On symmetric transformations in metric measured geometry

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Contents

Introduction

A mathematical object has symmetries if it can be moved around in such a way that, after all changes have been done, it looks about the same. Given a family of objects, our comprehension of it often improves drastically by studying those objects, inside the family, which have symmetries. Two interesting questions are that of how well- or badly-behaved is the structure representing all symmetries of a given object, and how can the knowledge on symmetries improve our understanding of the objects that we set up to study in the first place.

The central objects of study in this thesis are metric measure spaces. These are metric spaces which are endowed with a reference measure and enriched with basic topological, geometric and measure theoretical properties. The objective of the first part of the work is to study the existence of a differential structure on symmetry groups of metric measure spaces. The second part is concerned with the analysis of the induced geometry of spaces admitting non-trivial symmetries. We make this statements more precise in the remainder.

We analyze in §1 the group of isomorphisms (measure-preserving isometries) and the group of isometries; two noteworthy automorphism groups of a metric measure space. We consider a class of metric measure spaces in which tangent cones are well behaved. Within this class we provide in Theorem A a characterization of spaces whose automorphism groups are—possibly 0-dimensional—smooth manifolds, namely Lie groups.

In §1.1 we study those spaces, inside the class described earlier, for which automorphism groups contain small subgroups. Therein we reinterpret the "size" of a subgroup, which is by definition intrinsically connected to the distance, in a measure-theoretic fashion. More precisely we interpret this "size" in terms of a certain relative measure of fixed point sets. Since Gleason and Yamabe [Gle52, Yam53] showed that a locally compact topological group G is a Lie group if and only if G does not contain small subgroups, we deduce the identification of spaces explained in the previous paragraph.

The result is used in Theorem B in §1.2 to show that automorphism groups are smooth in spaces with good optimal transport properties. Examples of spaces that satisfy these transport properties are Riemannian manifolds, Alexandrov spaces of curvature bounded below, limits of weighted Riemannian manifolds with a uniform lower bound on the Ricci curvature, and all their Finsler counterparts. This compliments classical results of Myers-Steenrod [MS39], Fukaya-Yamaguchi [FY94], Cheeger-Colding-Naber [CC00, CN12], and Deng-Hou [DH02]. Most notably, in some situations spaces satisfying generalized notions of Ricci curvature lower bounds enjoy these properties as well.

Curvature-dimension conditions, which developed from work of Lott-Sturm-Villani [LV09, Stu06a, Stu06b], define notions of lower Ricci curvature bounds for metric measure spaces. Roughly stated, these conditions require the convexity of an entropy functional on the space of probability measures associated to an m.m. space; different choices of *entropy* and different types of *convexity* describe alternative versions of the conditions.

Theorem (1.2.9, 1.2.10, 1.2.11)**.** *Essentially non-branching spaces with well-behaved tangents that satisfy a finite dimensional curvature-dimension condition have a smooth isomorphism group and a smooth isometry group.*

In particular, metric measure spaces that fit into the assumptions of the theorem above are finite dimensional RCD* spaces and, granted they have well-behaved tangents, their Finsler counterparts: strong CD spaces; strong CD* spaces; and essentially non-branching MCP spaces. We end the first chapter by illustrating that the assumption of the theorem on geodesics essentially not branching is necessary; we construct a finite dimensional highly branching MCP space with well-behaved tangents but with a non-smooth group of symmetries.

Next we turn our focus towards submetries, a metric analogue of Riemannian submersions. In [O'N66, BGP92] O'Neill and Burago-Gromov-Perelman showed that lower sectional curvature bounds are preserved—possibly in a synthetic manner—under such maps. However, examples of Riemannian submersions that do not preserve lower bounds on the Ricci curvature tensor have been constructed by Pro and Wilhelm in [PW14]. On the other hand, by looking at a weighted, and eventually, a synthetic interpretation of Ricci curvature Lott and Lott-Villani [Lot03, LV09] achieved positive partial results in this direction. The purpose of §2 is to show the corresponding curvature stability results in full generality for synthetic lower Ricci curvature bounds.

Theorem E. *Curvature-dimension conditions are preserved by metric measure submetries with bounded leaves.*

Metric measure submetries are particular submetries that respect the structure of the spaces that we study; they are in correspondence with a special type of foliations called bounded metric measure foliations. In Theorem C we show that a quotient space, induced by such a foliation, inherits from the original space its synthetic Ricci curvature bounds. To prove this claim, in §2.1 we construct an isometry between the 2-Wasserstein space on the quotient space onto a subset of the 2-Wasserstein space in the original space. After showing a corresponding isometry of Sobolev spaces, the result follows.

We conclude in §2.2 by showing in Theorem D that quotient maps which are induced by isomorphic actions of compact groups are metric measure submetries; hence curvature-dimension conditions are stable under such quotient maps. As a consequence of Theorems C, D, E we obtain new constructions of examples of MCP, CD, CD*, and RCD* spaces.

We now take the opportunity to say that results drawn from the first chapter were presented in [Sos16]. The results of the second chapter come from collaborative work with Galaz-García, Kell, and Mondino, during which [GGKMS17] was assembled.

We will begin by explaining our terminology, and stating background results in §$\frac{1}{2}$.

Chapter $\frac{1}{2}$

(Some basics you might want to know) Before you go on

We explain our notation and develop basic concepts and results to make the text accessible. We intend to give a self-contained yet not superfluous presentation.

Metric measure spaces

The objects of our study are metric measure spaces which we write sometimes in short as m.m. space. A metric measure space, $(X, \mathsf{d}, \mathfrak{m})$, is a triple where

(X, d) is a *complete, separable metric space* and,

$\mathfrak{m} \neq 0$ is a *non-negative Borel measure finite on every bounded set*.

A pointed metric measure space, $(X, \mathsf{d}, \mathfrak{m}, x)$, is a m.m. space together with a base point $x \in X$. In the text a geodesics is a map, $\gamma\colon [0,1] \to X$, such that:

$$\mathsf{d}(\gamma_r, \gamma_s) = (r - s)\mathsf{d}(\gamma_0, \gamma_1) \qquad \text{for all } 0 \leq s \leq r \leq 1$$

where $\gamma_t := \gamma(t)$. We write $\mathrm{Geo}(X)$ for the space of all geodesics on X endowed with topology of uniform convergence. A metric space is called a *geodesic space* if for every given pair of points $x, y \in (X, \mathsf{d})$ there exists a geodesic that joins x and y. For $t \in [0,1]$ define the *evaluation map*, $\mathrm{e}_t : \mathrm{Geo}(X) \to X$, as $\mathrm{e}_t(\gamma) := \gamma_t$ for $\gamma \in \mathrm{Geo}(X)$. The *restriction map*, $\mathsf{rest}_s^t\colon \mathrm{Geo}(X) \to \mathrm{Geo}(X)$, is defined as $\mathsf{rest}_s^t(\gamma) := \gamma \circ f_s^t$ for $s, t \in [0,1], \gamma \in \mathrm{Geo}(X)$ and the real function $f_s^t(x) := (t - s)x + s$.

Two m.m. spaces $(X_1, \mathsf{d}_1, \mathfrak{m}_1)$, $(X_2, \mathsf{d}_2, \mathfrak{m}_2)$ are isomorphic if there exists an isometry

$$\begin{aligned} &f : \mathrm{supp}(\mathfrak{m}_1) \to X_2 \qquad \text{such that} \\ &(f)_{\#}\mathfrak{m}_1 = \mathfrak{m}_2. \end{aligned} \tag{$\frac{1}{2}$.1}$$

We use the word *isometry* to make reference to usual metric isometries. In contrast, we refer to maps satisfying ($\frac{1}{2}$.1) as *measure-preserving isometries* or *isomorphisms*. Particularly, we note that an isometry is defined on the whole space X_1 and does not necessarily satisfy ($\frac{1}{2}$.1). By definition $(X, \mathsf{d}, \mathfrak{m})$ is always isomorphic to $(\mathrm{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$. This induces a canonical equivalence class of isometric metric measure spaces where **only the Support of the measure is relevant**. In this work we assume that

$$\mathrm{supp}(\mathfrak{m}) = X,$$

which is a natural restriction in the class of isomorphisms of m.m. spaces. We write $\mathrm{ISO}_{\mathfrak{m}}(X)$ and $\mathrm{ISO}(X)$ to denote the group of isomorphisms of $(X, \mathsf{d}, \mathfrak{m})$, and the group of isometries of $(X, \mathsf{d}, \mathfrak{m})$ respectively. As usual the group operation is given by the composition of functions. We endow the groups $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ with the compact-open topology making them topological groups, see [KN63] pp.46. We write in the remainder G$\in \{\mathrm{ISO}(X), \mathrm{ISO}_{\mathfrak{m}}(X)\}$ to denote one of these two groups.

Remark $\frac{1}{2}$.2 (Topology on $\mathrm{ISO}_{\mathfrak{m}}(X)$). We explain and motivate our choice of topology on $\mathrm{ISO}_{\mathfrak{m}}(X)$. For locally compact metric spaces it's natural to endow $\mathrm{ISO}(X)$ with the compact-open topology since the structure under study is of pure metric nature. In addition, in this context, the rigidity of the isometries assures that pointwise convergence implies convergence w.r.t. the compact-open topology.[1] Alternatively, on m.m. spaces there is additional structure of interest, namely, the measure structure. However, as we explain below, the rigidity of the measure-preserving isometries guarantee that a reasonable choice of topology on $\mathrm{ISO}_{\mathfrak{m}}(X)$ coincides with the compact-open topology.

We first observe that a topology that only considers the measure structure is too coarse for our purposes because it doesn't see metric properties. A logical way to proceed would be to couple a measure-wise and a metric-wise topology. However, the weakest metric convergence, the pointwise convergence, coincides with the compact-open convergence. On the other hand, in Lemma 1.1.1 we show that the compact-open convergence of a sequence of measure-preserving isometries, (f_n), implies the weak convergence of the pushforward measures $(f_n)_\#(\mathfrak{m})$ in a locally compact m.m. space.

We will study group actions on sequences of pointed metric spaces. In this framework the *pointed Gromov-Hausdorff* (pGH) and *pointed equivariant Gromov-Hausdorff convergence* (peGH) provide canonical types of convergence. We present a quick remainder of these concepts and refer to [DBI01, Fuk86, FY92, Har16] for more details.

Let us first denote the set of isometry classes of compact metric measure spaces by $\mathcal{M}^{\mathrm{c}}$. We also consider triples $(X, \mathsf{d}, \mathsf{H})$, where (X, d) is a compact metric space and $\mathsf{H} \leq \mathrm{ISO}(X)$ is a closed subgroup and say that two triples are equivalent if they are equivariantly isomorphic up to automorphisms of the groups. We denote the equivalence classes of these triples by $\mathcal{M}^{\mathrm{c}}_{\mathrm{eq}}$.

Next we define ϵ-approximations.

Definition $\frac{1}{2}$.3. *Let $(X, \mathsf{d}_X), (Y, \mathsf{d}_Y) \in \mathcal{M}^{\mathrm{c}}$ be metric spaces. A* Gromov-Hausdorff

[1]Rigorously we would have to justify the use of sequences to compare topologies. This can be done because $\mathrm{ISO}(X)$ is second-countable which can be concluded from the fact that X is a locally compact metric space. Consult for instance [KN63] pp.46.

ϵ-approximation *is a function* $f : X \to Y$ *such that, for all* $p, q \in X$ *it holds that* $|\mathsf{d}_X(p,q) - \mathsf{d}_Y(f(p), f(q))| \le \epsilon$ *and an* ϵ*-neighborhood of* $f(X)$ *covers all of* Y*.*

Let $(X, \mathsf{d}_X, \mathsf{H}_X), (Y, \mathsf{d}_Y, \mathsf{H}_G) \in \mathcal{M}^{\mathrm{c}}_{\mathrm{eq}}$*. An* equivariant Gromov-Hausdorff ϵ-approximation *is a triple of functions* (f, φ, ψ) *where* $f : X \to Y$*,* $\varphi : \mathsf{H}_X \to \mathsf{H}_Y$ *and* $\psi : \mathsf{H}_Y \to \mathsf{H}_X$ *such that*

- f *is a Gromov-Hausdorff* ϵ*-approximation;*
- *if* $h_X \in \mathsf{H}_X$*, and* $x \in X$*, then* $\mathsf{d}(f(h_X x), \varphi(h_X) f(x)) < \epsilon$*; and*
- *if* $h_Y \in H_Y$ *and* $x \in X$*, then* $\mathsf{d}(f(\psi(h_Y)x), h_Y f(x)) < \epsilon$*.*

The distances are defined as follow.

Definition $\frac{1}{2}$.4 ((equivariant) Gromov-Hausdorff distance)**.** *The* Gromov-Hausdorff distance d_{GH} *between two compact metric spaces* (X, d_x) *and* (Y, d_Y) *is defined as the infimum of all* ϵ*'s such that there are Gromov-Hausdorff* ϵ*-approximations from* $X \to Y$ *and from* $Y \to X$*.*

The equivariant Gromov-Hausdorff distance *between two* $(X, \mathsf{d}_x, \mathsf{H}_X)$ *and* $(Y, \mathsf{d}_Y, \mathsf{H}_Y)$ *is defined as the infimum of all* ϵ*'s such that there exist equivariant Gromov-Hausdorff* ϵ*-approximations from* $X \to Y$ *and from* $Y \to X$*.*

GH-convergence for non-compact proper pointed spaces can also be defined requiring the approximations to preserve the base point. In this case we say that (X, d_X, x) converges in the pointed Gromov-Hausdorff topology to (Y, d_X, y) if $(\bar{B}_r(x), \mathsf{d}_X) \xrightarrow{\mathsf{GH}} (\bar{B}_r(y), \mathsf{d}_Y)$ for all radius r, under the extra requirement that $f(x) = y$ for all Gromov-Hausdorff ϵ-approximations. We will not need to define a GH-convergence for non-compact spaces, however we will need to keep track of base points. For this purpose we call $\mathcal{M}^{\mathrm{c}}_{\mathrm{eq,p}}$ the set of equivalence classes of quadruples $(M, \mathsf{d}, \mathsf{H}, x)$, for $(M, \mathsf{d}, \mathsf{H}) \in \mathcal{M}^{\mathrm{c}}_{\mathrm{eq}}$ and $x \in M$, under the equivalence given by equivariant isomorphisms (up to group automorphisms) that fix the base point. The pointed eGH-convergence is then defined in terms of eGH ϵ-approximations that fix base points.

In [FY92, Proposition 3.6] the following useful property is proved. Let $\mathcal{X} = \{(X_n, \mathsf{d}_n, \mathsf{H}_n, x_n)\}_{n\in\mathbb{N}} \subset \mathcal{M}^{\mathrm{c}}_{\mathrm{eq,p}}$ be a sequence for which the corresponding sequence of underlying metric spaces GH-converge to (Y, d_Y, y), then there exist a closed subgroup $\mathsf{H}_Y \le \mathrm{ISO}(Y)$ and a subsequence of $\mathcal{X}$ which converges in the pointed eGH-topology to $(Y, \mathsf{d}_Y, y, \mathsf{H}_Y)$.

Lastly let us remark that in the framework of m.m. spaces the ad hoc convergence is given by the *pointed measured Gromov-Hausdorff* convergence. This convergence couples, in an appropriate fashion, the pGH-convergence with the weak convergence of pushforward measures. However, we do not go in depth since we do not directly deal with these types of sequences nor topology. A good reference for this topic is [GMS13].

Any pGH-limit of a sequence of scaled spaces, $(X, \frac{1}{r_i}\mathsf{d}, x) \overset{\mathsf{pGH}}{\to} (Y^\infty, \mathsf{d}_\infty, x_\infty)$ for $r_i \to 0$, is called a (metric) GH-tangent cone of X at x.[2] We denote the set of all tangent cones of

[2] The canonical concept of a tangent spaces, in the class of m.m. spaces, are m.m. spaces that appear as measured pGH-limits of rescalings of the metric and measure. Nonetheless we consider simply metric tangent cones since this suffices for our purposes.

X at x by $\mathsf{Tan}(X,x)$:= $\{(X_\infty, \mathsf{d}_\infty, x_\infty)$ is a pGH-limit as above$\}$. In general, tangent cones need not exist nor be unique.[3] We call the set of points of X with unique tangent cones the *regular set* $\mathcal{R}$ of X. Specifically,

$$\mathcal{R} := \left\{x \in X \,\|\, \text{there exist a unique } (Y_x^\infty, \mathsf{d}^{Y_x^\infty}, y^{Y_x^\infty}) \in \mathsf{Tan}(X,x)\right\}. \qquad (\tfrac{1}{2}.5)$$

Note that the limit space Y_x^∞ might depend on the point $x \in X$. The collection of spaces, up to isometry, that appear as tangents of the regular set $\mathcal{R}$ is written as $\mathsf{Tan}(\mathcal{R})$:= $\cup_{x\in\mathcal{R}}\mathsf{Tan}(X,x)$. For fixed ϵ, $\delta > 0$ the set $(\mathcal{R})_{\epsilon,\delta}$ is defined as the set of points $x \in X$ such that there exists $(Y^\infty, \mathsf{d}^\infty, y^\infty) \in \mathsf{Tan}(\mathcal{R})$ for which

$$\mathsf{d}_{\mathsf{GH}}(B_s(x), B_s^{Y^\infty}(y^\infty)) \leq s\,\epsilon \qquad \forall\, 0 < s < \delta, \qquad (\tfrac{1}{2}.6)$$

where $B_s^{Y^\infty}(y^\infty) \subset (Y^\infty, \mathsf{d}^\infty, y^\infty)$ is the metric ball of radius s around y^∞. The *ϵ-regular set* $\mathcal{R}_\epsilon$ is the set $\mathcal{R}_\epsilon := \cup_\delta(\mathcal{R})_{\epsilon,\delta}$. We make the next observations to understand the relations between these sets.

- In general $\mathcal{R}_{\epsilon,\delta} \not\subset \mathcal{R}$ for some ϵ, δ; and then
- In general $\mathcal{R}_\epsilon \not\subset \mathcal{R}$ for some ϵ;
- In general $\mathcal{R} \not\subset \mathcal{R}_{\epsilon,\delta}$ for some ϵ, δ; although
- for every $x \in \mathcal{R}$ and $\epsilon > 0$ there exist δ_ϵ such that $x \in \mathcal{R}_{\epsilon,\delta_\epsilon}$;
- $\mathcal{R} = \cap_{\epsilon>0}\mathcal{R}_\epsilon$;

We say that $(X, \mathsf{d}, \mathfrak{m})$ has $\mathfrak{m}$-*almost everywhere unique tangents* if

$$\mathfrak{m}(X \setminus \mathcal{R}) = 0.$$

It follows from the observations made above that for every $\epsilon > 0$ the measure $\mathfrak{m}(X \setminus \mathcal{R}_\epsilon) = 0$ if X has $\mathfrak{m}$-almost everywhere unique tangents.

Sobolev spaces in m.m. spaces

Sobolev spaces in m.m. spaces can be defined in several ways, see for instance the review [Hei07] and references therein. However, under mild regularity assumptions on the m.m. space, which we later specify, they turn out to be equivalent (for results in this direction see [Che99, AGS14a, AGS13, Kel15, Kel14]). We present a simple approach which suffices for our purposes, always having in mind that the spaces that we will study are sufficiently regular for all approaches to be equivalent.

[3] A rather wild example of the non-uniqueness of tangent cones was shown by Chen and Rossi in [CR14]. Therein they constructed a metric space that has any other compact metric space as a tangent cone at all points!

Given a metric space (X, d), we denote with $\mathsf{LIP}(X, \mathsf{d})$ the space of Lipschitz functions, i.e. the space of those functions $f : X \to \mathcal{R}$ such that

$$\sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} < \infty.$$

Recall that the left hand side is called the *Lipschitz constant* of f. The *upper asymptotic Lipschitz constant* $\mathsf{Lip} f \colon X \to \mathbb{R} \cup \{+\infty\}$ of a function $f \colon X \to \mathbb{R}$ is defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{r\to 0} \sup_{y\in B_r(x)} \frac{|f(y) - f(x)|}{r}. \tag{$\frac{1}{2}$.7}$$

An easy observation is that $\operatorname{Lip} f(x) \leq L$ if f is Lipschitz continuous with Lipschitz constant at most L. If (X, d) is a geodesic space then the converse statement is also valid. We will write Lip^Y when we want to stress the dependence on the domain Y of the functions.

