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An Obstacle Problem for a fractional power of the Laplace Operator

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Abstract

We will deal with a paper of Luis Silvestre [14] that is based on his Ph.D. Thesis. He worked on the following problem:
For a given function φ and $s \in (0,1)$, he tried to find the solution u of the following obstacle problem:

- $u \geq \varphi$ in $\mathbb{R}^n$
- $(-\Delta)^s u \geq 0$ in $\mathbb{R}^n$
- $(-\Delta)^s u(x) = 0$ for x such that: $u(x) > \varphi(x)$
- $\lim_{|x|\to\infty} u(x) = 0$

Our main goal will be to understand the paper and fill the gaps in his proofs. We will focus on the basic properties of the Operator $(-\Delta)^s$, the existence of the solution of the beforementioned obstacle problem when φ is continuous and has compact support and his first regularity results, especially the proof that the solution is continuous.

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Chapter 1.

Introduction

The first chapter will deal with the statement of our problem and its mathematical classification.

1.1. From the Dirichlet Problem to the Obstacle Problem

It is well known that the Dirichlet problem for the Laplace operator

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

in the Sobolev space $W^{1,2}(\Omega)$ is equivalent to a minimum problem for the energy functional

$$J(u) = \frac{1}{2}\int_\Omega \sum_{i=1}^{n} |\partial_i u(x)|^2 dx - \int_\Omega f(x)u(x)dx.$$

But let's go a bit into details

1. The boundary value problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

has, for given $f \in L_2(\Omega)$ and $g \in W^{1,2}(\Omega)$, exactly one weak solution $u \in W^{1,2}(\Omega)$. That means there exists exactly one $u \in W^{1,2}(\Omega)$ with $u - g \in W_0^{1,2}(\Omega)$ and

$$a(u,v) = \int_\Omega \sum_{j=1}^{n} \partial_j u \partial_j v dx = \int_\Omega f(x)v(x)dx = b(v) \text{ for every } v \in W_0^{1,2}(\Omega). \tag{1.1}$$

This is also called a variational equality.

2. The above u is the solution of the following variational problem

$$\min_{u\in A} J(u) =!$$

with

$$J(u) = \frac{1}{2}\int_\Omega \sum_{j=1}^{n} |\partial_j u|^2 dx - \int_\Omega f u dx$$

and

$$A = \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}.$$

We note that A is a affine subspace of $W^{1,2}(\Omega)$. In the case $g = 0$, we get $A = W_0^{1,2}(\Omega)$, that is the space of $W^{1,2}(\Omega)$ functions with generalized zero boundary values.

The physical interpretation is that Ω is the idle state of a membrane that is fixed on $\partial\Omega$ (in the case $g = 0$). This membrane is subjected to an outer force f and we are looking for the equilibrium position u.
Now we assume there is an obstacle described by a function ψ on an open set $\Omega_1 \subset \Omega$. This obstacle forces the membrane to stay above it, i.e. the height function u must satisfy the unilateral condition $u \geq \psi$ on Ω_1.
We suppose that $g = 0$. In the presence of the obstacle ψ, instead of

$$\min_{u \in A} J(u) =!$$

we now consider the restricted minimum problem

$$\min\{J(v) : v \in \mathbb{K}\} \tag{1.2}$$

where $\mathbb{K} = \{v \in W_0^{1,2}(\Omega) : v \geq \psi \text{ a.e. on } \Omega_1\}$. Then a necessary and sufficient condition for $u \in \mathbb{K}$ to be a solution of (1.2) is the variational inequality

$$\int_\Omega \sum_{j=1}^n \partial_j u \partial_j (v-u) dx \geq \int_\Omega f(v-u)dx \text{ for all } v \in \mathbb{K}$$

or, in an abstract setting

$$a(u, v-u) \geq b(v-u) \text{ for all } v \in \mathbb{K} \tag{1.3}$$

where a is a bilinear form and b is a functional. Both problems are called variational inequality. Remark that (1.3) is a nonlinear problem even though the differential operator involved, i.e. $(-\Delta)$, is a linear operator.
Let us also consider the variational inequality (1.3) where the data f, g are given as above while $\mathbb{K}$ is given by

$$\mathbb{K} = \{v \in W^{1,2}(\Omega) : v \geq 0 \text{ a.e. on } \partial\Omega\}$$

This problem is called a variational inequality with thin obstacle since the dimension of $\partial\Omega \subset \mathbb{R}^n$ is $n-1$. In contrast, sometimes the constraint $u \geq \psi$ on Ω_1 is called a thick obstacle. Thin obstacle problems are in general more involved than thick obstacle problems.
In case of elasticity, the corresponding problem is called Signori's problem.

1.2. Regularity

In case of variational equation we, roughly spoken, have the following regularity result for the solution of (1.1).

$$f \in W^{k,2}(\Omega) \Rightarrow u \in W^{k+2,2}(\Omega).$$

Thus by Sobolev-Morrey embedding theorem, for k large enough, we may get classical (smooth) solutions.
This is definitely false for the variational inequalities. There is a threshold for regularity that cannot be surpassed in general. For thick obstacle problems we have in general $u \notin C^2$, see [6, p. 47]. In fact the optimal regularity is $W^{2,\infty}$, see [12, p. 163]. For thin obstacle problems there is a counterexample by Shamir that the threshold is $C^{1,\frac{1}{2}}$, which is cited in [12, p. 279].
Regularity is not only interesting from a mathematical point of view, but also for users (physicists, engineers, ...) who want to know how good their solutions are. Also numerical algorithms converge in general faster for regular solutions.

