sqrt{h}}\int_{\{z_1\lessgtr 0\}} G_h(z)\int \eta(x)\,(\chi(x+z)-\chi(x))_\pm\,dx\,dz\\ &\quad= \frac{1}{\sqrt{h}}\int G_h(z)\int \eta(x)\,(|\chi(x+z)-\chi(x)| - |\chi^*(x+z)-\chi^*(x)|)\,dx\,dz\\ &\qquad - \frac{1}{\sqrt{h}}\int \operatorname{sign}(z_1)\,G_h(z)\int \eta(x)\,((\chi^*-\chi)(x+z)-(\chi^*-\chi)(x))\,dx\,dz\\ &\quad\le \varepsilon^2 - \frac{1}{\sqrt{h}}\int \operatorname{sign}(z_1)\,G_h(z)\int (\eta(x)-\eta(x-z))\,(\chi^*-\chi)(x)\,dx\,dz,\end{aligned}$$

where we used again $<,+$ and $>,-$, respectively as a short notation for the sum of the two integrals. Now we can apply a Taylor expansion for η around x, i. e. write $\eta(x)-\eta(x-z) = \nabla\eta(x)\cdot z + O(|z|^2)$, where the constant in the $O(|z|^2)$-term depends linearly on $\left\|\nabla^2\eta\right\|_\infty$. By symmetry, the first-order term is

$$\begin{aligned}&\frac{1}{\sqrt{h}}\int \operatorname{sign}(z_1)\,z\,G_h(z)\,dz\cdot\int \nabla\eta(x)(\chi^*-\chi)(x)\,dx\\ &\qquad= \int \frac{|z_1|}{\sqrt{h}}G_h(z)\,dz\int \partial_1\eta(x)(\chi^*-\chi)(x)\,dx.\end{aligned}$$

Note that the right-hand side can be controlled by

$$\|\partial_1\eta\|_\infty\int_{B_{2r}} |\chi-\chi^*|\,dx \lesssim \frac{1}{r}\int_{B_{2r}} |\chi-\chi^*|\,dx \le \varepsilon^2.$$

The second-order term is controlled by

$$\left\|\nabla^2\eta\right\|_\infty \frac{1}{\sqrt{h}}\int |z|^2 G_h(z)\,dz = \left\|\nabla^2\eta\right\|_\infty \sqrt{h}\int |z|^2 G(z)\,dz \lesssim \sqrt{h}\frac{1}{r^2},$$

which completes the proof of (68).

Argument for (69). For the first arguments let w. l. o. g. $h=1$. The first ingredient is the identity

$$\partial_1(G*\chi)(x) = \int |z_1|G(z)\,|\chi(x+z)-\chi(x)|\,dz - 2\int_{\{z_1\lessgtr 0\}} |z_1|G(z)\left(\chi(x+z)-\chi(x)\right)_\pm dz, \tag{81}$$

where the last term is the sum of the two integrals. Indeed, since $\partial_1 G(z) = -z_1 G(z)$ is odd in z_1,

$$\partial_1(G*\chi)(x) = \int \partial_1 G(z)\chi(x-z)\,dz = \int z_1 G(z)\left(\chi(x+z)-\chi(x)\right)dz$$

and splitting the integrand in the form $u = |u| - 2u_-$ on the set $\{z_1>0\}$ and $-u = |u|-2u_+$ on $\{z_1<0\}$, respectively, we derive

$$\begin{aligned}
\partial_1(G*\chi)(x) = &\int_{\{z_1>0\}} |z_1|G(z)\,|\chi(x+z)-\chi(x)|\,dz + \int_{\{z_1<0\}} |z_1|G(z)\,|\chi(x+z)-\chi(x)|\,dz\\
&- 2\int_{\{z_1>0\}} |z_1|G(z)\left(\chi(x+z)-\chi(x)\right)_- dz\\
&- 2\int_{\{z_1<0\}} |z_1|G(z)\left(\chi(x+z)-\chi(x)\right)_+ dz,
\end{aligned}$$

which is (81).
The second ingredient for (69) is

$$\int |z_1|G(z)|\chi(x+z)-\chi(x)|dz \gtrsim \left(\int G(z)|\chi(x+z)-\chi(x)|dz\right)^2. \tag{82}$$

To obtain (82), we estimate

$$\begin{aligned}
\int |z_1|G(z)|\chi(x+z)-\chi(x)|dz \geq &\int_{\{|z_1|\geq\epsilon\}} |z_1|G(z)|\chi(x+z)-\chi(x)|dz\\
\geq &\epsilon\int_{\{|z_1|\geq\epsilon\}} G(z)|\chi(x+z)-\chi(x)|dz\\
= &\epsilon\int G(z)|\chi(x+z)-\chi(x)|dz\\
&- \epsilon\int_{\{|z_1|<\epsilon\}} G(z)|\chi(x+z)-\chi(x)|dz.
\end{aligned}$$

We recall that we G factorizes in a one-dimensional Gaussian G^1 and a $(d-1)$-dimensional Gaussian G^{d-1}, i. e. $G(z) = G^1(z_1)\,G^{d-1}(z')$ so that the second integral can be estimated from above by $2G^1(0)\epsilon$. Therefore we have

$$\int |z_1| G(z)|\chi(x+z)-\chi(x)|dz \geq \epsilon \int G(z)|\chi(x+z)-\chi(x)|dz - 2G^1(0)\epsilon^2.$$

Optimizing in ϵ yields (82).
Using the fact that $\chi \in \{0,1\}$,

$$\int G(z)|\chi(x+z)-\chi(x)|dz = (1-\chi)(x)(G*\chi)(x) + \chi(x)(G*(1-\chi))(x)$$

implies the third ingredient:

$$\int G(z)\,|\chi(x+z)-\chi(x)|\,dz \geq (G*\chi)(x) \wedge (1-G*\chi)(x). \tag{83}$$

Combining (81), (82) and (83), one finds a positive constant $\overline{c}$ such that

$$\partial_1(G*\chi)(x) \geq 18\,\overline{c}\,[(G*\chi)(x)\wedge(1-G*\chi)(x)]^2 - 2\int_{\{z_1 \lesseqgtr 0\}} |z_1| G(z)\,(\chi(x+z)-\chi(x))_{\pm}\,dz,$$

where we recall that the last term is the sum of the two integrals. We consider the "bad" set

$$E := \left\{ x\colon \int_{\{z_1 \lesseqgtr 0\}} |z_1| G(z)\,(\chi(x+z)-\chi(x))_{\pm}\,dz \geq \frac{\overline{c}}{2} \right\}.$$

By construction of E we have a good estimate on E^c:

$$\partial_1(G*\chi)(x) \geq 18\,\overline{c}\,[\min\{(G*\chi)(x),\,(1-G*\chi)(x)\}]^2 - \overline{c} \quad \text{on } E^c,$$

and thus we obtain strict monotonicity of $G*\chi$ in e_1-direction outside E as long as the first term on the left-hand side dominates the second term:

$$\partial_1(G*\chi) \geq \overline{c} \quad \text{on } E^c \cap \left\{ \frac{1}{3} \leq G*\chi \leq \frac{2}{3} \right\}.$$

Therefore

$$\int_{\left\{\frac{1}{3}\leq G*\chi\leq\frac{2}{3}\right\}} \eta\,(\partial_1 G*\chi - \overline{c})_-^2\,dx = \int_{E\cap\left\{\frac{1}{3}\leq G*\chi\leq\frac{2}{3}\right\}} \eta\,(\partial_1 G*\chi - \overline{c})_-^2\,dx \lesssim \int_E \eta\,dx.$$

We introduce the parameter h again. Then this turns into

$$\frac{1}{\sqrt{h}} \int_{\left\{\frac{1}{3}\leq G_h*\chi\leq\frac{2}{3}\right\}} \eta\left(\sqrt{h}\partial_1 G_h*\chi - \overline{c}\right)_-^2 dx \lesssim \frac{1}{\sqrt{h}} \int_{E_h} \eta\,dx,$$

with now

$$E_h := \left\{ x\colon \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} \frac{|z_1|}{\sqrt{h}} G_h(z) \left(\chi(x+z) - \chi(x)\right)_\pm dz \geq \frac{\overline{c}}{2} \right\}.$$

By construction of E and since $|z| G_h(z) \lesssim \sqrt{h}\, G_h(\frac{z}{2})$, we have

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int_{E_h} \eta \, dx \lesssim & \frac{1}{h} \int_{\{z_1 \lesseqgtr 0\}} |z_1| G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
\lesssim & \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z/2) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
\lesssim & \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
& + \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+2z) - \chi(x+z)\right)_\pm dx\, dz
\end{aligned} \tag{84}$$

by a change of coordinates $z \mapsto 2z$ and the subadditivity of the functions $u \mapsto u_\pm$. The last term can be handled using a Taylor expansion of η around x:

$$\begin{aligned}
& \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+2z) - \chi(x+z)\right)_\pm dx\, dz \\
= & \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z) \int \eta(x-z) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
= & \frac{1}{\sqrt{h}} \int_{\{z_1 \lesseqgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz + O\big(\sqrt{h}\,\big),
\end{aligned}$$

where the constant in the $O(\sqrt{h})$-term depends linearly on $E_h(\chi)$ and $\|\nabla \eta\|_\infty$. Indeed, the error in the equation above is - up to a constant times $\|\nabla \eta\|_\infty$ - estimated by

$$\int \frac{|z|}{\sqrt{h}} G_h(z) \int |\chi(x+z) - \chi(x)| \, dx\, dz \lesssim \int G_h(\frac{z}{2}) \int |\chi(x+z) - \chi(x)| \, dx\, dz \overset{(18)}{\lesssim} \sqrt{h} E_h(\chi).$$

Using (68), we obtain

$$\frac{1}{\sqrt{h}} \int_{E_h} \eta \, dx \lesssim \varepsilon^2 + \sqrt{h} \frac{1}{r^2} + \sqrt{h} \frac{1}{r} E_h(\chi)$$

and thus (69) holds. □

Proof of Lemma 4.5. By the same argument as in Corollary 4.3, we can ignore the cut-off η and the parameter $h > 0$ and reduce the claim to the following version:

$$\begin{aligned}
\int_I |\chi - \tilde{\chi}| \, dx_1 \lesssim & \int_{\{|u_1 - \frac{1}{2}| \lesssim s\}} (\partial_1 u_1 - \overline{c})_-^2 \, dx_1 + \frac{1}{s} \int_I \sum_{j \geq 3} [u_j \wedge (1 - u_j)] \, dx_1 \\
& + s + \frac{1}{s^2} \int_I |u - \tilde{u}|^2 \, dx_1.
\end{aligned} \tag{85}$$

We will prove

$$\{\chi \neq \tilde{\chi}\} \subset \Big\{|u_1 - \tfrac{1}{2}| \lesssim s\Big\} \cup \Big\{\sum_{j\geq 3}[u_j \wedge (1-u_j)] \gtrsim s\Big\} \cup \{|u-\tilde{u}| \gtrsim s\}. \tag{86}$$

Then (85) follows from the one-dimensional case in the form of (78) for the first right-hand side set and Chebyshev's inequality for the second and third right-hand side set. The fact that we replaced the 1 in (78) by the universal constant $\overline{c} > 0$ can be justified by a simple rescaling.

In order to prove (86), we fix $i \in \{1,\dots,P\}$ and define the functions

$$v := \min_{j\neq i} \phi_j - \phi_i \in C^{0,1}(I)$$

and $\tilde{v}$ in the same way, so that $\chi_i = \mathbf{1}_{\{v>0\}}$, $\tilde{\chi}_i = \mathbf{1}_{\{\tilde{v}>0\}}$ and

$$\{\chi_i \neq \tilde{\chi}_i\} \subset \{|v| < s\} \cup \{|v-\tilde{v}| \geq s\}. \tag{87}$$

We clearly have

$$|v-\tilde{v}| \leq \big|\phi_i - \tilde{\phi}_i\big| + \big|\min_{j\neq i}\phi_j - \min_{j\neq i}\tilde{\phi}_j\big| \leq \sum_{i=1}^{P}\big|\phi_i - \tilde{\phi}_i\big| \lesssim |u-\tilde{u}|,$$

which together with Chebyshev's inequality yields the desired bound on the measure of the second right-hand side set of (87). Therefore our goal is to prove

$$|u_1 - \tfrac{1}{2}| \lesssim s \quad \text{or} \quad \sum_{j\geq 3}[u_j \wedge (1-u_j)] \gtrsim s \qquad \text{on } \{|v| < s\}, \tag{88}$$

which then implies (86).

Now we give the argument for (88). First, we decompose the set $\{|v| < s\}$ in the following way:

$$\{|v|<s\} = \bigcup_{j\neq i} E_j, \quad E_j := \big\{\, |\phi_i - \phi_j| < s,\ \phi_j = \min_{k\neq i}\phi_k\big\}.$$

We claim that

$$u_i,\, u_j \leq \frac{1}{2} + \frac{s}{2\sigma_{ij}}, \quad u_k \leq \frac{1}{2},\ k \notin \{i,j\} \quad \text{on } E_j. \tag{89}$$

Indeed, plugging in the definition of ϕ, using the triangle inequality for the surface tensions and $\sum_l u_l = 1$, for $k \notin \{i,j\}$, we have on E_j

$$\begin{aligned}\phi_j \leq \phi_k = \sum_{l\neq k}\sigma_{kl}u_l \leq \sum_{l\neq k}\sigma_{jl}u_l + \sigma_{jk}\sum_{l\neq k}u_l &= \phi_j - \sigma_{jk}u_k + \sigma_{jk}(1-u_k)\\ &= \phi_j + \sigma_{jk}(1-2u_k).\end{aligned}$$

Subtracting ϕ_j on both sides, we obtain $u_k \leq \frac{1}{2}$. Since $\phi_j - s \leq \phi_i$ on E_j with the same chain of inequalities as before we obtain

$$-s \leq \sigma_{ij}(1-2u_i).$$

The same inequality holds for u_j since $\phi_i - s \leq \phi_j$, which concludes the argument for (89).

On the one hand, (89) gives us the upper bound for u_1 on $\{|v|<s\}$. On the other hand, since $u \wedge (1-u) = u-(2u-1)_+$ for any u we infer from (89) that on the set $\{u_1 \leq \frac{1}{2} - Cs\} \cap \{|v|<s\}$ we have

$$\sum_{j\geq 3} [u_j \wedge (1-u_j)] = 1 - u_1 - u_2 - \sum_{j\geq 3} (2u_j-1)_+ \geq \left(C - \frac{1}{\sigma_{\min}}\right) s \gtrsim s,$$

if $C \geq \frac{2}{\sigma_{\min}}$. This concludes the argument for (88) and therefore the proof of the lemma. □

Proof of Corollary 4.6. By Lemma 4.4, the claim follows from the obvious inequality

$$u_j \wedge (1-u_j) = G_h * \chi_j \wedge G_h * (1-\chi_j) \leq (1-\chi)\, G_h * \chi_j + \chi\, G_h * (1-\chi_j)\,.$$

□

Proof of Lemma 4.7. We recall the definition of the inner variation of $-E_h(\chi - \tilde\chi)$ in (33): We have for any pair of admissible functions $\chi, \tilde\chi$ and any test function $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$

$$\begin{aligned}
-\delta E_h(\,\cdot\, - \tilde\chi)(\chi,\xi) =& \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int (\chi_i - \tilde\chi_i)\, G_h * (\xi \cdot \nabla \chi_j)\, dx \\
=& \frac{2}{\sqrt{h}} \sum_{ij} \sigma_{ij} \int (\chi_i - \tilde\chi_i) \left[\nabla G_h * (\xi\, \chi_j) - G_h * ((\nabla \cdot \xi)\, \chi_j)\right] dx.
\end{aligned}$$

In our case, after integration in time, this yields

$$\begin{aligned}
&\int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t), \xi(t))\, dt \\
&\quad = \sum_{i,j} \sigma_{ij} h \sum_{n=1}^N \frac{2}{\sqrt{h}} \int \left(\chi_i^n - \chi_i^{n-1}\right) \left[\nabla G_h * \left(\overline{\xi}^n \chi_j^n\right) - G_h * \left(\left(\nabla \cdot \overline{\xi}^n\right) \chi_j^n\right)\right] dx,
\end{aligned}$$

where

$$\overline{\xi}^n := \frac{1}{h} \int_{nh}^{(n+1)h} \xi(t)\, dt$$

denotes the time average of ξ over a microscopic time interval $[nh,(n+1)h)$.
Now we prove step by step that

1. the $(\nabla\cdot\xi)$-term is negligible as $h\to 0$;
2. we can freeze mesoscopic time for ξ, that is, substitute $\overline{\xi}^n$ by some nearby value $\xi(l_n\tau)$ at the expense of an $o(1)$-term;
3. we can smuggle in η at the expense of an $o(1)$-term;
4. we can freeze mesoscopic time for χ^h and substitute χ^n in the second factor by the mean $\frac{1}{2}\left(\chi^h((l_n-1)\tau)+\chi^h(l_n\tau)\right)$, which is the main step;
5. we can get rid of η again at the expense of an $o(1)$-term; and finally
6. we can pull ξ out of the convolution at the expense of an $o(1)$-term.

Note that Step 3 and Step 5 are just auxiliary steps for Step 4.

Step 1: The $(\nabla\cdot\xi)$-term vanishes as $h\to 0$. By Jensen's inequality, for any pair i,j we have

$$\begin{aligned}
&\left|h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\left(\chi_i^n-\chi_i^{n-1}\right)G_h*\left(\left(\nabla\cdot\overline{\xi}^n\right)\chi_j^n\right)\,dx\right|\\
&\qquad\le\|\nabla\xi\|_\infty T\frac{1}{\sqrt{h}}\frac{1}{N}\sum_{n=1}^N\int\left|G_h*\left(\chi_i^n-\chi_i^{n-1}\right)\right|dx\\
&\qquad\lesssim\|\nabla\xi\|_\infty T\frac{1}{\sqrt{h}}\left(\frac{1}{N}\sum_{n=1}^N\int\left|G_h*\left(\chi^n-\chi^{n-1}\right)\right|^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Since the L^2-norm of G_h*u is decreasing in h and by the energy-dissipation estimate (10), the error is controlled by

$$\|\nabla\xi\|_\infty T\frac{1}{\sqrt{h}}\left(\frac{1}{N}\sqrt{h}E_0\right)^{\frac{1}{2}}\le\|\nabla\xi\|_\infty E_0^{\frac{1}{2}}T^{\frac{1}{2}}h^{\frac{1}{4}}=o(1).$$

Step 2: Time freezing for ξ. We can approximate $\overline{\xi}^n$ by a nearby value $\xi(l_n\tau)$, where $l_n\in\{1,\dots L\}$ is chosen such that $K(l_n-1)<n\le Kl_n$. Note that $|\overline{\xi}^n-\xi^{l_n}|\le\tau\|\partial_t\xi\|_\infty$. Therefore, by Jensen's inequality, we have for any pair i,j

$$\begin{aligned}
&\left|h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\left(\chi_i^n-\chi_i^{n-1}\right)\nabla G_h*\left(\left(\xi^{l_n}-\overline{\xi}^n\right)\chi_j^n\right)dx\right|\\
&\qquad\le\alpha\|\partial_t\xi\|_\infty T\frac{1}{N}\sum_{n=1}^N\int\left|\nabla G_h*\left(\chi_i^n-\chi_i^{n-1}\right)\right|dx\\
&\qquad\lesssim\alpha\|\partial_t\xi\|_\infty T\left(\frac{1}{N}\sum_{n=1}^N\int\left|\nabla G_h*\left(\chi^n-\chi^{n-1}\right)\right|^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

But $\sqrt{h}\|\nabla G_h * u\|_{L^2} \lesssim \|G_{h/2} * u\|_{L^2}$ yields

$$\int \left|\nabla G_h * \left(\chi^n - \chi^{n-1}\right)\right|^2 dx \lesssim \frac{1}{h} \int \left[G_{h/2} * \left(\chi^n - \chi^{n-1}\right)\right]^2 dx.$$

Using the energy-dissipation estimate (10), the error is controlled by

$$\alpha \|\partial_t \xi\|_\infty T \left(\frac{1}{N} \frac{1}{\sqrt{h}} E_0\right)^{\frac{1}{2}} = \alpha \|\partial_t \xi\|_\infty E_0^{\frac{1}{2}} T^{\frac{1}{2}} h^{\frac{1}{4}} = o(1).$$

Step 3: Smuggling in η. We claim

$$\begin{aligned} h \sum_{n=1}^N \frac{1}{\sqrt{h}} & \int \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_h * \left(\xi(l_n\tau)\chi_j^n\right) dx \\ &= h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta\, G_{h/2} * \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_{h/2} * \left(\xi(l_n\tau)\chi_j^n\right) dx + o(1) \quad \text{as } h \to 0. \end{aligned}$$

Using $\nabla G_h = G_{h/2} * \nabla G_{h/2}$, the left-hand side is equal to

$$h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int G_{h/2} * \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_{h/2} * \left(\xi(l_n\tau)\chi_j^n\right) dx.$$

Note that since $\eta \equiv 1$ on the support of ξ and $|z| \left|\nabla G_{1/2}(z)\right| \lesssim |z|^2 G(z)$ has finite integral, we have for any $\chi \in \{0,1\}$,

$$\begin{aligned} \left|(1-\eta)\nabla G_{h/2} * (\xi\chi)\right| &= \left|\int \nabla G_{h/2}(z)(\eta(x+z) - \eta(x))\xi(x+z)\chi(x+z)\, dz\right| \\ &\lesssim \|\nabla\eta\|_\infty \|\xi\|_\infty \int |z| \left|\nabla G_{h/2}(z)\right| dz \lesssim \frac{1}{r}\|\xi\|_\infty. \end{aligned}$$

Thus, using the Cauchy-Schwarz inequality and the energy-dissipation estimate (10), the error is controlled by

$$\begin{aligned} h^{\frac{1}{4}} \left(\sum_{n=1}^N \frac{1}{\sqrt{h}} \int \left|G_{h/2} * \left(\chi^n - \chi^{n-1}\right)\right|^2 dx\right)^{\frac{1}{2}} & \left(h \sum_{n=1}^N \left(\frac{1}{r}\|\xi\|_\infty\right)^2\right)^{\frac{1}{2}} \\ & \lesssim E_0^{\frac{1}{2}} T^{\frac{1}{2}} \frac{1}{r} \|\xi\|_\infty h^{\frac{1}{4}} = o(1). \end{aligned}$$

Step 4: Time freezing for χ^h. We claim that for any pair of indices i, j

$$\begin{aligned} h \sum_{n=1}^N \frac{2}{\sqrt{h}} & \int \eta\, G_{h/2} * \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_{h/2} * \left(\xi(l_n\tau)\chi_j^n\right) dx \\ &\approx h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta\, G_{h/2} * \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_{h/2} * \left(\xi(l_n\tau) \left(\chi_j^h((l_n-1)\tau) + \chi_j^h(l_n\tau)\right)\right) dx, \end{aligned}$$

in the sense that the error is controlled by

$$\|\xi\|_\infty \left(\frac{1}{\alpha}\tau \sum_{l=1}^{L} \varepsilon^2(\chi^{Kl-1}) + \alpha^{\frac13} \iint \eta \, d\mu_h + \alpha^{\frac13} r^{d-1} T \right) + o(1), \quad \text{as } h \to 0.$$

Here, we assumed that Phases 1 and 2 are the majority phases in the support of η. Indeed, we can control the error using the Cauchy-Schwarz inequality by

$$\left(\sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \eta^2 \left| G_{h/2} * \left(\chi^n - \chi^{n-1} \right) \right|^2 dx \right)^{\frac12} \times$$
$$\left(\tau \sum_{l=1}^{L} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\sqrt{h}} \int \left[\sqrt{h} \nabla G_{h/2} * \left(\xi(l\tau) \left[\chi_j^{K(l-1)+k} - \tfrac12 \left(\chi_j^{K(l-1)} + \chi_j^{Kl} \right) \right] \right) \right]^2 dx \right)^{\frac12}.$$

Since $0 \leq \eta \leq 1$, the term in the first parenthesis is controlled by $\iint \eta \, d\mu_h$. For the term in the second parenthesis, we fix the mesoscopic block index l and the microscopic time step index k and sum at the end. Let $l = 1$ and write ξ instead of $\xi(l\tau)$ for notational simplicity. We use the L^2-convolution estimate and introduce η in the second integral, which is equal to 1 on the support of ξ:

$$\begin{aligned}
&\frac{1}{\sqrt{h}} \int \left[\sqrt{h} \nabla G_{h/2} * \left(\xi(l\tau) \left[\chi_j^k - \tfrac12 \left(\chi_j^0 + \chi_j^K \right) \right] \right) \right]^2 dx \\
&\quad \leq \frac{1}{\sqrt{h}} \left(\int \left| \sqrt{h} \nabla G_{h/2} \right| dz \right)^2 \int |\xi|^2 \left[\chi_j^k - \tfrac12 \left(\chi_j^0 + \chi_j^K \right) \right]^2 dx \\
&\quad \lesssim \|\xi\|_\infty^2 \left(\frac{1}{\sqrt{h}} \int \eta \left| \chi^k - \chi^0 \right| dx + \frac{1}{\sqrt{h}} \int \eta \left| \chi^K - \chi^k \right| dx \right).
\end{aligned}$$

With Lemma 4.5 in the integrated form and Corollary 4.6, we can estimate these terms in the following way. We set for abbreviation

$$\alpha^2(k,k') := \frac{1}{\sqrt{h}} \int \eta \left(G_h * \left(\chi^k - \chi^{k'} \right) \right)^2 dx.$$

By Minkowski's triangle inequality w. r. t. the measure $\eta \, dx$, we see that α also satisfies a triangle inequality. Thus, thanks to Jensen's inequality,

$$\alpha^2(k-1,-1) \leq \left(\sum_{n=0}^{k-1} \alpha(n,n-1) \right)^2 \leq k \sum_{n=0}^{k-1} \alpha^2(n,n-1) \leq K \sum_{n=0}^{K-1} \alpha^2(n,n-1).$$

Therefore, by integrating Lemma 4.5 over the tangential directions $x_2, \ldots, x_d$ and using Corollary 4.6, we have

$$\frac{1}{\sqrt{h}} \int \eta \left| \chi^k - \chi^0 \right| dx \lesssim \frac{1}{s} \varepsilon^2(\chi^{-1}) + s r^{d-1} + \frac{1}{s^2} K \sum_{n=0}^{K-1} \alpha^2(n,n-1) + o(1). \tag{90}$$

By (15) we have $\sum_n \alpha^2(n,n-1) \leq \iint \eta \, d\mu_h$ and the relation $K\tau = \alpha^2$, we have

$$\tau \sum_{l=1}^{L} \alpha^2(Kl-1, K(l-1)-1) \leq \alpha^2 \iint \eta \, d\mu_h. \tag{91}$$

This justifies the name $\alpha^2(k,k')$, since the term arising from $\alpha^2(k,k')$ is estimated in (91) by α^2, the square of the fudge factor in the definition of the mesoscopic time scale $\tau = \alpha\sqrt{h}$. Therefore, after summation over the mesoscopic block index l, (90) in conjunction with (91) yields

$$\tau \sum_{l=1}^{L} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\sqrt{h}} \int \eta \left| \chi^{Kl+k} - \chi^{Kl} \right| dx \lesssim \frac{1}{s} \tau \sum_{l=1}^{L} \varepsilon^2(\chi^{Kl-1}) + s r^{d-1} T + \frac{1}{s^2} \alpha^2 \iint \eta \, d\mu_h.$$

Using Young's inequality, the total error in this step is controlled by $\|\xi\|_\infty$ times

$$\begin{aligned}
&\left(\iint \eta \, d\mu_h \right)^{\frac{1}{2}} \left(\frac{1}{s} \tau \sum_{l=1}^{L} \varepsilon^2(\chi^{Kl-1}) + s r^{d-1} T + \left(\frac{\alpha}{s}\right)^2 \iint \eta \, d\mu_h \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\alpha} \tau \sum_{l=1}^{L} \varepsilon^2(\chi^{Kl-1}) + \frac{s^2}{\alpha} r^{d-1} T + \frac{\alpha}{s} \iint \eta \, d\mu_h.
\end{aligned}$$

If we now choose $s = \alpha^{\frac{2}{3}} \ll 1$, this is the desired error term.

Step 5: Getting rid of η again. As in Step 3, we can estimate

$$\begin{aligned}
&h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \eta \, G_{h/2} * \left(\chi_i^n - \chi_i^{n-1} \right) \nabla G_{h/2} * \left[\xi(l_n \tau) \left(\chi_j^h((l_n-1)\tau) + \chi_j^h(l_n \tau) \right) \right] dx \\
&= h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \left(\chi_i^n - \chi_i^{n-1} \right) \nabla G_h * \left[\xi(l_n \tau) \left(\chi_j^h((l_n-1)\tau) + \chi_j^h(l_n \tau) \right) \right] dx + o(1),
\end{aligned}$$

as $h \to 0$.

Step 6: Pulling out ξ. First, fix l and write $\xi = \xi(l\tau)$. For simplicity of the formula, we will ignore l and formally set $l = 1$. Note that since ∇G is antisymmetric, we have for any two functions χ, v,

$$\int v \left[\xi \cdot \nabla G_h * \chi - \nabla G_h * (\xi \, \chi) \right] dx = \int \nabla G_h(z) \int v(x+z) \, \chi(x) \left(\xi(x+z) - \xi(x) \right) dx \, dz.$$

Set $K(z) := z \otimes z \, G(z)$, take a Taylor-expansion of ξ around x: $\xi(x+z) - \xi(x) = \nabla\xi(x)\, z + O(|z|^2)$, where the constant in the $O(|z|^2)$-term is depending linearly on $\|\nabla^2 \xi\|_\infty$. Then the error

on this single time interval splits into two terms.
The one coming from the first-order term in the expansion of ξ is

$$\left|\frac{1}{K}\sum_{k=1}^{K}\frac{1}{\sqrt{h}}\int \nabla\xi \colon K_h * \left(\chi_i^k-\chi_i^{k-1}\right)\left(\chi_j^0+\chi_j^K\right)dx\right| \\ \lesssim \|\nabla\xi\|_\infty \frac{1}{\sqrt{h}}\left(\frac{1}{K}\sum_{k=1}^{K}\int\left|K_h*\left(\chi^k-\chi^{k-1}\right)\right|^2dx\right)^{\frac{1}{2}},$$

where we used Jensen's inequality. Since $K_h = h\nabla^2 G_h + G_h\, Id$, $\|h\nabla^2 G_h * u\|_{L^2} \lesssim \|G_{h/2}*u\|_{L^2}$ for any u and since the L^2-norm of $G_h * u$ is non-increasing in h, we have for any function v

$$\int |K_h*v|^2\,dx \le \int \left|h\nabla^2 G_h * v\right|^2dx + \int (G_h*v)^2\,dx \lesssim \int \left(G_{h/2}*v\right)^2dx.$$

Plugging this into the inequality above with v playing the role of $\chi_i^k-\chi_i^{k-1}$, multiplying by τ, summing over the block index l and using Jensen's inequality, we can control the contribution to the error coming from the first-order term by

$$T\|\nabla\xi\|_\infty\frac{1}{\sqrt{h}}\left(\frac{1}{N}\sum_{n=1}^{N}\int\left|G_{h/2}*\left(\chi^n-\chi^{n-1}\right)\right|^2dx\right)^{\frac{1}{2}} \le \|\nabla\xi\|_\infty E_0^{\frac{1}{2}}T^{\frac{1}{2}}h^{\frac{1}{4}} = o(1).$$

By Lemma 2.5, the contribution coming from the second-order term in the expansion of ξ is controlled by

$$\|\nabla^2\xi\|_\infty h\sum_{n=1}^{N}\int\left(\frac{|z|}{\sqrt{h}}\right)^3 G_h(z)\int\left|\chi^n-\chi^{n-1}\right|dx\,dz \\ \lesssim \|\nabla^2\xi\|_\infty\int_0^T\int\left|\chi^h(t)-\chi^h(t-h)\right|dx\,dt \lesssim \|\nabla^2\xi\|_\infty E_0(1+T)\sqrt{h} = o(1).$$

Finally, we note that by the time freezing in Step 4, we constructed a telescope sum: Rewriting the summation over the microscopic time step index $n=1,\dots,N$ as the double sum over the microscopic time step index $k=1,\dots,K$ in the respective mesoscopic time intervals and the mesoscopic block index $l=1,\dots,L$, we have for each l,

$$\sum_{k=1}^{K}\left(\chi_i^{K(l-1)+k}-\chi_i^{K(l-1)+k-1}\right)\xi(l\tau)\cdot\nabla G_h*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right) \\ = \left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)\xi(l\tau)\cdot\nabla G_h*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right).$$

Thus we obtain

$$
\begin{aligned}
&h\sum_{n=1}^{N}\frac{1}{\sqrt{h}}\int\left(\chi_i^n-\chi_i^{n-1}\right)\nabla G_h*\left(\xi(l_n\tau)\left(\chi_j^h((l_n-1)\tau)+\chi_j^h(l_n\tau)\right)\right)dx\\
&\qquad=\tau\sum_{l=1}^{L}\frac{1}{\tau}\int\left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)\xi(l\tau)\cdot\left(\sqrt{h}\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx+o(1),
\end{aligned}
$$

which concludes the proof. □

Proof of Lemma 4.8. Step 1: Rough estimate for minority phases. We first argue that if $\{i,j\}\neq\{1,2\}$, that is if the product involves at least one minority phase, then we can estimate this term. Let us first assume that $j\notin\{1,2\}$. By a manipulation as in the proof of Lemma 4.7 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left|\tau\sum_{l=1}^{L}\int\frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx\right|\\
&\lesssim\|\xi\|_\infty\left(\sum_{l=1}^{L}\int\eta\left|G_{h/2}*\left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)\right|^2dx\right)^{\frac12}\left(\sum_{l=0}^{L}\int\eta\left|\sqrt{h}\nabla G_{h/2}*\chi_j^{Kl}\right|^2dx\right)^{\frac12}+o(1),
\end{aligned}
$$

as $h\to0$. Note that for any characteristic function $\chi\in\{0,1\}$, since $\int\nabla G(z)\,dz=0$,

$$
\begin{aligned}
\frac{1}{\sqrt{h}}\int\eta\left|\sqrt{h}\nabla G_{h/2}*\chi\right|^2dx&\lesssim\frac{1}{\sqrt{h}}\int\eta\left|\sqrt{h}\nabla G_{h/2}*\chi\right|dx\\
&\lesssim\frac{1}{\sqrt{h}}\int\left|\sqrt{h}\nabla G_{h/2}(z)\right|\int\eta(x)\left|\chi(x+z)-\chi(x)\right|dx\,dz\\
&\lesssim\frac{1}{\sqrt{h}}\int G_h(z)\int\eta(x)\left|\chi(x+z)-\chi(x)\right|dx\,dz\\
&=\frac{1}{\sqrt{h}}\int\eta\left[(1-\chi)\,G_h*\chi+\chi\,G_h*(1-\chi)\right]dx.
\end{aligned}
$$

Treating the metric term as in the proof of Lemma 4.7 with the triangle inequality and Jensen's inequality afterwards, we obtain the bound $\|\xi\|_\infty$ times

$$
\left(\iint\eta\,d\mu_h\right)^{\frac12}\left(\tau\sum_{l=1}^{L}\varepsilon^2(\chi^{Kl})\right)^{\frac12}+o(1)\leq\left(\frac{1}{\alpha}\tau\sum_{l=1}^{L}\varepsilon^2(\chi^{Kl})+\alpha\iint\eta\,d\mu_h\right)+o(1).
$$

If instead $i\notin\{1,2\}$, using a discrete integration by parts, the antisymmetry of ∇G and a manipulation as in the proof of Lemma 4.7 for ξ, we can exchange the roles of the two phases, Phase i and

Phase j:

$$
\begin{aligned}
&\tau \sum_{l=1}^{L} \int \frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx\\
&\qquad= -\tau \sum_{l=1}^{L} \int \frac{\chi_j^{Kl}-\chi_j^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right)dx + o(1).
\end{aligned}
$$

Thus, we can use the above argument also in this case and the only terms contributing to the sum as $h\to 0$ are the terms involving both majority phases.
In the following we assume that $i=1$ and $j=2$. In the other case, we can just exchange the roles of χ_1 and χ_2 in the following steps and use $-e_1$ as the approximate normal to χ_2 instead and the proof is the same.

Step 2: Substituting χ_2 by $1-\chi_1$. We claim that

$$
\begin{aligned}
&\tau \sum_{l=1}^{L} \int \frac{\chi_1^{Kl}-\chi_1^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_2^{K(l-1)}+\chi_2^{Kl}\right)dx\\
&\qquad= -\tau \sum_{l=1}^{L} \int \frac{\chi_1^{Kl}-\chi_1^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_1^{K(l-1)}+\chi_1^{Kl}\right)dx + o(1).
\end{aligned}
$$

Since $\nabla G * 1 = 0$, the claim is clearly equivalent to proving that we can replace χ_2 by $1-\chi_1$ in the second left-hand side term of the claim. But by $\sum_i \chi_i = 1$ and linearity the resulting error term is

$$
\left|\sum_{j=3}^{P} \tau \sum_{l=1}^{L} \int \frac{\chi_1^{Kl}-\chi_1^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(\sqrt{h}\,\nabla G_h\right)*\left(\chi_j^{K(l-1)}+\chi_j^{Kl}\right)dx\right|,
$$

which can be handled by Step 1.

Step 3: Substitution of ∇G. We want to replace the convolution with ∇G on the left-hand side of the claim by a convolution with the anisotropic kernel

$$
K(z) := \operatorname{sign}(z_1)\, z\, G(z).
$$

To that purpose, we claim that for any characteristic function $\chi\in\{0,1\}$,

$$
\frac{1}{\sqrt{h}}\int \eta \left|\sqrt{h}\,\nabla G_h*\chi - \left(\chi\, K_h*(1-\chi)+(1-\chi)K_h*\chi\right)\right| dx \lesssim \varepsilon^2 + \frac{\sqrt{h}}{r^2} + \frac{\sqrt{h}}{r}E_h(\chi). \tag{92}
$$

Here,

$$\varepsilon^2 := \inf_{\chi^*} \left\{ \frac{1}{\sqrt{h}} \int \eta \left[\chi\, G_h * (1-\chi) + (1-\chi)\, G_h * \chi \right] dx \right.$$
$$\left. - \frac{1}{\sqrt{h}} \int \eta \left[\chi^*\, G_h * (1-\chi^*) + (1-\chi^*)\, G_h * \chi^* \right] dx + \frac{1}{r} \int_{B_{2r}} |\chi - \chi^*| \, dx \right\},$$

where the infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x_1>\lambda\}}$ in direction e_1. Using this inequality for $\chi_1^{K(l-1)}$ and χ_1^{Kl}, we can substitute those two summands and the error is estimated as desired:

$$\frac{1}{\alpha} \|\xi\|_\infty \tau \sum_{l=0}^{L} \frac{1}{\sqrt{h}} \int \eta \left| \sqrt{h}\, \nabla G_h * \chi_1^{Kl} - \left[\chi_1^{Kl}\, K_h * \left(1-\chi_1^{Kl}\right) + \left(1-\chi_1^{Kl}\right) K_h * \chi_1^{Kl} \right] \right| dx$$
$$\lesssim \frac{1}{\alpha} \|\xi\|_\infty \tau \sum_{l=0}^{L} \varepsilon^2(\chi^{Kl}) + o(1), \quad \text{as } h \to 0.$$

Now we give the argument for (92). By measuring length in terms of $\sqrt{h}$, we may assume that $h=1$. Since $\int \nabla G \, dz = 0$ and $\nabla G(z) = -z\, G(z)$, using the identities $u = |u| - 2u_-$ and $u = -|u| + 2u_+$ on the sets $\{z_1 > 0\}$ and $\{z_1 < 0\}$, respectively,

$$\begin{aligned} \nabla G * \chi &= \int z\, G(z) \left(\chi(x+z) - \chi(x)\right) dz \\ &= \int_{\{z_1>0\}} K(z) \left|\chi(x+z) - \chi(x)\right| dz - 2 \int_{\{z_1>0\}} z\, G(z) \left(\chi(x+z) - \chi(x)\right)_- dz \\ &\quad + \int_{\{z_1<0\}} K(z) \left|\chi(x+z) - \chi(x)\right| dz + 2 \int_{\{z_1<0\}} z\, G(z) \left(\chi(x+z) - \chi(x)\right)_+ dz. \end{aligned}$$

Using $|\chi_1 - \chi_2| = (1-\chi_1)\chi_2 + \chi_1(1-\chi_2)$ for $\chi_1,\ \chi_2 \in \{0,1\}$, this implies the pointwise identity

$$\nabla G * \chi = \chi\, K * (1-\chi) + (1-\chi) K * \chi - 2 \int_{\{z_1 \lesseqgtr 0\}} \operatorname{sign}(z_1)\, z\, G(z) \left(\chi(x+z) - \chi(x)\right)_\pm dz,$$

where the last term stands for the sum of the two integrals. Integration w. r. t. $\eta\, dx$ now yields:

$$\int \eta \Big| \nabla G * \chi - \left(\chi K * (1-\chi) + (1-\chi) K * \chi\right) \Big| dx$$
$$\lesssim \int_{\{z_1 \lesseqgtr 0\}} |z| G(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz.$$

As in the argument for (69), we can follow the lines from (84) on so that (68) yields (92).

Step 4: An identity for K. We claim that for any two characteristic functions $\chi,\,\tilde\chi\in\{0,1\}$, we have the pointwise identity

$$\begin{aligned}&(\chi-\tilde\chi)\left(\chi\,K_h*(1-\chi)+(1-\chi)K_h*\chi+\tilde\chi\,K_h*(1-\tilde\chi)+(1-\tilde\chi)K_h*\tilde\chi\right)\\&\qquad=2c_0\,e_1\,(\chi-\tilde\chi)-|\chi-\tilde\chi|\,K_h*(\chi-\tilde\chi)\,.\end{aligned}$$

Indeed, by scaling, we may w. l. o. g. assume $h=1$ and start with

$$\begin{aligned}&(\chi-\tilde\chi)\,\tilde\chi\,K*(1-\tilde\chi)+(\chi-\tilde\chi)\,(1-\tilde\chi)\,K*\tilde\chi\\&\quad=(\chi-1)\,\tilde\chi\left(\int K-K*\tilde\chi\right)+\chi\,(1-\tilde\chi)\,K*\tilde\chi\\&\quad=(\chi-1)\,\tilde\chi\left(\int K\right)+\big((1-\chi)\,\tilde\chi+\chi\,(1-\tilde\chi)\big)K*\tilde\chi\\&\quad=(\chi-1)\,\tilde\chi\left(\int K\right)+|\chi-\tilde\chi|\,K*\tilde\chi.\end{aligned}$$

Exchanging the roles of χ and $\tilde\chi$, one obtains for the second part

$$(\chi-\tilde\chi)\,\chi\,K*(1-\chi)+(\chi-\tilde\chi)\,(1-\chi)\,K*\chi=-\,(\tilde\chi-1)\;\chi\left(\int K\right)-|\chi-\tilde\chi|\,K*\chi.$$

Using the factorization property of G and the symmetry $\int z'G^{d-1}(z')dz'=0$, one computes that for any vector $\xi\in\mathbb{R}^d$

$$\begin{aligned}\xi\cdot\int K=&\int\operatorname{sign}(z_1)\int\left(\xi_1\,z_1+\xi'\cdot z'\right)G^{d-1}(z')\,dz'\,G^1(z_1)\,dz_1=\xi_1\int|z_1|\,G^1(z_1)\,dz_1\\=&2\,\xi_1\int_0^\infty z_1\,G^1(z_1)\,dz_1=2\,\xi_1\int_0^\infty-\frac{d}{dz_1}G^1(z_1)\,dz_1\\=&2\,\xi_1G^1(0)=2\,\xi_1\frac{1}{\sqrt{2\pi}}=2\,c_0\xi_1.\end{aligned}$$

Hence the identity follows from $(\chi-1)\,\tilde\chi-(\tilde\chi-1)\,\chi=\chi-\tilde\chi$.

Step 5: Conclusion. Applying Steps 1 and 2, using the identity in Step 3 for the remaining two terms involving Phases 1 and 2, we end up with the right-hand side of the claim. The error is controlled by $\|\xi\|_\infty$ times

$$\frac{\tau}{\alpha}\sum_{l=1}^{L}\varepsilon^2(\chi^{Kl})+\alpha\iint\eta\,d\mu_h+\tau\sum_{l=1}^{L}\frac{1}{\tau}\int\eta^2\big|\chi^{Kl}-\chi^{K(l-1)}\big|\,|K_h|*\big|\chi^{Kl}-\chi^{K(l-1)}\big|\,dx+o(1),$$

as $h\to0$. Note that $|K|=k$, where k is the kernel defined in the statement of the lemma. It remains to argue that η can be equally distributed on both copies of $\big|\chi^{Kl}-\chi^{K(l-1)}\big|$. For this, note that for $u = \left|\chi^{Kl} - \chi^{K(l-1)}\right| \in [0,1]$,

$$\begin{aligned}\frac{1}{\sqrt{h}}&\left|\int \eta^2 u\, k_h * u\, dx - \int \eta\, u\, k_h * (\eta\, u)\, dx\right| \\ &\leq \frac{1}{\sqrt{h}} \int k_h(z) \int \eta(x)\, u(x)\, u(x+z)\, |\eta(x+z) - \eta(x)|\, dx\, dz \\ &\leq \|\nabla \eta\|_\infty \int \frac{|z|}{\sqrt{h}} k_h(z)\, dz \int \eta\, u\, dx \lesssim \frac{1}{r} \int u\, dx.\end{aligned}$$

Thus, in our case, we can use Lemma 2.6 and bound the error by

$$\frac{1}{\alpha}\|\xi\|_\infty \frac{1}{r} \tau \sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim \frac{1}{\alpha^{\frac{1}{2}}} \|\xi\|_\infty \frac{1}{r} E_0 T h^{\frac{1}{4}} = o(1). \qquad \square$$

Proof of Lemma 4.9. First, we note that it is enough to prove the following similar statement for a fixed mesoscopic time interval:

$$\frac{1}{\tau} \int \eta \left|\chi^K - \chi^0\right|\, k_h * \left(\eta \left|\chi^K - \chi^0\right|\right) dx \lesssim \frac{1}{\alpha^2} \varepsilon^2(\chi^{-1}) + \alpha^{\frac{1}{9}} r^{d-1} + \alpha^{\frac{1}{9}} \frac{1}{\tau} \int_0^\tau \int \eta\, d\mu_h. \tag{93}$$

Indeed, if we multiply (93) by τ and sum over the mesoscopic block index l we obtain the statement.