We use the upper asymptotic Lipschitz constant to define the *Cheeger energy* of a function $f : X \to \mathbb{R}$ by

$$\operatorname{Ch}_2(f) := \frac{1}{2} \inf \left\{ \liminf_{n\to\infty} \int_X \operatorname{Lip} f_n(x)^2 \mathrm{d}\mathfrak{m} \;\|\; \{f_n\}_{n\in\mathbb{N}} \subset \operatorname{Lip}(X),\ f_n \stackrel{L^2}{\to} f \right\}. \tag{$\frac{1}{2}$.8}$$

The quantity $\operatorname{Ch}_2(f)$ should be tought of as an "L^2-norm of the gradient of f". The domain of the Cheeger energy is $D(\operatorname{Ch}_2) := \{f \,\|\, \operatorname{Ch}_2(f) < \infty\}$. One can show that for any $f \in D(\operatorname{Ch})$ there exists a unique function $|\nabla f|_2 \in L^2(X, \mathsf{d}, \mathfrak{m})$ that minimizes equation ($\frac{1}{2}$.8), i.e. $2\operatorname{Ch}_2(f) := \int_X |\nabla f|_2^2 \mathrm{d}\mathfrak{m}$. We call $|\nabla f|_2$ the *minimal weak upper gradient* or *minimal relaxed slope* of f; this object acts as an $\mathfrak{m}$-almost everywhere "norm of the gradient".[4] It is the best notion that one expects for first order derivatives in a general setting.[5]

It follows from the definition that Ch_2 is convex and lower semicontinuous with respect to convergence in $L^2(\mathfrak{m})$. In particular, it induces a complete norm defined by

$$\|f\|_{W^{1,2}} = \left(\|f\|_{L^2}^2 + 2\operatorname{Ch}_2(f)\right)^{\frac{1}{2}}$$

on the space

$$W^{1,2}(X, \mathsf{d}, \mathfrak{m}) = W^{1,2}(\mathfrak{m}) = L^2(\mathfrak{m}) \cap D(\operatorname{Ch}_2).$$

We call the Banach space $(W^{1,2}(\mathfrak{m}), \|\cdot\|_{W^{1,2}})$ the *Sobolev space* of $(X, \mathsf{d}, \mathfrak{m})$. In the remainder, and only when working with more than one m.m. space, to stress the dependence on the space under consideration we write Ch_2^Y for the Cheeger energy of functions on $(Y, \mathsf{d}, \mathfrak{n})$.

By definition we have that $|\nabla f|_2 \leq \operatorname{Lip} f$ for $f \in \operatorname{LIP}(X, \mathsf{d})$. We emphasize that there exist spaces where the equality is not achieved for some Lipschitz function. However, the following was shown by Cheeger.

[4]As a word of warning, keep in mind that the different names for $|\nabla f|_2$ come from different approaches to its definition, so in general the concepts with these names do not coincide. However, as already stated, these approaches are equivalent under weak assumptions which are fulfilled in our setting.

[5]As a matter of fact, the *norm of the differential* is a better analogy. For instance check the last comment of pp. 5 in [Gig15]. Compare also with [Che99] where, among other things, a notion of an actual differential is constructed.

Theorem $\frac{1}{2}$.9 ([Che99]). *Let $(M, \mathsf{d}, \mathfrak{m})$ be a m.m. space satisfy a doubling condition and admitting a weak local 1-1 Poincaré inequality.*[6] *Then:*

- *The upper asymptotic Lipschitz constant and the minimal relaxed slope agree $\mathfrak{m}$-a.e. for every locally Lipschitz function in $W^{1,2}(\mathfrak{m})$; and*
- *Lipschitz functions are dense in $W^{1,2}(\mathfrak{m})$.*

Let us also recall that $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is not a Hilbert space in general, example in the case of Finsler manifolds which are not Riemannian manifolds. An m.m. space is said to be *infinitesimally Hilbertian* if $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Hilbert space, see [AGS14a, AGS14b, Gig15]. The infinitesimally Hilbertian condition is equivalent to the validity of the parallelogram rule for the Cheeger energy,

$$\mathrm{Ch}_2(f+g) + \mathrm{Ch}_2(f-g) = 2\Big(\mathrm{Ch}_2(f) + \mathrm{Ch}_2(g)\Big), \quad \forall f, g \in W^{1,2}(\mathfrak{m}). \qquad (\tfrac{1}{2}.10)$$

Optimal transport theory

In order to state curvature conditions for m.m. spaces let us begin by giving a rapid review of optimal transport theory. We present concepts and results without giving proofs. Two references to comprehensive works in which all material (up to the last definition) is contained, and to which we refer to for proofs, are [Vil09, AG13]. Definition $\frac{1}{2}$.15 comes originally from [RS14].

Let $\mathcal{P}(X)$ be the space of probability measures on (X, d) and write $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ for the subspace of measures with finite second moments. We also write $\mathcal{P}^{ac}(M) := \{\mu \in \mathcal{P}(M) : \mu \ll \mathfrak{m}\}$ for the subset of absolutely continuous measures with respect to $\mathfrak{m}$ and define $\mathcal{P}_2^{ac}(M) := \mathcal{P}_2(M) \cap \mathcal{P}^{ac}(M)$. For $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ the 2-Wasserstein distance is defined as

$$W_2(\mu_0, \mu_1)^2 := \min_{\sigma \in \mathsf{Adm}(\mu_0, \mu_1)} \int_{X \times X} \mathsf{d}(x, y)^2 \mathrm{d}\sigma(x, y). \qquad (\tfrac{1}{2}.11)$$

The minimum taken over the set of *admissible couplings* between μ_0 and μ_1, $\mathsf{Adm}(\mu_0, \mu_1) \subset \mathcal{P}(X \times X)$, defined as all measures $\sigma \in \mathcal{P}(X \times X)$ with first and second marginals equal to μ_0 and μ_1 respectively. $(\mathcal{P}_2(X), W_2)$ is a complete separable metric space which is geodesic if (X, d) is geodesic.

Measures in $\mathsf{Adm}(\circ, \circ)$ are called *couplings* or *plans*. Minimizers in ($\frac{1}{2}$.11) are referred to as *optimal couplings/plans* and we denote the space of all optimal couplings between the measures $\mu_0, \mu_1 \in \mathcal{P}(X)$ as $\mathsf{OptAdm}(\mu_0, \mu_1)$. If there exists a measurable function $T : X \to X$ such that the measure $\sigma = (\mathbb{I}, T)_{\#}\mu_0$ is an optimal plan we call σ an *optimal map*. It is a fundamental result in optimal transport that under weak assumptions on the cost function, which are fulfilled in our setting, the minimum in ($\frac{1}{2}$.11) is achieved. However, optimal maps rarely exist.

It is useful to give a geodesic interpretation of these concepts which we now discuss. Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ the set $\mathrm{OptGeo}(\mu_0, \mu_1) \subset \mathcal{P}(\mathrm{Geo}(X))$ is defined as the set of all measures π

[6] This is valid even under weaker assumptions, however this will be enough for us.

such that the pushforward $(\mathrm{e}_0, \mathrm{e}_1)_{\#}\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$. A measure $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is called an *optimal geodesic plan* and if there exists a measurable function $T^G : X \to \mathrm{Geo}(X)$ such that $\pi = (T^G)_\sharp \mathfrak{m}$ we call the measure π an *optimal geodesic map*. Any geodesic $\{\mu_t\}_{t\in[0,1]} \subset (\mathcal{P}_2(M), W_2)$ can be lifted to a measure $\pi \in \mathcal{P}(\mathrm{Geo}(M))$ in the sense that $(\mathrm{e}_t)_\sharp \pi = \mu_t$ for all $t \in [0, 1]$. Thus, the set $\mathsf{OptAdm}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathcal{P}_2(M)$ if (M, d) is a geodesic space.

It is convenient to use as well the dual formulation of the optimal transport problem so let us explain it. First we give the next

Definition $\frac{1}{2}$.12 (c_2-transform, c_2-concave function, c_2-superdifferential)**.** *Let $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$ be any function. Its* c_2-transform *ψ^{c_2} is defined as*

$$\begin{aligned} \psi^{c_2} : X &\to \mathbb{R} \cup \{\pm\infty\} \\ x &\mapsto \psi^{c_2}(x) := \inf_{y\in X} \left(\mathsf{d}^2(x, y) - \psi(y)\right). \end{aligned} \tag{$\frac{1}{2}$.13}$$

We say that ψ is c_2-concave *if there exists $\phi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\psi = \phi^{c_2}$. The c_2-superdifferential of a c_2-concave function φ is defined as the set*

$$\partial^{c_2}\varphi := \{(x, y) \in X \times X \,|\, \varphi(x) + \varphi^{c_p}(y) = \mathsf{d}^2(x, y)\}.$$

The dual formulation of the optimal transport problem ($\frac{1}{2}$.11), rather than seeking for minimizers in $\mathsf{Adm}(\circ, \circ)$, proposes to maximize *pairs of functions*.

$$W_2(\mu_0, \mu_1)^2 = \sup_{(\phi,\psi)} \int_X \phi \, d\mu_0 + \int_X \psi \, d\mu_1, \tag{$\frac{1}{2}$.14}$$

the supremum taken over all the pairs (ϕ, ψ) with $\phi \in L^1(\mu_0), \psi \in L^1(\mu_1)$ such that

$$\phi(x) + \psi(y) \le \mathsf{d}^2(x, y), \quad \forall x, y \in X.$$

It is known that the supremum in the dual problem ($\frac{1}{2}$.14) is always attained by a maximizing couples of the form (ϕ, ϕ^{c_2}) for some c_2-concave function ϕ, called *c_2-Kantorovich potential*.

The optimal solutions to the original and dual statements are related in the following manner. Let $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$ be an optimal coupling, then there exists a c_2-Kantorovich potential ϕ such that

$$\phi(x) + \phi^{c_2}(y) = \mathsf{d}(x, y)^2, \quad \text{for } \pi\text{-a.e. } (x, y).$$

In this case we say that (ϕ, ϕ^{c_2}) is a dual solution corresponding to π.

Lastly, we need to define one more concept of relevance in what follows. A set $\Gamma \subset \mathrm{Geo}(X)$ is called *non-branching* if for any $\gamma^1, \gamma^2 \in \Gamma$ the existence of a $t \in (0, 1)$ such that $\gamma^1_t = \gamma^2_t$ implies that $\gamma^1 = \gamma^2$. Moreover, we say that a measure $\Pi \in \mathcal{P}(\mathrm{Geo}(X))$ is *concentrated on a set of non-branching geodesics* if there exists a non-branching Borel set $\Gamma \subset \mathrm{Geo}(X)$ such that $\Pi(\Gamma) = 1$.

Definition $\frac{1}{2}$.15. *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is* essentially non-branching *if for every $\mu_0, \mu_1 \in \mathcal{P}^{ac}(X)$ any $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.*

Curvature-dimension conditions

Curvature-dimension type conditions use optimal mass transport theory to define a notion of lower Ricci curvature bounds for metric measure spaces. Roughly stated, these conditions require *the convexity* of an *entropy functional* on the space of probability measures associated to an m.m. space; different choices of *entropy* and different types of *convexity* describe alternative non-equivalent versions of the condition. These notions are variations of an original condition introduced independently by Lott and Villani and by Sturm in [LV09, Stu06a, Stu06b]. Important contributors to these developments were made by L. Ambrosio, K. Bacher, M. Erbar, K. Kuwada, N. Gigli, A. Mondino, T. Rajala, G. Savaré, and K.T. Sturm. For a historical recount one can consult for example the introductions to [MN14,EKS15] and references therein.

To formulate these curvature properties we start by introducing the following distortion coefficients. Let $(K,N) \in \mathbb{R} \times [0,\infty)$, we set for $(t,\theta) \in [0,1] \times (0,\infty)$

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leq 0 \text{ and } N > 0, \end{cases} \qquad (\tfrac{1}{2}.16)$$

and for $(K,N) \in \mathbb{R} \times [1,\infty)$ and $(t,\theta) \in [0,1] \times (0,\infty)$ we write

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}. \qquad (\tfrac{1}{2}.17)$$

We also require the definition of the Shannon relative entropy functional $\mathrm{Ent}_{\mathfrak{m}} : \mathcal{P}(M) \to [-\infty,+\infty]$.

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int_M \rho \log \rho \, d\mathfrak{m}, & \text{if } \mu = \rho\mathfrak{m} \text{ and } (\rho \log \rho)^+ \in L^1(\mathfrak{m}) \\ +\infty & \text{otherwise,} \end{cases} \qquad (\tfrac{1}{2}.18)$$

where $(f)^+$ denotes the positive part of a real valued function, and $0 \log(0) := 0$.

Definition $\frac{1}{2}$.19 (Curvature-dimension conditions)**.** *Let $(M,\mathsf{d},\mathfrak{m})$ be a metric measure space and fix $(K,N) \in \mathbb{R} \times [1,\infty)$. We say that $(M,\mathsf{d},\mathfrak{m})$ satisfies the*

- $\mathsf{CD}(K,\infty)$-condition *if for each pair of measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(M)$ there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}$ along which $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex, i.e.*

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leq (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0,\mu_1)^2, \qquad (\tfrac{1}{2}.20)$$

holds for all $t \in [0,1]$

- CD(K,N)-condition *if for each pair of measures* $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathcal{P}_2^{ac}(M)$ *there exists a* W_2*-geodesic* $\{\mu_t = \rho_t\mathfrak{m}\}_{t\in[0,1]} \subset \mathcal{P}_2^{ac}(M)$ *such that*

$$\int_M \rho_t^{1-\frac{1}{N'}} d\mathfrak{m} \geq \int_{M\times M} \left[\tau_{K,N'}^{(1-t)}\big(\mathsf{d}(x,y)\big)\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}\big(\mathsf{d}(x,y)\big)\rho_1(y)^{-\frac{1}{N'}}\right] d\pi(x,y) \tag{$\frac{1}{2}$.21}$$

holds for all $t \in [0,1]$ *and* $N' \in [N,\infty)$. *Where* π *is the optimal coupling induced by* $\{\mu_t\}_{t\in[0,1]}$.

- CD$^*(K,N)$-condition *if for each pair of measures* $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathcal{P}_2^{ac}(M)$ *there exists a* W_2*-geodesic* $\{\mu_t = \rho_t\mathfrak{m}\}_{t\in[0,1]} \subset \mathcal{P}_2^{ac}(M)$ *such that*

$$\int_M \rho_t^{1-\frac{1}{N'}} d\mathfrak{m} \geq \int_{M\times M} \left[\sigma_{K,N'}^{(1-t)}\big(\mathsf{d}(x,y)\big)\rho_0(x)^{-\frac{1}{N'}} + \sigma_{K,N'}^{(t)}\big(\mathsf{d}(x,y)\big)\rho_1(y)^{-\frac{1}{N'}}\right] d\pi(x,y) \tag{$\frac{1}{2}$.22}$$

holds for all $t \in [0,1]$ *and* $N' \in [N,\infty)$. *Where* π *is the optimal coupling induced by* $\{\mu_t\}_{t\in[0,1]}$.

- MCP(K,N)-condition *if for each* $x \in M$ *and* $\mu_0 = \rho_0\mathfrak{m} \in \mathcal{P}_2^{ac}(M)$, *writing* $\mu_1 = \delta_x$, *there exists a* W_2*-geodesic* $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}_2(M)$ *such that*

$$\int_M \rho_t^{1-\frac{1}{N'}} d\mathfrak{m} \geq \int_M \tau_{K,N'}^{(1-t)}\big(\mathsf{d}(x,y)\big)\rho_0(x)^{1-\frac{1}{N'}}\, d\mathfrak{m}(y), \tag{$\frac{1}{2}$.23}$$

holds for all $t \in [0,1]$ *and* $N' \in [N,\infty)$. *Where we write* $\mu_t = \rho_t\mathfrak{m} + \mu_t^s$ *with* $\mu_t^s \perp \mathfrak{m}$.

We also say that $(M,\mathsf{d},\mathfrak{m})$ *satisfies the* strong CD(K,∞)-condition *if* ($\frac{1}{2}$.20) *holds for every* W_2*-geodesic between any given pair* $\mu_0, \mu_1 \in P_2^{ac}(M)$. *The* strong CD$^*(K,N)$-condition, *and the* strong CD(K,N)-condition *are defined in an analogous manner.*

One more curvature condition we consider is the *Riemannian curvature-dimension condition*. Recall that the concept of infinitesimally Hilbertianity was defined in ($\frac{1}{2}$.10).

Definition $\frac{1}{2}$.24 (RCD(K,∞) and RCD$^*(K,N)$ conditions)**.** *Let* $(M,\mathsf{d},\mathfrak{m})$ *be a m.m. space and fix* $(K,N) \in \mathbb{R}\times[1,\infty)$. *We say that* $(M,\mathsf{d},\mathfrak{m})$ *satisfies the:*

- RCD(K,∞)-condition *if it is infinitesimally Hilbertian and it satisfies the* CD(K,∞)-*condition.*

- RCD$^*(K,N)$-condition *if it is infinitesimally Hilbertian and satisfies the* CD$^*(K,N)$-*condition.*

We now enunciate some properties that spaces satisfying curvature-dimension conditions enjoy. Let $K \in \mathbb{R}$, and $N \in [1, +\infty)$.

The curvature conditions just defined are closed under pointed measured Gromov-Hausdorff convergence, limit spaces satisfying the same condition with the same parameters that the members of the sequences. Furthermore, the CD, CD*, and RCD* conditions are compatible with the smooth counterpart: an N-dimensional smooth Riemannian manifold (M, g) has a lower Ricci curvature bound K if and only if the associated metric measure space $(M, \mathsf{d}_g, d\,\mathrm{vol}_g)$ satisfies any of those curvature-dimension conditions with the same parameters. On the other hand, the curvature-dimension conditions are not always equivalent. However, there is an inclusion relation between them:

$$\text{RCD}^*\text{-spaces} \subsetneq \text{CD}^*\text{-spaces} \subsetneq \text{CD-spaces} \subsetneq \text{MCP-spaces},$$

where all the inclusions are proper and the parameters K, N might vary. In [CM16] it was recently shown that the conditions CD(K, N) and CD$^*(K, N)$ are equivalent in essentially non-branching spaces with finite total measure. On the other hand, this statement is not necessarily true if the essentially non-branching condition is dropped [Raj13]. Additionally, the next inclusions are valid as well: CD$(K, N < \infty) \subset$ CD(K, ∞) and (writing $\mathfrak{CD}$ to denote any of the curvature conditions defined for finite N) $\mathfrak{CD}(K, N) \subset \mathfrak{CD}(K', N')$ where $K \geq K'$ and $N \leq N'$.

Closed and bounded sets in CD(K, N)-spaces are compact [Stu06b, Corollary 2.4] (a metric space with this property is called proper). In particular, the completeness and separability of m.m. spaces imply that CD(K, N)-spaces are locally compact, and geodesic metric spaces.

The next theorem guarantees the existence of optimal maps in m.m. spaces that satisfy a curvature-dimension type condition where not too many geodesics branch, recall Definition $\frac{1}{2}$.15 of the essentially non-branching condition.

Theorem $\frac{1}{2}$.25 (Existence of optimal maps. Cavalleti-Gigli-Kell-Mondino-Rajala-Sturm [GRS15, CM17, Kel17])**.** *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching* MCP(K, N) *space. Then, for every $\mu_0 (\ll \mathfrak{m}), \mu_1 \in \mathcal{P}_2(X)$, there exist a unique optimal geodesic plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$. Furthermore, such π is given by a map. In particular, there exists $\Gamma \subset \mathrm{Geo}(X)$ with $\pi(\Gamma) = 1$ such that the map $e_t : \Gamma \to X$ is injective for all $t \in [0, 1)$.*
More generally the same property holds for essentially non-branching geodesic spaces with qualitatively non-degenerate measure $\mathfrak{m}$.

Additionally in [RS14, Corollary 1.2] it is shown that RCD(K, ∞)-spaces are essentially non-branching. Hence RCD(K, ∞)-spaces are strong CD(K, ∞)-spaces and RCD$^*(K, N)$-spaces are strong CD$^*(K, N)$-spaces [GRS15, Theorem 1.2].

As a matter of fact an m.m. space satisfies *strong* CD(K, N) if and only if it is essentially non-branching and satisfies CD(K, N), [RS14, Theorem 1.1] and [CM17, Corollary 5.3]. The analogous statement is valid for, *strong* CD$^*(K, N)$-spaces.

Infinitesimal behavior of RCD*-spaces is rather well understood.

Theorem $\frac{1}{2}$.26 ($\mathfrak{m}$-*a.e.* Euclidean tangents in RCD$^*(K, N)$-spaces. Mondino-Naber (2014) [MN14], Gigli-Mondino-Rajala (2013) [GMR15])**.** *Let $K, N \in \mathbb{R}$, $N \geq 1$ and $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$^*(K, N)$ *space. Then X has $\mathfrak{m}$-a.e. Euclidean (metric) tangents.*

Remark $\frac{1}{2}$.27. In section §1.2 we will work with Ohta's definition of the $\mathsf{MCP}(K,N)$-condition. This definition of the MCP-condition is alternative to the one previously given in Definition $\frac{1}{2}$.23. Both conditions impose restrictions on the contraction properties of arbitrary measures to final δ-measures, but implemented in a different way. Ohta's definition is implied by $\frac{1}{2}$.23, and in essentially non-branching spaces they are equivalent [CM16, Lemma 6.11., Proposition 9.1.]. Accordingly, we will always refer to Definition $\frac{1}{2}$.23 when considering essentially non-branching spaces, and to this definition otherwise (that is, in §1.2).

We simply enunciate the condition for the specific parameters that we will consider.

Definition $\frac{1}{2}$.28 ($\mathsf{MCP}(2,3)$-condition). *An m.m. space, $(X,\mathsf{d},\mathfrak{m})$, has the $(2,3)$-measure contraction property, $\mathsf{MCP}(2,3)$, if for every point $x\in X$ and a measurable set $A\subset X$ with $0<\mathfrak{m}(A)<\infty$ and $A\subset B_\pi(x)$ there exists a probability measure $\pi\in\mathcal{P}(\mathrm{Geo}(X))$ such that $(e_0)_\#\pi=\delta_x$, $(e_1)_\#\pi=\mathfrak{m}(A)^{-1}\mathfrak{m}|_A$, and*

$$(e_t)_\#\left(t\frac{\sin^2(t\,l(\gamma))}{\sin^2(l(\gamma))}\mathfrak{m}(A)\mathrm{d}\pi(\gamma)\right)\leq \mathrm{d}\mathfrak{m}\qquad t\in[0,1]. \tag{$\frac{1}{2}$.29}$$

Ohta's condition is as well stable with respect to pointed measured GH-convergence [Oht14].

