Isospectral nearly Kähler manifolds

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Chapter 0

Introduction

Nearly Kähler manifolds were first discovered and intensively studied by Alfred Gray in his works [Gr] and [Gr2] towards the notion of *weak holonomy*.[1] They also appear naturally as a class of almost Hermitian manifolds in the celebrated Grey-Hervella classification [Gr3]. In this context, they are defined as follows:

Definition 0.0.1. An almost Hermitian manifold (M, g, J) is said to be (strict) *nearly Kähler* if $(\nabla_X J)X = \nabla_X(JX) - J(\nabla_X X) = 0$ for any vector field $X \in \Gamma(TM)$ and it has no non-trivial local Kähler de Rham factor. Here ∇ denotes the Riemannian connection and $J : \Gamma(TM) \longrightarrow \Gamma(TM)$ denotes the underlying almost complex structure.

Although nearly Kähler manifolds are well-known to exhibit prominent geometric properties, this class of manifolds is not well understood. Deep results of Clayton and Swann concerning G-structures enabled P.A. Nagy to give a proof in [Na] that a complete simply connected nearly Kähler manifold is a product of

(i) twistor spaces of quaternionic Kähler manifolds of positive scalar curvature,

(ii) homogeneous spaces, and

(iii) six-dimensional nearly Kähler manifolds.

The lack of inhomogeneous examples is the most prominent problem to investigate this rich type of geometry further. Indeed, the only non- (locally) homogeneous examples that are known up to date are the cohomogeneity one manifolds constructed very recently by Marc Haskins and Lorenzo Foscolo [FoHa].

On the one hand, the twistor construction gives just finitely many examples (all of them homogeneous) of nearly Kähler manifolds, see page 110 in [LS]

Theorem 0.0.2. *There are finitely many isometry classes of positive quaternion Kähler manifolds up to homothety in each quaternionic dimension.*

On the other, there is a complete list of homogeneous nearly Kähler manifolds obtained by Francisco M. Cabrera and José C. Dávila in higher dimensions [CG, CG2] and

[1] This notion was later on shown not to lead to new geometric structures.

J. Butruille [But, But2] in dimension six. The scarcity of examples of nearly Kähler manifolds is thus apparent and constitute the main difficulty to attack even the most elementary questions concerning the spectrum of geometrically significant operators defined on these manifolds. Our aim in this thesis is *the explicit constructions of isospectral nearly Kähler pairs.*

The construction of isospectral nearly Kähler manifolds having different local geometries is already an intractable problem. For this reason, it is sensible for our purposes to consider manifolds having the same local geometry, i.e. finite quotients of a given (compact) nearly Kähler manifold that inherit the underlying geometric structure of their common covering. In this spirit we have a double task. First of all, we must build a good list of nearly Kähler manifolds having non-trivial fundamental group and have the same universal covering, and then we must assure that the pair of quotients we chose are isospectral for the operator in consideration and are not (holomorphically) isometric.

This thesis is organized as follows. In Chapter 1 we introduce the material on finite group representations and cohomology that will serve us later on to construct suitable groups within the transformation group of a nearly Kähler structure that is yet to be chosen. Besides describing the unitary dual of the symmetric group and results that allow us to compute symmetric characters in a systematical manner, we introduce the notion of a spin structure for a finite group representation (ρ, Γ) in Section 1.3 and give the obstruction for such representation to be *spinnable*, see Theorem 1.3.1 and Section 1.3 for the definition of this term.

Theorem 0.0.3. *Consider a representation* $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ *of a finite group* Γ. *The following statements are equivalent:*

(i) The canonical extension L *of* (ρ, Γ) *is trivial.*

(ii) The cohomology class $w_2(\rho, \Gamma) \in H^2(\Gamma, \mathbb{Z}_2)$ *vanishes.*

(iii) The representation $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ *is spinnable.*

This result is not available in the literature to the best knowledge of the author. However, it should be essentially known.

Note at this point that we have aimed to make a reasonably self-contained exposition of the material concerning finite groups. Reason for this is the fact that the underlying theory of this chapter can hardly be considered part of the common knowledge of the reading audience this thesis is targeted at, and so an expository effort was made. After recalling some basic facts on Clifford algebras and spin geometry in the setting of (compact) Lie groups, we introduce the notion of *Sunada Lie group*, see Definition 2.2.2, and proceed in Section 2.2 to use all the machinery developed in Chapter 1 to show one of the main technical results of this thesis:

Theorem 0.0.4. *Let* $\Gamma = \mathrm{Sym}(m)$ *for* $6 \neq m \geq 4$ *and let* $\rho : \Gamma \longrightarrow \mathrm{O}(n)$ *be a faithful irreducible representation of dimension* $n = n(\rho)$ *whose character* $\chi = \chi(\rho)$ *fulfills the* *following conditions*

$$\chi(x) = n \mod 8, \tag{1}$$
$$\chi(xy) = n \mod 8, \tag{2}$$
$$M(z) := \frac{1}{l}\sum_{k=0}^{l-1}(-1)^k\chi(z^k) > 0, \tag{3}$$

where $x, y \in \Gamma$ *are any disjoint transpositions and* $z \in \Gamma$ *is an arbitrary odd permutation with order* $l = l(z) = \mathrm{ord}(z)$. *The following statements hold.*

(a) *The group* $\mathrm{Spin}(n)$ *is Sunada provided* $n = n(\rho)$ *is odd.*

(b) *The group* $\mathrm{Pin}(n)$ *is Sunada.*

The hypotheses within Theorem 0.0.4 are rather restrictive. However, there is a long list of irreducible representations of the symmetric group satisfying such conditions, see Example 2.2.7.

In contrast to higher dimensional spin groups, lower dimensional ones turn out not to contain almost conjugate finite subgroups that are not conjugate. This is analyzed in Section 2.3, where we show that *almost conjugate*, see Definition 2.2.1, subgroups of Spin(4) must be conjugate. Chapter 2 ends up with the proof of a formula to compute the volume of a compact Lie group that is based on Weyl integration formula and whose proof has a more differential geometric flavor than the ones available in the literature.

Proposition 0.0.5. *Let* G *be a compact, connected Lie group, let* $T \subset G$ *be a maximal torus and* $W(G)$ *be the Weyl group of* G. *The volume of* G *with respect to a metric induced by an Ad-invariant scalar product* B *on* $\mathfrak{g}$ *is given by*

$$\mathrm{vol}\,(G,B) = 2^{\dim(T)}|W(G)|\pi^{\frac{\dim(T)+\dim(G)}{2}}\left(\sqrt{|\det(b^{-1}(\epsilon_\mu,\epsilon_\nu))|}\; e^{-\frac{1}{4}\Delta}|_0\delta_{\mathfrak{g}}\right)^{-1},$$

where (ϵ_μ) *is a basis of* $\mathfrak{t}^*$ *dual to a basis of* $\frac{1}{2\pi}P(G)$, *where* $P(G) = \ker(\exp : \mathfrak{t} \longrightarrow T)$ *is the weight lattice, and* $\delta_{\mathfrak{g}}$ *denotes the product of the product of the roots of* $\mathfrak{g}^{\mathbb{C}}$ *and* b^{-1} *is the scalar product of* $\mathfrak{t}^*$ *induced by* B. Δ *is the unique second order linear differential operator with constant coefficients acting on the polynomial algebra* $\mathbb{C}[\varepsilon_1,\dots,\varepsilon_m]^{\mathrm{Sym}(m)}$ *in the simple roots* $\varepsilon_1,\dots,\varepsilon_m \in \mathfrak{t}^*$ *of* $\mathfrak{g}^{\mathbb{C}}$, *such that* $-\Delta\varepsilon_j^2 = 2b^{-1}(\varepsilon_j,\varepsilon_j)$ *and* $\Delta\varepsilon_j^{2k+1} = 0$ *for any root* $\varepsilon_j \in \mathfrak{t}^*$.

Chapter 3 starts by making a choice of the compact homogeneous nearly Kähler manifold on which we would like to find isospectral finite quotients and calculating its transformation group, see Lemma 3.2.2. We proceed in Section 3.2.2 to put all the pieces we have manufactured in Chapter 1 and 2 to show the main statement of this thesis.

Theorem 0.0.6. *There is a strictly increasing sequence of numbers* $(d_n)_{n\in\mathbb{N}}$ *such that for each* $n \in \mathbb{N}$ *there is a pair* $M_1^{d_n}$ *and* $M_2^{d_n}$ *of non-isometric nearly Kähler manifolds that are isospectral for the Dirac and the Hodge-Laplace operator* Δ^k *for* $k = 0, 1, \dots, \dim(M)$.

The chapter ends up with an approximation to the problem of finding isospectral pairs in dimension six. As a consequence of our classification of almost conjugate subgroups of Spin(4) and our volume formula for compact Lie groups obtained in Chapter 2, we conclude that the ansatz to construct isospectral pairs in higher dimensions does in fact *not* produce isospectral pairs in dimension six. More exactly, we show the following result by considering heat invariants of the underlying locally homogeneous nearly Kähler manifolds, see Proposition 3.2.7.

Proposition 0.0.7. *Let M_Γ and $M'_{\Gamma'}$ be a pair of Laplace isospectral locally homogeneous nearly Kähler manifolds in dimension six with $M = S^3 \times S^3$. Then M' and M are holomorphically isometric. Moreover, Sunada isospectral pairs M_{Γ_i} with $\Gamma_i \subset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \{\mathrm{Id}\}$ are holomorphically isometric.*

This thesis concludes with Chapter 4. This chapter serves as an overlook in which we explain further research directions, current research that is being carried on by the author at the moment, and partial results obtained already. The aim of this chapter is to put the results of this thesis in perspective for a longer research project.

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Chapter 1

The world of finite groups

This chapter aims to introduce the most relevant notions on representation theory and the cohomology of finite groups, that will play a fundamental role in the construction of isospectral examples in Chapter 3. In this chapter we will assume nonetheless basic concepts on group representations, like irreducibility, complete irrducibility, etc.

We begin by exposing the general theory of group characters in Section 1.1 and then specialize our presentation in Section 1.2 on the symmetric group by describing all its (complex) irreducible representations and characters using the formalism of Young diagrams. Section 1.3 is devoted to the presentation of group cohomology as a tool to understand when a finite group representation is *spinnable*, see this definition in Section 1.3. This chapter ends up by giving a short account on the classification of finite groups of Spin(3) and Spin(4).

Our presentation in Sections 1.1 and 1.2 is notably based on the book of Sagan [Sa] and the standard reference [FH], whereas the material presented in Sections 1.3 and 1.4 is based on [CV], [MiSt] and [Br] respectively.

1.1 Character theory

A good amount of information of a finite dimensional (complex) representation of a (finite) group is encoded in its character. In this section we describe the most fundamental facts on characters of group representations in zero characteristic, of which we will make use later on. Throughout this section G will denote a finite group and $G^\sharp$ the set of conjugacy classes of elements in G.

Definition 1.1.1. The character of a (complex) representation $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is the function $\chi : G \longrightarrow \mathbb{C}$ defined by

$$\chi(g) = \chi_\rho(g) = \mathrm{Tr}(\rho(g)), \quad g \in G.$$

In addition, we say that χ is an irreducible character provided the underlying complex representation is irreducible.

Invariance of the trace under conjugation and elementary arguments give us the following properties of characters:

Lemma 1.1.2. *Let $\chi, \chi' : G \longrightarrow \mathbb{C}$ be the character of equivalent finite dimensional complex representations (ρ, V) and (ρ', V') of a group G. That is, there is an isomorphism $T : V \longrightarrow V'$ intertwining the representations ρ and ρ', i.e.*

$$T(\rho(g)v) = \rho'(g)T(v), \quad \forall (g, v) \in G \times V.$$

Then the following statements hold.

(i) The value of the character at the identity element $e \in G$ is the dimension of the underlying representation space

$$\chi(e) = \dim(V).$$

(ii) The character $\chi : G \longrightarrow \mathbb{C}$ of the representation $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is a class function, i.e. $\chi(g) = \chi(h)$ whenever $g, h \in G$ are conjugate to each other.

(iii) Character of equivalent representations are the same:

$$\chi = \chi'.$$

As a consequence of Lemma 1.1.2, we can order characters of G of inequivalent irreducible representations in an array that is usually called *character table.*

Definition 1.1.3. Let G be a group. The character table of G is an array with rows indexed by inequivalent irreducible characters of G and columns indexed by conjugacy classes. The entry χ_K corresponding to a pair (χ, K) is simply the value of the irreducible character $\chi : G \longrightarrow \mathbb{C}$ at a point in a given conjugacy class $K \in G^\sharp$.

Although the set of conjugacy classes $G^\sharp$ of a finite group G is clearly finite, the character table of a finite group might be *a priori* infinite. This is fortunately not the case. In fact, we will see in Proposition 1.1.9 that there are precisely $|G^\sharp| \in \mathbb{N}$ inequivalent (complex) irreducible representations of a given finite group G. Let us assume this fact in order to give our first description of all irreducible representations of a finite group.

Example 1.1.4. Let $G = \mathbb{Z}_n$, case in which $|G^\sharp| = n$. The underlying complex irreducible representations ϕ_l of G are all one dimensional

$$\phi_l([x]) = \omega_{lx} \in \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$$

and have character $\chi_l([x]) = \omega_{lx}$, where $\omega_x = e^{\frac{2\pi i x}{n}}$. The character table of $G = \mathbb{Z}_n$ is a $n \times n$ complex matrix given by

l/x	0	1	$\dots$	$n-1$
0	ω_0	ω_0	$\dots$	ω_0
1	ω_0	ω_1	$\dots$	ω_{n-1}
$\dots$	ω_0	$\dots$	$\dots$	$\dots$
$n-1$	ω_0	ω_{n-1}	$\dots$	ω_{n^2-2n+1}

Table 1.1: Character table of the cyclic group

Amongst the representations of a finite group G there is a fundamental one, which will be referred as the *regular representation* or *group algebra*. This representation is obtained as follows: let $\mathbb{C}[G]$ be the complex span of the elements of $G = \{g_1, ...g_n\}$. The group G acts (from the left) in a natural manner on $\mathbb{C}[G]$ by letting

$$g_j \cdot (a_1 g_1 + \cdots + a_n g_n) = a_1 g_j g_1 + \cdots + a_n g_j g_n \in \mathbb{C}[G], \quad a_j \in \mathbb{C}.$$

In the sequel, we will refer to this structure when we refer to $\mathbb{C}[G]$ as a (left) G-module.

Example 1.1.5. The character χ^{reg} of $\mathbb{C}[G]$ is easily computed by taking the standard basis $\mathcal{B} = \{g_1, ..., g_n\}$, case in which we see that we only need to count the fixed points of the left action of G on itself. Since the latter action is free, this implies that

$$\chi^{reg}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e. \end{cases}$$

1.1.1 Orthogonality relations

Irreducible characters of a finite group G satisfy some orthogonality relations with respect to a scalar product defined on the space $L^2(G, \mathbb{C})$ of complex valued functions $f : G \longrightarrow \mathbb{C}$. These relations were discovered first by Schur and it's proof is (essentially) a combination of linear algebra arguments and Schur's lemma[1], see Theorem 1.9.3. [Sa].

Definition 1.1.6. Let $\varphi, \psi : G \longrightarrow \mathbb{C}$ be complex valued functions defined on G. The (normalized) inner (or L^2) product of φ and ψ is given by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}. \tag{1.1}$$

In the particular case when we take the L^2-product of two characters $\chi, \chi' : G \longrightarrow \mathbb{C}$, formula (1.1) becomes

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{K \in G^\sharp} |K| \chi_K \overline{\chi}'_K,$$

where χ_K and χ'_K denote the values of the characters $\chi . \chi' : G \longrightarrow \mathbb{C}$ at a given conjugacy class $K \in G^\sharp$. In the sequel we will make solely use of type I orthogonality relations:

Theorem 1.1.7. *Let $\chi : G \longrightarrow \mathbb{C}$ and $\chi' : G \longrightarrow \mathbb{C}$ be irreducible characters of G. The following relation holds*

$$\langle \chi, \chi' \rangle = \delta_{\chi, \chi'},$$

where $\langle \cdot\, , \cdot \rangle$ denotes the scalar product defined in relation (1.1).

One of the reasons type I orthogonality relations play a fundamental role in the description of representations of a finite group G is the classical result of Maschke.

[1]Schur's lemma states that intertwiners $T : V_1 \longrightarrow V_2$ (in the sense of Lemma 1.1.2) between (complex) irreducible representations (G, V_i) are the zero map or an isomorphism. In particular, if $V_1 = V_2$, the intertwiner $T = \lambda \cdot$, for some complex number $\lambda \in \mathbb{C}$.

Every representation of a finite group G is completely reducible, i.e. it is a sum of G-irreducible modules (Maschke).

The proof of Maschke's result for *complex* representations is a simple application of Weyl's unitarian trick in the context, see [Sa]. However, the result is also valid for arbitrary representations $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{F}}(V)$, when $\gcd(\mathrm{Char}(\mathbb{F}), |G|) = 1$, see [Led] pages 21-23.

Theorem 1.1.7 and Maschke's result give few interesting consequences.

Corollary 1.1.8. *Let $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ be a complex representation of G and let*

$$V = m_1 V^{(1)} \oplus m_2 V^{(2)} \oplus \cdots \oplus m_n V^{(n)}$$

be its decomposition into irreducible components $(\rho^{(l)}, V^{(l)})$ each appearing with multiplicity $m_l \in \mathbb{N} \cup \{0\}$ for $l = 1, ..., n$. Then,

(i) The character $\chi : G \longrightarrow \mathbb{C}$ of the representation $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is a linear combination of the irreducible characters $\chi^{(l)} : G \longrightarrow \mathbb{C}$ of its irreducible components $\rho^{(l)} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$, i.e.

$$\chi = m_1 \chi^{(1)} + \cdots + m_n \chi^{(n)}.$$

(ii) The multiplicity $m_l \in \mathbb{N} \cup \{0\}$ of a given irreducible component $\rho^{(l)}$ is given by

$$m_l = \left\langle \chi, \chi^{(l)} \right\rangle,$$

where $\langle \cdot \, , \, \cdot \rangle$ is the inner product given in (1.1). In particular, the respresentation $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

(iii) Two representations $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ and $\rho' : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V')$ are equivalent if and only if their characters $\chi : G \longrightarrow \mathbb{C}$ and $\chi' : G \longrightarrow \mathbb{C}$ coincide.

As mentioned before, each complex representation of a finite group G is *unitarizable*. In particular, item (iii) of Corollary 1.1.8 tells us that the number of irreducible characters of G is exactly the cardinality of its *unitary dual* $\widehat{G}$, i.e. the set of irreducible unitary representations of G. Another important consequence of Maschke's theorem and type I orthogonality relations, see Theorem 1.1.7, is the following result.

Proposition 1.1.9. *The cardinality of the unitary dual $|\widehat{G}|$ is indeed $|G^\sharp| < \infty$.*

Sketch of the proof. The idea of the proof of this statement is to analyze the irreducible decomposition of the group algebra $\mathbb{C}[G]$. Indeed, if we have

$$\mathbb{C}[G] = \bigoplus_i m_i V^{(i)}, \tag{1.2}$$

where the sum runs over *all* pairwise inequivalent irreducible representations. Item (ii) of Corollary 1.1.8 implies that each irreducible G-module occurs in the decomposition (1.2) exactly $m_i = \dim(V^{(i)})$ times. Because the dimension of $\mathbb{C}[G]$ is finite, this implies that there are just finitely many (unitary) irreducible representations of G, say N. On the other hand, the $\mathbb{C}[G]$- algebras $\mathrm{End}(\mathbb{C}[G])$ and $\mathbb{C}[G]$ are isomorphic via the natural right action:

$$\phi : \mathbb{C}[G] \ni v \longmapsto \phi_v \in \mathrm{End}(\mathbb{C}[G]) : w \mapsto w \cdot v.$$

The map $\phi : \mathbb{C}[G] \longrightarrow \mathrm{End}(\mathbb{C}[G])$ induces an anti-automorphism of the centers of the algebras in question, and hence $\dim(Z_{\mathbb{C}[G]}) = \dim(Z_{\mathrm{End}\,\mathbb{C}[G]}) = N$, where the last equality is a consequence of Schur's lemma. The claim follows from Lemma 1.1.10.[2] □

Lemma 1.1.10. *Suppose that G has k conjugacy classes $K_1, ..., K_k$ and set $z_i = \sum_{g \in K_i} g$, for $i = 1, ..., k$. Then the center $Z_{\mathbb{C}[G]} = \mathrm{span}_{\mathbb{C}}\{z_1, ..., z_k\}$.*

Another important consequence of the previous analysis comes from counting dimensions in the isotypical decomposition (1.2) and Example 1.1.5.

Corollary 1.1.11. *Let G be a finite group and consider the decomposition* (1.2) *of* $\mathbb{C}[G]$. *Then it holds that*

$$\sum_i \dim(V^{(i)})^2 = |G|.$$

In general, the process of computing irreducible characters of a given finite group is very algorithmic and reduces to arithmetic and combinatorial considerations, as we will see in Section 1.2 where we illustrate the use of the machinery developed in this section on concrete examples. The importance of arithmetic expressions as the one stated in Corollary 1.1.11 comes in this vein.

1.2 The symmetric group

Perhaps the universal finite group is the symmetric group $G = \mathrm{Sym}(n)$, i.e. the group of bijections of a set $\overline{n} = \{1, 2, ..., n\}$ of n objects for any fixed $n \in \mathbb{N}$. In this section we would like to introduce the combinatorial framework that allows us, on the one hand, to give a complete description of (complex) irreducible representations of the symmetric group and its conjugacy classes. Furthermore, we will review a combinatorial rule due to Murnaghan and Nakayama, see Theorem 4.10.2 in [Sa], to compute irreducible characters of the symmetric group in an efficient algorithmic manner.

1.2.1 Cycle types and conjugacy classes

We have seen in Proposition 1.1.9 that in order to characterize all (complex) irreducible representations of G we need to have information about the set of conjugacy classes $G^\sharp$. We parametrize conjugacy classes for the symmetric group $G = \mathrm{Sym}(n)$.

Each permutation $\pi \in G$ in the symmetric group $G = \mathrm{Sym}(n)$ factorizes uniquely (up to permutation of the factors) as a products of disjoint cycles $(u_1, ..., u_l) \in G$ of a given length $l \in \mathbb{N}$, that is

$$\pi = (u_1, u_2, ..., u_l) \cdots (u_m, u_{m+1}, ..., u_n) \in G, \tag{1.3}$$

[2] See Proposition 1.10.1 in [Sa] for more a more detailed explanation.

where $u_k \in \{1, 2, ..., n\}$. In particular, with each permutation $\pi \in G$ we can associate a tuple encoding the information of the decomposition (1.3), which will be referred in the sequel as the *cycle decomposition* of the permutation $\pi \in G$.

Definition 1.2.1. The *cycle type*, or simply the *type*, of a permutation $\pi \in \mathrm{Sym}(n)$ is an expression of the form $\lambda_\pi = (1^{m_1}, ..., n^{m_n})$, where m_k is the amount of cycles of length $k \in \mathbb{N}$ appearing in the decomposition (1.3) of the permutation $\pi \in G$.

There are few other ways to represent cycle types. One of them is by means of partitions of natural numbers:

Definition 1.2.2. A *partition* $\lambda = (\lambda_1, ..., \lambda_l)$ of a natural number $n \in \mathbb{N}$, denoted by $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ or simply $\lambda \vdash n$, is a sequence of weakly decreasing numbers $\lambda_i \in \mathbb{N}$ such that $\lambda_1 + \cdots + \lambda_l = n$.

Cycle types and partitions are in bijection: a cycle type determines a partition by arranging the length of the cycles appearing in the cycle decomposition of a permutation in a weakly decreasing manner.

Example 1.2.3. The permutation $\pi = (135)(246)(7) \in \mathrm{Sym}(7)$ has (cycle) type $(3^2, 1)$ and determines the partition $\lambda = (3, 3, 1)$ of the natural number 7.