1.3. Fractional Laplace Operator

Now we will ask the question why fractional powers of the Laplace operator are of any interest. We will briefly discuss a classical example that leads to $(-\Delta)^{\frac{1}{2}}$, the Dirichlet-to-Neumann-operator, in the case of the upper half space.
The key is Green's formula: Suppose $u, v \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \sum_{j=1}^n \partial_j u \partial_j v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v dS - \int_\Omega (\Delta u)\, v dx.$$

If $\Delta u = 0$, this reads as

$$\int_\Omega \sum_{j=1}^n \partial_j u \partial_j v dx = \left\langle \frac{\partial u}{\partial \mathbf{n}}, v \right\rangle. \tag{1.4}$$

Here $\langle ., . \rangle$ denotes the duality pairing between $W^{-\frac{1}{2},2}(\partial\Omega)$ and $W^{\frac{1}{2},2}(\partial\Omega)$. This immediately leads to the idea to write the Dirichlet form $a(u,v) = \int_\Omega \sum_{j=1}^n \partial_j u \partial_j v dx$ as a bilinear form on $\partial\Omega$. From now on Ω shall be the upper half space $\mathbb{R}^n_+$. We consider the following classical Dirichlet problem:
For given $g \in C_0^\infty(\mathbb{R}^{n-1})$ find $u \in W^{1,2}(\mathbb{R}^n_+)$ with

$$\begin{aligned} \Delta u &= 0 \text{ in } \mathbb{R}^n_+ \\ u &= g \text{ on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}. \end{aligned} \tag{1.5}$$

Then we define $Pg = \frac{\partial u}{\partial \mathbf{n}}$ on $\partial\mathbb{R}^n_+$. Partial Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$ shows that the bounded solution of (1.5) is

$$u(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} e^{-x_n|\xi'|} \mathcal{F}[g](\xi') d\xi'.$$

Thus

$$(Pg)(x') = -\frac{\partial u}{\partial x_n}(x', 0) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} |\xi'| \mathcal{F}[g](\xi') d\xi'$$

Therefore $P : C_c^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1})$ can be viewed as a pseudodifferential operator with symbol $|\xi'|$. Because $-\Delta$ is represented by $|\xi'|^2$ in the Fourier space we denote $P = (-\Delta)^{\frac{1}{2}}$.
When we apply this approach to the thin obstacle problem with $\Omega = \mathbb{R}^n$ and $\mathbb{K} = \{v \in W^{1,2}(\Omega) : v \geq 0 \text{ a.e. on } \partial\Omega\}$ we get a thick obstacle problem on $G = \partial\Omega = \mathbb{R}^{n-1}$ with $\mathbb{K}_1 = \{v \in W^{\frac{1}{2},2}(G) : v \geq 0 \text{ a.e. on } G\}$. The variational inequality reads as follows

$$\langle Pu, v - u \rangle \geq b(u - v)$$

for every $v \in \mathbb{K}_1$. The price for that is that we replace our well-known Laplace-Operator with a hard to handle pseudodifferential operator P.
Now we want to briefly discuss the reasons why the fractional Laplacian got so much attention in the past years. To do that we cite [13, p.2].

> "Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete applications, since these operators arise in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves."

Next we want to give a brief impression of the diversity of problems containing the fractional Laplacian.

- In [13] "equations driven by non-local integrodifferential operators [...] with homogeneous Dirichlet boundary conditions" [13, p. 1] are discussed and, among other results, an existence theorem for the problem

$$\begin{aligned}(-\Delta)^s u - \lambda u &= f(x,u) \text{ in } \Omega\\ u &= 0 \text{ in } \mathbb{R}^n \backslash \Omega\end{aligned}$$

  is derived.

- In [19] Vazquez "describes two models of flow in porous media including nonlocal diffusion effects"[19, p.1]. In the first model he uses the inverse of the fractional Laplacian and gives, among other things, an application for the obstacle problem, see [19, 5.2]. The second model is a nonlinear heat equation with fractional diffusion, i.e. an equation of the form

$$\partial_t u + (-\Delta)^s(u^m) = 0.$$

  For the applications we quote Vazquez

  > "Interest in studying the nonlinear model we propose is two-fold: on the one hand, experts in the mathematics of diffusion want to understand the combination of fractional operators with porous medium type propagation. On the other hand, models of this kind arise in statistical mechanics when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling..."([19, p.19]

- In [22] the author derives lower bounds for an integral involving the fractional Laplacian and uses that to prove existence and uniqueness of solutions of the generalized Navier-Stokes equation, i.e. the equation

$$\partial_t u + u \cdot \nabla u + \nabla P = -\nu(-\Delta)^s u$$

  in Besov spaces.

- In [9] the author proves the existence of global weak solutions in time for the 2D critical dissipative surface quasi-geostrophic equation, i.e. the equation

$$\partial_t \theta(x,t) + u \cdot \nabla\theta + (-\Delta)^{\frac{s}{2}}\theta = 0$$

  where θ is the potential temperature and

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$$

  where $-\mathcal{R}_j$ is the Riesz transform.

1.4. Conclusion

Now we face the task to examine the variational inequality and to some extend its regularity when we exchange the Laplace operator with $(-\Delta)^s$. In the case $s = \frac{1}{2}$ we can orientate ourselves by the work of Frehse, Kinderlehrer, Uralzeva and Caffarelli.
The approach of Caffarelli and Silvestre is to use and improve potential theoretical methods that originate from Landkof.
In chapter 2, we will investigate basic properties of the operator $(-\Delta)^s$, among other things we will get a representation as a singular integral, find the fundamental solution of $(-\Delta)^s$, which we will modify to get a bounded $C^{1,1}$ function Γ that is easier to work with and lastly we will characterize supersolutions

of $(-\Delta)^s$.
In chapter 3 we prove the existence of a solution for the obstacle problem of $(-\Delta)^s$ if the obstacle is continuous and has compact support. To accomplish that we use a variational ansatz in an appropriate Sobolev space and show that this solution is also continuous. Then we prove regularity-like properties for the case of stricter requirements on φ. One of the main results there will be to prove the existence of a supporting plane with an error of $1+\alpha$ in the case that $\varphi \in C^{1,\alpha}(\mathbb{R}^n)$.
The first two appendices will be used to gather facts about rapidly decreasing functions and the Fourier transformation that will be used throughout the work. The 3rd appendix consists of a couple propositions and lemmas used in chapter 2 and in the 4th appendix we try to work around the problem that $(-\Delta)^s$ cannot technically be viewed as a pseudodifferential operator.