Lie Groups

We recall the following useful classical result.

Theorem $\frac{1}{2}$.30 (van Dantzig and van der Waerden (1928) [vDvdW28])**.** *Let (X,d) be a connected, locally compact metric space. Then $\mathrm{ISO}(X)$ is locally compact with respect to the compact-open topology. Furthermore if X is compact, then $\mathrm{ISO}(X)$ is compact.*

Let H be a topological group, and denote by H_0 the identity component of H, that is, the largest connected set containing the identity element $\mathbb{I}$. The following is the definition we adopt for Lie groups.

Definition $\frac{1}{2}$.31. *We say that H is a Lie group if and only if H/H_0 is discrete and the identity component H_0 is a Lie group in the usual smooth sense. In particular, we also consider discrete groups as 0-dimensional Lie groups.*

In Remark $\frac{1}{2}$.35 we discuss the cardinality of H/H_0. In short we conclude that in the worst cases H is a disjoint union of countably many copies of a smooth Lie group which do not accumulate. Next we define a property of topological groups that proved to be rather characteristic, as can be seen in the remarkable theorem that follows.

Definition $\frac{1}{2}$.32 (No small subgroups)**.** *A topological group has H has the* no small subgroup property *if there exists a neighborhood of the identity with no non-trivial subgroup.*

Theorem $\frac{1}{2}$.33 (Gleason (1952) [Gle52], Yamabe (1953) [Yam53])**.** *Let* H *be a locally compact, topological group. Then* H *is a Lie group if and only if it has the no small subgroups property.*

Remark $\frac{1}{2}$.34. In [Yam53] Yamabe generalizes Gleason's theorem to the infinite dimensional case, however, H is assumed to be connected. We present an argument due to a very friendly yet shy Russian mathematician, which shows that non-connected groups can be considered as well.

An equivalent way of stating Theorem $\frac{1}{2}$.33 is: Assuming the same hypothesis, then there exists an open subgroup $\mathsf{H}' < \mathsf{H}$ such that for every neighborhood of the identity $U \subset \mathsf{H}$ there exists a normal subgroup $(U \supset) K \trianglelefteq \mathsf{H}$ that makes H'/K a Lie Group [Tao14]. If H has the no small subgroup property the only small normal subgroup is $\mathbb{I}$ itself, thus making H' a Lie group which by definition means that H'/H_0 is discrete. However, this implies that H/H_0 is discrete since H/H' is also discrete and $\mathsf{H}/\mathsf{H}' = (\mathsf{H}/\mathsf{H}_0)\big/(\mathsf{H}'/\mathsf{H}_0)$.

Remark $\frac{1}{2}$.35. We make another observation regarding the cardinality of H/H_0. In principle the given definition of a Lie group does not exclude the possibility of the group of components being uncountable, fortunately, we can also discard this behavior. Assume that H is a second-countable Lie group. By definition H/H_0 is discrete which is equivalent to H_0 being open. In turn, this implies that the quotient map is open and it follows that H/H_0 is second-countable since by assumption H is second-countable. A second-countable space is separable, and discrete separable spaces are countable. Finally, we recall that $\mathrm{ISO}(X)$ is second-countable for a locally compact, connected metric space.

Foliations, submetries, group actions

Let $(X, \mathsf{d}, \mathfrak{m})$ be a geodesic metric measure space.

Metric foliations, submetries

Definition $\frac{1}{2}$.36 (Foliation, metric foliation)**.** *A partition $\mathcal{F}$ of a metric space (X, d) into closed subsets, called* leaves*, is called a* foliation*. In case that every leaf is bounded we say that the foliation is* bounded.

A foliation $\mathcal{F}$ of a metric space for which

$$\mathsf{d}_{\mathsf{H}}(F, G) = \mathsf{d}(x, G), \quad \textit{for all } F, G \in \mathcal{F} \textit{ and } x \in F, \tag{$\frac{1}{2}$.37}$$

it's called a metric foliation*. (Where d_{H} is the Hausdorff distance between subsets of X.) In case that each leaf is bounded we call it a* bounded metric foliation*.*

That is, a metric foliation $\mathcal{F}$ is a foliation of a metric space for which the distance from a point $x \in F \in \mathcal{F}$ to a leaf $G \in \mathcal{F}$ is independent of the choice of point in the leaf F.

Remark $\frac{1}{2}$.38. Riemannian foliations induce metric foliations in the sense above (see [Wal92]).

Given a foliation $\mathcal{F}$ on a metric space X, the space $X^* = X/\sim$ is the set of equivalence classes under the equivalence relation

$$x \sim y \quad \text{if and only if} \quad \mathcal{F}_x = \mathcal{F}_y,$$

where $\mathcal{F}_x$ denotes the leaf containing x. That is, X^* is the *leaf space* of the foliation. Space to which, making a slight abuse of language, we will also refer to as the *quotient* or the *orbit space*. We denote the projection onto the quotient by $\mathsf{p} : X \to X^*$ and elements of X^* with $\mathsf{p}(x) = x^* \in X^*$. Note that for every $x^* \in X^*$ there is a canonical associated leaf $\mathcal{F}_{x^*} \in \mathcal{F}$, namely, the unique leaf such that $\mathsf{p}(\mathcal{F}_{x^*}) = x^*$; we can then write the foliation as $\mathcal{F} = \{\mathcal{F}_{x^*}\}_{x^* \in M^*}$.

Another notion of that we will study is that of a *submetry*, which generalizes to metric spaces the concept of Riemannian submersions.

Definition $\frac{1}{2}$.39 (Submetry)**.** *A map $f : X \to Y$ between metric spaces is called* submetry *if, for all $x \in X$ and $r > 0$,*

$$f(B_r(x)) = B_r(f(x)).$$

We prove in Proposition 2.2.6 that the concepts of metric foliation and submetry are equivalent in the sense that the projection map is a submetry and that the fibers induce a foliation of X.

Group actions

Under weak assumptions, the orbits of an action of a group H on a space X define a foliation.

Definition $\frac{1}{2}$.40 (Group action)**.** *Let H be a compact Lie group. We say that a map $\mathsf{H} \times X \to X$, $(g, x) \mapsto gx$ is an* action by isomorphisms *of $(X, \mathsf{d}, \mathfrak{m})$ if the following properties are satisfied:*

- *$(gh)\,x = g\,(hx)$ for all $x \in X$ and all $g, h \in \mathsf{H}$;*
- *$\mathbb{I}\,x = x$, for all $x \in X$, where $\mathbb{I}$ is the identity element of the group H;*
- *for every fixed $x \in X$, the map $\star_x : \mathsf{H} \to X$ given by $g \mapsto gx$ is continuous;*
- *for every fixed $g \in \mathsf{H}$, the map $\tau_g : X \to X$ given by $x \mapsto \tau_g(x) := gx$ is an isomorphism of m.m. spaces, i.e. τ_g is an isometry and moreover $(\tau_g)_\sharp \mathfrak{m} = \mathfrak{m}$.*

Moreover, the action is called effective if the intersection of all isotropy subgroups of the action is trivial. The set $\mathsf{H}(x) = \{y \in X \parallel y = gx \text{ for some } g \in \mathsf{H}\}$ for an $x \in X$ is called the orbit *of x.*

It follows from the assumptions made on the action that the set of orbits $\mathcal{H} = \{\mathsf{H}(x)\}_{x \in X}$ defines a bounded metric foliation. Correspondingly, we denote the *space of orbits* by $X^* := X/\mathsf{H}$ and write $\mathsf{p} \colon X \to X^*$ for the projection onto the orbit space. Elements of X^* will be denoted by $x^* = \mathsf{p}(x) = \mathsf{p}(\{gx \mid g \in \mathsf{H}\})$.

We say that a Borel measure μ is *invariant* under the action of H if $\mu(\tau_g(B)) = \mu(B)$ for any $g \in \mathsf{H}$ and any Borel set $B \subset X$. Note that μ being invariant is equivalent to $g_\sharp\mu = \mu$ for every $g \in \mathsf{H}$ since H acts by isomorphisms of m.m. spaces.

Let $\mathcal{F}$ a metric foliation of $(X, \mathsf{d}, \mathfrak{m})$ be an m.m. space (either induced by a group action or of a more general kind). We now endow the orbit space X^* with a metric measure structure. Set the following distance on X^*:

$$\mathsf{d}^*(x^*, y^*) := \inf_{x' \in \mathcal{F}_{x^*}} \mathsf{d}_{\mathsf{H}}(x', \mathcal{F}_{y^*}) = \inf_{x' \in \mathcal{F}_{x^*}, y' \in \mathcal{F}_{y^*}} \mathsf{d}(x', y') = \mathsf{d}_{\mathsf{H}}(\mathcal{F}_{x^*}, \mathcal{F}_{y^*}), \qquad (\tfrac{1}{2}.41)$$

for $x^*, y^* \in X^*$.[7] Condition ($\frac{1}{2}$.37) implies that the quotient distance d^* is well-defined. By definition of d^* and the completeness of (X, d) it follows that the metric space (X^*, d^*) is complete as well. While the continuity of p and separability of (X, d) implies that (X^*, d^*) is separable. Additionally, the quotient (X^*, d^*) is a geodesic or proper space granted that (X, d) is geodesic or proper respectively.
We use the pushforward under the quotient map p of $\mathfrak{m}$ to define a measure $\mathfrak{m}^*$ on the quotient space X^*:

$$\mathfrak{m}^* := \mathsf{p}_\sharp \mathfrak{m}. \qquad (\tfrac{1}{2}.42)$$

We call $\mathfrak{m}^*$ the *quotient measure*. Because $\mathfrak{m}$ is a non-negative σ-finite Borel measure and leaves are bounded we conclude that $\mathfrak{m}^*$ is a non-negative σ-finite Borel measure over the complete and separable metric space (X^*, d^*). Explicitly, the quotient space $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is a metric measure space.

We conclude with the following version of a Disintegration Theorem of measure that will become important for our work.

Theorem $\frac{1}{2}$.43 (Disintegration Theorem of Pachl)**.** *Let $\mathcal{F}$ be a bounded metric foliation of $(X, \mathsf{d}, \mathfrak{m})$ and denote the quotient m.m. space by $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$. Then there exist an $\mathfrak{m}^*$-essentially unique disintegration of $\mathfrak{m}$ over $\mathfrak{m}^*$ consistent with the quotient map $\mathsf{p} : X \to X^*$. Specifically, the following is satisfied:*

There exist an $\mathfrak{m}^*$-almost everywhere unique family of probability measures $\{\mathfrak{m}_{x^*}\}_{x^* \in X^*} \subset \mathcal{P}(X)$, called conditional measures *or* elements of the disintegration*, such that:*

- *$\mathfrak{m}_{x^*}$ is concentrated on $\mathsf{p}^{-1}(x^*) \subset X$;*
- *the assignment $x^* \mapsto \mathfrak{m}_{x^*}(B) \in \mathcal{P}(X)$ is measurable for any Borel set $B \subset X$;*
- *$\mathfrak{m}(B) = \int_{X^*} \mathfrak{m}_{x^*}(B) d\mathfrak{m}^*(x^*)$ for any Borel set $B \subset X$.*

In particular, for any measurable function $f : X \to \mathbb{R}$ it holds that

$$\int_X f(x) d\mathfrak{m}(x) = \int_{X^*} \int_{\mathsf{p}^{-1}(x^*)} f(x) d\mathfrak{m}_{x^*}(x) d\mathfrak{m}^*(x^*)$$

This version, which has been fitted to our setting, comes from Pachl's Disintegration Theorem which can be found in [Fre00, Theorem 452I].

[7] Keep in mind that, in case the foliation arises from a group action, orbits and leaves coincide $\mathcal{F}_x = \mathsf{H}(x) = \mathsf{p}^{-1}(x^*)$ for every $x \in X$.

Chapter 1

Symmetry groups of metric measure spaces

M.m. spaces with smooth isomorphism groups

The purpose of the first part of this chapter is to prove Main Theorem A below; the second section will focus on examples and applications of the theorem.

In the reminder we let $\mathsf{G} \in \{\mathrm{ISO}(X), \mathrm{ISO}_{\mathfrak{m}}(X)\}$ and endow both groups with the compact-open topology. Recall that by a metric measure space we mean a complete, separable metric space endowed with a non-negative Borel measure which is finite on bounded sets and such that $\mathrm{supp}(\mathfrak{m}) = X$. The precise meaning of *well-behaved tangents* can be found in Definition 1.1.4.

Theorem A (M.m. spaces with smooth G)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact m.m. space where every closed ball coincides with the closure of its respective open ball. Assume that X has well-behaved tangents. Then G is a Lie Group if and only if

(a) There exist a point $x \in X$ and constants $0 < s$, and $0 < \mathsf{FIX} < \mathfrak{m}(B_s(x))$, such that for every $(\mathbb{I} \neq) g \in \mathsf{G}$

$$\mathfrak{m}(\mathrm{Fix}(g) \cap B_s(x)) < \mathsf{FIX}.$$

Moreover $\mathrm{ISO}_{\mathfrak{m}}(X)$ is a Lie group granted that $\mathrm{ISO}(X)$ is so as well.

Let us make some observations about Theorem A.

To the best of the author's knowledge the approach of studying the group of isomorphisms (measure-preserving maps) of a metric measure space is a new one. This point of view was consider as well in [GSR16], where Guijarro-Santos-Rodríguez showed that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is smooth for RCD*-spaces. In §1.2 Basic Examples we show that, in general, is not true that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is a Lie group granted that $\mathrm{ISO}(X)$ is also a Lie group nor the reversed statement. This is a motivation to consider the groups $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ separately in Theorem A.

Given that tangent cones don't behave *too wildly*, the conclusion of the theorem is valid even when considering m.m. spaces whose tangent cones fail to be Euclidean. Such situation arises, for example, when tangent cones are normed spaces or Carnot groups[1] with a uniform positive bound on the size of their subgroups of isometries. Accordingly, we are able to study spaces with different geometries in addition to Riemannian ones.

From a theorem of van Danzig and van der Waerden [vDvdW28] and Lemma 1.1.1 we conclude that $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ are compact if X is compact.

In the second part we study examples of metric measure spaces that satisfy condition (a). We confirm classical theorems asserting that the following spaces have a smooth isometry group $\mathrm{ISO}(X)$. Observe that in the majority of these examples the automorphism groups $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ coincide.

Weighted Riemannian manifolds [MS39], and

Finsler manifolds with either the Holmes-Thompson or Busemann volume measure [DH02].

Alexandrov spaces with curvature bounded below with the Hausdorff measure [FY94].

Additionally, we extend these results to a larger class of spaces characterized by having good optimal transport properties, see Definition 1.2.3 of the *good transport behavior*.

Theorem B (Spaces with good transport behavior)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact, length metric measure space. Assume that X has *good optimal transport behavior*. Then condition (a) is satisfied.
In particular, if X has well-behaved tangent cones then G is a Lie group.

Corollary (Corollaries 1.2.9, 1.2.10, 1.2.11)**.** *The group of isometries and the group of isomorphisms are smooth for the next classes of spaces. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$.*

$\mathsf{RCD}^*(K, N)$*-spaces;*

Strong $\mathsf{CD}(K, N)$*-spaces and strong* $\mathsf{CD}^*(K, N)$*-spaces with* well-behaved tangents*;*

Essentially non-branching $\mathsf{MCP}(K, N)$*-spaces with* well-behaved tangents

A class of Busemann-Kell concave spaces defined by Kell [Kel16], which are a Finsler version of Alexandrov spaces of non-negative curvature.

The CD, CD*, and MCP conditions allow for non-Riemannian geometries which include, but are not restricted to, Finsler manifolds. For example, any corank 1 Carnot group of dimension $(k+1)$ equipped with a left-invariant measure is an essentially non-branching MCP-space with unique non-Euclidean tangents by Rizzi [Riz16]; it follows that their automorphism groups are Lie groups.

[1] A Carnot group is nilpotent stratified Lie group equipped with a left-invariant subFinsler distance with the first stratum as horizontal distribution. These groups have a simple metric characterization: they are those locally compact, geodesic homogeneous metric spaces that admit dilatations [LD15].

Examples of RCD*-spaces are Alexandrov and Ricci limit spaces with the Hausdorff measure, generalized cone constructions over RCD*-spaces, and limits of weighted manifolds with a uniform lower bound on the Bakry-Emery Ricci tensor [Pet11, LV09, Stu06a, Stu06b, Ket15]. However, it is not known whether the class of RCD*-spaces is strictly bigger than that of weighted Ricci limit spaces. Additionally, in §2 we show that quotients of RCD*-spaces arising from isomorphic group actions and metric measure bounded foliations are also RCD*-spaces. A last class of examples of RCD*-spaces is presented by orbifolds and orbispaces with lower bounds on Ricci curvature as shown in [GGKMS17].

We conclude the chapter by illustrating that, on the other hand, not all curvature-dimension conditions are sufficiently restrictive to guarantee smooth isomorphism groups.

Proposition 1.0.1. *There exists an* $\mathsf{MCP}(2,3)$*-space for which neither* $\mathrm{ISO}_{\mathfrak{m}}(X)$ *nor* $\mathrm{ISO}(X)$ *are Lie groups.*

1.1 Proof of main result

We begin by showing that G fulfills the assumptions of Theorem $\frac{1}{2}$.30, which states that a topological group is not a Lie group if and only if that group contains small subgroups. Thereon, in Propositions 1.1.2 and 1.1.8, we study those m.m. spaces with well-behaved tangents for which G contains small subgroups and reinterpret the "small size" of a subgroup H $\leq$ G (related intrinsically to the distance d since G is endowed with the compact-open topology) in a measure-theoretic fashion. From there it will be rather simple to conclude Theorem A.

Lemma 1.1.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a connected, locally compact m.m. space. Then* $\mathrm{ISO}_{\mathfrak{m}}(X)$ *is a locally compact closed subgroup of* $\mathrm{ISO}(X)$ *with respect to the compact-open topology.*

Proof. We show that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is closed. The local compactness of $\mathrm{ISO}_{\mathfrak{m}}(X)$ follows from the fact that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is a closed subgroup of a locally compact group. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{ISO}_{\mathfrak{m}}(X)$ be a converging sequence w.r.t. the compact-open topology with limit $f := \lim_{n\to\infty} f_n$. It is easy to see that f is an isometry. Thus, to finish the proof, it remains to check that $(f)_\#\mathfrak{m} = \mathfrak{m}$. This follows from the regularity of the measure, as we argue below.

Indeed, since the measures $(f_n)_\#\mathfrak{m} = \mathfrak{m}$ are all equal, they trivially converge weakly to $\mathfrak{m}$. On the other hand we will show that the pushforward of $\mathfrak{m}$ under f_n weakly converges to the measure $(f)_\#\mathfrak{m}$. Therefore, $(f)_\#\mathfrak{m} = \mathfrak{m}$ by uniqueness of the limit. By using the definition of the pushforward and the continuity of $g \circ f_n$, it is enough to verify that for every bounded continuous function with bounded support, $g : X \to \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \int_X g \circ f_n \, \mathrm{d}\mathfrak{m} = \int_X g \circ f \, \mathrm{d}\mathfrak{m},$$

to show that $(f_n)_\#\mathfrak{m} \xrightarrow{w} (f)_\#\mathfrak{m}$. After the following observation it will be clear that this last equality holds.

Assume that g is as above. We can construct an $\mathfrak{m}$-integrable function, G, such that $|g \circ f_n(x)| \leq G(x)$ for all $x \in X$ and make use of the dominated convergence theorem.

Take for example the multiple of the characteristic function $G := k_g\, \chi|_{B_r(y)}$, where k_g is a bound on g and $r \in \mathbb{R}$ and $y \in X$ are such that $\cup_{n\in\mathbb{N}} \operatorname{supp}(g \circ f_n) \subset B_r(y)$. The existence of such a pair $\{r, y\}$ is guaranteed because g has bounded support, and because $f_n \to f$ converges uniformly in compact subsets. The integrability of G follows from $\mathfrak{m}$ being finite on bounded sets. □

The next two propositions contain the key ideas used in the proof of the main result. First let us see that we can generate small subgroups from the existence of automorphisms with large fixed point sets.

Proposition 1.1.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact m.m. space where every closed ball coincides with the closure of the open ball. Then G has the small subgroups property if for every $x \in X$, $0 < s$, and $0 < \xi' < 1$ there exists a non-trivial subgroup $\Lambda = \Lambda_{x,s,\xi'} \subset \mathsf{G}$ such that for every $g \in \Lambda$*

$$\mathfrak{m}(X \setminus \operatorname{Fix}(g) \cap B_s(x)) \leq \xi'\, \mathfrak{m}(B_s(x)).$$

Proof. We give a sequence $\{\xi_N\}_{N\in\mathbb{N}} \subset (0,1)$ which generates, according to the hypothesis, a sequence of non-trivial subgroups $\{\Lambda_{x,N,\xi_N}\}_{N\in\mathbb{N}} \leq \mathsf{G}$ such that $\Lambda_N \subset U_N (\ni \mathbb{I})$ for every $N \in \mathbb{N}$, where $\{U_N\}_{N\in\mathbb{N}} \subset \mathsf{G}$ is a local basis of the compact-open topology at $\mathbb{I}$. Thus proving the existence of small subgroups of G.