Moreover, conjugacy classes of elements in $G = \mathrm{Sym}(n)$ are in turn parametrized by cycle types (or partitions):

Proposition 1.2.4. *Consider $G = \mathrm{Sym}(n)$ and let $K_\pi \in G^\sharp$ be the conjugacy class of the element*

$$\pi = (u_1, u_2, ..., u_l)\cdots(u_m, u_{m+1}, \cdots, u_n) \in G = \mathrm{Sym}(n).$$

Then we have

(i) for any $\sigma \in G$

$$\sigma \circ \pi \circ \sigma^{-1} = (\sigma(u_1), \sigma(u_2), ..., \sigma(u_l))\cdots(\sigma(u_m), ..., \sigma(u_n)) \in G.$$

In particular, two permutations are conjugate if and only if they define the same cycle decomposition.

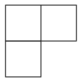
(ii) If the permutation $\pi \in G$ has type $\lambda_\pi = (1^{m_1}, ..., n^{m_n})$, then the cardinality of the conjugacy class of π is given by

$$|K_\pi| = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!}$$

1.2.2 Complex irreducible representations

A diagrammatic way to to represent partitions of natural numbers, and hence conjugacy classes of the corresponding symmetric group, see Proposition 1.2.4, is by means of *Young diagrams*. In this section we will give an account on how these combinatorial gadgets parametrize all (complex) irreducible representations of the symmetric group $G = \mathrm{Sym}(n)$.

Definition 1.2.5. A *Young* (or *Ferrer*) *diagram* of shape $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ is a left justified array of boxes t^λ, so that the length of the ith row of this array is λ_i. A *Young tableau* of shape λ (or a λ-*tableau*) is an array obtained from a Young diagram t^λ by filling in its boxes with the numbers $\{1, 2..., n\}$ bijectively.

Example 1.2.6. A Young diagram of shape $\lambda = (2, 1)$ is $t^\lambda =$  and it gives rise to $3! = (2 + 1)! = 6$ choices of Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Let us denote by $\mathrm{Tab}^\lambda(n)$ the set of all Young tableaux of shape $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ and define an equivalence relation on this set by declaring two tableaux $t, s \in \mathrm{Tab}^\lambda(n)$ to be equivalent precisely if there is an element

$$\sigma = \sigma_1 \times \sigma_2 \times ... \times \sigma_l \quad \text{with} \quad \sigma_i \in \mathrm{Sym}(\lambda_i)$$

so that $\sigma \cdot t = s \in \mathrm{Tab}^\lambda(n)$. Here the permutation $\sigma = \sigma_1 \times \sigma_2 \times ... \times \sigma_l$ acts on the tableau $t \in \mathrm{Tab}^\lambda(n)$ by permuting the elements in the ith row according to its ith component $\sigma_i \in \mathrm{Sym}(\lambda_i)$. An equivalence class $\{t\}$ of a tableau $t \in \mathrm{Tab}^\lambda(n)$ of shape λ is called a *tabloid* of shape λ, or shortly a λ-tabloid. We will not make any notational difference between λ-tabloids and λ-tableaux when performing explicit computations, as it will be clear from the context what it is meant.

The natural action of the symmetric group $\mathrm{Sym}(n)$ on $\mathrm{Tab}^\lambda(n)$ is well-defined at the level of tabloids and gives rise to a complex module

$$M^\lambda = \mathrm{span}_\mathbb{C}\{\{t\} \ : \ t \in \mathrm{Tab}^\lambda(n)\} \tag{1.4}$$

of dimension

$$\dim(M^\lambda) = \frac{n!}{\lambda_1!...\lambda_l!},$$

number which corresponds to the number of λ-tabloids, i.e. the number of λ-tableaux divided by the number of tableaux in an equivalence class.

The module M^λ defined in identity (1.4) for a given partition $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ is in general *not irreducible*:

Example 1.2.7. Let $\lambda = (1, 1, ..., 1) \vdash n$. In this case M^λ is nothing but the regular representation $\mathbb{C}[\mathrm{Sym}(n)]$ of the symmetric group $\mathrm{Sym}(n)$, see Section 1.1.

The importance of the modules $(M^\lambda \ : \ \lambda \vdash n)$ defined by identity (1.4) lies in the fact that each (complex) irreducible module of $\mathrm{Sym}(n)$ appears in the irreducible decomposition of some M^λ. To see this we need to refine our choice of tabloids.

Definition 1.2.8. Define the *row and column stabilizers* of a given tableau $t \in \mathrm{Tab}^\lambda(n)$ with rows $R_1, R_2, ..., R_l$ and columns $C_1, C_2, ..., C_k$, respectively by

$$R_t = \mathrm{Sym}(R_1) \times \mathrm{Sym}(R_2) \times ... \times \mathrm{Sym}(R_l), \tag{1.5}$$

$$C_t = \mathrm{Sym}(C_1) \times \mathrm{Sym}(C_2) \times ... \times \mathrm{Sym}(C_k). \tag{1.6}$$

For a given tableau $t \in \mathrm{Tab}^\lambda(n)$ of shape $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$, the row and column stabilizers can be naturally realized as subgroups of $\mathrm{Sym}(n)$. Moreover, with these subgroups we can associate the following elements in the regular representation $\mathbb{C}[\mathrm{Sym}(n)]$ by declaring

$$C_t^+ = \sum_{\pi \in C_t} \pi, \quad C_t^- = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi, \quad R_t^+ = \sum_{\pi \in R_t} \pi, \quad R_t^- = \sum_{\pi \in R_t} \mathrm{sgn}(\pi)\pi.$$

The element C_t^- plays a distinctive role in the construction of the (complex) irreducible modules of $\mathrm{Sym}(n)$, and so it deserves a name:

Definition 1.2.9. Let $t \in \mathrm{Tab}^\lambda(n)$. The element $e_t = C_t^-\{t\} \in M^\lambda$ is the *polytabloid* associated with the tableau t of shape $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$.

Example 1.2.10. The column stabilizer of the Young tableau $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$ is given by

$$C_t = \mathrm{Sym}(\{3,4\}) \times \mathrm{Sym}(\{1,5\}).$$

The element C_t^- is given by $C_t^- = (1) - (3,4) - (1,5) + (1,5)(3,4) \in \mathbb{C}[\mathrm{Sym}(5)]$ and its associated polytabloid is nothing but

$$e_t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \in M^{(3,2)}.$$

Let us summarize the most important (and elementary) properties of the row and columns stabilizers of a tableau, see Lemma 2.3.3 in [Sa].

Lemma 1.2.11. *Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ and R_t, C_t be the row and column stabilizers of a tableau $t \in \mathrm{Tab}^\lambda(n)$ and $e_t \in M^\lambda$ be the polytabloid associated with the the tableau $t \in \mathrm{Tab}^\lambda(n)$ of shape $\lambda = (\lambda_1, ..., \lambda_l)$. The following statements hold.*

(i) $R_{\pi \cdot t} = \pi \cdot R_t \cdot \pi^{-1}$,

(ii) $C_{\pi \cdot t} = \pi \cdot C_t \cdot \pi^{-1}$,

(iii) $C_{\pi \cdot t}^- = \pi \cdot C_t^- \cdot \pi^{-1}$.

(iv) $e_{\pi \cdot t} = \pi \cdot e_t$.

It is a consequence of item (iv) of Lemma 1.2.11 that the natural action of the symmetric group $\mathrm{Sym}(n)$ is well defined on polytabloids. Consequently, we can define new modules by considering the natural module structure associated with the set of all polytabloids of a given shape:

Definition 1.2.12. The *Specht module* associated with a partition $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ is given by

$$S^\lambda = \mathrm{span}_{\mathbb{C}}\{e_t \ : \ t \in \mathrm{Tab}^\lambda(n)\}, \tag{1.7}$$

where $e_t \in M^\lambda$ denotes the polytabloid associated with the tableau t of shape $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$.

The Specht module S^λ associated with a given partition $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ turns out to be irreducible, see Theorem 2.4.4 in [Sa]. Since there are as many Specht modules as partitions, the family $(S^\lambda \ : \ \lambda \vdash n)$ exhaust all complex irreducible modules of $\mathrm{Sym}(n)$, see Proposition 1.1.9 and item (i) of Proposition 1.2.4.

We can summarize the whole discussion in this section in the following theorem.

Theorem 1.2.13. *There is a one to one correspondence between partitions and equivalence classes of irreducible complex representations of* $\mathrm{Sym}(n)$*, namely the one that assigns to each partition* $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ *its corresponding Specht module* S^λ*, see Definition* (1.2.12)*.*

1.2.3 The Murnaghan-Nakayama rule

There is a powerful and computationally efficient combinatorial algorithm to compute symmetric characters. It bears the label of Murnaghan-Nakayama rule and we devote this section to describe it. It should be stressed here that we are interested in this rule purely from a user perspective. The reader interested in a deeper theoretical understanding of the Murnaghan-Nakayama rule can consult the original works [Mu] and [Nak], or the exposition in [Sa].

Let us begin the discussion by introducing the notion of *skew diagrams.*

Definition 1.2.14. Let $\mu = (\mu_1, ..., \mu_m)$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ be partitions and $\mathcal{Y}_\lambda$ and $\mathcal{Y}_\mu$ be their corresponding Young diagrams. If the relation $\mathcal{Y}_\mu \subset \mathcal{Y}_\lambda$ [3] holds, then the corresponding *skew diagram* (or *shape*) is the set of cells

$$\lambda/\mu = \{c \ : \ c \in \mathcal{Y}_\lambda \ \wedge \ c \notin \mathcal{Y}_\mu\}.$$

If $\zeta = \lambda/\mu$, we write $\lambda \backslash \zeta$ for μ. In addition, a skew diagram is said to be *normal* if $\mathcal{Y}_\mu = \varnothing$.

Skew diagrams will be distinguished from Young diagrams by inserting a bullet in each slot of the diagram.

Example 1.2.15. If $\lambda = (4,4,3)$ and $\mu = (2,2)$ then the skew diagram λ/μ is simply

$\lambda/\mu =$

		•	•
		•	•
•	•	•	

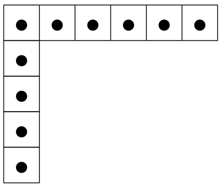
There is a special kind of skew diagram that are important in Murnaghan-Nakyama rule, the so called *rim hooks.*

Definition 1.2.16. A *rim hook* ζ is a skew diagram obtained by taking all cells on a finite lattice path with steps one unit northward or eastward. Further, we define the length $l(\zeta)$ of the rim hook ζ by declaring

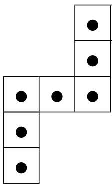
$$l(\zeta) = (\text{Number of rows of } \zeta) - 1. \tag{1.8}$$

[3] This inclusion is meant in the natural sense.

The name rim hook comes from the fact that such diagram can be obtained by projecting a hook along diagonals onto the boundary of a shape. For example, the hook

$L_{1,1} =$ 

inside the Young diagram $\mathcal{Y}_{(6,3,3,1,1)}$ projects to a rim hook of the form

$\bar{L}_{1,1} =$ 

We are in position to state the Murnaghan-Nakayama rule.

Theorem 1.2.17. *Let $\alpha, \lambda \vdash n$ be partitions. Then, the value χ^{λ}_{α} of the irreducible character $\chi^{\lambda} : \mathrm{Sym}(n) \longrightarrow \mathbb{C}$ corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ at the conjugacy class $C_{\alpha} \in \mathrm{Sym}(n)^{\sharp}$ determined by the partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ is given by*

$$\chi^{\lambda}_{\alpha} = \sum_{\zeta} (-1)^{l(\zeta)} \chi^{\lambda \setminus \zeta}_{\alpha \setminus \alpha_1}, \tag{1.9}$$

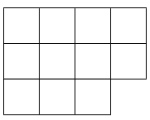
where we have used the notation $\alpha \setminus \alpha_1 = (\alpha_2, \ldots, \alpha_m)$ and the sum is over all rim hooks ζ of $\lambda \in \mathrm{Tab}^{\lambda}(n)$ having $\alpha_1 \in \mathbb{N}$ cells.

To calculate a value of an irreducible character at a given conjugacy class, the rule stated in Theorem 1.2.17 must be used recursively. First removing a rim hook from $\lambda \in \mathrm{Tab}^{\lambda}(n)$ with $\alpha_1 \in \mathbb{N}$ cells in all possible ways until what is left is in normal shape, and then proceed recursively according to formula (1.9) on the irreducible characters in the right hand side of identity (1.9). It should be mentioned that formula (1.9) is understood to be zero provided there is no rim hook of a given length.

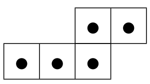
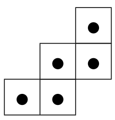
Example 1.2.18. Let us calculate the value of the irreducible character χ^{λ}_{α} of the irreducible character corresponding to the partition λ at the conjugacy class C_{α} for $\lambda = (4,4,3)$ and $\alpha = (5,4,2)$. The first iteration of formula (1.9) gives

$$\chi^{(4,4,3)}_{(5,4,2)} = -\chi^{(4,2)}_{(4,2)} + \chi^{(3,2,1)}_{(4,2)}. \tag{1.10}$$

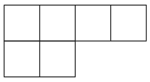
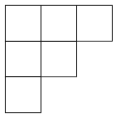
In fact, the Young diagram corresponding to $\lambda = (4,3,3)$ is

$\mathcal{Y}_{\lambda} =$ 

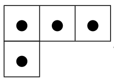
and the length 5 rim hooks of $\lambda = (4,3,3)$ correspond to

$\zeta_1 =$  and $\zeta_2 =$  .

The skew hooks ζ_i have lengths $l(\zeta_1) = 1$ and $l(\zeta_2) = 2$, see formula (1.8). Let us now set $\lambda_1 = \lambda\backslash\zeta_1 = (4,2)$ and $\lambda_2 = \lambda\backslash\zeta_2 = (3,2,1)$, partitions which correspond to the Young diagrams

$\mathcal{Y}_{\lambda_1} =$  and $\mathcal{Y}_{\lambda_2} =$  .

The value of $\chi^{(3,2,1)}_{(4,2)}$ in formula (1.10) is clearly zero, as $\mathcal{Y}_{\lambda_2}$ does not contain rim hooks of length 4. On the other hand, the Young diagram $\mathcal{Y}_{\lambda_1}$ contains exactly one rim hook of length 4, namely

$\zeta_3 =$  ,

which has length $l(\zeta_3) = 1$. Consequently, we have $\chi^{(4,2)}_{(4,2)} = -\chi^{(1,1)}_{(2)}$. Finally, set $\lambda_3 = \lambda_1\backslash\zeta_3 = (1,1)$ and observe that λ_3 has precisely one rim hook of length 2, and so $\chi^{(1,1)}_{(2)} = -1$. Inserting these calculations in formula (1.10), we get

$$\chi^{(4,4,3)}_{(5,4,2)} = -\chi^{(4,2)}_{(4,2)} + \chi^{(3,2,1)}_{(4,2)} = -(-\chi^{(1,1)}_{(2)}) + 0 = -1.$$

As we have seen in Example 1.2.18, the calculation of symmetric characters might be cumbersome. Fortunately enough, the combinatorial rule in Theorem 1.2.17 has been implemented in algebraic software, like MAGMA. In the sequel, we shall use this software to perform explicit computations of symmetric characters.

1.3 Group cohomology

Before introducing the relevant group cohomological concepts we will make use of, it is sensible to motivate the use of this cohomology theory for our purposes. We will assume some previous knowledge on higher homotopy theory and the theory of classifying spaces, for which we refer to [MiSt].

The main question we address in this section is the following.

When does a given representation $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ *of a finite group* Γ *lift to* $\mathrm{Spin}(n)$*?*

In case the answer of the question above is affirmative, we say that the representation (ρ, Γ) is *spinnable*.

The term *spinnable* comes from the remarkable resemblance of the aforementioned lifting problem in this context and the standard topological problem to determine whether a smooth (oriented) manifold M is spin. That is, to determine when the cocycles

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(n)$$

defining the tangent bundle TM lift to $\mathrm{Spin}(n)$.

First of all we would like to understand the obstruction for a representation (ρ, Γ) to be spinnable, and if possible, we would like to pack this information in a class in an appropriate cohomology theory, just as in the Riemannian case, where such obstruction is known to be a characteristic torsion class $w_2(M) = w_2(TM) \in H^2(M, \mathbb{Z}_2)$ called the second Stiefel-Whitney class.

Let us associate a topological problem with our group theoretical situation. To do this, let us recall the existence of a functor

$$B : \mathtt{TopGr} \longrightarrow \mathtt{hTop}, \qquad B : G \longmapsto BG,$$

where $\mathtt{TopGr}$ denotes the category of topological groups and $\mathtt{hTop}$ denotes the category whose objects are homotopy classes of topological spaces and whose morphisms are homotopy classes of continuous maps. The functor $B : G \longmapsto BG$ assigns to each topological group G a topological space BG (unique up to homotopy) called *classifying space* and it is referred in the literature as the *classifying space functor*. The space BG is in turn the quotient of a weakly contractible[4] G-space EG by the action of G. In particular, there is a natural fibration $E\pi : EG \longrightarrow BG$ associated with BG, which will be referred as the *classifying fibration*.

The name *classifying space* comes from the fact that G-principal bundles over manifolds are classified by BG: given any principal bundle $p : P \longrightarrow M$ over a smooth manifold M with structure group G, there is a map (unique up to homotopy) $f : M \longrightarrow BG$, so that the pullback bundle $EG \times_{(f, E\pi)} M$ is isomorphic to P (as principal bundles). The map $f : M \longrightarrow BG$ is said to be the *classifying map* of $p : P \longrightarrow M$.

Suppose the representation $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ is spinnable, i.e. there is a group homomorphism $\bar{\rho} : \Gamma \longrightarrow \mathrm{Spin}(n)$ so that $\pi \circ \bar{\rho} = \rho$, where $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the spin covering. Then, at the level of classifying spaces we have the following commutative diagram:

$$\begin{array}{ccccc} & \nearrow & B\mathrm{Spin}(n) & & \\ & & \big\downarrow B\pi & & \\ B\Gamma & \xrightarrow{B\rho} & B\mathrm{SO}(n) & \longrightarrow & B^2\mathbb{Z}_2 \end{array}$$

where the arrow $B\mathrm{SO}(n) \longrightarrow B^2\mathbb{Z}_2 = K(\mathbb{Z}_2, 2)$ above, which we will denote by w_2, is the classifying arrow of the $B\mathbb{Z}_2$ fibration

$$B\mathbb{Z}_2 \longrightarrow B\mathrm{Spin}(n) \longrightarrow B\mathrm{SO}(n),$$

and $B^2\mathbb{Z}_2 = K(\mathbb{Z}_2, 2)$ denotes the second Eilenberg-MacLane space over $\mathbb{Z}_2$, i.e. $B^2\mathbb{Z}_2$ is a connected space so that its second homotopy group $\pi_2(K(\mathbb{Z}_2, 2)) = \mathbb{Z}_2$ and whose other homotopy groups vanish. The existence of the lift $B\bar{\rho} : B\Gamma \longrightarrow B\mathrm{Spin}(n)$ at this level is equivalent to the composition $w_2 \circ B\rho : B\Gamma \longrightarrow B^2\mathbb{Z}_2$ being homotopically trivial.

By using the properties of the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$, we can realize the

[4] A space is weakly contractible if all its homotopy groups are trivial.

homotopy class $[w_2 \circ B\rho] \in [B\Gamma, B^2\mathbb{Z}_2]$ as a cohomology class $w_2(\rho,\Gamma) \in H^2(B\Gamma,\mathbb{Z}_2)$, which is the analogue of the second Stiefel-Whithney class in the present situation. In particular, we see the necessity to have

$$w_2(\rho,\Gamma) = 0 \in H^2(B\Gamma,\mathbb{Z}_2), \tag{1.11}$$

in order for the representation (ρ,Γ) to be spinnable. The classifying space functor $B : \texttt{TopGr} \longrightarrow \texttt{hTop}$ is *not* fully faithful, and so homotopy classes of maps $[B\Gamma, B\mathrm{Spin}(n)]$ might not come from a group homomorphism $\Gamma \longrightarrow \mathrm{Spin}(n)$. However, it turns out that condition (1.11) is also sufficient to guarantee that (ρ,Γ) is spinnable:

Theorem 1.3.1. *Consider a representation $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ of a finite group Γ. The following statements are equivalent:*

(i) The canonical extension L of (ρ,Γ) is trivial.

(ii) The cohomology class $w_2(\rho,\Gamma) \in H^2(\Gamma,\mathbb{Z}_2)$ vanishes.

(iii) The representation $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ is spinnable.

The key ingredient of the proof of this statement is to realize the cohomology class $w_2(\rho,\Gamma) \in H^2(B\Gamma,\mathbb{Z}_2)$ as an element of a more suitable cohomology theory: *group cohomology.*

To introduce group cohomology in a harmonic way, let us approach the question on the existence of a lift of (ρ,Γ) in a different manner: consider the canonical extension of Γ defined by (ρ,Γ). That is, the unique group L so that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\rho^*} & \mathrm{Spin}(n) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Gamma & \xrightarrow{\rho} & \mathrm{SO}(n) \end{array}$$

commutes. A sufficient condition for (ρ,Γ) to be spinnable is clearly to require the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} L \xrightarrow{\pi} \Gamma \longrightarrow 1.$$

to be trivial, i.e. $\iota : \mathbb{Z}_2 \longrightarrow L$ admits a section $L \longrightarrow \mathbb{Z}_2$. We see that in order for us to be successful recognizing whether a representation is spinnable using this approach, we need to describe all isomorphism classes of short exact sequences of the above form. However, the second cohomology group $H^2(B\Gamma,\mathbb{Z}_2)$ parametrizes such isomorphism classes of enlargements of Γ by $\mathbb{Z}_2$, see [AM] and Section 1.8.3 in [Zi], as we will see in a moment.

The following definition clarifies the link between our problem and group cohomology.

Definition 1.3.2. The *ith group cohomology group* $H^i(\Gamma, \mathbb{Z}_2)$ of a finite group Γ with coefficients in the abelian group $\mathbb{Z}_2$ is defined to be one of the following groups.

(i) $H^i(B\Gamma, \mathbb{Z}_2)$.

(ii) $\mathrm{Ext}^i_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \mathbb{Z}_2)$,

where $\mathbb{Z}[\Gamma] = \{\sum_{i=1}^n a_i\gamma_i \ : \ a_i \in \mathbb{Z}, \ \gamma_i \in \Gamma\}$, see Section 1.1.

The explicit correspondence between elements in $H^2(\Gamma, \mathbb{Z}_2)$ and isomorphism classes of enlargements

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \bar{\Gamma} \xrightarrow{\pi} \Gamma \longrightarrow 1.$$

plays a role in the proof of Theorem 2.2.6 in Chapter 2. It is our interest to describe this correspondence in the most tractable way. For that, we use a realization of the second cohomology group $H^2(\Gamma, \mathbb{Z}_2)$ in terms of cocycles a coboundaries, see Proposition 1.8.39 in [Zi]:

Proposition 1.3.3. *The cohomology group $H^2(\Gamma, \mathbb{Z}_2)$ is isomorphic to the quotient of the group of 2-cocycles $B^2(\Gamma, \mathbb{Z}_2)$ and 2-coboundaries $Z^2(\Gamma, \mathbb{Z}_2)$, where*

$$Z^2(\Gamma, \mathbb{Z}_2) = \{f : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2 : \ g_2 \cdot f(g_3, g_4) - f(g_2g_3, g_4) + f(g_2, g_3g_4) - f(g_2, g_3) = 0\}$$
$$B^2(\Gamma, \mathbb{Z}_2) = \{f : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2 : \ \exists\sigma : \Gamma \longrightarrow \mathbb{Z}_2 \ s.t. \ f(g_2, g_3) = g_2 \cdot \sigma(g_3) - \sigma(g_2g_3) + \sigma(g_2)\},$$

and $\cdot : \Gamma \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ denotes a suitable action of Γ on $\mathbb{Z}_2$.[5]

Let $\mathcal{E}(\Gamma, \mathbb{Z}_2)$ be the set of enlargements of a group Γ by a group $\mathbb{Z}_2$, and let $(\iota, \pi) \in \mathcal{E}(\Gamma, \mathbb{Z}_2)$ be an enlargement of Γ by $\mathbb{Z}_2$

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \bar{\Gamma} \xrightarrow{\pi} \Gamma \longrightarrow 1.$$

The pair (ι, π) defines a cocycle $\varphi = \varphi_{(\iota,\pi)} : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2$ as follows: take a set theoretical section $s : \Gamma \longrightarrow \bar{\Gamma}$ of $\pi : \overline{\Gamma} \longrightarrow \Gamma$ with $s(e) = \bar{e} \in \bar{\Gamma}$ and define a map $\varphi : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2$ by letting

$$\varphi(g, h) = s(gh)^{-1}s(g)s(h), \quad (g, h) \in \Gamma \times \Gamma.$$

The action $\cdot : \Gamma \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ meant in Proposition 1.3.3 can be described at this level as

$$g \cdot a = s(g)as(g)^{-1} \qquad (g, a) \in \Gamma \times \iota(\mathbb{Z}_2), \tag{1.12}$$

where we have identified $\mathbb{Z}_2$ and $\iota(\mathbb{Z}_2)$. It is then an elementary exercise to verify that the map $\varphi : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2$ defines a cocycle in the sense of Proposition 1.3.3 and that the corresponding cohomology class with coefficients in $\mathbb{Z}_2$ is independent of the (set theoretical) lift $s : \Gamma \longrightarrow \bar{\Gamma}$.