Chapter 2.

Properties of the Fractional Laplace Operator

In this chapter we will adress the definition and some properties of the fractional Laplace operator. It basically is defined using the Fourier transformation as a Fourier multiplier similar to a pseudodifferential operator, but the multiplier $|\xi|^{2s}$ is not smooth in $\xi = 0$. For the usual definition as a pseudodifferential operator, one has to use a cut off function. We will further examine this in appendix A.4.
For the later application, a integral representation will be more useful. Thus we will establish a representation as a principal value integral and a representation using a "normal" integral, where we have to use second differences.
The basic idea is to use the fundamental solution $\Psi(x) = \frac{1}{|x|^{n-2s}}$ of the fractional Laplace operator. The singularity in $x = 0$ complicates the work with the before mentioned integrals. Lankof had the idea to simply cut off the fundamental solution around zero. We will do it differently. We will modify Ψ in a neighbourhood of zero to get a function Γ, which is still $C^{1,1}$ on $\mathbb{R}^n$, a fact that is not proven by Silvestre, but we will show it. This will have the advantage that $(-\Delta)^s\Gamma$ is a continuous function, which we will use instead of the fundamental solution. For that reason we will investigate it closely.
Lastly we will characterize supersolutions of $(-\Delta)^s$, i.e. functions u such that $(-\Delta)^s u \geq 0$, for that the function Γ will play an important role. In the course of that, we will find a generalization of the mean value property of superharmonic functions and some kind of comparison principle.

2.1. Definitions and Properties

First of all, we want to motivate the definition of the operator $(-\Delta)^s u$. Using proposition A.2.3 we have for $f \in \mathcal{S}$:

$$\mathcal{F}[\partial_{x_j x_j} f] = -\xi_j^2 \mathcal{F}[f].$$

Applying this to the Laplace Operator, we get:

$$\mathcal{F}[-\Delta f] = |\xi|^2 \mathcal{F}[f].$$

To finish it off, we apply the inverse Fouriertransformation and get in the end:

$$-\Delta f = \mathcal{F}^{-1}[|\xi|^2 \mathcal{F}[f](\xi)].$$

Therefore, we define:

2.1.1 Definition. Given $s > -\frac{n}{2}$ and $f \in \mathcal{S}$ we define $(-\Delta)^s f$ as

$$(-\Delta)^s f(x) = \mathcal{F}^{-1}\left[|\xi|^{2s}(\mathcal{F}[f])(\xi)\right](x)$$

This definition is strongly reminiscent of the definition of a pseudodifferential operator, compare A.4.2. Unfortunately, the multiplier $|\xi|^{2s}$ does not fulfill the necessary smoothness criteria. We will go into detail later on. In this context we would like to refer to the book of Abels [1].

For starters, we will check whether or not this definition is reasonable. To do that, we need the following two propositions.

2.1.2 Proposition. *For $s \in \mathbb{R}$ with $s > -\frac{n}{2}$ and $f \in \mathcal{S}$ we have for $g(\xi) = |\xi|^{2s}\mathcal{F}[f](\xi)$ that $g \in L_1(\mathbb{R}^n)$.*

Proof. (i) First $s \geq 0$.

$$\begin{aligned}\|g|L_1(\mathbb{R}^n)\| &= \int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}[f](\xi)|d\xi \\ &\leq C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_{\mathbb{R}^n} |\xi|^{2s}\frac{1}{(1+|\xi|)^{2\lceil s\rceil+n+1}}d\xi.\end{aligned}$$

Here we used the fact that $f \in \mathcal{S}$ implies $\mathcal{F}[f] \in \mathcal{S}$; see proposition A.2.5. $\lceil s\rceil$ is the ceiling function, so the smallest integer that is not smaller than s.

Now we'll use spherical coordinates.

$$\begin{aligned}&= \Xi(n)C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_0^\infty \frac{\rho^{2s}}{(1+\rho)^{2\lceil s\rceil+n+1}}\rho^{n-1}d\rho \\ &= \Xi(n)C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_0^\infty \frac{\rho^{2s+n-1}}{(1+\rho)^{2\lceil s\rceil+n+1}}d\rho \\ &\leq \Xi(n)C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_0^\infty \frac{(1+\rho)^{2s+n-1}}{(1+\rho)^{2\lceil s\rceil+n+1}}d\rho \qquad (1)\\ &= \Xi(n)C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_1^\infty \rho^{2s+n-1-2\lceil s\rceil-n-1}d\rho \\ &= \Xi(n)C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_1^\infty \rho^{2(s-\lceil s\rceil)-2}d\rho.\end{aligned}$$

An integral of that type converges if the exponent is smaller than -1. So:

$$\begin{aligned}&-1 > 2(s-\lceil s\rceil)-2 \\ \iff\ &1 > 2(s-\lceil s\rceil)\end{aligned}$$

The last inequality is true since $(s-\lceil s\rceil) \leq 0$. The estimate in (1) is admissible because $s \geq 0$ implies $n-1+2s \geq 0$.

In the end we get $\|g|L_1(\mathbb{R}^n)\| < \infty$.