In the proof of (93), we will exploit the convolution in the normal direction e_1 in Step 1, which will allow us in Step 2 to make use of the quadratic structure of this term.

Step 1: We can estimate the kernel k by a kernel that factorizes in two kernels k^1, k' in normal- and tangential direction, respectively, which are of the form

$$\begin{aligned} k^1(z_1) :=& (1 + z_1^2)^{\frac{1}{2}}\, G^1(z_1), \\ k'(z') :=& (1 + |z'|^2)^{\frac{1}{2}}\, G^{d-1}(z'). \end{aligned}$$

Let us still denote the kernel by k. We have

$$k_h * \left(\eta|\chi^K - \chi^0|\right) \leq \sup_{x_1} \left\{k_h' *' k_h^1 *_1 \left(\eta|\chi^K - \chi^0|\right)\right\} \leq k_h' *' \sup_{x_1} \left\{k_h^1 *_1 \left(\eta|\chi^K - \chi^0|\right)\right\}.$$

The second factor in the right-hand side convolution can be estimated in two ways:

$$\begin{aligned}\sup_{x_1} &\left\{k_h^1 *_1 \left(\eta|\chi^K - \chi^0|\right)\right\} \\ &\leq \min\left\{\int k_h^1\, dz_1 \sup_{x_1} \left(\eta|\chi^K - \chi^0|\right),\ \left(\sup_{x_1} k_h^1\right) \int \eta \left|\chi^K - \chi^0\right| dx_1\right\} \\ &\lesssim \min\left\{1,\ \frac{1}{\sqrt{h}} \int \eta \left|\chi^K - \chi^0\right| dx_1\right\}.\end{aligned}$$

Therefore, we obtain a quadratic term with two copies of $\frac{1}{\sqrt{h}}\int \eta \left|\chi^K - \chi^0\right| dx_1$:

$$\begin{aligned}&\frac{1}{\tau}\int \eta\left|\chi^K-\chi^0\right| k_h * \left(\eta\left|\chi^K-\chi^0\right|\right) dx\\ &\quad\lesssim \frac{1}{\alpha}\int\left(\frac{1}{\sqrt{h}}\int \eta\left|\chi^K-\chi^0\right| dx_1\right) k_h' *' \left(1\wedge \frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^0\right| dx_1\right) dx'. \end{aligned}\tag{94}$$

Step 2: Now we use Lemma 4.5 before integration in x'. We write $\varepsilon^2(x')$ for the first error term in (70) and set

$$\alpha^2(x') := \frac{1}{\sqrt{h}}\int \eta \left|G_h * \left(\chi^{K-1}-\chi^{-1}\right)\right|^2 dx_1,$$

so that the statement of Lemma 4.5 turns into

$$\frac{1}{\sqrt{h}}\int \eta\left|\chi^K-\chi^0\right| dx_1 \lesssim \frac{1}{s}\varepsilon^2(x') + s\mathbf{1}_{\{|x'|<2r\}} + \frac{1}{s^2}\alpha^2(x').$$

We recall the link between the function $\alpha^2(x')$ and the fudge factor α as mentioned in (91) but now before summation over the mesoscopic block index l:

$$\int \alpha^2(x')\, dx' \leq \frac{\alpha^2}{\tau}\int_0^\tau\int \eta\, d\mu_h. \tag{95}$$

Then for any two parameters $s, \tilde{s} \ll 1$ the right-hand side of (94) is estimated by

$$\frac{1}{\alpha}\int\left(\frac{1}{s}\varepsilon^2(x') + s + \frac{1}{s^2}\alpha^2(x')\right) k_h' *' \left(1\wedge\left(\frac{1}{\tilde{s}}\varepsilon^2(x') + \tilde{s}\mathbf{1}_{\{|x'|<2r\}} + \frac{1}{\tilde{s}^2}\alpha^2(x')\right)\right) dx'. \tag{96}$$

For the first and the last summand in the first factor, $\frac{1}{s}\varepsilon^2(x')$ and $\frac{1}{s^2}\alpha^2(x')$, we use the 1 in the minimum on the right. For the second summand on the left, s, we use the second term in the minimum for the pairing. Using the L^1-convolution estimate and (95) we can control (96) by

$$\frac{1}{\alpha}\left(\frac{1}{s}+\frac{s}{\tilde{s}}\right)\int \varepsilon^2(x')\, dx' + \frac{s\tilde{s}}{\alpha} r^{d-1} + \left(\frac{\alpha s}{\tilde{s}^2} + \frac{\alpha}{s^2}\right)\frac{1}{\tau}\int_0^\tau\int \eta\, d\mu_h.$$

By Corollary 4.6 we can estimate $\int \varepsilon^2(x')\, dx'$ as desired and thus obtain (93) by choosing $\tilde{s} = \alpha^{\frac{2}{3}} \ll 1$ and then $s = \alpha^{\frac{4}{9}} \ll 1$. $\square$

Proof of Lemma 4.10. Thanks to the convergence assumption (8), we can apply Proposition 3.1. Using the Euler-Lagrange equation (34) for χ^n and (36), we can identify the first term on the left-hand side as the limit of the first variation of the dissipation functional as $h \to 0$. Following Step

1 of the proof of Lemma 4.7 and then estimating directly as in Step 3, but for ξ instead of η, we obtain

$$
\begin{aligned}
c_0 \sum_{i,j} \sigma_{ij} \int_0^T \int \left(\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i\right) \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt \\
= \lim_{h\to 0} \sum_{i,j} \sigma_{ij} \sum_{n=1}^N \int \left[G_{h/2} * (\chi_i^n - \chi_i^{n-1})\right] \overline{\xi}^n \cdot \left[\left(\sqrt{h} \nabla G_{h/2}\right) * \chi_j^n\right] dx.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, for any pair i,j we have

$$
\begin{aligned}
&\left| \sum_{n=1}^N \int \left[G_{h/2} * (\chi_i^n - \chi_i^{n-1})\right] \overline{\xi}^n \cdot \left[\left(\sqrt{h} \nabla G_{h/2}\right) * \chi_j^n\right] dx \right| \\
&\lesssim \|\xi\|_\infty \left(\sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta \left[G_{h/2} * (\chi_i^n - \chi_i^{n-1})\right]^2 dx \right)^{\frac{1}{2}} \left(h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta \left[\sqrt{h} \nabla G_{h/2} * \chi_j^n\right]^2 dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

The first right-hand side factor is bounded by $\iint \eta \, d\mu_h$. As in Step 1 in the proof of Lemma 4.8, the second right-hand side factor can be controlled by

$$
h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta \left[\left(1 - \chi_j^n\right) G_h * \chi_j^n + \chi_j^n \, G_h * \left(1 - \chi_j^n\right)\right] dx \to 2c_0 \int_0^T \int \eta \, |\nabla \chi_j| \, dt,
$$

as $h \to 0$, where we used Lemma 2.8 to pass to the limit. Thus, using Young's inequality, we have

$$
\begin{aligned}
&\left| \sum_{i,j} \sigma_{ij} \int_0^T \int \left(\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt \right| \\
&\qquad \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha} \sum_{i=1}^P \int_0^T \int \eta \, |\nabla \chi_i| \, dt + \alpha \iint \eta \, d\mu \right).
\end{aligned}
$$

To estimate the second term in the lemma, note that by Young's inequality we have

$$
|\xi \cdot \nu_i \, V_i| \le \|\xi\|_\infty \eta \left(\frac{1}{\alpha} V_i^2 + \alpha \right).
$$

Integrating w. r. t. $|\nabla \chi_i| \, dt$ yields

$$
\left| \int_0^T \int \xi \cdot \nu_i \, V_i \, |\nabla \chi_i| \, dt \right| \le \|\xi\|_\infty \left(\int_0^T \int \eta \, V_i^2 \, |\nabla \chi_i| \, dt + \int_0^T \int \eta \, |\nabla \chi_i| \, dt \right),
$$

which concludes the proof. $\square$

5 Convergence

In Section 3, we identified the limit of the first variation of the energy; in Section 4, we identified the limit of first variation of the metric term up to an error that measures the local approximability by a half space. In this section, we show by soft arguments from Geometric Measure Theory that this error can be made arbitrarily small. Before that, we will state the main ingredients of the proof here.

Definition 5.1. Given $r > 0$, we define the covering

$$\mathscr{B}_r := \{B_r(i) \colon i \in \mathscr{L}_r\}$$

of $[0,\Lambda)^d$, where $\mathscr{L}_r = [0,\Lambda)^d \cap \frac{r}{\sqrt{d}}\mathbb{Z}^d$ is a regular grid of midpoints on $[0,\Lambda)^d$. By construction, for each $n \geq 1$ and each $r > 0$, the covering

$$\{B_{nr}(i) \colon i \in \mathscr{L}_r\} \quad \text{is locally finite,} \tag{97}$$

in the sense that for each point in $[0,\Lambda)^d$, the number of balls containing this point is bounded by a constant $c(d,n)$ which is independent of r. For given $\delta > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\} \in BV$, we define $\mathscr{B}_{r,\delta}$ to be the subset of $\mathscr{B}_r$ consisting of all balls B such that the following two conditions hold:

$$\inf_{\nu^*} \int \eta_{2B} \left|\nu - \nu^*\right|^2 \left|\nabla\chi\right| \leq \delta\, r^{d-1} \quad \text{and} \tag{98}$$

$$\int_{2B} \left|\nabla\chi\right| \geq \frac{1}{2}\omega_{d-1}(2r)^{d-1}, \tag{99}$$

where η_{2B} is a cut-off for $2B$.

Lemma 5.2. *For every $\varepsilon > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\}$, there exists an $r_0 > 0$ such that for all $r \leq r_0$ there exist unit vectors $\nu_B \in S^{d-1}$ such that*

$$\sum_{B \in \mathscr{B}_r} \frac{1}{2} \int \eta_{2B} \left|\nu - \nu_B\right|^2 \left|\nabla\chi\right| \lesssim \varepsilon^2 \int \left|\nabla\chi\right|.$$

The following lemma will be used to control the error terms obtained in Section 4 on the “bad” balls $B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}$.

Lemma 5.3. *For any $\delta > 0$ and any $\chi \colon [0,\Lambda)^d \to \{0,1\} \in BV$, we have*

$$\lim_{r\to 0} \sum_{B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}} \int_{2B} \left|\nabla\chi\right| = 0.$$

In a rescaled version, the following lemma can be used to control the error terms on the “good” balls $B \in \mathscr{B}_{r,\delta}$.

Lemma 5.4. *Let η be a radially symmetric cut-off for the unit ball B. Then for any $\varepsilon > 0$ there exists $\delta = \delta(d,\varepsilon) > 0$ such that for any $\chi\colon [0,\Lambda)^d \to \{0,1\}$ with*

$$\int \eta \, |\nu - e_1|^2 \, |\nabla \chi| \le \delta^2 \tag{100}$$

there exists a half space χ^* in direction e_1 such that

$$\left| \int_B (|\nabla \chi| - |\nabla \chi^*|) \right| \le \varepsilon^2, \quad \int_B |\chi - \chi^*| \, dx \le \varepsilon^2. \tag{101}$$

Lemma 5.5. *Let η be a cut-off for the unit ball B. Then for any $\varepsilon > 0$ there exists $\delta = \delta(d,P,\varepsilon) > 0$ such that for any $\chi\colon [0,\Lambda)^d \to \{0,1\}^P$ with $\sum_i \chi_i = 1$, the following statement holds: Whenever we can approximate each normal separately, i. e.*

$$\sum_{i=1}^P \inf_{\nu_i^*} \frac{1}{2} \int \eta \, |\nu_i - \nu_i^*|^2 \, |\nabla \chi_i| \le \delta^2,$$

then we can do so with one normal $\nu^* \in S^{d-1}$ and its inverse $-\nu^*$:

$$\min_{i\neq j} \inf_{\nu^*} \left\{ \sum_{k \notin \{i,j\}} \int_B |\nabla \chi_k| + \frac{1}{2} \int_B |\nu_i - \nu^*|^2 \, |\nabla \chi_i| + \frac{1}{2} \int_B |\nu_j + \nu^*|^2 \, |\nabla \chi_j| \right\} \le \varepsilon^2.$$

5.1 Proof of Theorem 1.4

Using Proposition 4.1 and the lemmas from above, we can give the proof of the main result. The proof consists of three steps:

1. Post-processing Propositions 3.1 and 4.1, using the Euler-Lagrange equation (34) and by making the half space time-dependent,
2. Estimates for fixed time and
3. Integration in time.

Proof of Theorem 1.4. Step 1: Post-processing Propositions 3.1 and 4.1. Let us first link the results we obtained in Sections 3 and 4. For any fixed vector $\nu^* \in S^{d-1}$ and any test function $\xi_B \in C_0^\infty((0,T)\times B, \mathbb{R}^d)$, supported in a space-time cylinder $(0,T)\times B$, we claim

$$\left| \sum_{i,j} \sigma_{ij} \int_0^T \int (\nabla \cdot \xi_B - \nu_i \cdot \nabla \xi_B \, \nu_i - 2\, \xi_B \cdot \nu_i \, V_i) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt \right| \tag{102}$$

$$\begin{aligned} \lesssim \|\xi_B\|_\infty \Bigg[\left(\min_{i \neq j} \int_0^T \left(\frac{1}{\alpha^2} \mathscr{E}_{ij}^2(\nu^*, t) + \alpha^{\frac{1}{9}} r^{d-1} \right) dt \right) \wedge \left(\frac{1}{\alpha} \sum_{i=1}^P \int_0^T \int_B |\nabla \chi_i| \, dt \right) \\ + \alpha^{\frac{1}{9}} \iint \eta_B \, d\mu + \alpha \sum_{i=1}^P \int_0^T \int \eta_B \, V_i^2 \, |\nabla \chi_i| \, dt \Bigg], \end{aligned}$$

where $\mathscr{E}_{ij}^2$ is defined via

$$\mathscr{E}_{ij}^2(\nu^*,t) := \sum_{k\notin\{i,j\}} \int \eta_{2B}\,|\nabla\chi_k(t)| + \int \eta_{2B}\,|\nu_i(t)-\nu^*|^2\,|\nabla\chi_i(t)|$$
$$+ \int \eta_{2B}\,|\nu_j(t)+\nu^*|^2\,|\nabla\chi_j(t)|$$
$$+ \inf_{\chi^*}\Bigg\{ \left|\int \eta_B\,(|\nabla\chi_i(t)| - |\nabla\chi^*|)\right| + \frac{1}{r}\int_{2B} |\chi_i(t)-\chi^*|\,dx$$
$$+ \left|\int \eta_B\,(|\nabla\chi_j(t)| - |\nabla\chi^*|)\right| + \frac{1}{r}\int_{2B} |\chi_j(t)-(1-\chi^*)|\,dx \Bigg\}.$$

The infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x\cdot\nu^*>\lambda\}}$ in direction ν^*.
Argument for (102): By symmetry, we may assume w. l. o. g. that the minimum on the right-hand side of (102) is realized for $i=1$ and $j=2$. The Euler-Lagrange equation (34) of the minimizing movements interpretation (5) links Proposition 3.1 with the metric term:

$$\lim_{h\to0}\int_0^T -\delta E_h(\,\cdot\,-\chi^h(t-h))(\chi^h(t),\xi_B(t))\,dt$$
$$= -c_0\sum_{i,j}\sigma_{ij}\int_0^T\int (\nabla\cdot\xi_B - \nu_i\cdot\nabla\xi_B\,\nu_i)\,\frac{1}{2}\,(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|)\,dt.$$

Before applying the results of Section 4 we symmetrize the second term on the left-hand side of (66): We claim that we can replace

$$\sigma_{12}\left(\int \xi_B\cdot\nu^*\,V_1\,|\nabla\chi_1| + \int \xi_B\cdot(-\nu^*)\,V_2\,|\nabla\chi_2|\right) \tag{103}$$

which appears on the left-hand side of (66) by the symmetrized term

$$\sum_{i,j}\sigma_{ij}\int \xi_B\cdot\nu_i\,V_i\frac{1}{2}\,(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|) \tag{104}$$

which appears in the weak formulation (6). Then using Proposition 4.1 and this symmetrization or the rough estimate Lemma 4.10 yields (102). Now we show how to replace (103) by (104).
We start by noting that the sum in (104) contains two terms involving only Phases 1 and 2. The contribution to the sum is

$$\sigma_{12}\int \xi_B\cdot(\nu_1\,V_1+\nu_2\,V_2)\,\frac{1}{2}\,(|\nabla\chi_1| + |\nabla\chi_2| - |\nabla(\chi_1+\chi_2)|)\,,$$

which can be brought into the form of (103) at the expense of an error which is controlled by $\|\xi_B\|_\infty$ times

$$\int \eta_B\,|\nu_1-\nu^*|\,|V_1|\,|\nabla\chi_1| + \int \eta_B\,|\nu_2+\nu^*|\,|V_2|\,|\nabla\chi_2|\,.$$

Note that by Young's inequality we have

$$|\nu_1 - \nu^*|\, |V_1| \leq \frac{1}{\alpha}\, |\nu_1 - \nu^*|^2 + \alpha V_1^2,$$

so that we can estimate both terms after integration in time by

$$\frac{1}{\alpha}\int_0^T \left(\int \eta_B\, |\nu_1 - \nu^*|^2\, |\nabla \chi_1| + \int \eta_B\, |\nu_2 + \nu^*|^2\, |\nabla \chi_2| \right) dt + \alpha \sum_{i=1}^P \int_0^T \int \eta_B\, V_i^2\, |\nabla \chi_i|\, dt.$$

We are left with estimating the summands in (104) with $\{i,j\} \neq \{1,2\}$. For those terms we can use Young's inequality in the following form

$$|\nu_i|\, |V_i| \leq \frac{1}{\alpha} + \alpha V_i^2$$

so that after integration in time these terms are controlled by $\|\xi_B\|_\infty$ times

$$\frac{1}{\alpha} \sum_{i=3}^P \int \eta_B\, |\nabla \chi_i| + \alpha \sum_{i=1}^P \int_0^T \int \eta_B\, V_i^2\, |\nabla \chi_i|\, dt,$$

which concludes the argument for the symmetrization and thus for (102).
Here, we see, why we needed to introduce extra terms in $\mathscr{E}_1$ compared to the terms that were already present in the definition of $\mathscr{E}_1$ in Section 4. These different terms are sometimes called *tilt-excess* and *excess energy*, respectively.

Now let $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$ be given. First, we localize ξ in space according to the covering $\mathscr{B}_r$ from Definition 5.1. To do so, we introduce a subordinate partition of unity $\{\varphi_B\}_{B\in\mathscr{B}_r}$ and set $\xi_B := \varphi_B \xi$. Then $\xi = \sum_{B\in\mathscr{B}_r} \xi_B$, $\xi_B \in C_0^\infty(B)$ and $\|\xi_B\|_\infty \leq \|\xi\|_\infty$. Given a radially symmetric and radially non-increasing cut-off η of $B_1(0)$ in $B_2(0)$, for each ball B in the covering, we can construct a cut-off η_B of B in $2B$ by shifting and rescaling. Given any measurable function $\nu^* \colon (0,T) \to S^{d-1}$ and any $\alpha \in (0,1)$ we claim

$$\begin{aligned}
&\left| \sum_{i,j} \sigma_{ij} \int_0^T \int \left(\nabla\cdot \xi_B - \nu_i \cdot \nabla \xi_B\, \nu_i - 2\, \xi_B \cdot \nu_i\, V_i \right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right) dt \right| \qquad (105)\\
&\lesssim \|\xi\|_\infty \left[\int_0^T \left(\frac{1}{\alpha^2} \mathscr{E}_B^2(\nu^*(t),t) + \alpha^{\frac{1}{9}} r^{d-1} \right) \wedge \left(\frac{1}{\alpha} \sum_{i=1}^P \int_B |\nabla \chi_i| \right) dt + \alpha^{\frac{1}{9}} \iint \eta\, d\mu \right.\\
&\qquad\qquad \left. + \alpha \sum_{i=1}^P \int_0^T \int \eta\, V_i^2\, |\nabla \chi_i|\, dt \right],
\end{aligned}$$

where $\mathscr{E}_B^2(\nu^*,t) := \min_{i\neq j} \mathscr{E}_{ij}^2(\nu^*,t)$ for $\nu^* \in S^{d-1}$.
Now we give the argument that (102) implies (105). We approximate the measurable function ν^*

in time by a piecewise constant function. Let $0 = T_0 < \cdots < T_M = T$ denote a partition of $(0,T)$ such that the approximation ν_M^* of ν^* is constant on each interval $[T_{m-1}, T_m)$. Since the measures on the left-hand side are absolutely continuous in time, we can approximate ξ_B by vector fields which vanish at the points T_m and both, the curvature and the velocity term converge. Therefore, we can apply (102) on each time interval (T_{m-1}, T_m). Lebesgue's Dominated Convergence gives us the convergence of the integral on the right-hand side and thus (105) holds.

Step 2: Estimates for fixed time. Let $t \in (0,T)$ be fixed. We will omit the argument t in the following. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$ (to be determined later). Let $\mathscr{B}_{r,\delta}$ be defined as the set of good balls in the lattice:

$$\mathscr{B}_{r,\delta} := \Bigg\{ B \in \mathscr{B}_r \colon \sum_{i=1}^P \inf_{\nu^*} \int \eta_{2B} \left|\nu_i - \nu^*\right|^2 \left|\nabla \chi_i\right| \leq \delta r^{d-1} \text{ and } \sum_{i=1}^P \int_{2B} \left|\nabla \chi_i\right| \geq \frac{1}{2} \omega_{d-1} (2r)^{d-1} \Bigg\}.$$

For $B \in \mathscr{B}_{r,\delta}$, and $i = 1, \ldots, P$, we denote by $\nu_{B,i}$ the vector ν^* for which the infimum is attained, so that

$$\sum_{i=1}^P \frac{1}{2} \int \eta_{2B} \left|\nu_i - \nu_{B,i}\right|^2 \left|\nabla \chi_i\right| \leq \delta\, r^{d-1}.$$

By a rescaling and since η is radially symmetric, we can upgrade Lemma 5.5, so that for given $\gamma > 0$, we can find $\delta = \delta(d,\gamma) > 0$ (independent of χ) and $\nu_B \in S^{d-1}$, such that

$$\min_{i \neq j} \left\{ \sum_{k \notin \{i,j\}} \int \eta_B \left|\nabla \chi_k\right| + \frac{1}{2} \int \eta_B \left|\nu_i - \nu_B\right|^2 \left|\nabla \chi_i\right| + \frac{1}{2} \int \eta_B \left|\nu_j + \nu_B\right|^2 \left|\nabla \chi_j\right| \right\} \leq \gamma\, r^{d-1}.$$

Rescaling Lemma 5.4, we can define $\gamma = \gamma(\varepsilon) > 0$ and a half space χ^* in direction ν_B, such that

$$\mathscr{E}_B^2(\nu_B, t) \leq \varepsilon^2 r^{d-1}.$$

These two steps give us the dependence of δ on ε. Using the lower bound on the perimeters on $B \in \mathscr{B}_{r,\delta}(t)$, we obtain

$$\sum_{B \in \mathscr{B}_{r,\delta}} \left(\frac{1}{\alpha^2} \mathscr{E}_B^2(\nu_B, t) + \alpha^{\frac{1}{9}} r^{d-1} \right) \lesssim \sum_{B \in \mathscr{B}_{r,\delta}} \left(\frac{1}{\alpha^2} \varepsilon^2 + \alpha^{\frac{1}{9}} \right) r^{d-1} \lesssim \left(\frac{1}{\alpha^2} \varepsilon^2 + \alpha^{\frac{1}{9}} \right) \sum_{i=1}^P \int \left|\nabla \chi_i\right|.$$

Note that for the balls $B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}$, we have by Lemma 5.3:

$$\sum_{B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}} \sum_{i=1}^P \int_B \left|\nabla \chi_i\right| \to 0, \quad \text{as } r \to 0. \tag{106}$$

The speed of convergence depends on χ and ε (through δ).

Step 3: Integration in time. Using Lebesgue's Dominated Convergence Theorem, we can integrate the pointwise-in-time estimates of Step 2. Recalling the decomposition $\xi = \sum_B \xi_B$ and using the finite overlap (97), we have

$$
\begin{aligned}
&\left|\sum_{i,j}\sigma_{ij}\int_0^T\int \left(\nabla\cdot\xi-\nu_i\cdot\nabla\xi\,\nu_i-2\,\xi\cdot\nu_i\,V_i\right)\left(|\nabla\chi_i|+|\nabla\chi_j|-|\nabla(\chi_i+\chi_j)|\right)dt\right|\\
&\lesssim \sum_{B\in\mathscr{B}_r}\left|\sum_{i,j}\sigma_{ij}\int_0^T\int \left(\nabla\cdot\xi_B-\nu_i\cdot\nabla\xi_B\,\nu_i-2\,\xi_B\cdot\nu_i\,V_i\right)\left(|\nabla\chi_i|+|\nabla\chi_j|-|\nabla(\chi_i+\chi_j)|\right)dt\right|\\
&\lesssim \|\xi\|_\infty\Bigg[\left(\frac{1}{\alpha^2}\varepsilon^2+\alpha^{\frac19}\right)\int_0^T\sum_{i=1}^P\int|\nabla\chi_i|\,dt+\int_0^T\sum_{B\in\mathscr{B}_r-\mathscr{B}_{r,\delta}(t)}\frac{1}{\alpha}\sum_{i=1}^P\int_B|\nabla\chi_i|\,dt\\
&\qquad\qquad+\alpha^{\frac19}\iint d\mu+\alpha\sum_{i=1}^P\int_0^T\int V_i^2\,|\nabla\chi_i|\,dt\Bigg].
\end{aligned}
$$

Since by the energy-dissipation estimate (10) we have $E(\chi(t))\le E_0$ and can control the first term. By Lebesgue's Dominated Convergence and (106), the second term vanishes as $r\to 0$. By (16) and Proposition 2.2, we can handle the last two terms. Thus we obtain

$$
\begin{aligned}
&\left|\sum_{i,j}\sigma_{ij}\int_0^T\int \left(\nabla\cdot\xi-\nu_i\cdot\nabla\xi\,\nu_i-2\,\xi\cdot\nu_i\,V_i\right)\left(|\nabla\chi_i|+|\nabla\chi_j|-|\nabla(\chi_i+\chi_j)|\right)dt\right|\\
&\qquad\lesssim\|\xi\|_\infty\left(\frac{1}{\alpha^2}\varepsilon^2E_0T+\alpha^{\frac19}(1+T)E_0\right).
\end{aligned}
$$

Taking first the limit ε to zero and then α to zero yields (6), which concludes the proof of Theorem 1.4. □

5.2 Proofs of the lemmas

Proof of Lemma 5.2. Let $\varepsilon>0$ be given and w. l. o. g. $\int|\nabla\chi|>0$. Since the normal ν is measurable, we can approximate it by a continuous vector field $\tilde\nu\colon[0,\Lambda)^d\to\overline{B}$ in the sense that

$$
\sum_{B\in\mathscr{B}_r}\frac12\int_B|\nu-\tilde\nu|^2\,|\nabla\chi|\lesssim\int|\nu-\tilde\nu|^2\,|\nabla\chi|\le\varepsilon^2\int|\nabla\chi|,
$$

where we have used the finite overlap property (97). Since $\tilde\nu$ is continuous, we can find $r_0>0$ such that for any $r\le r_0$ we can find vectors $\tilde\nu_B$ with $|\tilde\nu_B|\le1$ with

$$
\sum_{B\in\mathscr{B}_r}\frac12\int_B|\tilde\nu-\tilde\nu_B|^2\,|\nabla\chi|\le\varepsilon^2\int|\nabla\chi|.
$$

The only missing step is to argue that we can also choose $\nu_B \in S^{d-1}$. If $|\tilde{\nu}_B| \geq 1/2$, this is clear because then $|\nu - \tilde{\nu}_B/|\tilde{\nu}_B|| \leq 2\,|\nu - \tilde{\nu}_B|$. If $|\tilde{\nu}_B| \leq 1/2$, we have the easy estimate

$$|\nu - \tilde{\nu}_B| \geq \frac{1}{2} \geq \frac{1}{4}\left(|\nu| + |\nu_B|\right) \geq \frac{1}{4}\,|\nu - \nu_B|$$

for any $\nu_B \in S^{d-1}$. $\square$

Proof of Lemma 5.3. Let $\varepsilon, \delta > 0$ be arbitrary. Note that a ball in $\mathscr{B}_r - \mathscr{B}_{r,\delta}$ satisfies

$$\inf_{\nu^*} \int_{2B} |\nu - \nu^*|^2\,|\nabla\chi| \geq \delta r^{d-1} \quad \text{or} \tag{107}$$

$$\int_{2B} |\nabla\chi| \leq \frac{1}{2}\omega_{d-1} r^{d-1}. \tag{108}$$

Step 1: Balls satisfying (107). By Lemma 5.2, for any $\gamma > 0$, to be chosen later, there exists $r_0 = r_0(\gamma, \delta, \chi) > 0$, such that for every $r \leq r_0$ we can find vectors $\nu_B \in S^{d-1}$ such that

$$\sum_{B \in \mathscr{B}_r} \int_{2B} |\nu - \nu_B|^2\,|\nabla\chi| \lesssim \gamma\delta \int |\nabla\chi|\,. \tag{109}$$

Thus we have

$$\#\left\{B\colon \int_{2B} |\nu - \nu_B|^2\,|\nabla\chi| \geq \delta r^{d-1}\right\} \leq \sum_B \frac{1}{\delta r^{d-1}} \int_{2B} |\nu - \nu_B|^2\,|\nabla\chi| \overset{(109)}{\lesssim} \frac{\gamma}{r^{d-1}} \int |\nabla\chi|\,. \tag{110}$$

Using that the covering is locally finite and De Giorgi's Structure Theorem, we have

$$\sum_{B:(107)} \int_{2B} |\nabla\chi| \lesssim \int_{\bigcup_{(107)} 2B} |\nabla\chi| = \mathscr{H}^{d-1}\Big(\partial^*\Omega \cap \bigcup_{(107)} 2B\Big).$$

Since $\partial^*\Omega$ is rectifiable, we can find Lipschitz graphs Γ_n such that $\partial^*\Omega \subset \bigcup_{n=1}^\infty \Gamma_n$. Therefore,

$$\mathscr{H}^{d-1}\Big(\partial^*\Omega \cap \bigcup_{(107)} 2B\Big) \leq \sum_{n=1}^N \mathscr{H}^{d-1}\Big(\Gamma_n \cap \bigcup_{(107)} 2B\Big) + \mathscr{H}^{d-1}\Big(\partial^*\Omega - \bigcup_{n \leq N} \Gamma_n\Big).$$

Note that for any ball B

$$\mathscr{H}^{d-1}\left(\Gamma_n \cap 2B\right) \lesssim \left(1 + \mathrm{Lip}\ \Gamma_n\right) r^{d-1}$$

and thus

$$\begin{aligned}\mathscr{H}^{d-1}\Big(\Gamma_n \cap \bigcup_{(107)} 2B\Big) &\leq \sum_{B:(107)} \mathscr{H}^{d-1}\left(\Gamma_n \cap 2B\right)\\ &\lesssim \Big(1+\max_{n\leq N} \mathrm{Lip}\ \Gamma_n\Big)\, r^{d-1} \#\left\{B\colon (107)\right\}.\end{aligned}$$

Using (110), we have

$$\sum_{B:(107)} \int_{2B} |\nabla \chi| \lesssim N\Big(1+\max_{n\leq N} \mathrm{Lip}\ \Gamma_n\Big)\, \gamma \int |\nabla\chi| + \mathscr{H}^{d-1}\Big(\partial^*\Omega - \bigcup_{n\leq N} \Gamma_n\Big).$$

Now, choose N large enough such that

$$\mathscr{H}^{d-1}\Big(\partial^*\Omega - \bigcup_{n\leq N} \Gamma_n\Big) \leq \varepsilon^2.$$

Then, choose $\gamma > 0$ small enough, such that

$$N\Big(1+\max_{n\leq N} \mathrm{Lip}\ \Gamma_n\Big)\, \gamma \int |\nabla\chi| \leq \varepsilon^2.$$

Step 2: Balls satisfying (108). By De Giorgi's Structure Theorem (Theorem 4.4 in [41]), we may restrict to balls B which in addition satisfy $\partial^*\Omega \cap 2B \neq \emptyset$ and pick $x \in \partial^*\Omega \cap 2B$. Note that since B has radius r we have

$$B_{2r}(x) \subset 4B \subset B_{6r}(x).$$

Therefore, if (108) holds,

$$\int_{B_{2r}(x)} |\nabla\chi| \leq \int_{4B} |\nabla\chi| \leq \frac{1}{2}\omega_{d-1}(2r)^{d-1}.$$

For $x \in \partial^*\Omega$ we have

$$\liminf_{r\to 0} \frac{1}{r^{d-1}} \int_{B_r(x)} |\nabla\chi| \geq \omega_{d-1}$$

and thus in particular

$$\mathbf{1}\Big(\Big\{x \in \partial^*\Omega\colon \int_{B_r(x)} |\nabla\chi| \leq \frac{1}{2}\omega_{d-1} r^{d-1}\Big\}\Big) \to 0$$

pointwise as $r \to 0$. By De Giorgi's Structure Theorem (Theorem 4.4 in [41]), the finite overlap and Lebesgue's Dominated Convergence Theorem, we thus have

$$\sum_{B:(108)} \int_{2B} |\nabla\chi| \lesssim \mathscr{H}^{d-1}\Big(\partial^*\Omega \cap \bigcup_{B:(108)} 2B\Big) \to 0$$

as $r \to 0$. □

Proof of Lemma 5.4. Let us first prove that for any χ satisfying (100), we have

$$(1-\delta)\int \eta\,|\nabla\chi| \le \left|\int \chi\,\nabla\eta\,dx\right| + \delta. \tag{111}$$

Indeed, we have

$$\left|\int \eta\,\nu\,|\nabla\chi|\right| \ge \left|\int \eta\,e_1\,|\nabla\chi|\right| - \left|\int \eta\,(\nu-e_1)\,|\nabla\chi|\right| = \int \eta\,|\nabla\chi| - \left|\int \eta\,(\nu-e_1)\,|\nabla\chi|\right|.$$

By Young's inequality we have $|\nu-e_1| \le \frac{1}{\delta}\,|\nu-e_1|^2+\delta$, so that by (100) we can estimate the last right-hand side term

$$\left|\int \eta\,(\nu-e_1)\,|\nabla\chi|\right| \le \int \eta\,|\nu-e_1|\,|\nabla\chi| \overset{(100)}{\le} \delta + \delta\int \eta\,|\nabla\chi|\,.$$

Therefore

$$\left|\int \eta\,\nu\,|\nabla\chi|\right| \ge (1-\delta)\int \eta\,|\nabla\chi| - \delta,$$

which is (111).

Now we give an indirect argument for the lemma. Suppose there exists an $\varepsilon>0$ and a sequence $\{\chi_n\}_n$ such that

$$\int \eta\,|\nu_n-e_1|^2\,|\nabla\chi_n| \le \frac{1}{n^2} \tag{112}$$

while for all half spaces χ^* in direction e_1,

$$\int_B |\nabla\chi_n| \ge \varepsilon^2 + \int_B |\nabla\chi^*|\,, \quad \int_B |\nabla\chi^*| \ge \varepsilon^2 + \int_B |\nabla\chi_n|\,, \quad \text{or} \quad \int_B |\chi_n-\chi^*|\,dx \ge \varepsilon^2. \tag{113}$$

By (112), we can use (111) for χ_n and obtain:

$$\int \eta\,|\nabla\chi_n| \le \frac{1}{1-1/n}\left(\int |\nabla\eta|\,dx + \frac{1}{n}\right) \quad \text{stays bounded as } n\to\infty.$$

Therefore, after passage to a subsequence and a diagonal argument to exhaust the open ball $\{\eta > 0\}$, we find χ such that

$$\chi_n \to \chi \quad \text{pointwise a. e. on } \{\eta>0\}. \tag{114}$$

By (112) we have

$$2\int \eta\,|\nabla\chi_n| - 2\int \nabla\eta\cdot e_1\;\chi_n\,dx = \int \eta\,|\nu_n-e_1|^2\,|\nabla\chi_n| \le \frac{1}{n^2} \to 0.$$

Since the first term on the left-hand side is lower semi-continuous and the second one is continuous, we can pass to the limit in the above inequality and obtain

$$\int \eta \, |\nu - e_1|^2 \, |\nabla \chi| = 2 \int \eta \, |\nabla \chi| - 2 \int \nabla \eta \cdot e_1 \, \chi \, dx \leq 0.$$

Hence

$$\nu = e_1 \quad |\nabla \chi| \text{-a. e. in } \{\eta > 0\}.$$

A mollification argument shows that there exists a half space χ^* in direction e_1 such that

$$\chi = \chi^* \quad \text{a. e. in } \{\eta > 0\}.$$

Because of (114), this rules out

$$\int_B |\chi_n - \chi^*| \geq \varepsilon^2$$

on the one hand. On the other hand, by lower semi-continuity of the perimeter, also

$$\int_B |\nabla \chi^*| \geq \varepsilon^2 + \int_B |\nabla \chi_n|$$

is ruled out. To obtain a contradiction also w. r. t. the first statement in (113), let $\tilde{\eta} \leq \eta$ be a cut-off for B in $(1+\delta)B$. Since (111) holds also for $\tilde{\eta}$ instead of η, we have

$$\begin{aligned} \varepsilon^2 + \int_B |\nabla \chi^*| \overset{(113)}{\leq} \int_B |\nabla \chi_n| \leq \int \tilde{\eta} \, |\nabla \chi_n| \overset{(111)}{\leq} \frac{1}{1 - 1/n} \left(\left| \int \chi_n \, \nabla \tilde{\eta} \, dx \right| + \frac{1}{n} \right) \\ \overset{(114)}{\to} \left| \int \chi^* \, \nabla \tilde{\eta} \, dx \right| = \left| \int \tilde{\eta} \, \nabla \chi^* \right| \leq \int_{(1+\delta)B} |\nabla \chi^*| \, . \end{aligned}$$

Since χ^* is a half space and therefore has no mass on ∂B, we have

$$\int_{(1+\delta)B} |\nabla \chi^*| \to \int_B |\nabla \chi^*| \, , \quad \text{as } \delta \to 0,$$

which is a contradiction. □

Proof of Lemma 5.5. We give an indirect argument. Assume there exists a sequence of characteristic functions $\{\chi^n\}_n$ with $\sum_i \chi_i^n = 1$ a. e., a number $\varepsilon > 0$ such that we can find approximate normals $\nu_i^{*n} \in S^{d-1}$ with

$$\sum_{i=1}^P \frac{1}{2} \int \eta \, |\nu_i^n - \nu_i^{*n}|^2 \, |\nabla \chi_i^n| \leq \frac{1}{n^2}$$

while for all $\nu^* \in S^{d-1}$, $n \in \mathbb{N}$ and any pair of indices $i \neq j$, we have

$$\sum_{k \notin \{i,j\}} \int_B |\nabla \chi_k^n| + \frac{1}{2} \int_B |\nu_i^n - \nu^*|^2 |\nabla \chi_i^n| + \frac{1}{2} \int_B \left|\nu_j^n + \nu^*\right|^2 \left|\nabla \chi_j^n\right| \geq \varepsilon^2. \qquad (115)$$

Since S^{d-1} is compact, we can find vectors $\nu^* \in S^{d-1}$, such that, after passing to a subsequence if necessary, $\nu_i^{*n} \to \nu_i^*$ as $n \to \infty$. Following the lines of the proof of Lemma 5.4, we find

$$\int \eta \, |\nabla \chi_i^n| \leq \frac{1}{1 - 1/n} \left(\int |\nabla \eta| \, dx + \frac{1}{n} \right) \quad \text{stays bounded as } n \to \infty$$

so that there exist $\chi_i \in \{0, 1\}$ with

$$\chi_i^n \to \chi_i \quad \text{pointwise a. e. on } \{\eta > 0\} \qquad (116)$$

and

$$\frac{1}{2} \int \eta \, |\nu_i - \nu_i^*|^2 \, |\nabla \chi_i| \leq \liminf_{n \to \infty} \frac{1}{2} \int \eta \, |\nu_i^n - \nu_i^{*n}|^2 \, |\nabla \chi_i^n| = 0.$$

Therefore, $\nu_i = \nu_i^*$ $|\nabla \chi_i|$- a. e. and each $\chi_i = \chi_i^*$ is a half space in direction ν_i^*. Continuing in our setting now, we note that the condition $\sum_i \chi_i^n = 1$ carries over to the limit: $\sum_i \chi_i^* = 1$. Therefore there exists a pair of indices $i \neq j$ (w. l. o. g. $i = 1$, $j = 2$) such that for all $k \geq 3$ $\chi_k^* = 0$ in B. Then the other two half spaces are complementary, $\chi_2^* = (1 - \chi_1^*)$ and in particular $\nu_1^* = -\nu_2^* =: \nu^*$. As in the proof of Lemma 5.4, we have

$$\int_B |\nabla \chi_i^n| \to \int_B |\nabla \chi_i^*| \, .$$

Together with (116), we can take the limit $n \to \infty$ in (115) and obtain

$$\sum_{k \geq 3} \int_B |\nabla \chi_1^*| + \frac{1}{2} \int_B |\nu_1^* - \nu^*|^2 \, |\nabla \chi_1^*| + \frac{1}{2} \int_B |\nu_2^* + \nu^*|^2 \, |\nabla \chi_2^*| \geq \varepsilon^2,$$

which is a contradiction since the left-hand side vanishes by construction. □

Chapter 2

Brakke's inequality for the thresholding scheme

This chapter describes the forthcoming result [52] with Felix Otto. We establish the convergence of the thresholding scheme to Brakke's motion by mean curvature in the two-phase case. As in Chapter 1 our result is just a conditional convergence result in the sense that we assume the time-integrated energies to converge. In a forthcoming work we will generalize this result to the multi-phase case.

1 Introduction

Our proof is based on the observation that thresholding does not only have a *global* minimizing movements interpretation, but indeed solves a *family* of *localized* minimization problems. In Section 2 we state our main results, in particular Theorem 2.2. We use De Giorgi's variational interpolation for these localized minimization problems to derive an *exact* energy-dissipation relation and pass to the limit in the according terms with help of our strengthened convergence. Section 4 provides the tools for these results. We first recall the known results from the abstract framework of gradient flows in metric spaces (cf. Chapter 3 in [7]). Then we pass to the limit $h \to 0$ in these terms with help of our strengthened convergence.

Let us recall the definition of the thresholding scheme in the two-phase case and the basic notation.

Algorithm 1.1. *Given the phase Ω^{n-1} at time $t = (n-1)h$, obtain the evolved phase Ω^n at time $t = nh$ by:*

1. *Convolution step:* $\phi := G_h * \mathbf{1}_{\Omega^{n-1}}$.
2. *Thresholding step:* $\Omega^n := \left\{\phi > \frac{1}{2}\right\}$.

Here

$$G_h(z) := \frac{1}{(2\pi h)^{d/2}} \exp\left(-\frac{|z|^2}{2h}\right)$$

denotes the heat kernel at time h. For convenience we will work with periodic boundary conditions, i.e. on the flat torus $[0,\Lambda)^d$. We write $\int dx$ short for $\int_{[0,\Lambda)^d} dx$ and $\int dz$ short for $\int_{\mathbb{R}^d} dz$. We write $\chi^n := \mathbf{1}_{\Omega^n}$ to denote the characteristic function of the phase Ω^n at time step n and denote its piecewise constant interpolation by

$$\chi^h(t) := \chi^n = \mathbf{1}_{\Omega^n} \quad \text{for } t \in [nh, (n+1)h).$$

However, we will mostly use a nonlinear interpolation which will be introduced later. Selim Esedoğlu and Felix Otto [33] showed that thresholding preserves the gradient flow structure of (multi-phase) mean curvature flow in the sense that it can be viewed as a minimizing movements scheme

$$\chi^n = \arg\min_u \left\{ E_h(u) + \frac{1}{2h} \mathrm{d}_h^2(u, \chi^{n-1}) \right\}, \tag{1}$$

where the dissipation functional

$$\frac{1}{2h} \mathrm{d}_h^2(u, \chi) := \frac{1}{\sqrt{h}} \int \left[G_{h/2} * (u - \chi)\right]^2 dx \tag{2}$$

is the square of a metric and the energy is

$$E_h(u) := \frac{1}{\sqrt{h}} \int (1-u)\, G_h * u \, dx, \tag{3}$$

an approximation of the perimeter functional. Indeed, the functionals E_h Γ-converge to

$$E(\chi) := c_0 \int |\nabla \chi|\,, \qquad \text{for } \chi\colon [0,\Lambda)^d \to \{0,1\}.$$

Here $c_0 = \frac{1}{\sqrt{2\pi}}$ is some universal constant. Throughout this chapter, we will work with periodic boundary conditions, i. e. on the flat torus $[0,\Lambda)^d$. We write $A \lesssim B$ to express that $A \leq CB$ for a generic constant $C < \infty$ that only depends on the dimension d and on the size Λ of the domain.