Now it is time to prove our characterization of spinnable representations of finite groups, see Theorem 1.3.1.

[5] The meant action is defined in (1.12) at the level of enlargements.

Proof of Theorem 1.3.1. It is well known that the cohomology ring of the classifying space of the special orthonormal group with $\mathbb{Z}_2$ coefficients is polynomial in the Stiefel-Whithney classes $w_2, ..., w_n$. That is, we have that

$$H^\bullet(B\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, ..., w_n],$$

where the generators w_i have degree $\deg(w_i) = i$ for $2 \geq i \leq n$. In particular, condition (1.11) and the induced diagram on the level of cohomology

$$\begin{array}{ccc} H^\bullet(B\mathrm{SO}(n), \mathbb{Z}_2) & \xrightarrow{H^2(B\rho, \mathbb{Z}_2)} & H^\bullet(B\Gamma, \mathbb{Z}_2) \\ \downarrow{=} & & \downarrow{=} \\ H^\bullet(\mathrm{SO}(n), \mathbb{Z}_2) & \xrightarrow{H^2(\rho, \mathbb{Z}_2)} & H^\bullet(\Gamma, \mathbb{Z}_2) \end{array}$$

imply that the induced map $H^\bullet(\rho, \mathbb{Z}_2) : H^\bullet(\mathrm{SO}(n), \mathbb{Z}_2) \longrightarrow H^\bullet(\Gamma, \mathbb{Z}_2)$ is trivial at degree two. On the other hand, the pullback diagram embeds into the bigger diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & L & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\rho^*} & & \downarrow{\rho} & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\iota^*} & L^* & \xrightarrow{\pi^*} & \rho(\Gamma) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\iota} & \mathrm{Spin}(n) & \xrightarrow{\pi} & \mathrm{SO}(n) & \longrightarrow & 1 \end{array}$$

where the short exact sequence below corresponds to the standard covering of $\mathrm{SO}(n)$, the vertical arrows in the part below of the previous diagram are inclusions, and ρ^*, π^* are the induced homomorphisms by taking pullback.

Stated in different words, the homomorphism $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$ induces a map at the level of enlargements $\mathcal{E}(\mathrm{SO}(n), \mathbb{Z}_2) \longrightarrow \mathcal{E}(\Gamma, \mathbb{Z}_2)$. It is an easy exercise to check that this map corresponds to the induced map $H^2(\rho, \mathbb{Z}_2) : H^2(\mathrm{SO}(n), \mathbb{Z}_2) \longrightarrow H^2(\Gamma, \mathbb{Z}_2)$ at the cohomology level, see Chapter 4 page 94 in [Br]. However, by hypothesis $w_2(\rho, \Gamma) = 0 \in H^2(\Gamma, \mathbb{Z}_2)$, and so the map $H^2(\rho, \mathbb{Z}_2)$ is trivial. Consequently, condition (1.11) implies triviality of the canonical extension L and so the representation (ρ, Γ) is spinnable. The necessity to have $(ii) \Rightarrow (iii)$ is clear and it was discussed when introducing $w_2(\Gamma, \mathbb{Z}_2)$. $\square$

The explicit calculation of the group cohomology of a given finite group has been an active research topic in algebra and topology during the last century. Perhaps the group whose cohomology is most understood by now is the symmetric group. We will thus make the choice $\Gamma = \mathrm{Sym}(m)$ for $m \geq 4$. To the best knowledge of the author, the explicit calculation of the cohomology ring of Γ is due to Nakaoka, see the reference in Chapter

VI of [AM]. The interesting fact for us is that the second cohomology group $H^2(\Gamma, \mathbb{Z}_2)$ is independent of the amount of letters $m \geq 4$.

Example 1.3.4. The second cohomology group $H^2(\Gamma, \mathbb{Z}_2)$ of $\Gamma = \mathrm{Sym}(m)$ for $m \geq 4$ is isomorphic to the Klein Vierergruppe $V_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

There are various ways to see the validity of this statement, but they require additional tools to the one we presented in this section. On the one hand, we can define a dual theory to group cohomology, i.e. a homology theory based on the usual homology of the classifying space of a group. A similar interpretation as the one of $H^2(\Gamma, \mathbb{Z}_2)$ holds at the homology level: elements in the *Schur multiplier* $H_2(\Gamma, \mathbb{Z})$ are in one to one correspondence with 2-fold coverings of Γ. It is a standard result of Schur himself that $H_1(\Gamma, \mathbb{Z}) = H_2(\Gamma, \mathbb{Z}) = \mathbb{Z}_2$. In particular, the statement follows from the identity

$$H^2(\Gamma, \mathbb{Z}_2) \cong \mathrm{Hom}(H_2(\Gamma, \mathbb{Z}), \mathbb{Z}_2) \oplus \mathrm{Ext}_{\mathbb{Z}[\Gamma]}(H_1(\Gamma, \mathbb{Z}), \mathbb{Z}_2),$$

which follows from a suitable version of the (dual) universal coefficient theorem for group cohomology, see [Br]. The same statement also follows from Nakaoka's calculation of the cohomology of $\Gamma = \mathrm{Sym}(m)$.

The three non-trivial elements L_1, L_2 and L_3 in $H^2(\Gamma, \mathbb{Z}_2)$ for $m \geq 4$ can be explicitly described using the fact that $\Gamma = \mathrm{Sym}(m)$ admits a distinctive presentation. Indeed, this presentation of Γ allowed Schur to understand the coverings of Γ in a very succinct manner. The third non trivial element in $H^2(\Gamma, \mathbb{Z}_2)$ is a combination of the other two non trivial elements in a group cohomological sense. We postpone this discussion to the proof of Theorem 2.2.6 in order to keep the course of this proof uninterrupted.

1.4 Finite subgroups of Spin(3) and Spin(4)

In this section we aim to recall the classification of finite groups of $\mathrm{Spin}(3) = \mathrm{SU}(2)$ and $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ together with some generalities of the appearing finite groups of which we will use in Section 2.3. In what follows we shall identify SU(2) with the group of unit quaternions.

Finite subgroups of SU(2) are central extensions of finite groups of oriented isometries of the 3-dimensional euclidean space. Any finite subgroup of SU(2) is in fact conjugate to one of the so-called ADE groups, see e.g. Theorem 1.2.4 in [To]. These groups are described in Table 1.2.

Label	Name	Order	Generators	Elements
$\mathbb{A}_{n-1}$	$\mathbb{Z}_n$	n	$e^{\frac{i2\pi}{n}}$	$e^{\frac{2\pi i x}{n}},\ x \in \mathbb{Z}$
$\mathbb{D}_{n+2}$	$2\mathrm{D}_{2n}$	$4n$	$j, e^{\frac{i\pi}{n}}$	$e^{\frac{i\pi x}{n}}, je^{\frac{i\pi x}{n}},\ x \in \mathbb{Z}$
$\mathbb{E}_6$	2T	24	$\frac{1}{2}(1+i)(1+j), \frac{1}{2}(1+j)(1+i)$	$2\mathrm{D}_4 \cup \left\{\frac{\pm 1 \pm i \pm j \pm k}{2}\right\}$
$\mathbb{E}_7$	2O	48	$\frac{1}{2}(1+i)(1+j), \frac{1}{\sqrt{2}}(1+i)$	$2\mathrm{T} \cup e^{\frac{i\pi}{4}}\, 2\mathrm{T}$
$\mathbb{E}_8$	2I	120	$\frac{1}{2}(1+i)(1+j), \frac{1}{2}(\phi + \phi^{-1} i + j)$	$q^j\, 2\mathrm{T},\ j \leq 4$

Table 1.2: Finite subgroups of SU(2).

Here $n \geq 2$, $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio and $q = \frac{1}{2}\left(\phi + \phi^{-1} i + j\right)$. The subgroups of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ are in turn obtained from the finite groups of SU(2) by means of Goursat's lemma, see Theorem 2.1 in [CV].

Lemma 1.4.1. *Let G_1, G_2 be groups. There is a one-to-one correspondence between subgroups $C \subset G_1 \times G_2$ and quintuples $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$, where $A_0 \lhd A \subset G_1$, $B_0 \lhd B \subset G_2$ and $\theta : {}^{A}/_{A_0} \longrightarrow {}^{B}/_{B_0}$ is an isomorphism.*

The process of obtaining subgroups of the product of two copies of SU(2) from a given quintuple as stated in Lemma 1.4.1 will be frequently used in the proof of Proposition 2.3.1. Therefore, we give a proof of this statement for convenience of the reader.

Proof. A quintuple (A, A_0, B, B_0, θ) with $A_0 \lhd A \subset \mathrm{SU}(2)$ and $B_0 \lhd B \subset \mathrm{SU}(2)$ defines the subgroup of Spin(4) as a fibered product:

$$\mathcal{G}(A, A_0, B, B_0, \theta) = \{(a, b) \in A \times B |\ \alpha(a) = \beta(b)\}. \tag{1.13}$$

Here the group homomorphisms

$$\alpha : A \longrightarrow {}^{A}/_{A_0} \xrightarrow{\ \theta\ } {}^{B}/_{B_0} \qquad \text{and} \qquad \beta : B \longrightarrow {}^{B}/_{B_0}$$

are the natural ones. Conversely, given a group $C \subset \mathrm{Spin}(4)$ one can define a quintuple $\mathcal{Q}(C) = \{A, A_0, B, B_0, \theta\}$ by letting

$$A = \pi_1(C) \subset \mathrm{SU}(2) \quad , \quad B = \pi_2(C) \subset \mathrm{SU}(2), \tag{1.14}$$

$$A_0 = \pi_1(\mathrm{Ker}(\pi_2|_C)), \quad B_0 = \pi_2(\mathrm{Ker}(\pi_1|_C)), \tag{1.15}$$

and $\theta(aA_0) = bB_0$, where $(a, b) \in C$. It is easy to check that the group homomorphism $\theta : A/A_0 \longrightarrow B/B_0$ is well defined and the quintuple $\mathcal{Q}(C)$ defined above satisfies all hypotheses of Lemma 1.4.1. □

In order for use to use Lemma 1.4.1 effectively, we need to describe normal subgroups of ADE groups and the automorphisms of the quotients. The former groups together with the isomorphism type of the corresponding quotients are described in Tables 1.3 and 1.4, which are borrowed from [FD].

$A_0 \lhd A$	A/A_0
$\mathbb{Z}_k \lhd \mathbb{Z}_{kl}$	$\mathbb{Z}_l$
$\mathbb{Z}_{2k} \lhd 2\mathrm{D}_{2kl}$	D_{2l}
$\mathbb{Z}_{2k+1} \lhd 2\mathrm{D}_{2l(2k+1)}$	$2\mathrm{D}_{2l}$
$\mathbb{Z}_{2k+1} \lhd 2\mathrm{D}_{2(2k+1)}$	$\mathbb{Z}_4$
$2\mathrm{D}_{2k} \lhd 2\mathrm{D}_{4k}$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \lhd 2\mathrm{T}$	T

Table 1.3: Subgroups I.

$A_0 \lhd A$	A/A_0
$2\mathrm{D}_4 \lhd 2\mathrm{T}$	$\mathbb{Z}_3$
$\mathbb{Z}_2 \lhd 2\mathrm{O}$	O
$2\mathrm{D}_4 \lhd 2\mathrm{O}$	D_6
$2\mathrm{T} \lhd 2\mathrm{O}$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \lhd 2\mathrm{I}$	I

Table 1.4: Subgroups II.

Here T, O, I denote the usual polyhedral groups and D_{2n} the dihedral group.

1.4.1 Automorphisms of quotient groups of ADE groups

The present section comprises descriptions of automorphisms groups of quotients of ADE groups that are relevant in the forthcoming sections. The main reference is [CV], but the material will be however adapted to our needs.

(1) The group of outer automorphisms of $\mathbb{Z}_n$ is given by

$$\mathrm{Out}(\mathbb{Z}_n) = \{\varphi(r)\ :\ \gcd(r,n) = 1\},$$

where $\varphi(r)$ denotes the map $\mathbb{Z}_n \ni x \mapsto x^r \in \mathbb{Z}_n$ in multiplicative notation.

(2) To describe the outer automorphism group of a dihedral group D_{2n}, consider the following presentation of D_{2n}

$$\mathrm{D}_{2n} = \left\langle x, y\ :\ x^2 = y^n = (xy)^2 = 1\right\rangle = \{y^p :\ 0 \le p < n\} \cup \{xy^p :\ 0 \le p < n\}.$$

Observe $\mathrm{D}_2 = \mathbb{Z}_2$, so we can assume that $n > 1$. The case $n = 2$ is also special as D_4 is isomorphic to the Klein Vierergruppe. The automorphism group of D_4 isomorphic to Sym(3) and acts by permutations of the 3 non trivial involutions. The outer automorphism group of D_{2n} for $n > 2$ is

$$\mathrm{Out}(\mathrm{D}_{2n}) = \langle \tau_{a,b}\ :\ (a,b) \in \mathbb{Z}_n^\times \times \mathbb{Z}_n\rangle \cong \mathbb{Z}_n^\times \ltimes \mathbb{Z}_n,$$

where the action of the affine group $\mathbb{Z}_n^\times \ltimes \mathbb{Z}_n$ on D_{2n} is given by

$$\tau_{a,b}(y^p) = y^{ap} \quad , \quad \tau_{a,b}(xy^p) = xy^{ap+b}.$$

(Here $\mathbb{Z}_n$ denotes the additive group and $\mathbb{Z}_n^\times$ the multiplicative group of units in the ring $\mathbb{Z}_n$.)

(3) Since $2\mathrm{D}_2 \cong \mathbb{Z}_4$, we consider $2\mathrm{D}_{2n}$ only for $n > 1$. We have the following presentation:

$$2\mathrm{D}_{2n} = \left\langle s, t :\ s^2 = t^n = (st)^2\right\rangle = \{t^p :\ 0 \le p < 2n\} \cup \{st^p\ :\ 0 \le p < 2n\}.$$

In fact, we can take $s = je^{i\frac{\pi}{n}}$ and $t = e^{i\frac{\pi}{n}}$ when $2\mathrm{D}_{2n}$ is realized as a subgroup of SU(2). The outer automorphism group of $2\mathrm{D}_{2n}$ for $n > 2$ is also an affine group:

$$\mathrm{Out}(2\mathrm{D}_{2n}) = \langle \tau_{a,b}\ :\ (a,b) \in \mathbb{Z}_{2n}^\times \times \mathbb{Z}_{2n}\rangle \cong \mathbb{Z}_{2n}^\times \ltimes \mathbb{Z}_{2n},$$

where the action on $2\mathrm{D}_{2n}$ is given by

$$\tau_{a,b}(t^p) = t^{ap} \quad , \quad \tau_{a,b}(st^p) = st^{ap+b}.$$

We need to make a distinction for $n = 2$. Any automorphism of $2\mathrm{D}_4 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathrm{SU}(2)$ is obtained via conjugation with an element in 2O modulo $\mathbb{Z}_2 = \{\pm 1\}$. The pointwise action of ${}^{2\mathrm{O}}\!/\mathbb{Z}_2$ on $2\mathrm{D}_4$ is described below.

	i	j	k		i	j	k
$[i]$	i	$-j$	$-k$	$[\frac{1}{\sqrt{2}}(1-i)]$	i	$-k$	j
$[j]$	$-i$	j	$-k$	$[\frac{1}{\sqrt{2}}(j+k)]$	$-i$	k	j
$[k]$	$-i$	$-j$	k	$[\frac{1}{\sqrt{2}}(j-k)]$	$-i$	$-k$	$-j$
$[\frac{1}{2}(1+i+j+k)]$	j	k	i	$[\frac{1}{\sqrt{2}}(i+k)]$	k	$-j$	i
$[\frac{1}{2}(1-i-j-k)]$	k	i	j	$[\frac{1}{\sqrt{2}}(1-k)]$	$-j$	i	k
$[\frac{1}{2}(1+i-j-k)]$	$-j$	k	$-i$	$[\frac{1}{\sqrt{2}}(i-k)]$	$-k$	$-j$	$-i$
$[\frac{1}{2}(1+i+j-k)]$	$-k$	i	$-j$	$[\frac{1}{\sqrt{2}}(i+j)]$	j	i	$-k$
$[\frac{1}{2}(1-i+j-k)]$	$-j$	$-k$	i	$[\frac{1}{\sqrt{2}}(1+j)]$	$-k$	j	i
$[\frac{1}{2}(1-i-j+k)]$	j	$-k$	$-i$	$[\frac{1}{\sqrt{2}}(1-j)]$	k	j	$-i$
$[\frac{1}{2}(1-i+j+k)]$	$-k$	$-i$	j	$[\frac{1}{\sqrt{2}}(1+k)]$	j	$-i$	k
$[\frac{1}{2}(1+i-j+k)]$	k	$-i$	$-j$	$[\frac{1}{\sqrt{2}}(i-j)]$	$-j$	$-i$	$-k$
$[\frac{1}{\sqrt{2}}(1+i)]$	i	k	$-j$				

Table 1.5: Action of ${}^{2\mathrm{O}}/_{\mathbb{Z}_2}$ on $2\mathrm{D}_4$.

(4) The outer automorphism group of $2\mathrm{T} \subset \mathrm{SU}(2)$ is generated by an involution that exchanges the generators $s = \frac{1}{2}(1+i)(1+j), t = \frac{1}{2}(1+j)(1+i)$, which satisfy the relations $s^3 = t^3 = (st)^3$. This automorphism is given by conjugation with $\frac{1+j}{\sqrt{2}} \in 2\mathrm{O} \subset \mathrm{SU}(2)$.

(5) The outer automorphism group of $2\mathrm{O} \subset \mathrm{SU}(2)$ is generated by an involution φ fixing s and sending t to $-t$, where $s = \frac{1}{2}(1+i+j+k)$ and $t = e^{\frac{i\pi}{4}}$ generate 2O.

(6) The outer automorphism group of $2\mathrm{I} \subset \mathrm{SU}(2)$ is generated by an involution ψ which fixes s and sends t to $\frac{-\phi^{-1}-\phi i+k}{2}$, where $s = \frac{1}{2}(1+i+j+k)$ and $t = \frac{\phi+\phi^{-1}i+j}{2}$ generate 2I. The action of the automorphisms $\varphi \in \mathrm{Out}(2\mathrm{O})$ and $\psi \in \mathrm{Out}(2\mathrm{I})$ on conjugacy classes $\mathcal{C}(x)$, for $x \in 2\mathrm{O}$ or $x \in 2\mathrm{I}$ respectively, is described in the following tables.

Representative	Size	Real parts
1	1	1
-1	1	-1
s	8	$\frac{1}{2}$
t	6	$\frac{1}{\sqrt{2}}$
s^2	8	$-\frac{1}{2}$
t^2	6	0
t^3	6	$-\frac{1}{\sqrt{2}}$
st	12	0

Table 1.6: Conjugacy classes in 2O .

$\mathcal{C}(x)$	$\varphi(\mathcal{C}(x))$	$\mathrm{Re}(\varphi(x))$
$\mathcal{C}(1)$	$\mathcal{C}(1)$	1
$\mathcal{C}(-1)$	$\mathcal{C}(-1)$	-1
$\mathcal{C}(s)$	$\mathcal{C}(s)$	$\frac{1}{2}$
$\mathcal{C}(t)$	$\mathcal{C}(t^3)$	$-\frac{1}{\sqrt{2}}$
$\mathcal{C}(s^2)$	$\mathcal{C}(s^2)$	$-\frac{1}{2}$
$\mathcal{C}(t^2)$	$\mathcal{C}(t^2)$	0
$\mathcal{C}(t^3)$	$\mathcal{C}(t)$	$\frac{1}{\sqrt{2}}$
$\mathcal{C}(st)$	$\mathcal{C}(st)$	0

Table 1.7: Action of φ.

Representative	Size	Real parts
1	1	1
-1	1	-1
t	12	$\frac{1+\sqrt{5}}{4}$
t^2	12	$-\frac{1-\sqrt{5}}{4}$
t^3	12	$\frac{1-\sqrt{5}}{4}$
t^4	12	$-\frac{1+\sqrt{5}}{4}$
s	20	$\frac{1}{2}$
s^4	20	$-\frac{1}{2}$
st	30	0

Table 1.8: Conjugacy classes in 2I.

$\mathcal{C}(x)$	$\psi(\mathcal{C}(x))$	$\mathrm{Re}(\psi(x))$
$\mathcal{C}(1)$	$\mathcal{C}(1)$	1
$\mathcal{C}(-1)$	$\mathcal{C}(-1)$	-1
$\mathcal{C}(t)$	$\mathcal{C}(t^3)$	$\frac{1-\sqrt{5}}{4}$
$\mathcal{C}(t^2)$	$\mathcal{C}(t^4)$	$-\frac{1+\sqrt{5}}{4}$
$\mathcal{C}(t^3)$	$\mathcal{C}(t)$	$\frac{1+\sqrt{5}}{4}$
$\mathcal{C}(t^4)$	$\mathcal{C}(t^2)$	$-\frac{1-\sqrt{5}}{4}$
$\mathcal{C}(s)$	$\mathcal{C}(s)$	$\frac{1}{2}$
$\mathcal{C}(s^4)$	$\mathcal{C}(s^4)$	$-\frac{1}{2}$
$\mathcal{C}(st)$	$\mathcal{C}(st)$	0

Table 1.9: Action of ψ.

(7) The tetrahedral group T is isomorphic to the alternating group Alt(4), which has automorphism group Sym(4), acting by conjugation on the normal subgroup Alt(4). This corresponds to the action of the octahedral group O $\cong$ Sym(4) on its normal subgroup T, which is induced by the action of 2O on the normal subgroup 2T. In fact, it can be derived from Table 1.5 that the image of $\mathrm{O} = {}^{2\mathrm{O}}/_{\mathbb{Z}_2}$ in $\mathrm{Aut}(\mathrm{T}) = \mathrm{Aut}({}^{2\mathrm{T}}/_{\mathbb{Z}_2})$ is isomorphic to Sym(4).

(8) Every automorphism of O is inner.

(9) The outer automorphism group Out(I) of the icosahedral group I is isomorphic to $\mathbb{Z}_2$. The non-trivial outer automorphism of 2I induces an automorphism $\varphi \in \mathrm{Aut(I)}$ by realizing I as $2\mathrm{I}/Z(2\mathrm{I})$, where $Z(2\mathrm{I})$ denotes the center of 2I. Observe now that two elements $[x],[y] \in \mathrm{SU}(2)/Z(\mathrm{SU}(2))$ are conjugate to one another if and only if either $\mathrm{Re}(x) = \mathrm{Re}(y)$ or $\mathrm{Re}(x) = -\mathrm{Re}(y)$. It follows from this observation that $\varphi \in \mathrm{Aut(I)}$ can not be inner, see Table 1.9.