Accordingly we fix $x \in X$, $N \in \mathbb{N}$, and define

$$\xi'_N := \mathfrak{m}(B_N(x))^{-1} \inf_{y \in B_N(x)} \{\mathfrak{m}(B_{1/N}(y) \cap B_N(x))\}.$$

We claim that $0 < \xi'_N$. Indeed, choose a converging sequence[2] $y_m \to y_\infty \in \overline{B}_N(x)$ such that $\liminf_{m\to\infty} \mathfrak{m}(B_N(x))^{-1}\mathfrak{m}(B_{1/N}(y_m) \cap B_N(x))) = \xi'_N$.[3] Since the measure $\mathfrak{m}$ has full support there exists a small ball, $B_\tau(y_\infty)$, with $\mathfrak{m}(B_\tau(y_\infty) \cap B_N(x)) > 0$ which is a lower bound of $\mathfrak{m}(B_{1/N}(y_m) \cap B_N(x))$ for large enough m, hence, validating the claim. We take $0 < \xi_N < \xi'_N$ and write $\Lambda_N := \Lambda_{x,N,\xi_N}$ for the non-trivial subgroup given by the hypothesis for the triple (x, N, ξ_N). By construction we verify that

$$\mathfrak{m}(X \setminus \operatorname{Fix}(f) \cap B_N(x)) < \mathfrak{m}(B_{1/N}(y) \cap B_N(x)) \tag{1.1.3}$$

for every $y \in B_N(x)$ and $f \in \Lambda_N$.

Observe now that if $\mathsf{d}(z, g(z)) > 2t$ then $B_t(z) \cap B_R(x) \subset X \setminus \operatorname{Fix}(g) \cap B_R(x)$ for $g \in \mathsf{G}$, $z \in X$, and numbers $t, R \in \mathbb{R}^+$. Therefore, we conclude from (1.1.3) that for every $y \in B_N(x)$ and $f \in \Lambda_N$ we have that $d(y, f(y)) \leq 2/N$. Hence Λ_N is contained in the neighborhood of the identity:

$$U_N := \left\{ g \in \mathsf{G} \,\|\, \mathsf{d}(y, g(y)) < 3/N \text{ for every } y \in \overline{B}_N(x) \right\}.$$

Accordingly, the proof is complete considering that the choice of N was arbitrary. □

[2] Using a subsequence if necessary.

[3] The existence of such subsequence is guaranteed from the fact that locally compact, complete metric spaces for which the closure of open balls coincides with closed balls are proper.

To show that the reverse statement to the previous Proposition 1.1.2 is ture, we first give a couple of definitions.

We define, for $r > 0$, $x \in X$, and a subgroup $\Lambda \leq \mathsf{G}$:

$$D_\Lambda(r, x) := \sup_{g \in \Lambda} \sup_{y \in B_{\frac{r}{2}}(x)} \mathsf{d}(y, g(y)).$$

For fixed Λ, the function $D_\Lambda(r, x)$ is continuous in r and x as long as every closed ball in X is the closure of its respective open ball [CC00]. This holds true when X is a length space, for example.

Recall that we denote all pGH-tangent cones of X at y by $\mathsf{Tan}(X, y)$, that the regular set is defined $\mathcal{R} \subset X$ as all points of X that have a unique tangent cone, and that we write $\mathsf{Tan}(\mathcal{R}) = \cup_{y \in \mathcal{R}} \mathsf{Tan}(X, y)$. We say that X has $\mathfrak{m}$-almost everywhere unique tangents if $\mathfrak{m}(X \setminus \mathcal{R}) = 0$. Details can be found in the Introductory §$\frac{1}{2}$.

Definition 1.1.4 (Well-behaved tangents)**.** *We say that a m.m. space has* well-behaved tangents *if it has $\mathfrak{m}$-almost everywhere unique tangents, the set of spaces appearing as tangents $\mathsf{Tan}(\mathcal{R})$ is compact and there exist a constant $0 < \mathsf{k}_o$ such that*

$$D_{\mathsf{H}^\infty}(y^\infty, 1) > \mathsf{k}_o, \qquad \textit{for all } (\mathbb{I} \neq) H^\infty \leq \mathrm{ISO}(Y^\infty),\ (Y^\infty, \mathsf{d}_{Y^\infty}, y^\infty) \in \mathsf{Tan}(\mathcal{R}). \quad (1.1.5)$$

Remark 1.1.6. The definition of well-behaved tangents is fairly general and will allow us to study a wide range of types of metrics. Metric measure spaces with a unique space appearing $\mathfrak{m}$-almost everywhere as tangent cone have well-behaved tangents, this is the case of smooth manifolds and spaces of curvature bounded above and below. More examples are presented by m.m. spaces for which $\mathsf{Tan}(\mathcal{R})$ is any finite union of Euclidean spaces, normed spaces, and Carnot groups with a uniform positive bound on the size of the subgroups of isometries.

A result of relevance in this direction is due to Le Donne [LD11].

Theorem 1.1.7 (Le Donne [LD11])**.** *Geodesic spaces with a doubling measure that have $\mathfrak{m}$-a.e. unique tangents have $\mathfrak{m}$-a.e. Carnot groups as tangents.*

In the coming proposition we do not assume a fully supported measure.

Proposition 1.1.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m. space where every closed ball coincides with the closure of the open ball and such that $\mathfrak{m} \neq 0$. Assume that X has well-behaved tangents.*

If G has the small subgroups property, then for every $x \in X$, $0 < s$, and $0 < \xi < 1$ there exists a non-trivial subgroup $\Lambda = \Lambda_{x,s,\xi} \subset \mathsf{G}$ such that for every $g \in \Lambda$

$$\mathfrak{m}(\mathrm{Fix}(g) \cap B_s(x)) \geq \xi\, \mathfrak{m}(B_s(x)). \quad (1.1.9)$$

Proof. We assume that $\mathfrak{m}(B_s(x)) > 0$ since the inequality above trivially holds true otherwise. Let $0 < \mathsf{k}_o$ be the size constant given by the well-behaved tangents property of X. We argue by contradiction. The strategy is the following: assuming that inequality

(1.1.9) doesn't hold we will find for every $\epsilon > 0$ a quadruple $(\delta_\epsilon, r_\epsilon, x_\epsilon, \Lambda_\epsilon) \in (\mathbb{R}^+)^2 \times X \times 2^{\mathsf{G}}$ with the following properties:

$$\begin{aligned} &\bullet\ 0 < r_\epsilon \leq \delta_\epsilon; \\ &\bullet\ x_\epsilon \in (\mathcal{R})_{\epsilon,\delta_\epsilon}; \\ &\bullet\ \Lambda_\epsilon \leq \mathsf{G} \text{ is a subgroup;} \\ &\bullet\ D_{\Lambda_\epsilon}(r_\epsilon, x_\epsilon) = \frac{r_\epsilon\, \mathsf{k_o}}{2}. \end{aligned} \tag{1.1.10}$$

The existence of such a family of quadruples would lead to a contradiction thus, would prove the proposition. Indeed, observe that if for every $\epsilon > 0$ there exists a quadruple as above, then for a sequence $\epsilon_n \to 0$ there exists a subsequence ϵ_n (denoted in the same way) such that the scaled spaces below converge, in the eGH-sense, to

$$\left(B_{r_\epsilon}(x_\epsilon), \frac{1}{r_\epsilon}\mathrm{d}, \Lambda_\epsilon\right) \xrightarrow{\mathsf{eGH}} (B_1(y^\infty) \subset Y^\infty, \mathrm{d}_{Y^\infty}, \Lambda_\infty),$$

where $Y^\infty \subset \mathsf{Tan}(\mathcal{R})$, and $\Lambda_\infty \leq \mathrm{ISO}(Y^\infty)$ is a non-trivial subgroup satisfying $D_{\Lambda_\infty}(1, y^\infty) = \frac{\mathsf{k_o}}{2}$.[4] This creates the contradiction since, by hypothesis, X has well-behaved tangents which implies that every non-trivial H subgroup of $\mathrm{ISO}(Y^\infty)$ satisfies $D_{\mathsf{H}}(1, y^\infty) > \frac{\mathsf{k_o}}{2}$.

We proceed to construct a family of quadruples satisfying conditions (1.1.10).

Suppose that (1.1.9) does not hold. That is, there exist $x \in X$, $0 < s$, and $0 < \xi < 1$ such that for every non-trivial subgroup $\mathsf{K} \subset \mathsf{G}$ there exists an $f \in \mathsf{K}$ where

$$(\mathrm{Fix}(f) \cap B_s(x)) < \xi\, \mathfrak{m}(B_s(x)). \tag{1.1.11}$$

Note that necessarily$f \neq \mathbb{I}$. Let $\epsilon > 0$ and choose small enough $\delta_\epsilon \in \mathbb{R}$ so that $0 < \delta_\epsilon < s$, and

$$\xi\, \mathfrak{m}(B_s(x)) < \mathfrak{m}((\mathcal{R})_{\epsilon,\delta_\epsilon} \cap B_s(x)). \tag{1.1.12}$$

The $\mathfrak{m}$-almost everywhere unique tangents of X, together with the continuity from below of the measure and the fact that $\mathcal{R} \subset (\mathcal{R})_\epsilon = \cup_{\delta>0}(\mathcal{R})_{\epsilon,\delta}$ make possible the choice of such a δ_ϵ. Indeed, since for $\delta' \leq \delta''$ it holds that $(\mathcal{R})_{\epsilon,\delta''} \subset (\mathcal{R})_{\epsilon,\delta'}$ we can write $(\mathcal{R})_\epsilon = \cup_{n\in\mathbb{N}}(\mathcal{R})_{\epsilon,1/n}$ as a countable union of sets. Now just notice that

$$\begin{aligned} \mathfrak{m}(B_s(x)) &= \mathfrak{m}(B_s(x) \cap \mathcal{R}_\epsilon) = \mathfrak{m}(B_s(x) \cap (\cup_{n\in\mathbb{N}}(\mathcal{R})_{\epsilon,1/n})) \\ &= \lim_{n\to\infty} \mathfrak{m}(\cup_{j\leq n}(B_s(x) \cap (\mathcal{R})_{\epsilon,1/j})) \\ &= \lim_{n\to\infty} \mathfrak{m}(B_s(x) \cap (\mathcal{R})_{\epsilon,1/n}). \end{aligned}$$

Choose $n \in \mathbb{N}$ sufficiently big and take $\delta_\epsilon < \min\{s, 1/n\}$.

[4] In more detail, the definition of the sets $(\mathcal{R})_{\epsilon,\delta_\epsilon}$ and compactness of $\mathsf{Tan}(\mathcal{R})$ imply that there exist a subsequence ϵ'_n for which $(B_{r_{\epsilon'_n}}(x_\epsilon), \frac{1}{r_{\epsilon'_n}}\mathrm{d}) \xrightarrow{\mathsf{GH}} (B_1(y^\infty) \subset Y^\infty, \mathrm{d}_{Y^\infty})$ for some $(Y^\infty, \mathsf{d}_{y^\infty}, y^\infty) \in \mathsf{Tan}(\mathcal{R})$. Then [FY92, Proposition 3.6] guarantees the existence of another subsequence for which the claimed convergence holds. The last claim made (stating that $D_{\Lambda_\infty}(1, y^\infty) = \frac{\mathsf{k_o}}{2}$) is justified by the continuity of $\mathsf{D} : \mathcal{M}^{\mathrm{c}}_{\mathrm{eq,p}} \to \mathbb{R} : (X_n, \mathsf{d}_n, \Lambda_n, x_n) \mapsto D_{\Lambda_n}(1, x_n)$ under (pointed) eGH-convergence, which follows straight from the definitions. For more details on equivariant GH-convergence see Definition $\frac{1}{2}$.4 and the discussion that follows.

Inequalities (1.1.11) and (1.1.12) imply that for every non-trivial $\mathsf{K} \leq \mathsf{G}$ there exist $f(\neq \mathbb{I}) \in \mathsf{K}$ such that the set $B_s(x) \cap (\mathcal{R})_{\epsilon,\delta_\epsilon} \setminus \mathrm{Fix}(f)$ is not empty.

In view of the small subgroups property of G, we can find a non-trivial small subgroup

$$
\begin{aligned}
\Lambda_\epsilon \subset U_\epsilon &:= \left\{ g \in \mathsf{G} \parallel \sup_{y \in B_{2s}(x)} \mathrm{d}(y, g(y)) < \frac{\delta_\epsilon \,\mathsf{k_o}}{2} \right\} \\
&= \left\{ g \in \mathsf{G} \parallel g(y) \in B_{\delta_\epsilon \,\mathsf{k_o}/2}(y) \text{ for all } y \in \overline{B_{2s}}(x) \right\}
\end{aligned}
$$

In particular, there exist $g(\neq \mathbb{I}) \in \Lambda_\epsilon$ and $x_\epsilon \in B_s(x)$ such that

$$
\begin{aligned}
x_\epsilon \quad &\in \quad B_s(x) \cap (\mathcal{R})_{\epsilon,\delta_\epsilon} \setminus \mathrm{Fix}(g) \qquad \text{and} \\
0 \quad &< \quad \mathsf{d}(x_\epsilon, g(x_\epsilon)) < \frac{\delta_\epsilon \,\mathsf{k_o}}{2}.
\end{aligned}
$$

Denote by $\theta = \theta(x_\epsilon) := 2/\mathsf{k_o}\, \mathsf{d}(x_\epsilon, g(x_\epsilon)) < \delta_\epsilon$. By construction it follows that

$$
\begin{aligned}
\frac{\mathsf{k_o}}{2}\theta \leq & D_{\Lambda_\epsilon}(\theta, x_\epsilon) \\
& D_{\Lambda_\epsilon}(\delta_\epsilon, x_\epsilon) \leq D_{\Lambda_\epsilon}(4s, x) < \frac{\mathsf{k_o}}{2}\delta_\epsilon.
\end{aligned}
$$

Finally, the continuity of $D_{\Lambda_\epsilon}(\circ, x_\epsilon)$ and the intermediate value theorem imply that there exists $r_\epsilon \in \mathbb{R}$ that satisfies $D_{\Lambda_\epsilon}(r_\epsilon, x_\epsilon) = \frac{\mathsf{k_o}}{2} r_\epsilon$ for some $\theta \leq r_\epsilon < \delta_\epsilon$. Hence we have shown that for every $\epsilon > 0$ there exists a quadruple $(\delta_\epsilon, r_\epsilon, x_\epsilon, \Lambda_\epsilon)$ satisfying (1.1.10). $\square$

We prove now the main theorem.

Main Theorem A. Being the groups of isometries and of measure-preserving isometries locally compact spaces (Theorem $\frac{1}{2}$.30, and Lemma 1.1.1) we can rely on Gleason and Yamabe's characterization of Lie groups. That is to say, $\mathsf{G} \in \{\mathrm{ISO}(X), \mathrm{ISO}_{\mathfrak{m}}(X)\}$ is a Lie group if and only if G does not have the small subgroup property. Note that the contrapositive statements to Propositions 1.1.2 and 1.1.8 show that G not having the small subgroups property is equivalent to:

(a′) *There exist $x \in X$, $0 < s$, $0 < \xi < 1$ such that for every non-trivial subgroup $\Lambda \subset \mathrm{ISO}(X)$ there exists an isometry $g \in \Lambda$ with*

$$
\mathfrak{m}(\mathrm{Fix}(g) \cap B_s(x)) < \xi \,\mathfrak{m}(B_s(x)).
$$

It is clear that (a) implies (a′). The implication in the other direction follows after observing that the existence of an isomorphism $\mathbb{I} \neq g \in \mathsf{G}$ with $\mathfrak{m}(\mathrm{Fix}(g) \cap B_s(x)) \geq \xi \,\mathfrak{m}(B_s(x)) =: \mathsf{Fix}$ implies that the measure of the fixed point set of every element in the subgroup generated by g, $\langle g \rangle \neq \mathbb{I}$, is greater than or equal to Fix. This proves the first part of the theorem.

Finally, note that granted that $\mathrm{ISO}(X)$ has the no small subgroup property, then $\mathrm{ISO}_{\mathfrak{m}}(X)$ has the same property since they both are endowed with the compact-open topology. This last argument together with the local compactness of $\mathrm{ISO}_{\mathfrak{m}}(X)$ show that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is a Lie group if $\mathrm{ISO}(X)$ is a Lie group.

$\square$

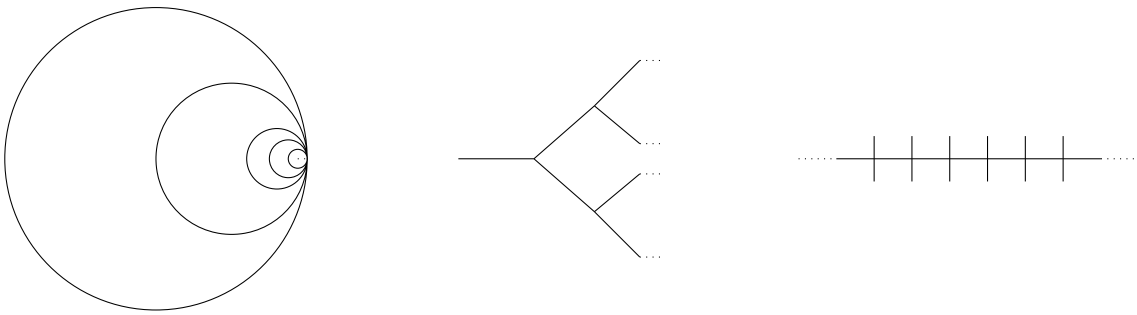

Figure 1.2.1: Hawaiian earring $\mathbb{H}$ and branching spaces.

1.2 Examples and application to synthetic Ricci curvature

We present examples of spaces that satisfy the assumptions of Main Theorem A. We start with basic examples of spaces whose automorphism group G is not a Lie group to develop intuition about the relationship between G not being a Lie group, G having small subgroups, and the branching of geodesics inside X.

From there in Lemma 1.2.5 and Theorem B we show that having *good optimal transport properties* guarantees that condition (a) will be satisfied. Afterwards we enunciate in Corollaries 1.2.9, 1.2.10, 1.2.11 which spaces, satisfying a curvature-dimension condition, enjoy these transport properties and therefore have smooth automorphism groups.

We conclude the chapter with the construction of a first Example 1.2.15 of an MCP-space, for which for which neither $\mathrm{ISO}(X)$ nor $\mathrm{ISO}_{\mathsf{m}}(X)$ are Lie groups. Hence not all curvature-dimension conditions are restrictive enough to guarantee that automorphism groups are Lie groups.

Basic examples

Example 1.2.1 (Spaces with non-smooth G).
Non-smooth $\mathrm{ISO}_{\mathsf{m}}(X)$ and non-smooth $\mathrm{ISO}(X)$. Denote the circle of radius r by S_r. The Hawaiian earring, $\mathbb{H}$, is the space we obtain after gluing the circles $\{S_{\frac{1}{n^2}} \parallel n \in \mathbb{N}\}$ by identifying one point of every circle, see Figure 1.2.1. Endow $\mathbb{H}$ with the arc-length distance $\mathsf{d}_{\mathbb{H}}$ and the 1-dimensional Hausdorff measure $\mathcal{H}^1$. This makes $(\mathbb{H}, \mathsf{d}_{\mathbb{H}}, \mathcal{H}^1)$ a compact, geodesic metric measure space with finite measure. Observe that $\mathrm{ISO}(H) = \mathrm{ISO}_{\mathsf{m}}(H) = \Pi^{\infty}\{\pm 1\}$ where the compact-open topology coincides with the product topology. Hence $\mathrm{ISO}(X)$ is totally disconnected but not discrete. By definition, $\mathrm{ISO}(X)$ is not a Lie group since $\mathrm{ISO}(X)/\mathrm{ISO}(X)_0$ is not discrete.

In the same manner we can show that an infinite branching tree and a line with a countable number of segments crossing it, shown in Figure 1.2.1, have non-smooth automorphism groups.

Observe as well that it's not guaranteed that the relation $\mathrm{ISO}(X) \iff \mathrm{ISO}_{\mathsf{m}}(X)$ is valid nor an implication in any direction.

Non-smooth $\mathrm{ISO}_{\mathfrak{m}}(X)$ **but smooth** $\mathrm{ISO}(X)$**.** Let $(\mathbb{R}, \mathsf{d}_{\mathrm{E}}, \mathfrak{m}_{\mathbb{Q}})$ be the m.m. space where d_E is the Euclidean distance and $\mathfrak{m}_{\mathbb{Q}} = \sum_{q\in\mathbb{Q}} \delta_q$ is a sum of δ-measures supported on each $q \in \mathbb{Q}$. We have that $\mathrm{ISO}_{\mathfrak{m}}(X) \cong O(1) \times \mathbb{Q} \ncong O(1) \times \mathbb{R} = \mathrm{ISO}(X)$. Observe that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is neither a locally compact group, nor a closed subgroup of $\mathrm{ISO}(X)$. In Theorem A the implication to the right side is achieved relying on the regularity of the measure $\mathfrak{m}$.

Smooth $\mathrm{ISO}_{\mathfrak{m}}(X)$ **but non-smooth** $\mathrm{ISO}(X)$**.** This situation is more drastic and it's easy to construct examples. For instance, let $(Y, \mathsf{d}, \mathfrak{m})$ be the m.m. space that we obtain after identifying a point of a space X with non-smooth $\mathrm{ISO}(X)$ with any point of an homogeneous manifold (M, g) and setting the measure $\mathfrak{m} = \mathrm{vol}_g + \mathfrak{n}$, where vol_g is the Riemannian volume of (M, g) and $\mathfrak{n}$ is a non-symmetric measure with support $\mathrm{supp}(\mathfrak{n}) = X$. In this case we have $\mathrm{ISO}_{\mathfrak{m}}(Y)$ is the subgroup group $\mathrm{ISO}(M)$ which fixes the point where the gluing has been done, but $\mathrm{ISO}(X) \subset \mathrm{ISO}(Y)$ is not smooth.