2 Brakke's inequality

The main statement of this work is Theorem 2.2 below. Assuming there was no drop of energy as $h \to 0$, i.e.

$$\int_0^T E_h(\chi^h)\, dt \to \int_0^T E(\chi)\, dt, \tag{4}$$

it states that the limit of the approximate solutions satisfies a BV-version of Brakke's inequality [14].

Brakke's inequality is a weak formulation of motion by mean curvature $V = \frac{H}{2}$ and is motivated by the following characterization of the normal velocity of a smoothly evolving set. Given a smoothly evolving hypersurface $\partial\Omega(t) = \Sigma(t)$ with normal velocity V we have

$$\frac{d}{dt}\int_\Sigma \zeta \leq \int_\Sigma (-\zeta H\, V - V\,\nabla\zeta\cdot\nu + \partial_t\zeta) \tag{5}$$

for any smooth test function $\zeta \geq 0$. Here ν denotes inner normal of $\partial\Omega$ and we take the convention $V > 0$ for an expanding Ω. The converse is also true: Given a function $V\colon \Sigma \to \mathbb{R}$ such that (5) holds for any such test function $\zeta \geq 0$ then V is the normal velocity of Σ. In the pioneering work [14], Brakke uses this inequality as a definition for the equation $V = \frac{H}{2}$ to extend the concept of motion by mean curvature to general varifolds. We recall his definition in our more restrictive setting of finite perimeter sets.

Definition 2.1. We say that $\chi\colon (0,T)\times[0,\Lambda)^d \to \{0,1\}$ moves by mean curvature if there exists a $|\nabla\chi|\,dt$-measurable function $H\colon (0,T)\times[0,\Lambda)^d \to \mathbb{R}$ with

$$\int_0^T\int H^2\,|\nabla\chi|\,dt < \infty,$$

which is the mean curvature in the sense that for all test vector fields $\xi \in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^d)$

$$\int_0^T\int (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,|\nabla\chi|\,dt = \int_0^T\int H\,\xi\cdot\nu\,|\nabla\chi|\,dt, \tag{6}$$

such that for any test function $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$ with $\zeta \geq 0$ we have

$$\int_0^T\int\left(\partial_t\zeta - \zeta\frac{1}{2}H^2 - \frac{1}{2}H\nu\cdot\nabla\zeta\right)|\nabla\chi|\,dt \geq 0. \tag{7}$$

Theorem 2.2 (Brakke's inequality)**.** *Given initial data $\chi^0\colon [0,\Lambda)^d \to \{0,1\}$ with $E(\chi^0) < \infty$ and a finite time horizon $T < \infty$, for any sequence there exists a subsequence $h \downarrow 0$ such that the approximate solutions given by Algorithm 1.1 converge to a limit $\chi\colon (0,T)\times[0,\Lambda)^d \to \{0,1\}$. Given the convergence assumption (4), χ moves by mean curvature in the sense of Definition 2.1.*

Remark 2.3. Given initial conditions χ^0 with $E(\chi^0) < \infty$ the compactness in Chapter 1 yields a subsequence such that $\chi^h \to \chi$ a.e. for a function χ with $\sup_t E(\chi(t)) \leq E(\chi^0)$.

This statement is similar to our result in Chapter 1. There we proved the convergence of thresholding towards a distributional formulation of (multi-phase) mean-curvature flow. Under the same assumption (4) we constructed a measurable function $V\colon (0,T)\times[0,\Lambda)^d \to \mathbb{R}$ with

$$\int_0^T\int V^2\,|\nabla\chi|\,dt < \infty,$$

which is the normal velocity in the sense that

$$\int_0^T \int \partial_t \zeta \, \chi \, dx \, dt = -\int_0^T \int \zeta \, V \, |\nabla \chi| \, dt$$

for all $\zeta \in C_0^\infty((0,T) \times [0,\Lambda)^d)$, such that $V = \frac{H}{2}$ in the following distributional sense:

$$\int_0^T \int \left(\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu - 2 \, \xi \cdot \nu \, V \right) |\nabla \chi| \, dt = 0 \tag{8}$$

for all $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$.

The connection of (8) to the strong equation $V = \frac{H}{2}$ comes from the integration by parts rule for smooth hypersurfaces:

$$\int_\Sigma (\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu) = \int_\Sigma H \, \xi \cdot \nu.$$

Without any regularity assumption, none of the two formulations is stronger in the sense that it implies the other. Nevertheless (8) requires more regularity as it is formulated for sets of finite perimeter, whereas Brakke's inequality naturally extends to general varifolds.

3 De Giorgi's variational interpolation

It is a well-appreciated fact that a classical gradient flow $\dot{u}(t) = -\nabla E(u(t))$ of a smooth energy E on a Hilbert space can be characterized by the optimal rate of dissipation of the energy E along the solution u:

$$\frac{d}{dt} E(u(t)) \leq -\frac{1}{2} |\dot{u}(t)|^2 - \frac{1}{2} |\nabla E(u(t))|^2. \tag{9}$$

This is the guiding principle in generalizing gradient flows to metric spaces where one replaces $|\dot{u}|$ by the metric derivative and $|\nabla E(u)|$ by some upper gradient, e.g. the *local slope* $|\partial E(u)|$, see (14) for a definition in our context.

Mean curvature flow can be viewed as a gradient flow in the sense that for a smooth evolution $\Sigma = \Sigma(t)$ the energy, which in this case is the surface area $|\Sigma(t)|$, satisfies the inequality

$$2 \frac{d}{dt} |\Sigma| = \int_\Sigma H \, 2V \leq -\frac{1}{2} \int_\Sigma H^2 - \frac{1}{2} \int_\Sigma (2V)^2.$$

While in the abstract framework, one measures the dissipation of the energy w.r.t. both terms $|\dot{u}|^2 \hat{=} \int_\Sigma (2V)^2$ and $|\partial E(u)|^2 \hat{=} \int_\Sigma H^2$, Brakke measures the rate only in terms of the local slope $\int_\Sigma H^2$ but asks for the *localized* version (7).

The main result of this section and the second main result of this work is the approximate version of Brakke's inequality, Lemma 3.1 below. In view of the minimizing movements interpretation (1) it should be feasible to obtain at least the *global* inequality

$$2 \frac{d}{dt} |\Sigma| \leq -\int_\Sigma H^2$$

but the localized inequality (7) would be still out of reach. The lemma states that thresholding does not only solve the *global* minimization problem (1) but a whole *family* of *local* minimization problems.

Lemma 3.1 (Local minimization)**.** *Let χ^n be obtained from χ^{n-1} by one iteration of Algorithm 1.1 and $\zeta \geq 0$ an arbitrary test function. Then*

$$\chi^n = \arg\min_u \left\{ E_h(u,\chi^{n-1};\zeta) + \frac{1}{2h}\mathrm{d}_h^2(u,\chi^{n-1};\zeta)\right\}, \tag{10}$$

where the minimum runs over all $u\colon [0,\Lambda)^d \to [0,1]$. With $\mathrm{d}_h(u,\chi;\zeta)$ we denote the localization of $\mathrm{d}_h(u,\chi)$ given by

$$\frac{1}{2h}\mathrm{d}_h^2(u,\chi;\zeta) := \frac{1}{\sqrt{h}}\int \zeta\left[G_{h/2} * (u-\chi)\right]^2 dx, \tag{11}$$

which is again a metric on the space of all such u's as above and in particular satisfies a triangle inequality. With $E_h(u,\chi;\zeta)$ we denote the localized (approximate) energy incorporating the localization error:

$$\begin{aligned} E_h(u,\chi;\zeta) := \frac{1}{\sqrt{h}}\int \zeta\,(1-u)\,G_h * u\,dx + \frac{1}{\sqrt{h}}\int (u-\chi)\,[\zeta, G_h*]\,(1-\chi)\,dx \\ + \frac{1}{\sqrt{h}}\int (u-\chi)\left[\zeta, G_{h/2}*\right] G_{h/2} * (u-\chi)\,dx. \end{aligned} \tag{12}$$

Here and throughout the chapter

$$[\zeta, G_h*]\,u := \zeta\,G_h * u - G_h * (\zeta\,u) \approx -\nabla\zeta \cdot h\nabla G_h * u$$

denotes the commutator of the multiplication with the function ζ and the convolution with the kernel G_h.

Let us comment on the structure of the localized energy E_h. The first integral is an approximation of the localized surface energy $c_0\int_\Sigma \zeta$. Expanding ζ, as $h \to 0$ the leading-order term of the second integral in the definition of $E_h(\chi^n,\chi^{n-1};\zeta)$ is

$$\int \frac{\chi^n - \chi^{n-1}}{h}\,\nabla\zeta\cdot\sqrt{h}\nabla G_h * \chi^{n-1}\,dx,$$

which at least formally converges to $c_0\int_\Sigma V\ \nabla\zeta\cdot\nu$ and hence we expect to recover the "transport term" $\frac{c_0}{2}\int_\Sigma H\ \nabla\zeta\cdot\nu$ in Brakke's inequality (7). We will see later that the last integral in the definition of E_h, the commutator in the metric term, is negligible in the limit $h \to 0$. By definition of E_h we have

$$E_h(u,u;\zeta) = E_h(u;\zeta) := \frac{1}{\sqrt{h}}\int \zeta\,(1-u)\,G_h * u\,dx \quad \text{and} \quad E_h(u,\chi;1) = E_h(u)$$

so that in particular we recover the minimizing movements interpretation [33] in the case $\zeta \equiv 1$.

Thanks to the above *local* minimization property of the thresholding scheme we can apply the abstract framework of De Giorgi, cf. Chapters 1–3 in [7], to these localized energies. As for any minimizing movements scheme, the comparison of χ^n to the previous time step χ^{n-1} in the minimization (10) yields an energy-dissipation inequality which works well as an a priori estimate, but which is however not sharp. To obtain a *sharp* inequality we follow the ideas of De Giorgi. We introduce the following "interpolation" u^h of χ^n and χ^{n-1}: For $t \in (0,h]$ and $n \in \mathbb{N}$ we let

$$u^h((n-1)h+t) := \arg\min_u \left\{ E_h(u,\chi^{n-1};\zeta) + \frac{1}{2t}\mathrm{d}_h^2(u,\chi^{n-1};\zeta) \right\}.$$

Comparing $u^h(t)$ with $u^h(t+\delta t)$ in this minimization problem and taking the limit $\delta t \to 0$ while keeping h fixed, one obtains the sharp energy-dissipation inequality along this interpolation, the following approximate version of Brakke's inequality (7).

Corollary 3.2 (Approximate Brakke inequality)**.** *For any test function $\zeta \geq 0$, a time-step size $h > 0$ and $T = Nh$ we have*

$$\frac{h}{2}\sum_{n=1}^N \left|\partial E_h(\,\cdot\,,\chi^{n-1};\zeta)\right|^2(\chi^n) + \frac{1}{2}\int_0^{T-h} \left|\partial E_h(\,\cdot\,,\chi^h(t);\zeta)\right|^2(u^h(t))\,dt$$
$$+ \sum_{n=1}^N \left(E_h(\chi^n,\chi^{n-1};\zeta) - E_h(\chi^n,\chi^n;\zeta)\right) \leq E_h(\chi^0,\chi^0;\zeta) - E_h(\chi^N,\chi^N;\zeta), \quad (13)$$

where $\left|\partial E_h(\,\cdot\,,\chi;\zeta)\right|(u)$ is the local slope of $E_h(\,\cdot\,,\chi;\zeta)$ at u defined by

$$\left|\partial E_h(\,\cdot\,,\chi;\zeta)\right|(u) := \limsup_{v\to u} \frac{(E_h(u,\chi;\zeta) - E_h(v,\chi;\zeta))_+}{\mathrm{d}_h(u,v;\zeta)}. \quad (14)$$

The convergence $v \to u$ is in the sense of the metric d_h.

Our goal is to derive Brakke's inequality (7) from its approximate version (13), i.e. we want to relate the limits of the expressions in (13) with the terms appearing in (7), cf. Propositions 4.6 and 4.7.

4 Some lemmas

Because of the localization, our energy (12) depends on the configuration at the previous time step. However, we can apply the abstract framework (cf. Chapter 3 of [7]) to this case if we only follow one time step. Both, h and ζ are fixed parameters when applying these results.

We start by defining the above mentioned "interpolation" of the approximations χ^h.

Definition 4.1. Given χ we define the *Moreau-Yosida approximation* $E_{h,t}$ of E_h by

$$E_{h,t}(\chi;\zeta) := \min_u \left\{ E_h(u,\chi;\zeta) + \frac{1}{2t}\mathrm{d}_h^2(u,\chi;\zeta) \right\} \tag{15}$$

and furthermore the (not necessarily unique) *variational interpolation* $u^h(t)$ of χ and $\chi^1 = u^h(h)$ by

$$u^h(t) := \arg\min_u \left\{ E_h(u,\chi;\zeta) + \frac{1}{2t}\mathrm{d}_h^2(u,\chi;\zeta) \right\}. \tag{16}$$

As t decreases we have a stronger penalization. Thus we expect $u^h(t)$ to be "closer" to $\chi = u^h(0)$ than $\chi^1 = u^h(h)$ which justifies the name "interpolation". We will make this statement more rigorous later. Note that $E_h(u,\chi;\zeta)$ and $\mathrm{d}(u,\chi;\zeta)$ are because of the smoothing property of the kernel G_h both weakly continuous in u and χ.

The following theorem monitors the evolution of the (approximate) energy along the interpolation $u^h(t)$ in terms of the distances at different time instances measured by the metric d_h and gives a lower bound in terms of the local slope $|\partial E_h|$ of E_h, cf. (14).

Theorem 4.2 (Theorem 3.1.4 and Lemma 3.1.3 in [7])**.** *For every $\chi\colon [0,\Lambda)^d \to \{0,1\}$ the map $t \mapsto E_{h,t}(\chi;\zeta)$ is locally Lipschitz in $(0,h]$ and continuous in $[0,h]$ with*

$$\frac{d}{dt}E_{h,t}(\chi;\zeta) = -\frac{\mathrm{d}_h^2(u^h(t),\chi;\zeta)}{2t^2} \tag{17}$$

and furthermore we have

$$|\partial E_h(\,\cdot\,,\chi;\zeta)|\,(u^h(t)) \le \frac{\mathrm{d}_h(u^h(t),\chi;\zeta)}{t}. \tag{18}$$

In particular

$$\begin{aligned} &\frac{t}{2}\,|\partial E_h(\,\cdot\,,\chi;\zeta)|^2\,(u^h(t)) + \frac{1}{2}\int_0^t |\partial E_h(\,\cdot\,,\chi;\zeta)|^2\,(u^h(s))\,ds \\ &\qquad \le \frac{1}{2t}\mathrm{d}_h^2(u^h(t),\chi;\zeta) + \int_0^t \frac{\mathrm{d}_h^2(u^h(s),\chi;\zeta)}{2s^2}ds = E_h(\chi,\chi;\zeta) - E_h(u^h(t),\chi;\zeta). \end{aligned} \tag{19}$$

The following a priori estimate follows immediately from the minimality the above theorem.

Corollary 4.3 (Energy-dissipation estimate)**.** *Given initial conditions $\chi^0\colon [0,\Lambda)^d \to \{0,1\}$ with $E(\chi^0) < \infty$, a time-step size $h > 0$ and $T = Nh$ we have that*

$$\sup_t E_h(u^h(t)), \quad h\sum_{n=1}^N \frac{\mathrm{d}_h^2(\chi^n,\chi^{n-1})}{2h^2} \quad \textit{and} \quad \int_0^T \frac{\mathrm{d}_h^2(u^h(t),\chi^h(t))}{2h^2}dt \quad \textit{stay bounded as } h \downarrow 0. \tag{20}$$

While the above statements are a mere application of the abstract theory [7] and did not use the structure of our problem, we will now use the particular character of thresholding, i.e. the structure of the energy (12) and the metric term (11) in order to pass to the limit in the approximate Brakke inequality (13).

We recall the following proposition from Chapter 1 which will allow us to pass to the limit in the Brakke inequality for the approximate solutions.

Proposition 4.4. *Given $u^h \to \chi$ and $E_h(u^h) \to E(\chi)$, a test function $\zeta \in C^\infty([0,\Lambda)^d)$ and a test matrix field $A \in C^\infty([0,\Lambda)^d, \mathbb{R}^{d\times d})$ we have*

$$\frac{1}{\sqrt{h}} \int \zeta \big(1-u^h\big) G_h * u^h \, dx \to c_0 \int \zeta \, |\nabla \chi| \quad \textit{and} \tag{21}$$

$$\frac{1}{\sqrt{h}} \int A \colon \big(1-u^h\big) h \nabla^2 G_h * u^h \, dx \to c_0 \int A \colon \nu \otimes \nu \, |\nabla \chi| \,. \tag{22}$$

In Chapter 1 we used the above proposition to pass to the limit in the first variation of the energy

$$\delta E_h(u,\xi) := \frac{d}{ds}\Big|_{s=0} E_h(u_s),$$

where the inner variations u_s of u along a vector field ξ are given by the transport equation

$$\partial_s u_s + \xi \cdot \nabla u_s = 0 \quad u_s|_{s=0} = u. \tag{23}$$

Proposition 4.5 (Proposition 3.2, Remark 3.3 and Lemma 3.4 in Chapter 1)**.** *Given $u \colon [0,\Lambda)^d \to [0,1]$ we have*

$$\delta E_h(u,\xi) = \frac{1}{\sqrt{h}} \int \nabla \xi \colon \, (1-u) \big(G_h \, Id - h \nabla^2 G_h\big) * u \, dx + o(1), \tag{24}$$

where the constant in the $o(1)$-term only depends on u through $E_h(u)$. In particular if $u^h \to \chi \in \{0,1\}$ and $E_h(u^h) \to E(\chi) < \infty$ we have

$$\delta E_h(u^h,\xi) \to \delta E(\chi,\xi) = c_0 \int \nabla \xi \colon \, (Id - \nu \otimes \nu) \, |\nabla \chi| \,. \tag{25}$$

Although the proof is contained in Chapter 1, we will repeat the short argument for the proposition in this two-phase context based on (21) and (22) for the convenience of the reader in the following section.

Without the localization, i.e. if $\zeta \equiv 1$, we can show

$$\frac{c_0}{2} \int H^2 \, |\nabla \chi| \le \liminf_{h\to 0} |\partial E_h|^2 \left(u^h\right)$$

whenever $u^h \to \chi$ in L^1 and $E_h(u^h) \to E(\chi)$. In the following proposition we prove that with the localization $\zeta \geq 0$ we obtain a similar estimate for the local slope $\left|\partial E_h(\,\cdot\,,\chi^h;\zeta)\right|^2(u^h)$ after integration in time if we have the following *quantitative* proximity of $u^h(t)$ to $\chi^h(t)$ in L^1 after mollification:

$$\int_0^T \int \left| G_{h/2} * \left(u^h - \chi^h\right)\right| dx\,dt = o(\sqrt{h}).$$

In our case where $u^h(t)$ is the variational interpolation (16) or the approximate solution $\chi^h(t+h)$ itself, this rate is a direct consequence of the energy-dissipation estimate (20).

Proposition 4.6. *Let $\zeta \geq 0$ be smooth, $\chi^h(t)$ the approximate solution obtained by Algorithm 1.1 and let $u^h(t)$ be either the variational interpolation* (16) *or the approximate solution $\chi^h(t+h)$ at time $t+h$. Given the convergence assumption* (4) *we have*

$$\frac{c_0}{2}\int_0^T \int \zeta\, H^2\, |\nabla \chi|\, dt \leq \liminf_{h\to 0} \int_0^T \left|\partial E_h(\,\cdot\,,\chi^h;\zeta)\right|^2 (u^h)\, dt. \tag{26}$$

The above tools are enough to pass to the limit in the approximate Brakke inequality (13) if the test function ζ is constant. However, for a non-constant test function $\zeta \geq 0$ we have to pass to the limit in the extra term $\sum_{n=1}^N \left(E_h(\chi^n,\chi^{n-1};\zeta) - E_h(\chi^n,\chi^n;\zeta)\right)$ in (13). In the following proposition we prove the convergence towards the "transport" term $\frac{c_0}{2}\int_\Sigma H\,\nu\cdot\nabla\zeta$ in Brakke's inequality under the convergence assumption (4).

Proposition 4.7. *Given the convergence assumption* (4) *and let $T = Nh$. Then we have*

$$\sum_{n=1}^N \left(E_h(\chi^n,\chi^{n-1};\zeta) - E_h(\chi^n,\chi^n;\zeta)\right) \to \frac{c_0}{2}\int_0^T \int H\,\nu\cdot\nabla\zeta\,|\nabla\chi|\,dt \quad \textit{as } h\to 0.$$

In order to to apply Proposition 4.4 and Proposition 4.5 in the situation of Proposition 4.6 and Proposition 4.7 we need the following lemma.

Lemma 4.8. *Given the convergence assumption (4), for a subsequence, we also have the pointwise property*

$$E_h(\chi^h) \to E(\chi) \quad \textit{a.e. in } (0,T) \tag{27}$$

and furthermore for the variational interpolation u^h given by (16)

$$E_h(u^h) \to E(\chi) \quad \textit{a.e. in } (0,T) \tag{28}$$

and in particular the integrated version

$$\int_0^T E_h(u^h)\,dt \to \int_0^T E(\chi)\,dt. \tag{29}$$

5 Proofs

We first give the proofs of the main results, Theorem 2.2, Lemma 3.1 and Corollary 3.2 with help of the auxiliary statements in Section 4.

Proof of Theorem 2.2. Step 1: Time-freezing for ζ. We claim that it is enough to prove

$$\int_0^T \int \left(\zeta \frac{1}{2} H^2 + \frac{1}{2} H\nu \cdot \nabla \zeta\right) |\nabla \chi| \, dt \leq \int \zeta \, |\nabla \chi(0)| - \int \zeta \, |\nabla \chi(T)| \tag{30}$$

for any time-independent test function $\zeta = \zeta(x) \geq 0$ and a.e. T.

In order to reduce (7) to (30) we fix a time-dependent test function $\zeta = \zeta(t,x) \geq 0$ and two time instances $0 \leq s < t$. It is no restriction to assume $s = 0$. Writing $t = T$ for the time horizon we take a regular partition $0 = T_0 < \cdots < T_M = T$ of $(0,T)$ of fineness $\tau = T/M$. We write ζ_M for the piecewise constant interpolation of ζ:

$$\zeta_M(t) := \zeta(T_{m-1}) \quad \text{if } t \in [T_{m-1}, T_m).$$

Writing $\partial^{-\tau} \zeta_M(t) := \frac{1}{\tau}\left(\zeta_M(t) - \zeta_M(t-\tau)\right)$ for the discrete (backwards) time derivative we have

$$\zeta_M \to \zeta, \quad \nabla \zeta_M \to \nabla \zeta \quad \text{and} \quad \partial^{-\tau}\zeta_M \to \partial_t \zeta \quad \text{uniformly as } M \to \infty. \tag{31}$$

Using (30) for ζ_M on each interval $[T_{m-1}, T_m)$ we have

$$\begin{aligned} \int_0^T \int \left(\zeta_M \frac{1}{2} H^2 + \frac{1}{2} H\nu \cdot \nabla \zeta_M\right) |\nabla \chi| \, dt & \\ \leq \sum_{m=1}^{M} \int \zeta(T_{m-1}) \, |\nabla \chi(T_{m-1})| - \int \zeta(T_{m-1}) \, |\nabla \chi(T_m)| \, . & \end{aligned} \tag{32}$$

The right-hand side can be written as

$$\int \zeta(0) \, |\nabla \chi(0)| - \int \zeta(T_{M-1}) \, |\nabla \chi(T)| + \sum_{m=1}^{M-1} \int \left(\zeta(T_m) - \zeta(T_{m-1})\right) |\nabla \chi(T_m)| \, .$$

In order to pass to the limit in this expression we introduce the piecewise constant interpolation

$$\chi_M(t) := \chi(T_{m-1}) \quad \text{if } t \in [T_{m-1}, T_m).$$

Then we clearly have $\chi_M \to \chi$ a.e. and furthermore since $\int |\nabla \chi(t)|$ is monotone in t we have

$$\int |\nabla \chi_M(t)| \to \int |\nabla \chi(t)| \quad \text{for a.e. } t \quad \text{as } M \to \infty. \tag{33}$$

Therefore the right-hand side of (32) is equal to

$$\int \zeta(0)\,|\nabla\chi(0)| - \int \zeta(T_{M-1})\,|\nabla\chi(T)| + \int_0^T \int \partial^{-\tau}\zeta_M\,|\nabla\chi_M|\,dt,$$

which converges to

$$\int \zeta(0)\,|\nabla\chi(0)| - \int \zeta(T)\,|\nabla\chi(T)| + \int_0^T \int \partial_t\zeta\,|\nabla\chi|\,dt$$

as $M \to \infty$. Indeed, also the last term converges since

$$\begin{aligned}\left|\int_0^T \int \partial^{-\tau}\zeta_M\,|\nabla\chi_M|\,dt - \int_0^T \int \partial_t\zeta\,|\nabla\chi|\,dt\right| \leq & \|\partial^{-\tau}\zeta_M - \partial_t\zeta\|_\infty \int_0^T \int \left|\nabla\chi^M\right| dt \\ & + \left|\int_0^T \int \partial_t\zeta\,(|\nabla\chi_M| - |\nabla\chi|)\,dt\right|.\end{aligned}$$

The first term vanishes as $M \to \infty$ by (31). The second term vanishes by the convergence of the perimeters (33).

Step 2: Proof of (30). Given a test function $\zeta = \zeta(x) \geq 0$ and $T > 0$, we want to prove (30). We may assume that $T = Nh$ is a multiple of the time step size h. Furthermore by (4) we may assume that $E_h(\chi^h(T)) \to E(\chi(T))$. We pass to the limit in (13) to prove (7).

By (29) in Lemma 4.8 we may apply Proposition 4.6 to obtain

$$\frac{c_0}{4}\int_0^T \int \zeta\,H^2\,|\nabla\chi|\,dt \leq \liminf_{h\to 0} \frac{h}{2}\sum_{n=1}^N \left|\partial E_h(\,\cdot\,,\chi^{n-1};\zeta)\right|^2(\chi^n),$$

as well as

$$\frac{c_0}{4}\int_0^T \int \zeta\,H^2\,|\nabla\chi|\,dt \leq \liminf_{h\to 0} \frac{1}{2}\int_0^T \left|\partial E_h(\,\cdot\,,\chi^h(t);\zeta)\right|^2(u^h(t))\,dt.$$

In addition we may apply Proposition 4.7 for the transport term and after division by the common prefactor c_0 we obtain (30). □

Proof of Lemma 3.1. Given initial conditions $\chi \in \{0,1\}$ and a time-step size $h > 0$, one iteration of the thresholding scheme yields $\chi^1 = \mathbf{1}_{\{G_h*\chi>\frac{1}{2}\}}$. Then χ^1 clearly minimizes

$$(1-u)\,G_h*\chi + u\,G_h*(1-\chi)$$

among all $u \in [0,1]$ pointwise a.e. This expression is equal to

$$(1-u)\,G_h*u + (u-\chi)\,G_h*(u-\chi) + \left[-(1-\chi)\,G_h*(u-\chi) + u\,G_h*(1-\chi)\right].$$

The term in the parenthesis can be rewritten as

$$(u-\chi)\, G_h * (1-\chi) - (1-\chi)\, G_h * (u-\chi) + \chi\, G_h * (1-\chi)\,,$$

where the last summand is independent of u and thus irrelevant for the minimization. Multiplying with $\zeta \geq 0$ and integrating shows that χ^1 minimizes

$$\int \zeta\left[(u-\chi)\, G_h * (1-\chi)\,(u-\chi)\, G_h * (1-\chi) - (1-\chi)\, G_h * (u-\chi)\right] dx + \text{const.}$$

Dividing by $\sqrt{h}$ and recalling the definitions (11) and (12) of the localized distance and energy yields (10). □

Proof of Corollary 3.2. For any $n = 1, \ldots, N$ we apply Theorem 4.2 with $\chi = \chi^{n-1}$ and $t = h$, sum over n and hence obtain the claim. □

Now we prove the auxiliary statements of Section 4 which we used for the proofs of the main theorems. The first part of this section is contained in Chapter 3 of [7].

Proof of Proposition 4.5. The first variation of E_h along the vector field ξ is given by

$$\begin{aligned}
\delta E_h(u,\xi) =& \frac{1}{\sqrt{h}} \int -\xi \cdot \nabla(1-u)\, G_h * u - (1-u)\, G_h * (\xi \cdot \nabla u)\, dx \\
=& \frac{1}{\sqrt{h}} \int \xi \cdot \left[(1-u)\, \nabla G_h * u\right] - (1-u)\, \nabla G_h * (\xi\, u)\, dx \\
&+ \frac{1}{\sqrt{h}} \int (\nabla \cdot \xi)\,(1-u)\, G_h * u + (1-u)\, G_h * ((\nabla\cdot\xi)\, u)\, dx.
\end{aligned}$$

This can be compactly rewritten as

$$\delta E_h(u,\xi) = \frac{1}{\sqrt{h}} \int 2\,(\nabla\cdot\xi)\,(1-u)\, G_h * u + (1-u)\left[\nabla G_h *, \xi\cdot\right] u + (1-u)\left[G_h *, \nabla\cdot\xi\right] u\, dx.$$

We expand the first commutator

$$\left[\nabla G_h *, \xi\cdot\right] u = \nabla\xi \colon \left(\frac{z}{\sqrt{h}} \otimes \frac{z}{\sqrt{h}}\right) G_h * u + O\left(\sqrt{h}\left(\frac{|z|}{\sqrt{h}}\right)^3 G_h * u\right).$$

The second commutator can be estimated by $\left[G_h *, \nabla\cdot\xi\right] u = O(\sqrt{h}\frac{|z|}{\sqrt{h}} G_h * u)$ and by the identity $\nabla^2 G(z) = G\,(z\otimes z - Id)$ we indeed obtain (24) with an error of order $\sqrt{h}\, E_h(u)$. □

Proof of Proposition 4.6. We let the variations u_s defined in (23) play the role of v in the definition of the local slope (14) so that we obtain the inequality

$$|\partial E_h(\,\cdot\,, \chi^h;\zeta)|(u^h) \geq \limsup_{s\to 0} \frac{\left(E_h(u^h,\chi^h;\zeta) - E_h(u^h_s,\chi^h;\zeta)\right)_+}{\mathrm{d}_h(u^h_s, u^h;\zeta)}.$$

As $s \to 0$ we expand the numerator in the following way

$$E_h(u_s^h, \chi^h; \zeta) = E_h(u^h, \chi^h; \zeta) + s\, \delta E_h(\,\cdot\,, \chi^h; \zeta)(u^h, \xi) + o(s),$$

where $\delta E_h(\,\cdot\,, \chi^h; \zeta)(u^h, \xi) := \frac{d}{ds}\big|_{s=0} E_h(u_s^h, \chi^h; \zeta)$ denotes the first variation of the localized energy along ξ. For the denominator we use

$$\frac{1}{2h} \mathrm{d}_h^2(u_s^h, u^h; \zeta) = \frac{s^2}{2} \frac{2}{\sqrt{h}} \int \zeta \left(G_{h/2} * \left(\xi \cdot \nabla u^h \right) \right)^2 dx + o(s^2)$$

as $s \to 0$. Taking the limit $s \to 0$ we obtain

$$|\partial E_h(\,\cdot\,, \chi^h; \zeta)|(u^h) \geq \frac{\delta E_h(\,\cdot\,, \chi^h; \zeta)(u^h, \xi)}{\sqrt{2\sqrt{h} \int \zeta \left(G_{h/2} * (\xi \cdot \nabla u^h) \right)^2 dx}} \quad \text{for all } \xi. \tag{34}$$

Now we expand ζ and ξ to analyze the leading order terms as $h \to 0$. Using (23) we can compute the first variation $\delta E_h(u, \xi)$ of E_h:

$$\begin{aligned} &\frac{d}{ds}\Big|_{s=0} E_h(u_s, \chi; \zeta) \\ &\quad = \frac{1}{\sqrt{h}} \int - \zeta\, \xi \cdot \nabla\, (1-u)\, G_h * u - \zeta\, (1-u)\, G_h * (\xi \cdot \nabla u) \\ &\qquad\qquad - \xi \cdot \nabla u\, [\zeta, G_h *]\, (1-u) - \xi \cdot \nabla u\, [\zeta, G_h *]\, (u - \chi) \\ &\qquad\qquad - \xi \cdot \nabla u\, \left[\zeta, G_{h/2} *\right] G_{h/2} * (u-\chi) + \xi \cdot \nabla u\, G_{h/2} * \left[\zeta, G_{h/2}\right] (u - \chi)\, dx. \end{aligned}$$

The fourth term in the sum comes from replacing $(1-\chi)$ by $(1-u)$ in the third term, while for the last term we used the antisymmetry $\int u \left[\zeta, G_{h/2} *\right] v\, dx = -\int v \left[\zeta, G_{h/2} *\right] u\, dx$. Note that due to the symmetry of G there is a cancellation between the second and third term in this sum:

$$\begin{aligned} \int -\zeta\, (1-u)\, G_h * (\xi \cdot \nabla u) - \xi \cdot \nabla u\, [\zeta, G_h *]\, (1-u)\, dx &= \int -\zeta\, \xi \cdot \nabla u\, G_h * (1-u)\, dx \\ &= \int - (1-u)\, G_h * (\zeta\, \xi \cdot \nabla u\,)\, dx. \end{aligned}$$

A direct computation based on the semi-group property $G_h = G_{h/2} * G_{h/2}$ yields

$$- [\zeta, G_h *]\, v - \left[\zeta, G_{h/2} *\right] G_{h/2} * v + G_{h/2} * \left[\zeta, G_{h/2} *\right] v = -2 \left[\zeta, G_{h/2} *\right] G_{h/2} * v \tag{35}$$

so that the last three terms in the first variation of E_h above simplify, and we get

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} E_h(u_s, \chi; \zeta) = &\frac{1}{\sqrt{h}} \int -\zeta\, \xi \cdot \nabla\, (1-u)\, G_h * u - (1-u)\, G_h * (\zeta\, \xi \cdot \nabla u\,)\, dx \\ &+ \frac{2}{\sqrt{h}} \int -\xi \cdot \nabla u \left[\zeta, G_{h/2} *\right] G_{h/2} * (u - \chi)\, dx. \end{aligned} \tag{36}$$

Note that the first right-hand side integral is exactly $\delta E_h(u, \zeta\,\xi)$. Now we plug $u = u^h$ into the above formula. Since $u^h \to \chi$ in L^1 and $E_h(u^h) \to E(\chi)$ for a.e. t using Proposition 4.5 for $\zeta\,\xi$ playing the role of ξ, for a.e. t, along the sequence u^h the first right-hand side integral converges to

$$\delta E(\chi, \zeta\,\xi) = c_0 \int \nabla\left(\zeta\,\xi\right) :\, \left(Id - \nu\otimes\nu\right)|\nabla\chi| \overset{(6)}{=} c_0 \int \zeta\, H\nu\cdot\xi\,|\nabla\chi|\,.$$

Now we give the argument that the second integral in (36) is negligible:

$$\frac{2}{\sqrt{h}}\int -\xi\cdot\nabla u^h\left[\zeta, G_{h/2}*\right] G_{h/2}*\left(u^h-\chi^h\right)dx \to 0 \quad \text{in } L^1(0,T). \tag{37}$$

In order to prove (37) we use $-\xi\cdot\nabla u = -\nabla(\xi\cdot u) + (\nabla\cdot\xi)\,u$ and obtain one term of lower order:

$$\left|\int \left(\nabla\cdot\xi\right)u^h\frac{1}{\sqrt{h}}\left[\zeta, G_{h/2}*\right] G_{h/2}*\left(u^h-\chi^h\right)dx\right| \lesssim \|\nabla\xi\|_\infty\|\nabla\zeta\|_\infty\int|u^h-\chi^h|\,dx,$$

which vanishes in $L^1(0,T)$ and the leading-order term

$$\frac{2}{\sqrt{h}}\int -u^h\xi\cdot\nabla\left[\zeta, G_{h/2}*\right] G_{h/2}*\left(u^h-\chi^h\right)dx.$$

Using $\nabla\left[\zeta, G_{h/2}*\right]v = \left[\zeta, \nabla G_{h/2}*\right]v + \nabla\zeta\,G_{h/2}*v$ and estimating the commutator we have

$$\begin{aligned}\int\left|\nabla\left[\zeta, G_{h/2}*\right]v\right|dx &\le \|\nabla\zeta\|_\infty\left(\int\left(|z||\nabla G_h|\right)*|v|dx + \int G_{h/2}*|v|dx\right)\\ &\lesssim \|\nabla\zeta\|_\infty\int|v|\,dx\end{aligned}$$

and therefore

$$\left|\frac{2}{\sqrt{h}}\int u^h\xi\cdot\nabla\left[\zeta, G_{h/2}*\right] G_{h/2}*\left(u^h-\chi^h\right)dx\right| \lesssim \|\xi\|_\infty\|\nabla\zeta\|_\infty\frac{1}{\sqrt{h}}\int\left|G_{h/2}*\left(u^h-\chi^h\right)\right|dx.$$

Using Jensen's inequality and the energy-dissipation estimate (20) we obtain (37). Therefore we have proven the following convergence of the first variation of the localized energy (12):

$$\delta E_h(u^h, \xi\,\zeta) \to c_0\int\zeta\,H\nu\cdot\xi\,|\nabla\chi| \quad \text{in } L^1(0,T). \tag{38}$$

With the same methods we can handle the term in the expansion of the metric term $\mathrm{d}_h(u_s^h, \chi^h)$: We claim that

$$\begin{aligned}2\sqrt{h}\int\zeta\left(G_{h/2}*\left(\xi\cdot\nabla u^h\right)\right)^2dx =& \frac{2}{\sqrt{h}}\int\zeta\left(\xi\otimes\xi\right):\,\left(1-u^h\right)\left(h\nabla^2G_h\right)*u^h\,dx + o(1)\\ \to& 2c_0\int\zeta\left(\xi\cdot\nu\right)^2|\nabla\chi| \quad \text{as } h\to 0 \quad \text{for a.e. } t.\end{aligned} \tag{39}$$

To this end we plug $\xi \cdot \nabla u = \nabla \cdot (\xi u) - (\nabla \cdot \xi) u$ into the quadratic term on left-hand side and expand the square. First we note that only the term

$$2\sqrt{h} \int \zeta \left(G_{h/2} * \left(\nabla \cdot (\xi u^h) \right) \right)^2 dx = 2\sqrt{h} \int \zeta \left(\nabla G_{h/2} * \left(\xi u^h \right) \right)^2 dx$$

survives in the limit $h \to 0$. Indeed, we have

$$2\sqrt{h} \int \zeta \left(G_{h/2} * \left((\nabla \cdot \xi) \, u^h \right) \right)^2 dx \lesssim \|\nabla \xi\|_\infty^2 \sqrt{h} \int |\zeta| \, dx$$

and for the mixed term we have

$$\left| 4\sqrt{h} \int \zeta \, G_{h/2} * \left((\nabla \cdot \xi) u^h \right) \nabla G_{h/2} * \left(\xi u^h \right) dx \right| \lesssim \|\nabla \xi\|_\infty \|\zeta\|_\infty \sqrt{h} \int \left| \nabla G_{h/2} * (\xi \, u^h) \right| dx. \tag{40}$$

To bound this term we use

$$\sqrt{h} \int \left| \nabla G_{h/2} * (\xi \, u^h) \right| dx \le \sqrt{h} \int \left| \xi \cdot \nabla G_{h/2} * u^h \right| dx + O(\sqrt{h}).$$

Note that $\int \nabla G(z) \, dz = 0$ implies

$$\begin{aligned} \sqrt{h} \int \left| \nabla G_{h/2} * u \right| dx \le & \sqrt{h} \int |\nabla G_{h/2}(z)| \int \left| u^h(x) - u^h(x-z) \right| dx \, dz \\ & \lesssim \int G_h(z) \int \left| u^h(x) - u^h(x-z) \right| dx \, dz. \end{aligned}$$

Since $0 \le u \le 1$ implies $|u(x) - u(x-z)| \le (1-u)(x) \, u(x-z) + u(x) \, (1-u)(x-z)$ we thus have

$$\sqrt{h} \int \left| \nabla G_{h/2} * u^h \right| dx \lesssim \sqrt{h} \, E_h(u^h) \overset{(20)}{=} O(\sqrt{h})$$

and therefore the mixed term (40) vanishes in the limit $h \to 0$ and indeed we have

$$2\sqrt{h} \int \zeta \big(G_{h/2} * \left(\xi \cdot \nabla u^h \right) \big)^2 dx = 2\sqrt{h} \int \zeta \big(\nabla G_{h/2} * \left(\xi \, u^h \right) \big)^2 dx + o(1).$$

Using the antisymmetry of ∇G and in particular $\int \nabla G(z) \, dz = 0$ we may add a lower-order term:

$$\begin{aligned} 2\sqrt{h} \int \zeta \big(\nabla G_{h/2} * \left(\xi \, u^h \right) \big)^2 dx &= 2\sqrt{h} \int u^h \, \xi \cdot \nabla G_{h/2} * \big(\zeta \nabla G_{h/2} * \left(-\xi \, u^h \right) \big) dx \\ &= 2\sqrt{h} \int u^h \, \xi \cdot \nabla G_{h/2} * \big(\zeta \nabla G_{h/2} * \left(\xi \, (1 - u^h) \right) \big) dx + O(\sqrt{h}). \end{aligned}$$

The term involving $\nabla G_{h/2} * \big(\zeta \nabla G_{h/2} * \xi \big)$ is indeed of lower order since both gradients may be put on the test functions ζ and ξ. Since the operator norms of the commutators $\sqrt{h} \, [\xi \cdot, \nabla G_h *]$ and $\sqrt{h} \, [\zeta, \nabla G_h *]$ are of order $\sqrt{h}$ we can rewrite this last integral as

$$\frac{2}{\sqrt{h}} \int \zeta \, (\xi \otimes \xi) : \left(1 - u^h \right) (h \nabla^2 G_h) * u^h \, dx + o(1).$$

Then (39) follows from the convergence of the energies (cf. Lemma 4.8) and Proposition 4.4.