Chapter 2

Lie groups

The manifolds we will encounter from now on are locally isometric to a (compact) Lie group G endowed with a left invariant metric. We start in Section 2.1 by recalling some spin geometric facts in the context of Lie groups, where the standard reference we used for this material is [LM]. Then we proceed in Sections 2.2 and 2.3 of this chapter to analyze the existence of Sunada triples for the spin group, see Section 3.1. The main technical original result in this regard is Theorem 2.2.6, which relies on all the material introduced in Chapter 1. This chapter finishes with Section 2.4, where we give a formula to compute the volume of a compact Lie group that will be used in Chapter 3 to compute spectral invariants of locally homogeneous six dimensional nearly Kähler manifolds. Here, and in Section 3.1, we will assume some general knowledge of the classical theory of Lie groups for which we refer to [BrD].

2.1 Spin structures and the Dirac operator

Let G be a compact Lie group of dimension $n \in \mathbb{N}$ endowed with a left invariant Riemannian metric $g \in \mathrm{Sym}^2_+(TG)$ and denote the associated bundle of orthonormal oriented frames by $\mathrm{SO}(G)$.

Definition 2.1.1. A *spin structure* on the Lie group G is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(G) \longrightarrow G$ together with a bundle homomorphism $\varphi : \mathrm{Spin}(G) \longrightarrow \mathrm{SO}(G)$ which restricts fiberwise to the standard covering $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$, i.e. $\varphi(g{\cdot}u) = \pi(g){\cdot}\phi(u)$ for all $u \in \mathrm{Spin}(G)$ and $g \in \mathrm{Spin}(n)$.

A manifold which admits a spin structure is called *spin manifold*. Just as we have discussed in Section 1.3 from the viewpoint of classifying spaces, the obstruction for the existence of a spin structure is a torsion class $w_2(G) \in H^2(G, \mathbb{Z}_2)$ called *second Stiefel-Whitney class*. Spin structures on the Lie group G are in turn parametrized by elements in the first cohomology group $H^1(G, \mathbb{Z}_2)$.

Example 2.1.2. Any n-dimensional Lie group G admits a spin structure, since its frame bundle is trivial: $\mathrm{SO}(G) = G \times \mathrm{SO}(n)$. Moreover, if the Lie group G is in addition simply connected then the spin structure is also trivial, i.e. it is given by

$$G \times \mathrm{Spin}(n) \ni (g, u) \longmapsto (g, \pi(u)) \in G \times \mathrm{SO}(n) = \mathrm{SO}(G),$$

where $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the standard covering.

Since the spin group $\mathrm{Spin}(n)$ is a covering of the special orthogonal group $\mathrm{SO}(n)$, any representation of $\mathrm{SO}(n)$ gives rise to an representation of $\mathrm{Spin}(n)$. There is however a representation (γ, Σ_n) of the spin group $\mathrm{Spin}(n)$ that does *not* come from a representation of the special orthogonal group. This irreducible representation is called *spin representation* and turns out to extend to the Clifford algebra $\mathcal{C}l(\mathbb{R}^n) \supset \mathrm{Spin}(n)$, see [Fr] for an explicit realization. The *spinor module* Σ_n has dimension $2^{[\frac{n}{2}]}$ and turns out to be irreducible if $n \in \mathbb{N}$ is odd, whereas it splits as sum $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ of irreducible $\mathrm{Spin}(n)$-modules $\Sigma_n^\pm$, when $n \in \mathbb{N}$ is even.

Elements $\varphi \in \Sigma_n$ in the spinor module Σ_n are called (algebraic) *spinors*, and the element $v \cdot \varphi \in \Sigma_n$ obtained from the action of a vector $v \in \mathbb{R}^n \subset \mathcal{C}l(\mathbb{R}^n)$ on an algebraic spinor $\varphi \in \Sigma_n$ is called *Clifford multiplication* (or contraction) of $v \in \mathbb{R}^n$ with $\varphi \in \Sigma_n$. Observe at this stage that one can endow the representation space Σ_n with a Hermitian scalar product $\langle \cdot\ ,\ \cdot \rangle_\Sigma$ so that the action $v \cdot \varphi = \gamma(v)\varphi \in \Sigma_n$ is unitary, by applying Weyl unitarian trick.

The algebraic concepts we have just introduced generalize to global objects on the Riemannian manifold (G, g): the *bundle of spinors* of G is the geometric[1] vector bundle $(\Sigma(G), \nabla^\Sigma)$, where $\Sigma(G)$ is the associated bundle $\Sigma(G) = \mathrm{Spin}(G) \times_{\mathrm{Spin}(n)} \Sigma_n$ and ∇^Σ is the connection induced by the Levi-Civita connection form, i.e. the matrix valued one form $\omega = (\omega_\alpha^\beta)$ defined locally by the equation

$$\nabla e_\alpha = \sum_{\beta=1}^{n} \omega_\alpha^\beta(\cdot) \otimes e_\beta,$$

where $(e_1, ..., e_n)$ is a local frame of TG. Ths bundle $\Sigma(G)$ comes equipped with a Hermitian metric

$$G \ni m \longmapsto \langle \cdot\ ,\ \cdot \rangle_m^{\Sigma(G)} \in \mathrm{Sym}_+^2(T_m G), \tag{2.1}$$

which depends smoothly on the base point $m \in G$ and a smooth representation map

$$\gamma : TM \longrightarrow \mathrm{Aut}(\Sigma(G)); \quad X \longmapsto \gamma(X) : \varphi \mapsto \gamma(X)\varphi = X \cdot \varphi,$$

which resembles fiberwise the Clifford action $\gamma : \mathbb{R}^n \longrightarrow \mathrm{Aut}(\Sigma_n)$ and is unitary with respect to the Hermitian metric $\langle \cdot\ ,\ \cdot \rangle^{\Sigma(G)}$ given in (2.1). We will sometimes abuse the notation by writing γ to mean the map $\mathbb{R}^n \times \Sigma_n \longrightarrow \Sigma_n$ obtained from the Clifford action $\mathbb{R}^n \longrightarrow \mathrm{Aut}(\Sigma_n)$.

Finally, we define the most natural operator associated with a Riemannian spin manifold.

Definition 2.1.3. Let (G, g) be a compact Lie group endowed with a left invariant metric $g \in \mathrm{Sym}_+^2(G)$. The Dirac operator $D_g : \Sigma(G) \longrightarrow \Sigma(G)$ relative to the metric g

[1] A geometric vector bundle over (G, g) is a pair (E, ∇), where E is a vector bundle obtained from the frame bundle $\mathrm{SO}(G)$ by means of a representation $\rho : \mathrm{SO}(n) \longrightarrow \mathrm{Aut}(V)$ and ∇ is the connection on E induced by a chosen connection form on $\mathrm{SO}(G)$.

and the underlying spin structure of G is defined to be the composition:

$$D_g = \gamma \circ ((\cdot)^\sharp \times \mathrm{Id}_{\Sigma(G)}) \circ \nabla^\Sigma : \Sigma(G) \longrightarrow T^*G \otimes \Sigma(G) \longrightarrow TG \otimes \Sigma(G) \longrightarrow \Sigma(G),$$

where $\sharp : T^*G \longrightarrow TG$ denotes the musical isomorphism with respect to the metric $g \in \mathrm{Sym}^2_+(TG)$. In a local frame $(e_1, ..., e_n)$ of the tangent bundle TG we have

$$D_g\varphi = \sum_{i=1}^{n} \gamma(e_i)\nabla^\Sigma_{e_i}\varphi, \qquad \varphi \in \Sigma(G).$$

The Dirac operator can be shown to be self-adjoint with respect to the L^2-metric induced by the Hermitian product (2.1) and its symbol to be given by $\sigma_\zeta(D_g) = \gamma(\zeta)(\cdot)$. Since the Clifford algebra has no zero divisors, the Dirac operator is elliptic[2]. Being G compact, general elliptic theory assures that the spectrum of $D_g : \Sigma(G) \longrightarrow \Sigma(G)$ consists in (real) eigenvalues and each eigenspace is finite dimensional.

2.2 Sunada Lie groups

The fundamental notion of this section is the one of *almost conjugacy*, which is motivated by the classical result of Sunada, see Theorem 3.1.1.

Definition 2.2.1. A pair of finite subgroups Γ_i of a group G are *almost conjugate* if there is a bijection $\phi : \Gamma_1 \longmapsto \Gamma_2$ that preserves conjugacy classes of elements in G, i.e. any $\gamma \in \Gamma_1$ is conjugate to $\phi(\gamma)$ in G by an element $x = x(\gamma) \in G$.

Clearly two conjugate subgroups of a given compact Lie group G are almost conjugate. However, two almost conjugate subgroups of G are not necessarily conjugate, as we shall see shortly. To distinguish between almost conjugacy and conjugacy is a problem that has many roots in pure mathematics. In our context, the importance of this subtle difference relies in the fact that it allows us to decide whether certain finite quotients are isometric or not, see Lemma 3.2.3. There is also an interesting relation between the aforementioned distinction between conjugacy and almost conjugacy with the theory of automorphic forms, see [Bl], that served M. Larsen as a motivation to study this question, see [Lar, Lar2]. He constructed a pair of non-conjugate almost conjugate subgroups by considering *element-wise conjugate* homomorphisms:

Definition 2.2.2. Let G be a Lie group and let Γ be a finite group. Two homomorphisms $\phi_i : \Gamma \longmapsto G$ are *element-wise* or *almost* conjugate, if for any $\gamma \in \Gamma$ there is an element $x(\gamma) \in G$ so that

$$\phi_1(\gamma) = x(\gamma)\phi_2(\gamma)x(\gamma)^{-1}. \tag{2.2}$$

In addition, a compact Lie group G is called *Sunada* if there is a finite group Γ and a pair of element-wise conjugate homomorphisms $\phi_i : \Gamma \longrightarrow G$ so that the groups $\phi_i(\Gamma) \subset G$ are not related by any inner automorphism of G.

[2]A differential operador D is elliptic provided it symbol $\sigma_\zeta(D)$ is an isomorphism for any non-zero covector $\zeta \in TM'$.

Remark. It is worthwhile to mention that being a Sunada Lie group is a stronger condition than the one of just admitting almost conjugate finite subgroups. Indeed, Spin(6) admits such subgroups while it is not Sunada, see [Lar].

Clearly, if a compact Lie group G is Sunada and $\phi_i : \Gamma \longrightarrow G$ are homomorphisms satisfying the conditions described in Definition 2.2.2, then the groups $\phi_i(\Gamma)$ are almost conjugate subgroups of G that are not conjugate. M. Larsen showed in Proposition 2.6 of [Lar2] that Spin(35) is Sunada. His proof is nonetheless rather ad-hoc and contains a little mistake that is *not* harmful. In order to turn out Larsen's idea into an Ansatz to produce almost conjugate groups, we need some technical lemmata.

Lemma 2.2.3. *Let* $\Gamma = \mathrm{Sym}(m)$ *and let* $\rho : \Gamma \longrightarrow \mathrm{O}(n)$ *be a representation with character* χ. *Denote by* $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ *the standard two-fold covering and let* $x, y \in \Gamma$ *be any disjoint transpositions. The following statements hold.*

(a) ρ *takes values in* $\mathrm{SO}(n)$ *if and only if* $\chi(x) = n \mod 4$.

(b) Suppose that $\rho(z) \in \mathrm{SO}(n)$ *for* $z \in \{x, xy\}$. *The element* $\rho(z)$ *lifts to an element of order two in* $\mathrm{Spin}(n)$ *if and only if* $\chi(z) = n \mod 8$.

(c) Suppose that $\rho(z) \in \mathrm{SO}(n)$ *for* $z \in \{x, xy\}$. *The element* $\rho(z)$ *lifts to an element of order four in* $\mathrm{Spin}(n)$ *if and only if* $\chi(z) = n \mod 4$ *and* $\chi(z) \neq n \mod 8$.

Proof. Let $x = x_1 ... x_N$ be a product of disjoint transpositions $x_i \in \Gamma = \mathrm{Sym}(m)$. Since the operators $\rho(x_i) \in \mathrm{O}(n)$ commute, the element $\rho(x)$ is an involution and it has eigenvalues 1 and -1. The multiplicities $p = \dim \mathrm{Eig}(\rho(x), 1)$ and $q = \dim \mathrm{Eig}(\rho(x), -1)$ satisfy the following arithmetic relations

$$\chi(x) = p - q, \quad n = p + q, \quad \det(\rho(x)) = (-1)^q, \tag{2.3}$$

from which it can be derived that $2q = n - \chi(x)$.

As the group Γ is generated by transpositions, the representation $\rho : \Gamma \longrightarrow \mathrm{O}(n)$ factorizes through $\mathrm{SO}(n)$ precisely when $\det(\rho(x)) = 1$ for any transposition $x \in \Gamma$. However, from the identity $2q = n - \chi(x)$, we see that the this is equivalent to

$$\chi(x) = n \mod 4$$

for any transposition $x \in \Gamma = \mathrm{Sym}(m)$, which is claim (a). Suppose now that ρ factorizes through $\mathrm{SO}(n)$ and let $q = 2q'$. The element $\rho(x) = \rho(x_1 ... x_N) \in \mathrm{SO}(n)$ conjugates to the element $s_x \in T_{\mathrm{SO}(n)}$ in the standard maximal torus $T_{\mathrm{SO}(n)}$ given by

$$s_x = \mathrm{diag}(\underbrace{A, ..., A}_{q'-times}, \mathrm{Id}_p) \in T_{\mathrm{SO}(n)}\ , \quad A = \exp_{\mathfrak{so}(2)} \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} \in \mathrm{SO}(2),$$

which can be written in the standard orthonormal basis $(e_1, ... e_n)$ of $\mathbb{R}^n$ as

$$\mathrm{SO}(n) \ni s_x = \exp_{\mathfrak{so}(n)}(\pi(e_1 \wedge e_2 + ... + e_{q'-1} \wedge e_{q'})).$$

Let $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ be the standard covering of $\mathrm{SO}(n)$ and $\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Clifford multiplication. Recall that the differential of the projection $d\pi : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ is a Lie algebra isomorphism that sends $\frac{1}{2}e_i \cdot e_j \in \mathfrak{spin}(n)$ to $e_i \wedge e_j \in \mathfrak{so}(n)$. The elements in the fiber $\pi^{-1}(s_x) \subset \mathrm{Spin}(n)$ of $s_x \in T_{\mathrm{SO}(n)}$ are hence of the form

$$\bar{s}_{x,\varepsilon} = \varepsilon \exp_{\mathfrak{spin}(n)}\left(\frac{\pi}{2}(e_1 \cdot e_2 + \ldots + e_{q'-1} \cdot e_{q'})\right),$$

where $\varepsilon \in Z(\mathrm{Spin}(n))$ is a central element in $\mathrm{Spin}(n)$, and direct calculation yields

$$\bar{s}^2_{x,\varepsilon} = \exp^2_{\mathfrak{spin}(n)}\left(\frac{\pi}{2}(e_1 \cdot e_2 + \ldots + e_{q'-1} \cdot e_{q'})\right) = (e_1 \cdot e_2)^2 \ldots (e_{q'-1} \cdot e_{q'})^2 = \varepsilon^{q'}\,\mathrm{Id}\,.$$

Claims (b) and (c) follow, as we have seen that $4q' = n - \chi(x)$. □

Lemma 2.2.4. *Let $\pi : \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n)$ be the usual covering and $x \in \mathrm{SO}(n)$ be an element such that $\mu(x,-1) = \dim \mathrm{Eig}(x,-1) > 0$. Then the following statements hold.*

(a) If n is odd, then the elements in $\pi^{-1}(x)$ are conjugate to one another in $\mathrm{Spin}(n)$.

(b) The elements in $\pi^{-1}(x)$ are conjugate to one another in $\mathrm{Pin}(n)$.

Proof. If $n = 2m+1$, then the two elements in the fiber of an element $x \in \mathrm{SO}(n)$ with $\mu(x,-1) > 1$ are conjugate to each other in $\mathrm{Spin}(n)$, see Lemma 3.9 in [Lar], and *a fortiori* in $\mathrm{Pin}(n)$. So we can suppose that $n = 2m$. An element $x \in \mathrm{SO}(n)$ with $\mu(x,-1) > 0$ conjugates to the following element $s_x \in T_{\mathrm{SO}(n)}$ in the standard maximal torus $T_{\mathrm{SO}(n)}$ of $\mathrm{SO}(n)$

$$s_x = \mathrm{diag}(R(\theta_1), R(\theta_2), \ldots, R(\theta_m)) \in T_{\mathrm{SO}(n)},$$

where $R(\theta) \in \mathrm{SO}(2)$ is a rotation with angle $\theta \in [0, 2\pi]$ and $\theta_1 = \pi$. The element in the maximal torus $s_x \in T_{\mathrm{SO}(n)}$ that corresponds to $x \in \mathrm{SO}(n)$ has fiber

$$\pi^{-1}(s_x) = \{\bar{s}_{x,\varepsilon} \in T_{\mathrm{Spin}(n)} \quad : \quad \varepsilon \in Z(\mathrm{Spin}(n))\} \subset \mathrm{Spin}(n),$$

where $\varepsilon \in Z(\mathrm{Spin}(n))$ is a central element and

$$\bar{s}_{x,\varepsilon} = \varepsilon \prod_{j=1}^{m} \mu(\theta_j)\,, \qquad \mu(\theta_j) = \cos\left(\frac{\theta_j}{2}\right) + (e_{2j-1} \cdot e_{2j}) \sin\left(\frac{\theta_j}{2}\right) \in T_{\mathrm{Spin}(n)}.$$

Conjugation with $e_1 \in \mathrm{Pin}(n)$ exchanges $e_1 \cdot e_2 \in \mathrm{Spin}(n)$ and $\varepsilon e_1 \cdot e_2 \in \mathrm{Spin}(n)$ and fixes $e_j \cdot e_{j+1} \in \mathrm{Spin}(n)$ for any $2 \leq j < m$, and so it exchanges the elements in the fiber of $s_x \in T_{\mathrm{SO}(n)}$. □

In what follows, we will use the standard parametrization of conjugacy classes and irreducible representations of the symmetric group by using Young diagrams, as described in Sections 1.2.1 and 1.2.2.

Lemma 2.2.5. *Two generators of the cyclic group generated by an arbitrary permutation $z \in \mathrm{Sym}(m)$ are conjugate to each other in* $\mathrm{Sym}(m)$.

Proof. We must show that z^r is conjugate to z for any $r \in \mathbb{Z}_l^\times$, where $l = \operatorname{ord}(z)$. Decompose z as a product of n disjoint cycles Z_j of length m_j and note that $1 = \gcd(r, m_j)$ for any $j \leq n$. The r-fold composition of each individual cycle Z_j leads to another cycle of length m_j. It follows that z and z^r have the same cycle decomposition and are thus conjugate, see Proposition 1.2.4. □

Theorem 2.2.6. *Let* $\Gamma = \operatorname{Sym}(m)$ *for* $6 \neq m \geq 4$ *and let* $\rho : \Gamma \longrightarrow \mathrm{O}(n)$ *be a faithful irreducible representation of dimension* $n = n(\rho)$ *whose character* $\chi = \chi(\rho)$ *fulfills the following conditions*

$$\chi(x) = n \mod 8, \tag{2.4}$$

$$\chi(xy) = n \mod 8, \tag{2.5}$$

$$M(z) := \frac{1}{l} \sum_{k=0}^{l-1} (-1)^k \chi(z^k) > 0, \tag{2.6}$$

where $x, y \in \Gamma$ *are any disjoint transpositions and* $z \in \Gamma$ *is an arbitrary odd permutation with order* $l = l(z) = \operatorname{ord}(z)$. *The following statements hold.*

(a) The group $\operatorname{Spin}(n)$ *is Sunada provided* $n = n(\rho)$ *is odd.*

(b) The group $\operatorname{Pin}(n)$ *is Sunada.*

Proof. By Lemma 2.2.3 (a), we can suppose that $\rho : \Gamma \longrightarrow \mathrm{SO}(n)$. Let us consider the canonical central extension $\hat{\Gamma}$ of Γ by $\mathbb{Z}_2$, i.e. the central extension $\hat{\Gamma}$ making the natural diagram

$$\begin{array}{ccc} \hat{\Gamma} & \xrightarrow{\rho^*} & \operatorname{Spin}(n) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\ \Gamma & \xrightarrow{\rho} & \mathrm{SO}(n) \end{array}$$

commute. Here $\pi : \operatorname{Pin}(n) \longrightarrow \mathrm{O}(n)$ is the standard covering of $\mathrm{O}(n)$ and π^* and ρ^* are the homomorphisms induced by the pullback.

Claim. The cohomology class $w_2(\Gamma, \mathbb{Z}_2) \in H^2(\Gamma, \mathbb{Z}_2)$ is trivial.

Let us first recall that isomorphism classes of central extensions of Γ are in one to one correspondence with elements in the second cohomology group $H^2(\Gamma, \mathbb{Z}_2)$, as explained in Section 1.3. We saw in Example 1.3.4 that the second cohomology group $H^2(\Gamma, \mathbb{Z}_2)$ is isomorphic to the Klein Vierergruppe for $m \geq 4$. Its non-trivial elements correspond to group coverings $2 \cdot \Gamma^\pm$ and an additional extension which we denote by $L = 2 \cdot \Gamma^+ + 2 \cdot \Gamma^-$.[3] The description of the first two extensions is obtained from the fact that $\Gamma = \operatorname{Sym}(m)$ admits a Coxeter presentation, see Section 2.8 in [Wi]:

$$\Gamma = \left\langle t_1, \cdots, t_{m-1} \;:\; t_i^2 = 1,\ (t_i t_j)^2 = 1 \text{ if } |i-j| > 1,\ (t_i t_{i+1})^3 = 1 \right\rangle. \tag{2.7}$$

[3] It will be clear from the presentation of these groups (or Table 2.1) that these groups indeed correspond to non-trivial elements in $H^2(\Gamma, \mathbb{Z}_2)$.

The double covers $2 \cdot \Gamma^{\pm}$ admit also a presentation in m generators $z, s_1, ..., s_{m-1}$, which collapse to (2.7) once projecting to Γ, see [Sch] page 244. The relations for $2 \cdot \Gamma^{-}$ read

$$z^2 = 1, \tag{2.8}$$
$$s_i^2 = z, \qquad 1 \le i \le m-1, \tag{2.9}$$
$$s_{i+1} \cdot s_i \cdot s_{i+1} = s_i \cdot s_{i+1} \cdot s_i, \qquad 1 \le i \le m-2, \tag{2.10}$$
$$s_j \cdot s_i = s_i s_j z, \qquad 1 \le i < j \le m-1, \quad |i-j| \ge 2, \tag{2.11}$$

whereas for $2 \cdot \Gamma^{+}$ we have

$$z^2 = 1, \tag{2.12}$$
$$s_i \cdot z = z \cdot s_i, \qquad 1 \le i \le m-1, \tag{2.13}$$
$$s_i^2 = 1, \qquad 1 \le i \le m-1, \tag{2.14}$$
$$s_{i+1} \cdot s_i \cdot s_{i+1} = s_i \cdot s_{i+1} \cdot s_i z, \qquad 1 \le i \le m-2, \tag{2.15}$$
$$(s_i \cdot s_j)^2 = z, \qquad 1 \le i < j \le m-1, \quad |i-j| \ge 2. \tag{2.16}$$

Let $x, y \in \Gamma = \mathrm{Sym}(m)$ be disjoint transpositions. Any transposition $x \in \Gamma$ is conjugate to a Coxeter generator $t_i \in \Gamma$, and those generators lift to elements of order two or four depending on the isomorphism type of the canonical extension $\hat{\Gamma}$. Let us now consider the order of the lifts of elements $x, xy \in \Gamma$. In case the canonical extension $\hat{\Gamma}$ is trivial, these elements lift to involutions. If $\hat{\Gamma}$ is isomorphic to one of the coverings $2 \cdot \Gamma^{\pm}$, the order of the lifted elements can be read off from the presentations of such coverings and it is either two or four, see identity (2.9), (2.11) and (2.14) and (2.16). Finally, the additional non-trivial element in $H^2(\Gamma, \mathbb{Z}_2)$ can be explicitly read off from the discussion in Section 1.3: indeed, if $\varphi_{\pm} : \Gamma \times \Gamma \longrightarrow \mathbb{Z}_2$ are the cocycles obtained from $2 \cdot \mathrm{Sym}(m)^{\pm}$ respectively, then a cocycle defining L is $\varphi_L = \varphi_+ + \varphi_-$. A little experimentation to make the function φ_L satisfy the cocyle condition, see Proposition 1.3.3, tells us that L is generated by elements $z, s_1, ..., s_{m-1}$ subject to relations

$$z^2 = 1, \tag{2.17}$$
$$s_i^2 = z, \qquad 1 \le i \le m-1, \tag{2.18}$$
$$s_{i+1} \cdot s_i \cdot s_{i+1} = s_i \cdot s_{i+1} \cdot s_i z, \qquad 1 \le i \le m-2, \tag{2.19}$$
$$s_j \cdot s_i = s_i s_j, \qquad 1 \le i < j \le m-1, \quad |i-j| \ge 2. \tag{2.20}$$

It follows from identities (2.18) and (2.20), that $x \in \Gamma$ lifts to an element of order 4 and $xy \in \Gamma$ lifts to an element of order 2. Table 2.1 summarizes the previous analysis.