Example 1.2.2 (Spaces with smooth G).

Riemannian manifolds equipped with a weighted volume measure, Finsler manifolds endowed with either the Holmes-Thompson or Busemann volume measure, and Alexandrov spaces of curvature bounded below with the Hausdorff measure satisfy condition (a) of Theorem A, and thus G is a Lie group. This is a consequence of the non-branching of geodesics in this spaces and the measure being absolutely continuous with respect to the Hausdorff measure.

Generalized Ricci curvature and good transport behavior

The next definition is of central interest in this section.

Definition 1.2.3. *[Good transport behavior] A metric measure space $(M, \mathsf{d}, \mathfrak{m})$ has good transport behavior* GTB*, if for all $\mu, \nu \in \mathcal{P}_2(M)$ with $\mu \ll \mathfrak{m}$ any optimal transport plan between μ and ν is induced by a map.*

The term *good transport behavior* was coined by Martin Kell in the writing of [GGKMS17] where the property is used in the context of group actions on spaces with generalized Ricci curvature bounds. As the next theorem recalls, a large class of spaces have good transport behavior.

Theorem 1.2.4. *[Cavalletti-Huesmann, Cavalletti-Mondino, Gigli-Rajala-Sturm, Kell [CH13, CM17, GRS15, Kel17] The following spaces have* GTB_p*:*

- *Essentially non-branching* $\mathsf{MCP}(K, N)$*-spaces for $K \in \mathbb{R}$, and $N \in [1, \infty)$. In particular, this includes, essentially non-branching* $\mathsf{CD}^*(K, N)$*-spaces, essentially non-branching* $\mathsf{CD}(K, N)$*-spaces, and* $\mathsf{RCD}^*(K, N)$*-spaces.*

- *Essentially non-branching spaces with qualitatively non-degenerate measure $\mathfrak{m}$.*

A space with GTB has favorable geometrical properties, see for instance the recent developments in [GGKMS17, Kel17]. Intuitively the property tells us that in average geodesics don't branch. It follows directly from the definition and the fact that the convex combination of two optimal plans between the same initial and terminal measure is optimal that: *optimal dynamical plans are unique in spaces with* GTB. This is a standard argument in optimal transport theory. It also follows from the definition that: *for every* $x \in X$ *and* $\mathfrak{m}$-*a.e.* $y \in X$ *there exists a unique geodesic joining* x *to* y.

Observe the similarity of the concept of GTB to that of essentially non-branching. This is no coincidence, granted that the space satisfies a curvature-dimension-type condition, these two concepts are equivalent see [Kel17] and [CM17, GRS15] for previous results.

We now see that having a good transport behavior is sufficient to guarantee that non-trivial isometries have fixed-point sets of measure zero.

Lemma 1.2.5 (Zero measure of the fixed point set). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a geodesic metric measure space with* GTB *and let* $f \neq \mathbb{I}$ *be an isometry of* X. *Then* $\mathfrak{m}(\mathrm{Fix}(f)) = 0$.

Proof. We proceed by contradiction. Suppose that there exist $\mathbb{I} \neq f \in \mathrm{ISO}(X)$, and a set $A \subset \mathrm{Fix}(f)$ with positive measure. Let $x \in X \setminus \mathrm{Fix}(f)$ and define the probability measures $\mu_0 := \mathfrak{m}(A)^{-1}\mathfrak{m}|_A$ and $\mu_1 := \frac{1}{2}(\delta_x + \delta_{f(x)})$. We denote by $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ the unique geodesic plan between μ_0 and μ_1 given by the map T. Let $\Gamma := T(A) \subset \mathrm{Geo}(X)$ and note that π is concentrated in Γ and that e_0 is injective in the same set. Set:

$$\begin{aligned} \Gamma_1 &:= \{\gamma \in \Gamma \mid \mathrm{e}_1(\gamma) = x\} \\ \Gamma_2 &:= \{\gamma \in \Gamma \mid \mathrm{e}_1(\gamma) = f(x)\} \\ A_i &:= \mathrm{e}_0(\Gamma_i) \qquad i = 1, 2. \end{aligned}$$

Γ_1 (Γ_2) is the subset of geodesics of Γ that end in x ($f(x)$) and A_i is the projection of Γ_i onto the set A. We have that none of these sets are empty, that the measures $\pi(\Gamma \setminus (\Gamma_1 \cup \Gamma_2)) = 0 = \mathfrak{m}(A \setminus (A_1 \cup A_2))$ and that $A_1 \cap A_2 = \emptyset$. The last fact is a consequence of the injectivity of e_0. We define now the measure $\pi' \in \mathcal{P}(\mathrm{Geo}(X))$ as

$$\pi' := (\hat{f})_{\#}\pi|_{\Gamma_1} + (\widehat{f^{-1}})_{\#}\pi|_{\Gamma_2}, \tag{1.2.6}$$

where the bijection of $\mathrm{Geo}(X)$, $\gamma \mapsto g \circ \gamma$, induced by some $g \in \mathrm{ISO}(X)$ is written as $\hat{g} : \mathrm{Geo}(X) \to \mathrm{Geo}(X)$. The measure π' is a symmetric analog of π but $\pi' \neq \pi$. Indeed, note that $\pi'(\hat{f}(\Gamma_1)) = 1/2 \neq 0 = \pi(\hat{f}(\Gamma_1))$ because $\hat{f}(\Gamma_1) \cap \Gamma_1 = \emptyset$ by construction.

We claim that $\pi' \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is also a dynamical plan. This would contradict the hypothesis of the uniqueness of π and finish the proof of the lemma. We proceed to verify the claim.

We need to show that π' minimizes $\int_{\mathrm{Geo}(X)} l(\gamma) d\rho$. The minimum taken over all measures $\rho \in \mathcal{P}(\mathrm{Geo}(X))$ such that $(e_i)_{\#}\rho = \mu_i$ for $i = 0, 1$. We check that the pushforwards of π' under the evaluation map are as above. For this we observe that for $g \in \mathrm{ISO}(X)$, $B \subset X$, and $t \in [0, 1]$

$$\begin{aligned} \hat{g} \circ \mathrm{e}_t^{-1}(B) &= \hat{g}\left(\{\gamma \in \mathrm{Geo}(X) \mid \mathrm{e}_t(\gamma) \in B\}\right) \\ &= \{\gamma \in \mathrm{Geo}(X) \mid \mathrm{e}_t(\gamma) \in g(B)\} = \mathrm{e}_t^{-1} \circ g(B), \end{aligned}$$

and that $\hat{g}^{-1} = \widehat{g^{-1}}$. Next we compute the pushforward of π' under e_t:

$$\begin{aligned}(\mathrm{e}_t)_\#\pi' &= (e_t \circ \hat{f})_\#\pi|_{\Gamma_1} + (e_t \circ \widehat{f^{-1}})_\#\pi|_{\Gamma_2} \\ &= (f)_\#(e_t)_\#\pi|_{\Gamma_1} + (f^{-1})_\#(e_t)_\#\pi|_{\Gamma_2}.\end{aligned}$$

Then $(\mathrm{e}_0)_\#\pi' = \mu_0$ since $f|_A = \mathbb{I}|_A$. As for the other pushforward we have that $(\mathrm{e}_1)_\#\pi' = (f)_\#(\frac{1}{2}\delta_x) + (f^{-1})_\#(\frac{1}{2}\delta_{f(x)}) = \frac{1}{2}(\delta_x + \delta_{f(x)}) = \mu_1$. To finish we see that $\pi' \in \mathrm{OptGeo}(\mu_0, \mu_1)$ by showing that the value of $\int_{\mathrm{Geo}(X)} l(\gamma)\mathsf{d}\pi'$ is the minimum of the functional.

$$\begin{aligned}\int_{\mathrm{Geo}(X)} l^2(\gamma)\mathsf{d}\pi'(\gamma) &= \int_{\mathrm{Geo}(X)} l^2(\gamma)\mathsf{d}\left((\hat{f})_\#\pi|_{\Gamma_1} + (\widehat{f^{-1}})_\#\pi|_{\Gamma_2}\right)(\gamma) \\ &= \int_{\mathrm{Geo}(X)} l^2 \circ \hat{f}(\gamma) \cdot \chi_{\Gamma_1}(\gamma)\mathsf{d}\pi(\gamma) + l^2 \circ \widehat{f^{-1}}(\gamma) \cdot \chi_{\Gamma_2}(\gamma)\mathsf{d}\pi(\gamma) \\ &= \int_{\mathrm{Geo}(X)} l^2(\gamma) \cdot (\chi_{\Gamma_1} + \chi_{\Gamma_2})(\gamma)\mathsf{d}\pi(\gamma) = \int_{\mathrm{Geo}(X)} l^2(\gamma)\mathsf{d}\pi(\gamma).\end{aligned}$$

□

Remark 1.2.7. The hypothesis in Lemma 1.2.5 can be we weakened. We may require the existence of the unique geodesic plan only for final measures satisfying $\mu_1 \ll \mathfrak{m}$ rather than for an arbitrary $\mu_1 \in \mathcal{P}_2(X)$. We can repeat the proof choosing as final measure

$$\mu_1 := \frac{1}{2}(\mathfrak{m}(B_r(x))^{-1}\mathfrak{m}|_{B_r(x)} + \mathfrak{m}(f(B_r(x)))^{-1}\mathfrak{m}|_{f(B_r(x))})$$

where $B_r(x) \subset X \setminus \mathrm{Fix}(f)$ is a sufficiently small ball.
Congruently, we can require that X is essentially non-branching rather than X having GTB.

Accordingly with the works of Gigli-Rajala-Sturm and Cavalleti-Mondino Theorem $\frac{1}{2}$.25 we obtain the next

Corollary 1.2.8. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching* $\mathsf{MCP}(K, N)$*-space and* $f \in \mathrm{ISO}(X)$*. If* $\mathfrak{m}(\mathrm{Fix}(f)) > 0$ *then* $f = \mathbb{I}$.

In particular, this holds true for RCD^**-spaces, essentially non-branching* CD^**-spaces, and essentially non-branching* CD*-spaces.*

Since a locally compact, complete length space is a geodesic space we have proved the following result, which provides a large class of examples of metric measure spaces that have smooth automorphism groups.

Theorem B. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a locally compact, length metric measure space. Assume that* X *has* GTB *or that it is essentially non-branching. Then condition* (a) *is satisfied. In particular, if* X *has well-behaved tangent cones then* G *is a Lie group.*

Theorems 1.2.4 and B imply the next

Corollary 1.2.9 (Automorphisms of $\mathsf{RCD}^*(K, N)$-spaces)**.** *Let* $K \in \mathbb{R}$*,* $N \in [1, \infty)$*, and* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}^*(K, N)$*-space. Then the groups* $\mathrm{ISO}(X)$ *and* $\mathrm{ISO}_{\mathfrak{m}}(X)$ *are Lie groups.*

More generally, we consider spaces satisfying different curvature-dimension conditions. Recall that an m.m. space is essentially non-branching and satisfies the CD^*-condition if and only if it satisfies the strong CD^*-condition. The corresponding statement for the CD-condition is valid as well. (For a comment on this see the second paragraph after Theorem $\frac{1}{2}$.25 in §$\frac{1}{2}$.)

Corollary 1.2.10 (Automorphisms of CD-, CD^*-, and MCP-spaces)**.** *Let K and N be as above. The groups $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ are Lie groups for essentially non-branching ($\mathsf{CD}(K,N)$-)$\mathsf{CD}^*(K,N)$-spaces and essentially non-branching $\mathsf{MCP}(K,N)$-spaces that have well-behaved tangents.*

Corollary 1.2.11 (Automorphisms of m.m. space with weak quantitative MCP)**.** *The groups $\mathrm{ISO}(X)$ and $\mathrm{ISO}_{\mathfrak{m}}(X)$ are Lie groups for essentially non-branching spaces with qualitatively non-degenerate measure $\mathfrak{m}$ that have well-behaved tangents.*

Example 1.2.12 ((Finsler) Ricci limit spaces)*.*

The compactness of the RCD^*-spaces in the pointed measured GH-topology and Corollary 1.2.9 assure that Ricci Limits, and (pointed) measured Gromov-Hausdroff limits of weighted manifolds with lower bounds on the Bakry-Emery Ricci tensor have automorphism groups that are smooth. By using strong results involving a certain type of connectedness of the regular set of Ricci Limit spaces, this conclusion was reached in the non-collapsed case by Cheeger-Colding [CC00] and in the collapsed case by Colding-Naber [CN12]. Our approach takes into consideration as well the group of measure-preserving isometries and *measured limits* of weighted manifolds.

Accordingly from Corollary 1.2.10 we conclude the corresponding result for spaces arising as limits of Finsler manifolds with a uniform lower bound on the weighted Ricci curvature (as the weighted trace of the flag curvature).

Corollary 1.2.10 brings up the following interesting, yet difficult-to-answer, question.

Question: Which m.m. spaces arise as tangents cones of MCP, CD^*, and CD spaces?

See Ketterer-Rajala [KR15] for a discussion in this direction in the case of MCP-spaces.

Example 1.2.13 (Busemann-Kell concave spaces)*.* In [Kel16] Kell defines a Finsler version of non-negative sectional curvature bounds for metric spaces by considering an analogous formulation to Busemann's non-positive curvature condition. In particular, Alexandrov spaces of non-negative curvature, and normed spaces are examples of this type. Also in [Kel16] it is shown that if the Hausdorff measure $\mathcal{H}^n$ of a complete Busemann-Kell concave space (Y, d) is non-trivial, then the metric measure space $(Y, \mathsf{d}, \mathcal{H}^n)$ is an $\mathsf{MCP}(K, n)$-space. Observe that $n \in \mathbb{N}$ and that tangent cones are unique if (Y, d) is finite dimensional. Since complete Busemann-Kell concave spaces are non-branching it follows that the isometry group is a Lie group for this class of spaces, granted that the measure $\mathcal{H}^n$ is not trivial.

To end the section we prove that not all curvature-dimension conditions are restrictive enough to guarantee the smoothness of the automorphism groups.

Proposition 1.2.14. *There exists an $\mathsf{MCP}(2,3)$-space for which neither $\mathrm{ISO}_{\mathfrak{m}}(X)$ nor $\mathrm{ISO}(X)$ are Lie groups.*

Example 1.2.15 (The fancy necklace)*.*

Given $n \in \mathbb{N}$, an n-necklace $(\mathcal{N}^n, \mathsf{d}_n, \mathfrak{m}_n) \subset \mathbb{R}^2$ is a m.m. space with n diamond-shaped figures, whose definition is inspired by a construction done by Ketterer and Rajala in [KR15]. A *fancy necklace* $(\mathcal{FN}, \mathsf{d}_{\mathcal{FN}}, \mathfrak{m}_{\mathcal{FN}})$ is then a measured GH-limit of a sequence of n-necklaces $\{(\mathcal{N}^n, \mathsf{d}_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$. We begin by defining inductively the sets $\mathcal{N}^n \subset \mathbb{R}^2$ and then endowing them with a metric measure structure.

Given a sequence $\{(r_n, x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ (consistency conditions will be specified below), for $k \in \mathbb{N}$, we write $I_k = [x_k - \frac{1}{4} r_k, x_k + \frac{1}{4} r_k]$ and the diamond-shaped sets:

$$D_k := \left\{(x, y) \in \mathbb{R}^2 \;\; \| \;\; |y| \leq \frac{1}{9}(\frac{1}{4}\, r_k - |x - x_k|)\right\}.$$

Set $\mathcal{N}^0 := [0, \pi/2] \times \{0\} \subset \mathbb{R}^2$. For $n \in \mathbb{N}$, construct the n-necklace $\mathcal{N}^n$ by replacing, in the $(n-1)$-necklace $\mathcal{N}^{n-1}$, the segment $I_n \times \{0\} \subset \mathcal{N}^{n-1}$ with the diamond D_n. (See Figure 1.2.2.) To have a consistent construction we require that the sequence $\{(r_n, x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ satisfies:

$$\begin{aligned} &0 < r_n \leq 1, \quad \frac{1}{4}\, r_n \leq x_n \leq \frac{\pi}{2} - \frac{1}{4}\, r_n, \text{ and} \\ &I_k \cap I_j = \emptyset \text{ for } k < j. \end{aligned} \tag{1.2.16}$$

The first condition assures that we have the correct size and scaling of our figures, while the second condition assures that different diamonds do not intersect.

We proceed now to give the n-necklace $\mathcal{N}^n$ a metric measure structure. For $n \in \mathbb{N} \cup \{0\}$, endow $\mathcal{N}^n$ with the distance, $\mathsf{d}_n = \mathsf{d}_{L^\infty}$, induced from the L^∞-norm in $\mathbb{R}^2$. To set a measure on the n-necklace we start by defining $\mathfrak{m}_{D_n} \ll \mathcal{L}^2$ on D_n by

$$\frac{\mathrm{d}\mathfrak{m}_{D_n}}{\mathrm{d}\mathcal{L}^2}(x) := \left[\frac{2}{9}(\frac{1}{4} r_n - |x - x_n|)\right]^{-1} \cos^2(x)\, \chi|_{D_n}(x) \qquad \text{for } n \in \mathbb{N},$$

and $\chi|_A$ the characteristic function of the set A. Denote by $\mathsf{D}^n = \cup_{1\leq k\leq n} D_k$ the union of all diamonds $D_k \subset \mathcal{N}^n$, by $\mathsf{L}^n := \mathcal{N}^n \setminus \mathsf{D}^n$ its complement, and we write $L^0 := \mathcal{N}^0$. We set on $\mathcal{N}^n$ the measure $\mathfrak{m}_n$ defined as

$$\begin{aligned} \mathrm{d}\mathfrak{m}_n &:= \mathrm{d}\mathfrak{m}_{\mathsf{D}^n} + \cos^2(x)\, \mathrm{d}\mathcal{H}^1|_{\mathsf{L}^n}, \qquad \text{where} \\ \mathfrak{m}_{\mathsf{D}^n} &:= \sum_{1\leq k\leq n} \mathfrak{m}_{D_k}. \end{aligned}$$

In words, the measure $\mathfrak{m}_n$ has a 2-dimensional contribution coming from D^n, which has constant density for fixed x-coordinate, and a 1-dimensional contribution coming from L^n, which is absolutely continuous w.r.t. the 1-dimensional Hausdorff measure. Finally, we define the fancy necklace as the measured Gromov-Hausdorff limit $(\mathcal{FN}, \mathsf{d}_{\mathcal{FN}}, \mathfrak{m}_{\mathcal{FN}}) := \mathsf{mGH}\text{-}\lim_{n\to\infty}(\mathcal{N}^n, \mathsf{d}_n, \mathfrak{m}_n)$. Since in Lemma 1.2.17 we will show that n-necklaces satisfy the $\mathsf{MCP}(2, 3)$-condition, for all $n \in \mathbb{N}$, the existence of the limit is guaranteed by the compactness of MCP-spaces.

It will be convenient to fix some notation before presenting our next Lemma. Given a sequence $\{(r_i, x_i)\}_{n\in\mathbb{N}}$ consider the m-necklace $(\mathcal{N}^m, \mathsf{d}_m, \mathfrak{m}_m)$ constructed from it. We will call "projected $(m-1)$-necklace", denoted by $(P\mathcal{N}_k^{m-1}, \mathsf{d}_{m-1}, \mathfrak{m}'_{m-1})$, the $(m-1)$-necklace constructed from the sequence $\{(r_i, x_i)\}_{i\neq k}$ for $1 \leq k \leq m$. That is, $P\mathcal{N}_k^{m-1}$ is the

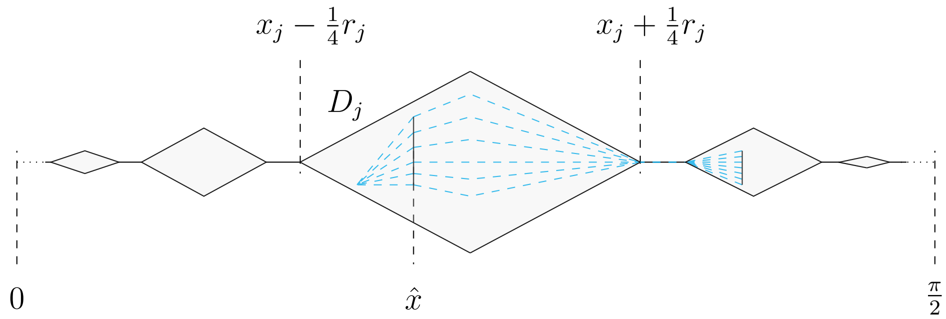

Figure 1.2.2: Fancy necklace $\mathcal{FN}$.

necklace with $(m-1)$ diamonds obtained by removing the kth-diamond from $\mathcal{N}^m$. The x-coordinate of vertices of diamonds D_n will be denoted by $x_n^{\pm} = x_n \pm 1/4\, r_n$, and for $x \in \mathcal{N}^n$ and $B \subset \mathcal{N}^n$, we define the height as $h(w, B) := \mathcal{H}^1(B \cap \{x = w\})$. Moreover, we define the following set of geodesics,

$$\Upsilon(B_0, B_1) \;\; := \;\; \big\{\gamma \in \mathrm{Geo}(\mathcal{N}^n) \,\|\, \gamma \text{ is a line segment with } \gamma_i \in B_i,\ i = 0, 1\big\}.$$

The set $\Upsilon(B_0, B_1)$ consists of Euclidean geodesics that go from B_0 to B_1. (Take into consideration that there exist many non-Euclidean geodesics in $(\mathcal{N}^m, \mathsf{d}_m, \mathfrak{m}_m)$.) Lastly, for $|y| \le r_k/36$ and $k \in \mathbb{N}$ define $\gamma^{k,y} \in \mathrm{Geo}(\mathcal{N}^n)$ as the geodesic obtained after gluing $\Upsilon((x_k^-, 0), (x_k, y))$ with $\Upsilon((x_k, y), (x_k^+, 0))$ and reparametrizing. The image of $\gamma^{k,y}$ is the union of a line segment going from the left vertex of D_k, $(x_k^-, 0)$, to (x_k, y) with its reflection over $\{x = x_k\}$. Define M^k as the set of all such geodesics for $|y| \le r_k/36$.