Using (38) for the numerator and (39) for the denominator of the right-hand side of (34) we obtain by Fatou's Lemma in t

$$\liminf_{h\to0}\int_0^T |\partial E_h|^2(u^h)\,dt \geq \frac{c_0}{2}\int_0^T \left(\sup_\xi \frac{\int \zeta\, H\,\nu\cdot\xi\,|\nabla\chi|}{\sqrt{\int\zeta\,|\xi|^2\,|\nabla\chi|}}\right)^2 dt = \frac{c_0}{2}\int_0^T\int \zeta\, H^2\,|\nabla\chi|\,dt,$$

which concludes the proof. □

Proof of Proposition 4.7. We first note that by definition

$$\begin{aligned}E_h(\chi^n,\chi^{n-1})-E_h(\chi^n) =& \frac{1}{\sqrt{h}}\int\left(\chi^n-\chi^{n-1}\right)[\zeta,G_h*]\left(1-\chi^{n-1}\right)dx\\ &+\frac{1}{\sqrt{h}}\int\left(\chi^n-\chi^{n-1}\right)\left[\zeta,G_{h/2}*\right]G_{h/2}*\left(\chi^n-\chi^{n-1}\right)dx.\end{aligned}$$

We replace $(1-\chi^{n-1})$ by $(1-\chi^n)$ on the right-hand side, and by the simple manipulation (35) of the commutator we obtain that the above expression is equal to

$$\frac{1}{\sqrt{h}}\int\left(\chi^n-\chi^{n-1}\right)[\zeta,G_h*]\,(1-\chi^n)+\left(\chi^n-\chi^{n-1}\right)\left[G_{h/2}*,\zeta\right]G_{h/2}*\left(\chi^n-\chi^{n-1}\right)dx.$$

Now we prove the proposition in two steps. First, we show that the first term converges to the right-hand side of the claim:

$$\int_0^T\int \partial_t^{-h}\chi^h\,\frac{1}{\sqrt{h}}\,[\zeta,G_h*]\left(1-\chi^h\right)dx \to c_0\int_0^T\int H\,\nu\cdot\nabla\zeta\,|\nabla\chi|\,dt \quad \text{as } h\to0, \tag{41}$$

where $\partial_t^{-h}\chi^h=\frac{\chi^h-\chi^h(\,\cdot\,-h)}{h}$ denotes the discrete backwards time-derivative of χ^h. Then we prove that the second term is negligible:

$$\int_0^T\sqrt{h}\int\partial_t^{-h}\chi^h\left[G_{h/2}*,\zeta\right]G_{h/2}*\partial_t^{-h}\chi^h\,dx\,dt\to0 \quad \text{as } h\to0. \tag{42}$$

Step 1: Argument for (41). Expanding the commutator to second order

$$\frac{1}{\sqrt{h}}[\zeta,G_h*]\,v=-\sqrt{h}\nabla G_h*(\nabla\zeta\, v)+\frac{\sqrt{h}}{2}\left(G_h\,Id+h\nabla^2G_h\right)*\left(\nabla^2\zeta\, v\right)+O\left(h\,k_h*|v|\right), \tag{43}$$

where the kernel k_h is given by $k(z)=|z|^3G(z)$, we obtain for the first-order term

$$\begin{aligned}&h\sum_{n=1}^N\int\frac{\chi^n-\chi^{n-1}}{h}\sqrt{h}\nabla G_h*\left(-\nabla\zeta\,(1-\chi^n)\right)\,dx\\ &\qquad=-\frac{h}{2}\sum_{n=1}^N\delta\left(\frac{1}{2h}\mathrm{d}_h^2(\,\cdot\,,\chi^{n-1})\right)(\chi^n,\nabla\zeta)+o(1),\end{aligned}$$

where $\delta\frac{1}{2h}\mathrm{d}_h^2$ is the first variation of the dissipation functional:

$$\delta\left(\frac{1}{2h}\mathrm{d}_h^2(\,\cdot\,,\chi^{n-1})\right)(\chi^n,\xi) = \frac{2}{\sqrt{h}}\int\left(\chi^n - \chi^{n-1}\right)G_h * (-\xi\cdot\nabla\chi^n)\,dx.$$

Formally, this term converges to $c_0\int_\Sigma V\,\nabla\zeta\cdot\nu$ but we want to obtain the term $\frac{c_0}{2}\int_\Sigma H\,\nabla\zeta\cdot\nu$ instead. Therefore we use the minimizing movements interpretation (1) in form of the Euler-Lagrange equation

$$\delta E_h(\chi^n,\xi) + \delta\left(\frac{1}{2h}\mathrm{d}_h^2(\,\cdot\,,\chi^{n-1})\right)(\chi^n,\xi) = 0 \quad \text{for all } \xi\in C^\infty([0,\Lambda)^d,\mathbb{R}^d).$$

We thus have

$$h\sum_{n=1}^N\int\frac{\chi^n-\chi^{n-1}}{h}\sqrt{h}\nabla G_h * (\nabla\zeta\,\chi^n)\;dx = \frac{h}{2}\sum_{n=1}^N\delta E_h(\chi^n,\nabla\zeta) + o(1).$$

By the convergence of the energies (4) and Proposition 4.4 we may pass to the limit $h\to 0$ and obtain

$$\frac{1}{2}\int_0^T\delta E(\chi,\nabla\zeta)\,dt = \frac{c_0}{2}\int_0^T\int\nabla^2\zeta\colon\,(Id-\nu\otimes\nu)\,|\nabla\chi|\,dt \overset{(6)}{=} \frac{c_0}{2}\int_0^T\int H\,\nu\cdot\nabla\zeta\,|\nabla\chi|\,dt.$$

Now we conclude the argument for (41) by showing that the second- and third-order terms in the expansion (43) are negligible in the limit $h\to 0$. The second-order term is estimated as follows

$$\begin{aligned}&\int_0^T\int\partial_t^{-h}\chi^h\frac{\sqrt{h}}{2}\left(G_h\,Id + h\nabla^2G_h\right)*\left(\nabla^2\zeta\left(1-\chi^h\right)\right)\,dx\,dt\\ &\le\left(\int_0^T\sqrt{h}\int\left|\left(G_h\,Id+h\nabla^2G_h\right)*\partial_t^{-h}\chi^h\right|^2dx\,dt\right)^{\frac12}\left(\int_0^T\sqrt{h}\int\left|\nabla^2\zeta\left(1-\chi^h\right)\right|^2dx\,dt\right)^{\frac12}.\end{aligned}$$

The second right-hand side integral is bounded by $T\Lambda^d\|\nabla^2\zeta\|_\infty^2\sqrt{h}$ while the first right-hand side integral can be estimated by

$$\int\left|\left(G_h\,Id+h\nabla^2G_h\right)*\partial_t^{-h}\chi^h\right|^2dx \lesssim \int\left(G_h*\partial_t^{-h}\chi^h\right)^2 + \left|h\nabla^2G_h*\partial_t^{-h}\chi^h\right|^2dx,$$

which by the semi-group property $\nabla^2G_h = \nabla^2G_{h/2}*G_{h/2}$ is bounded by a constant times $\int\left(G_{h/2}*\partial_t^{-h}\chi^h\right)^2dx$. Therefore the time integral stays bounded by the energy-dissipation estimate (20).

The third-order term is controlled by

$$\int_0^T\int h\left|\partial_t^{-h}\chi^h\right|dx\,dt = \int_0^T\int\left|\chi^h(t)-\chi^h(t-h)\right|dx\,dt$$

which converges to zero by the strong convergence of χ^h. This concludes the proof of (41).

Step 2: Argument for (42). We expand the commutator to first order

$$\left[G_{h/2}*, \zeta\right] v = \frac{h}{2}\nabla G_{h/2} * (\nabla\zeta\, v) + O\left(h\left(\frac{|z|}{\sqrt{h}}\right)^2 G_{h/2} * |v|\right), \tag{44}$$

the contribution of the first-order term to (42) is

$$\frac{h}{2}\int_0^T \sqrt{h}\int \partial_t^{-h}\chi^h \nabla G_{h/2} * \left(\nabla\zeta\, G_{h/2} * \partial_t^{-h}\chi^h\right) dx\, dt.$$

Using the antisymmetry of ∇G, the chain rule and integration by parts this is equal to

$$\begin{aligned} &-\frac{h}{2}\int_0^T \sqrt{h}\int \nabla\left(G_{h/2} * \partial_t^{-h}\chi^h\right)\cdot\nabla\zeta\left(G_{h/2} * \partial_t^{-h}\chi^h\right) dx\, dt \\ &\quad = -\frac{h}{2}\int_0^T \sqrt{h}\int \nabla\zeta\cdot\nabla\left(\frac{1}{2}\left(G_{h/2} * \partial_t^{-h}\chi^h\right)^2\right) dx\, dt \\ &\quad = \frac{h}{4}\int_0^T \sqrt{h}\int \Delta\zeta\left(G_{h/2} * \partial_t^{-h}\chi^h\right)^2 dx\, dt. \end{aligned}$$

By the energy-dissipation estimate (20) this term is $O(h)$ as $h \to 0$.

The second-order term coming from the expansion (44) is controlled by

$$\int_0^T \sqrt{h}\int \left|\partial_t^{-h}\chi^h\right| h\left(\frac{|z|}{\sqrt{h}}\right)^2 G_{h/2} * \left|G_{h/2} * \partial_t^{-h}\chi^h\right| dx\, dt \lesssim \int_0^T \sqrt{h}\int \left|G_{h/2} * \partial_t^{-h}\chi^h\right| dx\, dt,$$

which is $O(\sqrt{h})$ by Jensen's inequality and the energy-dissipation estimate (20). □

Proof of Lemma 4.8. First we note that (29) follows from (28) and Lebesgue's Dominated Convergence Theorem since $E_h(u^h) \le E_0$. The argument for (27) is given in the proof of Lemma 2.8 in Chapter 1. (28) in turn follows from (27) since we have the monotonicity property $E_h(\chi^h(t)) \le E_h(u^h(t)) \le E_h(\chi^h(t+h))$. □

Chapter 3

Variants of thresholding schemes

In this chapter we present the work [55] with Drew Swartz. We prove convergence results for three variants of the thresholding scheme. The schemes considered here all incorporate either a local force coming from an energy in the bulk, or a non-local force coming from a volume constraint.

1 Introduction

We first establish the convergence of a scheme proposed by Ruuth and Wetton [80] for approximating volume-preserving mean-curvature flow in Section 2. The main ingredient of our proof is an L^2-bound on the Lagrange multiplier coming from the constraint, which corresponds to the following quantitative estimate on the threshold value $\lambda_h(t)$:

$$\left|\lambda_h - \tfrac{1}{2}\right| = O(\sqrt{h}) \quad \text{in } L^2(0,T).$$

In Section 3 we study a scheme incorporating external forces which is based on an idea of Mascarenhas [60]. In Section 4 we consider a thresholding scheme for simulating grain growth in a polycrystal incorporating boundary effects. The large-scale simulations [29] for grain growth as well as our convergence proof in Chapter 1 assume periodic boundary conditions and are therefore restricted to the interior behavior in a polycrystal. Taking into account boundary effects on the solid-vapor interface is more difficult. A widely accepted model for the evolution of the surface of a polycrystal is *surface diffusion*, a fourth order flow. However, computational simulations involving fourth order flows present various challenges. In Section 4 we discuss a simpler algorithm proposed by Esedoğlu and Jin in [8] for approximating these effects. They consider a scheme which replaces surface diffusion, the fourth order local motion law on the outer boundary of the polycrystal, by volume-preserving mean-curvature flow, a second order but non-local equation. This is plausible because both motions are volume preserving and (due to the gradient flow structure) energy dissipative flows for the area functional. Simulations for this model have been performed in [8], demonstrating that the model is reasonable and captures the typical effect of surface grooving. However it is admittedly not perfect, as it is also shown that for relatively large numbers of grains ($\sim 10^3$), non-physical phenomena are observed in the simulations. In Theorem 4.8 we show that the proof in Chapter 1 can also be applied in this situation under some moderate modeling assumptions. The limiting motion is shown to be mean curvature flow on the grain boundaries, and volume-preserving mean-curvature flow on the solid-vapor interface.

Our starting point in Chapter 1 was the minimizing movements interpretation of Esedoğlu and Otto [33], which means that thresholding preserves the gradient flow structure of (multi-phase) mean curvature flow. We show in Lemma 2.7 that this structural property is conserved in the case of the scheme for volume-preserving mean-curvature flow in [80] as well. In particular, we have the important a priori estimate (14). Most recently Mugnai, Seis and Spadaro [69] studied a volume-preserving variant of the minimizing movements scheme of Almgren, Taylor and Wang [3], and Luckhaus and Sturzenhecker [57]. They proved a conditional convergence result in the same way as Luckhaus and Sturzenhecker. In the proof of Theorem 2.11 we face similar issues as the ones in that work. Bellettini, Caselles, Chambolle and Novaga [11] studied anisotropic versions of mean curvature flow starting from *convex* sets. In particular they proved convergence of the thresholding scheme with uniformly bounded forcing terms. Furthermore, they considered a variant of the volume-preserving scheme [80] where the volume is not precisely preserved in the approximation but still in the limit when the time-step size goes to zero. They were able to prove uniform bounds on the resulting forcing term. In contrast, we work with the exact constraint on the volume and only work with an L^2-bound on the forcing term coming from the Lagrange multipliers associated to the volume constraint. We establish this bound in Proposition 2.12. In Lemma 2.19 we generalize the one-dimensional estimate Lemma 4.2 and Corollary 4.3 in Chapter 1 to our situation where the threshold value may differ from $\frac{1}{2}$.

2 Volume-preserving mean-curvature flow

In this section, we discuss a scheme for volume-preserving mean-curvature flow, here Algorithm 2.1, which was introduced by Ruuth and Wetton in [80]. We first state the algorithm and fix the notation, and present the main result of this section in Theorem 2.11. Following this we give the details of the proof of the theorem.

2.1 Algorithm and notation

The following algorithm by Ruuth and Wetton [80] produces a sequence of phases Ω^n which preserve the volume *exactly*, cf. Figure 3.1 for an example.

Algorithm 2.1. *Given the phase Ω, i. e. an open, bounded set in $\mathbb{R}^d$, with $|\Omega| = 1$ at time $t = (n-1)h$, obtain the evolved phase Ω' at time $t = nh$ by:*

1. *Convolution step: $\phi := G_h * \mathbf{1}_\Omega$.*

2. *Defining threshold value: Pick λ such that $|\{\phi > \lambda\}| = 1$.*

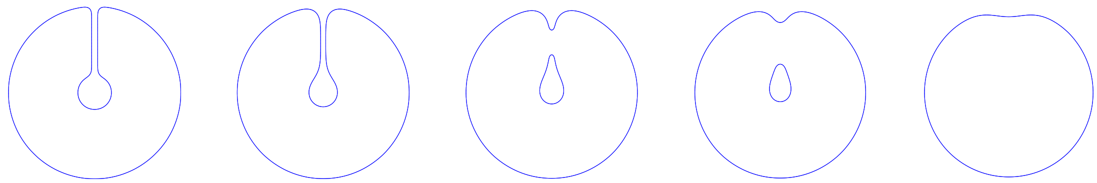

Figure 3.1: The evolution of a "Thüringer Bratwurst". Computation based on a variant of the code provided by Esedoğlu [31]. The threshold value was computed exactly by sorting the grid points according to their ϕ-values.

3. *Thresholding step:* $\Omega' := \{\phi > \lambda\}$.

Here and throughout the chapter

$$G_h(z) := \frac{1}{(4\pi h)^{d/2}} \exp\left(-\frac{|z|^2}{4h}\right)$$

denotes the heat kernel at time h.

Remark 2.2. In general, the threshold value λ is not necessarily a regular value of ϕ, so that a priori we cannot say that the function $s \mapsto |\{\phi > s\}|$ will attain the value 1 for any $s \in [0,1]$. Since by Sard's Lemma a. e. value of ϕ is a regular value, this practically does not happen in simulations. Therefore, as in [80], we ignore this fact in stating the algorithm. Our analysis also works if one replaces the second step of the scheme by defining λ via

$$\lambda := \inf\{s > 0\colon\ |\{\phi > s\}| < 1\}$$

and then chooses the updated set in the following way:

$$\{\phi > \lambda\} \subset \Omega' \subset \{\phi \geq \lambda\} \quad \text{such that} \quad \left|\Omega'\right| = 1.$$

Notation 2.3. We denote the characteristic function of Ω^n at the n-th time step by χ^n, i. e.

$$\chi^n := \mathbf{1}_{\Omega^h}\big|_{t=nh} \equiv \mathbf{1}_{\Omega^n}$$

and interpolate these functions piecewise constantly in time, i. e.

$$\chi^h(t) := \chi^n \quad \text{for } t \in [nh,(n+1)h).$$

As in [33], here for the two-phase case, we define the following approximate energies

$$E_h(\chi) := \frac{1}{\sqrt{h}} \int (1-\chi)\, G_h * \chi \, dx, \tag{1}$$

for $\chi\colon \mathbb{R}^d \to \{0,1\}$ and the approximate dissipation functionals as

$$D_h(\omega) := \frac{1}{\sqrt{h}} \int \omega\, G_h * \omega\, dx \tag{2}$$

for any $\omega\colon \mathbb{R}^d \to \{-1,0,1\}$.

Remark 2.4. As $h \to 0$, the approximate energies E_h Γ-converge to the perimeter functional

$$E(\chi) := \frac{1}{\sqrt{\pi}} \int |\nabla \chi|$$

w. r. t. the L^1-topology. Esedoğlu and Otto proved in [33] that this Γ-convergence which has already been established by Miranda et al. in [65] is a consequence of pointwise convergence of the functionals, namely

$$E_h(\chi) \to E(\chi) \quad \text{for any } \chi \in \{0,1\}, \tag{3}$$

and the following approximate monotonicity: For any $0 < h \leq h_0$ and any $\chi \in \{0,1\}$,

$$E_h(\chi) \geq \left(\frac{\sqrt{h_0}}{\sqrt{h} + \sqrt{h_0}} \right)^{d+1} E_{h_0}(\chi). \tag{4}$$

Our main result of this section, Theorem 2.11, establishes the convergence of the scheme towards the following weak formulation of volume-preserving mean-curvature flow which was also used by Mugnai, Seis and Spadaro [69] and is the analogue of the formulation used by Luckhaus and Sturzenhecker without the volume constraint [57].

Definition 2.5 (Volume-preserving motion by mean curvature)**.** We say that $\chi : (0,T) \times \mathbb{R}^d \to \{0,1\}$ is a *solution to the volume-preserving mean-curvature flow equation with initial data* χ^0 if there exists a function $V\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ with $V \in L^2(|\nabla\chi|\, dt)$ such that

$$\int_0^T \int (\nabla \cdot \xi - \nu \cdot \nabla \xi\, \nu)\, |\nabla \chi|\, dt = \int_0^T \int (V + \Lambda)\; \xi \cdot \nu\, |\nabla \chi|\, dt \tag{5}$$

for any $\xi \in C_0^\infty((0,T) \times \mathbb{R}^d)$ and

$$\int_0^T \int \partial_t \zeta\, \chi\, dx\, dt + \int \zeta(0)\, \chi^0\, dx = - \int_0^T \int \zeta\, V\, |\nabla \chi|\, dt \tag{6}$$

for all $\zeta \in C_0^\infty([0,T) \times \mathbb{R}^d)$, where $\Lambda \in L^2(0,T)$ is the average of the generalized mean curvature $H \in L^2(|\nabla\chi|\, dt)$ of χ:

$$\Lambda := \langle H \rangle = \frac{\int H\, |\nabla \chi|}{\int |\nabla \chi|}. \tag{7}$$

Remark 2.6. For our convergence proof we assume the following convergence of the energies which is not guaranteed by the a priori estimates we have at hand:

$$\int_0^T E_h(\chi^h)\,dt \to \int_0^T E(\chi)\,dt. \tag{8}$$

In the following we prove Theorem 2.11 using the techniques from Chapter 1. Throughout this section, we write $A \lesssim B$ if there exists a constant $C = C(d) < \infty$ such that $A \leq CB$. Combining (3) and (4), we have

$$E_0 := E(\chi^0) \geq E_h(\chi^0). \tag{9}$$

Furthermore by scaling we can normalize the prescribed volume $|\Omega^0| = \int \chi^0\,dx = 1$.

2.2 Minimizing movements interpretation

In the following lemma we elaborate the interpretation of Algorithm 2.1 as a minimizing movements scheme which is the starting point of the convergence proof.

Lemma 2.7 (Minimizing movements interpretation)**.** *Given $\chi^0 \in \{0,1\}$ with $\int \chi^0\,dx = 1$, let ϕ, λ and χ^1 be obtained by Algorithm 2.1. Then χ^1 solves*

$$\min \quad E_h(\chi) + D_h(\chi - \chi^0) + \frac{2\lambda - 1}{\sqrt{h}} \int \chi\,dx, \tag{10}$$

where the minimum runs over all $\chi\colon \mathbb{R}^d \to \{0,1\}$. Or equivalently

$$\min \quad E_h(\chi) + D_h(\chi - \chi^0) \quad \text{s. t.} \quad \int \chi\,dx = 1, \quad \chi \in \{0,1\}. \tag{11}$$

Proof. First we show that (10) is equivalent to minimizing the 'linearized energy'

$$L_{\lambda,h}(\phi,\chi) := \frac{1}{\sqrt{h}} \int (1-\chi)\,\phi + \chi\,(2\lambda - \phi)\,dx, \tag{12}$$

over $\chi\colon \mathbb{R}^d \to \{0,1\}$. Indeed, this is just a consequence of the fact that

$$E_h(\chi) + D_h(\chi - \chi^0) + \frac{2\lambda - 1}{\sqrt{h}} \int \chi\,dx = L_{\lambda,h}(\phi,\chi) + \text{Terms depending only on } \chi^0, \tag{13}$$

Second we show that (11) is equivalent to minimizing $L_{\lambda,h}(\phi,\chi)$ over $\chi\colon \mathbb{R}^d \to \{0,1\}$ such that $\int \chi\,dx = 1$. This again follows from (13) and the fact that $\frac{2\lambda-1}{\sqrt{h}} \int \chi\,dx$ is a constant in this case.

Finally we show that χ^1 as obtained through Algorithm 2.1 minimizes the linearized energy $L_{\lambda,h}(\phi,\chi)$ over $\chi\colon \mathbb{R}^d \to \{0,1\}$ (and therefore also minimizes $L_{\lambda,h}(\phi,\chi)$ over this class when the unit volume constraint is enforced). To see this, note that the integrand is clearly bounded below by $\phi \wedge (2\lambda - \phi)$ for any $\chi \in \{0,1\}$. And by definition, χ^1 admits this minimum pointwise:

$$\left(1-\chi^1\right)\phi + \chi^1\left(2\lambda - \phi\right) = \phi \wedge \left(2\lambda - \phi\right). \qquad \square$$

The following a priori estimate is a direct consequence of the minimizing movements interpretation but is a very important tool to prove compactness of the approximate solutions.

Lemma 2.8 (Energy-dissipation estimate)**.** *The approximate solutions χ^h satisfy the following energy-dissipation estimate*

$$E_h(\chi^N) + \sum_{n=1}^{N} D_h(\chi^n - \chi^{n-1}) \leq E_0. \tag{14}$$

Proof. As a direct consequence of the minimization procedure (11) we obtain

$$E_h(\chi^n) + D_h(\chi^n - \chi^{n-1}) \leq E_h(\chi^{n-1}).$$

Iterating this estimate from $n = 1$ to N together with (9) yields the claim. $\square$

Above we used the minimizing movements interpretation to derive an easy a priori estimate by comparing the solution χ^n to its predecessor χ^{n-1}. Now we use this interpretation to derive an optimality condition, the Euler-Lagrange equation associated to the functional

$$E_h(\chi) + D_h(\chi - \chi^0) + \frac{2\lambda - 1}{\sqrt{h}} \int \chi\, dx.$$

This will be an important component of our convergence proof. To state this precisely, let us first define the notion of first variation of $E_h(\cdot)$ and $D_h(\cdot - \chi^0)$. Since we are considering characteristic functions of sets, which induces the "constraint" $\chi \in \{0,1\}$, the correct variations are *inner* variations, i. e. variations of the independent variable. Geometrically this corresponds to a deformation of the phase Ω.

Definition 2.9 (First variation)**.** For any $\chi \in \{0,1\}$ and $\xi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ let χ_s be generated by the flow of ξ, i. e. χ_s solves the following distributional equation:

$$\partial_s \chi_s + \xi \cdot \nabla \chi_s = 0.$$

We denote the first variation along this flow by

$$\delta E_h(\chi, \xi) := \frac{d}{ds} E_h(\chi_s)\big|_{s=0}, \quad \delta D_h(\,\cdot\, - \tilde{\chi})(\chi, \xi) := \frac{d}{ds} D_h(\chi_s - \tilde{\chi})\big|_{s=0},$$

where $\tilde{\chi} \in \{0,1\}$ is fixed.

Corollary 2.10 (Euler-Lagrange equation)**.** *Given $\chi^0 \in \{0,1\}$, let χ^1 be obtained by Algorithm 2.1 with threshold value λ. Then χ^1 solves the Euler-Lagrange equation associated to (10):*

$$\delta E_h(\chi^1,\xi) + \delta D_h(\,\cdot\, - \chi^0)(\chi^1,\xi) + \frac{2\lambda-1}{\sqrt{h}} \int (\nabla\cdot\xi)\,\chi^1\,dx = 0. \tag{15}$$

Equation (15) follows directly from the minimizing movements interpretation (10) and can be regarded as an approximate version of the weak formulation (5). One can easily compute the formal limit of each single term. A formal expansion suggests that with H denoting the mean curvature of $\partial\Omega^1$ and V denoting the normal velocity moving $\partial\Omega^0$ to $\partial\Omega^1$ in time h we have

$$\delta E_h(\chi^1,\xi) \approx \frac{1}{\sqrt{\pi}} \int_{\partial\Omega^1} H\,\xi\cdot\nu \quad \text{and} \quad \delta D_h(\,\cdot\, - \chi^0)(\chi^1,\xi) \approx -\frac{1}{\sqrt{\pi}} \int_{\partial\Omega^1} V\,\xi\cdot\nu.$$

Therefore, at least formally, (15) is similar to the desired equation $V = H - \langle H\rangle$. In our rigorous justification we will interpret the terms in a weak sense and use the strategy of Chapter 1. Following the lines of the proof of Lemma 3.4, we can also compute the first variation δE_h of the energy rigorously and obtain

$$\begin{aligned}\delta E_h(\chi,\xi) =& \frac{1}{\sqrt{h}} \int \xi\cdot\nabla\chi\, G_h * \chi - (1-\chi)\, G_h * (\xi\cdot\nabla\chi)\,dx \\ =& \frac{1}{\sqrt{h}} \int \xi\cdot[(1-\chi)\,\nabla G_h * \chi] - (1-\chi)\,\nabla G_h * (\xi\,\chi)\,dx \\ &+ \frac{1}{\sqrt{h}} \int (\nabla\cdot\xi)\,(1-\chi)\,G_h*\chi + (1-\chi)\,G_h * ((\nabla\cdot\xi)\,\chi)\,dx.\end{aligned} \tag{16}$$

Expanding $\xi(x) - \xi(x-z) = (z\cdot\nabla)\,\xi(x) + O(|z|^2)$ for the first right-hand side integral, and $(\nabla\cdot\xi)\,(x-z) = (\nabla\cdot\xi)\,(x) + O(|z|)$ for the second we obtain

$$\delta E_h(\chi,\xi) = \frac{1}{\sqrt{h}} \int \nabla\xi\colon\ (1-\chi)\,(G_h\, Id - 2h\nabla G_h) * \chi\,dx + o(1), \tag{17}$$

as $h\to 0$. The integral on the right hand side formally converges to $\frac{1}{\sqrt{\pi}}\int \nabla\xi\colon\ (Id - \nu\otimes\nu)\,|\nabla\chi|$, and can be made rigorous. We will discuss this fact below in Proposition 2.17. For the first variation of the dissipation we can expand ξ again and obtain

$$\delta D_h(\,\cdot\, - \chi^0)(\chi^1,\xi) = -2\int \frac{\chi^1-\chi^0}{h}\xi\cdot\sqrt{h}\nabla G_h * \chi^1\,dx + o(1),$$

where the first factor in the right-hand side integral is a finite difference and formally converges to $\partial_t\chi = V\,|\nabla\chi|$, and the second factor formally converges to $\frac{1}{2\sqrt{\pi}}\nu$. The rigorous justification of this fact is more involved since one has to pass to the limit in a product of two weakly converging terms. We will show how to overcome this difficulty in the following.

2.3 Main result

From (15) we establish convergence to the weak formulation of volume-preserving mean- curvature flow in Definition 2.5. The central novelties of this section are establishing the equivalence of (15) to Algorithm 2.1, which was done above, and to show that the threshold value λ remains close to $\frac{1}{2}$ in a certain sense, which is done in Proposition 2.12 below. The latter property plays an important role in showing that each of the three terms of (15) converges to its respective limit. The mean curvature is recovered as the limit of the first variation δE_h of the energies (cf. Proposition 2.17), and the normal velocity is recovered as the limit of the first variation δD_h of the dissipation (cf. Proposition 2.18). Doing so is similar to results in Chapter 1, however technical difficulties must be overcome due to the fact that the threshold parameter λ may vary (as opposed to being fixed at $\frac{1}{2}$ in the original MBO scheme). The averaged mean curvature is recovered as the limit of the Lagrange multipliers, cf. proof of Theorem 2.11.

We now state and prove the main result of this section, Theorem 2.11 below. Under the same convergence assumption as in Chapter 1 which is inspired by the assumption in [57] we can prove the convergence of the scheme. For clarity of presentation, the given proof merely highlights the main ideas involved in establishing the convergence of (15) to (5). The more technical aspects of the proof are then postponed to later subsections.

Theorem 2.11. *Let $T < \infty$ and $\chi^0 \in \{0,1\}$ with $E(\chi^0) < \infty$ and $\{\chi^0 = 1\} \subset\subset \mathbb{R}^d$. After passage to a subsequence, the functions χ^h obtained by Algorithm 2.1 converge to a function χ in $L^1((0,T)\times\mathbb{R}^d)$. Under the convergence assumption (8), χ is a solution of the volume-preserving mean-curvature flow equation in the sense of Definition 2.5.*

Proof of Theorem 2.11. By Proposition 2.13 the approximate solutions χ^h converge to some limit χ after passage to a subsequence. The strategy of our proof for (5) is to pass to the limit in the Euler-Lagrange equation (15) after integration in time.

By Proposition 2.12, after passing to a further subsequence, we can find a function $\Lambda \in L^2(0,T)$ such that

$$\frac{2\lambda_h - 1}{\sqrt{h}} \rightharpoonup \frac{1}{\sqrt{\pi}}\Lambda \quad \text{in } L^2(0,T).$$

Since the integrals converge strongly,

$$\int (\nabla\cdot\xi)\,\chi^h\,dx \to \int (\nabla\cdot\xi)\,\chi\,dx \quad \text{in } L^2(0,T),$$

we can pass to the limit $h \to 0$ in the product. This is one of the three terms of the Euler-Lagrange equation. In Proposition 2.17 we recover the mean curvature from the first variation of the energy, i. e. the first term in (15). In Proposition 2.18 we recover the normal velocity from the second term in (15), the first variation of the dissipation. Therefore, the limit solves (5). Furthermore, V solves (6) by construction. Note that since Λ, $V \in L^2(|\nabla\chi|\,dt)$ we have a generalized mean curvature

$H \in L^2(|\nabla\chi|\,dt)$. We are left with proving (7). Note that $t \mapsto \int \chi(t)\,dx \in H^1(0,T)$ with

$$\frac{d}{dt}\int \chi\,dx = \int V\,|\nabla\chi|\,.$$

Indeed, given $f \in C_0^\infty(0,T)$ and $g \in C_0^\infty(\mathbb{R}^d)$ with $g \equiv 1$ on B_{R^*} with $R^* = R^*(d,E_0,T)$ from Proposition 2.14, setting $\zeta(x,t) := f(t)g(x)$, we have

$$-\int_0^T f'(t)\int \chi(t)\,dx\,dt = -\int_0^T\int \partial_t\zeta\,\chi\,dx\,dt = \int_0^T\int \zeta\,V\,|\nabla\chi|\,dt = \int_0^T f(t)\int V\,|\nabla\chi|\,dt.$$

Since $\int \chi^h\,dx$ is constant in time, also $\int \chi\,dx$ is constant in time. Using (5) as a pointwise a. e. statement in time, we have

$$0 = \frac{d}{dt}\int \chi\,dx = \int V\,|\nabla\chi| \overset{(5)}{=} \int (H-\Lambda)\,|\nabla\chi| = \int H\,|\nabla\chi| - \Lambda\int|\nabla\chi|$$

almost everywhere in $(0,T)$. Solving for Λ yields (7). □

2.4 L^2-estimate for Lagrange multipliers

The following proposition gives a quantitative estimate on the closeness of the threshold values λ_n to $\frac{1}{2}$ in the natural topology coming from the gradient flow structure and the appearance of $\frac{2\lambda_n-1}{\sqrt{h}}$ as a Lagrange multiplier. Roughly speaking, the lemma states that $\left|\lambda_h - \frac{1}{2}\right| = O(\sqrt{h})$ in L^2. This is the analogue of Corollary 3.4.4 in [69] but our proof works in a different way. While they couple the bound on the Lagrange multiplier and the growth rate of the sets via the estimate (3.28) in [69], we prove the bound on the Lagrange multipliers first, independently of the growth rate. The main difference is that we construct our test function ξ via some elliptic problem in Step 3 of the proof below so that we can obtain estimates by using elliptic regularity theory, in particular the Calderón-Zygmund inequality, cf. Theorem 9.9 in [40].

Proposition 2.12 (L^2-estimate for Lagrange multipliers)**.** *Given the approximate solutions χ^h obtained by Algorithm 2.1 with threshold values λ_h, for $h \ll \frac{1}{E_0^2}$ we have*

$$\int_0^T \left(\lambda_h - \tfrac{1}{2}\right)^2 dt \lesssim (1+T)\left(1+E_0^4\right) h.$$

Here $h \ll \frac{1}{E_0^2}$ means that there exists a generic constant $C = C(d) < \infty$ such that the statement holds for $h < \frac{1}{CE_0^2}$. We recall that $A \lesssim B$ means $A \le C\,B$ for some generic constant $C = C(d) < \infty$.

Proof. Squaring the Euler-Lagrange equation (15), we obtain

$$\frac{1}{h}\left(\lambda_n - \tfrac{1}{2}\right)^2\left(\int (\nabla\cdot\xi)\,\chi^n\,dx\right)^2 \lesssim \left[\delta E_h(\chi^n,\xi)\right]^2 + \left[\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi)\right]^2 \tag{18}$$

for any $\xi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$. In order to prove the proposition, we first estimate the right-hand side for an arbitrary test vector field ξ, cf. Step 1 for the first and Step 2 for the second term. In Step 3 we construct a specific vector field such that the integral on the left-hand side is bounded from below.

Step 1: Estimates on $\delta E_h(\chi,\xi)$. For any $\chi \in \{0,1\}$ and any $\xi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we have

$$|\delta E_h(\chi,\xi)| \lesssim \|\nabla \xi\|_\infty E_h(\chi). \tag{19}$$

Argument: Starting from the computation (16) we see that the second integral on the right-hand side is clearly controlled by $\|\nabla\xi\|_\infty E_h(\chi)$, whereas the first integral on the right-hand side can be estimated via

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\int \xi \cdot [(1-\chi)\,\nabla G_h * \chi] - (1-\chi)\,\nabla G_h * (\xi\chi)\,dx \\
&\quad= \frac{1}{\sqrt{h}}\int -\frac{z}{2h}G_h(z)\cdot\int (\xi(x)-\xi(x-z))\,(1-\chi)\,(x)\chi(x-z)\,dx\,dz \\
&\quad\le \|\nabla\xi\|_\infty \frac{1}{\sqrt{h}}\int \frac{|z|^2}{2h}G_h(z)\int (1-\chi)\,(x)\chi(x-z)\,dx\,dz.
\end{aligned}$$

Using $|z|^2 G_1(z) \lesssim G_2(z)$ we thus have

$$|\delta E_h(\chi,\xi)| \lesssim \|\nabla\xi\|_\infty \left(E_{2h}(\chi) + E_h(\chi)\right)$$

and the approximate monotonicity of the energy (4) yields (19).

Step 2: Estimates on $\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi)$. We have

$$h\sum_{n=1}^{N}\left[\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi_n)\right]^2 \lesssim \sup_n \|\xi_n\|_{W^{1,\infty}}^2\left(1+E_0^2\right). \tag{20}$$

Argument: For any $\xi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and any $n \in \{1,\dots,N\}$, we have

$$\begin{aligned}
\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi) &= \frac{2}{\sqrt{h}}\int \left(-\xi\cdot\nabla\chi^n\right) G_h * \left(\chi^n - \chi^{n-1}\right)dx \\
&= \frac{2}{\sqrt{h}}\int \chi^n \xi\cdot\nabla G_h * \left(\chi^n-\chi^{n-1}\right) + \left(\nabla\cdot\xi\right)\chi^n G_h * \left(\chi^n-\chi^{n-1}\right)dx.
\end{aligned}$$

Setting (compare to the *dissipation measures* μ_h, Definition 2.7 in Chapter 1)

$$\mu_n := \frac{1}{\sqrt{h}}\int \left[G_{h/2} * \left(\chi^n - \chi^{n-1}\right)\right]^2 dx$$

and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\big[\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi)\big]^2 \\
&\qquad \lesssim \left(\frac{1}{h}\int \sqrt{h}\nabla G_{h/2} * (\chi^n \xi)\, G_{h/2} * \big(\chi^n - \chi^{n-1}\big)\, dx\right)^2 \\
&\qquad\qquad + \|\nabla\xi\|_\infty^2 \left(\frac{1}{\sqrt{h}}\int G_{h/2} * \chi^n \left|G_{h/2} * \big(\chi^n - \chi^{n-1}\big)\right| dx\right)^2 \\
&\qquad \le \frac{1}{h}\left(\frac{1}{\sqrt{h}}\int \Big[\sqrt{h}\nabla G_{h/2} * (\chi^n\xi)\Big]^2 dx\right)\mu_n + \frac{1}{\sqrt{h}}\|\nabla\xi\|_\infty^2 \int \chi^n\, dx\ \mu_n.
\end{aligned}$$

For the first right-hand side term, we first observe that for any $\chi \in \{0,1\}$ and any $\xi \in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$, by $|\xi(x+z) - \xi(x)| \le \|\nabla\xi\|_\infty |z|$ we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\int |\sqrt{h}\nabla G_{h/2}(z)| \int \big|\xi(x+z) - \xi(x)\big|\chi(x+z)\left|\sqrt{h}\nabla G_{h/2} * (\chi\xi)\right|(x)\, dx\, dz \\
&\qquad \le \|\xi\|_\infty \|\nabla\xi\|_\infty \int \chi\, dx \left(\int |z||\nabla G_{h/2}(z)|\, dz\right)\left(\int |\sqrt{h}\nabla G_{h/2}(z)|\, dz\right),
\end{aligned}$$

where the last two integrals are uniformly bounded in h. Thus, in our case where $\chi = \chi^n$ with $\int \chi^n\, dx = 1$, we obtain an estimate on the error when commuting the multiplication with ξ and the convolution with the kernel $\sqrt{h}\nabla G_{h/2}$ in one of the factors:

$$\frac{1}{\sqrt{h}}\int \big[\sqrt{h}\nabla G_{h/2} * (\chi\xi)\,\big]^2 dx = \frac{1}{\sqrt{h}}\int \xi\cdot \sqrt{h}\nabla G_{h/2} * \chi\big[\sqrt{h}\nabla G_{h/2} * (\chi\xi)\,\big] dx + O(\|\xi\|_{W^{1,\infty}}^2).$$

Since ∇G is antisymmetric and since $|z|\,G(z) \lesssim G_2(z)$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\int \xi\cdot\sqrt{h}\nabla G_{h/2} * \chi\left[\sqrt{h}\nabla G_{h/2} * (\chi\xi)\right] dx \\
&\qquad = \frac{1}{\sqrt{h}}\int \xi\cdot\sqrt{h}\nabla G_{h/2} * (\chi - 1)\left[\sqrt{h}\nabla G_{h/2} * (\chi\xi)\right] dx \\
&\qquad \lesssim \|\xi\|_\infty^2 \frac{1}{\sqrt{h}}\int G_h * (1-\chi)\, G_h * \chi\, dx \lesssim \|\xi\|_\infty^2 E_h(\chi).
\end{aligned}$$

Thus, we have

$$\big[\delta D_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi)\big]^2 \lesssim \frac{1}{h}\left(\|\xi\|_\infty^2 E_0 + \|\xi\|_{W^{1,\infty}}^2 + \sqrt{h}\,\|\nabla\xi\|_\infty^2\right)\mu_n,$$

which is (20) after integration in time and using the energy-dissipation estimate (14) once more.

Step 3: Choice of ξ. For any $E_0 > 0$, any $0 < h \ll 1/E_0^2$ and any $\chi \in \{0,1\}$ with $\int \chi\, dx = 1$, $\operatorname{supp}\chi \subset\subset \mathbb{R}^d$ and $E_h(\chi) \le E_0$ there exists $\xi \in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ with

$$\int (\nabla\cdot\xi)\,\chi\, dx \ge \frac{1}{2} \quad \text{and} \tag{21}$$

$$\|\xi\|_{W^{1,\infty}} \lesssim 1 + E_0. \tag{22}$$

Argument: Set $\varepsilon^2 = \frac{1}{CE_0^2}$. We will determine the constant $C = C(d)$ later. Set $\chi_\varepsilon := \varphi_\varepsilon * \chi$ for some standard mollifier $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^d}\varphi_1(\frac{z}{\epsilon})$ with $0 \le \varphi_1 \le 1$, $\int \varphi_1 \, dz = 1$, $\varphi_1 \lesssim G_1$ and $\int |\nabla \varphi_1| \, dz \lesssim 1$. Then $\chi_\varepsilon \in C_0^\infty(\mathbb{R}^d, [0,1])$. Let u denote the solution of

$$\Delta u = \chi_\varepsilon$$

given by the Newtonian potential $u = \Gamma * \chi_\varepsilon$. We define $\xi := \nabla u = \Gamma * \nabla \chi_\varepsilon$ and claim that ξ satisfies (21). Indeed, since $|\chi_\varepsilon - \chi| = \chi\,(1-\chi_\varepsilon) + (1-\chi)\,\chi_\varepsilon$ for $\chi \in \{0,1\}$ and $0 \le \chi_\varepsilon \le 1$, we can use the approximate monotonicity (4) such that for any $0 < h \le \varepsilon^2$ we have

$$\begin{aligned}\int |\chi_\varepsilon - \chi| \, dx &= 2\int (1-\chi)\,\varphi_\varepsilon * \chi \, dx \\ &\lesssim \int (1-\chi)\, G_{\varepsilon^2} * \chi \, dx \overset{(4)}{\le} \varepsilon \left(\frac{\varepsilon + \sqrt{h}}{\varepsilon}\right)^{d+1} E_h(\chi) \lesssim \varepsilon E_0.\end{aligned}$$

Thus, if we pick the constant $C(d)$ in the definition of ε large enough, we have

$$\int (\nabla \cdot \xi)\,\chi \, dx = \int \chi_\varepsilon \chi \, dx \ge \int \chi \, dx - \int |\chi_\varepsilon - \chi| \, dx \ge \frac{1}{2},$$

which is (21). Now we give an argument for (22). The Calderón-Zygmund inequality yields

$$\int_{\mathbb{R}^d} |\nabla \xi|^p dx \lesssim_p \int |\chi_\varepsilon|^p dx \le 1 \tag{23}$$

for any $1 < p < \infty$, where we write $\lesssim_p$ to stress that the constant depends not only on the dimension d but also on the parameter p. Since χ_ε is smooth, we can differentiate the equation:

$$\Delta \xi = \nabla \chi_\varepsilon.$$

Thus by the Calderón-Zygmund inequality and Jensen's inequality

$$\int_{\mathbb{R}^d} |\nabla^2 \xi|^p dx \lesssim_p \int |\nabla \chi_\varepsilon|^p dx \le \left(\int |\nabla \varphi_\varepsilon| \, dz\right)^p \int |\chi|^p \, dx \lesssim \frac{1}{\varepsilon^p} \tag{24}$$

for any $1 < p < \infty$. Now we want to bound the 0-th order term of ξ. Let $R > 0$ be big enough such that $\operatorname{supp} \chi_\varepsilon \subset B_{\frac{R}{2}}$ and take $\eta \in C_c^\infty(B_{2R})$ to be a cut-off function for B_R in B_{2R} with $|\nabla \eta| \lesssim \frac{1}{R}$. Then we have

$$\int |\nabla(\eta\,\xi)|^p dx \lesssim_p \int \eta |\nabla \xi|^p dx + \int |\nabla \eta|^p |\xi|^p dx \overset{(23)}{\lesssim_p} 1 + \frac{1}{R^p}\int_{B_{2R}\setminus B_R} |\xi|^p dx.$$

Note that for any $x \in \mathbb{R}^d \setminus B_R$, since then $\operatorname{dist}(x, \operatorname{supp} \chi_\varepsilon) \gtrsim R$, we have

$$|\xi(x)| \le \int |\nabla \Gamma(x-y)|\, \chi_\varepsilon(y) \, dy \lesssim \frac{1}{R^{d-1}} \int \chi_\varepsilon(y) \, dy = \frac{1}{R^{d-1}}.$$

Thus,

$$\int |\nabla(\eta\,\xi)|^p dx \lesssim_p 1 + R^{d(1-p)}. \tag{25}$$

Now we fix some $p = p(d) \in (\frac{d}{2}, d)$. Since $\eta\,\xi$ has compact support, we can apply the Gagliardo-Nirenberg-Sobolev inequality, so that

$$\int_{B_R} |\xi|^{p^*} dx \le \int |\eta\,\xi|^{p^*} dx \lesssim \left(\int |\nabla(\eta\,\xi)|^p dx\right)^{p^*/p} \overset{(25)}{\lesssim} \left(1 + R^{d(1-p)}\right)^{d/(d-p)},$$

where $p^* = \frac{pd}{d-p} > d$ is the Sobolev conjugate of p. Taking the limit $R \to \infty$, we obtain

$$\int |\xi|^{p^*} dx \lesssim 1. \tag{26}$$

Since $p^* > d$, by Morrey's inequality and the above estimates (23), (24) with p^* playing the role of p and (26), we have

$$\|\xi\|_{W^{1,\infty}(\mathbb{R}^d)} \lesssim \|\xi\|_{W^{2,p^*}(\mathbb{R}^d)} \lesssim 1 + \frac{1}{\varepsilon} \sim 1 + E_0.$$

Step 4: Conclusion. We apply Step 3 on $\chi = \chi^n$ and find $\xi^n \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with

$$\int (\nabla \cdot \xi^n)\,\chi^n\,dx \ge \frac{1}{2}$$

$$\|\xi^n\|_{W^{1,\infty}} \lesssim 1 + E_0.$$

Plugging $\xi = \xi^n$ into (18), summing over n and using the estimates in Steps 1 and 2, we obtain

$$\sum_{n=1}^{N} \left(\lambda_n - \tfrac{1}{2}\right)^2 \lesssim \sup_n \|\xi^n\|_{W^{1,\infty}}^2 \left(TE_0^2 + 1 + E_0^2\right) \lesssim (1+T)(1+E_0^4),$$

which is the desired estimate. □

2.5 Compactness

Proposition 2.13 (Compactness)**.** *For any sequence there exists a subsequence $h \searrow 0$ and a limit $\chi \in L^1((0,T) \times \mathbb{R}^d, \{0,1\})$ such that*

$$\chi^h \longrightarrow \chi \quad \textit{in } L^1((0,T) \times \mathbb{R}^d). \tag{27}$$

Moreover,

$$\chi^h \longrightarrow \chi \quad \textit{a. e. in } (0,T) \times \mathbb{R}^d \tag{28}$$

and $\chi(t) \in BV(\mathbb{R}^d, \{0,1\})$, $\int \chi(t)\,dx = 1$ for a. e. $t \in (0,T)$.

Proof. As in Lemmas 2.5 and 2.4 in Chapter 1 we can prove that

$$\int_0^T \int \left|\chi^h(x+\delta\, e, t+\tau) - \chi^h(x,t)\right| dx\, dt \lesssim (1+T)\, E_0 \left(\delta + \tau + \sqrt{h}\right). \tag{29}$$

The proposition follows then from the arguments in Proposition 2.1 in Chapter 1 in conjunction with Proposition 2.14 below. Indeed, in Chapter 1, we showed that this can be done by adapting the proof of the Riesz-Kolmogorov compactness theorem. Since we work in $\mathbb{R}^d$ and not on a periodic domain as in Chapter 1 we need to guarantee that no mass escapes to infinity. The proposition below establishes precisely this. □

Take $R_0 > 0$ such that $\Omega^0 \subset B_{R_0}$. For subsequent n we take a sequence of radii $R_n \geq R_{n-1}$ such that $\Omega^n \subset B_{R_n}$. The focus of this section will be to show that we can choose the radii R_n such that they are uniformly bounded for $n \in \{1, \ldots, N\}$, independent of the time step h.