Extension	$\Gamma \times \mathbb{Z}_2$	Γ^+	Γ^-	L
$\mathrm{Ord}(\bar{x})$	2	2	4	4
$\mathrm{Ord}(\overline{xy})$	2	4	4	2

Table 2.1: Order of the lifts of $x \in \Gamma$ and $xy \in \Gamma$.

Conditions (2.4)-(2.5) assure that the elements $x, xy \in \Gamma$ lift to elements of order 2 in $\hat{\Gamma}$, see Lemma 2.2.3, and so the extension $\hat{\Gamma}$ must be trivial and the claim follows from

Theorem 1.3.1.

As discussed in Section 1.3, the last claim implies that we can choose a homomorphism $\phi_1 : \Gamma \longrightarrow \mathrm{Spin}(n)$ lifting the representation ρ, i.e. $\pi \circ \phi_1 = \rho$, where $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the standard covering. Put

$$\phi_2(\gamma) = \eta(\gamma)\phi_1(\gamma), \quad \gamma \in \Gamma,$$

where $\eta : \Gamma \longrightarrow \mathbb{Z}_2$ is the natural homomorphism, and $\mathbb{Z}_2$ acts as multiplication with a central element $\pm 1 \in \mathrm{Spin}(n)$.

Claim. The groups $\phi_i(\Gamma)$ are almost conjugate in $\mathrm{Pin}(n)$ for any $n \in \mathbb{N}$, and in $\mathrm{Spin}(n)$ for odd $n \in \mathbb{N}$.

For this, it suffices the check that $\rho(z)$ has -1 as an eigenvalue, for any odd permutation $z \in \Gamma$, see Lemma 2.2.4. Consider the cyclic group $\mathbb{Z}_{2l} \subset \Gamma$ generated by an odd permutation $z \in \Gamma$. Denote by $\mathcal{Y}|_{\langle z \rangle}$ the restriction of ρ to $\mathbb{Z}_{2l}$ and decompose it into isotypic components

$$V^{\mathcal{Y}|_{\langle z \rangle}} = \bigoplus_{\lambda \in \mathbb{Z}_{2l}} M_\lambda R_\lambda = \bigoplus_{d|2l} \left(\bigoplus_{\gcd(\lambda,2l)=d} M_\lambda R_\lambda \right), \tag{2.21}$$

where R_λ is the representation space of the representation of $\mathbb{Z}_{2l}$ with character $\chi_\lambda(\cdot) = e^{\frac{2\pi i \lambda(\cdot)}{2l}}$ and M_λ is its multiplicity in $\mathcal{Y}|_{\langle z \rangle}$. Any two generators of $\mathbb{Z}_{2l} = \langle z \rangle$ are conjugate in Γ, see Lemma 2.2.5. In particular, the action of any element $\sigma \in \mathrm{Out}(\mathbb{Z}_{2l})$ leaves invariant the character of $\mathcal{Y}|_{\langle z \rangle}$ and the left-hand side of (2.21). In accordance with this, we must have $M_\lambda = M_{\sigma(\lambda)}$ for any $\lambda \in \mathbb{Z}_{2l}$. The representation spaces R_λ in the right-hand side of (2.21) are in turn permuted according to the action of the element $\sigma \in \mathrm{Out}(\mathbb{Z}_{2l})$, so we must have $M_\lambda = M_d$ for any $\lambda \in \mathbb{Z}_{2l}$ with $\gcd(\lambda, 2l) = d$. Consequently, relation (2.21) becomes

$$V^{\mathcal{Y}|_{\langle z \rangle}} = \bigoplus_{d|2l} M_d \left(\bigoplus_{\gcd(\lambda,2l)=d} R_\lambda \right). \tag{2.22}$$

In view of (2.22), the characteristic polynomial of $\rho(z)$ is given by

$$\det(w - \rho(z)) = \prod_{d|2l} \left(\prod_{\gcd(\lambda,2l)=d} \left(w - e^{\frac{2\pi i \lambda}{2l}} \right)^{M_d} \right) = \prod_{d|2l} \Phi_{2l/d}^{M_d}(w),$$

where $\Phi_{2l/d}$ denotes the $(2l/d)$th cyclotomic polynomial. We conclude that -1 is an eigenvalue for $\rho(z)$ precisely when $M_l > 0$. The explicit formula for M_l is given by (2.6) which is a consequence of the standard orthogonality relations for irreducible characters, see Theorem 1.1.7.

Claim. The groups $\phi_i(\Gamma)$ are not conjugate in $\mathrm{Pin}(n)$. Or equivalently, there is no automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ and $g \in \mathrm{Pin}(n)$ such that

$$\mathrm{Ad}_g\, \phi_1(\sigma(\gamma)) = \phi_2(\gamma), \qquad \forall \gamma \in \Gamma. \tag{2.23}$$

Let $\pi : \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n)$ be the standard covering and suppose that there is an element $g \in \mathrm{Pin}(n)$ so that relation (2.23) holds. Any automorphism of $\Gamma = \mathrm{Sym}(m)$ is inner for $6 \neq m \geq 4$. So, we can choose without loss of generality $\sigma = \mathrm{Id}_\Gamma$ and conclude that $\pi(g)$ centralizes $\rho(\Gamma)$ by projecting relation (2.23) to $\mathrm{O}(n)$. Since ρ is irreducible, we must have $\pi(g) \in \{\pm \mathrm{Id}\}$. In particular, $g \in \{\pm 1, \pm\omega\}$ where $\omega = e_1 \cdot e_2 \cdot \ldots \cdot e_n$ is the volume element in the corresponding Clifford algebra. It follows that $g \in \mathrm{Pin}(n)$ commutes with any element in $\mathrm{Spin}(n)$, and thus $\phi_1 = \phi_2$, which is absurd. □

Remark. The attentive reader might have noticed that condition (2.4) alone does *not* assure that the canonical extension associated with a representation of $\mathrm{Sym}(m)$ is trivial, as claimed in [Lar2]. For instance, the group $\mathrm{GL}(2, \mathbb{F}_3)$ is an example of a non-trivial extension of $\mathrm{Sym}(4)$ associated with a representation that satisfies condition (2.4) and *not* condition (2.5). The argument in the proof of Proposition 2.6 in [Lar2] is nonetheless *correct* as the representation of $\mathrm{Sym}(10)$ associated with the partition $(2, 2, 1, \cdots, 1)$ fulfills conditions (2.4)-(2.6).

The utility of Theorem 2.2.6 relies on the fact that its hypotheses can be checked algorithmically: given a representation $(\rho, \mathrm{Sym}(m))$ one can verify conditions (2.4) and (2.5) using the Murnaghan-Nakayama rule, see Section 1.2.3, and then create an algorithm to verify condition (2.6) manually on all faithful irreducible representations satisfying the former conditions. This procedure yields a wealth of examples of representations that fulfill the hypothesis of Theorem 2.2.6:

Example 2.2.7. The table below displays a list consisting of two numbers $n, m \in \mathbb{N}_{\geq 2}$ and a partition $(m_1, m_2, \ldots, m_l) \vdash m$ that defines a representation $\rho : \mathrm{Sym}(m) \longrightarrow \mathrm{O}(n)$ satisfying all hypothesis in Theorem 2.2.6.

$\mathcal{P}$	n	m
(3,2,1)	16	6
(4,1,1,1,1)	35	8
(5,2,1)	64	
(2, 2, 1, ⋯, 1)	35	
(3, 2, 1, 1, 1, 1, 1)	160	
(7, 2, 1)	160	
(5, 4, 1)	288	
(3, 2, 2, 2, 1)	288	
(6, 3, 1)	315	10
(6, 2, 1, 1)	350	
(5, 2, 1, 1, 1)	448	
(3, 3, 2, 1, 1)	450	
(5, 3, 1, 1)	567	
(3, 1, ⋯, 1)	45	
(4, 1, ⋯, 1)	120	
(8, 1, 1, 1)	120	11
(7, 4)	165	
(7, 1, 1, 1, 1)	210	

$\mathcal{P}$	n	m
(3, 3, 3, 2)	462	
(4, 4, 1, 1, 1)	825	
(4, 2, 2, 1, 1, 1)	1232	11
(6, 3, 1, 1)	1232	
(4, 4, 2, 1)	1320	
(4, 3, 2, 2)	1320	
(4, 1, 1, ⋯ , 1)	165	
(3, 2, 1, ⋯, 1)	320	
(9, 2, 1)	320	
(8, 1, 1, 1, 1)	330	
(3, 3, 3, 3)	462	
(6, 1, 1, 1, 1, 1, 1)	462	12
(3, 3, 1, ⋯, 1)	616	
(8, 2, 2)	616	
(6, 5, 1)	1155	
(5, 5, 2)	1320	
(3, 3, 2, 2, 2)	1320	
(3, 2, 2, 2, 1, 1, 1)	1408	

Remark.

(i) The size of the symmetric group that is embedded in $G = \mathrm{Spin}(35)$ using the representation associated with the partition $\mathcal{P}$ is not optimal as there is a representation of $\mathrm{Sym}(8)$ that satisfies the conditions of Theorem 2.2.6.

(ii) The representation of $\Gamma = \mathrm{Sym}(6)$ associated with the partition $(3, 2, 1)$ does not lead to almost conjugate groups of $\mathrm{Pin}(16)$. This is due to the fact that Γ is the only symmetric group having non-trivial outer automorphisms, and as there is just one 16-dimensional representation of Γ, the pullback of this representation with a non-trivial outer automorphism can be checked to be equivalent to the former one by means of a transformation in $\mathrm{SO}(16)$.

The following conjecture is motivated by the list of irreducible representations of the symmetric group fulfilling the hypotheses of Theorem 2.2.6 that were found in Example 2.2.7.

Conjecture. Let $N(n)$ be the number of equivalence classes of irreducible representations of $\mathrm{Sym}(n)$ satisfying the conditions stated in Theorem 2.2.6. Then, $N(n) \geq n$, for each natural number $n \geq 10$.

2.3 Almost conjugate subgroups of low dimensional spin groups

This section is primarily devoted to the classification of almost conjugate subgroups of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, as this will be needed in Section 3.2.3 where we look for Sunada pairs of locally homogeneous six dimensional nearly Kähler manifolds. Our classification is obtained after a by a case by case analysis in the spirit of Goursat's lemma 1.4.1, using the standard ADE classification, see Table 1.2.

Proposition 2.3.1. *The following statements hold.*

(a) Two almost conjugate subgroups of $\mathrm{SU}(2)$ *are conjugate.*

(b) Two almost conjugate subgroups of $\mathrm{Spin}(4)$ *are conjugate.*

(c) The groups $\mathrm{O}(6), \mathrm{Spin}(6)$ *and* $\mathrm{Pin}(6)$ *admit almost conjugate subgroups that are not conjugate.*

Proof. Identify $G = \mathrm{SU}(2)$ with the sphere of unit quaternions in the usual way and recall that two elements in G are conjugate to each other precisely if their real parts are the same. In particular, if we have two almost conjugate subgroups $\Gamma_i \subset G$, then $\mathrm{Re}(\Gamma_1) = \mathrm{Re}(\Gamma_2)$. This together with $|\Gamma_1| = |\Gamma_2|$ implies (a) in view of the standard ADE classification, see Table 1.2. This shows claim (a).

Suppose now that $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and let $\Gamma_i \subset G$ be almost conjugate subgroups defined by quintuples (A, A_0, B, B_0, θ) and $(A', A_0', B', B_0', \theta')$ respectively, see Lemma 1.4.1.

Claim. The pair of almost conjugate groups $\Gamma_i \subset G$ are conjugate to groups defined by quintuples of the form $\mathcal{Q}(\Gamma_i) = (A, A_0, B, B_0, \theta_i)$.

Proof of the claim. The almost conjugacy condition between the groups $\Gamma_i \subset G$ implies that $A_0 = \{a \in A \ : \ (a,1) \in \Gamma_1\}$ and $A_0' = \{a \in A' \ : \ (a,1) \in \Gamma_2\}$ are such that $\mathrm{Re}(A_0) = \mathrm{Re}(A_0')$ and $|A_0| = |A_0'|$. It follows that the groups $A_0, A_0' \subset \mathrm{SU}(2)$ are conjugate in $\mathrm{SU}(2)$ and that

$$|A| = |\pi_1(\Gamma_1)| = \frac{|\Gamma_1|}{|A_0|} = \frac{|\Gamma_2|}{|A_0'|} = |\pi_1(\Gamma_2)| = |A'|.$$

Almost conjugacy of the groups $\Gamma_i \subset G$ implies that $\mathrm{Re}(A) = \mathrm{Re}(A')$, and hence that $A \subset \mathrm{SU}(2)$ and $A' \subset \mathrm{SU}(2)$ are conjugate in $\mathrm{SU}(2)$. Consequently, we can assume that $\Gamma_1 = \mathcal{G}(A, A_0, B, B_0, \theta_1)$ and $\Gamma_2 = \mathcal{G}(A, A_0', B, B_0', \theta_2)$, see Section 1.4 where we introduced the latter notation.

The groups A_0 and A_0' are isomorphic normal subgroups of a given ADE group $A \subset \mathrm{SU}(2)$ satisfying the relation $\mathrm{Re}(A_0) = \mathrm{Re}(A_0')$. The following assertion shows that we can assume that $A_0 = A_0'$ without loss of generality, and hence the claim.[4] □

Assertion. Two isomorphic normal subgroups $A_0, A_0' \lhd A$ of a given ADE group $A \subset \mathrm{SU}(2)$ that satisfy $\mathrm{Re}(A_0) = \mathrm{Re}(A_0')$ must be equal, unless $A = 2\mathrm{D}_4$ and $A_0 \cong \mathbb{Z}_4$, In the latter case, the groups A_0 and A_0' are exchanged by conjugation with an element $g \in \mathrm{SU}(2)$ which leaves A invariant.

Proof of the assertion. Two isomorphic normal subgroups of either 2T, 2O or 2I are verified to be equal. Let $A = \mathbb{Z}_{mn} = \left\langle e^{\frac{2\pi i}{mn}} \right\rangle$ and $A_0 \lhd A$ be isomorphic to $\mathbb{Z}_m$. Each normal subgroup of $A \subset \mathrm{SU}(2)$ isomorphic to $\mathbb{Z}_m$ induces a surjective homomorphism $A \longrightarrow \mathbb{Z}_n$, and conversely, any normal subgroup of $A \subset \mathrm{SU}(2)$ isomorphic to $\mathbb{Z}_m$ arrises in this manner. However, there are exactly $\Phi(n)$ such homomorphisms $\alpha(r)(e^{\frac{2\pi i}{mn}}) = e^{\frac{2\pi i r}{n}}$, where $r \in \mathbb{Z}_n^\times$ and Φ is the Euler function. These homomorphisms have kernel exactly $\mathbb{Z}_m = \left\langle e^{\frac{2\pi i}{m}} \right\rangle$.

We are left with the case $A = 2\mathrm{D}_{2l}$ for $l \geq 2$. The group $A = 2\mathrm{D}_4$ has exactly four normal subgroups. Three of them are isomorphic to $\mathbb{Z}_4$ and generated by the imaginary units i, j, k, and the remaining normal subgroup which is its center. The three normal subgroups with four elements are related by conjugation with an element in 2O, see Table 1.5.

Let us suppose that $A = 2\mathrm{D}_{2l(2k+1)}$ for $k \geq 1$, $l > 1$ and that $A_0 \lhd A$ is isomorphic to $\mathbb{Z}_{2k+1}$. We assert that $A_0 = \left\langle e^{\frac{2\pi i}{2k+1}} \right\rangle$. For this we just need to consider (surjective) homomorphisms $2\mathrm{D}_{2l(2k+1)} \longrightarrow 2\mathrm{D}_{2l} \cong A/A_0$. It is not difficult to see that there are exactly $2l\Phi(2l)$ surjective homomorphisms $\alpha(x,r) : A \longrightarrow 2\mathrm{D}_{2l}$, where $0 \leq x < 2l$ is an integer, $r \in \mathbb{Z}_{2l}^\times$ and Φ is the Euler function. The automorphisms $\alpha(x,r)$ map the generators of $A = 2\mathrm{D}_{2l(2k+1)}$ as follows

$$\alpha(x,r)(j) = je^{\frac{i\pi x}{l}}, \quad \alpha(x,r)\left(e^{\frac{i\pi}{l(2k+1)}}\right) = e^{\frac{i\pi r}{l}}.$$

[4]See Table 1.3 and 1.4 for information on normal subgroups of ADE groups.

The condition

$$\alpha(x,r)\left(e^{\frac{i\pi y}{l(2k+1)}}\right) = 1$$

implies that $y = 0 \mod l$, and so we get $\operatorname{Ker}(\alpha(x,r)) = \left\langle e^{\frac{2\pi i}{2k+1}} \right\rangle$. Let us suppose that $l = 1$ and that $\alpha : 2\mathrm{D}_{2(2k+1)} \longrightarrow 2\mathrm{D}_2$ is a surjective homomorphism with $\operatorname{Ker}(\alpha)$ isomorphic to $\mathbb{Z}_{2k+1}$. Since the order $\operatorname{ord}(je^{\frac{i\pi x}{2k+1}}) = 4$ for any integer $x \in \mathbb{Z}$, the condition $\operatorname{Ker}(\alpha) \cong \mathbb{Z}_{2k+1}$ implies that no element of this form can be mapped to the trivial element. So we must have that $\operatorname{Ker}(\alpha) = \left\langle e^{\frac{2\pi i}{2k+1}} \right\rangle$.

Let us consider $A = 2\mathrm{D}_{2kl}$ and $A_0 \triangleleft A$ to be isomorphic to $\mathbb{Z}_{2k}$. Surjective homomorphisms from $A = 2\mathrm{D}_{2kl}$ to D_{2l} are of the form $\alpha(x,r)$ for an integer $0 \le x < l$ and a unit $r \in \mathbb{Z}_l^\times$. Those automorphisms are given by

$$\alpha(x,r)(j) = je^{\frac{i\pi x}{l}}\mathbb{Z}_{2k}, \quad \alpha(x,r)\left(e^{\frac{i\pi}{lk}}\right) = e^{\frac{i\pi r}{l}}\mathbb{Z}_{2k},$$

where we realized D_{2l} as the quotient $2\mathrm{D}_{2kl}/\mathbb{Z}_{2k}$. Just as before $\operatorname{Ker}(\alpha(x,r)) = \left\langle e^{\frac{2\pi i}{2k}} \right\rangle$, and so $A_0 = \left\langle e^{\frac{\pi i}{k}} \right\rangle$. Lastly, let $A = 2\mathrm{D}_{4k}$ and suppose that $A_0 \triangleleft A$ is isomorphic to $2\mathrm{D}_{2k}$. The quotient A/A_0 is isomorphic to $\mathbb{Z}_2$, and so we must consider homomorphisms $2\mathrm{D}_{4k} \longrightarrow \mathbb{Z}_2$. These are of the form $\alpha(r,s)$ and their action on the generators of $2\mathrm{D}_{4k}$ is as follows

$$\alpha(r,s)(j) = r, \quad \alpha(r,s)\left(e^{\frac{i\pi}{2k}}\right) = s,$$

where $r, s \in \mathbb{Z}_2 = \{1, -1\}$. There is only one of these homomorphisms having kernel isomorphic to $2\mathrm{D}_{2k}$. □

The last assertion tells us that it suffices to consider almost conjugate groups $\Gamma_i \subset G$ with quintuples of the form $\mathcal{Q}(\Gamma_i) = (A, A_0, B, B_0, \theta_i)$. In order to show claim (b) we proceed considering the following cases separately

(I) $A_0 = A$ and $B_0 = B$,

(II) $A_0 = B_0 = \{1\}$,

(III) $A_0 = \{1\}$ and $B_0 \neq \{1\}$,

(IV) $B_0 \neq \{1\}$ and $A_0 \neq \{1\}$.

Case I. The Subgroups $\Gamma_i \subset G$ considered here are of the form $A_i \times B_i$ and clearly any two such subgroups that are almost conjugate must be conjugate.

Case II. $A_0 = B_0 = \{1\}$.

We proceed by considering all possibilities for $A \subset \mathrm{SU}(2)$.

1. Let $\underline{A = \mathbb{Z}_n}$. The groups $\Gamma_i \subset G$ are of the form

$$\Gamma_j = \{(e^{\frac{2\pi i x}{n}}, e^{\frac{2\pi i r_j x}{n}}) \; : \; x \in \mathbb{Z}\}, \qquad r_j \in \mathbb{Z}_n^\times.$$

The almost conjugacy condition between the groups $\Gamma_j \subset G$ reads as follows: for any integer $x \in \mathbb{Z}$ there is $y = y(x) \in \mathbb{Z}$ such that

$$\cos\left(\frac{2\pi r_1 x}{n}\right) = \cos\left(\frac{2\pi r_2 y}{n}\right) \qquad , \qquad \cos\left(\frac{2\pi x}{n}\right) = \cos\left(\frac{2\pi y}{n}\right).$$

It follows that $r_2 = \varepsilon r_1 \mod n$ for some value $\varepsilon \in \{1, -1\}$. Conjugation with e.g. $(1, j) \in G$ exchanges the groups Γ_1 and Γ_2.

2. Let $\underline{A = 2\mathrm{D}_{2n}}$. That is, we consider almost conjugate subgroups of the form

$$\Gamma_i = \{(e^{\frac{\pi i x}{n}}, e^{\frac{\pi i a_k x}{n}}), (je^{\frac{\pi i x}{n}}, je^{\frac{\pi i (a_k x + b_k)}{n}}) \; : \; a_k \in \mathbb{Z}_{2n}^{\times}, \; b_k \in \mathbb{Z}, \; x \in \mathbb{Z}\}.$$

If the groups $\Gamma_i \subset G$ are almost conjugate, then a similar reasoning as in (1) implies that either $a_2 = \varepsilon a_1 \mod 2n$ for some $\varepsilon \in \{1, -1\}$. In case $\varepsilon = 1$, then the groups $\Gamma_i \subset G$ are related by conjugation with the element $(1, e^{\frac{i\pi(b_1 - b_2)}{2n}}) \in G$, whereas if $\varepsilon = -1$ conjugation with $(1, je^{\frac{i\pi(b_1 + b_2)}{2n}}) \in G$ will exchange them.

3. Let $\underline{A \in \{2\mathrm{T}, 2\mathrm{O}, 2\mathrm{I}\}}$ and recall that the outer automorphism group of A is generated by a single involution $\varphi \in \mathrm{Out}(A)$, see Section 1.4.1. For $A = 2\mathrm{T}$ this involution is obtained by conjugation with an element in $\mathrm{SU}(2)$, which is not the case for $A = 2\mathrm{O}, 2\mathrm{I}$. Observe there is an element $a \in A$ such that $\varphi(a) \in \mathrm{SU}(2)$ is not conjugate to $a \in \mathrm{SU}(2)$, see Tables 1.9 and 1.6. In particular, if the groups $\Gamma_i = \mathcal{G}(A, (1), A, (1), \theta_i)$ are almost conjugate, then either $\theta_i \in \mathrm{Inn}(\mathrm{SU}(2))$ or $\theta_i \in \mathrm{Inn}(\mathrm{SU}(2)) \cdot \varphi$. In both cases the resulting groups are easily checked to be conjugate.