(ii) Now $s \in \left(-\frac{n}{2}, 0\right)$. Then we have:

$$\begin{aligned}\|g|L_1(\mathbb{R}^n)\| &= \int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}[f](\xi)|d\xi \\ &= \int_{|x|\leq 1} |\xi|^{2s}|\mathcal{F}[f](\xi)|d\xi + \int_{|x|>1} |\xi|^{2s}|\mathcal{F}[f](\xi)|d\xi. \qquad (2)\end{aligned}$$

We have for the first integral:

$$\begin{aligned}\int_{|x|\leq 1} |\xi|^{2s}|\mathcal{F}[f](\xi)|d\xi &\leq C_{0,0}(\mathcal{F}[f]) \int_{|x|\leq 1} |\xi|^{2s}d\xi \\ &= \Xi(n)C_{0,0}(\mathcal{F}[f]) \int_0^1 \rho^{2s+n-1}d\rho.\end{aligned}$$

Again we were using the fact that from $f \in \mathcal{S}$ it follows that $\mathcal{F}[f] \in \mathcal{S}$ and after that we changed to spherical coordinates.
The integral we get exists if the exponent is bigger than -1.

$$\begin{aligned} & -1 < 2s + n - 1 \\ \iff & \quad 0 < 2s + n \\ \iff & -n < 2s \\ \iff & -\frac{n}{2} < s. \end{aligned}$$

This is true by our assumptions.
For the second integral we have:

$$\begin{aligned} \int_{|x|>1} |\xi|^{2s} |\mathcal{F}[f](\xi)| d\xi &\leq C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_{|x|>1} \frac{|\xi|^{2s}}{(1+|\xi|)^{2\lceil s\rceil+n+1}} d\xi \\ &= \Xi(n) C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_1^\infty \frac{\rho^{2s+n-1}}{(1+\rho)^{2\lceil s\rceil+n+1}} d\rho \\ &\leq \Xi(n) C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_1^\infty \frac{\rho^{2s+n-1}}{\rho^{2\lceil s\rceil+n+1}} d\rho \\ &= \Xi(n) C_{0,2\lceil s\rceil+n+1}(\mathcal{F}[f]) \int_1^\infty \rho^{2s+n-1-2\lceil s\rceil-n-1} d\rho. \end{aligned}$$

The convergence of this integral was already shown in (i).
Finally we have seen that both integrals in equation (2) exist and, therefore, we conclude that $\|g|L_1(\mathbb{R}^n)\| < \infty$.

□

2.1.3 Proposition. *For $s > -\frac{n}{2}$ we have*

$$(-\Delta)^s : \mathcal{S} \to C_b^0(\mathbb{R}^n).$$

This especially means that there exists a constant $c = c(f, n)$ such that $|(-\Delta)^s f(x)| < c \ \forall x \in \mathbb{R}^n$.

Proof. In proposition 2.1.2 we have seen that $g \in L^1$. Thus $\mathcal{F}^{-1}[g] \in C_b^0(\mathbb{R}^n)$ by Proposition A.2.2. This allows us to conclude

$$(-\Delta)^s f = \mathcal{F}^{-1}[g] \in C_b^0(\mathbb{R}^n).$$

□

2.1.4 Remark. *Proposition 2.1.3 shows us that the definition 2.1.1 is reasonable. But in our application we are mostly interested in the case $s \in (0, 1)$, alternatively $s \in (-1, 1)$, as we will see in a minute.*

2.1.5 Remark. *We will shortly discuss the regularity of the operator. Unfortunately $(-\Delta)^s f \notin \mathcal{S}$, because its Fourier transform, i.e. the function $|\xi|^{2s}\mathcal{F}[f](\xi)$, is not in $\mathcal{S}$. The problem here is the factor $|\xi|^{2s}$, at least its second derivatives have a singularity at the origin. So $|\xi|^{2s}\mathcal{F}[f](\xi)$ lacks differentiability and this translates into a lack of rapid decay of $(-\Delta)^s f$, according to remark A.2.4. Luckily, $(-\Delta)^s f$ is still in C^∞, we show that in A.4.*

2.1.6 Proposition. *We can form the composition of the Operator:*

$$(-\Delta)^{s_1} \circ (-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}.$$

Proof. Let $f \in \mathcal{S}$ be arbitrary. Then

$$((-\Delta)^{s_1} \circ (-\Delta)^{s_2})f = \mathcal{F}^{-1}\left[|\xi|^{2s_1}\mathcal{F}[\mathcal{F}^{-1}[|\xi|^{2s_2}\mathcal{F}[f](\xi)]]\right] \tag{1}$$

We showed in Proposition 2.1.2 that $|\xi|^{2s_2}\mathcal{F}[f](\xi)$ is integrable, in proposition A.1.5 that $L^1(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and finally in proposition A.2.11 that Fourier inversion is true on $\mathcal{S}'$. This gives us

$$\mathcal{F}[\mathcal{F}^{-1}[|\xi|^{2s_2}\mathcal{F}[f]]] = |\xi|^{2s_2}\mathcal{F}[f]$$

and therefore

$$\begin{aligned}(1) &= \mathcal{F}^{-1}\left[|\xi|^{2s_1}|\xi|^{2s_2}\mathcal{F}[f]\right] \\ &= \mathcal{F}^{-1}\left[|\xi|^{2(s_1+s_2)}\mathcal{F}[f](\xi)\right] \\ &= (-\Delta)^{s_1+s_2}f.\end{aligned}$$

□

2.1.7 Remark. *We deliberately formulated the last proposition a bit vaguely, i.e. we did not specify what s_1 and s_2 are allowed, because we won't actually need the composition of the operator. Except at one point to derive the fundamental solution and there we have $s_1 = -s_2$.*

The definition for the fractional Laplacian might be sensible, but it is not easy to work with. We need a representation that allows us to make actual computations and estimations. For that purpose, we will derive a representation as a singular integral. In the proof we will need n-dimensional spherical coordinates in direction of a vector $\xi \in \mathbb{R}^n$, so we will briefly review their definition without going too much into detail, see [8].