Proposition 2.14 (Tightness)**.** *There is a finite radius $R^* = R^*(d, E_0, T)$, independent of h such that*

$$\Omega^h(t) \subset B_{R^*} \quad \textit{for all } t \in [0,T].$$

We separate the indices n into 'good' and 'bad' iterations. A 'good' iteration is taken to mean that $|\lambda_n - \frac{1}{2}| < \frac{1}{4}$, and a bad iteration will be taken to mean that $|\lambda_n - \frac{1}{2}| \geq \frac{1}{4}$. The L^2-bounds in Proposition 2.12 give us a suitable level of control over the number of 'bad' iterations. Indeed, Chebyshev's inequality implies that the number of 'bad' iterations is controlled by $(1+T)(1+E_0^4)$.

In the next lemma we show that in the worst case scenario, the radii R_n grow exponentially over consecutive iterations.

Lemma 2.15. *R_n may be chosen such that $R_n \leq 3R_{n-1}$.*

Proof. In order to reduce the notation we may assume $n = 1$ and write $\phi = G_h * \chi^0$, $R := R_0$, $\chi = \chi^1$ and $\lambda = \lambda_1$. We first claim that

$$\min_{\overline{B}_R} \phi > \max_{\mathbb{R}^d \setminus B_{3R}} \phi. \tag{30}$$

This follows immediately from the definition of ϕ using $\{\chi^0 = 1\} \subset B_R$ and the obvious inequality

$$|x - z| < 2R < |y - z| \quad \text{for all } x \in B_R,\ y \in \mathbb{R}^d \setminus B_{3R} \text{ and } z \in B_R.$$

Now suppose that $U := \Omega \setminus B_{3R}$ has positive measure. This being the case, we may construct a new set, call it $\widetilde{\Omega}$, by deleting the volume U from $\Omega \setminus B_{3R}$ and filling it into B_R. Indeed, since $|\Omega| = \left|\Omega^0\right|$, we can find a set $\widetilde{U} \subset B_R$ of the same volume as U such that $\widetilde{U} \cap \Omega = \emptyset$. Then we set $\widetilde{\Omega} := (\Omega \setminus U) \cup \widetilde{U}$ and $\tilde{\chi} = \mathbf{1}_{\widetilde{\Omega}}$. Recall the definition of L_h in (12). We claim that $\tilde{\chi}$ has lower linearized energy $L_h(\phi, \cdot)$ than χ, which is a contradiction. By $\int \tilde{\chi}\, dx = \int \chi\, dx$ and (30) we have

$$L_h(\phi, \chi) - L_h(\phi, \tilde{\chi}) = \frac{2}{\sqrt{h}} \int \phi\, (\tilde{\chi} - \chi)\, dx = \frac{2}{\sqrt{h}} \int \phi \left(\mathbf{1}_{\tilde{U}} - \mathbf{1}_U\right) dx > 0.$$

Thus we conclude that the minimizer of the linearized energy $L_h(\phi,\cdot)$ cannot contain any volume outside B_{3R}. $\square$

Next we show that over 'good' iterations, i. e. $|\lambda_n - \frac{1}{2}| < \frac{1}{4}$, the growth of R_{n-1} to R_n is $O(|\lambda_n - \frac{1}{2}|\sqrt{h})$, which in terms of Proposition 2.12 can be interpreted as 'linear growth'.

Lemma 2.16. *There exists a universal constant $C < \infty$ such that over 'good' iterations we have*

$$R_n \leq R_{n-1} + C\sqrt{h}|\lambda_n - \tfrac{1}{2}|.$$

Proof. Given $|\lambda_n - \frac{1}{2}| < \frac{1}{4}$, we want to find a constant $C < \infty$ so that for any direction $e \in S^{d-1}$ we have $\phi < \lambda_n$ and therefore $\chi^n = 0$ in $\{x \cdot e > R_{n-1} + C\sqrt{h}|\lambda_n - \frac{1}{2}|\}$. We prove this by comparing to a half space $H = \{x \cdot e < R_{n-1}\}$ whose boundary is tangent to $\partial B_{R_{n-1}}$. By rotational symmetry we may assume w. l. o. g. that $e = e_1$ so that at a point $x = (x_1, x')$, thanks to the factorization property of G, we can estimate

$$\phi = G_h * \chi^{n-1} \leq G_h * \mathbf{1}_H = \int_{-\infty}^{\infty} G_h^1(z_1)\mathbf{1}_{x_1+z_1<R_{n-1}} dz_1 = \frac{1}{2} - \int_0^{x_1-R_{n-1}} G_h^1(z_1)\, dz_1.$$

We observe that the right-hand side expression is monotone decreasing in x_1 and find the upper bound for $R_n \geq R_{n-1}$ simply by setting the right-hand side to be equal to λ_n for $x_1 = R_n$:

$$|\lambda_n - \frac{1}{2}| = \int_0^{\frac{1}{\sqrt{h}}(R_n - R_{n-1})} G^1(z_1)\, dz_1.$$

There exists a universal $C < \infty$ such that $\int_0^C G^1(z_1)\, dz_1 = \frac{1}{4}$. Thus, since $|\lambda_n - \frac{1}{2}| < \frac{1}{4}$, we have $\frac{R_n - R_{n-1}}{\sqrt{h}} < C$. In turn this gives

$$\frac{R_n - R_{n-1}}{\sqrt{h}} \min_{|z_1|\leq C} G^1(z_1) < |\lambda_n - \tfrac{1}{2}|,$$

which is the desired estimate. $\square$

Proof of Proposition 2.14. The result follows by iterating the estimate of the previous two lemmas. Indeed, over 'good' iterations we have the estimate

$$R_n \leq R_{n-1} + C\sqrt{h}|\lambda_n - \tfrac{1}{2}|.$$

And over 'bad' iterations we have the estimate

$$R_n \leq 3R_{n-1}.$$

Iterating these two estimates and keeping in mind that we have at most a finite number $\sim (1 + T)(1 + E_0^4)$ of 'bad' iterations we obtain

$$R_N \leq C(d,T,E_0) \left(R_0 + \sum_{n=1}^{N} \sqrt{h}|\lambda_n - \tfrac{1}{2}| \right).$$

Finally we note that by Jensen's inequality and Proposition 2.12

$$\sum_{n=1}^{N} \sqrt{h}|\lambda_n - \tfrac{1}{2}| \leq \left(h \sum_{n=1}^{N} \frac{|\lambda_n - \frac{1}{2}|^2}{h} \right)^{\frac{1}{2}} T^{\frac{1}{2}} \leq C(d, E_0, T).$$

The constant $C(d, E_0, T)$ yields the estimate on R^*. Note that our proof does not give a linear growth estimate in time. Indeed, the upper bound R^* growth exponentially in T. Nevertheless, for our purpose, this is enough. $\square$

2.6 Convergence

In this section we give the details of the proof of Theorem 2.11. We can directly apply Proposition 3.1 of Chapter 1 to our situation, which we state in Proposition 2.17. In Proposition 2.18 we prove that we can change the proof of Proposition 4.1 of 1 so that it applies in our situation. For this part we need Proposition 2.12 to apply the one-dimensional lemma, Lemma 2.19 stated below.

Proposition 2.17 (Energy and mean curvature; Prop. 3.1 in Chapter 1)**.** *Under the convergence assumption (8) we have*

$$\lim_{h\to 0} \int_0^T \delta E_h(\chi^h, \xi)\, dt = \frac{1}{\sqrt{\pi}} \int_0^T \int (\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu)\, |\nabla \chi|\, dt$$

for any $\xi \in C_0^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$.

Proof. The proof of Proposition 3.1 in Chapter 1 only uses the convergence that we deduced here in Proposition 2.13 and the convergence assumption. However, we briefly highlight the line of proof here. We observe that the expansion (17) of the first variation of the energy is already in the same form as the limit: multiplication with the anisotropic kernel $G_h\, Id - 2h\nabla G_h$ corresponds to multiplication with $Id - \nu \otimes \nu$, i.e. projection onto the tangent space. More precisely, evaluated at a fixed configuration χ, the right-hand side of (17) converges to the correct quantity. Under the strengthened convergence (8) this holds true also along the sequence χ^h. $\square$

Proposition 2.18 (Dissipation and normal velocity)**.** *There exists a function* $V\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ *which is a normal velocity in the sense of (6). Given the convergence assumption (8),* $V \in L^2(|\nabla \chi|\, dt)$ *and for any* $\xi \in C_0^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$ *we have*

$$\lim_{h\to 0} \int_0^T \delta D_h(\,\cdot\,, \chi^h(t-h))(\chi^h(t), \xi(t))\, dt = -\frac{1}{\sqrt{\pi}} \int_0^T \int V\, \xi \cdot \nu\, |\nabla \chi|\, dt. \tag{31}$$

Proof. Since we have the same energy-dissipation estimate, namely (14), with the volume constraint as in Chapter 1 without a constraint, we can directly apply most of the techniques. In Lemma 2.19, we show that for most of the iterations we can also apply the finer estimate, Lemma

4.2 in Chapter 1 when changing the threshold value from $\frac{1}{2}$ to λ as in Step 2 of Algorithm 2.1. To make this applicable we need the L^2-estimate in Proposition 2.12.

Step 1: Construction of the normal velocity and (6). We construct the normal velocity V exactly as in Lemma 2.1 in Chapter 1. First, one proves that the distributional time derivative $\partial_t\chi$ of χ is a Radon measure using only the energy-dissipation estimate, in our case (14). Using the convergence assumption, for us (8), this measure turns out to be absolutely continuous w. r. t. $|\nabla\chi|\,dt$, so that one can define V to be the density of $\partial_t\chi$ w. r. t. $|\nabla\chi|\,dt$ and prove higher integrability, $V\in L^2(|\nabla\chi|\,dt)$. Then V satisfies (6) by construction.

Step 2: Argument for (31). One of the key ideas in Chapter 1 is to introduce a mesoscopic time scale $\alpha\sqrt{h}$. In Step 2 of the proof of Proposition 4.1 there, one chooses a shift of the mesoscopic time slices so that one has control over the error terms. We can make use of this degree of freedom to make sure that in addition the mesoscopic time steps are 'good' iterations. Given $N=T/h$, $K=\alpha/\sqrt{h}$, $L=N/K$, for any function $\varepsilon^2\colon\{1,\dots,N\}\to[0,\infty)$ we can find $k_0\in\{1,\dots,K\}$, such that in addition to

$$\frac{1}{L}\sum_{l=1}^{L}\varepsilon^2(Kl+k_0)\le 4\frac{1}{N}\sum_{n=1}^{N}\varepsilon^2(n) \tag{32}$$

as in Chapter 1 we furthermore have

$$\frac{1}{L}\sum_{l=1}^{L}\left(\lambda_{Kl+k_0}-\tfrac{1}{2}\right)^2\le 4\frac{1}{N}\sum_{n=1}^{N}\left(\lambda_n-\tfrac{1}{2}\right)^2 \quad\text{and} \tag{33}$$

$$\left|\lambda_{Kl+k_0}-\tfrac{1}{2}\right|\le\tfrac{1}{8}\quad(1\le l\le L). \tag{34}$$

We give a short counting argument for this. By Proposition 2.12

$$\begin{aligned}
&\#\left\{k_0\colon \text{(32) is violated, (33) is violated, or (34) is violated for some } l\right\}\\
&\le\#\left\{k_0\colon\text{(32) is violated}\right\}+\#\left\{k_0\colon\text{(33) is violated}\right\}+\sum_{l=1}^{L}\#\left\{k_0\colon\text{(34) is violated for } l\right\}\\
&\le\frac{K}{4}+\frac{K}{4}+8^2\sum_{l=1}^{L}\sum_{k=1}^{K}\left(\lambda_{Kl+k}-\tfrac{1}{2}\right)^2\le\frac{K}{2}+C
\end{aligned}$$

for some constant $C=C(d,E_0,T)$. Therefore, we can adapt the proof of Proposition 4.1 in Chapter 1 so that indeed we can link the first variation of the dissipation with the normal velocity. Furthermore, the localization argument in Section 5 of Chapter 1 applies one-to-one so that we have (31). □

One of the main tools of the proof in Chapter 1 are Lemma 4.2 and its rescaled version, Corollary 4.3. Roughly speaking, this lemma establishes control over the distance of the super level sets $\{u > \frac{1}{2}\}$ and $\{\tilde{u} > \frac{1}{2}\}$ in terms of the L^2-distance of two functions $u, \tilde{u} \colon \mathbb{R} \to \mathbb{R}$, provided at least one of the two functions is sufficiently monotone around the threshold value $\frac{1}{2}$, which is measured by the term $\frac{1}{\sqrt{h}} \int_{\frac{1}{3} \leq u \leq \frac{2}{3}} \left(\sqrt{h}\, \partial_1 u - \overline{c} \right)_-^2$; see Lemma 2.19 below for the precise statement with more general threshold values, which however reduces to the statement in Chapter 1 when $\lambda = \tilde{\lambda} = \frac{1}{2}$. Note that such an estimate would clearly fail without such an extra term on the right-hand side.

In order to motivate the lemma let us streamline its application to the thresholding scheme. To this purpose let us ignore the localization η. We apply the one-dimensional estimate to the thresholding scheme in a fixed direction $\nu^* \in S^{d-1}$ with $\chi = \chi^h(t)$ and $\tilde{\chi} = \chi^h(t+\tau)$ for some $\tau = \alpha\sqrt{h}$. We think of the fudge factor α as small, but independent of h. After dividing by α and integrating the resulting estimate over the further $d-1$ directions and over the time variable we obtain an estimate for the difference quotient $\iint \left| \partial_t^\tau \chi^h \right| dx\, dt$ in terms of $\iint \sqrt{h} (G_{h/2} * \partial_t^\tau \chi^h)^2 dx\, dt$, the above term measuring the monotonicity of $G_h * \chi^h(t-h)$ in direction ν^* and a term involving the L^2-norm of $\lambda_h - \frac{1}{2}$. The constant $\overline{c}$ in the term measuring the monotonicity is chosen such that if χ^h was a half space in direction ν^* this term would vanish. One can indeed prove, cf. Lemma 4.4 in Chapter 1, that this term is bounded by the energy-excess

$$\varepsilon^2 := \int_0^T E_h(\chi^h) - E_h(\chi^*)\, dt, \quad \text{for some half space } \chi^* \text{ in direction } \nu^*.$$

This term in turn is small (after localization) by our strengthened convergence (8) and the local flatness of the limit — which is guaranteed by De Giorgi's Structure Theorem. The second term, $\iint \sqrt{h} (G_{h/2} * \partial_t^\tau \chi^h)^2 dx\, dt$, is bounded by the dissipation and is thus finite by the energy-dissipation estimate (14). Therefore we obtain the following estimate for the discrete time derivative

$$\int_0^T \int \left| \partial_t^\tau \chi^h \right| dx\, dt \lesssim \frac{1}{\alpha} \left(\varepsilon^2 + sT \right) + \frac{1}{s^2} \alpha^2 E_0 + \frac{1}{\alpha s^2} \frac{1}{\sqrt{h}} \int_0^T \left(\lambda_h - \frac{1}{2} \right)^2 dt,$$

which differs from the estimate in Chapter 1 only by the last right-hand side term involving the threshold value. However, this term is of order $\sqrt{h}$ by our L^2-estimate, cf. Proposition 2.12. We apply a localized version of this estimate and sum over a partition of unity with fineness $r > 0$. Sending first h to zero, the first right-hand side term converges to the the energy-excess on each patch, while the other terms stay uniformly bounded in r if the patches have finite overlap. Then we take the limit $r \to 0$ so that the first right-hand side term vanishes by De Giorgi's Structure Theorem. Optimizing the additional parameter s and then sending α to zero, the right-hand side stays uniformly bounded. The resulting estimate resembles

$$\int_0^T \int \left| \partial_t^\tau \chi^h \right| dx\, dt = O(1) \quad \text{for } \tau = o(\sqrt{h}).$$

In comparison, the analogous estimate coming from (29) only holds for larger time scales $\tau \sim \sqrt{h}$.

Lemma 2.19. *Let* $u,\ \tilde u \in C^\infty(\mathbb{R})$, $|\lambda-\frac12|<\frac18$, $\chi=\mathbf{1}_{\{u>\lambda\}}$, $\tilde\chi=\mathbf{1}_{\{\tilde u>\tilde\lambda\}}$ *and let* $\eta\in C_0^\infty(-2r,2r)$ *be a radially non-increasing cut-off for* $(-r,r)$ *inside* $(-2r,2r)$*. Then*

$$\frac{1}{\sqrt h}\int \eta\,|\chi-\tilde\chi|\,dx_1 \lesssim \frac{1}{\sqrt h}\int_{\{\frac13\le u\le\frac23\}}\eta\left(\sqrt h\,\partial_1 u-\overline{c}\right)_-^2 dx_1+s + \frac{1}{s^2}\frac{1}{\sqrt h}\int\eta\,(u-\tilde u)^2\,dx_1+\frac{r}{s^2}\frac{\left(\lambda-\tilde\lambda\right)^2}{\sqrt h}$$

for any $s\ll 1$.

Proof of Lemma 2.19. The lemma follows from Corollary 4.3 in Chapter 1 with a shifting argument to make the threshold value λ appear. Set $v:=u-\lambda+\frac12$ so that $\chi=\mathbf{1}_{v>\frac12}$ (and analogously with $\tilde v$) and Corollary 4.3 in Chapter 1 applies for $v,\ \tilde v$: For any $s>0$, we have

$$\frac{1}{\sqrt h}\int\eta|\chi-\tilde\chi|\lesssim\frac{1}{\sqrt h}\int_{\{|v-\frac12|\le s\}}\eta\left(\sqrt h\,\partial_1 v-\overline{c}\right)_-^2+s+\frac{1}{s^2}\frac{1}{\sqrt h}\int\eta\,(v-\tilde v)^2\,. \tag{35}$$

Now we can resubstitute $v=u-\lambda+\frac12$ and $\tilde v=\tilde u-\tilde\lambda+\frac12$ on the right-hand side. Then the integrand of the first integral stays unchanged since λ is constant. If $|\lambda-\frac12|<\frac18$ and $s\ll1$, the domain of integration is

$$\left\{\left|v-\tfrac12\right|<s\right\}=\{|u-\lambda|<s\}\subset\left\{\tfrac13<u<\tfrac23\right\}.$$

Since $(v-\tilde v)^2\lesssim(u-\tilde u)^2+(\lambda-\tilde\lambda)^2$, also the second integral is in the form of the claim. □

3 Mean curvature flow with external force

The following algorithm is based on an idea of Mascarenhas in [60] but we allow the forcing term to be space-time dependent.

3.1 Algorithm and main result

Algorithm 3.1. *Given the phase* Ω *at time* $t=(n-1)h$*, obtain the evolved phase* Ω' *at time* $t=nh$ *by:*

1. *Convolution step:* $\phi:=G_h*\mathbf{1}_\Omega$.
2. *Thresholding step:* $\Omega':=\{\phi>\frac12-\frac{1}{2\sqrt\pi}f(x,nh)\sqrt h\}$.

The following weak formulation of mean-curvature flow with an external force has already been introduced in [57].

Definition 3.2 (Motion by mean curvature with external force)**.** We say that $\chi : (0,T) \times \mathbb{R}^d \to \{0,1\}$ *moves by mean curvature with external force* $f \in C^\infty([0,T] \times \mathbb{R}^d)$ *and initial data* χ^0 if there exists a function $V\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ with $V \in L^2(|\nabla\chi|\,dt)$, which is the normal velocity in the sense of (6), such that

$$\int_0^T \int (\nabla \cdot \xi - \nu \cdot \nabla\xi\, \nu)\, |\nabla\chi|\, dt = \int_0^T \int (V - f)\ \xi \cdot \nu\, |\nabla\chi|\, dt \tag{36}$$

for any $\xi \in C_0^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$.

It is easy to see that also Algorithm 3.1 can be interpreted as a minimizing movements scheme. In fact, as in Lemma 2.7 we add a linear functional as a correction.

Lemma 3.3 (Minimizing movements interpretation)**.** *Given* $\chi^0 \in \{0,1\}$*, let* χ^1 *be obtained by Algorithm 3.1. Then* χ^1 *solves*

$$\min \quad E_h(\chi) + D_h(\chi - \chi_0) - \frac{1}{\sqrt{\pi}} \int f(nh,x)\, \chi\, dx, \tag{37}$$

where the minimum runs over all $\chi\colon \mathbb{R}^d \to \{0,1\}$.

Corollary 3.4 (Euler-Lagrange equation)**.** *Given* $\chi^0 \in \{0,1\}$*, let* χ^1 *be obtained by Algorithm 3.1. Then* χ^1 *solves the Euler-Lagrange equation*

$$\delta E_h(\chi^1,\xi) + \delta D_h(\,\cdot\, - \chi^0)(\chi^1,\xi) - \frac{1}{\sqrt{\pi}} \int \nabla \cdot (f(nh,x)\, \xi)\, \chi^1\, dx = 0. \tag{38}$$

We can prove a conditional convergence result for Algorithm 3.1 under the same assumption as in Section 2.

Theorem 3.5. *Let* $T < \infty$*,* $\chi^0 \in \{0,1\}$ *with* $E(\chi^0) < \infty$ *and* $\{\chi^0 = 1\} \subset\subset \mathbb{R}^d$ *and* $f \in C^\infty([0,T] \times \mathbb{R}^d)$*. After passage to a subsequence, the functions* χ^h *obtained by Algorithm 3.1 converge to a function* χ *in* $L^1((0,T) \times \mathbb{R}^d)$*. Under the convergence assumption (8),* χ *moves by mean curvature with external force* f *in the sense of Definition 3.2.*

We follow the same strategy as in Section 2 to prove the theorem. From the Euler-Lagrange equation (38), the mean curvature and normal velocity will be recovered from the limits of the first variations of the energy and dissipation, respectively. The convergence of the third term in this algorithm is much easier. f is a smooth function in time and space so the convergence of the third term is an immediate consequence of the compactness of the χ^h (cf. Proposition 3.8). As before we write $A \lesssim B$ if there exists a constant $C = C(d) < \infty$ such that $A \le CB$ and note that we also have (9).

3.2 Compactness

Since there are no 'bad' iterations as in Section 2, the argument in Lemma 2.16 yields the following linear growth estimate and is sufficient to prove the boundedness of the sets. Here we even have the optimal growth rate of the radii w. r. t. the time horizon T.

Proposition 3.6. *There exists a universal constant $C < \infty$ such that for any $n = 1, \dots, N$*

$$R_n \leq R_{n-1} + Ch\|f\|_\infty.$$

In particular, if $\Omega^0 \subset B_R$ and the sets $\Omega^h(t)$ are obtained by Algorithm 3.1, then $\Omega^h(t) \subset B_{R^*}$ for all $t \leq T$, where $R^* = R(1 + CT\|f\|_\infty)$ for some universal constant $C < \infty$.

The following lemma states the a priori estimate coming from the minimizing movements interpretation. Here, we obtain extra terms coming from the forcing term which did not appear in Section 2 due to the special structure of the equation there.

Lemma 3.7 (Energy-dissipation estimate)**.** *The approximate solutions χ^h constructed in Algorithm 3.1 satisfy*

$$E_h(\chi^N) + \sum_{n=1}^N D_h(\chi^n - \chi^{n-1}) \leq E_0 + C\left(\|f\|_\infty + \int_0^T \int |\partial_t f|\, dx\, dt\right). \tag{39}$$

Proof. Comparing χ^n to χ^{n-1}, we have

$$E_h(\chi^n) + D_h(\chi^n - \chi^{n-1}) - \frac{1}{\sqrt{\pi}} \int f(nh)\, \chi^n\, dx \leq E_h(\chi^{n-1}) - \frac{1}{\sqrt{\pi}} \int f(nh)\, \chi^{n-1}\, dx.$$

Iterating this estimate yields

$$E_h(\chi^N) + \sum_{n=1}^N D_h(\chi^n - \chi^{n-1}) \leq E_h(\chi^0) + \frac{1}{\sqrt{\pi}} \sum_{n=1}^N \int f(nh) \left(\chi^n - \chi^{n-1}\right) dx. \tag{40}$$

We handle the second right-hand side term by a discrete integration by parts,

$$\sum_{n=1}^N f(nh) \left(\chi^n - \chi^{n-1}\right) = f(Nh)\, \chi^N - f(0)\, \chi^0 - \sum_{n=0}^{N-1} \left(f((n+1)h) - f(nh)\right) \chi^n,$$

so that by Proposition 3.6 the right-hand side of (40) is estimated by

$$E_0 + \frac{1}{\sqrt{\pi}} \|f\|_\infty \int \left(\chi^0 + \chi^N\right) dx + \frac{1}{\sqrt{\pi}} \int_0^T \int |\partial_t f|\, dx\, dt \lesssim E_0 + \|f\|_\infty + \int_0^T \int |\partial_t f|\, dx\, dt,$$

which concludes the proof. $\square$

Now we can apply the same argument as in Section 2 to prove the relative compactness of the approximate solutions.

Proposition 3.8 (Compactness)**.** *Let $T < \infty$ and $\chi^0 \in \{0,1\}$ with $E(\chi^0) < \infty$. Then for any sequence there exists a subsequence $h \searrow 0$ and a function $\chi \in \{0,1\}$ such that $\chi^h \to \chi$ in $L^1((0,T) \times \mathbb{R}^d)$ and the convergence holds almost everywhere in $(0,T) \times \mathbb{R}^d$.*

3.3 Convergence

Proof of Theorem 3.5. By Proposition 3.8 we have compactness. Our a priori estimate (39) and the strengthened convergence (8) allow us to proceed as in Step 1 of the proof of Theorem 2.11 above to construct the normal velocity and establish the integrability.

As in Section 2, we can apply Proposition 2.17 because of our strengthened convergence (8) so that we recover the mean curvature from the first variation of the energy. To prove the analogue of Proposition 2.18, i. e. convergence of the first variation of the dissipation towards $\int V\,\xi\cdot\nu\,|\nabla\chi|$ we use Lemma 3.9 below to apply the proof in Chapter 1. This turns out to be easier compared to the proof in Section 2 since there are no 'bad' iterations and we do not have to take special care of the shift of the mesoscopic time slices as in Step 2. $\square$

The following lemma is the analogue of Lemma 2.19 but adapted to to the setting of this problem. There are two major differences. On the one hand, here the threshold values are not constant in space so that we obtain an extra term coming from the first right-hand side integral in (35) which gives an error term measuring the spatial variation of f. But on the other hand, the mild bound on the threshold value, $|\lambda-\frac12|<\frac18$ in Lemma 2.19, is here automatically satisfied if the time step h is small enough.

Lemma 3.9. *Let $u,\,\tilde u,\,f,\,\tilde f\in C^\infty(\mathbb{R})$ and $\chi=\mathbf{1}_{\{u>\frac12-\frac{1}{2\sqrt{\pi}}f\sqrt{h}\}}$, $\tilde\chi=\mathbf{1}_{\{\tilde u>\frac12-\frac{1}{2\sqrt{\pi}}\tilde f\sqrt{h}\}}$ and furthermore let $\eta\in C_0^\infty(-2r,2r)$ be a radially non-increasing cut-off for $(-r,r)$ inside $(-2r,2r)$. Then*

$$\frac{1}{\sqrt{h}}\int\eta\,|\chi-\tilde\chi|\,dx_1\lesssim\frac{1}{\sqrt{h}}\int_{\{\frac13\le u\le\frac23\}}\eta\left(\sqrt{h}\,\partial_1 u-\overline{c}\right)_-^2dx_1+s+\frac{1}{s^2}\frac{1}{\sqrt{h}}\int\eta\,(u-\tilde u)^2\,dx_1$$
$$+\sqrt{h}^3\int\eta\,(\partial_1 f)^2\,dx_1+\frac{1}{s^2}\sqrt{h}\int\eta\left(f-\tilde f\right)^2dx_1$$

for any $s\ll1$ and $h\ll\frac{1}{\|f\|_\infty^2}$.

Proof. The lemma follows immediately from Corollary 4.3 in Chapter 1 applied to the shifted functions $v:=u+\frac{1}{2\sqrt{\pi}}f\sqrt{h}$ and $\tilde v:=\tilde u+\frac{1}{2\sqrt{\pi}}\tilde f\sqrt{h}$. $\square$

4 Grain growth in polycrystals

In this section we present and study a thresholding algorithm for simulating grain growth in polycrystals including boundary effects. Especially for thin films this is very important since then these effects become more important.

4.1 Preliminaries

The energy that we are interested in is the following weighted sum of interfacial energies

$$E(\Omega_1, \ldots, \Omega_P) = \sum_{i,j} \sigma_{ij} \left|\Sigma_{ij}\right| + 2\sigma_0 \left|\Sigma_0\right|, \tag{41}$$

where the phases $\Omega_1, \ldots, \Omega_P$ represent the different grains and are assumed to be pairwise disjoint, bounded, open sets of finite perimeter in $\mathbb{R}^d$, and the interfaces

$$\Sigma_{ij} := \partial^* \Omega_i \cap \partial^* \Omega_j, \quad \Sigma_0 := \partial^* \left(\Omega_1 \cup \cdots \cup \Omega_P\right).$$

Here, $\partial^*\Omega$ denotes the (reduced) boundary of a set Ω. The number σ_{ij} is the surface tension between Phase i and Phase j and σ_0 the surface tension between the crystal and the air which is an additional modeling parameter. The equation we want to study is the gradient flow of the energy (41) subject to the volume constraint

$$\left|\Omega_1 \cup \cdots \cup \Omega_P\right| = \text{constant.}$$

In particular we analyze a thresholding algorithm (Algorithm 4.1) and in Theorem 4.8 we prove a (conditional) convergence result for a very general class of surface tensions that has been introduced in [33]. Esedoğlu and Otto showed that this class includes the 2-d and 3-d Read-Shockley formulas which are very prominent models for grain boundaries with a small mismatch in the angle. As in Chapter 1, we need slightly stronger assumptions for the convergence proof. We ask the matrix $\sigma = (\sigma_{ij})_{ij=1}^P$ of surface tensions to satisfy

$$\sigma_{ii} = 0, \quad \sigma_{ji} = \sigma_{ij} > 0 \text{ for all } i \neq j \tag{42}$$

and furthermore the following triangle inequality

$$\sigma_{ij} < \sigma_{ik} + \sigma_{kj} \quad \text{for all pairwise different } i, j, k. \tag{43}$$

For the dynamics, it is natural to assume that there exists a positive constant $\underline{\sigma} > 0$ such that

$$\sigma \leq -\underline{\sigma} < 0 \quad \text{on } (1, \ldots, 1)^\perp \tag{44}$$

as a bilinear form. Given a matrix of surface tension σ, the only modeling assumption on the parameter σ_0, the surface tension between the crystal and the air, is the the lower bound

$$\sigma_0 > \frac{1}{2} \max_{i,j} \sigma_{ij}. \tag{45}$$

In the following, we will normalize this parameter $\sigma_0 = 1$ by rescaling the other surface tensions $\sigma_{ij} \mapsto \frac{\sigma_{ij}}{\sigma_0}$ so that this modeling assumption turns into an additional assumption on the matrix of (normalized) surface tensions between the grains:

$$\sigma_{ij} < 2 \quad \text{for all } i, j. \tag{46}$$

Note that given this additional assumption, the extended matrix of surface tensions given by the $(P+1) \times (P+1)$-block matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \sigma & \\ 1 & & & \end{pmatrix} \tag{47}$$

satisfies all the assumptions mentioned before and in particular (44) with $\underline{\sigma}$ replaced by $\underline{\sigma} \wedge 2$. The resulting equation then becomes

$$V_{ij} = H_{ij} \tag{48}$$

on the smooth part of the interface Σ_{ij}, $(i, j \geq 1)$ and

$$\sigma_{ij}\nu_{ij}(p) + \sigma_{jk}\nu_{jk}(p) + \sigma_{ki}\nu_{ki}(p) = 0, \tag{49}$$

whenever p is a triple junction between the phases i, j and k, and

$$V_0 = H_0 - \langle H_0 \rangle \tag{50}$$

on the smooth part of the outer boundary Σ_0.

Esedoğlu and Otto showed in [33] that - up to a constant - the energy E in (41) can be approximated by

$$E_h(\chi) := \frac{1}{\sqrt{h}} \sum_{i,j \geq 1} \sigma_{ij} \int \chi_i \, G_h * \chi_j \, dx + \frac{2}{\sqrt{h}} \int (1 - \chi_0) \, G_h * \chi_0 \, dx \tag{51}$$

for *admissible* χ, i. e.

$$\chi = (\chi_0, \chi_1, \ldots, \chi_P) : \mathbb{R}^d \to \{0, 1\}^{P+1}, \quad \text{s. t.} \sum_{i=1}^{P} \chi_i = 1 - \chi_0. \tag{52}$$

Indeed, they proved that the functionals E_h Γ-converge to $\frac{1}{\sqrt{\pi}} E$ as $h \to 0$ when identifying the sets Ω_i with their characteristic functions $\chi_i = \mathbf{1}_{\Omega_i}$ and defining the area of the interface Σ_{ij} between Phases i and j via the term $\int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_i| - |\nabla \chi_i + \chi_j| \right)$ so that the energy E then becomes

$$E(\chi) = \frac{1}{\sqrt{\pi}} \sum_{i,j \geq 1} \sigma_{ij} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_i| - |\nabla \chi_i + \chi_j| \right) + \frac{2}{\sqrt{\pi}} \int |\nabla \chi_0| \, .$$

In the following we will w. l. o. g. assume that the total volume of the crystal is normalized to 1, i. e.

$$|\Omega_1 \cup \cdots \cup \Omega_P| = 1.$$

4.2 Algorithm and notation

The following algorithm was proposed in [8] to model grain growth in thin polycrystals. Similar to Algorithm 2.1, here the total volume of the polycrystal is preserved by the right choice of the threshold value.

Algorithm 4.1. *Given the phases $\Omega_1, \dots, \Omega_P$ with total volume 1 at time $t = (n-1)h$ and write $\Omega_0 := \mathbb{R}^d \setminus (\Omega_1 \cup \dots \cup \Omega_P)$, obtain the evolved phases $\Omega_1', \dots, \Omega_P'$ at time $t = nh$ by:*

1. *Convolution step:*
$$\phi_0 := G_h * \left(\sum_{j\geq 1} \mathbf{1}_{\Omega_j} \right), \quad \phi_i := G_h * \left(\sum_{j\geq 1} \sigma_{ij} \mathbf{1}_{\Omega_j} + \mathbf{1}_{\Omega_0} \right), \quad i \geq 1.$$
2. *Defining threshold value: Find λ such that*
$$\left| \bigcup_{i\geq 1} \{\phi_i < \phi_0 + \lambda\} \right| = 1.$$
3. *Thresholding step: For $i = 1, \dots, P$ set*
$$\Omega_i' := \{\phi_i < \phi_j \textit{ for all } j \neq i,\, j \geq 1\} \cap \{\phi_i < \phi_0 + \lambda\}$$
and $\Omega_0' := \mathbb{R}^d \setminus (\Omega_1' \cup \dots \cup \Omega_P')$.

4.3 Minimizing movements interpretation

With a similar argument as before, using the linearized energy

$$L_h(\phi, \chi) := \frac{2}{\sqrt{h}} \sum_{i=0}^{P} \int \chi_i \, \phi_i \, dx, \tag{53}$$

we can interpret Algorithm 4.1 as a minimizing movements scheme for the approximate energies E_h defined in (51) and dissipation $-E_h(\omega)$. Here the matrix of surface tensions σ_{ij} is extended as in (47).

Lemma 4.2 (Minimizing movements interpretation)**.** *Given any admissible χ^0, let ϕ, λ and χ^1 be obtained by Algorithm 4.1. Then χ^1 solves*

$$\min \quad E_h(\chi) - E_h(\chi - \chi^0) - \frac{2\lambda}{\sqrt{h}} \int (1 - \chi_0) \, dx, \tag{54}$$

where the minimum runs over (52). Or equivalently,

$$\min \quad E_h(\chi) - E_h(\chi - \chi^0) \quad \text{s. t.} \quad \int (1 - \chi_0) \, dx = 1, \tag{55}$$

where the minimum runs over (52) and is additionally constrained by the volume constraint.

Proof. Indeed, for any admissible χ in the sense of (52) we have

$$\sum_{i=0}^{P} \chi_i \, \phi_i - \lambda \, (1-\chi_0) = \chi_0 \, (\phi_0 + \lambda) + \sum_{i=1}^{P} \chi_i \, \phi_i - \lambda \overset{(52)}{\geq} \min \{ \phi_0 + \lambda, \phi_1, \dots, \phi_P \} - \lambda.$$

For χ^1 obtained by Algorithm 4.1 in turn we have equality in the above inequality so that χ^1 minimizes the left-hand side pointwise. In particular, after integration we see that χ^1 minimizes the functional

$$\frac{2}{\sqrt{h}} \sum_{i=0}^{P} \int \chi_i \, \phi_i \, dx - \frac{2\lambda}{\sqrt{h}} \int (1-\chi_0) \, dx = L_h(\chi, \phi) - \frac{2\lambda}{\sqrt{h}} \int (1-\chi_0) \, dx.$$

By the quadratic nature of the functional E_h we have

$$L_h(\phi, \chi) = E_h(\chi) - E_h(\chi - \chi_0) + \text{Terms depending only on } \chi^0,$$

which proves the first claim (54). Since the last term in (54) is constant for χ with the volume constraint, we also have (55). □

Again, as a direct consequence of the minimizing movements interpretation, we obtain an a priori estimate by comparing the solution to its predecessor.

Lemma 4.3 (Energy-dissipation estimate)**.** *The approximate solutions χ^h satisfy*

$$E_h(\chi^N) - \sum_{n=1}^{N} E_h(\chi^n - \chi^{n-1}) \leq E_0. \tag{56}$$

Note that as in Chapter 1 our assumption (44) guarantees that $\sqrt{-E_h}$ defines a norm on the process space $\{\omega \colon \sum_i \omega_i = 0\}$ in the same spirit as $\sqrt{D_h}$ in the previous two sections.

Definition 4.4 (First variation)**.** For any admissible $\chi \in \{0,1\}^P$ and $\xi \in C_0^\infty(D, \mathbb{R}^d)$ let χ_s be generated by the flow of ξ, i. e. $\chi_{i,s}$ solves the following distributional equation:

$$\partial_s \chi_{i,s} + \xi \cdot \nabla \chi_{i,s} = 0.$$

We denote the first variation along this flow by

$$\delta E_h(\chi, \xi) := \frac{d}{ds} E_h(\chi_s) \big|_{s=0}, \quad \delta E_h(\,\cdot\, - \tilde{\chi})(\chi, \xi) := \frac{d}{ds} E_h(\chi_s - \tilde{\chi}) \big|_{s=0},$$

where $\tilde{\chi} \in \{0,1\}$ is fixed.

Corollary 4.5 (Euler-Lagrange equation)**.** *Given an admissible $\chi^0 \in \{0,1\}^P$, let χ^1 be obtained by Algorithm 4.1 with threshold value λ. Then χ^1 solves the Euler-Lagrange equation*

$$\delta E_h(\chi^1, \xi) - \delta E_h(\,\cdot\, - \chi^0)(\chi^1, \xi) - \frac{2\lambda}{\sqrt{h}} \int (\nabla \cdot \xi) \left(1 - \chi_0^1\right) dx = 0. \tag{57}$$

The idea underlying the convergence proof now follows the framework laid out in Section 2. The first variation of the approximate energy will be shown to converge to the mean curvature of the crystal/grain boundary in a weak sense. The first variation of the dissipation will be shown to converge to the velocity in a weak sense. And the first variation of the Lagrange multiplier term will converge to zero on the inner grain boundaries, and the average of the mean curvature over the outer solid-vapor interface. The precise limit is formulated in the next definition.

The following definition is similar to the notion for multi-phase mean-curvature flow as described in Chapter 1 but incorporates an additional constraint on the total volume.

Definition 4.6. Fix some finite time horizon $T < \infty$, a matrix of surface tensions σ as above and initial data $\chi^0 \colon \mathbb{R}^d \to \{0,1\}^P$ with $E_0 := E(\chi^0) < \infty$. We say that

$$\chi = (\chi_1, \dots, \chi_P) : (0,T) \times \mathbb{R}^d \to \{0,1\}^P$$

with $\chi_0 := 1 - \sum_i \chi_i \in \{0,1\}$ a. e. and $\chi(t) \in BV(\mathbb{R}^d, \{0,1\}^P)$ for a. e. t moves by *total-volume preserving mean-curvature flow* if

$$\begin{aligned}\sum_{i,j=1}^P \sigma_{ij} \int_0^T \int (\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i - \xi \cdot \nu_i \, V_i) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt \\ + 2 \int_0^T \int (\nabla \cdot \xi - \nu_0 \cdot \nabla \xi \, \nu_0 - \xi \cdot \nu_0 \, (V_0 + \Lambda)) \, |\nabla \chi_0| \, dt = 0 \end{aligned} \tag{58}$$

for all $\xi \in C_0^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$, where the functions $V_i \colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ are normal velocities in the sense that

$$\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt + \int \zeta(0) \chi_i^0 \, dx = - \int_0^T \int \zeta \, V_i \, |\nabla \chi_i| \, dt \tag{59}$$

for all $\zeta \in C^\infty([0,T] \times \mathbb{R}^d)$ with $\zeta(T) = 0$ and $\operatorname{supp} \zeta(t) \subset\subset \mathbb{R}^d$ and all $i \in \{0, 1, \dots, P\}$ and if the Lagrange multiplier $\Lambda \colon (0,T) \to \mathbb{R}$ is such that the volume of the solid phase $(1 - \chi^0)$ is preserved:

$$\sum_{i=1}^P \int \chi_i(t) \, dx = \text{constant}. \tag{60}$$

Remark 4.7. We assume the following convergence of the energies defined in (51).

$$\int_0^T E_h(\chi^h) \, dt \to \int_0^T E(\chi) \, dt. \tag{61}$$

4.4 Main result

Theorem 4.8. *Let $T<\infty$ be a finite time horizon, $\chi^0=(\chi_1^0,\dots,\chi_P^0)$ be admissible initial data with $E(\chi^0)<\infty$ and $\{\sum_i \chi_i^0=1\}\subset\subset\mathbb{R}^d$ and let the matrix of surface tensions σ satisfy the assumptions (42)-(45). After passage to a subsequence, the approximate solutions χ^h constructed in Algorithm 4.1 converge to an admissible χ in $L^1((0,T)\times\mathbb{R}^d)$. Given the convergence assumption (61), χ moves by total-volume preserving mean-curvature flow according to Definition 4.6.*

One of the main ingredients – as in Section 2 – is the following estimate on the Lagrange multiplier.

Proposition 4.9. *Let χ_0 be admissible. Given the approximate solutions χ^h obtained by Algorithm 2.1 with thresholding values λ_h, we have the estimate*

$$\int_0^T \lambda_h^2\,dt \lesssim (1+T)\left(1+E_0^4\right)h.$$

Proof. We can adapt the proof of Proposition 2.12. We square the Euler-Lagrange equation and obtain an equation similar to (18) but with χ^n replaced by $1-\chi_0^n$ on the left-hand side. The estimates on δE and δD, i. e. Steps 1 and 2 work analogously with help of the a priori estimate (56). In Step 3 we choose the test vector field ξ to satisfy

$$\int (\nabla\cdot\xi)\,(1-\chi_0)\,dx \geq \frac{1}{2} \quad \text{and}$$
$$\|\xi\|_{W^{1,\infty}} \lesssim 1+E_0.$$

The construction of ξ is the same as there but with χ replaced by $1-\chi_0$, which has a fixed volume $\int(1-\chi_0)\,dx=1$. □

4.5 Compactness

Proposition 4.10 (Compactness)**.** *There exists a subsequence $h\searrow 0$ and an admissible vector of characteristic functions $\chi\in L^1((0,T)\times\mathbb{R}^d,\{0,1\}^P)$ such that*

$$\chi^h \longrightarrow \chi \quad \textit{in } L^1((0,T)\times\mathbb{R}^d). \tag{62}$$

Moreover,

$$\chi^h \longrightarrow \chi \quad \textit{a. e. in } (0,T)\times\mathbb{R}^d \tag{63}$$

and $\chi(t)\in BV(\mathbb{R}^d,\{0,1\}^{P+1})$, $\int(1-\chi_0)\,dx=1$ and $1-\chi_0\subset\subset\mathbb{R}^d$ for a. e. $t\in(0,T)$.

As in Section 2, this follows from Section 2 of Chapter 1 and the following two lemmas, which guarantee that the phases stay in a bounded region. In the proofs, we will reduce the statements until we can apply Lemma 2.15 and Lemma 2.16, respectively to conclude.