As $\mathbb{Z}_2$ has no non trivial outer automorphisms, any pair of almost conjugate subgroups $\Gamma_i = \mathcal{G}(A, A_0, B, B_0, \theta_i)$ such that the quotient A/A_0 is isomorphic to $\mathbb{Z}_2$ must be conjugate. Such cases require don't require further attention, and so they will be omitted in the sequel.

Case III. Suppose that $A \triangleright A_0 \supset \mathbb{Z}_2$ and $B_0 = (1)$.

We analyze case by case for $B \subset \mathrm{SU}(2)$ according Tables 1.3 and 1.4.

1. Let $\underline{B = \mathbb{Z}_k}$. Suppose the groups $\Gamma_i = \mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_l, \mathbb{Z}_k, (1), \theta(r_i))$ are almost conjugate for some choice of units $r_i \in \mathbb{Z}_k^{\times}$. A similar argument as in Case II (1) reveals that $r_2 = \varepsilon r_1 \mod k$ for some $\varepsilon \in \{1, -1\}$. So the groups $\Gamma_i \subset G$ are conjugate to each other.

2. Let $\underline{B = 2\mathrm{D}_{2l}}$ and suppose that $\underline{l > 1}$. Similarly as in Case II (2), almost conjugate subgroups of the form

$$\Gamma_i = \mathcal{G}(2\mathrm{D}_{2l(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2l}, (1), \theta_{a_i, b_i}) \subset G$$

are such that $a_2 = \varepsilon a_1 \mod 2l$ for some $\varepsilon \in \{1, -1\}$, and hence conjugate. The case in which $\underline{l = 1}$ stands apart, since $2\mathrm{D}_2 = \mathbb{Z}_4$. We can choose $2\mathrm{D}_2 = \langle j \rangle \subset \mathrm{SU}(2)$ and observe that the groups Γ_i are conjugate via an element $(e^{\frac{\pi i}{2(2k+1)}}, w)$, where $w \in 2\mathrm{O}$ is an element such that $c_w(j) = -j$, see Table 1.5.

3. The cases involving $\underline{B = \mathbb{Z}_3}$ will be analyzed in item (3) of Case IV.

Case IV. $B_0 \neq \{1\}$ and $A_0 \neq \{1\}$.

Here we discriminate according the isomorphism type of $F = A/A_0$.

1. Let $\underline{F \in \{\mathrm{T}, \mathrm{O}\}}$. The automorphisms of $F \cong {}^{A}/_{\mathbb{Z}_2}$ are induced by conjugation with an element in $\mathrm{SU}(2)$ for $A \in \{2\mathrm{O}, 2\mathrm{T}\}$. In consequence, a pair of subgroups of the form $\Gamma_i = \mathcal{G}(A, \mathbb{Z}_2, A, \mathbb{Z}_2, \theta_i)$ with $\theta_i \in \mathrm{Aut}(F)$, are conjugate.

2. Let $\underline{F = \mathrm{I}}$ and let $\Gamma_i = \mathcal{G}(2\mathrm{I}, \mathbb{Z}_2, 2\mathrm{I}, \mathbb{Z}_2, \theta_i)$ be a pair of almost conjugate subgroups of G. Let us consider the standard representation $\mathrm{I} \subset \mathrm{SO}(3)$ and recall that for the non-trivial outer automorphism $\varphi \in \mathrm{Out}(\mathrm{I})$ there an element $a \in \mathrm{I}$ which is not conjugate to $\phi(a) \in \mathrm{I}$ in $\mathrm{SO}(3)$, see Table 1.9 and item (9) of Section 1.4.1. This observation allows us to conclude that a necessary condition to make the groups $\Gamma_i \subset G$ almost conjugate, is to have either $\theta_i \in \mathrm{Inn}(F)$ or $\theta_i \in \mathrm{Inn}(F) \cdot \varphi$. In both cases the groups Γ_i are readily seen to be conjugate.

3. Let $\underline{F = \mathbb{Z}_3}$. We consider first the quintuples $(A, A_0, \mathbb{Z}_{3n}, \mathbb{Z}_n, \theta)$ for $n \geq 1$. Groups of the form $\Gamma_i = \mathcal{G}(\mathbb{Z}_{3n}, \mathbb{Z}_n, 2\mathrm{T}, 2\mathrm{D}_4, \theta_i)$, where $\theta_i \in \mathrm{Aut}(\mathbb{Z}_3)$, are given by

$$\Gamma_1 = \Big\{ (e^{\frac{2\pi i x}{n}}, z_0), (e^{\frac{2\pi i(1+3y)}{3n}}, z_1), (e^{\frac{2\pi i(2+3z)}{3n}}, z_2) \ : \ z_0 \in 2\mathrm{D}_4,\ z_1 \in \frac{1-i-j-k}{2} 2\mathrm{D}_4, \\ z_2 \in \frac{1+i+j+k}{2} 2\mathrm{D}_4,\ x, y, z \in \mathbb{Z} \Big\}$$

and

$$\Gamma_2 = \Big\{ (e^{\frac{2\pi i x}{n}}, z_0), (e^{\frac{2\pi i(1+3y)}{3n}}, z_1), (e^{\frac{2\pi i(2+3z)}{3n}}, z_2) \ : \ z_0 \in 2\mathrm{D}_4,\ z_1 \in \frac{1+i+j+k}{2} 2\mathrm{D}_4, \\ z_2 \in \frac{-1+i+j+k}{2} 2\mathrm{D}_4,\ x, y, z \in \mathbb{Z} \Big\}.$$

The groups Γ_i are conjugate to each other via elements $(1, w)$, where $w \in 2\mathrm{O}$ is so that the conjugation mapping $c_w \in \mathrm{Aut}(\mathrm{SU}(2))$ satisfies the following conditions:

$$c_w\left(\frac{1+i+j+k}{2} 2\mathrm{D}_4\right) = \frac{-1+i+j+k}{2} 2\mathrm{D}_4,$$

$$c_w\left(\frac{-1+i+j+k}{2} 2\mathrm{D}_4\right) = \frac{1+i+j+k}{2} 2\mathrm{D}_4,$$

see Table 1.5.

The groups $\Gamma_i = \mathcal{G}(\mathbb{Z}_{3k}, \mathbb{Z}_k, \mathbb{Z}_{3k'}, \mathbb{Z}_{k'}, \theta_i)$ with $\theta_i \in \mathrm{Aut}(\mathbb{Z}_3)$ are not almost conjugate unless $\theta_1 = \theta_2 \in \mathrm{Aut}(\mathbb{Z}_3)$, case in which they are indeed conjugate. This is simply because elements of the form

$$(e^{\frac{2\pi i(1+3x)}{3k}}, e^{\frac{2\pi i(1+3y)}{3k'}}), \quad x, y \in \mathbb{Z}$$

can not be conjugate to either

$$(e^{\frac{2\pi i(2+3x_1)}{3k}}, e^{\frac{2\pi i(1+3y_1)}{3k'}}) \quad \text{or} \quad (e^{\frac{2\pi i(1+3x_1)}{3k}}, e^{\frac{2\pi i(2+3y_1)}{3k'}})$$

for any $x_1, y_1 \in \mathbb{Z}$. Subgroups of the form $\Gamma_i = \mathcal{G}(2\mathrm{T}, 2\mathrm{D}_4, 2\mathrm{T}, 2\mathrm{D}_4, \theta_i) \subset G$ are checked to be conjugate via an element of the form $(w, w') \in 2\mathrm{O} \times 2\mathrm{O} \subset G$.

4. Let $\underline{F = \mathbb{Z}_4}$. Just as in Case III (1), one checks that almost conjugate subgroups $\Gamma_i \subset G$ of the form $\Gamma_i = \mathcal{G}(\mathbb{Z}_{4k}, \mathbb{Z}_k, \mathbb{Z}_{4k'}, \mathbb{Z}_{k'}, \theta_i)$ must be conjugate. Let us consider subgroups $\Gamma_i = \mathcal{G}(\mathbb{Z}_{4k}, \mathbb{Z}_k, 2\mathrm{D}_{2(2k'+1)}, \mathbb{Z}_{2k'+1}, \theta_i)$ with $\theta_i \in \mathbb{Z}_4^\times = \mathbb{Z}_2$. The groups in question are given by

$$\Gamma_1 = \Big\{(e^{\frac{2\pi i x_0}{k}}, e^{\frac{2\pi i y_0}{2k'+1}}),\ (e^{\frac{i\pi(1+4x_1)}{2k}}, e^{\frac{i\pi(1+2y_1)}{2k'+1}}),\ (e^{\frac{i\pi(1+2x_2)}{k}}, je^{\frac{2i\pi y_2}{2k'+1}}),$$
$$(e^{\frac{i\pi(3+4x_3)}{2k}}, je^{\frac{i\pi(2y_3+1)}{2k'+1}}) \ :\ x_j, y_j \in \mathbb{Z}\Big\},$$

$$\Gamma_2 = \Big\{(e^{\frac{2\pi i x_0}{k}}, e^{\frac{2\pi i y_0}{2k'+1}}),\ (e^{\frac{i\pi(1+4x_1)}{2k}}, je^{\frac{2i\pi y_1}{2k'+1}}),\ (e^{\frac{i\pi(1+2x_2)}{k}}, e^{\frac{i\pi(1+2y_1)}{2k'+1}}),$$
$$(e^{\frac{i\pi(3+4x_3)}{2k}}, je^{\frac{i\pi(2y_3+1)}{2k'+1}}) \ :\ x_j, y_j \in \mathbb{Z}\Big\}.$$

The almost conjugacy condition implies that for any $x \in \mathbb{Z}$ there is a $y \in \mathbb{Z}$ so that

$$\cos\left(\frac{\pi x}{2k}\right) = \cos\left(\frac{\pi y}{2k}\right).$$

This implies that for some value $\varepsilon \in \{1, -1\}$ we have

$$x = \varepsilon y \mod 4. \tag{2.24}$$

In particular, the groups $\Gamma_i \subset G$ are not almost conjugate, for if they were, then the elements

$$(e^{\frac{i\pi(1+4x)}{2k}}, e^{\frac{i\pi(1+2y)}{2k'+1}}) \in \Gamma_1, \qquad (e^{\frac{i\pi(1+2z)}{k}}, e^{\frac{i\pi(1+2w)}{2k'+1}}) \in \Gamma_2$$

have to be conjugate to one another for some value of $x, y, z, w \in \mathbb{Z}$, which is clearly impossible in view of identity (2.24). Let

$$\Gamma_i = \mathcal{G}(2\mathrm{D}_{2(2k+1)}, \mathbb{Z}_{2k+1}, 2\mathrm{D}_{2(2k'+1)}, \mathbb{Z}_{2k'+1}, \theta(r_i)) \subset G,$$

where $r_1 = 1$ and $r_2 = 3$. These groups are not almost conjugate, as one can check there is an element $(x, y) \in \Gamma_1$ such that $\mathrm{Re}(x) = \mathrm{Re}(y) = 0$, but there is no such element in Γ_2.

5. Let $\underline{F = \mathbb{Z}_l}$ for $\underline{l \neq 2, 3, 4}$. Consider $\Gamma_i = \mathcal{G}(\mathbb{Z}_{kl}, \mathbb{Z}_k, \mathbb{Z}_{pl}, \mathbb{Z}_p, \theta(r_i)) \subset G$ with $r_i \in \mathbb{Z}_l^\times$. In this case

$$\Gamma_i = \{(e^{\frac{2\pi i x}{kl}}, e^{\frac{2\pi i}{pl}(r_i x + l y)}) \ :\ x, y \in \mathbb{Z}\}.$$

The almost conjugacy condition between the groups Γ_i reads as follows: for any $x, y \in \mathbb{Z}$, there are integers $x' = x'(x,y)$ and $y' = y'(x,y)$ such that

$$\cos\left(\frac{2\pi x}{kl}\right) = \cos\left(\frac{2\pi x'}{kl}\right), \ \cos\left(\frac{2\pi}{pl}(r_1 x + ly)\right) = \cos\left(\frac{2\pi}{pl}(r_2 x' + ly')\right).$$

Equivalently

$$r_1 x + ly = \varepsilon r_2 x' + \varepsilon l y' \mod pl, \tag{2.25}$$

$$x = \varepsilon' x' \mod kl, \tag{2.26}$$

for some $\varepsilon, \varepsilon' \in \{1, -1\}$. From equation (2.25), we must have that

$$r_1 x = \varepsilon r_2 x' \mod l. \tag{2.27}$$

Equation (2.26) and (2.27) imply that either $r_1 = r_2 \mod l$ or $r_1 = -r_2 \mod l$. In the former case, the groups Γ_i are the same, whereas in the latter they are related by conjugation, e.g. with an element $(1, j) \in G$.

This finishes the proof of claim (b). At last, for claim (c) consider the following almost conjugate diagonal subgroups of SO(6), which correspond to certain linear codes on $\mathbb{Z}_2^6$ that appear first in [CS]

$$\begin{aligned}\Gamma_1 = \{&(1,1,1,1,1,1),(-1,-1,-1,-1,-1,-1),(-1,-1,1,1,1,1),\\ &(-1,1,-1,1,1,1),(1,-1,-1,1,1,1),(-1,1,1,-1,-1,-1),\\ &(1,-1,1,-1,-1,-1),(1,1,-1,-1,-1,-1)\},\end{aligned} \tag{2.28}$$

$$\begin{aligned}\Gamma_2 = \{&(1,1,1,1,1,1),(-1,-1,-1,-1,-1,-1),(-1,-1,1,1,1,1),\\ &(1,1,-1,-1,1,1),(1,1,1,1,-1,-1),(-1,-1,-1,-1,1,1),\\ &,(-1,-1,1,1,-1,-1),(1,1,-1,-1,-1,-1)\},\end{aligned} \tag{2.29}$$

It was noted in Example 2.4 of [RSW], that these subgroups are not conjugate in SO(6). The groups given in (2.28) and (2.29) are also not conjugate in O(6). To see this, write $O(6) = SO(6) \rtimes \mathbb{Z}_2$, where the $\mathbb{Z}_2$-action on SO(6) is given by conjugation with the diagonal matrix $(-1,1,1,1,1,1)$. Since the given $\mathbb{Z}_2$-action leaves the groups $\Gamma_i \subset SO(6)$ invariant, these groups are conjugate in O(6) precisely when they are so in SO(6), which is not the case. Let $\pi : \mathrm{Spin}(6) \longrightarrow SO(6)$ be the standard covering and consider the groups $\overline{\Gamma}_i = \pi^{-1}(\Gamma_i) \subset \mathrm{Spin}(6)$. These groups $\overline{\Gamma}_i \subset \mathrm{Spin}(6)$ have the same cardinality and can be easily seen to be almost conjugate in Spin(6). If $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ were related by conjugation in Pin(6), then $\Gamma_i \subset SO(6)$ would be conjugate in O(6), which is again not the case. □

2.4 The volume of a Lie group

The problem of finding the volume of a compact Lie group is a classical one. As such, there are already some formulae to compute this volume, which appear in different contexts in which this problem manifests itself. In a physical context the most credible treatment of this problem is to the best of the author's knowledge the one of Marinov [M], whereas in a more mathematical framework, the most sound contribution is due to Macdonald [Mac]. The core of the proof of Macdonald's formula relies prominently on Weyl integration's formula and the rest of his argument involves several non-trivial facts on compact Lie groups.[5] The aim in this section is to give an alternative formula to compute the volume of a compact Lie group. The arguments to deduce our formula are also supported on Weyl integration's formula. However, the rest of the proof can be thought as a simplification of Macdonald's argument.

Proposition 2.4.1. *Let G be a compact, connected Lie group, let $T \subset G$ be a maximal torus and $W(G)$ be the Weyl group of G. The volume of G with respect to a metric induced by an Ad-invariant scalar product B on $\mathfrak{g}$ is given by*

$$\mathrm{vol}\,(G,B) = 2^{\dim(T)}|W(G)|\pi^{\frac{\dim(T)+\dim(G)}{2}}\left(\sqrt{|\det(b^{-1}(\epsilon_\mu,\epsilon_\nu))|}\; e^{-\frac{1}{4}\Delta}|_0\delta_{\mathfrak{g}}\right)^{-1}, \tag{2.30}$$

where (ϵ_μ) is a basis of $\mathfrak{t}^*$ dual to a basis of $\frac{1}{2\pi}P(G)$, where $P(G) = \ker(\exp : \mathfrak{t} \longrightarrow T)$ is the weight lattice, and $\delta_{\mathfrak{g}}$ denotes the product of the product of the roots of $\mathfrak{g}^{\mathbb{C}}$ and b^{-1} is the scalar product of $\mathfrak{t}^*$ induced by B. Δ is the unique second order linear differential operator with constant coefficients acting on the polynomial algebra $\mathbb{C}[\varepsilon_1,\dots,\varepsilon_m]^{\mathrm{Sym}(m)}$ in the simple roots $\varepsilon_1,\dots,\varepsilon_m \in \mathfrak{t}^*$ of $\mathfrak{g}^{\mathbb{C}}$, such that $-\Delta\varepsilon_j^2 = 2b^{-1}(\varepsilon_j,\varepsilon_j)$ and $\Delta\varepsilon_j^{2k+1} = 0$ for any root $\varepsilon_j \in \mathfrak{t}^*$.

Proof. The formula is clear for a torus with respect to the metric induced by B. That is,

$$\mathrm{vol}\,(T,B) = (2\pi)^{\dim(T)}\;|\det(b^{-1}(\epsilon_\mu,\epsilon_\nu))|^{-1/2}.$$

Define a function f on $\mathfrak{g}$ by $f(X) = e^{-B(X,X)}$ and observe that

$$\pi^{\frac{\dim(G)}{2}} = \int_{\mathfrak{g}} f(X)d_B(X) = \int_{\mathfrak{g}^{reg}} f(X)d_B(X), \tag{2.31}$$

where $\mathfrak{g}^{reg} \subset \mathfrak{g}$ is the generic set consisting of regular elements, and d_B denotes the Riemannian volume of the metric B. Choose an open Weyl chamber $\mathcal{C}$ and consider the map

$$\mathfrak{g}^{reg} \ni X \longmapsto y(X) \in \mathcal{C}, \tag{2.32}$$

where $y(X) \in \mathrm{Ad}_G(X) \cap \mathcal{C}$. The metric B on $\mathfrak{g}$ induces metrics on $\mathcal{C}$, $\mathfrak{t}$, $\mathfrak{g}^{reg}$ and $\mathfrak{t}^{reg}$, and the map defined in (2.32) becomes a Riemannian submersion with respect to these metrics.

Let us denote by d_b the Riemannian volume of the metric b on $\mathfrak{t}$ and by d_{E_X} the one on the adjoint orbit of a regular element $X \in \mathfrak{g}^{reg}$ relative to the submersion (2.32). Integration over the submersion (2.32) yields

$$\begin{aligned}\int_{\mathfrak{g}^{reg}} f(X)d_B(X) &= \frac{1}{|W(G)|}\int_{\mathfrak{t}^{reg}}\left(\int_{\mathrm{Ad}_G(X)} f(Y)d_{E_X}(Y)\right)d_b(X)\\ &= \frac{1}{|W(G)|}\int_{\mathfrak{t}} f(X)\left(\int_{\mathrm{Ad}_{G/T}(X)} d_{E_X}(Y)\right)d_b(X),\end{aligned} \tag{2.33}$$

[5] Indeed his proof has been already simplified in [Has].

where in the last equality we have used the Ad-invariance of $f(\cdot)$ and the fact that T is a maximal torus whose Lie algebra contains $X \in \mathfrak{t}^{reg}$. On the other hand, the orbit map

$$\mathrm{Ad}_{G/T} : gT \longmapsto \mathrm{Ad}_g(X)$$

provides an isometric immersion of $(G/T, \mathrm{Ad}^*_{G/T} B)$ into $(\mathfrak{g}^{reg}, B)$. Comparison of the volumes of the flag manifold G/T relative to its submersion metric B and $\mathrm{Ad}^*_{G/T} B$ gives

$$\mathrm{vol}\,(G/T, \mathrm{Ad}^*_{G/T} B) = \det(\mathrm{ad}_X|_{\mathfrak{t}^\perp})\,\mathrm{vol}\,(G/T, B) = \delta_{\mathfrak{g}}(X)\frac{\mathrm{vol}\,(G,B)}{\mathrm{vol}\,(T,B)},$$

where $\delta_{\mathfrak{g}}$ is the product of the roots of $\mathfrak{g}^{\mathbb{C}}$. As a consequence, we have

$$\int_{\mathrm{Ad}_{G/T}(X)} d_{E_X}(Y) = \frac{\mathrm{vol}\,(G,B)}{\mathrm{vol}\,(T,B)}\delta_{\mathfrak{g}}(X), \tag{2.34}$$

and identities (2.31),(2.33) and (2.34) give

$$\pi^{\frac{\dim(G)}{2}} = \int_{\mathfrak{g}} f(X) d_B(X) = \frac{\mathrm{vol}\,(G,B)}{\mathrm{vol}\,(T,B)}\,\frac{1}{|W(G)|}\int_{\mathfrak{t}} f(X)\delta_{\mathfrak{g}}(X) d_b(X). \tag{2.35}$$

Formula (2.30) is then a consequence of identity (2.35) and the following claim.

Claim. The following formula holds for any polynomial $P \in \mathbb{C}[\varepsilon_1, ..., \varepsilon_m]^{\mathrm{Sym}(m)}$ in the simple roots $\varepsilon_1, ..., \varepsilon_m \in \mathfrak{t}^*$.

$$\frac{1}{\sqrt{\pi^{\dim(T)}}}\int_{\mathfrak{t}} f(X)P(X)d_b(X) = e^{-\frac{1}{4}\Delta}|_0 P. \tag{2.36}$$

The standard polarization argument [Ge], tells us that it suffices to verify formula (2.36) for $P = \alpha^k$ for any $k \in \mathbb{N}$ fixed and any functional $\alpha = \varepsilon_1 + ... + \varepsilon_j \in \mathfrak{t}^*$, where $\varepsilon_i \in \mathfrak{t}^*$ is a simple root of $\mathfrak{g}^{\mathbb{C}}$ and $j \le m$. First observe that

$$\begin{aligned}\sum_{k\ge 0}\frac{t^k}{k!}\cdot\frac{1}{\sqrt{\pi^{\dim(T)}}}\int_{\mathfrak{t}} e^{-b(X,X)}\alpha^k(X)d_b(X) &= \frac{1}{\sqrt{\pi^{\dim(T)}}}\int_{\mathfrak{t}} e^{-b(X,X)}\cdot e^{t\alpha(X)}d_b(X)\\ = e^{\frac{t^2}{4}b^{-1}(\alpha,\alpha)} = \sum_{k\ge 0}\frac{t^{2k}}{k!}\cdot\frac{b^{-1}(\alpha,\alpha)^k}{4^k}.\end{aligned}$$

The left hand side of identity (2.36) vanishes for $P = \alpha^{2k+1}$ and

$$\frac{1}{\sqrt{\pi^{\dim(T)}}}\int_{\mathfrak{t}} e^{-b(X,X)}\alpha^{2k}(X)d_b(X) = \frac{b^{-1}(\alpha,\alpha)^k}{4^k}\cdot\frac{(2k)!}{k!}.$$

The right hand side of (2.36) also vanishes whenever $P = \alpha^{2k+1}$ and

$$e^{-\frac{1}{4}\Delta}|_0\alpha^{2k} = \sum_{k\ge l\ge 0}\frac{1}{4^l}\frac{b^{-1}(\alpha,\alpha)^l\alpha^{2k-2l}(0)}{l!}\cdot\frac{(2k)!}{(2k-2l)!} = \frac{b^{-1}(\alpha,\alpha)^k}{4^k}\cdot\frac{(2k)!}{k!}.$$

The claim follows. □

Let us illustrate the use of Proposition 2.4.1 by performing some explicit calculations.