2.1.8 Remark. *Let $x \in \mathbb{R}^n$, then it's components are given by*

$$\begin{aligned}x_1 &= |x|\cos(\theta_1) \\ x_2 &= |x|\sin(\theta_1)\cos(\theta_2) \\ &\vdots \\ x_{n-1} &= |x|\sin(\theta_1)\ldots\sin(\theta_{n-2})\cos(\theta_{n-1}) \\ x_n &= |x|\sin(\theta_1)\ldots\sin(\theta_{n-1}).\end{aligned}$$

Where we choose θ_1 to be the angle between x and ξ. We have $0 \leq \theta_i < \pi$ for $i = 1, \ldots, n-2$ and $0 \leq \theta_{n-1} < 2\pi$. The Jacobian is given by $J(f) = |x|^{n-1}\sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\ldots\sin(\theta_{n-2})$.

2.1.9 Proposition. *If $f \in \mathcal{S}$ and $s \in (0,1)$, then:*

$$(-\Delta)^s f(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy$$

where PV stands for the principle value of the integral and is defined as

$$PV \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n - B_\epsilon(x)} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy$$

For $n > 2s > 0$ we can also compute:

$$(-\Delta)^{-s} f(x) = c_{n,-s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} dy.$$

Proof. We start by proving the first equation, which is also the more important one for us. For starters we note, by using Fourier Inversion, as seen in A.2.6

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \mathcal{F}[f](\xi) d\xi \qquad f(x+z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{iz\xi} \mathcal{F}[f](\xi) d\xi.$$

This leads to

$$f(x+z) - f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \left(e^{iz\xi} - 1\right) \mathcal{F}[f](\xi) d\xi.$$

Now we divide by $|z|^{n+2s}$ and integrate z over $K_{\epsilon,\infty} = \mathbb{R}^n - B_\epsilon(0)$, with $\epsilon > 0$:

$$\int_{K_{\epsilon,\infty}} \frac{f(x+z) - f(x)}{|z|^{n+2s}} dz = (2\pi)^{-\frac{n}{2}} \int_{K_{\epsilon,\infty}} \left(\int_{\mathbb{R}^n} e^{ix\xi} \frac{\left(e^{iz\xi} - 1\right)}{|z|^{n+2s}} \mathcal{F}[f](\xi) d\xi \right) dz.$$

We first note that:

$$\begin{aligned} \left| \int_{K_{\epsilon,\infty}} \int_{\mathbb{R}^n} e^{ix\xi} \frac{\left(e^{iz\xi} - 1\right)}{|z|^{n+2s}} \mathcal{F}[f](\xi) d\xi dz \right| &\leq 2 \int_{K_{\epsilon,\infty}} \int_{\mathbb{R}^n} \frac{|\mathcal{F}[f](\xi)|}{|z|^{n+2s}} d\xi dz \\ &= 2 \underbrace{\left(\int_{K_{\epsilon,\infty}} \frac{dz}{|z|^{n+2s}} \right)}_{(1)} \underbrace{\left(\int_{\mathbb{R}^n} |\mathcal{F}[f](\xi)| d\xi \right)}_{(2)}. \end{aligned}$$

(1) converges since $\epsilon > 0$ and $n + 2s > n$. (2) converges because of proposition A.1.2. So Fubini allows us to change the order of integration.

$$\begin{aligned} \int_{K_{\epsilon,\infty}} \frac{f(x+z) - f(x)}{|z|^{n+2s}} dz &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \left(\int_{K_{\epsilon,\infty}} \frac{e^{iz\xi} - 1}{|z|^{n+2s}} dz \right) \mathcal{F}[f](\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} \left(\int_{K_{\epsilon,\infty}} \frac{e^{iz\xi} - 1}{|\xi|^{2s}|z|^{n+2s}} dz \right) \mathcal{F}[f](\xi) d\xi. \end{aligned}$$

Now we take a closer look at the integral

$$\int_{K_{\epsilon,\infty}} \frac{e^{iz\xi} - 1}{|\xi|^{2s}|z|^{n+2s}} dz.$$

We will see that it is independent of ξ and real valued. To do that, we first use the substitution $w = z \cdot |\xi|$, for $\xi \neq 0$, that gives us $dw = |\xi|^n dz$.

$$\begin{aligned} \int_{K_{\epsilon,\infty}} \frac{e^{iz\xi} - 1}{|\xi|^{2s}|z|^{n+2s}} dz &= \int_{K_{\epsilon,\infty}} \frac{e^{iw\frac{\xi}{|\xi|}} - 1}{|\xi|^{n+2s}|z|^{n+2s}} dw \\ &= \int_{K_{\epsilon,\infty}} \frac{e^{iw\frac{\xi}{|\xi|}} - 1}{|w|^{n+2s}} dw. \end{aligned}$$

We start with the imaginary part of this integral, we first state that:

$$\left| \int_{K_{\epsilon,\infty}} \frac{\sin\left(w\frac{\xi}{|\xi|}\right)}{|w|^{n+2s}} dw \right| \leq \int_{K_{\epsilon,\infty}} \frac{1}{|w|^{n+2s}} dw$$

$$
\begin{aligned}
&= \Xi(n) \int_{\epsilon}^{\infty} \frac{\rho^{n-1}}{\rho^{n+2s}} d\rho \\
&= \Xi(n) \int_{\epsilon}^{\infty} \frac{1}{\rho^{1+2s}} d\rho < \infty.
\end{aligned}
$$