Lemma 4.11. *R_n may be chosen such that $R_n \le 3R_{n-1}$.*

Proof. For the sake of notational simplicity we will assume w. l. o. g. $n = 1$. We want to give a similar, energy-based argument as in the proof of Lemma 2.15. Let $1 - \chi_0^0$, the crystal at time 0, be located inside B_R. We write $\Omega_1, \dots, \Omega_P$ for the update in Algorithm 4.1, write $\chi_i = \mathbf{1}_{\Omega_i}$ and assume that $U := \Omega_1 \setminus B_{3R}$ has positive volume and construct $\widetilde{U} \subset B_R$ with the same volume as U as in the proof of Lemma 2.15. Then we define the competitor $\tilde{\chi}$ by setting $\widetilde{\Omega}_1 := (\Omega_1 \setminus U) \cup \widetilde{U}$ leaving the phases Ω_i, $i \ge 2$ unchanged so that $\widetilde{\Omega}_0 := (\Omega_0 \setminus \widetilde{U}) \cup U$. Recalling the linearized energy defined in (53), we see that

$$L_h(\phi, \chi) - L_h(\phi, \tilde{\chi}) = \frac{2}{\sqrt{h}} \int (\chi_0 - \tilde{\chi}_0)\, \phi_0 + (\chi_1 - \tilde{\chi}_1)\, \phi_1 \, dx.$$

By construction we have $\chi_0 - \tilde{\chi}_0 = -(\chi_1 - \tilde{\chi}_1) = \mathbf{1}_{\widetilde{U}} - \mathbf{1}_U$. Rewriting ϕ_1 in the form

$$\phi_1 = \left(1 - \sum_{j \ge 1} G_h * \chi_j^0\right) + \sum_{j \ge 1} \sigma_{1j}\, G_h * \chi_j^0,$$

we thus have

$$\begin{aligned} L_h(\phi, \chi) - L_h(\phi, \tilde{\chi}) =& \frac{2}{\sqrt{h}} \int (\phi_0 - \phi_1) \left(\mathbf{1}_{\widetilde{U}} - \mathbf{1}_U\right) dx \\ =& \frac{2}{\sqrt{h}} \sum_{j=1}^{P} (2 - \sigma_{1j}) \int G_h * \chi_j^0 \left(\mathbf{1}_{\widetilde{U}} - \mathbf{1}_U\right) dx. \end{aligned}$$

Note that by the normalization (46), which guarantees the strict triangle inequality for the extended surface tensions, each prefactor in the sum is strictly positive, furthermore we have (30) for $G_h * \chi_j^0$ playing the role of ϕ there and by construction of $\widetilde{U}$ the right-hand side term is positive which gives the desired contradiction. $\square$

Lemma 4.12. *Over 'good' iterations we have the estimate*

$$R_n \le R_{n-1} + C\sqrt{h}|\lambda_n|.$$

Proof. As before, we can ignore the index n and set $n = 1$ for convenience. Let $1 - \chi_0^0$, the crystal at time 0, be located inside some ball B_{R_0}. As in the proof of Lemma 2.16, via a comparison argument, we want to prove that $1 - \chi_0$, the crystal at time h, does not intersect the half space $\{x \cdot e > R_0 + C\sqrt{h}\}$ for any choice of $e \in S^{d-1}$. That means, we want to prove the existence of a constant $C < \infty$ such that

$$\phi_0 + \lambda < \phi_i \quad \text{for all } i \ge 1 \quad \text{in } \{x \cdot e > R_0 + C\sqrt{h}\}.$$

By rotational symmetry we may again restrict to the case $e = e_1$. Since we may relabel the phases inside the crystal, we may also prove the inequality only for $i = 1$. In that case, writing $x = (x_1, x') \in \mathbb{R}^d$, we have

$$(\phi_0 - \phi_1)(x) \le G_h * \left(\sum_{i \ge 1} \chi_i^0 - \chi_0^0 \right).$$

Thus, writing $\chi^0 := \sum_{i \ge 1} \chi_i^0$, we reduced the problem to the two-phase analogue which we handled in Lemma 2.16. Indeed, using the same comparison argument, i. e. using $\chi^0 \le \mathbf{1}_H$, where $H = \{x_1 < R_0\}$ is a half space tangent to ∂B_{R_0} we find

$$(\phi_0 - \phi_1)(x) \le 2 \int_0^{x_1 - R_0} G_h^1(z_1)\, dz_1.$$

Since for a 'good' iteration λ is bounded, as in the proof of Lemma 2.16 we can find a constant $C < \infty$, so that

$$(\phi_0 - \phi_1)(x) \le 2 \frac{R_1 - R_0}{\sqrt{h}} \min_{|z_1| \le C} G^1(z_1) \le |\lambda|$$

which concludes the proof. □

4.6 Convergence

The following lemma is the main technical ingredient of the convergence proof. It is slightly more general than our set-up here since it allows for several Lagrange-multipliers so that the order parameter becomes $\sigma\, u + \lambda$ instead of $\sigma\, u$, where $u = G * \chi$ and $\lambda \in \mathbb{R}^P$. The changes in the statement w. r. t. Lemma 4.5 in Chapter 1 are of the same form as before in Lemma 2.19 except for a lower order term, $|\lambda|$, which can be absorbed by the term $\frac{r}{s^2} \frac{|\lambda - \tilde{\lambda}|^2}{\sqrt{h}}$ and terms of order $\sqrt{h}$.

Lemma 4.13. *Let $I \subset \mathbb{R}$ be an interval, $h > 0$, $\eta \in C_0^\infty(\mathbb{R})$, $0 \le \eta \le 1$, radially non-increasing and $u, \tilde{u} \colon I \to \mathbb{R}^P$ be two maps into the standard simplex $\{U_i \ge 0, \sum_i U_i = 1\} \subset \mathbb{R}^P$. Let $\sigma \in \mathbb{R}^{P \times P}$ be admissible in the sense of (42)-(44) and λ, $\tilde{\lambda} \in \mathbb{R}^P$ with $|\lambda| \le \frac{1}{8}$. Define $\phi_i := \sum_j \sigma_{ij}\, u_j + \lambda_i$, $\chi_i := \mathbf{1}_{\{\phi_i > \phi_j\ \forall j \ne i\}}$ and $\tilde{\phi}$, $\tilde{\chi}_i$ in the same way. Then*

$$\begin{aligned} \frac{1}{\sqrt{h}} \int \eta\, |\chi - \tilde{\chi}|\, dx_1 \lesssim & \frac{1}{\sqrt{h}} \int_{\{\frac{1}{3} \le u_1 \le \frac{2}{3}\}} \left(\sqrt{h} \partial_1 u_1 - \overline{c} \right)_-^2 dx_1 + \frac{1}{s} \frac{1}{\sqrt{h}} \sum_{j \ge 3} \int \eta\, [u_j \wedge (1 - u_j)]\, dx_1 \\ & + s + |\lambda| + \frac{1}{s^2} \frac{1}{\sqrt{h}} \int \eta\, |u - \tilde{u}|^2\, dx_1 + \frac{r}{s^2} \frac{|\lambda - \tilde{\lambda}|^2}{\sqrt{h}} \end{aligned}$$

for any $s \ll 1$.

Proof. As in the proof of Lemma 4.5 in Chapter 1 by scaling we can assume $h = 1$ and by taking convex combinations, we may assume $\eta = \mathbf{1}_I$ for some interval $I \subset \mathbb{R}$:

$$\int_I |\chi - \tilde{\chi}|\, dx_1 \lesssim \int_{\{|u_1 - \frac{1}{2}| \le s + |\lambda|\}} (\partial_1 u_1 - \overline{c})_-^2\, dx_1 + \frac{1}{s} \sum_{j\ge 3} \int_I [u_j \wedge (1-u_j)]\, dx_1$$
$$+ s + |\lambda| + \frac{1}{s^2} \int_I |u - \tilde{u}|^2 + \frac{|I|}{s^2} |\lambda - \tilde{\lambda}|^2.$$

We will prove

$$\{\chi \ne \tilde{\chi}\} \subset \Big\{|u_1 - \tfrac{1}{2}| \lesssim s + |\lambda|\Big\} \cup \Big\{\sum_{j\ge 3} [u_j \wedge (1-u_j)] \gtrsim s\Big\} \cup \{|u - \tilde{u}| + |\lambda| \gtrsim s\}. \quad (64)$$

We fix $i \in \{1, \dots, P\}$ and define $v := \min_{j\ne i} \phi_j - \phi_i$ as in the proof of Lemma 4.5. Then $\chi_i = \mathbf{1}_{v>0}$ and

$$\{\chi_i \ne \tilde{\chi}_i\} \subset \{|v| < s\} \cup \{|v - \tilde{v}| \ge s\}.$$

We clearly have

$$|v - \tilde{v}| \lesssim |u - \tilde{u}| + |\lambda - \tilde{\lambda}|$$

so that our goal is to prove

$$|u_1 - \tfrac{1}{2}| \lesssim s + |\lambda| \quad \text{or} \quad \sum_{j\ge 3} [u_j \wedge (1-u_j)] \gtrsim s \quad \text{on } \{|v| < s\}, \quad (65)$$

which then implies (64). In order to prove (65) we claim that

$$u_j \le \frac{1}{2} + \frac{s + |\lambda|}{\sigma_{\min}} \quad \text{on } \{|v| < s\}. \quad (66)$$

First we show that (66) implies (65). By (66) we have on the one hand

$$u_1 \le \frac{1}{2} + C(s + |\lambda|) \quad \text{on } \{|v| < s\}$$

and on $\{|v| < s\} \cup \{u_1 \le \frac{1}{2} - C(s + |\lambda|)\}$ we have

$$\sum_{j\ge 3} [u_j \wedge (1-u_j)] = \sum_{j\ge 3} u_j - \sum_{j\ge 3} (1 - 2u_j)_- \ge \left(C - \frac{1}{\sigma_{\min}} - 2P \frac{1}{\sigma_{\min}}\right) s \gtrsim s$$

if $C < \infty$ is large enough. This implies (65).

We are left with proving the inequality (66). As in the proof of Lemma 4.5 we decompose the set

$$\{|v| < s\} = \bigcup_{j\neq i} E_j, \quad E_j := \big\{\, |\phi_i - \phi_j| < s,\, \phi_j = \min_{k\neq i} \phi_k \big\}.$$

For $k \neq \{i,j\}$ by the triangle inequality for the surface tensions we have on E_j

$$\phi_j \leq \phi_k \leq \sigma_{jk}\,(1-2u_k) + \phi_j + \lambda_k - \lambda_j,$$

so that

$$u_k \leq \frac{1}{2} + \frac{\lambda_k - \lambda_j}{2\sigma_{jk}}.$$

For u_i we can use that $\phi_j - s \leq \phi_i$ on E_j so that using the same chain of inequalities we have

$$u_i \leq \frac{1}{2} + \frac{s + \lambda_i - \lambda_j}{2\sigma_{ij}}.$$

Since also $\phi_i - s \leq \phi_j$ on E_j we have the analogous inequality for u_j, which concludes (66). □

As in Chapter 1, we have the following convergence of the first variations of the (approximate) energies.

Proposition 4.14 (Energy and mean curvature; Prop. 3.1 in Chapter 1)**.** *Under the convergence assumption (61)*

$$\lim_{h\to 0}\int_0^T \delta E_h(\chi^h,\xi)\,dt$$
$$= \frac{1}{\sqrt{\pi}}\sum_{i,j=0}^{P}\sigma_{ij}\int_0^T\int (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right)dt$$

for any $\xi \in C_0^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$.

Since we have both, the estimate on the Lagrange multiplier λ in Proposition 4.9 and the important estimate Lemma 4.13, as in Section 2, we can adapt the techniques from Chapter 1 to recover the normal velocity from the first variation of the dissipation functional.

Proposition 4.15 (Dissipation and normal velocity)**.** *There exist functions $V_i\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}$ which are normal velocities in the sense of (59). Given the convergence assumption (61), $V_i \in L^2(|\nabla\chi|\,dt)$ and for any $\xi \in C_0^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ we have*

$$\lim_{h\to 0}\int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t),\xi)\,dt$$
$$= -\frac{1}{\sqrt{\pi}}\sum_{i,j=0}^{P}\sigma_{ij}\int_0^T\int \xi\cdot\nu_i\,V_i\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right)dt.$$

Proof. Step 1: Construction of the normal velocities and (59). As before in the two-phase case we can also adapt the proof of Chapter 1 in this case. Indeed, the argument there only makes use of the a priori estimate (56) and the strengthened convergence (61).

Step 2: Argument for (31). Our L^2-estimate on the Lagrange-multiplier λ allows us to choose the shift of the mesoscopic time slices as in Step 2 of the proof of Proposition 2.18 such that these slices are 'good' in the sense that $|\lambda| \leq \frac{1}{8}$. Now we may use our main technical ingredient, Lemma 4.13, for all mesoscopic time slices and hence we can apply the proof as in Section 2 before. □

These two propositions conclude the proof of Theorem 4.8.

Chapter 4

The vector-valued Allen-Cahn Equation

This chapter is contained in the work [54] with Thilo Simon. We prove similar conditional convergence results as in the previous chapters, but here for phase-field models. We consider the vector-valued Allen-Cahn Equation and later show how to incorporate external forces or a volume constraint. The results are conditional in the sense that we assume the time-integrated energies to converge to those of the limit.

1 Introduction

Our main result, Theorem 2.2, establishes the convergence of solutions of the Allen-Cahn Equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} \partial_u W(u_\varepsilon) \tag{1}$$

for a general class of potentials W and any space dimension.

Like the results of Luckhaus and Sturzenhecker [57], and the ones presented in the previous chapters, also ours for the Allen-Cahn Equation is only a *conditional* convergence result in the sense that we assume the time-integrated energy of the approximations to converge to the time-integrated energy of the limit, see (9). Although this is a very natural assumption, it is not guaranteed by the a priori estimates coming from the energy-dissipation equality (19). However, the verification of this assumption is non-trivial and even fails for certain initial data, cf. [19] for an example of higher multiplicity interfaces in the limit of the volume-preserving Allen-Cahn Equation.

The main idea of our proof is to multiply the Allen-Cahn Equation $\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} \partial_u W(u_\varepsilon)$ with $\varepsilon \, (\xi \cdot \nabla) \, u_\varepsilon$, integrate in space and time and pass to the limit $\varepsilon \downarrow 0$. To this end we extend the above mentioned argument of Luckhaus and Modica [56] to the multi-phase case and obtain the curvature-term $\int_\Sigma H \, \xi \cdot \nu$ from the right-hand side. The more delicate part, and the core of this chapter, is how to pass to the limit in the velocity-term $\int_\Sigma V \, \xi \cdot \nu$. The difficulty is that one has to pass to the limit in a *product* of weakly converging terms, the normal and the velocity. We overcome this difficulty by “freezing” the normal and introducing an appropriate approximation

(54) of the tilt-excess. After doing so it turns out that the new nonlinearity with the frozen normal can be written as a derivative of a compact quantity. The technique of freezing the normal is the same as in Chapter 1, where we introduce an approximation of the energy-excess. To work with the tilt-excess instead of the energy-excess seems very natural to us in this particular problem and might be interesting in other cases too. The only extra difficulty is that one has to pass to the limit in the nonlinear quantity (54). However, our problem seems to be much simpler than the one in Chapter 1 as we do not have to work on multiple time scales.

The structure of this chapter is as follows. In Section 2 we introduce the notation and state our main result, Theorem 2.2. In Section 3 we prove compactness of the solutions together with bounds on the normal velocities. We took care to be precise in this section but do not claim the originality of the results. We use a general chain rule of Ambrosio and Dal Maso [4] to identify the nonlinearities in the multi-phase case as derivatives. Furthermore, we repeat the application of De Giorgi's Structure Theorem from Chapter 1 to handle the excess. In Section 4 we pass to the limit in the equation. Since this is the most original part, we give a short overview over the idea of the proof first. We then present our extension of the Reshetnyak argument by Luckhaus and Modica [56] in Proposition 4.1 to handle the curvature-term and prove the convergence of the velocity-term in Proposition 4.5, which is the main novelty and the core of the chapter. We conclude the section with the proof of the main result, Theorem 2.2. In Section 5 we apply our method to the cases when external forces are present or a volume-constraint is active, see Theorems 2.4 and 2.5.

2 Main results

The Allen-Cahn Equation (1) describes a system of fast reaction and slow diffusion and is the (by the factor $\frac{1}{\varepsilon}$ accelerated) L^2-gradient flow of the Ginzburg-Landau Energy

$$E_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx. \tag{2}$$

For convenience we will work with periodic boundary conditions for u, i.e. on the flat torus $[0,\Lambda)^d$ for some $\Lambda > 0$ and write $\int\, dx$ short for $\int_{[0,\Lambda)^d}\, dx$.

Here the (unknown) order parameter $u_\varepsilon \colon \mathbb{R}^d \to \mathbb{R}^N$ is vector-valued and $W \colon \mathbb{R}^N \to [0,\infty)$ is a smooth multi-well potential with finitely many zeros at $u = \alpha_1, \dots, \alpha_P \in \mathbb{R}^N$. We will furthermore impose polynomial growth and convexity of W at infinity:

1. There exist constants $0 < c < C < \infty$, $R < \infty$ and an exponent $p \geq 2$ such that

$$c|u|^p \leq W(u) \leq C|u|^p \quad \text{for } |u| \geq R \tag{3}$$

and

$$|\partial_u W(u)| \leq C|u|^{p-1} \quad \text{for } |u| \geq R. \tag{4}$$

2. There exist smooth functions $W_{conv}, W_{pert} : \mathbb{R}^N \to [0,\infty)$ such that

$$W = W_{conv} + W_{pert}. \tag{5}$$

Here, the function W_{conv} is convex and W_{pert} has at most quadratic growth in the sense that there exists a constant $\tilde{C}$ such that we have

$$\left|\partial_u^2 W_{pert}(u)\right| \le \tilde{C}. \tag{6}$$

These assumptions seem to be very natural to us: The classical two-well potential $W(u) = (u^2 - 1)^2$ for $u \in \mathbb{R}$ clearly has these properties and they are compatible with polynomial potentials also in the case of systems.

By now it is a classical result due to Baldo [9] that these energies Γ-converge w.r.t. the L^1-topology to an *optimal partition energy* given by

$$E(\chi) = \frac{1}{2}\sum_{i,j}\sigma_{ij}\int \frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right), \tag{7}$$

for a partition $\chi_1,\ldots,\chi_P\colon [0,\Lambda)^d \to \{0,1\}$ satisfying the compatibility condition $\sum_i \chi_i = 1$ a.e. Note that for $\chi_i = \mathbf{1}_{\Omega_i}$ we can also rewrite the limiting energy in terms of the interfaces $\Sigma_{ij} := \partial^*\Omega_i \cap \partial^*\Omega_j$ between the phases, where ∂^* denotes the reduced boundary:

$$E(\chi) = \frac{1}{2}\sum_{i,j}\sigma_{ij}\left|\Sigma_{ij}\right|.$$

The link between u_ε and χ is given by

$$u_\varepsilon \to u := \sum_{i=1}^{P}\chi_i\alpha_i.$$

The constants σ_{ij} are the geodesic distances with respect to the metric $2W(u)\langle\cdot,\cdot\rangle$, i.e.

$$\sigma_{ij} = d_W(\alpha_i,\alpha_j),$$

where the geodesic distance is defined as

$$d_W(u,v) := \inf\left\{\int_0^1 \sqrt{2W(\gamma)}|\dot\gamma|ds : \gamma\colon [0,1]\to\mathbb{R}^n \text{ a } C^1 \text{ curve with } \gamma(0)=u, \gamma(1)=v\right\}. \tag{8}$$

The surface tensions satisfy the triangle inequality

$$\sigma_{ij} \le \sigma_{ik} + \sigma_{kj} \quad \text{for all } i,j,k$$

and clearly

$$\sigma_{ii} = 0, \quad \sigma_{ij} > 0 \quad \text{for } i \neq j, \quad \text{and} \quad \sigma_{ij} = \sigma_{ji}.$$

It is an interesting and non-trivial question to find an appropriate potential W which generates given surface tensions σ. In a recent paper, such potentials with multiple wells have been constructed by Bretin and Masnou [15] for a related class of energies. We will want to localize both the Ginzburg-Landau Energy and the optimal partition energy. Given $\eta \in C([0,\Lambda)^d)$ let

$$E_\varepsilon(u_\varepsilon, \eta) := \int \eta \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx,$$
$$E(\chi, \eta) := E(u, \eta) := \frac{1}{2} \sum_{i,j} \sigma_{ij} \int \eta \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right).$$

For our result we will impose

$$\int_0^T E_\varepsilon(u_\varepsilon)\, dt \to \int_0^T E(\chi)\, dt \tag{9}$$

ruling out a certain loss of surface area in the limit $\varepsilon \downarrow 0$. Under this assumption we will establish convergence towards the following distributional formulation of mean-curvature flow, see [57, 53].

Definition 2.1 (Motion by mean curvature)**.** Fix some finite time horizon $T < \infty$, a $P \times P$-matrix of surface tensions σ as above and initial data $\chi^0 \colon [0,\Lambda)^d \to \{0,1\}^P$ with $E_0 := E(\chi^0) < \infty$ and $\sum_{1 \le i \le P} \chi_i^0 = 1$. We say that

$$\chi \in C\Big([0,T]; L^2([0,\Lambda)^d; \{0,1\}^P)\Big)$$

with $\sup_t E(\chi) < \infty$ and $\sum_i \chi_i = 1$ *moves by mean curvature* if there exist densities V_i with

$$\int_0^T \int V_i^2 |\nabla \chi_i|\, dt < \infty \tag{10}$$

satisfying the following properties:

1. For all $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$

$$\sum_{i,j} \sigma_{ij} \int_0^T \int \left(\nabla \cdot \xi - \nu_i \cdot \nabla \xi\, \nu_i - V_i\, \xi \cdot \nu_i \right) \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right) dt = 0, \tag{11}$$

   where ν_i is the inner normal of χ_i, i.e. the density of $\nabla \chi_i$ with respect to $|\nabla \chi_i|$.

2. The functions V_i are the normal velocities of the interfaces in the sense that

$$\partial_t \chi_i = V_i |\nabla \chi_i| dt \text{ distributionally in } (0,T) \times [0,\Lambda)^d. \tag{12}$$

3. The initial data is achieved in the space $C([0,T];L^2([0,\Lambda)^d))$, i.e.

$$\chi_i(0) = \chi_i^0$$

in $L^2([0,\Lambda)^d)$ for all $1 \le i \le P$.

If the evolution is smooth one can integrate by parts and obtain the classical formulation of multi-phase mean-curvature flow consisting of the evolution law

$$V_{ij} = H_{ij} \quad \text{on } \Sigma_{ij}$$

together with Herring's well-known angle condition

$$\sum_{i,j} \sigma_{ij}\nu_{ij} = 0 \quad \text{at triple junctions.}$$

Comparing to the more general evolution law $V_{ij} = \sigma_{ij}\mu_{ij}H_{ij}$ we see that in our case the mobility μ_{ij} of the interface Σ_{ij} is given by $\mu_{ij} = \frac{1}{\sigma_{ij}}$. How to generate general mobilities seems not to be settled yet.

Our main result is the following theorem.

Theorem 2.2. *Let W satisfy the growth conditions* (3) *and* (4)*, as well as the convexity at infinity* (5)*. Let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon [0,\Lambda)^d \to \mathbb{R}^N$ approximating a partition χ^0, in the sense that*

$$u_\varepsilon^0 \to \sum_{i=1}^P \chi_i\alpha_i \quad \text{a.e. and} \quad E_0 := E(\chi^0) = \lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty, \tag{13}$$

there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of (1) *with initial datum u_ε^0 converge to a time-dependent partition $\chi \in C([0,T];L^2([0,\Lambda)^d;\{0,1\}^P))$. If the convergence assumption* (9) *holds, then χ moves by mean curvature according to Definition 2.1.*

Remark 2.3. For any partition $\chi^0 \in BV\left([0,\Lambda)^d;\{0,1\}^P\right)$ it is possible to choose u_ε^0 with $u_\varepsilon^0 \to \sum_i \chi_i\alpha_i$ in L^1 and $E_\varepsilon(u_\varepsilon^0) \to E_0(\chi)$ by the Γ-convergence result [9].

Using some adjustments of our argument we can also deal with external forces and a volume constraint.

Theorem 2.4. *Let W satisfy* (3), (4) *and* (5) *and let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon [0,\Lambda)^d \to \mathbb{R}^N$ approximating a partition χ^0, in the sense of* (13) *and forces $f_\varepsilon : [0,T]\times[0,\Lambda)^d \to \mathbb{R}^N$ such that*

$$\sup_{\varepsilon>0} \int_0^T \int |f_\varepsilon|^2 + |\partial_t f_\varepsilon|^2 + |\nabla f_\varepsilon|^2 dx\, dt < \infty$$

there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon) + \frac{1}{\varepsilon} f_\varepsilon \tag{14}$$

with $u_\varepsilon(0) = u_\varepsilon^0$ converge to a time-dependent partition $\chi \in C([0,T]; L^2([0,\Lambda)^d; \{0,1\}^P))$. Furthermore, the forces also have a limit $f_\varepsilon \to f$ in L^2. If the convergence assumption (9) *holds, then χ moves by forced mean curvature according to Definition 2.1 with equation* (11) *replaced by*

$$\sum_{i,j} \sigma_{ij} \int_0^T \int (\nabla\cdot\xi - \nu_i \cdot \nabla\xi\, \nu_i - V_i\, \xi\cdot\nu_i)\frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right) dt \tag{15}$$
$$= \sum_i \int_0^T \int f\cdot\alpha_i (\xi\cdot\nabla)\,\chi_i dt.$$

Since we allow f to be only of class $W^{1,2}$, the right-hand side of (15) has to be interpreted in the following distributional sense

$$\int_0^T \int (f\cdot\alpha_i)\,(\xi\cdot\nabla)\,\chi_i dt = -\int_0^T \int (\nabla\cdot\xi)\,(f\cdot\alpha_i)\,\chi_i + \xi\cdot\nabla\,(f\cdot\alpha_i)\,\chi_i\,dx\,dt.$$

In the volume preserving case we only deal with the scalar equation.

Theorem 2.5. *Let $N=1$. Let W satisfy* (3), (4) *and* (5) *with zeros at* 0 *and* 1, *i.e. we have $P=2$. Let $T<\infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0\colon [0,\Lambda)^d \to \mathbb{R}$ approximating a characteristic function χ^0, in the sense that*

$$u_\varepsilon^0 \to \chi^0 \quad \textit{a.e. and} \quad E_0 := E(\chi^0) = \lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty,$$

there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) + \frac{1}{\varepsilon}\lambda_\varepsilon \tag{16}$$

with $u_\varepsilon(0) = u_\varepsilon^0$

$$\lambda_\varepsilon := -\frac{1}{\Lambda^d}\int \varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)dx = \frac{1}{\Lambda^d}\int \frac{1}{\varepsilon}W'(u_\varepsilon)dx \tag{17}$$

converge to a time-dependent characteristic function $\chi \in C([0,T]; L^2([0,\Lambda)^d; \{0,1\}^P))$ with

$$\int \chi(t,x)dx \equiv \int \chi^0(x)dx.$$

Furthermore, we have

$$\sup_{\varepsilon>0}\int_0^T \lambda_\varepsilon^2\,dt < \infty$$

and there is a limit $\lambda_\varepsilon \rightharpoonup \lambda$ *in* $L^2(0,T)$. *If the convergence assumption* (9) *holds, then* χ *moves by volume preserving mean curvature according to Definition 2.1 with equation* (11) *replaced by*

$$\int_0^T \int (\nabla \cdot \xi - \nu \cdot \nabla \xi \, \nu - V \, \xi \cdot \nu) \, |\nabla \chi| dt = - \int_0^T \lambda \int (\nabla \cdot \xi) \chi \, dx \, dt. \tag{18}$$

Throughout the chapter we will make use of the following notations: The symbol ∂_t denotes the time-derivative, ∇ the spatial gradient of a function defined on real space $\mathbb{R}^d \ni x$, $\partial_u W(u)$ denotes the gradient of W at a point $u \in \mathbb{R}^N$ in state space. For the functions ϕ_i we will abuse the notation ∂_u in the sense given by the generalized chain rule below, see Lemma 3.8. We will write $A \lesssim B$ if there exists a generic constant $C < \infty$ depending only on d, N, Λ and W such that $A \leq C \, B$.

3 Compactness

3.1 Results

Before we turn to the actual compactness results, we specify the setting for the Allen-Cahn Equation and make sure that solutions actually exist.

Although solutions to the Allen-Cahn Equation (1) are smooth, we choose the weak setting for the following reasons:

1. The parabolic character of both the Allen-Cahn Equation and mean-curvature flow is much more explicit.

2. It is the natural setting when including forces, which we will do later on in Section 5.

3. Once one accepts the function spaces involved, the necessary compactness properties for forced equations and equations with a volume constraint and how to deal with initial conditions becomes very natural.

We will essentially view solutions as maps of $[0,T]$ into some function space, so that we will need to deal with Banach space-valued L^p and Sobolev spaces. However, the material covered in Chapter 5.9 of [34] is perfectly sufficient for our purposes.

Definition 3.1. We say that a function $u_\varepsilon \in C([0,T]; L^2([0,\Lambda)^d; \mathbb{R}^N))$ is a weak solution of the Allen-Cahn Equation (1) for $\varepsilon > 0$ with initial data $u_\varepsilon^0 \in L^2([0,\Lambda)^d, \mathbb{R}^N)$ if

1. the energy stays bounded:
$$\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t)) < \infty,$$

2. its weak time derivative satisfies
$$\partial_t u_\varepsilon \in L^2([0,T] \times [0,\Lambda)^d),$$

3. for a.e. $t \in [0,T]$ and $\xi \in L^p(0,T;\mathbb{R}^N) \cap W^{1,2}(0,T;\mathbb{R}^N)$ we have

$$\int \partial_t u_\varepsilon(t) \cdot \xi + \nabla u_\varepsilon(t) \colon \nabla \xi + \frac{1}{\varepsilon^2} \partial_u W(u_\varepsilon(t)) \cdot \xi \, dx = 0,$$

4. the initial conditions are achieved:

$$u(0) = u^0.$$

Remark 3.2. Note that due to the growth condition (4) of $\partial_u W$ we know that

$$|\partial_u W(u)|^{\frac{p}{p-1}} \lesssim |u|^{(p-1)\frac{p}{p-1}} = |u|^p.$$

Combining this with boundedness of the energy and the growth condition (3) of W at infinity we obtain $\partial_u W(u(t)) \in L^{\frac{p}{p-1}} = L^{p'}$ for almost all times.

Also note that boundedness of the energy and the bound on the time derivative are sufficient to have $u \in C^{\frac{1}{2}}([0,T]; L^2([0,\Lambda)^d))$, up to a set of measure zero in time, by the embedding

$$W^{1,2}([0,T]; L^2([0,\Lambda)^d)) \hookrightarrow C^{\frac{1}{2}}([0,T]; L^2([0,\Lambda)^d)).$$

See (43) for a short proof of a similar statement.

We first take a brief moment to mention the (not very surprising) fact that the Allen-Cahn Equation (1) in fact have global solutions. For the convenience of the reader we later give a proof which relies on De Giorgi's minimizing movements and thus carries over to related settings. We point out that the long-time existence critically depends on the gradient flow structure, as solutions to the reaction-diffusion equation

$$\partial_t u - \Delta u = u^2$$

generically blow up in finite time.

Lemma 3.3. *Let $u_\varepsilon^0 : [0,\Lambda)^d \to \mathbb{R}^N$ be such that $E_\varepsilon(u_\varepsilon^0) < \infty$. Then there exists a weak solution $u : [0,T] \times [0,\Lambda)^d \to \mathbb{R}^N$ to the Allen-Cahn Equation* (1) *with initial data u^0. Furthermore, the solution satisfies the following energy dissipation identity*

$$E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx \, dt = E_\varepsilon(u_\varepsilon(0)) \tag{19}$$

and we have $\partial_i \partial_j u, \partial_u W(u) \in L^2([0,T] \times [0,\Lambda)^d)$ for all $1 \le i,j \le d$. In particular, we can test the Allen-Cahn equations (1) *with ∇u.*

Remark 3.4. Here, the identity (19) plays the role of an a priori estimate, which makes the whole machinery work. It can be formally derived by differentiating the energy along the solution:

$$\begin{aligned} \frac{d}{dt} E_\varepsilon(u_\varepsilon) &= \int \varepsilon \nabla u_\varepsilon : \nabla \partial_t u_\varepsilon + \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \\ &= \int \varepsilon \left(-\Delta u_\varepsilon + \frac{1}{\varepsilon^2} \partial_u W(u_\varepsilon) \right) \cdot \partial_t u_\varepsilon \, dx \\ &\overset{(1)}{=} - \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx. \end{aligned}$$

Remark 3.5. Note that by choosing $W \equiv 0$ in this calculation, we get a similar estimate for the heat equation. The structure of this estimate (the energy is bounded in time, while the time-derivative is only L^2-integrable) naturally leads to the mixed spaces we consider here and is our main justification for working in the weak setting.

We also remark that the heat equation admits many different interpretations as a gradient flow. Here we chose to view it as an L^2-gradient flow w.r.t. to the energy $\int |\nabla u|^2 dx$ in order to compare it to the Allen-Cahn Equation. However, when proving existence results for the heat equations it is more beneficial to interpret it as an H^{-1}-gradient flow w.r.t. to the energy $\int u^2 dx$ as this choice allows to accommodate more general forces.

Remark 3.6. As the a priori estimate is a natural consequence of the gradient flow structure we expect to have similar estimates in the case of forced equations and volume constraints. In order to later deal with these more general equations we point out that the proofs of the following statements (Proposition 3.7, Lemma 3.9, Proposition 3.10 and Lemma 3.11) only rely on the a priori estimate (19) and not on the Allen-Cahn Equation (1) itself. To be more precise, they remain valid - with slightly different quantitative estimates - for functions $u_\varepsilon \in C([0,T]; L^2([0,\Lambda)^d; \mathbb{R}^N))$ satisfying the bound

$$\sup_{\varepsilon>0} \sup_{0\leq t\leq T} E_\varepsilon(u_\varepsilon(t)) + \int_0^T \varepsilon \left|\partial_t u_\varepsilon\right|^2 dt < \infty. \tag{20}$$

We now turn to the central question of compactness for the constructed solutions:

- Proposition 3.7 ensures that there exists a time-dependent limiting partition, whose motion we want to characterize later on.
- Lemma 3.9 upgrades the convergence of u_ε to $\sum_i \chi_i \alpha_i$ to strong $C\left([0,T]; L^2([0,\Lambda)^d)\right)$ convergence, in particular implying that the initial conditions are achieved.
- Proposition 3.10 states that the partition is regular enough in time to admit normal velocities.

The existence of a limiting partition is essentially contained in the classical Γ-convergence theorem by Baldo [9]. In particular, it is constructed by considering the limits of $\phi_i \circ u_\varepsilon$ with

$$\phi_i(u) := d_W(u, \alpha_i), \text{ where } d_W \text{ was defined in (8).} \tag{21}$$

The main difference is that we also want the partition to be well-behaved in time, which we will make sure by exploiting that the control of $\partial_t u_\varepsilon$ and ∇u_ε is similar.

Proposition 3.7. *Given initial data $u_\varepsilon^0 \to \sum_i \chi_i^0 \alpha_i$ with*

$$E_\varepsilon(u_\varepsilon^0) \to E(\chi^0) < \infty,$$

for any sequence there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of (1) *converge:*

$$u_\varepsilon \to u \quad \text{a.e. in } (0,T) \times [0,\Lambda)^d. \tag{22}$$

Here the limit is given by $u = \sum_i \chi_i \alpha_i$ *with a partition* $\chi \in BV((0,T)\times[0,\Lambda)^d;\{0,1\}^P)$. *Furthermore we have*

$$\sup_{0\le t\le T} E(\chi) \le E_0$$

and the compositions $\phi_i \circ u_\varepsilon$ *are uniformly bounded in* $BV((0,T)\times[0,\Lambda)^d)$ *and converge:*

$$\phi_i \circ u_\varepsilon \to \phi_i \circ u \quad \textit{in } L^1([0,T]\times[0,\Lambda)^d). \tag{23}$$

In the following lemma, we record some properties of the functions $\phi\circ u_\varepsilon$, such as the estimates going back to Modica and Mortola by which one deduces BV-compactness of these compositions. The main point is however that we will need more precise information about $\phi\circ u_\varepsilon$ than for the previously known Γ-convergence results, where one only needs upper bounds for $|\nabla(\phi\circ u_\varepsilon)|$.

Because our proof works by multiplying the Allen-Cahn equation (1) with $\varepsilon\xi\cdot\nabla u$, we will need to pass to the limit in non-linear quantities of u_ε, such as $\int \eta\sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon$. For scalar equations one can easily identify the limit by applying the chain rule to see that this non-linearity has the form $\nabla(\phi\circ u_\varepsilon)$, where the primitive ϕ is given by $\phi(u) := \int_{\alpha_1}^u \sqrt{2W(\tilde u)}\,d\tilde u$. In the multi-phase case, unfortunately, the classical chain rule does not apply anymore: Because there could be multiple geodesics between u and α_i, the geodesic distances $\phi_i(u)$, playing the roles of "primitives", are only locally Lipschitz-continuous in general.

Luckily, there is a chain rule for Lipschitz functions due to Ambrosio and Dal Maso [4]. The upshot is that given a Lipschitz function f and a function u there exists a bounded function $g(x,u)$, defined almost everywhere, such that

$$D(f\circ u)(x) = g(x,u)Du(x)$$

and the dependence of g on u is local in x, but not pointwise. See Theorem 3.13 in the proof of Lemma 3.8 for the precise formulation.

The following lemma mainly serves to fix and justify our somewhat abusive notation of these differentials.

Lemma 3.8. *Let* $u \in C([0,T];L^2([0,\Lambda)^d;\mathbb{R}^N))$ *with*

$$\sup_{0\le t\le T} E_\varepsilon(u) + \int_0^T\int \varepsilon|\partial_t u|^2 dx\,dt < \infty$$

for some $\varepsilon > 0$. *Then for all* $1 \le i \le P$ *there exists a map*

$$\partial_u\phi_i(u) : [0,T]\times[0,\Lambda)^d \to \mathrm{Lin}(\mathbb{R}^N;\mathbb{R})$$

such that the chain rule is valid with the pair $\partial_u\phi_i(u)$ *and* $(\partial_t,\nabla)u$*: For almost every* $(t,x) \in [0,T]\times[0,\Lambda)^d$ *we have*

$$\nabla(\phi_i\circ u) = \partial_u\phi_i(u)\nabla u \quad \textit{and} \quad \partial_t(\phi_i\circ u) = \partial_u\phi_i(u)\partial_t u. \tag{24}$$

Furthermore, we can control the modulus of $\partial_u \phi_i(u)$ almost everywhere in time and space:

$$|\partial_u \phi_i(u)| \leq \sqrt{2W(u)}. \tag{25}$$

Additionally, we have $\phi_i \circ u \in L^\infty\left([0,T]; W^{1,1}([0,\Lambda)^d)\right) \cap W^{1,1}([0,T] \times [0,\Lambda)^d)$ with the estimates

$$\sup_{0\leq t\leq T} \int |\phi_i \circ u| \, dx \lesssim 1 + \sup_{0\leq t\leq T} \varepsilon E_\varepsilon(u), \tag{26}$$

$$\sup_{0\leq t\leq T} \int |\nabla(\phi_i \circ u)| \, dx \lesssim \sup_{0\leq t\leq T} E_\varepsilon(u), \tag{27}$$

$$\int_0^T \int |\partial_t(\phi_i \circ u)| dx dt \lesssim T \sup_{0\leq t\leq T} E_\varepsilon(u) + \int_0^T \int \varepsilon |\partial_t u|^2 dx \, dt. \tag{28}$$

Next, we turn to the stronger compactness properties of u_ε. In the case of the Allen-Cahn Equation without forces or constraints, it mainly serves to ensure that the initial data is achieved. When including forces or constraints we will also need it in the proof of the actual convergence.

Lemma 3.9. *We have $\phi_i \circ u_\varepsilon \in W^{1,2}([0,T]; L^1([0,\Lambda)^d))$ with the estimate*

$$\left(\int_0^T \left(\int |\partial_t(\phi_i \circ u_\varepsilon)| dx \right)^2 dt \right)^{\frac{1}{2}} \lesssim E_\varepsilon(u_\varepsilon(0)). \tag{29}$$

Furthermore, the sequence u_ε is pre-compact in $C\left([0,T]; L^2([0,\Lambda)^d; \mathbb{R}^N)\right)$. In particular, we get that χ achieves the initial data in $C([0,T]; L^2([0,\Lambda)^d))$.

Note that the estimate (29) and the embedding $W^{1,2}([0,T]) \hookrightarrow C^{\frac{1}{2}}([0,T])$, see (43) for a short proof for Banach space-valued functions, imply the well-known $\frac{1}{2}$-Hölder continuity of the volumes of the phases.

The proof of this lemma makes the most detailed use of mixed spaces. Estimate (29) is a time-localized version of the BV-compactness in time (42). Uniform convergence in time of $\phi_i \circ u_\varepsilon$ then boils down to combining this estimate with the Arzelà-Ascoli theorem. However, passing this convergence to u_ε is a little delicate because we have no quantitative information about how quickly ϕ_i grows around α_i. Consequently, we have to make do with u_ε only converging in measure uniformly in time.

While the compactness statement, Proposition 3.7, did not rely on the convergence assumption (9) we will need to assume it in the following, starting with the existence of the normal velocities.

Proposition 3.10. *In the situation of Proposition 3.7, given the convergence assumption* (9)*, for every $1 \leq i \leq P$ the measure $\partial_t \chi_i$ is absolutely continuous w.r.t. $|\nabla \chi_i| \, dt$ and the density V_i is square-integrable:*

$$\int_0^T \int V_i^2 \, |\nabla \chi_i| \, dt \lesssim E_0. \tag{30}$$

Furthermore, equation (12) *holds.*

While we previously localized the BV-compactness in time (42), for this statement we need to localize it in space. Unfortunately, the argument is somewhat delicate as one first proves $\partial_t \chi_i \ll E(\,\cdot\,, u)dt$ and then is forced to prove that $\partial_t \chi_i$ is singular to the "wrong" parts of the energy.

Finally, the following lemma shows that – up to a further subsequence – the convergence assumption can be refined to pointwise a.e. in time and can be localized by a smooth test function in space. We furthermore argue that our convergence assumption assures equipartition of energy as $\varepsilon \downarrow 0$.

Lemma 3.11. *Given $u_\varepsilon \to u$ and the convergence assumption* (9)*, by passing to a further subsequence if necessary, we have*

$$\lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon) = E(u) \quad \textit{for a.e. } 0 \leq t \leq T \tag{31}$$

and for any smooth test function $\zeta \in C^\infty([0,\Lambda)^d)$ we have

$$\begin{aligned} E(u,\zeta) = \lim_{\varepsilon\downarrow 0} E_\varepsilon(u,\zeta) &= \lim_{\varepsilon\downarrow 0} \int \zeta\varepsilon \left|\nabla u_\varepsilon\right|^2 dx \\ &= \lim_{\varepsilon\downarrow 0} \int \zeta \frac{2}{\varepsilon} W(u_\varepsilon)\, dx = \lim_{\varepsilon\downarrow 0} \int \zeta \sqrt{2W(u_\varepsilon)}\, \left|\nabla u_\varepsilon\right| dx \end{aligned} \tag{32}$$

for a.e. $0 \leq t \leq T$.

A key ingredient for this lemma to work was already observed by Baldo, see Proposition 2.2 in [9]: the optimal partition energy (7) can be written as a (measure-theoretic) supremum using the "primitives" ϕ_i defined in (21). We will use this fact in the following form: Given $\varepsilon > 0$ there exists a scale $r > 0$ such that

$$\sum_{B\in\mathscr{B}_r} \left\{ E(u,\eta_B) - \max_{1\leq i\leq P} \int \eta_B \left|\nabla\left(\phi_i \circ u\right)\right| \right\} \leq \varepsilon, \tag{33}$$

where η_B is a cutoff for B in the ball $2B$ with the same center but with the double radius and the covering $\mathscr{B}_r$ is given by

$$\mathscr{B}_r := \{B_r(i) \colon i \in \mathscr{L}_r\} \tag{34}$$

of $[0,\Lambda)^d$, where $\mathscr{L}_r = [0,\Lambda)^d \cap \frac{r}{\sqrt{d}}\mathbb{Z}^d$ is a regular grid of midpoints on $[0,\Lambda)^d$. Let us note that each summand in (33) is non-negative:

$$0 \leq E(u,\eta_B) - \max_{1\leq i\leq P} \int \eta_B \left|\nabla\left(\phi_i \circ u\right)\right|.$$

This is the same covering as in Definition 5.1 in Chapter 1. A nice feature is that by construction, for each $n \geq 1$ and each $r > 0$, the covering

$$\{B_{nr}(i) \colon i \in \mathscr{L}_r\} \quad \text{is locally finite,} \tag{35}$$

in the sense that for each point in $[0,\Lambda)^d$, the number of balls containing this point is bounded by a constant $c(d,n)$ which is independent of r.

We will later also apply this covering to exploit that BV-partitions generically only have a single, essentially flat interface on small scales, where flatness is measured by the variation of the normal, i.e. the tilt-excess mentioned earlier. This is ensured by the following fact, which is a direct consequence of Lemma 5.2 and Lemma 5.3 in Chapter 1.

Lemma 3.12. *For every $\kappa > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\}^P$ with $\sum_{1\le i\le P}\chi_i = 1$, there exists an $r_0 > 0$ such that for all $r \le r_0$ the following holds : There exist unit vectors $\nu_B \in S^{d-1}$ for all $B \in \mathscr{B}_r$ such that*

$$\sum_{B\in\mathscr{B}_r} \min_{i\neq j} \left\{ \int \eta_B \left|\nu_i - \nu_B\right|^2 |\nabla\chi_i| + \int \eta_B \left|\nu_j + \nu_B\right|^2 |\nabla\chi_j| + \sum_{k\notin\{i,j\}} \int \eta_B |\nabla\chi_k| \right\} \lesssim \kappa E(\chi). \tag{36}$$

3.2 Proofs

Proof of Lemma 3.3. Step 1: Existence via minimizing movements. Since ε is fixed, we may set $\varepsilon = 1$ and denote the Ginzburg-Landau energy by E. For a fixed time-step size $h > 0$ and $n \in \mathbb{N}$ we inductively set

$$u^n = \arg\min_u \left\{ E(u) + \frac{1}{2h}\int \left|u - u^{n-1}\right|^2 dx \right\}.$$

The existence of minimizers u^n follows from the direct method since both E and the metric term $\frac{1}{2h}\int \left|u - u^{n-1}\right|^2 dx$ are lower semi-continuous w.r.t. weak convergence in H^1. Note however that some care needs to be taken in the term $\int W(u)$, as W is non-convex and and the Rellich compactness theorem is applicable in the case $p \ge 2^* = \frac{2d}{d-2}$. Using the decomposition (5) one can still deduce lower semi-continuity in these cases as W_{conv} is convex and the non-convexity in W_{pert} can be treated using Rellich's compactness theorem.