Example 2.4.2. Let $G = \mathrm{SU}(2)$ endowed with the biinvariant metric

$$B(X,Y) = -\frac{1}{2}\,\mathrm{tr}_{\mathbb{C}}(XY), \qquad X, Y \in \mathfrak{su}(2). \tag{2.37}$$

The Cartan algebra of $\mathfrak{g}$ is given by $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}(H)$ where

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2).$$

Since $B(H,H) = 1$, we have $b^{-1}(x,x) = 1$ for $x \in \mathfrak{t}^*$ such that $x(H) = 1$, where b^{-1} denotes the scalar product of $\mathfrak{t}^*$ induced by B. The product of the roots is $\delta_{\mathfrak{su}(2)} = (2ix)(-2ix) = 4x^2$ and $e^{-\frac{1}{4}\Delta}\delta_{\mathfrak{su}(2)}|_0 = 2$. It follows that

$$\mathrm{vol}\,(\mathrm{SU}(2), B) = 2 \cdot 2 \cdot \pi^{\frac{1+3}{2}}\frac{1}{2} = 2\pi^2 = \frac{2\pi^2}{\Gamma(2)} = \mathrm{vol}\,(S^3), \tag{2.38}$$

where Γ denotes the gamma function and $\mathrm{vol}\,(S^3)$ is the standard Lebesgue volume of the three dimensional sphere S^3.

Observe that the Riemannian volume of the normal homogeneous metric on the sphere

$$S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$$

induced by the biinvariant scalar product

$$B : \mathfrak{so}(n) \times \mathfrak{so}(n) \longrightarrow \mathbb{R}; \quad B(X,Y) = -\frac{1}{2}\,\mathrm{tr}_{\mathbb{R}}(XY)$$

is exactly the standard Lebesgue volume[6]

$$\mathrm{vol}\,(S^{n-1}, B) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)},$$

where $\Gamma(x)$ denotes the value of the gamma function at $x \in \mathbb{R}$. One could use this fact together with the fibrations

$$\begin{array}{ccc} \mathrm{SO}(n) \longrightarrow \mathrm{SO}(n+1) & \mathrm{SU}(n) \longrightarrow \mathrm{SU}(n+1) & \mathrm{Sp}(n) \longrightarrow \mathrm{Sp}(n+1) \\ \downarrow & \downarrow & \downarrow \\ S^n & S^{2n+1} & S^{4n+3} \end{array}$$

[6]Observe also that $\mathrm{vol}\,(S^6, B) = \frac{16}{15}\pi^3$, which is consistent with Table 3.1.

and Proposition 2.4.1 to obtain a recursive formula for the volumes of $\mathrm{Sp}(n), \mathrm{SO}(n)$ and $\mathrm{SU}(n)$. Using this argument, can calculate that

$$\mathrm{vol}\,(\mathrm{Sp}(2), B_1) = \frac{\pi^6}{12} \quad , \quad \mathrm{vol}\,(\mathrm{SU}(3), B_2) = \sqrt{3}\pi^5,$$

where the metrics for the complex and quaternionic spheres in question are the submersion metrics induced by the invariant scalar products

$$B_1(X,Y) = -\frac{1}{2}\,\mathrm{tr}_{\mathbb{H}}(XY), \quad X, Y \in \mathfrak{sp}(2),$$

and

$$B_2(X,Y) = -\frac{1}{2}\,\mathrm{tr}_{\mathbb{C}}(XY), \quad X, Y \in \mathfrak{su}(3)$$

respectively. We will compute the volumes of Sp(2) and SU(3) using formula (2.30) within the proof of Proposition 3.2.7 as a test for the validity of the volume formula we deduced. For the time being, we would like to compute the volume of SO(3) using formula 2.30 and compare it with our previous computation (2.38) of the volume of SU(2).

Example 2.4.3. Let us define matrices I, J, K generating the Lie algebra $\mathfrak{so}(3)$ by letting

$$I = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and $\hat{I}, \hat{J}, \hat{K}$ be the generators of the Lie algebra $\mathfrak{su}(2)$ given by

$$\hat{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

For these generators we have the following brackets relations

$$[I,J] = K, \qquad [J,K] = I, \qquad [K,I] = J,$$
$$[\hat{I},\hat{J}] = 2\hat{K}, \qquad [\hat{J},\hat{K}] = 2\hat{I}, \qquad [\hat{K},\hat{I}] = 2\hat{J}.$$

In particular, the map

$$\varphi : \mathfrak{so}(3) \ni u(a,b,c) = \begin{pmatrix} 0 & -a & c \\ a & 0 & -b \\ -c & b & 0 \end{pmatrix} \longmapsto \varphi(u(a,b,c)) = \begin{pmatrix} \frac{ia}{2} & -\frac{b+ic}{2} \\ \frac{b-ic}{2} & -\frac{ia}{2} \end{pmatrix} \in \mathfrak{su}(2)$$

is a Lie algebra isomorphism. Moreover, it is straightforward to check that

$$-\frac{1}{2}\,\mathrm{tr}(\varphi(u(a,b,c)^2) = \frac{a^2+b^2+c^2}{4} \quad \text{and} \quad -\frac{1}{2}\,\mathrm{tr}(u(a,b,c)^2) = a^2+b^2+c^2.$$

In other words $\varphi^* B = \frac{1}{4}\bar{B}$, where

$$\bar{B}(X,Y) = -\frac{1}{2}\,\mathrm{tr}(XY), \quad X, Y \in \mathfrak{so}(3).$$

The change of the volume of a compact Lie group G under homothety is easily seen from formula 2.30 to be

$$\mathrm{vol}\,(G, \lambda^2 B) = (\lambda^{-2\dim(\mathfrak{t})} \cdot \lambda^{-2(\dim(\mathfrak{g}) - \dim(\mathfrak{t}))})^{-1/2}\,\mathrm{vol}\,(G, B) = \lambda^{\dim(\mathfrak{g})}\,\mathrm{vol}\,(G, B),$$

where $\lambda > 0$.

Application of this fact to the present case yields

$$\mathrm{vol}\,(\mathrm{SO}(3), \varphi^* B) = \mathrm{vol}\,(\mathrm{SO}(3), 2^{-2}\bar{B}) = \frac{1}{8}\,\mathrm{vol}\,(\mathrm{SO}(3), \bar{B}) = \pi^2 = \frac{1}{2}\,\mathrm{vol}\,(\mathrm{SU}(2), B),$$

as expected.

Chapter 3

Isospectral examples

In this chapter we aim to construct (infinitely many) pairs of nearly Kähler manifolds isospectral for the Hodge Laplace and Dirac operator. We introduce in Section 3.1 the relevant isospectral criteria that we shall use in Section 3.2 to prove the main result of this thesis, see Theorem 3.2.5. This chapter ends up with an account to the lowest possible dimension in the nearly Kähler setting that reveals that the ansatz that was used successfully in higher dimensions does not produce isospectral pairs in dimension six.

3.1 Isospectrality criteria

We aim to construct isospectral pairs of nearly Kähler manifolds using Sunada's criterion for Laplace isospectrality [Su].

Theorem 3.1.1. *Let (M, g) be a compact Riemannian manifold and let G be a group acting freely and by isometries on M. If Γ_1 and Γ_2 are element-wise conjugate subgroups of G acting freely on M, then the quotients $M_{\Gamma_i} = \Gamma_i \backslash M$ are Laplace isospectral.*

A triple (G, Γ_1, Γ_2) giving rise to an isospectral pair M_{Γ_i} by means of Theorem 3.1.1 will be called *Sunada triple.* Examples of isospectral pairs that are constructed as quotients of a compact manifold M using Sunada's method, which include the ones presented in this work, are in fact strongly isospectral, see [G]. That is, for any natural bundle E of M and for any strongly elliptic natural operator $D : \Gamma(E) \longrightarrow \Gamma(E)$, the associated operators on the quotients D_{Γ_i} acting on sections of the bundles E_{Γ_i} respectively, have the same spectrum.

The manifolds on which we will apply Sunada's criterion are in fact simply connected Lie groups, and as such, they carry a (unique) spin structure. Spin structures of a finite quotient $G_\Gamma = \Gamma \backslash G$ are in one-to-one correspondence with homomorphisms $\varepsilon : \Gamma \longrightarrow \{1, -1\}$. The spinor bundle of (G_Γ, ε) is in turn given by $\Sigma_\varepsilon(G_\Gamma) = G \times_\Gamma \mathrm{Spin}(n)$, where the action of $\gamma \in \Gamma$ on the first component is the natural one and on $\mathrm{Spin}(n)$ it is by multiplication with the central element $\varepsilon(\gamma)$. Let $L^2_\varepsilon(G_\Gamma)$ be the space of locally square-integrable complex-valued (Γ, ε)-equivariant functions, i.e. the space consisting in functions $f \in L^2(G)$ such that $f(\gamma g) = \varepsilon(\gamma) f(g)$, for any $\gamma \in \Gamma$ and $g \in G$. As a Hilbert space $L^2_\varepsilon(G_\Gamma) \otimes_{\mathbb{C}} \Sigma_n$ can be identified with $L^2(\Sigma_\varepsilon(G_\Gamma))$ via

$$L^2_\varepsilon(G_\Gamma) \otimes_{\mathbb{C}} \Sigma_n \ni f \otimes s \longmapsto fs \in L^2(\Sigma_\varepsilon(G_\Gamma)),$$

and Clifford contraction corresponds under this identification to

$$X \cdot (f \otimes s) = f \otimes (X \cdot s), \qquad X \in \mathfrak{g},\ f \otimes s \in L^2_\varepsilon(G_\Gamma) \otimes_{\mathbb{C}} \Sigma_n.$$

Ammann and Bär observed that if the right G-modules $L^2_\varepsilon(G_\Gamma)$ and $L^2_{\varepsilon'}(G_{\Gamma'})$ are G-isomorphic, then the corresponding Dirac operators have the same spectrum, see Theorem 5.1 in [AB]. If we set the spin structures $(\varepsilon, \varepsilon') \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2) \times \mathrm{Hom}(\Gamma', \mathbb{Z}_2)$ to be the trivial ones, the latter condition is nothing but the assertion that the right regular representations of G_Γ and $G_{\Gamma'}$ are G-isomorphic, which is in turn equivalent to the Sunada condition by a classical result of Berard, see [Be, Be2]. In particular, we have the following.

Theorem 3.1.2. *Let G be a compact simply connected Lie group endowed with a left-invariant metric, and $\Gamma_i \subset G$ be a pair of finite almost conjugate subgroups. Endow the quotients G_{Γ_i} with the trivial spin-structure. Then the finite quotients G_{Γ_1} and G_{Γ_2} are Dirac isospectral.*

3.2 Isospectral nearly Kähler pairs

Our isospectral pairs will have the same local geometry as one fixed nearly Kähler manifold: a Ledger-Obata space.

3.2.1 The Ledger-Obata space and its transformation group

Let G be a compact simple Lie group with finite center. The manifold $M = M(G) = G \times G$ admits a left-invariant nearly Kähler structure obtained by means of the Ledger-Obata construction [LO]. More explicitly: identify M with the homogeneous space L/K, where $L = G \times G \times G$ and $K = \Delta(G)$ denotes the diagonally embedded $G \hookrightarrow L$. Observe that the group L has an obvious 3-symmetry $S \in \mathrm{Aut}(L)$ which stabilizes K and makes (L, K) a 3-symmetric pair. In fact, all homogeneous nearly Kähler manifolds are 3-symmetric.[1] Consequently, the Lie algebra $\Delta(\mathfrak{g}) \subset \mathfrak{l}$ admits a complement $\mathfrak{m} \cong T_eM$ which is invariant under the adjoint representation and the infinitesimal action of the 3-symmetry S_*. An appropriately rescaled version of the Cartan-Killing form of L

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)), \quad X, Y \in \mathfrak{l}, \tag{3.1}$$

restricted to the complement $\mathfrak{m} \subset \mathfrak{l}$ defines the invariant nearly Kähler metric $g = g_e$ on M. The nearly Kähler almost complex structure J is in turn an invariant tensor characterized on the chosen reductive complement $\mathfrak{m}$ in terms of the 3-symmetry S as below

$$S_* = -\frac{1}{2}\,\mathrm{Id} + \frac{\sqrt{3}}{2} J \quad \text{on } \mathfrak{m}. \tag{3.2}$$

[1] See [CG2] for an exhaustive list of homogeneous nearly Kähler manifolds.

The triple (M, g, J) defines a homogeneous nearly Kähler manifold and we regard this structure in the sequel as the *Ledger-Obata space* $M(G)$ associated with G.

The space $M(G)$ is well understood in many extents and it is important as it generalizes in the natural manner the notion of being symmetric and provides to be a rich source of invariant Einstein metrics [CNN].

The group of (holomorphic) isometries of $M = M(G)$ can be calculated using that its identity component is $G^3/\Delta(Z(G))$, where $\Delta(Z(G))$ denotes the diagonally embedded center of G, see Theorem 5.3 in [CG]. We include the following lemma for the convenience of the reader.

Lemma 3.2.1. *Let M be a compact Riemannian manifold and assume that the maximal connected subgroup G_0 of $G = \mathrm{I}(M)$ acts transitively on M with stabilizer K_0. Then $G = (G_0 \times \Gamma)/L$ as sets, where Γ is a subgroup of the quotient of*

$$\mathrm{Aut}(G_0, K_0) = \{\varphi \in \mathrm{Aut}(G_0) \;:\; \varphi(K_0) \subset K_0\}$$

and the group $\mathrm{Ad}(K_0)$ and L is the kernel of injectivity of the action of $G_0 \times \Gamma$.

Proof. Write $M = G_0/K_0$ and let K be the stabilizer of the action of G on M. Then $G/G_0 = K/K_0$ and we can write

$$G = \bigcup_i G_0 \cdot k_i$$

as sets, where (k_i) is a (finite) system of representatives of K/K_0.

Claim. The kernel of $\mathrm{Ad} : K \longrightarrow \mathrm{Aut}(G_0, K_0)$ is trivial.

Let U be a small neighborhood of $e \in G_0$ so that $U \subset G_0$, and let $\pi : G \longrightarrow G/K$ be the canonical projection. An element $h \in \mathrm{Ker}(\mathrm{Ad}) = C_G(G_0) \cap K$ fixes the coset xK for any $xK \in \pi(U)$ and its differential is the identity. So $h = e$ and Ad is injective.

We can then realize the system (k_i) as automorphisms of G_0 fixing K_0 defined up to elements in $\mathrm{Ad}(K_0)$. This ends up the proof as $\mathrm{Aut}(G_0, K_0)$ acts isometrically on G_0/K_0 in a natural manner. □

The utility of Lemma 3.2.1 relies on the explicit knowledge of the outer automorphism group $\mathrm{Out}(G)$. For instance, in the complex semisimple case, the latter corresponds to symmetries of the Dynkin diagram associated with $\mathfrak{g}$. Direct application of Lemma 3.2.1 on the Ledger-Obata space $M(G)$ leads to the following result.

Lemma 3.2.2. *Let G be a compact simply-connected simple Lie group having no non-trivial outer automorphisms and let $M = M(G)$ be the Ledger-Obata space associated with this group. The isometry group and the group of holomorphic isometries of M are*

$$\mathrm{I}(M) = G^3/\Delta(Z(G)) \rtimes \mathrm{Sym}(3) \quad \textit{and} \quad \mathrm{I}^h(M) = G^3/\Delta(Z(G)) \rtimes \mathrm{Alt}(3),$$

where the action of $\mathrm{Sym}(3)$ on $G^3/\Delta(Z(G))$ is the standard one and $\Delta(Z(G)) \subset G^3$ denotes the diagonally embedded center.

Proof. The isometry group and the group of holomorphic isometries of the nearly Kähler manifold (M, g) has identity component $L_0 = G^3/\Delta(Z(G))$, group which acts transitively on M with stabilizer $L_{\mathrm{st}} = \Delta(G)/\Delta(Z(G))$. In virtue of Lemma 3.2.1 we must distinguish the group automorphisms of L_0 preserving the diagonal $\Delta(G)$ and the nearly Kähler metric g. Since G has no non-trivial outer automorphisms, the automorphisms of L_0 are either inner or permutations of the components. The metric g is automatically preserved by those automorphisms, as it is obtained by restriction of the Killing form of L_0. Consequently, the subset $\Gamma \subset \mathrm{Aut}(L_0, L_{\mathrm{st}})$ stated in Lemma 3.2.1 is $\Gamma = \mathrm{Sym}(3) \times N_{G^3}(\Delta(G))$, where the normalizer $N_{G^3}(\Delta(G))$ acts on M by conjugation. Note also that the kernel of effectivty of the action of $N_{G^3}(\Delta(G))$ on M is $Z(G)^3$. Because $N_{G^3}(\Delta(G)) = Z(G)^3 \cdot \Delta(G)$, we get

$$\mathrm{I}(M) = G^3/\Delta(Z(G)) \times \mathrm{Sym}(3),$$

where the above equality is meant as a set. Let $\varphi_{[x]} \circ \varphi_\sigma \in \mathrm{I}(M)$ be the isometry induced by $([x_1, x_2, x_3], \sigma) \in L_0 \times \mathrm{Sym}(3)$. Then, for all cosets $[z_1, z_2, z_3] \in M$, $\sigma, \omega \in \mathrm{Sym}(3)$ and $x, y \in G^3$, we have

$$(\varphi_{[x]} \circ \varphi_\sigma \circ \varphi_{[y]} \circ \varphi_\omega)([z]) = (\varphi_{[(x_1,x_2,x_3)\cdot\sigma(y_1,y_2,y_3)]} \circ \varphi_{\sigma\circ\omega})([z]),$$

where $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. In other words,

$$([x], \sigma) \cdot ([y], \omega) = ([x \cdot \sigma(y)], \sigma \circ \omega) \in \mathrm{I}(M).$$

To finish the proof of the claim just note that $\mathrm{Alt}(3)$ is the permutation group that preserves the almost complex structure defined in identity (3.2). □

3.2.2 The main result

Let $M = M(G)$ be the Ledger-Obata space associated with a compact simply connected Lie group G. The groups obtained in Theorem 2.2.6 inject into the transformation group of M and the locally homogeneous quotients they give rise are easily seen to be nearly Kähler manifolds. To apply Theorem 3.1.1 to a pair of such quotients we need to tell apart non-isometric quotients. An straight-forward lifting argument yields a simple criterion to do so.

Lemma 3.2.3. *Let (M, g) be a compact simply connected Riemannian manifold with isometry group $\mathrm{I}(M)$. Let $\Gamma_i \subset \mathrm{I}(M)$ be finite subgroups acting freely on M. Then, the finite quotients M_{Γ_i} are isometric precisely when the groups Γ_i are conjugate to each other in $\mathrm{I}(M)$.*

Now we compare conjugacy of finite subgroups in G and conjugacy in the transformation group $\mathrm{I}(M) = G^3/\Delta(Z(G)) \rtimes \mathrm{Sym}(3)$ of the Ledger-Obata space $M = G^3/\Delta(G)$.

Lemma 3.2.4. *Let Γ_i be a pair of non-trivial finite subgroups of a compact Lie group G and $M = M(G)$ be the associated Ledger-Obata space. The subgroups $\Gamma_i \subset G$ are isomorphic to $\overline{\Gamma}_i = \{(\gamma, 1, 1)\Delta(Z(G)) \in \mathrm{I}(M) \; : \; \gamma \in \Gamma_i \} \subset \mathrm{I}(M)$ in a natural manner. Moreover, conjugacy of the groups $\overline{\Gamma}_i$ in $\mathrm{I}(M)$ is equivalent to conjugacy of the groups Γ_i in G.*

Proof. Let us now suppose that the groups $\bar{\Gamma}_i$ are conjugate in $\mathrm{I}(M)$. That is, for any $\gamma_1 \in \Gamma_1$ there is $\gamma_2 \in \Gamma_2$ such that

$$\varphi_{[x]} \circ \varphi_\sigma \circ \varphi_{[\gamma_1,1,1]} = \varphi_{[\gamma_2,1,1]} \circ \varphi_{[x]} \circ \varphi_\sigma$$

for some $([x], \sigma) \in \mathrm{I}(M)$. Evaluation of the latter equality at $\Delta(G) \in M$ yields

$$\varphi_\sigma([\gamma_1, 1, 1]) = [\mathrm{Ad}_{x^{-1}}(\gamma_2), 1, 1].$$

This is impossible unless $\sigma = (1) \in \mathrm{Sym}(3)$, case in which the groups Γ_1 and Γ_2 are conjugate in G. □

We are now in position to give a proof for the main result of this thesis.

Theorem 3.2.5. *There is a strictly increasing sequence of numbers $(d_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ there is a pair $M_1^{d_n}$ and $M_2^{d_n}$ of non-isometric nearly Kähler manifolds that are isospectral for the Dirac and the Hodge-Laplace operator Δ^k for $k = 0, 1, ..., \dim(M)$.*

Proof. For each $n \in \mathbb{N}$ choose an odd number $d(n)$ so that $G = \mathrm{Spin}(d(n))$ is Sunada, see Theorem 2.2.6 and the list of irreducible representations fulfilling the conditions on the latter theorem in Example 2.2.7. In particular, G admits almost conjugate subgroups Γ_i that are not conjugate. Denote the Ledger-Obata space associated with G by $M = M(G)$. The chosen pair of almost conjugate groups Γ_i of G act freely on $M(G) = G \times G$ by left multiplication in the first component and yield an isospectral pair M_i. Lemma 3.2.4 assures that the groups Γ_i are not conjugate in $\mathrm{I}(M)$, and so the quotients M_{Γ_i} are not isometric, see Lemma 3.2.3. This shows that a pair of almost conjugate subgroups Γ_i of the spin group $\mathrm{Spin}(d(n))$ yields a pair of isospectral nearly Kähler manifolds. Since there are infinitely many such pairs, see Proposition 2.11 in [V] or the proof of Proposition 2.6 in [Lar2], this shows the existence of an infinite family of pairs of nearly Kähler manifolds, which are isospectral for the Hodge Laplace operator. The claim for the Dirac operator follows then from Theorem 3.1.2 after endowing the finite quotients M_i with the trivial spin structures $\alpha_i(\gamma) = 1$, where $\gamma \in \Gamma_i$. □

Remark. Almost conjugate finite groups Γ_i constructed by means of Theorem 2.2.6 yield explicit examples of isospectral good Riemannian orbifolds with different *maximal isotropy orders*, see Corollary 2.6 in [RSW] and good isospectral spherical spin orbifolds $\Gamma_i \backslash S^{2n}$.

3.2.3 Dimension six

Sunada isospectral quotients are generic in the compact setting, see [Pes], and so it is natural to ask whether the method used in Theorem 3.2.5 gives isospectral pairs in dimension six. We aim in this section to show that this is not the case. More exactly, we will show the following statement.

Proposition 3.2.6. *Let M_Γ and $M'_{\Gamma'}$ be a pair of Laplace isospectral locally homogeneous nearly Kähler manifolds in dimension six with $M = S^3 \times S^3$. Then M' and M are holomorphically isometric. Moreover, Sunada isospectral pairs M_{Γ_i} with $\Gamma_i \subset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \{\mathrm{Id}\}$ are holomorphically isometric.*

The following proposition will be needed to prove Proposition 3.2.6.

Proposition 3.2.7. *The volume of the four homogeneous six dimensional nearly Kähler manifolds with respect to the unique invariant metric with scalar curvature κ is given by*

(M^6, g)	S^6	$\mathbb{C}P^3$	$S^3 \times S^3$	$F(1,2)$
vol (M^6)	$\left(\frac{30}{\kappa}\right)^3 \frac{16\pi^3}{15}$	$\left(\frac{30}{\kappa}\right)^3 \frac{\pi^3}{6}$	$\left(\frac{30}{\kappa}\right)^3 \frac{32\pi^4}{81\sqrt{3}}$	$\left(\frac{30}{\kappa}\right)^3 \frac{\pi^3}{2}$

Table 3.1: Volumes of homogeneous manifolds (M^6, g) with $\mathrm{scal}(g) = \kappa$.