We want to check what happens when ϵ goes to zero. The main problem is obviously the fact that $|w|^{n+2s}$ is not integrable around zero. On top of that, we would need a term of at least second order in the numerator to "alleviate" this singularity.
To bypass these problems we will use the fact that we are integrating a product of the odd function sin and the even function $|w|^{-n-2s}$, therefore the integral must be zero. We will use the transformation $z = -w$ with $dz = dw$.

$$
\begin{aligned}
\int_{K_{\epsilon,\infty}} \frac{\sin\left(w\frac{\xi}{|\xi|}\right)}{|w|^{n+2s}} dw &= \int_{K_{\epsilon,\infty}} \frac{\sin\left((-z)\frac{\xi}{|\xi|}\right)}{|-z|^{n+2s}} dz \\
&= -\int_{K_{\epsilon,\infty}} \frac{\sin\left(z\frac{\xi}{|\xi|}\right)}{|z|^{n+2s}} dz
\end{aligned}
$$

and thus

$$
2\int_{K_{\epsilon,\infty}} \frac{\sin\left(w\frac{\xi}{|\xi|}\right)}{|w|^{n+2s}} dw = 0.
$$

Therefore we get

$$
\int_{K_{\epsilon,\infty}} \frac{\sin\left(w\frac{\xi}{|\xi|}\right)}{|w|^{n+2s}} dw = 0.
$$

Next we will turn to the real part. The integral

$$
\left| \int_{\mathbb{R}^n \setminus B_r(0)} \frac{\cos\left(w\frac{\xi}{|\xi|}\right) - 1}{|w|^{n+2s}} dw \right|
$$

exists for the same reason like the imaginary part.
The real problem lies in the singularity in zero. But unlike the imaginary part, we have that

$$
\frac{\cos(w\frac{\xi}{|\xi|}) - 1}{|w|^2}
$$

is bounded if $|w| < 1$ because: The series $\sum_{j=1}^{\infty} \frac{|w|^2}{2j!}$ converges by ratio test, since

$$
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2j!}{(2j+2)!} = \lim_{n\to\infty} \frac{1}{4j^2+6j+2} = 0
$$

and since

$$
\left| \frac{|w|^{2j}\left(\frac{w\xi}{|w||\xi|}\right)^{2j}}{2j!} \right| \leq \frac{|w|^2}{2j!},
$$

because $\frac{|w\xi|}{|w||\xi|} \leq 1$ by Cauchy-Schwarz and $|w| < 1$,

$$
\sum_{j=1}^{\infty} (-1)^n \frac{|w|^{2j}\left(\frac{w\xi}{|w||\xi|}\right)^{2j}}{2j!}
$$

converges absolutely by comparison test. So we can conclude that

$$\begin{aligned}\left|\cos\left(w\frac{\xi}{|\xi|}\right)-1\right| &= \left|\sum_{j=1}^{\infty}(-1)^n\frac{|w|^{2j}\left(\frac{w\xi}{|w||\xi|}\right)^{2j}}{2j!}\right| \\ &\le \sum_{j=1}^{\infty}\left|\frac{|w|^{2j}\left(\frac{w\xi}{|w||\xi|}\right)^{2j}}{2j!}\right| \\ &\le \sum_{j=1}^{\infty}\frac{|w|^2}{2j!}.\end{aligned}$$

It finally follows that:

$$\frac{\left|\cos\left(w\frac{\xi}{|\xi|}\right)-1\right|}{|w|^2}\le\sum_{j=1}^{\infty}\frac{1}{2j!}\le C$$

with some constant C. Now it's easy to see that

$$\begin{aligned}\left|\int_{B_1(0)}\frac{\cos\left(w\frac{\xi}{|\xi|}\right)-1}{|w|^{n+2s}}dw\right| &\le \int_{B_1(0)}\frac{\left|\cos\left(w\frac{\xi}{|\xi|}\right)-1\right|}{|w|^2}\frac{1}{|w|^{n-2+2s}}dw \\ &\le C\int_{B_1(0)}\frac{1}{|w|^{n-2+2s}}dw \\ &= C\Xi(n)\int_0^1\frac{\rho^{n-1}}{\rho^{n-2+2s}}d\rho \\ &= C\Xi(n)\int_0^1\frac{1}{\rho^{-1+2s}}d\rho.\end{aligned}$$

The last integral exists because $-1+2s>-1$.
We want to show that the real part is actually independent from ξ. If we choose spherical coordinates in direction of ξ, as described in remark 2.1.8, we get:

$$\int_{K_{\epsilon,\infty}}\frac{\cos\left(w\frac{\xi}{|\xi|}\right)-1}{|w|^{n+2s}}dw =$$
$$\int_{r=\epsilon}^{\infty}\int_{\theta_1=0}^{\pi}\cdots\int_{\theta_{n-2}=0}^{\pi}\int_{\theta_{n-1}}^{2\pi}\frac{\cos(r\cos(\theta_1))-1}{r^{n+2s}}r^{n-1}\sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\ldots\sin(\theta_{n-2})drd\theta_1\cdots d\theta_{n-1}.$$

The right hand side of the equation is obviously independent from the choice of ξ. So we can assume that ξ points into the direction of the first unit vector and we get:

$$\int_{K_{\epsilon,\infty}}\frac{f(x+z)-f(x)}{|z|^{n+2s}}dz=\left(\int_{K_{\epsilon,\infty}}\frac{\cos(w_1)-1}{|w|^{n+2s}}dw\right)\left(\int_{\mathbb{R}^n}e^{ix\xi}|\xi|^{2s}\mathcal{F}[f](\xi)d\xi\right).$$