We interpolate these functions in a piecewise constant way: $u^h(t) := u^n$ for $t \in [nh,(n+1)h)$. By comparing u^n to its predecessor u^{n-1} we obtain for any $T = Nh$ the a priori estimate

$$E(u^h(T)) + \frac{1}{2}\int_0^T \int \left|\partial_t^h u^h\right|^2 dx\, dt \le E(u^0),$$

where $\partial_t^h u(t) = \frac{u(t+h)-u(t)}{h}$ denotes the discrete time-derivative of a function u. By the estimate

$$||u^h(t+nh) - u^h(t)||_{L^2} \le \int_t^{t+nh} \left\|\partial_t^h u(s)\right\|_{L^2} ds \le E(u^0)^{\frac{1}{2}}(nh)^{\frac{1}{2}}$$

for $t + (n+1)h \le T$ one can deduce compactness: There exists a sequence $h \downarrow 0$ and a limiting function $u \in C^{\frac{1}{2}}([0,T]; L^2([0,\Lambda)^d;\mathbb{R}^N))$ such that

$$u^h \to u \text{ in } L^2([0,\Lambda)^d) \text{ for all times } 0 \le t \le T$$

and furthermore

$$\sup_t \int |\nabla u|^2 \, dx, \quad \int_0^T \int |\partial_t u|^2 \, dx \, dt < \infty.$$

We want to pass to the limit $h \downarrow 0$ in the the Euler-Lagrange equation

$$\int_0^T \int \partial_t^{-h} \xi \cdot u^h + \Delta \xi \cdot u^h \, dx \, dt = \int_0^T \int \xi \cdot \partial_u W(u^h) \, dx \, dt$$

for all test vector fields $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^N)$. By the pointwise convergence we have

$$\partial_u W(u^h) \to \partial_u W(u) \quad a.e.$$

By the polynomial growth conditions (3) and (4) of W we have

$$|\partial_u W(u^h)|^{\frac{p}{p-1}} \lesssim |u^h|^p,$$

which implies that $\sup_h ||\partial_u W(u^h)||_{L^{\frac{p}{p-1}}} < \infty$. Thus the sequence $|\partial_u W(u^h)|$ is equi-integrable, which implies that $\partial_u W(u^h) \to \partial_u W(u)$ in L^1.

Step 2: We have $\partial_i \partial_j u, \partial_u W(u) \in L^2$. We provide a formal argument which can easily be turned into a rigorous proof by considering discrete difference quotients instead of their limits. Differentiating the equation in the i^{th} coordinate direction for $1 \le i \le d$ gives

$$\partial_t \partial_i u - \Delta \partial_i u = -\partial_u^2 W(u) \partial_i u.$$

By multiplying the equation with $\partial_i u$ and integrating we find

$$\frac{1}{2} \int |\partial_i u(T)|^2 dx + \int_0^T \int |\partial_i \nabla u|^2 dx \, dt = \frac{1}{2} \int |\partial_i u(0)|^2 dx - \int_0^T \int \partial_i u \cdot \partial_u^2 W(u) \partial_i u \, dx \, dt.$$

The second right-hand side term has two contributions, one from W_{conv} and one from W_{pert}, see (5). The contribution due to W_{conv} is negative by convexity. The contribution coming from W_{pert} is controlled by

$$\int_0^T \int |\partial_i u|^2 dx \, dt$$

because W_{pert} has bounded second derivative. Thus we get $\partial_i \partial_j u \in L^2([0,T] \times [0,\Lambda)^d)$. As $\partial_t u$ is in the same space, a quick look at the PDE (1) reveals that $\partial_u W(u)$ is as well.

Finally, the equality (19) follows from integrating the outcome of the computation in Remark 3.4 from 0 to T.

□

Proof of Proposition 3.7. Plugging the a priori estimate (19) into the estimates (26), (27) and (28) of Lemma 3.8 we see that

$$\sup_{\varepsilon} \int_0^T \int |\phi \circ u_\varepsilon| + |\nabla(\phi_i \circ u_\varepsilon)| + |\partial_t(\phi_i \circ u_\varepsilon)| \, dx \, dt < \infty.$$

By the Rellich compactness theorem, we thus find a subsequence $\varepsilon \downarrow 0$ and a function $v\colon (0,T) \times [0,\Lambda)^d \to \mathbb{R}$ such that

$$\phi_i(u_\varepsilon) \to v \quad \text{in } L^1([0,T] \times [0,\Lambda)^d). \tag{37}$$

Step 1: The limit v takes the form $\sum_j \phi_i(\alpha_j)\chi_j$ and the functions u_ε converge to $\sum_j \chi_j \alpha_j$. The convergence of u_ε to $\sum_j \chi_j \alpha_j$ is a part of the classical Γ-limit result [9]. However, we take this opportunity to provide a clarification of the argument based on the compactness argument by Fonseca and Tartar [37].

After passing to another subsequence we can assume that the sequence u_ε generates a Young measure $p_{t,x}$. We note that

$$\int_0^T \int W(u_\varepsilon) \, dx \, dt \to 0$$

implies that u_ε tends to the zeros of W in measure: For any $\delta > 0$ we have

$$|\{(x,t)\colon \operatorname{dist}(u_\varepsilon, \{\alpha_1, \ldots, \alpha_P\}) \geq \delta\}| \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence also the Young measure concentrates and we get

$$p_{t,x} = \sum_{j=1}^P p_{t,x}(\alpha_j) \delta_{\alpha_j}.$$

From this estimate we also get that no mass escapes to infinity, i.e. $\sum_j p_{t,x}(\alpha_j) = 1$.

By (37) for all $f \in C_c(\mathbb{R})$ also $f \circ \phi_i(u_\varepsilon)$ converges to $f \circ v$ strongly in L^1. Therefore Young measure theory gives us the following (a.e.) identity:

$$f(v) = \lim f(\phi_i(u_\varepsilon)) = \sum_{j=1}^P f(\phi_i(\alpha_j)) \, p_{t,x}(\alpha_j).$$

In particular we have

$$\delta_v = \sum_{j=1}^P p_{t,x}(\alpha_j) \delta_{\phi_i(\alpha_j)}.$$

Since a Dirac measure cannot be decomposed into a non-trivial convex combination of multiple Dirac measures, we get that for each $1 \le j \le P$ we have almost everywhere

$$\chi_j(t,x) := p_{t,x}(\alpha_j) \in \{0,1\} \text{ with } \sum_{j=1}^{P} \chi_j = 1 \tag{38}$$

and

$$v = \sum_{j=1}^{P} \phi_i(\alpha_j)\chi_j.$$

In order to get pointwise a.e. convergence of u_ε, note that since the Young measures concentrate by equation (38), we get that $u_\varepsilon \to \sum_j \chi_j \alpha_j$ in measure. By passing to a subsequence, we can upgrade this to pointwise almost everywhere convergence.

Step 2: $\chi_i \in BV$. A similar claim is proven to be true in Prop. 2.2 in [9]. For the convenience of the reader and later refinement we reproduce the proof.

Applying the Fleming-Rishel coarea formula in space and time we see for each $1 \le i \le P$ that

$$\begin{aligned} ||(\partial_t, \nabla)\phi_i \circ u_\varepsilon||_{TV} &= \int_{-\infty}^{\infty} \mathcal{H}^d\left(\partial^*\{(t,x) : \phi_i \circ u_\varepsilon(t,x) \le s\}\right) ds \\ &\ge \int_0^{d_i} \mathcal{H}^d\left(\partial^*\{(t,x) : \phi_i \circ u_\varepsilon(t,x) \le s\}\right) ds \\ &= d_i ||(\partial_t, \nabla)\chi_i||_{TV}, \end{aligned}$$

where we define $d_i := \min_{j \ne i} d_W(\alpha_i, \alpha_j)$. Thus $\chi_i \in BV([0,T] \times [0,\Lambda)^d)$.

For the statement $||E(\chi)||_{L^\infty([0,T])} \le E_0$ we refer the reader the energy-dissipation equality (19) and to the proof of the $\Gamma - \liminf$ inequality in [9].

Finally, recalling Remark 3.6 we notice that the Allen-Cahn Equation only played into the argument via the energy-dissipation estimate (19). □

Proof of Lemma 3.8. Step 1: The chain rule holds if u additionally is bounded in space and time. In this case ϕ_i is in fact Lipschitz continuous on the image of u. By the following Theorem 3.13 due to Ambrosio and Dal Maso we know that the chain rule is valid for the pair $D(\phi_i|_{T_{t,x}})$ and $(\partial_t, \nabla)u$, where $\dot{T}_{t,x} := \operatorname{span}(\{\partial_1 u, \ldots, \partial_d u, \partial_t u\})$ and $T_{t,x} := u(t,x) + \dot{T}_{t,x}$:

Theorem 3.13 (Ambrosio, Dal Maso [4]; Corollary 3.2)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $p \in [1,\infty]$, $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, and let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a Lipschitz continuous function such that $f(0) = 0$. Then $v := f \circ u \in W^{1,p}(\Omega;\mathbb{R}^k)$. Furthermore, for almost every $x \in \Omega$ the restriction of the function f to the affine space*

$$T_x^u := \left\{ y \in \mathbb{R}^n : y = u(x) + (z \cdot D)u \textit{ for some } z \in \mathbb{R}^d \right\}$$

is differentiable at $u(x)$ and

$$Dv = D(f|_{T_x^u})(u)Du \quad \textit{a.e. in } \Omega.$$

Let $\Pi(t,x)$ be the orthogonal projection in $\mathbb{R}^N$ onto the subspace $\dot{T}_{t,x}$ and let

$$\partial_u \phi_i(u)(t,x)v := D\big(\phi_i|_{T_{t,x}}\big)(u(t,x))\Pi(t,x)v.$$

Due the obvious fact that $\Pi(t,x)\nabla u(t,x) = \nabla u(t,x)$ the chain rule still holds for $\partial_u\phi_i(u)$ and $(\partial_t,\nabla)u$. Let (t,x) be a point such that $\phi_i|_{T_{t,x}}$ is differentiable in $u := u(t,x)$, let $v \in \dot{T}_{t,x}$ and $h > 0$. Using the triangle inequality of d and comparing the length of geodesics to straight lines we get

$$|\phi_i(u+hv) - \phi_i(u)| \leq d_W(u+hv,u) \leq \int_0^1 \sqrt{2W(u+thv)}h|v|\,dt.$$

Continuity of W implies that we can pass to the limit $h \to 0$ to get

$$\big|D\phi_i|_{T_{t,x}}(u)v\big| \leq \sqrt{2W(u)}|v|,$$

which for all vectors of the form $v = \Pi(t,x)\tilde{v}$ for some $\tilde{v} \in \mathbb{R}^N$ gives

$$|\partial_u\phi_i(u)| \leq \sqrt{2W(u)}.$$

Step 2: The lemma holds for general functions u with bounded energy and controlled dissipation. The idea is to approximate u with bounded functions. Let $M > 0$ and let $u_{M,j} := \operatorname{sign}(u_j)\,(M \wedge |u_j|)$ for all $1 \leq j \leq N$ be the componentwise truncation of u. We then know that $u_M \to u$ pointwise almost everywhere, which implies $\phi_i(u_M) \to \phi_i(u)$ pointwise almost everywhere. Next, we will strengthen this to L^1 convergence by finding an integrable dominating function.

By the triangle inequality for d we get for all $v \in \mathbb{R}^N$ that

$$\phi_i(v) \leq d_W(\alpha_i,0) + d_W(0,v), \tag{39}$$

so that it is sufficient to consider $d_W(0,v)$. By the growth condition (3) on W we see

$$d_W(0,v) \leq \int_0^1 \sqrt{2W(sv)}|v|ds \lesssim |v| + |v|^{\frac{p}{2}+1} \lesssim 1 + |v|^p \tag{40}$$

for all $v \in \mathbb{R}^N$. Thus we have

$$\phi_i\,(u_M) \lesssim 1 + |u_M|^p \leq 1 + |u|^p$$

and we only need to prove L^p-boundedness of u. This is a straightforward consequence of the coercivity assumption (3) and boundedness of the energy, as for almost all times $0 \leq t \leq T$ we have

$$\sup_{0\leq t\leq T}\int |u|^p dx \overset{(3)}{\lesssim} \sup_{0\leq t\leq T}\int 1 + W(u)dx \lesssim 1 + \sup_{0\leq t\leq T} \varepsilon E_\varepsilon(u). \tag{41}$$

Thus we can apply Lebesgue's Dominated Convergence Theorem to see that $\phi_i(u_M) \to \phi_i(u)$ in L^1. Consequently, we have that

$$(\partial_t, \nabla)(\phi_i \circ u_M) \to (\partial_t, \nabla)(\phi_i \circ u)$$

as distributions.

Note that estimates (39), (40) and (41) imply the L^1 estimate (26) we claimed to hold in the statement of the lemma.

By an elementary property of weakly differentiable functions we have that

$$(\partial_t, \nabla)u_{M,j} = (\partial_t, \nabla)u_j \text{ a.e. on } \{u_{M,j} = u_j\}\,.$$

Since the sets $\{u_M = u\}$ are non-decreasing in M we see that

$$|\{u_M \neq u, (\partial_t, \nabla)u_M \neq (\partial_t, \nabla)u\}| \to 0.$$

Because the definition of $\partial_u\phi_i$ only depends on the values of the pre-composed function and its derivatives, we see that $\partial_u\phi_i(u_M)$ eventually becomes stationary almost everywhere. We denote the limit by $\partial_u\phi_i(u)$. Furthermore, we still have

$$|\partial_u\phi_i(u)| \leq \sqrt{2W(u)} \quad \text{a.e.},$$

which proves (25). Finally, to check the chain rule all remains to be seen is that

$$\partial_u\phi(u_M)(\partial_t, \nabla)u_M \to \partial_u\phi(u)(\partial_t, \nabla)u$$

in L^1. This follows by Lebesgue's Dominated Convergence from the above pointwise convergences and the following widely known application of Young's inequality

$$|\partial_u\phi(u_M)\nabla u_M| \leq \sqrt{2W(u_M)}\,|\nabla u_M| \lesssim \sqrt{2W(u)}|\nabla u| \leq \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)$$

for the spacial gradient and, similarly,

$$|\partial_u\phi(u_M)\partial_t u_M| \leq \frac{\varepsilon}{2}|\partial_t u|^2 + \frac{1}{\varepsilon}W(u) \tag{42}$$

as the right hand side is integrable in space and time by assumption. Note that both inequalities also imply

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int |\nabla\phi_i \circ u|dx \lesssim \sup_{0\leq t\leq T} E_\varepsilon(u), \\
&\int_0^T\int |\partial_t\phi_i \circ u|dxdt \lesssim T\sup_{0\leq t\leq T} E_\varepsilon(u) + \int_0^T\int \varepsilon|\partial_t u|^2dx\,dt,
\end{aligned}$$

which provides the bounds (27) and (28). □

Proof of Lemma 3.9. Step 1: We have $\phi_i \circ u_\varepsilon \in W^{1,2}([0,T];L^1([0,\Lambda)^d))$. The fact that $\phi_i \circ u_\varepsilon \in L^2([0,T];L^1([0,\Lambda)^d))$ is an immediate consequence of estimate (26) of Lemma 3.8. For the estimate on the derivative we localize the previous estimate for $\partial_t(\phi_i \circ u_\varepsilon)$ in time. Let $\zeta \in L^2([0,T])$ be non-negative. Using the chain rule (24), the Lipschitz estimate (25) and the Cauchy-Schwarz inequality in the spacial integral, we obtain

$$\begin{aligned}\int_0^T \zeta \int |\partial_t(\phi_i \circ u_\varepsilon)| dx\, dt &\leq \int_0^T \zeta \int \sqrt{2W(u_\varepsilon)} |\partial_t u_\varepsilon| dx\, dt \\ &\leq \int_0^T \zeta \left(2 \int \frac{1}{\varepsilon} W(u_\varepsilon) dx \right)^{\frac{1}{2}} \left(\int \varepsilon |\partial_t u_\varepsilon|^2 dx \right)^{\frac{1}{2}} dt.\end{aligned}$$

Applying the energy dissipation estimate (19) and the Cauchy-Schwarz inequality in time we arrive at

$$\int_0^T \zeta \int |\partial_t(\phi_i \circ u_\varepsilon)| dx\, dt \lesssim E_\varepsilon(u_\varepsilon(0)) \left(\int_0^T \zeta^2 dt \right)^{\frac{1}{2}}.$$

Optimizing in ζ with $||\zeta||_{L^2} = 1$ gives the $L_t^2 L_x^1$-estimate (29).

Step 2: The sequence $\phi_i \circ u_\varepsilon$ is pre-compact in $L^\infty([0,T];L^1([0,\Lambda)^d))$. Due to a version of the Fundamental Theorem of Calculus for the Bochner integral, cf. Chapter 5.9, Theorem 2 in [34], we know for almost every $s, r \in [0,T]$ with $s \leq r$ that

$$\phi_i \circ u_\varepsilon(r) - \phi_i \circ u_\varepsilon(s) = \int_s^r \partial_t(\phi_i \circ u_\varepsilon)(t) dt.$$

Consequently, the Cauchy-Schwarz inequality gives

$$\int |\phi_i \circ u_\varepsilon(r) - \phi_i \circ u_\varepsilon(s)| dx \leq \int_s^r \int |\partial_t(\phi_i \circ u_\varepsilon)(t)| dx\, dt \lesssim (r-s)^{\frac{1}{2}} \left(E_\varepsilon(u_\varepsilon^0) \right)^{\frac{1}{2}}. \qquad (43)$$

By estimate (27) we also know that

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \int |\nabla(\phi_i \circ u_\varepsilon)| dx \lesssim 1 + E_\varepsilon(u_\varepsilon^0).$$

Since $\sup_\varepsilon E_\varepsilon(u_\varepsilon^0) < \infty$ we consequently know that (a modification of) $\phi \circ u_\varepsilon$ is equi-continuous in $C([0,T];L^1([0,\Lambda)^d))$. Additionally, lower semi-continuity of the BV-norm and the compact Sobolev embedding of $W^{1,1}$ into L^1 implies that for all times $t \in [0,T]$ the maps $\phi_i \circ u_\varepsilon(t)$ are pre-compact in $L^1([0,\Lambda)^d)$. The Arzelà-Ascoli theorem then gives the claim.

Step 3: The sequence u_ε converges to $\sum_i \chi_i \alpha_i$ in measure uniformly in time. By $d_W(\alpha_i, \alpha_i) = 0$ for all $1 \leq i \leq P$ and Step 2 we get

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \sum_{i=1}^P \int d_W(\alpha_i, u_\varepsilon(t,x)) \chi_i dx \leq \operatorname*{ess\,sup}_{0 \leq t \leq T} \sum_i \int |d_W(\alpha_i, u_\varepsilon(t,x)) - d_W(\alpha_i, u(t,x))| dx \to 0.$$

For every $\delta > 0$ and $1 \le i \le P$ we have by continuity of the map $v \to d_W(\alpha_i, v)$ that

$$\min\left\{d_W(\alpha_i, v); v \in \mathbb{R}^N, |v-\alpha_i| \ge \delta\right\} > 0.$$

As a result we get essentially uniform in time convergence in measure, i.e. for every $\delta > 0$ we have

$$\operatorname*{ess\,sup}_{0\le t\le T}\left|\left\{\left|u_\varepsilon - \sum_{i=1}^P \chi_i\alpha_i\right| \ge \delta\right\}\right| \to 0. \tag{44}$$

Since u_ε is continuous in time, we can replace the essential supremum by a "true" supremum.

Step 4: The sequence u_ε^2 is equi-integrable uniformly in time. If $p > 2$, then this follows immediately from the uniform L^p bound (41) of u_ε we proved in Lemma 3.8 by an application of the Hölder inequality: For any measurable set $A \subset [0,\Lambda)^d$ and any $\varepsilon > 0$ we have

$$\sup_{0\le t\le T}\int_A u_\varepsilon^2(t,x)dx \le \sup_{0\le t\le T}|A|^{\frac{2}{p'}}\left(\int |u_\varepsilon|^p dx\right)^{\frac{2}{p}} \lesssim |A|^{\frac{2}{p'}}\left(1+E_\varepsilon(u_\varepsilon^0)\right)^{\frac{2}{p}}. \tag{45}$$

As $E_\varepsilon(u_\varepsilon^0)$ is bounded uniformly in ε, we get the statement.

If $p = 2$ we get some slightly better integrability from a Sobolev embedding: We define the function $G(u) := (|u| - R)_+^2$, where $R > 0$ is the radius from the growth condition (3) of W. This function is C^1 with

$$\partial_u G(u) = 2\left(|u| - R\right)_+ \frac{u}{|u|}\chi_{|u|>R}$$

and thus satisfies the same bounds as ϕ_i, see (40) and (25), namely

$$G(u) \le |u|^2,\ |\partial_u G(u)| \le |u|.$$

Consequently, we can use the same approximation argument as in Lemma 3.8 to see that

$$\sup_{\varepsilon>0}\sup_{0\le t\le T}||G\circ u_\varepsilon(t)||_{W^{1,1}} < \infty.$$

The Sobolev embedding theorem can thus be applied to conclude

$$\sup_{\varepsilon>0}\sup_{0\le t\le T}||G\circ u_\varepsilon(t)||_{L^{\frac{d}{d-1}}} < \infty.$$

Recalling the definition of G we see that this implies

$$\sup_{\varepsilon>0}\sup_{0\le t\le T}||u_\varepsilon(t)||_{L^{2\frac{d}{d-1}}} < \infty,$$

from which we deduce the necessary equi-integrability of $|u_\varepsilon|^2$ as before.

Step 5: The sequence u_ε converges in $C([0,T];L^2([0,\Lambda)^d))$. Essentially, we wish to exploit the fact that convergence in measure and equi-integrability are equivalent to convergence in L^1. However, since we want the convergence to be uniform in time and instead of L^1 convergence we want L^2 convergence in space, we quickly reproduce the argument.

For any cut-off $M>0$ we can split the integral

$$\int |u_\varepsilon - u|^2 dx = \int_{\{|u_\varepsilon-u|\geq M\}} |u_\varepsilon - u|^2 dx + \int_{\{|u_\varepsilon-u|<M\}} |u_\varepsilon - u|^2 dx.$$

The first term on the right-hand side satisfies

$$\sup_{0\leq t\leq T}\int_{\{|u_\varepsilon-u|\geq M\}} |u_\varepsilon - u|^2 \lesssim \sup_{0\leq t\leq T}\int_{\{|u_\varepsilon-u|\geq M\}} (|u_\varepsilon|^2+1)\,dx \to 0 \quad \text{as } \varepsilon\to 0$$

by applying uniform convergence in measure (44) and uniform equi-integrability (45). For every $\delta>0$ the second term on the right hand side can be estimated by

$$\sup_{0\leq t\leq T}\int \min\left\{|u_\varepsilon-u|^2, M^2\right\}dx \leq \sup_{0\leq t\leq T} \Lambda^d\delta^2 + |\{|u_\varepsilon-u|>\delta\}|\,M^2 \to \Lambda^d\delta^2, \quad \text{as } \varepsilon\to 0.$$

Taking first $\varepsilon\to 0$ and then $\delta\to 0$ we have indeed

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\int |u_\varepsilon-u|^2 dx = 0. \qquad \square$$

Proof of Proposition 3.10. The strategy is the following:

1. We prove the easier fact $\partial_t(\phi_i\circ u)\ll E(u,\cdot)dt$.
2. We replace $\phi_i\circ u$ with u, i.e. we prove $\partial_t u \ll E(u,\cdot)dt$, using a suitable localization of Step 4 of the proof of Proposition 3.7, i.e. the Fleming-Rishel coarea formula.
3. We prove that $\partial_t\chi_i$ is singular to the "wrong" parts of $E(u,\cdot)dt$ in order to replace the right-hand side with $|\nabla\chi_i|dt$.

Equation (12) immediately follows.

Step 1: For all $1\leq i\leq P$ we have $\partial_t(\phi_i\circ u)\ll E(u,\cdot)dt$ and the corresponding density is square-integrable w.r.t. $E(u,\cdot)dt$. We localize with a smooth test function $\zeta\in C_0^\infty((0,T)\times[0,\Lambda)^d;\mathbb{R}^{1+d})$ and use the chain rule (3.8), the Lipschitz estimate (25) and the Cauchy-Schwarz inequality to obtain

$$\int_0^T\int \partial_t\phi_i(u_\varepsilon)\zeta\,dx\,dt \leq \left(\int_0^T\int \varepsilon\,|\partial_t u_\varepsilon|^2\,dx\,dt\right)^{\frac12}\left(\int_0^T\int \zeta^2\frac{2}{\varepsilon}W(u_\varepsilon)\,dx\,dt\right)^{\frac12}. \qquad (46)$$

By the convergence (23) of the composition and the equipartition of energy (32) we can pass to the limit in this inequality and obtain

$$\int_0^T \int \phi_i(u)\partial_t\zeta \, dx\, dt \leq \left(\liminf_{\varepsilon\downarrow 0} \int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx\, dt\right)^{\frac{1}{2}} \left(\int_0^T E_\varepsilon(u,\zeta^2)dt\right)^{\frac{1}{2}}. \tag{47}$$

By equation (19) the first factor on the right-hand side is controlled by $\sqrt{E_0}$. From this we see that indeed $|\partial_t(\phi_i \circ u)| \ll E(u,\,\cdot\,)dt$ and by taking the supremum over the test functions ζ we see that the density is square-integrable.

Step 2: We have $d_i|\partial_t\chi_i| \leq |\partial_t(\phi_i \circ u)|$ *where* $d_i := \min_{1\leq j\leq P, i\neq j} d_W(\alpha_i,\alpha_j)$. Basically, we want to use the argument of Step 4 in the proof of Proposition 3.7 for the partial derivative $\partial_t\chi_i$. This can be done by combining the slicing theorem, cf. Theorem 3.103 in [6], and with the previous argument at almost each point $x \in [0,\Lambda)^d$, which leads to

$$d_i|\partial_t\chi_i|(U) \leq |\partial_t(\phi_i \circ u)|(U)$$

for all open sets $U \subset [0,T]\times[0,\Lambda)^d$. This implies that for all $\xi \in C_c([0,T]\times[0,\Lambda)^d;[0,\infty)$ we have the inequality

$$d_i|\partial_t\chi_i|(\xi) \leq |\partial_t(\phi_i \circ u)|(\xi):$$

Indeed, we can approximate ξ by constants on sets whose boundaries are negligible w.r.t. the measures on both sides. We thus get that for all $1 \leq i \leq P$ we have $\partial_t\chi_i \ll E(u,\,\cdot\,)dt$ and the corresponding density V_i satisfies $V_i \in L^2(E(u,\,\cdot\,)dt)$.

Step 3: We have that $|(\partial_t,\nabla)\chi_i|$ *and* $\frac{1}{2}(|\nabla\chi_j|_d + |\nabla\chi_k|_d - |\nabla(\chi_j+\chi_k)|_d)\,dt$ *are singular for all pairwise different* $1 \leq i,j,k \leq P$. For a characteristic function $\chi : [0,T]\times[0,\Lambda)^d \to \mathbb{R}$ we write $|\nabla\chi|_{d+1}$ for the total variation in time and space of the partial spacial derivatives and $|\nabla\chi|_d$ for the total variation the spacial derivatives in space defined almost everywhere in time.

According to Theorem 4.17 in [6] one can decompose $\operatorname{supp}|(\partial_t,\nabla)\chi_i|$ into the pairwise disjoint sets $\tilde{\Sigma}_{i,l} := \partial^*\tilde{\Omega}_i \cap \partial^*\tilde{\Omega}_l$, $1 \leq l \leq P$, which are the intersections of the reduced boundaries in time and space. The exceptional sets are $\mathcal{H}^d$-negligible and hence can be ignored in all the derivatives $|(\partial_t,\nabla)\chi_m|$, $1 \leq m \leq P$. Thus we only have to prove that

$$\frac{1}{2}(|\nabla\chi_j|_d + |\nabla\chi_k|_d - |\nabla(\chi_j+\chi_k)|_d)\,dt\big(\tilde{\Sigma}_{il}\big) = 0$$

for all $1 \leq l \leq P$.

Since $j,k \neq i$ and the interfaces are pairwise disjoint we have that

$$|(\partial_t,\nabla)\chi_j|\left(\tilde{\Sigma}_{il}\right) = 0 \text{ or } |(\partial_t,\nabla)\chi_k|\left(\tilde{\Sigma}_{il}\right) = 0.$$

In the first case we have, since restriction commutes with the total variation,

$$\begin{aligned}&\frac{1}{2}(|\nabla\chi_j|_{d+1} + |\nabla\chi_k|_{d+1} - |\nabla(\chi_j+\chi_k)|_{d+1})\left(\tilde{\Sigma}_{il}\right)\\ &\qquad = \frac{1}{2}\Big(|\nabla\chi_k|_{d+1}\big(\Sigma_{il}\big) - |\nabla\chi_k|_{d+1}\big(\tilde{\Sigma}_{il}\big)\Big) = 0.\end{aligned}$$

The analogous argument gives the same result in the second case. Finally, a straightforward generalization of Theorem 3.103 in [6] to higher dimensional slicings implies

$$|\nabla \chi_l|_{1+d} = |\nabla \chi_l|_d \, dt,$$

which proves the claim.

Step 4: Conclusion of the L^2-estimate. Since $|\partial_t \chi_i| \le |(\partial_t, \nabla)\chi_i|$ as measures we get from Step 2 and Step 3 that $|\partial_t \chi_i| \ll |\nabla \chi_i|_d \, dt$. Step 3 also allows to replace $E(u, \cdot\,)\, dt$ by $|\nabla \chi_i|_d \, dt$ in the L^2-estimate.

We once more point out that we did not use the Allen-Cahn Equation (1) apart from the energy dissipation estimate (19). □

Proof of Lemma 3.11. The proof is already contained in Chapter 1 and [56]. For the convenience of the reader we reproduce the arguments here.

Step 1: Localization in time. We first show that the integrated assumption of the convergence of the energies (9) and the Γ-convergence of E_ε to E already imply the pointwise convergence (31) – at least up to a further subsequence. We will prove

$$\lim_{\varepsilon\downarrow 0} \int_0^T |E_\varepsilon(u_\varepsilon) - E(\chi)| \, dt = 0, \tag{48}$$

which after passage to a subsequence clearly implies (31).

To convince ourselves of (48) we rewrite the integral as

$$\int_0^T |E_\varepsilon(u_\varepsilon) - E(\chi)| \, dt = \int_0^T (E_\varepsilon(u_\varepsilon) - E(\chi)) \, dt + 2\int_0^T (E_\varepsilon(u_\varepsilon) - E(\chi))_- \, dt.$$

The first right-hand side integral vanishes as $\varepsilon \downarrow 0$ by (9). By the lower semi-continuity part of the Γ-convergence of E_ε to E, see [9], and by the convergence (22) of u_ε to u the integrand of the second right-hand side term tends to zero pointwise a.e. in $(0,T)$. By Lebesgue's Dominated Convergence also the integral vanishes in the limit and we proved (48).

Step 2: Localization in space. We claim that the convergence (31) of the energies implies

$$\lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon, \zeta) = E(u, \zeta) \quad \text{for a.e. } 0 \le t \le T \text{ and all } \zeta \in C^\infty([0,\Lambda)^d). \tag{49}$$

Indeed, if we assume that w.l.o.g. by linearity $0 \le \zeta \le 1$, using the $\liminf$-inequality of the Γ-convergence on the domains $\{\zeta > s\}$ and the layer cake representation $\zeta = \int_0^1 \mathbf{1}_{\{\zeta > s\}} \, ds$ we obtain the inequality

$$E(u, \zeta) \le \liminf_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon, \zeta).$$

But the same argument works for $0 \leq 1-\zeta \leq 1$ instead of ζ and by the convergence (31) we have

$$E(u) - E(u,\zeta) = E(u,1-\zeta) \leq \liminf_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon, 1-\zeta) \overset{(31)}{=} E(u) - \limsup_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon,\zeta),$$

which is the inverse inequality and thus (49) follows.

Step 3: Equipartition of energy. Now let us turn to (32). First we claim that (32) reduces to

$$\int \sqrt{2W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,dx \to E(u). \tag{50}$$

Indeed, setting $a_\varepsilon^2 := \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2$ and $b_\varepsilon^2 := \frac{1}{\varepsilon}W(u_\varepsilon)$, using $a_\varepsilon^2 - b_\varepsilon^2 = (a_\varepsilon + b_\varepsilon)(a_\varepsilon - b_\varepsilon)$ and Cauchy-Schwarz

$$\int \zeta \left|a_\varepsilon^2 - b_\varepsilon^2\right| dx \leq \|\zeta\|_\infty \left(\int (a_\varepsilon + b_\varepsilon)^2\,dx\right)^{\frac{1}{2}} \left(\int (a_\varepsilon - b_\varepsilon)^2\,dx\right)^{\frac{1}{2}}.$$

Since $(a_\varepsilon + b_\varepsilon)^2 \lesssim a_\varepsilon^2 + b_\varepsilon^2$ the first right-hand side integral stays bounded in the limit $\varepsilon \downarrow 0$ and it is enough to prove that the second right-hand side integral vanishes as $\varepsilon \downarrow 0$. Expanding the square and using the definition of a_ε and b_ε we see that the limit of the second right-hand side integral is equal to

$$E_\varepsilon(u_\varepsilon) - \int \sqrt{2W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,dx$$

and indeed the proof of (32) reduces to (50).

We conclude by proving (50). By lower semi-continuity and Young's inequality for any cutoff $0 \leq \eta \leq 1$ and any $1 \leq i \leq P$ we get

$$\begin{aligned}\int \eta\,|\nabla(\phi_i \circ u)| &\leq \liminf_{\varepsilon\downarrow 0} \int \eta\,|\nabla(\phi_i\circ u_\varepsilon)|\,dx \leq \liminf_{\varepsilon\downarrow 0}\int \eta\sqrt{2W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,dx \\ &\leq \liminf_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon,\eta) \overset{(49)}{=} E(\eta,\chi).\end{aligned}$$

Using a partition of unity subordinate to the covering (34) and choosing the index $1 \leq i \leq P$ according to estimate (33) we conclude.

□

4 Convergence

In Section 3 we proved that the solutions u_ε of the Allen-Cahn Equation (1) are pre-compact. In this section we pass to the limit in the Allen-Cahn Equation (1) and prove that the limit moves by mean curvature. Since this section is the core of the chapter, we give a short idea of the proof and then pass to the rigorous derivation in the subsequent parts, first for the curvature term, and afterwards for the velocity term.

4.1 Idea of the proof

To illustrate the idea of our proof we give a short overview in the simpler two-phase case. In this setting the convergence of the curvature-term

$$\lim_{\varepsilon\downarrow 0}\int_0^T\int\left(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}W'(u_\varepsilon)\right)\xi\cdot\nabla u_\varepsilon\,dx\,dt=\sigma\int\nabla\xi\colon\,(Id-\nu\otimes\nu)\,|\nabla\chi|\,dt \tag{51}$$

is by (31) literally contained in [56] and the only difficulty is to prove

$$\lim_{\varepsilon\downarrow 0}\int_0^T\int\partial_t u_\varepsilon\,\xi\cdot\varepsilon\nabla u_\varepsilon\,dx\,dt=\sigma\int_0^T\int V\,\xi\cdot\nu\,|\nabla\chi|\,dt. \tag{52}$$

Since $\partial_t u_\varepsilon \rightharpoonup V\,|\nabla\chi|\,dt$ and $\varepsilon\nabla u_\varepsilon\approx\nu$ only in a weak sense, we cannot directly pass to the limit in the product. The general idea to work around this problem is to follow the strategy of Chapter 1: Thinking of the test vector field ξ as a localization, we "freeze" the normal along the sequence to be the fixed direction $\nu^*\in S^{d-1}$ and estimate the error w.r.t. an approximation of the *tilt-excess*

$$\mathscr{E}:=\sigma\int_0^T\int|\nu-\nu^*|^2|\nabla\chi|\,dt, \tag{53}$$

measuring the (local) flatness of the reduced boundary $\partial^*\Omega$ of the limit phase $\Omega=\{\chi=1\}$. The main difference to the work presented in Chapter 1 is that we measure the error w.r.t. the tilt-excess $\mathscr{E}$ instead of the energy-excess

$$\int|\nabla\chi|-\int|\nabla\chi^*|\,,\quad\text{where } \chi^* \text{ is a half-space in direction } \nu^*.$$

After a localization, De Giorgi's Structure Theorem guarantees the smallness in both cases, see Section 5 in Chapter 1. Our approximation of the tilt-excess along the sequence is

$$\mathscr{E}_\varepsilon:=\int_0^T\int|\nu_\varepsilon-\nu^*|^2\,\varepsilon\,|\nabla u_\varepsilon|^2\,dx\,dt, \tag{54}$$

where $\nu_\varepsilon=\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ denotes the normal of the level sets of u_ε.

We will use the approximate tilt-excess to suppress oscillations of the *direction* of the term $\varepsilon\nabla u_\varepsilon$ on the left-hand side of (52) so that we can pass to the limit in the product. We replace the normal ν_ε by a constant direction $\nu^*\in S^{d-1}$ and control the difference

$$\int_0^T\int\partial_t u_\varepsilon\,\xi\cdot\varepsilon\nabla u_\varepsilon dx\,dt-\int_0^T\int\partial_t u_\varepsilon\,\xi\cdot(\varepsilon\,|\nabla u_\varepsilon|\,\nu^*)\,dx\,dt \tag{55}$$

by the following combination of the excess and the initial energy

$$\|\xi\|_\infty\left(\frac{1}{\alpha}\mathscr{E}_\varepsilon+\alpha E_\varepsilon(u^0_\varepsilon)\right)$$

for any (small) parameter $\alpha > 0$ – an immediate consequence of Young's inequality and the energy-dissipation estimate (19). It is easy to check that by the equipartition of energy (32) we can replace $\varepsilon\,|\nabla u_\varepsilon|$ in the second integral in (55) by $\sqrt{2W(u_\varepsilon)}$ up to an error that vanishes as $\varepsilon \downarrow 0$:

$$\int_0^T \int \partial_t u_\varepsilon\, \xi \cdot (\varepsilon\,|\nabla u_\varepsilon|\,\nu^*)\,dx\,dt = \int_0^T \int \partial_t u_\varepsilon\,\sqrt{2W(u_\varepsilon)}\,\xi \cdot \nu^* dx\,dt + o(1). \tag{56}$$

Identifying the nonlinear term

$$\partial_t u_\varepsilon\,\sqrt{2W(u_\varepsilon)} = \partial_t\,(\phi \circ u_\varepsilon)$$

as the derivative of the compact quantity $\phi \circ u_\varepsilon \to \phi \circ u$, where $\phi(u) = \int_0^u \sqrt{2W(s)}ds$, we can pass to the limit $\varepsilon \downarrow 0$ and obtain

$$\int_0^T \int \partial_t\,(\phi \circ u_\varepsilon)\,\xi \cdot \nu^* dx\,dt \to \sigma \int_0^T \int V\,\xi \cdot \nu^*\,|\nabla\chi|\,dt.$$

As before, but now at the level of the limit, by Young's inequality we can "un-freeze" the normal, i.e. replace ν^* by ν at the expense of

$$\|\xi\|_\infty \left(\frac{1}{\alpha}\mathscr{E} + \alpha \int_0^T \int V^2\,|\nabla\chi|\,dt\right).$$

While in the case of our treatment of the thresholding scheme in Chapter 1 the convergence assumption trivially implies the convergence of the (approximate) energy-excess, here we have to argue why we can pass to the limit in our nonlinear excess $\mathscr{E}_\varepsilon$ and connect it to $\mathscr{E}$.

Using the trivial equality $|\nu - \nu^*|^2 = 2(1 - \nu \cdot \nu^*)$ and the convergence assumption (9) this question reduces to

$$\int_0^T \int \varepsilon\,|\nabla u_\varepsilon|\,\nabla u_\varepsilon\,dx\,dt \to \int_0^T \int \nabla\,(\phi \circ u)\,dt. \tag{57}$$

Now the argument is similar to the one before for the time derivative. Using again the equipartition of energy (32) we can replace $\varepsilon\,|\nabla u_\varepsilon|$ by $\sqrt{2W(u_\varepsilon)}$. Identifying the nonlinearity

$$\sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon = \nabla\,(\phi \circ u_\varepsilon)$$

as a derivative yields the convergence of the excess.

Thus we arrive at the right-hand side of (52) – up to an error that we can handle: we localize on a scale $r > 0$ so that $\mathscr{E} \to 0$ as $r \downarrow 0$, while the second error term stays bounded by the L^2-estimate (30). We then recover the motion law (11) by sending $\alpha \downarrow 0$.

4.2 Convergence of the curvature-term

In the two-phase case, the convergence (51) of the curvature-term is contained in the work of Luckhaus and Modica [56]. In our setting, the convergence does not follow immediately from their work. We give an extension of this result by quantifying the Reshetnyak-argument.

Proposition 4.1. *Given a sequence $u_\varepsilon \to u = \sum_i \chi_i \alpha_i$ such that the energies converge in the sense of*

$$E_\varepsilon(u_\varepsilon) \to E(u). \tag{58}$$

Then also the first variations converge:

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right) \cdot (\xi\cdot\nabla)\, u_\varepsilon \, dx \\ = \frac{1}{2}\sum_{i,j\le} \sigma_{ij} \int \nabla\xi \colon\, (Id - \nu_i\otimes\nu_i)\, \frac{1}{2}\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right).\end{aligned} \tag{59}$$

Furthermore we have

$$\int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right) \cdot (\xi\cdot\nabla)\, u_\varepsilon \, dx \lesssim \|\nabla\xi\|_\infty E_\varepsilon(u_\varepsilon). \tag{60}$$

Proof. Following the lines of [56] we can rewrite the left-hand side of (59) by integrating the first term by parts and using the chain rule for the second term. With Einstein's summation convention and omitting the index ε we have

$$\int (\varepsilon\partial_i\partial_i u_k - \tfrac{1}{\varepsilon}\partial_k W)\, \xi_j\, \partial_j u_k \, dx = \int \left\{ -\varepsilon\, \partial_i u_k\, \partial_i \xi_j\, \partial_j u_k - \varepsilon\, \partial_i u_k\, \xi_j\, \partial_i\partial_j u_k - \tfrac{1}{\varepsilon}\partial_j (W(u))\, \xi_j \right\} dx. \tag{61}$$

We can now rewrite the second term on the right-hand side and integrate by parts to see

$$-\int \varepsilon\, \partial_i u_k\, \xi_j\, \partial_i\partial_j u_k \, dx = -\int \varepsilon\, \xi_j\, \partial_j \left\{ \frac{1}{2}\left(\partial_i u_k\right)^2 \right\} dx = \int (\nabla\cdot\xi)\, \frac{\varepsilon}{2}\, |\nabla u|^2 \, dx.$$

Plugging this into (61) the left-hand side of (59) is thus equal to

$$\int \nabla\xi \colon\, (Id - N^\varepsilon \otimes N^\varepsilon)\, \varepsilon\, |\nabla u_\varepsilon|^2 \, dx + \int (\nabla\cdot\xi) \left(\frac{1}{\varepsilon} W(u_\varepsilon) - \frac{\varepsilon}{2}\, |\nabla u_\varepsilon|^2 \right) dx,$$

where $N^\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \in \mathbb{R}^{P\times d}$ and $(N^\varepsilon \otimes N^\varepsilon)_{ij} := \sum_k N^\varepsilon_{ki} N^\varepsilon_{kj} \in \mathbb{R}^{d\times d}$, a slightly non-standard definition of this symbol. From this we immediately obtain (60). By the equipartition of energy (32), see also Remark 3.6, the second integral is negligible as $\varepsilon \to 0$ and up to another error that vanishes as $\varepsilon \to 0$ we can replace the first term by

$$\int \nabla\xi \colon\, (Id - N^\varepsilon \otimes N^\varepsilon)\, \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| \, dx.$$

Again by the equipartition of energy (32) it is enough to prove the convergence of the nonlinear term

$$\int A\colon N^\varepsilon \otimes N^\varepsilon \sqrt{2W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,dx \to \sum_{i,j} \sigma_{ij} \int A\colon \nu_i \otimes \nu_i \frac{1}{2}\left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) \tag{62}$$

for any smooth matrix field $A\colon [0,\Lambda)^d \to \mathbb{R}^{d\times d}$. By linearity we may assume w.l.o.g. $|A| \le 1$.

We prove (62) using the following two claims:

Claim 1: We choose a majority phase by introducing the function $\phi = \phi_i$ for some arbitrary $1 \le i \le P$ on the right-hand side of (62)*. The corresponding estimate is*

$$\limsup_{\varepsilon\to 0} \left| \int A\colon N^\varepsilon \otimes N^\varepsilon \sqrt{2W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,dx - \int A\colon \nu_\varepsilon \otimes \nu_\varepsilon\,|\nabla(\phi\circ u_\varepsilon)|\,dx \right| \lesssim E(u,\eta) - \int \eta\,|\nabla(\phi\circ u)|\,, \tag{63}$$

where $\nu_\varepsilon := \frac{\nabla\phi(u_\varepsilon)}{|\nabla\phi(u_\varepsilon)|} \in \mathbb{R}^d$.