Proof. The normal nearly Kähler metric on $M = G/K$, where $G = \mathrm{I}_0^h(M)$, is induced by an invariant scalar product B on $\mathfrak{g}$. Furthermore, the fibers of the Riemannian submersion $\pi : G \longrightarrow G/K$ have the same volume, and so

$$\mathrm{vol}\,(M) = \mathrm{vol}\,(G, B)\,\mathrm{vol}\,(K, B|_{\mathfrak{k}\times\mathfrak{k}})^{-1}.$$

Consequently, we can use Proposition 2.4.1 to calculate the volume of M with respect to a chosen normal metric. We choose our metric, so that the scalar curvature of M is 30, see Lemma 5.4 in [MS] and perform the explicit calculations just for $\mathbb{C}P^3$, $F(1,2)$ and $S^3 \times S^3$.

(i) Let us first consider the flag manifold $F(1,2) = \mathrm{SU}(3)/T$ with the normal metric induced by the invariant scalar product

$$B : \mathfrak{su}(3) \times \mathfrak{su}(3) \longrightarrow \mathbb{R}, \qquad B(X,Y) = -\frac{1}{2}\,\mathrm{tr}_{\mathbb{C}^3}(XY), \quad X, Y \in \mathfrak{su}(3).$$

Fix a maximal torus of SU(3) by declaring its Lie algebra to be

$$\mathfrak{t} = \{\mathrm{diag}(iX_1, iX_2, iX_3) \;:\; X_i \in \mathbb{R},\; X_1 + X_2 + X_3 = 0\} = \mathrm{span}(H_i \;:\; i \le 3),$$

where $H_1 = \mathrm{diag}(0, i, -i)$, $H_2 = \mathrm{diag}(-i, 0, i)$ and $H_3 = -H_1 - H_2 = \mathrm{diag}(i, -i, 0)$. Eventually, in the base (H_1, H_2) and $(\varepsilon_1, \varepsilon_2)$, we get

$$b(H_\mu, H_\nu) = \frac{3}{2}\left(\delta_{\mu\nu} - \frac{1}{3}\right) \quad \text{and} \quad b^{-1}(\varepsilon_\mu, \varepsilon_\nu) = 2\left(\delta_{\mu\nu} - \frac{1}{3}\right),$$

where $\varepsilon_j \in \mathfrak{t}^*$ are the functionals $\varepsilon_j(\mathrm{diag}(iX_1, iX_2, iX_3)) = X_j$ and $b^{-1} : \mathfrak{t}^* \times \mathfrak{t}^* \longrightarrow \mathbb{R}$ is the scalar product dual to $b = B|_{\mathfrak{t}\times\mathfrak{t}}$. The roots of $\mathfrak{su}(3)$ are

$$\pm i(\varepsilon_1 - \varepsilon_2), \pm i(\varepsilon_3 - \varepsilon_1), \pm i(\varepsilon_2 - \varepsilon_3),$$

and a straightforward calculation reveals that

$$-\Delta = \frac{4}{3}\left(\frac{\partial^2}{\partial^2\varepsilon_1^2} + \frac{\partial^2}{\partial^2\varepsilon_2^2}\right) - \frac{4}{3}\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2},$$

where Δ is the operator introduced in Proposition 2.4.1. The polynomial $\delta_{\mathfrak{su}(3)}$ is

$$\delta_{\mathfrak{su}(3)} = (\varepsilon_1 - \varepsilon_2)^2(2\varepsilon_2 + \varepsilon_1)^2(2\varepsilon_1 + \varepsilon_2)^2 = 4\varepsilon_1^6 + 12\varepsilon_1^5\varepsilon_2 - 3\varepsilon_1^4\varepsilon_2^2 - 26\varepsilon_1^3\varepsilon_2^3 - 3\varepsilon_1^2\varepsilon_2^4 + 12\varepsilon_1\varepsilon_2^5 + 4\varepsilon_1^4,$$

and hence

$$e^{-\frac{1}{4}\Delta}|_0\delta = \frac{1}{6\cdot 3^3}\left(\frac{\partial^2}{\partial\varepsilon_1^2}+\frac{\partial^2}{\partial\varepsilon_2^2}-\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\right)^3\delta_{\mathfrak{su}(3)}(0) = \frac{1944}{162} = 12.$$

Using the fact that $|W(\mathrm{SU}(3))| = |\,\mathrm{Sym}(3)| = 6$, formulae (2.30) and (2.35) yield

$$\mathrm{vol}\,(F(1,2),B) = |W(\mathrm{SU}(3))|(e^{-\frac{1}{4}\Delta}|_0\delta)^{-1}\pi^{\frac{\dim(\mathfrak{su}(3))-\dim(\mathfrak{t})}{2}} = \frac{1}{2}\pi^3.$$

(ii) Let us now consider $\mathbb{C}P^3 = \mathrm{Sp}(2)/\mathrm{U}(1)\times\mathrm{Sp}(1)$ with the submersion metric induced by

$$B(X,Y) = -\frac{1}{2}\mathrm{Re}\,\mathrm{tr}_{\mathbb{H}^2}(XY) \qquad X,Y\in\mathfrak{sp}(2).$$

Fix a maximal torus T in $\mathrm{Sp}(2)$ by declaring $\mathfrak{t} = \{\mathrm{diag}(i\theta_1,i\theta_2)\ :\ \theta_i\in\mathbb{R}\}$ and observe that

$$b = B|_{\mathfrak{t}\times\mathfrak{t}} = \frac{1}{2}(\theta_1\otimes\theta_1+\theta_2\otimes\theta_2),$$

where $\theta_i\in\mathfrak{t}^*$ are the functionals

$$\theta_i(\mathrm{diag}(i\tilde{\theta}_1,i\tilde{\theta}_2)) = \tilde{\theta}_i,\quad \mathrm{diag}(i\tilde{\theta}_1,i\tilde{\theta}_2)\in\mathfrak{t}.$$

Consequently, we have

$$b^{-1}(\theta_1,\theta_1) = b^{-1}(\theta_2,\theta_2) = 2 \quad\text{and}\quad b^{-1}(\theta_1,\theta_2) = 0. \tag{3.3}$$

On the other hand, the roots of $\mathfrak{sp}(2)$ are $\pm 2i\theta_1, \pm 2i\theta_2, \pm i\theta_1 \pm i\theta_2$ and

$$\delta_{\mathfrak{sp}(2)} = 16\ \theta_1^2\theta_2^2(\theta_1^2-\theta_2^2)^2.$$

From the identities in (3.3), we see that the operator described in Proposition 2.4.1 is in turn given by

$$\Delta = 4\left(\frac{\partial^2}{\partial\theta_1}+\frac{\partial^2}{\partial\theta_2}\right).$$

After a short calculation, we get $e^{\frac{-1}{4}\Delta}\delta|_0 = 192$. Due to the latter calculations and the fact that $|W(\mathrm{Sp}(2))| = 8$, formula (2.30) implies that

$$\mathrm{vol}\,(\mathrm{Sp}(2),B) = \frac{\pi^6}{12}.$$

We also note that

$$\mathrm{vol}\,(\mathrm{U}(1)\times\mathrm{Sp}(1),B|_{\mathfrak{u}(1)\oplus\mathfrak{sp}(1)}) = \frac{\pi^3}{2}.$$

It follows that $\mathrm{vol}\,(\mathbb{C}P^3,B) = \frac{\pi^3}{6}$.

(iii) Let us now consider $S^3 \times S^3 = \mathrm{SU}(2)^3/\Delta(\mathrm{SU}(2))$, where $\Delta(\mathrm{SU}(2))$ denotes the diagonally embedded $\mathrm{SU}(2)$ in $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$, endowed with the metric

$$B(X,Y) = -\frac{1}{3}\operatorname{tr}(XY), \qquad X,Y \in \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2).$$

We have $B(\Delta H,\Delta H) = 2$, and $B^{-1}(\Delta x,\Delta x) = \frac{1}{2}$, where

$$\Delta H = (H,H,H) \in \Delta(\mathfrak{su}(2)) \quad , \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2)$$

and $\Delta x \in (\Delta\mathfrak{su}(2))^*$ is the unique element such that $\Delta x(\Delta H) = 1$. The polynomial $\delta_{\Delta(\mathfrak{su}(2))} = 4\Delta^2 x$, and so the volume of $\Delta(\mathrm{SU}(2))$ is

$$\mathrm{vol}\left(\Delta(\mathrm{SU}(2), B|_{\Delta(\mathfrak{su}(2))\times\Delta(\mathfrak{su}(2))}\right) = 32\sqrt{2}\pi^2.$$

Similarly, one calculates that

$$\mathrm{vol}\left(\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2), B\right)^{1/3} = \frac{8\sqrt{2}\pi^2}{3\sqrt{3}},$$

and so

$$\mathrm{vol}\left(S^3\times S^3, B\right) = \frac{32\pi^4}{81\sqrt{3}}.$$

Table 3.1 summarizes the results. □

Proof of Proposition 3.2.6. Let $M = S^3 \times S^3$. We can distinguish the spectrum of the scalar Laplacian of two locally homogeneous six dimensional nearly Kähler manifolds by their associated heat kernel coefficients, i.e. the coefficients of the asymptotic expansion of the Laplacian Δ_g^0 acting on functions, where $(M'_{\Gamma'}, g)$ is a locally homogeneous nearly Kähler manifold. That is, we consider

$$\operatorname{tr}(e^{-s\Delta_g^0}) \;\sim\; (4\pi s)^{-3}\sum_{i=0}^{\infty} a_i^0(g)s^i \quad \text{for} \quad s \searrow 0.$$

For example, see Theorem 4.8.18 in [Gi], we have

$$a_0^0(g) = \mathrm{vol}\,(M'_{\Gamma'}), \quad a_1^0(g) = \frac{1}{6}\int_{M'_{\Gamma'}} \kappa(M'_{\Gamma'})d\,\mathrm{vol}\,(M'_{\Gamma'}) = \frac{1}{6}\kappa(M'_{\Gamma'})\,\mathrm{vol}\,(M'_{\Gamma'}),$$

where $\kappa(M'_{\Gamma'})$ is the (constant) scalar curvature of $(M'_{\Gamma'}, g)$ and g is it submersion metric. Observe that $\kappa(M) = \kappa(M_\Gamma) = \kappa(M'_{\Gamma'}) = \kappa(M')$, and so we can normalize the metrics in question so that $\kappa(M) = \kappa(M') = 30$. On the other hand, the volumes of the locally homogeneous manifolds M_Γ and $M'_{\Gamma'}$ are the same, hence

$$\frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(M')} = \frac{|\Gamma|}{|\Gamma'|} \in \mathbb{Q}.$$

According Proposition 3.2.7, the volume of $M = S^3 \times S^3$ is a multiple of $3^{-1/2}\pi^4$. Because π is transcendental, we must have $M' = S^3 \times S^3$. The second claim is a direct consequence of Proposition 2.3.1. □

Chapter 4

Open problems and current research

As described in the preamble of this work, our general theme was to investigate the question whether the spectrum of (locally homogeneous) nearly Kähler manifolds characterizes the geometry within this class of manifolds. The main result of this thesis, see Theorem 3.2.5, already gives us a light in this direction. However, nearly Kähler manifolds in dimension higher than six behave very differently from their six dimensional counterparts. In particular, new phenomena concerning spectra of nearly Kähler six manifolds are to be expected.

As observed in Section 3.2.3, Sunada's result does not produce isospectral examples for most of the known locally homogeneous six dimensional nearly Kähler manifolds with non-trivial fundamental group. This does not mean that isospectral examples do not exist, but rather that we have more work to do. The work to be yet completed has essentially two parts. Firstly, we must compute the spectrum of the four homogeneous examples, and secondly try to characterize isospectrality of quotients M_{Γ_i} of a given homogeneous nearly Kähler manifold M in a suitable way, that is, by means of a power series (Poincaré, spectral zeta series, etc).

The current project in collaboration with Gregor Weingart aims to attack the first problem we mentioned: the explicit description of the spectrum of the four homogeneous six dimensional nearly Kähler manifolds. The main results until now can be summarized in a single theorem, see [VW].

Theorem 4.0.1. *Let (M^6, g, J) be a six dimensional nearly Kähler manifold with scalar curvature $\kappa = \kappa(M)$, let ΣM be its spinor bundle and let $\bar{\nabla}$ be the canonical Hermitian connection. Then the following statements hold.*

(i) The spinor bundle ΣM is isomorphic to the bundle $S(M) = \mathcal{A}M \oplus TM$, where $\mathcal{A}M \subset \mathrm{End}(TM)$ is the algebra bundle spanned by the $\bar{\nabla}$-parallel sections Id *and J. Clifford multiplication in $S(M)$ corresponds to the following action*

$$X \bullet (s, S) = (-g_{\mathcal{A}}(X, S), sX + X \times S), \quad X, S \in \Gamma(TM),\ s \in \Gamma(\mathcal{A}M), \tag{4.1}$$

where $\times : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ is the cross product defined by the generic 3-*form underlying the geometry of M, and $g_{\mathcal{A}} : \Gamma(TM) \times \Gamma(TM) \longrightarrow \mathcal{A}$ is the natural* *skew-hermitian extension of* g*, i.e.*

$$g_{\mathcal{A}}(X,Y) = g(X,Y)\,\mathrm{Id} + g(JX,Y)J, \quad X,Y \in \Gamma(TM).$$

In particular, the action of $\bar{\nabla}$ *is well defined on the spinor bundle* $S(M)$*.*

(ii) *Let* $D_l : \Gamma(S(M)) \longrightarrow \Gamma(S(M))$ *be the standard one parameter deformation of Dirac operators acting on the presentation of the spinor bundle given in item (i) obtained by interpolating between* $\bar{\nabla}$ *and the Levi-civita connection* ∇*, i.e.*

$$D_l = \sum_{\mu} E_\mu \bullet \left(\bar{\nabla}_{E_\mu} + l\sqrt{\frac{\kappa(M)}{30}}(E_\mu \times \cdot) \right) \quad l \in \mathbb{R}, \tag{4.2}$$

where (E_μ) *is an orthonormal frame of* TM*. Then the following Konstant-Parthasarathy type identity holds.*

$$D_l^2 = \left(\Delta + \frac{2\kappa(M)}{pr_{TM}}\right) - \sqrt{\frac{2\kappa(M)}{15}}(3l-1)(\mathcal{D}+\mathcal{D}^*) + 9l^2\frac{2\kappa(M)}{15}pr_{\mathcal{A}M}, \tag{4.3}$$

where $\mathcal{D} : S(M) \longrightarrow S(M)$ *is the operator defined by*

$$\mathcal{D}(s,S) = (0, \sum_{\mu} (\bar{\nabla}_{E_\mu} s)^c E_\mu),$$

where (E_μ) *is an orthonormal frame of* TM *and* $^c : \Gamma(\mathcal{A}M) \longrightarrow \Gamma(\mathcal{A}M)$ *is the natural conjugation.*

(iii) *The Hermitian Laplace operator*[1]

$$\Delta = \bar{\nabla}^*\bar{\nabla} + \frac{1}{2}\sum_{\mu,\nu}(E_\mu \wedge E_\nu) \star \bar{R}_{E_\mu,E_\nu},$$

where $\star : \mathfrak{su}(M) \otimes \Gamma(S(M)) \longrightarrow \Gamma(S(M))$ *is the infinitesimal representation map coming from the nearly Kähler* $\mathrm{SU}(3)$*-structure, satisfies the following commutation relations*

$$[\Delta, D_l] = [\Delta, \mathcal{D}] = [\Delta, \mathcal{D}^*] = 0. \tag{4.4}$$

(iv) *The sections of the spinor bundle decomposes in three* Δ*-invariant,* L^2*-orthogonal pieces*

$$\Gamma(S(M)) = \Gamma_{const}(S(M)) \oplus \Gamma_{stable}(S(M)) \oplus \Gamma_{unstable}(S(M))$$

such that for each $l \in \mathbb{R}$ *we have* $D_l^2 = 9\frac{2\kappa(M)}{15}l^2\,\mathrm{Id}$ *on constant sections* $\Gamma_{const}(S(M))$*, and* $D_l^2 = \Delta + \frac{2\kappa(M)}{15}\mathrm{Id}$ *on stable sections* $\Gamma_{stable}(S(M))$*. Furthermore, the space of unstable sections* $\Gamma_{unstable}(S(M))$ *decomposes into two dimensional subspaces* $\mathrm{span}_{\mathbb{R}}\{f, \mathcal{D}f\}$ *parametrized by eigenfunctions* $f \in \Gamma(\mathcal{A}(M))$ *of the Hermitian Laplace operator* $\Delta f = Ef$ *with positive eigenvalue* $E > 0$ *so that* D_l^2 *takes two eigenvalues* $\lambda_i(E,\kappa,l)$*.*[2]

[1] This operator was introduced in [MS] to study infinitesimal deformations of nearly Kähler six manifolds.

[2] The formula for the numbers $\lambda_i(E,\kappa,l)$ is explicit but somewhat cumbersome.

The canonical Hermitian connection $\bar{\nabla}$ coincides with the normal connection on any homogeneous nearly Kähler manifold in the present dimension, and so the Hermitian Laplace operator acts fiberwise on the spinor bundle as a Casimir operator[3]. Consequently, Theorem 4.0.1 tells us that the problem of computing the spectrum of any (invertible) Dirac operator D_l in our family reduces to a (tractable) representation theoretical problem by means of the Peter-Weyl formalism. Loosely stated, Theoreom 4.0.1 says that the Dirac spectrum on M is characterized by the spectrum of the Hermitian Laplacian acting on functions and vector fields. The latter operator coincides with the Hodge-Laplace operator on functions. Moreover, we expect that 1-isospectrality will imply isospectrality for the Hermitian Laplace operator on vector fields. In particular, we are very interested in looking for Sunada pairs as quotients of a given homogeneous six dimensional nearly Kähler manifold M. Most of these quotients are obtained when $M = S^3 \times S^3$, where the transformation group is $G = G_0 \rtimes \mathrm{Alt}(3)$ for $G_0 = \mathrm{SU}(2)^3/\Delta(\mathbb{Z}_2)$, as we have seen in Lemma 3.2.2. In this spirit, it is meaningful to try to extend the classification results obtained in [CV].

Let us observe first that Lemma 3.1 in [CV] has a small mistake arising from the non effectiveness of the action of G on M. Therefore, any classification statement in [CV] concerns *just* groups $\Gamma \subset G$ that act freely and effectively on M. We rectify and extend this result.

Proposition 4.0.2. *Let $\Gamma \subset G$ be a finite subgroup and set*

$$\Gamma_0 = \Gamma \cap G_0, \quad \Gamma_{\mathrm{Alt}} = \Gamma \cap \mathrm{Alt}(3), \tag{4.5}$$
$$\Gamma_\sigma = \Gamma \cap \{([x_{123}], \sigma) \in G \ : \ x_{123} \in \mathrm{SU}(2)^3\}, \tag{4.6}$$

where $\sigma = (123), (132)$. Suppose there is no element $X \in \Gamma_{(123)} \cup \Gamma_{(132)}$ of order 3. Then, the group Γ acts non-freely on M precisely when one of the following conditions are met.

(a) There is a non trivial element $[x_{123}] \in \Gamma_0$ such that

$$\mathrm{Re}(x_1) = \mathrm{Re}(x_2) = \mathrm{Re}(x_3). \tag{4.7}$$

(b) There is an element $(x_{123}, \sigma) \in \Gamma_{(123)}$ such that $[x_{123}] \neq \Delta(\mathbb{Z}_2)$ and either of the following conditions is met

$$\mathrm{Re}(x_1x_2x_3) = \mathrm{Re}(x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}) \quad \forall \alpha \in \mathrm{Alt}(3), \tag{4.8}$$

(c) There is an element $(x_{123}, \sigma) \in \Gamma_{(132)}$ such that $[x_{123}] \neq \Delta(\mathbb{Z}_2)$ and

$$\mathrm{Re}(x_3x_2x_1) = \mathrm{Re}(x_2x_1x_3) = \mathrm{Re}(x_1x_3x_2). \tag{4.9}$$

(d) Γ_{Alt} is non trivial.

[3]See Section 5.1 in [MS] for a more detailed explanation of this statement.

Proof. The group $G = G_0 \rtimes \mathrm{Alt}(3)$ acts transitively on $M = S^3 \times S^3$ with stabilizer $K = \Delta(\mathrm{SU}(2))/\Delta(\mathbb{Z}_2) \rtimes \mathrm{Alt}(3)$, where the action of $\mathrm{Alt}(3)$ is the natural one. Let us suppose there is a element $X = ([x_{123}], \sigma) \in \Gamma$ fixing a coset $([y_{123}], \gamma)K = ([y_{123}], (1))K \in G/K$ and observe that X^2 and X^3 will also fix the same coset. Rewriting these conditions, we see that

$$y_1^{-1} x_1 y_{\sigma(1)} = y_2^{-1} x_2 y_{\sigma(2)} = y_3^{-1} x_3 y_{\sigma(3)}, \tag{4.10}$$

$$y_1^{-1} x_1 x_{\sigma(1)} y_{\sigma^2(1)} = y_2^{-1} x_2 x_{\sigma(2)} y_{\sigma^2(2)} = y_3^{-1} x_3 x_{\sigma(3)} y_{\sigma^2(3)}, \tag{4.11}$$

$$y_1^{-1} x_1 x_{\sigma(1)} x_{\sigma^2(1)} y_1 = y_2^{-1} x_2 x_{\sigma(2)} x_{\sigma^2(2)} y_2 = y_3^{-1} x_3 x_{\sigma(3)} x_{\sigma^2(3)} y_3. \tag{4.12}$$

The group Γ is a disjoint union of the sets defined in (4.5) and (4.6) and the element $X \in \Gamma$ must be contained in either of these disjoint parts. For instance, if $X \in \Gamma_0$ or $X \in \Gamma_{\mathrm{Alt}}$, then either condition (a) or (d) hold. Let us now suppose that $X \in \Gamma_\sigma$ for some element $(1) \neq \sigma \in \mathrm{Alt}(3)$ and recall that $\mathrm{ord}(X) \neq 3$ by hypothesis. The need to have either of the conditions stated in (b) or (c) follows directly from equation (4.12). Conversely, if either of the statements in (a) or (d) holds, then clearly Γ acts non-freely on G/K. Lastly, if either of the conditions stated in (b) or (c) is met, then the group Γ contains an element X so that $\Delta(\mathbb{Z}_2) \neq X^3 \in \Gamma_0$ fixes a coset in G/K. □

In seems unreachable at the moment to obtain a full list of finite subgroups of G. The reason for this is the fact that Usenko's classification [U] of general subgroups of a semi-direct product gives rise to group cohomological issues whose solution is not yet available in the literature. On the other hand, the conditions in Proposition 4.0.2 seem to be quite restrictive, and so they motivate the following conjecture.

Conjecture I Every finite group $\Gamma \subset G$ that acts freely on $M = S^3 \times S^3$ by nearly Kähler transformations must be contained in the maximal connected subgroup G_0 of the full isometry group $G = \mathrm{I}(S^3 \times S^3)$.

If this conjecture holds, it is natural to expect this stronger conjecture:

Conjecture II Six dimensional locally homogeneous nearly Kähler manifolds M_Γ are *spectrally characterized* in the class of nearly Kähler manifolds.

Showing this statement is our ultimate aim in this project.

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Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

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List of Publications

1. *Isospectral nearly Kähler manifolds*, submitted to the Abhandlungen aus dem mathematischen seminar der Universität Hamburg, arXiv:1607.01897 (math.DG).

   Comments. This publication contains the main results of this thesis and its arxiv version is out of date.

2. *Locally homogeneous nearly Kähler manifolds* (with Vicente Cortés), Annals of Global Analysis and Geometry (2015), Volume 48, pp 269-294, arXiv:1410.6912 (math.DG).

   Comments. This publication summarizes and extends results I obtained in my masters thesis and serves here as a reference for background material.

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