We are ready to prove our next lemma.

Lemma 1.2.17. *The m.m. space* $(\mathcal{FN}, \mathsf{d}_{\mathcal{FN}}, \mathfrak{m}_{\mathcal{FN}})$ *satisfies the* $\mathsf{MCP}(2,3)$*-condition.*

Proof. The stability of the MCP-condition assures that it's enough to show that $(\mathcal{N}^n, \mathsf{d}_n, \mathfrak{m}_n) \in \mathsf{MCP}(2,3)$ for every $n \in \mathbb{N}$ and every sequence $\{(x_k, r_k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ that satisfies (1.2.16). Accordingly, we fix $n \in \mathbb{N}$ and such sequence. We proceed using key ideas from a proof in [KR15].

Definition $\frac{1}{2}$.28 of the MCP condition requires that, for every $\tilde{z} = (\tilde{x}, \tilde{y})$ and $A \subset \mathcal{N}^n$ with $0 < \mathfrak{m}_n(A) < \infty$, we give a measure $\pi \in \mathcal{P}(\mathrm{Geo}(\mathcal{N}^n))$ such that $(\mathrm{e}_0)_\#\pi = \delta_{\tilde{z}}$, $(\mathrm{e}_1)_\#\pi = \mathfrak{m}_n(A)^{-1}\mathfrak{m}_n|_A$, and inequality $\frac{1}{2}$.29 is valid. Given $\tilde{z}$ and A we will choose a set of geodesics $\Gamma = \Gamma_{\tilde{z},A} \subset \mathrm{Geo}(\mathcal{N}^n)$ and define π as the optimal geodesic plan arising from the lift of the induced to the optimal transport going along geodesics in Γ. However, we reduce before the number of transports that we need to study.

To begin with, note that we can analyze separately the sets $A_{x'} = A \cap \{x = x'\}$ for a fixed x'. The simplification can be made because we will assure that the first coordinate contributes to the dilatation of the measure $\mathfrak{n}_t := (\mathrm{e}_t)_\#\pi$ a factor equal to t. We will achieve this by picking geodesics with projection $\mathrm{p}_1(\gamma(t)) = (1-t)\,\tilde{x} + t\,x'$ for $(x', y') = z' \in A$. Therefore the analysis reduces to estimating separately the dilatation of the sets $A_{x'}$ for every $x' \in \mathrm{p}_1(A)$. Accordingly, to verify the $\mathsf{MCP}(2,3)$-condition, it is enough to provide a set $\Gamma \subset \mathrm{Geo}(\mathcal{N}^n)$ such that $\mathrm{e}_0(\Gamma) = \tilde{z}$, $\mathrm{e}_1(\Gamma) \in A_{x'}$, and

$$\frac{\mathrm{d}\mathfrak{n}_t}{\mathrm{d}\mathfrak{m}_n}(\gamma_t) \leq \frac{\sin^2(l(\gamma))}{t\sin^2(t\,l(\gamma))}\frac{\mathrm{d}\mathfrak{n}_1}{\mathrm{d}\mathfrak{m}_n}(\gamma_1) \qquad \text{for all } t \in [0,1],\ x' \in \mathrm{p}_1(A),\ \gamma \in \Gamma. \tag{1.2.18}$$

Claim 1. It's sufficient to check that $(\mathcal{N}^m, \mathsf{d}_m, \mathfrak{m}_m) \in \mathsf{MCP}(2,3)$ *for* $m = 0, 1, 2$.

Proof. First note that if $\tilde{z}, z' \notin D_k$ for some $k \in \{1, ..., n\}$ then we can choose Γ in a way that makes the density of $\frac{\mathrm{d}\mathfrak{n}_t}{\mathrm{d}\mathfrak{m}_n}$ independent of $y \in \mathrm{p}_2(D_k)$, that is, $\frac{\mathrm{d}\mathfrak{n}_t}{\mathrm{d}\mathfrak{m}_n}((x,y)) = \frac{\mathrm{d}\mathfrak{n}_t}{\mathrm{d}\mathfrak{m}_n}((x,0))$ for $(x,y) \in D_k$. We can do this by choosing geodesics whose restriction to D_k is exactly the set M^k. This choice of Γ grants that the analysis of the transport of the measure inside $(\mathcal{N}^n, \mathsf{d}_n, \mathfrak{m}_n)$ is equivalent to the analysis of the transport of the measure inside the projected $(n-1)$-necklace $(P\mathcal{N}_k^{n-1}, \mathsf{d}_{n-1}, \mathfrak{m}'_{n-1})$. Furthermore, observe that if $n > 2$ there exist at least $(n-2)$ such diamonds, D_{k_i}, for every $\tilde{z}, z' \in \mathcal{N}^n$. Thus, for every transport inside $\mathcal{N}^n$ we can project at least $(n-2)$ times, reducing the task to checking the $\mathsf{MCP}(2,3)$-condition in the m-necklaces, for $m = 0, 1, 2$. ■

In [Stu06b] and [KR15] it is shown that the 0-necklace and 1-necklace satisfy the $\mathsf{MCP}(2,3)$-condition, this covers the cases of $m = 0, 1$ so we move to $m = 2$.[5] We assume, because of symmetry, that $\tilde{x} \leq x$ and fix $\tilde{z} \in \mathcal{N}^2$ and $A_{x'} \subset \mathcal{N}^2$. We conclude from the preceding claim that the only situation left to check is that of $\tilde{x} \in D_1$ and $x' \in D_2$. Let's first explain intuitively the way we transport the measure in this case. We start by expanding the measure uniformly from $\tilde{x}$ to a set $\hat{A}_{\hat{x}}$ with the same relative height as $A_{x'}$. Then we transport the measure from $\hat{A}_{\hat{x}}$ to x_1^+ without changing the relative height of the set $A_t := \mathrm{e}_t(\Gamma)$ with respect to $D_1 \cap \{x = \gamma_t\}$. We continue through L^2 and expand again keeping the heights ratio constant from x_2^- to $A_{x'}$. The image of a transporting geodesic is the union of segments of straight lines described below, see Figure 1.2.2. In detail, to define Γ first choose any set $\hat{A}_{\hat{x}} \subset D_1 \cap \{x = \hat{x}\}$ such that

$$\frac{h(\hat{x}, D_1)}{h(\hat{x}, \hat{A}_{\hat{x}})} = \frac{h(x', D_2)}{h(x', A_{x'})}, \tag{1.2.19}$$

for $\hat{x} = \frac{1}{5}(\frac{r_1}{4} + 4(\tilde{x} - x_1)) + x_1$. Write $\hat{t} := \frac{\hat{x} - \tilde{x}}{x - \tilde{x}}$, $t_1 := \frac{x_1^+ - \tilde{x}}{x - \tilde{x}}$, and $t_2 := \frac{x_2^- - \tilde{x}}{x - \tilde{x}}$ for the times at which the x-coordinate of any geodesic $\gamma \in \Upsilon(\tilde{x}, A_{x'})$ is equal to $\hat{x}$, x_1^+, and x_2^-. Geodesics in $\Upsilon(\tilde{x}, D_1 \cap \{x = \hat{x}\})$ have the same length. Now define Γ as the set of all geodesics satisfying the following: $\mathsf{rest}_0^{\hat{t}}(\gamma) \in \Upsilon(\hat{x}, \hat{A}_{\hat{x}})$, $\mathsf{rest}_{\hat{t}}^{t_1}(\gamma) \in M^1$, $\mathsf{rest}_{t_1}^{t_2}(\gamma) \in \Upsilon(x_1^+, x_2^-)$, and $\mathsf{rest}_{t_2}^1(\gamma) \in M^2|_{A_{x'}}$, where $M^2|_{A_{x'}}$ is the subset of geodesics of M^2 that cross through $A_{x'}$.

We now estimate the density of the corresponding measure, for $\gamma(t) = (x_t, y_t)$ we have that

$$\frac{\mathrm{d}\mathfrak{n}_t}{\mathrm{d}\mathfrak{m}}(\gamma_t) = \frac{1}{t}\frac{h(x_t, D_1)}{h(x_t, A_t)} = \frac{1}{t^2}\frac{h(x_t, D_1)}{h(\hat{x}, \hat{A}_{\hat{x}})}\frac{h(\hat{x}, D_1)}{h(\hat{x}, D_1)} = \frac{1}{t^2}\frac{h(x_t, D_1)}{h(x, D_1)}\frac{\mathrm{d}\mathfrak{n}_1}{\mathrm{d}\mathfrak{m}}(\gamma_1),$$

for $0 \leq t \leq \hat{t}$. The shape of the diamond D_1 allows to estimate $\frac{h(x_t, D_1)}{h(x, D_1)} \leq \left(\frac{5}{4} - \frac{t}{4}\right)$. We can bound the time when the geodesics reach $\hat{x}$ by $\hat{t} \leq r_1/5 \leq 1/5$, and the length of the geodesics is necessarily $l \leq \pi/2$. Moreover in [KR15] the estimate $\frac{5}{4} - \frac{t}{4} \leq t\,\frac{\sin^2(d)}{\sin^2(t\,d)}$

[5] More precisely, the proof in [KR15] can be repeated verbatim by doing minor modifications.

for all $(t, d) \in [0, 1/5] \times (0, \pi/2 + 1/4)$ is proved. Putting inequalities together we obtain inequality (1.2.18) for $t \in [0, \hat{t}]$.

To finish, note that for $t \in [\hat{t}, 1]$, the relative density of $\mathfrak{n}_t$ is independent of the y-coordinate. Thus, its density is equal to the one of the transport in the 0-necklace, which is a $\mathsf{MCP}(2, 3)$-space. This shows that inequality (1.2.18) is satisfied also for $t \in [\hat{t}, 1]$, hence, in the complete interval $t \in [0, 1]$. $\square$

Observe that the automorphism groups of $(\mathcal{FN}, \mathsf{d}_{\mathcal{FN}}, \mathfrak{m}_{\mathcal{FN}})$ are $\mathsf{G} = \Pi^\infty\{\pm 1\}$. This proves Proposition 1.2.14 and confirms that the measure contraction property, without extra assumptions, does not guarantee smoothness of the automorphism groups.

Chapter 2

Quotients and Ricci curvature

Stability of synthetic Ricci curvature lower bounds

In Theorem C we prove the stability of curvature-dimension conditions under quotients maps induced by *metric measure foliations*. We will see that *metric measure foliations* are particular types of foliations which preserve the structure of m.m. spaces in an *appropriate* fashion. To be precise in our statements let us first recall some notation.

We consider spaces satisfying a curvature-dimension condition with finite dimensional parameter; since these spaces are geodesic and proper, there is no loss of generality by making this assumption now.[1] Let $(X, \mathsf{d}, \mathfrak{m})$ be a geodesic proper m. m. space. A *partition* $\mathcal{F}$ of a metric measure space into closed subsets, called *leaves*, is called a *foliation*. The *quotient* space $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is the *space of leaves* endowed with the quotient distance and measure. Given that $\mathcal{F}$ is a metric foliation it follows that $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is a geodesic proper m.m. space. The quotient map $\mathsf{p} : X \to X^*$ maps points in X to their corresponding leaf. The Disintegration Theorem $\frac{1}{2}$.43 guarantees a disintegration of the measure $\mathfrak{m}$ over $\mathfrak{m}^*$ consistent with the quotient map p which we write as $\{\mathfrak{m}_{x^*}\}_{x^* \in X^*} \subset \mathcal{P}(X)$. We refer to the introductory §$\frac{1}{2}$ for more details.

We define the central objects under study in this chapter.

Definition 2.0.1. *[Bounded Metric Measure Foliation] A foliation $\mathcal{F}$ of $(X, \mathsf{d}, \mathfrak{m})$ is called a* bounded metric measure foliation*,* b.m.m. foliation *in short, if the next conditions are*

[1] A more precise statement is the following. The aforementioned assumptions will be used when we study the analysis of m.m. spaces in Subsection *Isometry of Wasserstein spaces* of §2.1. The results achieved in that Subsection will be used in RCD*-spaces of finite dimensional parameter, which are geodesic and proper.

satisfied:

$$\text{Leaves of } \mathcal{F} \text{ are bounded;}$$
$$\mathsf{d}_{\mathsf{H}}(F,G) = \mathsf{d}(x,G) \quad \text{for all } F,G \in \mathcal{F} \text{ and } x \in F; \text{ and} \tag{2.0.2}$$
$$W_2(\mathfrak{m}_{x*},\mathfrak{m}_{y*}) = \mathsf{d}^*(x^*,y^*) \quad \text{for } \mathfrak{m}^*\text{-a.e. } x^* \in X^*. \tag{2.0.3}$$

Metric foliations of Alexandrov spaces of curvature bounded below, and Riemannian foliations of Riemannian manifolds are examples of bounded metric measure foliations if leaves are bounded and the spaces are endowed with the Hausdorff measure. Many more examples of b.m.m. foliations arise from actions of compact Lie groups on m.m. spaces. As a matter of fact, b.m.m. foliations are precisely the foliations which are induced by submetries with bounded fibers that satisfy (2.0.3). We confirm these statements in Propositions 2.2.1 and 2.2.6.

In the remainder let $\mathcal{F}$ be a bounded metric measure foliation of the m.m. space $(X,\mathsf{d},\mathfrak{m})$. From now on all considered foliations are to be understood of this particular kind, even in the case in which we name them, for simplicity, just foliations. Furthermore, we say that a m.m. space $(X,\mathsf{d},\mathfrak{m})$ satisfies a *curvature-dimension condition* if it satisfies, for $K \in \mathbb{R}$ and $N \in [1,\infty)$, at least one of the following conditions: strong $\mathsf{CD}(K,N)$, strong $\mathsf{CD}(K,\infty)$, strong $\mathsf{CD}^*(K,N)$, $\mathsf{RCD}^*(K,N)$, or essentially non-branching $\mathsf{MCP}(K,N)$.

The main result of the chapter is the following.

Theorem C (Synthetic Ricci is stable under quotients)**.** Assume that $(X,\mathsf{d},\mathfrak{m})$ satisfies a curvature-dimension condition.
Then the quotient metric measure space $(X^*,\mathsf{d}^*,\mathfrak{m}^*)$ satisfies the same condition that $(X,\mathsf{d},\mathfrak{m})$ with the same parameters.

This result is one of the principal achievements of collaborative research of Fernando Galaz-García, Martin Kell, Andrea Mondino and the author, during which [GGKMS17] was assembled. Compare in particular with Theorem 8.8 and Corollary 8.10 within the reference and see Remark 2.0.4 below.

Additionally, from Proposition 2.2.1 and Proposition 2.2.6 we obtain the next results. A bounded metric measure submetry is a submetry that satisfies condition (2.0.3), see Definition 2.2.4.

Theorem D (Synthetic Ricci is stable: group actions [GGKMS17, Theorems 3.7 and 6.2])**.** Let G be a compact Lie group that acts by isomorphisms on $(X,\mathsf{d},\mathfrak{m})$. Assume that $(X,\mathsf{d},\mathfrak{m})$ satisfies a curvature-dimension condition.
Then the quotient metric measure space $(X/\mathsf{G},\mathsf{d}^*,\mathfrak{m}^*)$ satisfies the same condition that $(X,\mathsf{d},\mathfrak{m})$ with the same parameters.

Theorem E (Synthetic Ricci is stable: bounded m.m. submetries)**.** Curvature-dimension conditions are preserved with the same parameters by bounded metric measure submetries.

Remark 2.0.1 (Applications)**.** In [GGKMS17] results concerning geometric applications of Theorems C and D are studied. We rapidly mention some of these results, despite the fact that the aim of this section is different, to illustrate the practicality of the theorems themselves (precise statements are to be found in [GGKMS17]). Examples are: a generalization of Kobayashi's Classification Theorem of homogenous manifolds to $\mathsf{RCD}^*(K,N)$-spaces with essential infinitesimal dimension $n<N$; a structure theorem for $\mathsf{RCD}^*(K,N)$-spaces admitting actions by large compact groups; and geometric rigidity results for orbifolds such as Cheng's Maximal Diameter and Maximal Volume rigidity Theorems.

Remark 2.0.2 (Construction of new examples)**.** As a consequence of Theorems C, D, E we can enlarge the list of examples of $\mathsf{RCD}^*(K,N)$-spaces to include quotients by isometric group actions and foliations of Riemannian manifolds with $\mathrm{Ric}_g \geq Kg$, and more generally, of $\mathsf{RCD}^*(K,N)$-spaces. The analogous remarks are also valid for CD, CD^*, and MCP spaces.

Remark 2.0.3 (Extensions)**.** In Definition 2.0.1 it is possible to consider more general leaves, however, the measure $\mathfrak{m}^*$ might not be σ-finite nor unique. For this one may replace $\{\mathfrak{m}_{x^*}\}_{x^*\in X^*}$ (which serves as a natural lift for the family of measures $\delta_{x^*}\in\mathcal{P}(X^*)$) by a family of measures $\{\nu_{x^*}\}_{x^*\in X^*}$ supported on the corresponding leaves such that Equation (2.0.3) is satisfied, and whose naturally defined lifts preserve the entropy up to a fixed constant. For instance, this can be done in a obvious manner when fundamental domains of positive $\mathfrak{m}$-measure exist.

The same arguments that we present here show that the strong $\mathsf{CD}_p(K,N)$-condition defined by Kell in [Kel13] is satisfied by quotient spaces of strong $\mathsf{CD}_p(K,N)$-spaces. Additionally, granted that the group G is finite, also the intermediate p-Ricci lower curvature bounds in terms of optimal transport introduced by Ketterer and Mondino in [KM16] are preserved under quotients.

The isometry of Sobolev spaces shown in Proposition 2.1.8 can be extended to more general m.m. spaces, for example infinite dimensional spaces. See the next Remark for more on this direction.

Remark 2.0.4 (The $\mathsf{RCD}(K,\infty)$ case)**.** In this work we chose to present an alternative version of Theorems 3.7, 6.2, 8.8 and Corollary 8.10 in [GGKMS17] (corresponding to Theorems C and D here), the difference being that here we do not consider $\mathsf{RCD}(K,\infty)$-spaces.

A reason for this choice lies in that the work regarding analysis in m.m. spaces is simplified while retaining the fundamental idea: to show an isometric embedding of the Sobolev space $W^{1,2}(\mathfrak{m}^*)$ of the quotient space onto the Sobolev space $W^{1,2}(\mathfrak{m})$ of the original space. This is because $\mathsf{RCD}(K,\infty)$-spaces don't necessarily satisfy a doubling or Poincaré condition, conditions which improve significantly analysis in m.m. spaces. Moreover, we are able to maintain all other parts of the proof essentially the same (modulo walking through the proof in a different manner). Another reason is that [GGKMS17] had the purpose to show the aforementioned Sobolev embedding in a more general setting as a result of interest by it's own right; here our aim is different. One last argument, more from the motivational point of view, is that our Theorems A and B do not show directly

that $\mathsf{RCD}(K,\infty)$-spaces have smooth isomorphism groups, which was one of the initial motivations to approach this problem.
Summarizing, we opted for a lighter version since there is no loss of ideas although we might profit in clarity from a simpler, less general, exposition: we will be satisfied with proving Theorem C.

2.1 Proof of Main Theorem C

The argument is divided as we explain now.

i Define and show useful properties of *particular lifting maps* of relevant objects on the quotient space X^* to the respective type of objects in the original space X.

ii Show invariance of Wasserstein geometry under *lifts*.

iii Show invariance of Sobolev spaces under *lifts*.

iv Conclude Theorem C.

To reach the conclusion of the theorem we consider Wasserstein geodesics in the space of probability measures on the quotient space and their lifts to Wasserstein geodesics on the original space by using point ii. Since, by hypothesis, the condition of convexity of entropy is valid for the lifted geodesics, we will obtain the curvature-dimension inequalities for the Wasserstein geodesic on the quotient after showing that taking the Radon-Nikodym derivative and lifting commutes with lifting measures and taking the Radon-Nikodym derivative. This is roughly the way that we show the stability of the CD/CD*/MCP conditions. Subsequently, we extend the results to RCD*-spaces by relaying on point iii.

Lifts

Our objective is to compare optimal transport and analysis on the quotient space against corresponding notions in the original space. The first step is to define a way to lift objects of interest from the quotient, to the analogous type of objects in the original space. It will suffice to consider functions, measures and couplings.