Claim 2: We adapt the Reshetnyak argument in [56] to our setting by turning the qualitative statements there into a quantitative statement. Under the assumption (58) we claim

$$\limsup_{\varepsilon\to 0} \left| \int A\colon \nu_\varepsilon \otimes \nu_\varepsilon\,|\nabla(\phi\circ u_\varepsilon)|\,dx - \int A\colon \nu\otimes\nu\,|\nabla(\phi\circ u)| \right| \lesssim E(u,\eta) - \int \eta\,|\nabla(\phi\circ u)|\,. \tag{64}$$

In both cases we express the errors in terms of the "mild excess"

$$E(u,\eta) - \int \eta\,|\nabla\phi(u)|\,, \tag{65}$$

which measures the local difference of the multi-phase setting to the two-phase setting on the support of the matrix field A approximated with a cut-off η.

Decomposing an arbitrary matrix field A by a partition of unity and using the localization estimate (33) we obtain (62) and thus proved the proposition.

Proof of Claim 1: Introducing a majority phase. First we replace the matrix $N^\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ by $\pi_{u_\varepsilon} N^\varepsilon$, where $\pi_u = \frac{\partial_u\phi}{|\partial_u\phi|} \otimes \frac{\partial_u\phi}{|\partial_u\phi|}$. Note that then the additional sum in the definition of the symbol $\pi_{u_\varepsilon} N^\varepsilon \otimes \pi_{u_\varepsilon} N^\varepsilon$ collapses:

$$(\pi_u N \otimes \pi_u N)_{ij} = \sum_{k=1}^N (\pi_u N)_{ki}\,(\pi_u N)_{kj} = \left(\sum_{k=1}^N \frac{(\partial_u\phi)_k^2}{|\partial_u\phi|^2}\right) \frac{(\partial_u\phi\nabla u)_i}{|\partial_u\phi||\nabla u|}\frac{(\partial_u\phi\nabla u)_j}{|\partial_u\phi||\nabla u|}.$$

Furthermore, using the chain rule of Ambrosio and Dal Maso, Lemma 3.8, we see

$$A\colon\ (\pi_{u_\varepsilon} N^\varepsilon) \otimes (\pi_{u_\varepsilon} N^\varepsilon) = |\pi_{u_\varepsilon} N^\varepsilon|^2 A\colon \nu_\varepsilon \otimes \nu_\varepsilon.$$

Two errors arise in (63). The first error when replacing N^ε by ν_ε and the second when replacing $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$ by $|\nabla(\phi \circ u)|$.

The first error when introducing the projection π_u is bounded by

$$\int \eta\, |(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2\, \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| + \eta\left(1 - |\pi_{u_\varepsilon} N^\varepsilon|^2\right)^2 \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx. \qquad (66)$$

Since multiplication by π_u is an orthogonal projection in matrix-space and $\left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right| = |\pi_u N^\varepsilon| \leq 1$ we have

$$|(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2 = |N^\varepsilon|^2 - |\pi_{u_\varepsilon} N^\varepsilon|^2 \lesssim 1 - |\pi_{u_\varepsilon} N^\varepsilon| = 1 - \left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right|.$$

Multiplying this inequality with $\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$ and using the Lipschitz estimate for $\phi \circ u$ (25) we see that

$$\begin{aligned} |(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2\, \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| &\leq \left(1 - \left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right|\right) \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| \\ &\leq \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\partial_u \phi(u_\varepsilon) \nabla u_\varepsilon|\,. \end{aligned}$$

Plugging this into (66) and using the Ambrosio-Dal Maso chain rule (24) again, we see that the error is controlled by

$$E_\varepsilon(u_\varepsilon, \eta) - \int \eta\, |\nabla(\phi \circ u_\varepsilon)|\, dx.$$

By the convergence of the energies (58) and lower semi-continuity of the total variation we can pass to the limit $\varepsilon \to 0$ in this expression and obtain the upper bound

$$E(u, \eta) - \int \eta\, |\nabla(\phi \circ u)|\,.$$

Finally, we turn to the second error, when substituting $\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$ by $|\nabla(\phi \circ u_\varepsilon)|$ in (63). Since $|\nabla(\phi \circ u_\varepsilon)| \leq |\partial_u \phi||\nabla u_\varepsilon| \leq \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$, by Young's inequality this second error is estimated by

$$\int \eta\, \left|\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\nabla(\phi \circ u_\varepsilon)|\right| dx = \int \eta\left(\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\nabla(\phi \circ u_\varepsilon)|\right) dx,$$

which by the equipartition (32) and Remark 3.6 again passes to the limit as before and thus proves (63).

Proof of Claim 2: A quantitative Reshetnyak-argument for $\phi \circ u$. We could pass to the limit in the nonlinear expression $\int A\colon \nu \otimes \nu \, |\nabla \, (\phi \circ u)|$ by the classical Reshetnyak argument if we knew that the mass $\int |\nabla \, (\phi \circ u)|$ converged. In our case we unfortunately do not know if the total variation for each $\phi_i \circ u$ converges, but we can make the error small by localizing.

Our argument for (64) can be regarded as a quantitative analogue of the classical Reshetnyak-argument [75], see also [56].

By Banach-Alaoglu and a disintegration result for measures we can find a measure μ on $[0,\Lambda)^d$ and a family of probability measures $\{p_x\}_{x\in[0,\Lambda)^d}$ on S^{d-1} such that

$$\int \zeta(x,\nu_\varepsilon) \, |\nabla(\phi \circ u_\varepsilon)| \, dx \to \iint \zeta(x,\tilde{\nu}) \, dp_x(\tilde{\nu}) \, d\mu(x) \tag{67}$$

for all $\zeta \in C([0,\Lambda)^d \times S^{d-1})$ – at least after passage to a subsequence. But since we will identify the limit we may pass to subsequences. In particular we have

$$\int A\colon \nu_\varepsilon \otimes \nu_\varepsilon \, |\nabla(\phi \circ u_\varepsilon)| \, dx \to \int A(x)\colon \int \tilde{\nu} \otimes \tilde{\nu} \, dp_x(\tilde{\nu}) \, d\mu(x). \tag{68}$$

Our aim is to prove that – up to the "mild excess" (65) – the right-hand side of (68) is equal to

$$\int A\colon \nu \otimes \nu \, |\nabla(\phi \circ u)| \, .$$

On the one hand, by the lower semi-continuity of the total variation and (67) with $\zeta(x,\nu) = \eta(x) \geq 0$

$$\int \eta |\nabla(\phi \circ u)| \leq \liminf_{\varepsilon \downarrow 0} \int \eta |\nabla(\phi \circ u_\varepsilon)| \, dx = \iint \eta \, d\mu, \tag{69}$$

i.e. $|\nabla(\phi \circ u)|$ is dominated by μ.

On the other hand, by the assumption (58) the measure μ is dominated by the energy. Indeed, for any $\eta \geq 0$ we have by Young's inequality

$$\int \eta \, d\mu = \lim_{\varepsilon \downarrow 0} \int \eta \, |\nabla(\phi \circ u_\varepsilon)| \, dx \leq \liminf_{\varepsilon \downarrow 0} E_\varepsilon(u_\varepsilon, \eta) = E(\chi, \eta). \tag{70}$$

Using $|\tilde{\nu} \otimes \tilde{\nu} - \nu \otimes \nu| \leq 2 \, |\tilde{\nu} - \nu|$ and the relation (69) between the measures $|\nabla(\phi \circ u)|$ and μ we see

$$\begin{aligned}\left| \int A\colon \int \tilde{\nu} \otimes \tilde{\nu} \, dp_x(\tilde{\nu}) \, d\mu - \int A\colon \nu \otimes \nu \, |\nabla(\phi \circ u)| \right| &\lesssim \int \eta \, (d\mu - |\nabla(\phi \circ u)|) \\ &\quad + \int \eta \int |\nu - \tilde{\nu}| \, dp_x(\tilde{\nu}) \, |\nabla(\phi \circ u)| \, .\end{aligned}$$

By (70) the first right-hand side term is estimated by the "mild excess" (65).

We are left with proving

$$\int \eta \int |\nu - \tilde{\nu}|\, dp_x(\tilde{\nu})\, |\nabla(\phi \circ u)| \lesssim E(\chi, \eta) - \int \eta\, |\nabla(\phi \circ u)|\,. \tag{71}$$

But distributional convergence of $\nabla(\phi \circ u_\varepsilon)$ towards $\nabla(\phi \circ u)$ and (67) with $\zeta(x, \tilde{\nu}) = \xi(x) \cdot \tilde{\nu}$ yield an equality for the linear term

$$\begin{aligned}\int \xi \cdot \nu\, |\nabla(\phi \circ u)| = \int \xi \cdot \nabla(\phi \circ u) &= \lim_{\varepsilon \downarrow 0} \int \xi \cdot \nabla(\phi \circ u_\varepsilon)\, dx \\ &= \lim_{\varepsilon \downarrow 0} \int \xi \cdot \nu_\varepsilon\, |\nabla(\phi \circ u_\varepsilon)|\, dx \overset{(67)}{=} \int \xi \cdot \int \tilde{\nu}\, dp_x(\tilde{\nu})\, d\mu\end{aligned} \tag{72}$$

for any smooth test vector field $\xi \colon [0, \Lambda)^d \to \mathbb{R}^d$. This draws a connection between the normal ν and the expectation $\int \tilde{\nu}\, dp_x(\tilde{\nu})$ of the measures p_x.

Therefore for any such ξ with $|\xi| \leq \eta$ we get

$$\begin{aligned}\int \xi \cdot \int (\nu - \tilde{\nu})\, dp_x(\tilde{\nu})\, |\nabla(\phi \circ u)| &\overset{(72)}{=} \int \xi \cdot \int \tilde{\nu} dp_x(\tilde{\nu})\, (d\mu - |\nabla(\phi \circ u)|) \\ &\overset{(69)}{\leq} \|\xi\|_\infty \left(\int \eta\, d\mu - \int \eta\, |\nabla(\phi \circ u)| \right)\end{aligned}$$

and after taking the supremum over all such ξ we discover

$$\int \eta \int |\nu - \tilde{\nu}|\, dp_x(\tilde{\nu})\, |\nabla(\phi \circ u)| \leq \int \eta\, d\mu - \int \eta\, |\nabla(\phi \circ u)|\,. \tag{73}$$

Finally, notice that another application (70) proves the claim (71). □

Remark 4.2. The quantitative Reshetnyak argument (64) holds also for any other Lipschitz continuous function $f(x, \tilde{\nu})$ on S^{d-1} instead of $A(x) \colon \tilde{\nu} \otimes \tilde{\nu}$.

4.3 Convergence of the velocity-term

As in the proof of convergence in the two-phase case our main tool will be a suitable tilt-excess. However, because ∇u_ε now describes the direction of change both in physical space and in state space, some care needs to be taken in defining such an excess. It is apparent that the limiting equation only sees the direction of change in physical space explicitly. In contrast, the change of direction in state space only enters implicitly through the surface tensions, which are the lengths of geodesics connecting the wells. It is therefore natural to define an approximate tilt-excess which only fixes the change of direction in physical space.

Definition 4.3. Let $\nu^* \in S^{d-1}$ and $\eta \in C^\infty\left([0,T]\times[0,\Lambda)^d;[0,1]\right)$. For $\varepsilon > 0$ and a function $u_\varepsilon \in W^{1,2}([0,T]\times[0,\Lambda)^d;\mathbb{R}^n)$ the localized tilt-excess of the i-th phase, $1\le i\le N$, is given by

$$\mathscr{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon) := \int_0^T\int \eta\frac{1}{\varepsilon}\left|\varepsilon\nabla u_\varepsilon + \partial_u\phi_i(u_\varepsilon)\otimes\nu^*\right|^2 dxdt. \tag{74}$$

In the limit $\varepsilon = 0$ and for a partition $\chi_i = \mathbf{1}_{\Omega_i}\in BV\left([0,T]\times[0,\Lambda)^d;\{0,1\}\right)$ with $\sum_i\chi_i = 1$ we define the tilt-excess for $1\le i,j\le P$, $i\ne j$, to be

$$\mathscr{E}^{ij}(\nu^*;\eta,u) := \int_0^T\int\eta\left|\nu_i-\nu^*\right|^2\left|\nabla\chi_i\right|dt + \int_0^T\int\eta\left|\nu_j+\nu^*\right|^2\left|\nabla\chi_j\right|dt$$
$$+\sum_{k\notin\{i,j\}}\int_0^T\int\eta\left|\nabla\chi_k\right|dt, \tag{75}$$

where $u = \sum_{1\le i\le N}\alpha_i\chi_i$ and ν_i, as throughout the chapter, is the inner normal of Ω_i.

Note that the limiting excess measures two things: Firstly, the last term measures whether mostly the interface between the i-th and the j-th phase is present. Secondly, the first two terms measure how close the interface is to being flat.

A subtle point in the definition is that χ_i falls while moving out of the corresponding phase, while ϕ_i grows. Hence their differentials have opposite directions. We choose ν^* to be the approximate inner normal of χ_i, which leads to the positive sign in $\mathscr{E}^i_\varepsilon$ and the second term in $\mathscr{E}^{ij}$ and the negative one in the first term in $\mathscr{E}^{ij}$. For a similar reason the limiting excesses are not symmetric in i and j. Instead we have $\mathscr{E}^{ij}(\nu^*;\eta,u) = \mathscr{E}^{ji}(-\nu^*;\eta,u)$.

We first make sure that we can use $\mathscr{E}^{ij}(\nu^*;\eta,\chi)$ to asymptotically bound $\mathscr{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon)$.

Lemma 4.4. *Let u^ε satisfy the a priori estimate* (20) *and the convergence assumption* (9)*. Then for every $1\le i,j\le P$, $i\ne j$, $\nu^*\in S^{d-1}$ and $\eta\in C^\infty([0,T]\times[0,\Lambda)^d;[0,1])$ we have*

$$\limsup_{\varepsilon\to0}\mathscr{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon)\lesssim\mathscr{E}^{ij}(\nu^*;\eta,\chi). \tag{76}$$

Using this estimate, as in the two-phase case before, we prove (52) up to an error controlled by the tilt-excess (75).

Proposition 4.5. *Given u^ε satisfying the a priori estimate* (20) *and the convergence assumption* (9)*, there exists a finite Radon measure μ on $[0,T]\times[0,\Lambda)^d$, such that for any $1\le i,j\le P$, $i\ne j$, any parameter $\alpha>0$, any direction $\nu^*\in S^{d-1}$ and any test vector field $\xi\in C_0^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ we have*

$$\limsup_{\varepsilon\downarrow0}\left|\int_0^T\int\varepsilon(\xi\cdot\nabla)u_\varepsilon\cdot\partial_tu_\varepsilon\,dx\,dt - \sigma_{ij}\int_0^T\int\xi\cdot\nu_iV_i\frac{1}{2}\left(\left|\nabla\chi_i\right|+\left|\nabla\chi_j\right|-\left|\nabla(\chi_i+\chi_j)\right|\right)dt\right|$$
$$\lesssim\|\xi\|_\infty\left(\frac{1}{\alpha}\mathscr{E}^{ij}(\nu^*;\eta,u)+\alpha\mu(\eta)\right). \tag{77}$$

Here $\eta\in C^\infty([0,T]\times\mathbb{R}^d)$ is a smooth cut-off for the support of ξ, i.e. $\eta\ge0$ and $\eta\equiv1$ on $\operatorname{supp}\xi$.

Proof of Lemma 4.4. Expanding the square and exploiting that $|\nabla(\phi \circ u_\varepsilon)| \leq \sqrt{2W(u_\varepsilon)}$ we see that

$$\mathscr{E}_\varepsilon^i(\nu^*;\eta,u_\varepsilon) \leq \int_0^T \int \eta \left(\varepsilon |\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon} W(u_\varepsilon) + 2(\nu^* \cdot \nabla) u_\varepsilon \cdot \partial_u \phi_i(u_\varepsilon) \right) dx dt.$$

By the chain rule (24) we can rewrite the last term as

$$(\nu^* \cdot \nabla) u_\varepsilon \cdot \partial_u \phi_i(u_\varepsilon) = \nu^* \cdot \nabla(\phi_i \circ u_\varepsilon).$$

Thus we see using the convergence assumption (9) and the convergence (23) of $\phi_i \circ u_\varepsilon$ to $\phi_i \circ u$ that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \mathscr{E}_\varepsilon^i(\nu^*;\eta,u_\varepsilon) \leq & \limsup_{\varepsilon \to 0} 2 \int_0^T \int \eta \left(e_\varepsilon(u_\varepsilon) + \nu^* \cdot \nabla(\phi_i \circ u_\varepsilon) \right) dx\, dt \\ = & 2 \int_0^T \left(E(u,\eta) + \nu^* \cdot \int \eta \nabla(\phi_i \circ u) \right) dt. \end{aligned} \tag{78}$$

The second term can be rewritten as

$$\nu^* \cdot \nabla(\phi_i \circ u) = \nu^* \cdot \sum_{1 \leq k \leq P} \sigma_{ik} \nabla \chi_k \leq \sigma_{ij}\, \nu^* \cdot \nabla \chi_j + \sum_{k \notin \{i,j\}} \sigma_{ik} \left| \nabla \chi_k \right|,$$

while the first one can be estimated by

$$E(u,\eta) \leq \sigma_{ij} \int \eta \left| \nabla \chi_j \right| + C \sum_{k \notin \{i,j\}} \int \eta \left| \nabla \chi_k \right|$$

for some constant $C < \infty$ only depending on $\max_{ij} \sigma_{ij}$. Thus we can asymptotically bound the excess by

$$\limsup_{\varepsilon \to 0} \mathscr{E}_\varepsilon^i(\nu^*;\eta,u_\varepsilon) \leq \sigma_{ij} \int_0^T \int \eta\, 2\left(1 + \nu_j \cdot \nu^*\right) \left| \nabla \chi_j \right| dt + C \sum_{k \notin \{i,j\}} \int_0^T \int \eta \left| \nabla \chi_k \right| dt.$$

Since $2\left(1 + \nu_j \cdot \nu^*\right) = |\nu_j + \nu^*|^2$ in particular (76) holds. Note that we symmetrized the multi-phase excess (54) w.r.t. the two majority phases Ω_i and Ω_j which means we added an extra (non-negative) term. $\square$

Proof of Proposition 4.5. Step 1: Replacing ∇u_ε with $\partial_u \phi_i(u_\varepsilon) \otimes \nu^*$. Using the tilt-excess (74) and Young's inequality we see

$$\begin{aligned} & \left| \int_0^T \int \left(\varepsilon (\xi \cdot \nabla) u_\varepsilon + \xi \cdot \nu^* \partial_u \phi_i(u_\varepsilon) \right) \cdot \partial_t u_\varepsilon \, dx\, dt \right| \\ & \qquad \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha} \mathscr{E}_\varepsilon^i(\nu^*;\eta,u_\varepsilon) + \alpha \int_0^T \int \eta \varepsilon |\partial_t u_\varepsilon|^2 dx dt \right). \end{aligned} \tag{79}$$

By the energy-dissipation equality (19) the sequence $\varepsilon|\partial_t u_\varepsilon|^2$ is bounded in L^1 and thus, along a subsequence, has a weak*-limit μ as Radon measures. In the limit we get, applying Lemma 4.4 along the way,

$$\limsup_{\varepsilon\downarrow 0}\left|\int_0^T\int \left(\varepsilon(\xi\cdot\nabla)u_\varepsilon+\xi\cdot\nu^*\partial_u\phi_i(u_\varepsilon)\right)\cdot\partial_t u_\varepsilon\,dx\,dt\right| \lesssim \|\xi\|_\infty\left(\frac{1}{\alpha}\mathscr{E}^{ij}(\nu^*;\eta,u)+\alpha\mu(\eta)\right).$$

Step 2: Passing to the limit in the nonlinear term. In the second term on the left-hand side of (79) we may now use the chain rule again to see

$$\begin{aligned}-\int_0^T\int \xi\cdot\nu^*\partial_u\phi_i(u_\varepsilon)\cdot\partial_t u_\varepsilon dxdt &= -\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ u_\varepsilon\right)dxdt\\ &\to -\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ\sum_{k=1}^P\chi_k\alpha_k\right)dt.\end{aligned}$$

Step 3: Rewriting the limit in terms of the interface between χ_i and χ_j. We can rewrite this limit to read

$$\begin{aligned}\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ\sum_{k=1}^P\chi_k\alpha_k\right)dt \;&=\; -\int_0^T\int \xi\cdot\nu^*\sum_{k=1}^P\sigma_{ik}\partial_t\chi_k\\ &\overset{3.10}{=} -\int_0^T\int \xi\cdot\nu^*\sum_{k=1}^P\sigma_{ik}V_k|\nabla\chi_k|dt.\end{aligned}$$

Thanks to the tilt-excess (75) we can now get rid of all terms except the j-th one: With a little help from our friends Cauchy, Schwarz and Young we arrive at

$$\begin{aligned}&\left|-\int_0^T\int \xi\cdot\nu^*\sum_{k=1}^P\sigma_{ik}V_k|\nabla\chi_k|dt+\int_0^T\int \xi\cdot\nu^*\sigma_{ij}V_j\left|\nabla\chi_j\right|dt\right|\\ &\qquad\lesssim \|\xi\|_\infty\left(\frac{1}{\alpha}\mathscr{E}^{ij}(\nu^*;\eta,u)+\alpha\int_0^T\int\eta\sum_{k=1}^P V_k^2|\nabla\chi_k|dt\right)\end{aligned}$$

for a smooth cut-off η for the support of ξ. Here, due to the L^2-estimate Proposition 3.10, the right-hand side is an acceptable error term after redefining μ.

Hence we are left with a term only depending on the j-th phase which we can replace with (minus) the according term for the i-th phase: Indeed, using $\sum_k\chi_k=1$ the error in doing so is equal to

$$\begin{aligned}\left|\int_0^T\int \xi\cdot\nu^*\sigma_{ij}\left(V_j\left|\nabla\chi_j\right|+V_i\left|\nabla\chi_i\right|\right)dt\right| &= \left|\int_0^T\int \xi\cdot\nu^*\sigma_{ij}\partial_t\left(1-\sum_{k\notin\{i,j\}}\chi_k\right)dt\right|\\ &\lesssim \int_0^T\int|\xi|\sum_{k\notin\{i,j\}}\left|V_k\right|\left|\nabla\chi_k\right|dt,\end{aligned}$$

which by Young's inequality is controlled by the same right-hand side as before.

Exploiting $|\nu^* - \nu_i|\,|V_i| \lesssim \frac{1}{\alpha}|\nu^* - \nu_i|^2 + \alpha|V_i|$ we now use the tilt-excess once again to "unfreeze" the approximate normal ν^* and eliminate other interfaces:

$$\left|\int_0^T \int \xi \cdot \nu^* \sigma_{ij} V_i \,|\nabla \chi_i|\, dt - \int_0^T \int \xi \cdot \nu_i \sigma_{ij} V_i \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt\right|$$
$$\lesssim \|\xi\|_\infty \left(\frac{1}{\alpha}\mathscr{E}^{ij}(\nu^*; \eta, u) + \alpha \int_0^T \int \eta\, V_i^2 |\nabla \chi_i| dt\right).$$

Retracing our steps we see that we arrived at the desired estimate. □

We conclude this section with the proof of our main result.

Proof of Theorem 2.2. We found the limit u of the approximations u_ε in Proposition 3.7, verified the initial conditions in Lemma 3.9 and constructed the normal velocity with the according L^2-bounds in Proposition 3.10. We only have to prove the motion law (11). Given a smooth test vector field $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$, by Lemma 3.3 we may multiply the Allen-Cahn Equation (1) by $\varepsilon\,(\xi \cdot \nabla)\, u_\varepsilon$ and integrate w.r.t. space and time:

$$\int_0^T \int \varepsilon (\xi \cdot \nabla)\, u_\varepsilon \cdot \partial_t u_\varepsilon \,dx\,dt = \int_0^T \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right) \cdot (\xi \cdot \nabla)\, u_\varepsilon \,dx\,dt. \tag{80}$$

By Proposition 4.1 the convergence of the energies (31) imply the convergence of the first variations for a.e. t. Recall that by (60) and Lebesgue's Dominated Convergence the right-hand side of (80) converges:

$$\lim_{\varepsilon \downarrow 0} \int_0^T \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right) \cdot (\xi \cdot \nabla)\, u_\varepsilon \,dx\,dt$$
$$= \sum_{i,j} \sigma_{ij} \int_0^T \int \nabla \xi \colon \left(Id - \nu_i \otimes \nu_i\right) \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt.$$

In order to prove the convergence of the left-hand side, we proceed as in Chapter 1. We decompose $\xi = \sum_{B \in \mathscr{B}_r} \varphi_B \xi$ with a partition of unity underlying the covering $\mathscr{B}_r$ defined in (34). Using Proposition 4.5 for $\xi_B = \varphi_B \xi$ on time intervals $0 = T_1 < \ldots < T_K = T$ and passing to the limit $K \to \infty$ we obtain the error

$$\left|\int_0^T \int \varepsilon (\xi \cdot \nabla) u_\varepsilon \cdot \partial_t u_\varepsilon \,dx\,dt - \sum_{i,j} \sigma_{ij} \int_0^T \int V_i\, \xi \cdot \nu_i \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) dt\right|$$
$$\lesssim \|\xi\|_\infty \Bigg(\frac{1}{\alpha} \int_0^T \sum_{B \in \mathscr{B}_r} \min_{i,j} \min_{\nu^* \in S^{d-1}} \int \eta_B \,|\nu_i - \nu^*|^2\, |\nabla \chi_i| + \int \eta_B \,|\nu_j + \nu^*|^2\, |\nabla \chi_j|$$
$$+ \sum_{k \notin \{i,j\}} \int \eta_B\, |\nabla \chi_k|\, dt + \alpha \int_0^T \int \sum_{B \in \mathscr{B}_r} \eta_B \,d\mu\Bigg),$$

where for a ball B the function η_B denotes a cutoff for B in $2B$ as in equation (33). Because of the finite overlap (35) the last term is uniformly bounded in r. Using Lemma 3.12 we see that the first term vanishes as $r \to 0$. Then taking $\alpha \to 0$ we obtain the convergence of the velocity-term and thus verified the motion law (11). □

5 Forces and volume constraint

The proofs in Section 3 and Section 4 stem from the a priori estimate (19) and the convergence assumption (9). We mostly used the Allen-Cahn Equation (1) to prove this a priori bound. Besides that we made use of it only at one other point, in the proof of Theorem 2.2 in the form of (80) and the justification for testing the equation with $\varepsilon(\xi \cdot \nabla)u_\varepsilon$.

In this section we exploit this flexibility of our proof and apply it to the case when external forces are present or when a volume constraint is active, cf. Theorem 2.4 and Theorem 2.5, respectively.

5.1 External forces

Since the forces f_ε in equation (14) come from an extra energy-term we do not expect to have the same energy-dissipation equality as in the case above where $f_\varepsilon \equiv 0$. Indeed, one can view (14) as the (again by the factor $\frac{1}{\varepsilon}$ accelerated) L^2-gradient flow of the total energy

$$E_\varepsilon(u_\varepsilon) + \int f_\varepsilon \cdot u \, dx,$$

which is the sum of the "surface energy" $E_\varepsilon(u_\varepsilon)$ and the "bulk energy" $\int f_\varepsilon \cdot u \, dx$. Since the extra term is a compact perturbation in the static setting, these total energies Γ-converge to

$$E(u) + \int f \cdot u \, dx.$$

This energetic view-point seems also the most natural way to understand the scaling in ε for the forces f_ε in equation (14). Under our assumption on the forces f_ε in Theorem 2.4 we can control this bulk energy and get an *estimate* on the "surface energy" $E_\varepsilon(u_\varepsilon)$ and the dissipation, which is reminiscent of equality (19).

Lemma 5.1. *Let u_ε solve the forced Allen-Cahn Equation* (14). *Then for $\varepsilon \ll 1$ we have*

$$E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx \, dt$$
$$\lesssim \left(1 + e^{C\varepsilon T}\right) \left(1 + T + E_\varepsilon(u_\varepsilon(0)) + \frac{1}{T} \|f_\varepsilon\|_{L^2}^2 + (1+T) \|\partial_t f_\varepsilon\|_{L^2}^2\right).$$

With $\varepsilon \ll 1$ we mean that we assume $\varepsilon \leq \frac{1}{C}$ for some generic constant C. Note that the exponential prefactor is ~ 1 for small ε.

Proof of Lemma 5.1. We differentiate the energy E_ε along the trajectory of $t \mapsto u_\varepsilon(t)$ and integrate by parts

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(u_\varepsilon) &= \int \varepsilon \nabla u_\varepsilon : \nabla \partial_t u_\varepsilon + \frac{1}{\varepsilon}\partial_u W(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \\ &= \int \varepsilon \left(-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\right) \cdot \partial_t u_\varepsilon \, dx \\ &\overset{(14)}{=} -\int \varepsilon |\partial_t u_\varepsilon|^2 \, dx + \int f_\varepsilon \cdot \partial_t u_\varepsilon \, dx.\end{aligned}$$

We integrate from 0 to T and obtain

$$E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx \, dt = E_\varepsilon(u_\varepsilon(0)) + \int_0^T \int f_\varepsilon \cdot \partial_t u_\varepsilon \, dx \, dt. \tag{81}$$

Now we want to integrate the right-hand side integral by parts. First note that by the trace theorem for a.e. t we have

$$\int |f_\varepsilon(t)|^2 \, dx \lesssim \frac{1}{T}\int_0^T \int |f_\varepsilon|^2 \, dx \, dt + T \int_0^T \int |\partial_t f_\varepsilon|^2 \, dx \, dt,$$

which we may assume w.l.o.g. for $t = 0$ and $t = T$ so that by Young's inequality

$$\begin{aligned}\left|\int_0^T \int f_\varepsilon \cdot \partial_t u_\varepsilon \, dx \, dt\right| &\leq \int |f_\varepsilon(T)| \, |u_\varepsilon(T)| \, dx + \int |f_\varepsilon(0)| \, |u_\varepsilon(0)| \, dx + \int_0^T \int |\partial_t f_\varepsilon| \, |u_\varepsilon| \, dx \, dt \\ &\lesssim \int |u_\varepsilon(T)|^2 \, dx + \int |u_\varepsilon(0)|^2 \, dx + \int_0^T \int |u_\varepsilon|^2 \, dx \, dt \\ &\quad + \frac{1}{T}\int_0^T \int |f_\varepsilon|^2 \, dx \, dt + (1+T)\int_0^T \int |\partial_t f_\varepsilon|^2 \, dx \, dt.\end{aligned}$$

By the coercivity assumption (3) of W at infinity we have

$$\int |u_\varepsilon|^2 \, dx \lesssim 1 + \varepsilon E_\varepsilon(u_\varepsilon).$$

Plugging these two observations into (81), for $\varepsilon \ll 1$ we can absorb the term $\varepsilon E_\varepsilon(u_\varepsilon(T))$ and obtain

$$\begin{aligned}E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx \, dt \lesssim & 1 + T + E_\varepsilon(u_\varepsilon(0)) + \varepsilon \int_0^T E_\varepsilon(u_\varepsilon) \, dt \\ &+ \frac{1}{T}\int_0^T \int |f_\varepsilon|^2 \, dx \, dt + (1+T)\int_0^T \int |\partial_t f_\varepsilon|^2 \, dx \, dt\end{aligned}$$

and a Gronwall argument yields the claim. □

This estimate is indeed enough to apply our techniques to the case of (14).

Proof of Theorem 2.4. As noted in Remark 3.6, the a priori estimate, Lemma 5.1, allows us to apply the statements in Section 3 so that in particular we can find a convergent subsequence $u_\varepsilon \to u$ satisfying the initial conditions by Lemma 3.9, for some $u = \sum_i \chi_i \alpha_i$, and we can construct the normal velocities under the convergence assumption (9). The bounds for f_ε allow us to extract a further subsequence such that also the forces converge to some $f \in H^1((0,T)\times[0,\Lambda)^d, \mathbb{R}^N)$:

$$f_\varepsilon \to f \quad \text{in } L^2 \quad \text{and} \quad \nabla f_\varepsilon \rightharpoonup \nabla f \quad \text{in } L^2. \tag{82}$$

If we formally differentiate the equation (14) and use $\nabla f_\varepsilon \in L^2$ we can show as in Step 2 of the proof of Lemma 3.3 that $\partial_i \partial_j u_\varepsilon, \partial_u W(u_\varepsilon) \in L^2$. Hence we are allowed to test the equation for u_ε, here the forced Allen-Cahn Equation (14), with $\varepsilon\,(\xi\cdot\nabla)\,u_\varepsilon$ to obtain

$$\int_0^T \int \varepsilon(\xi\cdot\nabla)\,u_\varepsilon\cdot\partial_t u_\varepsilon\,dx\,dt = \int_0^T \int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right)\cdot(\xi\cdot\nabla)\,u_\varepsilon + f_\varepsilon\cdot(\xi\cdot\nabla)\,u_\varepsilon\,dx\,dt.$$

Integrating the last term by parts gives

$$\int_0^T \int f_\varepsilon\cdot(\xi\cdot\nabla)\,u_\varepsilon\,dx\,dt = -\int_0^T \int (\nabla\cdot\xi)\,f_\varepsilon\cdot u_\varepsilon + (\xi\cdot\nabla)\,f_\varepsilon\cdot u_\varepsilon\,dx\,dt.$$

Since $u_\varepsilon \to u = \sum_i \chi_i\alpha_i$ in L^2 and (82) we can pass to the limit $\varepsilon\to 0$ and obtain

$$\int_0^T \int (\nabla\cdot\xi)\,f\cdot u + (\xi\cdot\nabla)\,f\cdot u\,dx\,dt = \sum_{i=1}^P \int_0^T \int (f\cdot\alpha_i)\,(\xi\cdot\nu_i)\,|\nabla\chi_i|\,dt.$$

We can apply Proposition 4.1 to pass to the limit in the curvature-term. For the velocity-term we may apply Proposition 4.5 and follow the lines of the proof of Theorem 2.2 for the localization argument. We thus verified (15). □

5.2 Volume constraint

Again, our starting point is an energy-dissipation estimate. It is quite natural that the solution of the volume-preserving Allen-Cahn Equation (16) satisfies the same energy-dissipation equation as the solution of the Allen-Cahn Equation (1).

Lemma 5.2. *Let u_ε solve the volume-preserving Allen-Cahn Equation* (16). *Then*

$$E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon\,|\partial_t u_\varepsilon|^2\,dx\,dt = E_\varepsilon(u_\varepsilon(0)). \tag{83}$$

Proof of Lemma 5.2. We follow the lines of the proof of Lemma 5.1 until (81) with $f_\varepsilon(x,t)$ replaced by $\lambda_\varepsilon(t)$. Since λ_ε is independent of x for the second right-hand side integral in (81) we have

$$\int_0^T \int \lambda_\varepsilon \, \partial_t u_\varepsilon \, dx \, dt = \int_0^T \lambda_\varepsilon \frac{d}{dt} \int u_\varepsilon \, dx \, dt.$$

But by the choice of λ_ε integrating (16) gives $\frac{d}{dt} \int u_\varepsilon \, dx = 0$ and we obtain (83). □

Proof of Theorem 2.5. Since we have the same energy-dissipation estimate, Lemma 5.2, as in the unconstrained case, by Remark 3.6 we can apply the statements in Section 3 so that in particular we obtain a convergent subsequence $u_\varepsilon \to u$ as before and we can construct the normal velocities under the convergence assumption (9).

The Lagrange multiplier λ_ε does not depend on the space variable x and hence the same computation as in Step 2 in the proof of Lemma 3.3 yields $\partial_i \partial_j u_\varepsilon, \partial_u W(u_\varepsilon) \in L^2$ and we may test our equation (16) with $\varepsilon\,(\xi \cdot \nabla)\, u_\varepsilon$ and obtain

$$\begin{aligned}\int_0^T \int \varepsilon(\xi \cdot \nabla)\, u_\varepsilon \cdot \partial_t u_\varepsilon \, dx \, dt \\ = \int_0^T \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right) \cdot (\xi \cdot \nabla)\, u_\varepsilon \, dx \, dt + \int_0^T \lambda_\varepsilon \int (\nabla \cdot \xi)\, u_\varepsilon \, dx \, dt.\end{aligned}$$

We wish to pass to the limit in this weak formulation of (16).

By Proposition 4.1 we can pass to the limit in the first right-hand side term and the left-hand side term. Again, with Proposition 4.5 and the localization argument in the proof of Theorem 2.2 we can pass to the limit on the left-hand side. In order to pass to the limit in the second right-hand side term we use Proposition 5.3 below, which provides control of λ_ε in L^2. After passage to a further subsequence if necessary we have

$$\lambda_\varepsilon \rightharpoonup \lambda \quad \text{weakly in } L^2(0,T)$$

and since by Lemma 3.9

$$\int (\nabla \cdot \xi)\, u_\varepsilon \, dx \to \int (\nabla \cdot \xi)\, u \, dx \quad \text{strongly in } L^2(0,T)$$

we can pass to the limit in the product. This concludes the proof of the theorem. □

Proposition 5.3 (Estimates on Lagrange multiplier)**.** *Let* u_ε *solve* (16) *and let* λ_ε *be the Lagrange multiplier* (17)*. Then*

$$\limsup_{\varepsilon \to 0} \int_0^T \lambda_\varepsilon^2 \, dt \lesssim (1+T) E_0.$$

Proof of Proposition 5.3. We follow the idea of the proof of Proposition 2.12 in Chapter 3. For a given test vector field $\xi \in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^d)$ we first multiply (16) by $\varepsilon\,(\xi\cdot\nabla)\,u_\varepsilon$, integrate in space and take the square:

$$\lambda_\varepsilon^2\left(\int (\nabla\cdot\xi)\,u_\varepsilon\,dx\right)^2 \lesssim \left(\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right)\cdot(\xi\cdot\nabla)\,u_\varepsilon\,dx\right)^2 + \left(\int \varepsilon(\xi\cdot\nabla)\,u_\varepsilon\cdot\partial_t u_\varepsilon\,dx\right)^2.$$

With Cauchy-Schwarz we can estimate the second right-hand side term

$$\left(\int \varepsilon(\xi\cdot\nabla)\,u_\varepsilon\cdot\partial_t u_\varepsilon\,dx\right)^2 \lesssim \|\xi\|_\infty^2\left(\varepsilon\int|\partial_t u_\varepsilon|^2\,dx\right)E_\varepsilon(u_\varepsilon).$$

For the first right-hand side term we use (60) to obtain

$$\left(\int \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right)\cdot(\xi\cdot\nabla)\,u_\varepsilon\,dx\right)^2 \lesssim \|\nabla\xi\|_\infty^2 E_\varepsilon(u_\varepsilon)^2.$$

Since $\nabla\cdot\xi$ is orthogonal to constant functions we might subtract the average $\langle u_\varepsilon\rangle := \frac{1}{\Lambda^d}\int u_\varepsilon\,dx$ of u_ε on the left-hand side and obtain

$$\lambda_\varepsilon^2\left(\int(\nabla\cdot\xi)\,(u_\varepsilon-\langle u_\varepsilon\rangle)\,dx\right)^2 \lesssim \|\nabla\xi\|_\infty^2 E_\varepsilon(u_\varepsilon)^2 + \|\xi\|_\infty^2\left(\varepsilon\int|\partial_t u_\varepsilon|^2\,dx\right)E_\varepsilon(u_\varepsilon).$$

We integrate in time and apply the energy-dissipation estimate (5.2) on the right-hand side:

$$\int_0^T \lambda_\varepsilon^2\left(\int(\nabla\cdot\xi)\,(u_\varepsilon-\langle u_\varepsilon\rangle)\,dx\right)^2 dt \lesssim \sup_t\|\xi\|_{W^{1,\infty}}^2\,(1+T)\,E_0^2.$$

Hence it is enough to find a test field ξ such that we can bound the left-hand side integral from below while the right-hand side stays uniformly bounded:

$$\frac{1}{\Lambda^d}\int(\nabla\cdot\xi)\,(u_\varepsilon-\langle u_\varepsilon\rangle)\,dx \geq \frac{1}{2} \quad\text{and} \tag{84}$$

$$\|\xi\|_{W^{1,\infty}} \lesssim 1+E_0. \tag{85}$$

We now proceed by constructing a vector field ξ satisfying (84) and (85) in a similar manner as in Chapter 3. To this end we first fix some $t\in(0,T)$, convolve the limit $u=\lim u_\varepsilon$ with a standard mollifier $\varphi_\delta(x)=\frac{1}{\delta^d}\varphi(\frac{x}{\delta})$ on scale $\delta>0$ (to be chosen later) and write $u_\delta := \varphi_\delta * u$. Then we let $v\colon[0,\Lambda)^d\to\mathbb{R}$ denote the solution of

$$\Delta v = \varphi_\delta*(u-\langle u\rangle) = u_\delta - \langle u\rangle. \tag{86}$$

Note that since the right-hand side has vanishing integral, this problem is well-posed. We set $\xi := \nabla v$ and verify (84) which works by construction of ξ and (85) which boils down to elliptic estimates.

Step 1: Argument for the lower bound (84). By Lemma 3.9 we have $u_\varepsilon \to u$ in $C_t L^2_x$ as $\varepsilon \to 0$. Thus

$$\inf_t \int (\nabla \cdot \xi)\,(u_\varepsilon - \langle u_\varepsilon \rangle)\, dx = \inf_t \left\{ \int (u - \langle u \rangle)^2\, dx + \int (u_\delta - u)\,(u - \langle u \rangle)\, dx \right\} + o(1),$$

as $\varepsilon \to 0$. Since $u = \sum_i \chi_i \alpha_i$ we have for the first left-hand side integral

$$\int (u - \langle u \rangle)^2\, dx = \int \left(u - \langle u^0 \rangle\right)^2 dx \geq \text{dist}\left(\langle u^0 \rangle, \{\alpha_1, \alpha_2\}\right)^2 \Lambda^d.$$

The second left-hand side integral can be estimated with help of the energy (7):

$$\left| \int (u_\delta - u)\, u\, dx \right| \lesssim \int |u_\delta - u|\, dx \leq \delta \int |\nabla u| \lesssim \delta\, E(u) \leq \delta\, E_0.$$

Setting $\delta := \frac{1}{C} \frac{1}{E_0} \text{dist}\left(\langle u^0 \rangle, \{\alpha_1, \alpha_2\}\right)^2 \Lambda^d > 0$ for some sufficiently large constant $C < \infty$, we arrive at (84) for sufficiently small ε.

Step 2: Argument for the estimate (85). The upper bound (85) follows from basic elliptic regularity theory. We fix some exponent $q = q(d) > d$. Since $u = \sum_i \chi_i \alpha_i$ is uniformly bounded, the Calderón-Zygmund inequality yields

$$\int |\nabla \xi|^q\, dx \lesssim \int |u_\delta - \langle u_\delta \rangle|^q\, dx \lesssim 1.$$

Since the right-hand side is smooth, we can differentiate the equation (86) for v and obtain:

$$\Delta \xi = \nabla u_\delta$$

and we obtain again by Calderón-Zygmund

$$\left(\int \left|\nabla^2 \xi\right|^q dx \right)^{\frac{1}{q}} \lesssim \left(\int |\nabla u_\delta|^q\, dx \right)^{\frac{1}{q}} \lesssim \int |\nabla \varphi_\delta|\, dx \lesssim \frac{1}{\delta}.$$

Since $\langle \xi \rangle = 0$ we thus have by Poincaré's inequality $\|\xi\|_{W^{2,q}} \lesssim \frac{1}{\delta}$ and since $q > d$ Morrey's inequality yields

$$\|\xi\|_{W^{1,\infty}} \lesssim 1 + \frac{1}{\delta} \sim 1 + E_0,$$

which is precisely our claim (85). $\square$

Outlook

In this thesis we proved conditional convergence results for several schemes modeling multi-phase mean-curvature flow and related equations. However, various questions concerning the rigorous asymptotic analysis of such schemes remain open. We list some relevant open problems which might or might not be answered in the future.

1. Extensions of our techniques to anisotropic motions seem feasible.
2. The analysis in [85] shows that the convergence of the energies is guaranteed for the two-phase thresholding scheme as long as the evolution is smooth. Do the energies converge for mean convex initial conditions? A similar question has been raised by Ilmanen [46] for the Allen-Cahn Equation but seems to be still unresolved.
3. Ilmanen [46] proved the convergence of the Allen-Cahn Equation towards Brakke's varifold solution without any further assumptions. Whether we can use similar techniques to drop our assumption in the case of the (two-phase) thresholding scheme is an urging question. The main difficulty in Ilmanen's proof is the equipartition of energy
$$\int_0^T \int \zeta \left(\frac{\varepsilon}{2} \left|\nabla u_\varepsilon\right|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx\, dt \to 0 \quad \text{for all smooth } \zeta.$$
The analogue for the thresholding scheme is
$$\int_0^T \int \zeta \, \frac{1}{\sqrt{h}} (1-\chi^h) \left(G_h - h\Delta G_h\right) * \chi^h \, dx\, dt \to 0 \quad \text{for all smooth } \zeta.$$
4. A generalization of Ilmanen's proof to the multi-phase case is a long-standing open problem and seems still out of reach – let alone an unconditional convergence result for the multi-phase thresholding scheme. However, the methods of Kim and Tonegawa [50] might give new insights.
5. The generalization [79] of the thresholding scheme to higher codimensions, e.g. a filament in $\mathbb{R}^3$, preserves the gradient flow structure of the limiting motion in the sense that it comes with a minimizing movements interpretation. This might amount to a rigorous convergence proof similar to [12] for the Ginzburg-Landau Equation.

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Bibliographische Daten

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