Now we let ϵ go to 0 on both sides. The limit

$$\lim_{\epsilon\to 0}\int_{K_{\epsilon,\infty}}\frac{\cos(w_1)-1}{|w|^{n+2s}}dw$$

exists, because we already showed that

$$\int_{\mathbb{R}^n} \frac{|\cos(w_1) - 1|}{|w|^{n+2s}} dw < \infty.$$

Therefore

$$\begin{aligned} PV \int_{\mathbb{R}^n} \frac{f(x+z) - f(x)}{|z|^{n+2s}} dz &= \lim_{\epsilon \to 0} \int_{K_{\epsilon,\infty}} \frac{f(x+z) - f(x)}{|z|^{n+2s}} dz \\ &= (2\pi)^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} \frac{\cos(w_1) - 1}{|w|^{n+2s}} dw \right) \left(\int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} \mathcal{F}[f](\xi) d\xi \right). \end{aligned}$$

We substitute $y = x + z$ with $dy = dz$ and get

$$\begin{aligned} (-\Delta)^s f(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} \mathcal{F}[f](\xi) d\xi \\ &= c_{n,s} PV \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2s}} dy \end{aligned}$$

with

$$c_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(w_1)}{|w|^{n+2s}} dw \right)^{-1}.$$

To proof the formula for $(-\Delta)^{-s}$, we simply refer to [3, Definition 6.1.1]. For the sake of completeness we note that

$$c_{n,-s} = \pi^{-\frac{n}{2}} \frac{\Gamma_f\left(\frac{n-2s}{2}\right)}{\Gamma_f(s)}.$$

Here Γ_f stands for the Gamma function. □

2.1.10 Remark. *We should note that the integral representation is not really singular in the case $s \in (0, \frac{1}{2})$. In that case and for $u \in \mathcal{S}$ we have*

$$\begin{aligned} \int_{B_R(x)} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} dy &\leq L \int_{B_R(x)} \frac{|x-y|}{|x-y|^{n+2s}} dy \\ &= L \int_{B_R(x)} \frac{1}{|x-y|^{n+2s-1}} dy \\ &= L\Xi(n) \int_0^R \frac{\rho^{n-1}}{\rho^{n+2s-1}} d\rho \\ &= L\Xi(n) \int_0^R \frac{1}{\rho^{2s}} d\rho < \infty. \end{aligned}$$

Because now, $2s < 1$. L is the Lipschitz constant of u.

There is another integral representation for $(-\Delta)^s u$ that is not singular, but in that case we have to use second differences.

2.1.11 Proposition. *Let $s \in (0,1)$ and $f \in \mathcal{S}$, then we also have*

$$(-\Delta)^s f(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2f(x) - f(x+y) - f(x-y)}{|y|^{n+2s}} dy$$

$\forall x \in \mathbb{R}^n$.

Proof. See [18, Proposition 3.3] □

2.1.12 Conclusion. *Let $s \in (0,1)$. We can use the formula from proposition 2.1.11 to define $(-\Delta)^s f$ for $f \in C^{1,1}$, if f is bounded, i.e. $\sup_{x\in\mathbb{R}^n} |f(x)| = M$.*

Proof. Since $f \in C^{1,1}$ constants $M_1, \dots, M_n$ exist such that

$$|\partial_j f(x) - \partial_j f(y)| \leq M_j |x-y| \ \forall x, y \in \mathbb{R}^n \text{ and for } 1 \leq j \leq n.$$

So we can estimate:

$$\begin{aligned}
|f(x+y) - f(x) + f(x-y) - f(x)| &= |\nabla f(\xi_1)\cdot y - \nabla f(\xi_2)\cdot y| = |(\nabla f(\xi_1) - \nabla f(\xi_2))\cdot y| \\
&\leq |\nabla f(\xi_1) - \nabla f(\xi_2)||y| \\
&= \left(\sum_{j=1}^n |\partial_j f(\xi_1) - \partial_j f(\xi_2)|^2\right)^{\frac{1}{2}} |y| \\
&\leq \left(\sum_{j=1}^n M_j^2\right)^{\frac{1}{2}} |\xi_1 - \xi_2||y| \\
&\leq 2\left(\sum_{j=1}^n M_j^2\right)^{\frac{1}{2}} |y|^2.
\end{aligned}$$

Where we defined $\xi_1 = x + \theta_1 y$ and $\xi_2 = x - \theta_2 y$ for $\theta_1, \theta_2 \in (0,1)$ and so we can conclude $|\xi_1 - \xi_2| = |(\theta_1 + \theta_2)y| \leq 2|y|$.
This ensures the existence of the following integral for $r > 0$

$$\int_{B_r(0)} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|^{n+2s}} dy$$

because

$$\int_{B_r(0)} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|^{n+2s}} dy \leq 2\left(\sum_{j=1}^n M_j^2\right) \int_{B_r(0)} \frac{1}{|y|^{n+2(s-1)}} dy < \infty.$$

Because f is bounded, we can conclude

$$\int_{\mathbb{R}^n - B_r(0)} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|^{n+2s}} dy \leq 4M \int_{\mathbb{R}^n - B_r(0)} \frac{1}{|y|^{n+2s}} dy < \infty.$$

In summary we have seen that

$$(-\Delta)^s f(x) < \infty$$

for all $x \in \mathbb{R}^n$. □

We can use the representation of the inverse operator in 2.1.9 to find the fundamental solution of $(-\Delta)^s$.