In order to define *lifting maps* we begin by choosing certain optimal couplings between elements of the disintegration $\{\mathfrak{m}_{x^*}\}_{x^*\in X^*}$ of the measure $\mathfrak{m}$. Accordingly, we choose for every $x^*, y^* \in X^*$ an optimal plan π_{x^*,y^*} for $\mathfrak{m}_{x^*}$ and $\mathfrak{m}_{y^*}$,[2] that is to say,

$$\pi_{x^*,y^*} \in \mathsf{OptAdm}(\mathfrak{m}_{x^*}, \mathfrak{m}_{y^*}).$$

[2]It's important to say that this choice can be made in a measurable fashion; the following argument substantiates. Let $\pi_{x,y}$ be an optimal coupling between $\mathfrak{m}_{x^*}$ and $\mathfrak{m}_{y^*}$ for every $(x,y) \in \mathcal{OD}$. Condition (2.0.2) and the boundedness of leaves (in fact compactness since the space is proper by hypothesis) guarantee that, for every $x \in \mathcal{F}_{x^*}$, we can find a $y \in \mathcal{F}_{y^*}$ such that $(x,y) \in \mathcal{OD}$. It follows by a measurable selection argument that there exists a measurable assignment $(x^*,y^*) \mapsto (x'_{(x^*,y^*)}, y'_{(x^*,y^*)}) \in \mathcal{OD} \cap (\mathcal{F}_{x^*} \times \mathcal{F}_{y^*})$ and that $(x^*,y^*) \mapsto \pi_{(x'_{(x^*,y^*)}, y'_{(x^*,y^*)})}(A)$ is measurable for every measurable set $A \in X \times X$. We write $\pi_{x^*,y^*} = \pi_{(x'_{(x^*,y^*)}, y'_{(x^*,y^*)})}$.

From the inclusion $\mathrm{supp}(\pi_{x^*,y^*}) \subset \mathrm{supp}(\mathfrak{m}_{x^*}) \times \mathrm{supp}(\mathfrak{m}_{y^*})$ and the definition of the quotient distance d^* it follows that for all $x^*, y^* \in X^*$

$$W_2(\mathfrak{m}_{x^*}, \mathfrak{m}_{y^*})^2 = \int_{\mathcal{F}_{x^*} \times \mathcal{F}_{y^*}} \mathsf{d}(w,z)^2 \, d\pi_{x^*,y^*}(w,z) \geq \mathsf{d}^*(x^*, y^*)^2, \tag{2.1.1}$$

which, in view of assumption (2.0.3), implies that π_{x^*,y^*} is concentrated on $\mathcal{OD} \cap \big(\mathcal{F}_{x^*} \times \mathcal{F}_{y^*}\big)$. Where we have written $\mathcal{OD}$ for the subset of pairs of points in $X \times X$ that achieve the distance between leaves:

$$\mathcal{OD} := \{(x,y) \in X \times X \,|\, \mathsf{d}(x,y) = \mathsf{d}^*(x^*, y^*)\}. \tag{2.1.2}$$

Definition 2.1.3 (Lifting maps $\Lambda^F, \Lambda^M, \Lambda^\Pi$)**.** *Consider the following functions:*

$$\begin{aligned} \mathbf{\Lambda^F} : \{g \,|\, g : X^* \to \mathbb{R} \cup \{-\infty\}\} \ &\to \ \{h \,|\, h : X \to \mathbb{R} \cup \{-\infty\}\} \\ f \ &\to \ \hat{f}(x) := \Lambda^F(f)(x) = f \circ \mathsf{p}\,(x); \end{aligned} \tag{2.1.4}$$

$$\begin{aligned} \mathbf{\Lambda^M} : \mathcal{P}(X^*) \ &\to \ \mathcal{P}(X) \\ \mu \ &\to \ \hat{\mu} := \Lambda^M(\mu) = \int_{X^*} \mathfrak{m}_{x^*} \, d\mu(x^*); \end{aligned} \tag{2.1.5}$$

$$\begin{aligned} \mathbf{\Lambda^\Pi} : \mathcal{P}(X^*) \times \mathcal{P}(X^*) \ &\to \ \mathcal{P}(X) \times \mathcal{P}(X) \\ \pi \ &\to \ \hat{\pi} := \Lambda^\Pi(\pi) := \int_{X^* \times X^*} \pi_{x^*,y^*} \, d\pi(x^*, y^*). \end{aligned} \tag{2.1.6}$$

We denote by convention, abusing the notation, the image of an element under any lift Λ^* by placing a hat "$\hat{\ }$" over the element. For example, the images of a function $g : X^* \to \mathbb{R}$, a measure $\nu \in \mathcal{P}(X^*)$, and a coupling $\rho \in \mathcal{P}(X^*) \times \mathcal{P}(X^*)$ will be denoted by $\hat{f}$, $\hat{\nu}$, and $\hat{\rho}$ respectively. Moreover, since the risk of confusion is low, we will sometimes simply write Λ for any of the functions just defined.

Below we see that the maps defined in 2.1.3 enjoy advantageous properties (including the fact that they are actual *lifts*).

Lemma 2.1.1 (Properties of lifts Λ^F, Λ^M, and Λ^Π)**.** *Let $f : X^* \to \mathbb{R} \cup \{-\infty\}$, $\mu, \nu \in \mathcal{P}(X^*)$, and $\pi \in \mathcal{P}(X^*) \times \mathcal{P}(X^*)$ and denote by $\hat{f}$, $\hat{\mu}$, $\hat{\nu}$, $\hat{\pi}$ their respective lifts. Then the following holds:*

1. *The function $\tilde{f} : X^* \to \mathbb{R} : x^* \mapsto \hat{f}(x)$, with $x \in \mathsf{p}^{-1}(x^*)$, is well-defined and it coincides with f.*
   Moreover, a c_2-concave function $\phi = \psi^{c_2}$ is lifted to a c_2-concave function such that $\hat{\phi} = \hat{\psi}^{c_2}$.

2. *The pushforward of a lifted measure coincides with such measure: $\mathsf{p}_\sharp(\hat{\nu}) = \nu$.*
   Moreover, an absolutely continuous measure (with respect to $\mathfrak{m}^*$) $\mu = f\,\mathfrak{m}^*$ is lifted to an absolutely continuous (with respect to $\mathfrak{m}$) with Radon-Nikodym derivative $\mathfrak{m}$-almost everywhere equal to $\hat{f}$.

3. *The pushforward of a lifted coupling coincides with such coupling:* $(\mathsf{p}\times\mathsf{p})_\sharp(\hat\pi)=\pi$*. Moreover, the lift of* $\pi\in\mathsf{Adm}(\mu,\nu)$ *is an admissible coupling for the corresponding lifted measures* $\hat\pi\in\mathsf{Adm}(\hat\mu,\hat\nu)$*.*

Proof. 1.) The validity of the first part of *1.* is clear by the definition of Λ^F. To show the second part we let $\psi: X^*\to\mathbb{R}\cup\{-\infty\}$ be a c_2-concave function, i.e. such that $\phi=\psi^{c_2}$. Then

$$\inf_{y\in X}\left(\mathsf{d}(x,y)^2-\hat\psi(y)\right)=\inf_{y^*\in X^*}\left(\mathsf{d}^*(x^*,y^*)^2-\psi(y^*)\right)=\phi(x^*)=\hat\phi(x),$$

which shows that $\hat\phi$ is c_2-concave and that $\hat\phi=\hat\psi^{c_2}$.

2.) We verify the first part of the statement using the definition of Λ^M and of the pushforward. Indeed, for any measurable set $A\in\mathcal{B}(X^*)$ it holds that

$$\mathsf{p}_\sharp(\hat\nu)(A)=\int_{X^*}\mathfrak{m}_{x^*}(\mathsf{p}^{-1}(A))d\nu(x^*)=\int_{X^*}\chi_A(x^*)d\nu(x^*)=\nu(A).$$

For the second part of *2.* we let $\mu\in\mathcal{P}^{ac}(X^*)$ be an absolutely continuous measure with respect to $\mathfrak{m}^*$ and write $\mu=f\,\mathfrak{m}^*$. We readily check that for $\hat f(x):=f(\mathsf{p}(x))$ and every measurable set B it holds that

$$\begin{aligned}\hat\mu(B)=\int_{X^*}\int_B d\mathfrak{m}_{x^*}(x)d\mu(x^*)=\int_{X^*}\int_B f(x^*)d\mathfrak{m}_{x^*}(x)d\mathfrak{m}^*(x^*)=\\ \int_{X^*}\int_B \hat f(x)d\mathfrak{m}_{x^*}(x)d\mathfrak{m}^*(x^*)=\int_B\hat f(x)d\mathfrak{m}(x).\end{aligned}\tag{2.1.7}$$

The first equality is the definition of the lift Λ^μ (2.1.5), while the second and third are given by the absolute continuity of μ and definition of $\hat f$ respectively. The last equivalence is just a consequence of the disintegration of the measure $\mathfrak{m}$. In particular we have showed that the Radon-Nikodym derivative of $\hat\mu$ with respect to $\mathfrak{m}$ is $\hat f$, i.e. $\hat\mu=\hat f\,\mathfrak{m}$.

3.) The beginning of the statement is proved analogously to the begging of *2.* Therefore all that is left to check is that a coupling $\pi\in\mathsf{Adm}(\mu,\nu)$ between the measures μ and ν is lifted to an admissible coupling between the lifts $\hat\mu$ and $\hat\nu$. We let $A\in\mathcal{B}(X^*)$ and compute

$$\begin{aligned}\hat\pi(A\times X) &= \int_{X^*\times X^*}\pi_{x^*,y^*}(A\times X)d\pi(x^*,y^*)=\\ &\int_{X^*\times X^*}\mathfrak{m}_{x^*}(A)d\pi(x^*,y^*)=\int_{X^*}\mathfrak{m}_{x^*}(A)d\mu(x^*)=\hat\mu(A)\end{aligned}$$

This computation shows that $\mathsf{p}_1(\hat\pi)=\hat\mu$. The definitions of Λ^Π and of Λ^M explain the first and last equalities. After recalling that by construction $\pi_{x^*,y^*}\in\mathsf{Adm}(\mathfrak{m}_{x^*},\mathfrak{m}_{y^*})$ and that $\pi\in\mathsf{Adm}(\mu,\nu)$ it's clear that the remaining equalities are valid. With a similar computation the corresponding result for the second marginal is checked, hence, we obtain that $\hat\pi\in\mathsf{Adm}(\hat\mu,\hat\nu)$. □

Isometry of Wasserstein spaces

Proposition 2.1.8 (Λ^M is an Isometry)**.** *A lift Λ^M satisfying (2.1.5) is an isometric embedding into its image that preserves absolutely continuous measures.*
Specifically, for $\mu_0, \mu_1 \in \mathcal{P}(X^*)$, the map $\Lambda^M : \mathcal{P}(X^*) \hookrightarrow \Lambda^M(\mathcal{P}(X))$ satisfies:

1. $\Lambda^M(\mathcal{P}^{ac}(X^*)) = \mathcal{P}^{ac}(X) \cap \Lambda^M(\mathcal{P}(X^*))$;
2. $\Lambda^M(\mathcal{P}_2(X^*)) = \mathcal{P}_2(X) \cap \Lambda^M(\mathcal{P}(X^*))$;
3. $W_2(\hat{\mu}_0, \hat{\mu}_1) = W_2(\mu_0, \mu_1)$ *whenever* $\mu_0, \mu_1 \in \mathcal{P}_2(X^*)$.

In particular, lifts of W_2-geodesics in $\mathcal{P}_2(X^*)$ are W_2-geodesics in $\mathcal{P}_2(X)$.

Proof. The first claim is included in point *2.* of Lemma 2.1.1 so we move on to show that $\Lambda^\mu(\mathcal{P}_2(X^*)) \subset \mathcal{P}_2(X)$. Let $\mu_0, \mu_1 \in \mathcal{P}_2(X^*)$ and let $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$ be an optimal coupling between the pair. By construction the lift $\hat{\pi} \in \mathsf{Adm}(\hat{\mu}_0, \hat{\mu}_1)$ is an admissible coupling for the lifted pair $\hat{\mu_0}, \hat{\mu_1} \in \mathcal{P}(X)$, therefore, it holds that

$$\begin{aligned} W_2(\hat{\mu}_0, \hat{\mu}_1)^2 &\leq \int_{X \times X} \mathsf{d}(x, y)^2 d\hat{\pi}(x, y) \\ &= \int_{X^* \times X^*} \mathsf{d}^*(x^*, y^*)^2 d\pi(x^*, y^*) = W_2(\mu_0, \mu_1)^2. \end{aligned} \tag{2.1.9}$$

Where the first equality is verified using the definition of the lift $\hat{\pi}$ and the fact that $\operatorname{supp}(\pi_{x^*,y^*}) \subset \mathcal{OD} \cap \{\mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(y^*)\}$. Using inequality (2.1.9) and the fact that the leaves of the foliation are bounded we are able to show that the lifted measures $\hat{\mu}_i$ have finite second moment, for $i = 0, 1$,

$$\begin{aligned} \left(\int_X \mathsf{d}(x, y)^2 d\hat{\mu}_i(x)\right)^{1/2} &\leq W_2(\hat{\mu}_i, \mathfrak{m}_{y^*}) + W_2(\mathfrak{m}_{y^*}, \delta_y) \leq W_2(\mu_i, \delta_{y^*}) + \max_{y' \in \mathsf{p}^{-1}(y^*)} \mathsf{d}(y', y) \\ &= \int_{X^*} \mathsf{d}(x^*, y^*)^2 d\mu_i(x^*) + \max_{y' \in \mathsf{p}^{-1}(y^*)} \mathsf{d}(y', y) < \infty. \end{aligned}$$

This allows comparing Wasserstein distances between the original and the lifted pair of measures. With this aim, we let (ϕ, ψ) be a dual solution to $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$. Specifically, let

$$\phi(x^*) + \psi(y^*) \leq \mathsf{d}^*(x^*, y^*)^2 \quad \text{for all } x^*, y^* \in X^*, \tag{2.1.10}$$

and equality realized for π-almost every $(x^*, y^*) \in X^* \times X^*$. It turns out that the lifted pair $(\hat{\phi}, \hat{\psi})$, defined in (2.1.4), is an admissible pair for the dual transport problem with marginals $(\hat{\mu}_0, \hat{\mu}_1)$. Indeed, the integrability of $(\hat{\phi}, \hat{\psi})$ follows from the definition of the lift Λ^μ and the integrability of (ϕ, ψ). Moreover, we are able to verify that

$$\hat{\phi}(x) + \hat{\psi}(y) = \phi(x^*) + \psi(y^*) \leq \mathsf{d}^*(x^*, y^*)^2 \leq d(x, y)^2 \quad \text{for all } x, y \in X.$$

The right inequality is given by the definition of the quotient metric d^*. Therefore, we have that the next inequality is satisfied

$$\begin{aligned} W_2(\hat{\mu_0}, \hat{\mu_0})^2 &\geq \int_X \hat{\phi}(x) d\hat{\mu_0}(x) + \int_X \hat{\psi}(y) d\hat{\mu_1}(y) \\ &= \int_{X^*} \phi(x^*) d\mu_0(x^*) + \int_{X^*} \psi(y^*) d\mu_1(y^*) = W_2(\mu_0, \mu_1)^2. \end{aligned} \tag{2.1.11}$$

Hence we conclude Claim *3.* from inequalities (2.1.9) and (2.1.11). ☐

As a matter of fact, we have also shown that optimal plans and dual solutions are preserved by lifts since in this case the inequalities in (2.1.9) and (2.1.11) turn out, a posteriori, to be equalities.

Corollary 2.1.12. *Let $\mu_0, \mu_1 \in \mathcal{P}_2(X^*)$. Then for every optimal coupling $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$ the lifted coupling $\hat{\pi}$ defined in (2.1.6) is an optimal coupling of the lifted pair of measures $\hat{\pi} \in \mathsf{OptAdm}(\hat{\mu}_0, \hat{\mu}_1)$ for which $\mathsf{d}(x, y) = \mathsf{d}^*(x^*, y^*)$ for $\hat{\pi}$-almost every $(x, y) \in X \times X$.*

Furthermore, if (ϕ, ψ) is a dual solution corresponding to π then the lift $(\hat{\phi}, \hat{\psi})$ is a dual solution corresponding to $\hat{\pi}$ for which $\hat{\psi}(x) + \hat{\psi}(y) = \mathsf{d}(x, y)^2$ for $\hat{\pi}$-almost every $(x, y) \in X \times X$.

The following corollary is also concluded.

Corollary 2.1.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be essentially non-branching. Then the quotient space $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is essentially non-branching.*

Proof. Note that a branching geodesic in X^* lifts to a family of branching geodesics in X. Moreover, since absolutely continuous measures are lifted to absolutely continuous measures and any optimal dynamical coupling on X^* lifts to an optimal dynamical coupling on X, we see that any $\gamma \in \mathrm{OptGeo}(\mu_0, \mu_1)$ between $\mu_i \in \mathcal{P}_2^{ac}(X^*)$, $i = 0, 1$, must be concentrated on a set of non-branching geodesics. ☐

Remark 2.1.2. A result in the direction of Proposition 2.1.8, in the case that the foliation arises from a compact group action, had already been shown by Lott and Villani in [LV09, Lemma 5.36]. In comparison to their work, in addition to considering general bounded foliations, we have been more explicit in the construction of lifts of measures and optimal plans. This allows us to show that the natural lifts of dual solutions are dual solutions as well, Corollary 2.1.12. This information is necessary in the proof of Theorem C which in comparison to [LV09, Lemma 5.36] drops the assumption on the compactness of the space, considers general $K \in \mathbb{R}$ for finite N, and takes into account the RCD^*-condition.

Isometry of Sobolev spaces

We prove that, under our assumptions, the Sobolev space $W^{1,2}(\mathfrak{m}^*)$ on the quotient is isomorphic to the closed subspace of $W^{1,2}(\mathfrak{m})$-Sobolev functions on X which are constant on each leaf up to null-measure sets.

Proposition 2.1.14 (Embedding of $W^{1,2}$). *Let $(X, \mathsf{d}, \mathfrak{m})$ and $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ be locally doubling m.m. spaces which admit a Poincaré inequality. Then,*

1. *The lift*
$$\Lambda^F : W^{1,2}(\mathfrak{m}^*) \to W^{1,2}(\mathfrak{m}) \cap \Lambda^F(W^{1,2}(\mathfrak{m}^*))$$
is an isometric embedding whose image is the set of functions in $W^{1,2}(\mathfrak{m})$ which are $\mathfrak{m}_{x^*}$-almost everywhere constant on $\mathfrak{m}^*$-almost every leaf. Moreover, $\mathrm{Ch}_2^X(\hat{f}) = \mathrm{Ch}_2^{X^*}(f)$ holds for all $f \in D(\mathrm{Ch}_2^{X^*})$.

2. *$(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is infinitesimally Hilbertian granted that $(X, \mathsf{d}, \mathfrak{m})$ is so as well.*

We prove first the following auxiliary lemma.

Lemma 2.1.3 (More properties of Λ^F). *The following holds:*

1. *The upper asymptotic Lipschitz constant is preserved when lifting. That is, for all functions $f : X^* \to \mathbb{R}$ and all $x \in X$ it holds that*
$$\mathrm{Lip}\, \hat{f}(x) = \mathrm{Lip}\, f(x^*). \tag{2.1.15}$$
In particular, this implies that $\Lambda(\mathrm{LIP}(X^*, \mathsf{d}^*)) \subset \mathrm{LIP}(X, \mathsf{d})$.

2. *Λ^F is an isometric embedding of $L^2(\mathfrak{m}^*)$ into $L^2(\mathfrak{m})$, with image the convex closed subset of $\mathfrak{m}_{x^*}$-almost everywhere constant functions on $\mathfrak{m}^*$-almost every leaf.*

Proof. Observe that for a function $f : X \to \mathbb{R}$ and its lift $\hat{f} : X \to \mathbb{R}$ the next identity is valid
$$\sup_{y \in B_r(x)} \frac{|\hat{f}(y) - \hat{f}(x)|}{r} = \sup_{y^* \in B_r(x^*)} \frac{|f(y^*) - f(x^*)|}{r}. \tag{2.1.16}$$
Indeed, by definition of quotient metric, we have that $y^* \in B_r(x^*)$ if $y \in B_r(x)$ and moreover, if $y^* \in B_r(x^*)$ there exists a $y \in B_r(x) \cap \mathsf{p}^{-1}(y^*)$. Then the definition of the lifted function $\hat{f}$ shows the validity of the above identity. The first part of the lemma is shown directly using this last identity and the definition of the upper asymptotic constant $\mathrm{Lip}\, f$. Moreover, since (X, d) is geodesic (2.1.15) implies that Lipschitz functions of (X^*, d^*) lift to Lipschitz functions in (X, d).

For the second point of the lemma we let $f \in L^2(\mathfrak{m}^*)$. Observe that every function $\tilde{f}$ that agrees $\mathfrak{m}$-almost everywhere with the lift $\hat{f}$ is $\mathfrak{m}_{x^*}$-almost everywhere constant on $\mathfrak{m}^*$-almost every leaf. Otherwise there would exist a set of positive measure $A^* \subset X^*$ and, for every $x^* \in X^*$, subsets $A_{x^*} \subset \mathsf{p}^{-1}(x^*)$ of positive $\mathfrak{m}_{x^*}$-measure for which $\tilde{f} \neq \hat{f}$ which contradicts the assumption. To conclude notice that by definition of the quotient measure $\mathfrak{m}^* = \mathsf{p}_\sharp \mathfrak{m}$ it holds that
$$\|f\|^2_{L^2(\mathfrak{m}^*)} = \int_{X^*} f^2(x^*) d\mathfrak{m}^*(x^*) = \int_X (f \circ \mathsf{p})^2(x) d\mathfrak{m}(x) = \|\hat{f}\|^2_{L^2(\mathfrak{m})}.$$
□

We now show that Λ^F is an isometric embedding.

Proposition 2.1.14. In [Che99] Cheeger showed that under the assumptions of Proposition 2.1.14 the upper asymptotic Lipschitz constant and the minimal relaxed slope agree $\mathfrak{m}/\mathfrak{m}^*$-almost everywhere for every locally Lipschitz function in $W^{1,2}(\mathfrak{m})/W^{1,2}(\mathfrak{m}^*)$ respectively. Specifically,

$$\begin{aligned} \mathrm{Lip}^{X^*} g\,(x^*) = |\nabla^{X^*} g|_2(x^*) \quad & \mathfrak{m}^*\text{-a.e. } x^* \in X, \text{ for every locally Lipschitz } f \in W^{1,2}(\mathfrak{m}^*), \\ \mathrm{Lip}^{X} h\,(x) = |\nabla^{X} h|_2(x) \quad & \mathfrak{m}\text{-a.e. } x \in X, \text{ for every locally Lipschitz } h \in W^{1,2}(\mathfrak{m}). \end{aligned}$$

Let $f \in W^{1,q}(\mathfrak{m}^*) \cap \mathrm{LIP}(X^*, \mathsf{d}^*)$. From *Claim 1.* Lemma 2.1.3 we now that the upper asymptotic Lipschitz constant is preserved by lifts, therefore

$$2\,\mathrm{Ch}_2^{X^*}(f) = \int_{X^*} \mathrm{Lip}^{X^*} f\,(x^*)^2 d\mathfrak{m}^*(x^*) = \int_X \mathrm{Lip}^X \hat{f}\,(x)^2 d\mathfrak{m}(x) = 2\,\mathrm{Ch}_2^X(\hat{f}),$$

since Lipschitz functions in X^* lift to Lipschitz functions in X (*Claim 1.* Lemma 2.1.3). By *Claim 2.* Lemma 2.1.3 , we have $\|f\|_{L^2(\mathfrak{m}^*)} = \|\hat{f}\|_{L^2(\mathfrak{m})}$ thus,

$$\|f\|^2_{W^{1,2}(\mathfrak{m}^*)} = \|f\|^2_{L^2(\mathfrak{m}^*)} + \mathrm{Ch}_2^{X^*}(f) = \|\hat{f}\|^2_{L^2(\mathfrak{m})} + \mathrm{Ch}_2^X(\hat{f}) = \|\hat{f}\|^2_{W^{1,2}(\mathfrak{m})}.$$

Hence $W^{1,2}(\mathfrak{m}^*) \cap \mathrm{LIP}(X^*, \mathsf{d}^*)$ is isometric to the subspace of functions in $W^{1,2}(\mathfrak{m}) \cap \mathrm{LIP}(X, \mathsf{d})$ which are constant on leaves since any such function is a lift of a function in $W^{1,2}(\mathfrak{m}^*) \cap \mathrm{LIP}(X^*, \mathsf{d}^*)$. In view of Cheeger's work [Che99], the assumptions also imply that Lipschitz functions are dense in $W^{1,2}(\mathfrak{m})$ then the result is concluded using a standard approximation argument.
The second part of the proposition follows directly using the parallelogram characterization ($\frac{1}{2}$.10) of the infinitesimal Hilbertian property. □

In fact it is known that strong $\mathsf{CD}/\mathsf{CD}^*(K,N)$-spaces are locally doubling and Poincaré spaces, for $K \in \mathbb{R}, N \in [1,\infty)$, as shown in [BS10, Raj12].

Corollary 2.1.17. *Suppose that $(X, \mathsf{d}, \mathfrak{m})$ and $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ satisfy the $\mathsf{CD}/\mathsf{CD}^*$-condition with possibly different parameters but for some finite N, N^*. Then the hypotheses of Proposition 2.1.14 are fulfilled.*

Stability

We prove that convexity of the entropy in Wasserstein spaces is stable under quotients.

Main Theorem C. The proof for the $\mathsf{CD}/\mathsf{CD}^*(K,N)$-condition with $K \in \mathbb{R}, N \in [1,\infty]$, is very similar and therefore we only write a proof for the strong $CD(K,N)$ condition for finite N. Moreover, observe that once proven that the quotient space inherits from the original space the aforementioned conditions we can conclude that, for $N < \infty$, $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is an $\mathsf{RCD}^*(K,N)$-space granted that $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{RCD}^*(K,N)$ condition. Indeed, this follows from Corollary 2.1.17 and Proposition 2.1.14 which assure that infinitesimal Hilbertianity passes on to the quotient space.

Accordingly, let $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X^*)$ be measures on X^* and denote by $\{\mu_t\}_{t\in[0,1]} \in \mathcal{P}(X^*)$ a Wasserstein geodesic and by $\pi \in \mathsf{OptAdm}(\mu_0, \mu_1)$ an optimal coupling induced by $\{\mu_t\}_{t\in[0,1]}$.
The isometry $\Lambda^M : \mathcal{P}_2(X^*) \to \mathcal{P}_2(X) \cap \Lambda^M(\mathcal{P}_2(X^*))$ maps the geodesic $\{\mu_t\}_{t\in[0,1]} \mapsto \{\hat{\mu}_t\}_{t\in[0,1]}$. In particular $\{\mu_t\}_{t\in[0,1]}$ is a geodesic and $\hat{\mu}_t \in \mathcal{P}_2^{ac}(X)$ for every $t \in [0,1]$. We have that $\hat{\mu}_t \ll \mathfrak{m}$ for every $t \in [0,1]$ since $(X, \mathsf{d}, \mathfrak{m})$ satisfies the strong $\mathsf{CD}(K,N)$-condition, which in turn implies that $\mu_t \ll \mathfrak{m}^*$ for every $t \in [0,1]$. Indeed, the property of being absolutely continuous is preserved under the pushforward and furthermore, for every $t \in [0,1]$, the quotient measure coincides with the pushforward of the lift $\mathsf{p}_\sharp(\hat{\mu}_t) = \mu_t$. We write $\mu_t = \rho_t \mathfrak{m}^*$ and, we can assume that $\hat{\mu}_t = \hat{\rho}_t \mathfrak{m}^*$ from item *2.* of Lemma 2.1.1. Lastly, recall that the lift of π satisfies $\hat{\pi} \in \mathsf{OptAdm}(\hat{\mu}_0, \hat{\mu}_1)$, this was concluded in Corollary 2.1.12 and note that $\hat{\pi}$ coincides with the coupling induced by the geodesic $\hat{\mu}_t = \hat{\rho}_t \mathfrak{m}^*$.

We can show that the required convexity of the entropy is satisfied along $\{\mu_t\}_{t\in[0,1]}$, because it holds that

$$\int_{X^*} \rho_t^{1-\frac{1}{N'}}(x^*)\, d\mathfrak{m}^*(x^*) = \int_{X^*}\int_{\mathcal{F}_{x^*}} \hat{\rho}_t^{1-\frac{1}{N'}}(y)\, d\mathfrak{m}_{x^*}(y)\, d\mathfrak{m}^*(x^*) =$$

$$\int_X \hat{\rho}_t^{1-\frac{1}{N'}}(y)\, d\mathfrak{m}(y) \geq \int_{X\times X} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\hat{\rho}_0^{-\frac{1}{N'}}(x) + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\hat{\rho}_1^{-\frac{1}{N'}}(y)\right] d\hat{\pi}(x,y) =$$

$$\int_{X^*\times X^*} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}^*(x^*,y^*))\rho_0^{-\frac{1}{N'}}(x^*) + \tau_{K,N'}^{(t)}(\mathsf{d}^*(x^*,y^*))\rho_1^{-\frac{1}{N'}}(y^*)\right] d\pi(x^*,y^*) \qquad ,$$

for every $t \in [0,1]$ and $N' \geq N$. Indeed, the Green Inequality in the middle is the $\mathsf{CD}(K,N)$-condition which is valid in $(X, \mathsf{d}, \mathfrak{m})$ and, in the last equality we have used that $\mathsf{d}(x,y) = \mathsf{d}^*(x^*,y^*)$ for $\hat{\pi}$-almost all $(x,y) \in X \times X$. We find the desired inequality written in Purple on the sides.

□

Remark 2.1.4. An essentially non-branching m.m. space which satisfies $\mathsf{MCP}(K,N)$ has unique Wasserstein geodesic starting from $\mu_0 \in \mathcal{P}_2^{ac}(X, \mathsf{d}, \mathfrak{m})$ to an arbitrary measure $\mu_1 \in \mathcal{P}_2(X)$. Therefore, the verbatim argument above together with Corollary 2.1.13 proves that if $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching $\mathsf{MCP}(K,N)$-space then also $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ is essentially non-branching and satisfies the $\mathsf{MCP}(K,N)$-condition.

As a matter of fact, note that the essential property needed for the proof is that the curvature-dimension condition is satisfied along Wasserstein geodesics which have constant densities along leaves.

2.2 Applications

Compact group actions

Assume that G is a compact Lie group acting effectively by isomorphisms on the m.m. space $(X, \mathsf{d}, \mathfrak{m})$. We show that there exists a natural way of lifting measures by pushing forward the Haar measure of G onto X, which induces a bounded metric measure foliation. Accordingly, synthetic Ricci bounds are inherited by the orbit space.

Remark 2.2.1. The above conditions on G are met, for instance, when G is a compact subgroup of the group of isomorphisms $\mathrm{ISO}_{\mathfrak{m}}(X)$ of an RCD^*-space, or an essentially non-branching $\mathsf{CD}/\mathsf{CD}^*/\mathsf{MCP}$-space with well-behaved tangents. Indeed, this is the conclusion of our Main Theorem A. In this direction, recall that the isotropy group $\mathsf{G}_p \leq \mathrm{ISO}_{\mathfrak{m}}(X)$ of a point $p \in X$ is compact, and that $\mathrm{ISO}_{\mathfrak{m}}(X)$ is compact if X is a compact m.m. space.

Proposition 2.2.1 (Group actions)**.** *Let G be a compact Lie group acting effectively by isomorphisms on $(X, \mathsf{d}, \mathfrak{m})$. Then the induced foliation by the orbits $\mathcal{G} = \{\mathsf{p}^{-1}(x^*)\}_{x^* \in X^* = X/\mathsf{G}}$ is a bounded metric measure foliation.*

Proof.

The compactness of G and the continuity of the map $\star_x : \mathsf{G} \to X$ guarantee the compactness of the orbits $G(x)$ for every $x \in X$. Whereas the definition of quotient metric, the compactness of the orbits and the fact that G acts by isometries imply that $\mathcal{G}$ is a metric foliation, i.e.

$$\mathsf{d}(G(x), G(y)) = \mathsf{d}^*(x^*, y^*) = \mathsf{d}(x, G(y)).$$

Hence, it only remains to show the validity of equation (2.0.3) of Definition 2.0.1. For this, we define measurable assignments[3] $(X^* \ni) x^* \mapsto \nu_{x^*} \in \mathcal{P}(X)$ and $(X^* \times X^* \ni)(x^*, y^*) \mapsto \pi_{x^*,y^*} \in \mathcal{P}(X) \times \mathcal{P}(X)$ as follows,

$$\begin{aligned} \nu_{x^*} &:= (\star_x)_\sharp \nu_{\mathsf{G}} && (2.2.2)\\ \pi_{x^*,y^*} &:= (\star_{(x,y)})_\sharp \nu_{\mathsf{G}} \quad \text{for some } (x, y) \in \mathcal{OD} \cap \mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(y^*). && (2.2.3) \end{aligned}$$

Where where $\star_{(x,y)} : \mathsf{G} \to X \times X$ is the map $g \mapsto (gx, gy)$. Note that the measure $\pi_{x,y}$ is concentrated on $\mathcal{OD} \cap \mathsf{G}((x, y))$, and that $\pi_{x,y} = \pi_{gx,gy}$ for every $x, y \in \mathcal{OD}$, and $g \in \mathsf{G}$, thus π_{x^*,y^*} is well-defined for every $x^*, y^* \in X^*$.

Using these definitions the validity of equation (2.0.3) can be concluded from the next two **Claims.**

Claim 1. The conditional probability $\{m_{x^*}\}_{x^* \in X^*}$ coincides with $\{\nu_{x^*}\}_{x^* \in X^*}$ for $\mathfrak{m}^*$-almost every $x^* \in X^*$.

Claim 2. The measure $\pi_{x^*.y^*}$ is an admissible coupling, i.e. $\pi_{x^*.y^*} \in \mathsf{Adm}(\nu_{x^*}, \nu_{y^*})$, with $\mathrm{supp}(\pi_{x^*.y^*}) \subset \mathcal{OD} \cap (\mathsf{p}(x^*) \times \mathsf{p}(y^*))$.

Indeed, if the **Claims** were true we could conclude from the properties of $\pi_{x^*.y^*}$ that

$$\begin{aligned} W_2(\nu_x, \nu_y)^2 &\leq \int_{\mathcal{OD} \cap (\mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(y^*))} \mathsf{d}(w, z)^2 \, d\pi_{x^*.y^*}(w, z)\\ &= \int_{\mathcal{OD} \cap (\mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(y^*))} \mathsf{d}^*(\mathsf{p}(w), \mathsf{p}(z))^2 \, d\pi_{x^*.y^*}(w, z)\\ &= \mathsf{d}^*(w^*, z^*) = \mathsf{d}^*(x^*, y^*). \end{aligned}$$

[3]Similar arguments to the one included in footnote 2 show that we can in fact consider measurable assignments.

Moreover, for any optimal coupling $\tilde{\pi}_{x^*,y^*}\mathsf{OptAdm}(\nu_{x^*},\nu y,^*)$ it always holds that $\mathrm{supp}(\tilde{\pi}_{x^*,y^*}) \subset \mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(y^*)$. Therefore, by definition of the quotient metric d^* it holds that

$$W_2(\nu_{x^*},\nu_{y^*})^2 = \int_{(\mathsf{p}^{-1}(x^*)\times\mathsf{p}^{-1}(y^*))} \mathsf{d}(w,z)^2 d\tilde{\pi}_{x^*,y^*}(w,z) \geq \mathsf{d}^*(x^*,y^*).$$

Hence, **Claim 1.** together with these last inequalities show that

$$W_2(\mathfrak{m}_{x^*},\mathfrak{m}_{y^*}) = W_2(\nu_{x^*},\nu_{y^*}) = \mathsf{d}^*(x^*,y^*) \quad (\mathfrak{m}^* \times \mathfrak{m}^*)\text{-almost everywhere.}$$

Accordingly, we prove the **Claims** to conclude.

claim 1. By constriction ν_{x^*} is the unique G-invariant probability measure with $\mathrm{supp}(\nu_{x^*}) \subset \mathsf{p}^{-1}(x^*)$. The statement is concluded since G acts by measure-preserving isomorphisms and $\mathfrak{m}$ is G-invariant, thus the measure $\mathfrak{m}_{x^*}$ is a G-invariant probability measure with $\mathrm{supp}(\mathfrak{m}_{x^*}) = \mathrm{supp}(\nu_{x^*})$ for $\mathfrak{m}^*$-almost every $x^* \in X^*$. ∎

claim 2. The statement about the support of π_{x^*,y^*} holds by construction. To finish we show that for $A \subset X$ measurable $\pi_{x^*,y^*}(A\times X) = \nu_{x^*}(A)$ for every $x^*, y^* \in X^*$. The proof for the other marginal is carried out identically. Let $(x,y) \in \mathcal{OD} \cap (\mathsf{p}^{-1}(x^*) \times \mathsf{p}^{-1}(x^*))$ and A be as above, then

$$\begin{aligned}\pi_{x^*,y^*}(A\times X) = \int_{\star_{x,y}^{-1}(A\times X)} d\nu_{\mathsf{G}}(g) = \int_G \chi_{\{g\,\|\,(gx,gy)\in A\times X\}}(g) d\nu_{\mathsf{G}}(g) = \\ \int_G \chi_{\{g\,\|\,gx\in A\}}(g) d\nu_{\mathsf{G}}(g) = \nu_{x^*}(A)\end{aligned}$$

∎

□

Remark 2.2.2. In the present situation, as we just saw, we can explicitly write the elements of the disintegration of the measure $\mathfrak{m}$ by pushing-forward the Haar measure to G-invariant probability measures on X. Correspondingly, this implies that the lifts of measures on X^* by Λ^M defined in (2.1.5) are also G-invariant. Thus, writing $\mathcal{P}^{\mathsf{G}}(X) \subset \mathcal{P}(X)$ for the subspace of probability measures that are G-invariant, Proposition 2.1.8 reads as follows:

$\Lambda : \mathcal{P}_2(X^*) \hookrightarrow \mathcal{P}_2(X) \cap \mathcal{P}^{\mathsf{G}}(X)$ *is an isometric embedding which preserves absolutely continuous measures. In particular, lifts of W_2-geodesics in $\mathcal{P}_2(X^*)$ are G-invariant W_2-geodesics in $\mathcal{P}_2(X)$.*

From Proposition 2.2.1 and Main Theorem C we obtain the next

Theorem D (Synthetic Ricci is stable: group actions)**.** *Let G be a compact Lie group that acts by isomorphisms on $(X,\mathsf{d},\mathfrak{m})$. Assume that $(X,\mathsf{d},\mathfrak{m})$ satisfies one of the following conditions: strong $\mathsf{CD}(K,N)$, strong $\mathsf{CD}(K,\infty)$, strong $\mathsf{CD}^*(K,N)$, $\mathsf{RCD}^*(K,N)$, or essentially non-branching $\mathsf{MCP}(K,N)$-condition.*
Then the quotient metric measure space $(X^*,\mathsf{d}^*,\mathfrak{m}^*)$ satisfies the corresponding condition for the same parameters.

Remark 2.2.3 (Applications)**.** The conclusion of Theorem D is particularly useful to understand the structure of RCD^*-spaces with symmetries, in particular of spaces admitting an effective action by a *large group*. Indeed, this is due to the fact that low dimensional RCD-spaces are well-understood [KL16]. This approach was followed in [GGKMS17, Section 6], in particular see Theorems 6.7, 6.8, and 6.9.

Bounded metric measure submetries

More generally b.m.m. foliations are in correspondence with a particular type of submetries.

Definition 2.2.4 (bounded metric measure submetries)**.** *A map $f : X \to X^*$ is called a bounded metric measure submetry if the following conditions are satisfied.*

$$\begin{aligned} &\text{i)} \quad \textit{fibers } f^{-1}(x^*) \textit{ are bounded for every } x^* \in X^*; \\ &\text{ii)} \quad f(B_r(x)) = B_r(f(x)) \textit{ for every } 0 < r, \textit{ and } x \in X; \textit{ and} \\ &\text{iii)} \quad W_2(\mathfrak{m}_{x*}, \mathfrak{m}_{y*}) = \mathsf{d}^*(x^*, y^*) \quad \textit{for } \mathfrak{m}^*\textit{-a.e. } x^* \in X^*. \end{aligned} \tag{2.2.5}$$

From this equivalence, which is a consequence of the next result, we conclude Theorem E. We recall that a metric foliation (metric submetry) is a foliation (submetry) which satisfies only condition *ii)* of Definition 2.0.1 (condition *ii)* Definition 2.2.4).

Proposition 2.2.6 (Metric foliations are submetries)**.** *There is a one-to-one correspondence between metric foliations and submetries up to an isometry. Namely, the projection $\mathsf{p} : X \to X^*$ of a metric foliation is a submetry and, given a submetry $f : X \to N$, the foliation given by $\{f^{-1}(y)\}_{y\in N}$ is a metric foliation for which there is an isometry $i_f : N \to X^*$ with*

$$i_f \circ f = \mathsf{p}.$$

Proof. The fact that $\mathsf{p} : X \to X^*$ is a submetry follows directly from the definitions. Now consider a submetry $f : X \to N$. The continuity of f guarantees that $\mathcal{F}_f = \{f^{-1}(y)\}_{y\in N}$ is a foliation so we just have to check the equidistance property. This follows from the next observation. Let $F, G \in \mathcal{F}$ and suppose, for the sake of contradiction, that there exists $x \in F$ such that $\mathsf{d}(x, G) - 2\varepsilon \geq r := \mathsf{d}(F, G)$, for some $\varepsilon > 0$. Then there exist $x' \in F, y' \in G$ with $\mathsf{d}(x', y') < r + \varepsilon$ and the submetry assumption gives that

$$f(y') \in B_{r+\varepsilon}(f(x')) = B_{r+\varepsilon}(f(x)) = f(B_{r+\varepsilon}(x)).$$

Therefore, there exists $y \in G \cap B_{r+\varepsilon}(x)$, contradicting that $\mathsf{d}(x, G) \geq r + 2\varepsilon$. Next, by noting that $\mathsf{p} : X \to X^*$ is by construction independent of the representative $x \in F \in \mathcal{F}$, we see that the function $i_f : N \to X^*$ given by $i_f := \mathsf{p} \circ f^{-1}$ is a well defined isometry by using the definition of the quotient metric.

Finally, suppose that there exists another submetry, $g : X \to \tilde{N}$, which induces the same foliation of f, that is $\mathcal{F} = \mathcal{G} = \{g^{-1}(z)\}_{z\in\tilde{N}}$. Then we have that $i_f^{-1} \circ i_g : \tilde{N} \to N$ is an isometry and $g = i_g^{-1} \circ i_f \circ f$. Thus, up to isometries, f is unique. $\square$

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Symbols

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(Gerardo Sosa Garciamarín)