2.1.13 Proposition. *Let $s \in (0,1)$. The fundamental solution of $(-\Delta)^s$ is given by $\Psi(x) = \frac{c_{n,-s}}{|x|^{n-2s}}$. This means $(-\Delta)^s(\Psi * f)(x) = f(x)$. Where $(\Psi * f)$ is the convolution of both functions.*

Proof.

$$\begin{aligned}(\Psi * f)(x) &= \int_{\mathbb{R}^n} f(y)\Psi(x-y)dy \\ &= c_{n,-s}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}}dy \\ &= (-\Delta)^{-s}f(x).\end{aligned}$$

As seen in proposition 2.1.9. By using propostion 2.1.6 and remark 2.1.5 we get:

$$\begin{aligned}(-\Delta)^s(\Psi * f)(x) &= (-\Delta)^s(-\Delta)^{-s}f(x) \\ &= (-\Delta)^{s-s}f(x) \\ &= (-\Delta)^0 f(x) = f(x).\end{aligned}$$

□

For our application it will not be enough to have $(-\Delta)^s$ defined for functions in $C^{1,1}$. Thus we will further extend the domain of the operator.

2.1.14 Lemma. *For $x, y \in \mathbb{R}^n$ and $s > 0$ we have*

$$(1+|x|)^s(1+|y|)^{-s} \le (1+|x+y|)^s.$$

Proof. We begin with

$$\begin{aligned}(1+|z-y|) &\le (1+|z|+|y|) \\ &\le (1+|z|+|y|+|z||y|) \\ &= (1+|z|)(1+|y|),\end{aligned}$$

where $z \in \mathbb{R}^n$. Thus

$$(1+|z-y|)^s(1+|y|)^{-s} \le (1+|z|)^s.$$

Now when we define $z = x+y$ we get

$$(1+|x|)^s(1+|y|)^{-s} \le (1+|x+y|)^s.$$

□

2.1.15 Proposition. *For $f \in \mathcal{S}$ we have*

$$\sup_{x\in\mathbb{R}^n}(1+|x|^{n+2s})|(-\Delta)^s f(x)| < \infty.$$

See [14, p. 73].

Proof. We will use the representation formula given in 2.1.11 and show that

$$\sup_{x\in\mathbb{R}^n}\left|(1+|x|)^{n+2s}\int_{\mathbb{R}^n}\frac{2f(x)-f(x+y)-f(x-y)}{|y|^{n+2s}}dy\right| < \infty.$$

From this the proposed follows since

$$(1+|x|^{n+2s}) \le c(1+|x|)^{n+2s}.$$

We begin by integrating over all y with $|y| \geq 1$. We will show that

$$(1+|x|)^{n+2s} \int_{|y|\geq 1} \frac{2|f(x)| + |f(x+y)| + |f(x-y)|}{|y|^{n+2s}} dy \leq const.$$

To do that we estimate the individual integrals.
Because $|y| \geq 1$ we have that

$$|y| \geq \frac{1+|y|}{2}$$

and therefore

$$\frac{1}{|y|^{n+2s}} \leq c_1(n,s) \frac{1}{(1+|y|)^{n+2s}}.$$

Thus we have

$$\begin{aligned}
\int_{|y|\geq 1} \frac{(1+|x|)^{n+2s}|f(x+y)|}{|y|^{n+2s}} dy &\leq c_1 \int_{|y|\geq 1} \frac{(1+|x|)^{n+2s}|f(x+y)|}{(1+|y|)^{n+2s}} dy \\
&\leq c_1 \int_{|y|\geq 1} (1+|x+y|)^{n+2s}|f(x+y)| dy \\
&\leq c_1 \int_{\mathbb{R}^n} (1+|z|)^{n+2s}|f(z)| \frac{(1+|z|)^{n+1}}{(1+|z|)^{n+1}} dz \\
&\leq c_1 C_{0,2n+3}(f) \int_{\mathbb{R}^n} \frac{1}{(1+|z|)^{n+1}} dz \\
&= const.
\end{aligned}$$

Here inequality 2.1.14 was used in the second line and in the 4th line the fact that $f \in \mathcal{S}$ and that $2s < 2$. Analogously one can show that

$$\int_{|y|\geq 1} \frac{(1+|x|)^{n+2s}|f(x-y)|}{|y|^{n+2s}} dy \leq const.$$

because one can simply exchange y with $-y$ in 2.1.14. Finally we have

$$\begin{aligned}
\int_{|y|\geq 1} \frac{(1+|x|)^{n+2s}|f(x)|}{|y|^{n+2s}} dy &\leq C_{0,n+2}(f) \int_{|y|\geq 1} \frac{1}{|y|^{n+2s}} dy \\
&= const.
\end{aligned}$$

All that is left to show is that we can find an estimation for the integral over the ball. For that we will use that one can find a $\theta \in (-1,1)$ that depends on x, y such that

$$|2f(x) - f(x+y) - f(x-y)| \leq c_2(n) \max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)||y|^2.$$

So now we have

$$\begin{aligned}
&(1+|x|)^{n+2s} \int_{|y|<1} \frac{|2f(x) - f(x+y) - f(x-y)|}{|y|^{n+2s}} dy \\
&\leq c_2 (1+|x|)^{n+2s} \int_{|y|<1} \frac{\max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)||y|^2}{|y|^{n+2s}} dy \\
&= c_2 (1+|x|)^{n+2s} \left\{ \sup_{|y|<1} \max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)| \right\} \int_{|z|<1} \frac{1}{|z|^{n-2(1-s)}} dz \\
&= c_3(n,s) \sup_{|y|<1} (1+|x|)^{n+2s} \max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)|
\end{aligned}$$

$$
\begin{aligned}
&\overset{(1)}{\leq} c_3 \sup_{|y|<1} (1+|\theta y|)^{n+2s}(1+|x+\theta y|)^{n+2s} \max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)| \\
&\leq c_4(n,s) \sup_{|y|<1} (1+|x+\theta y|)^{n+2s} \max_{i,j=1,\dots,n} |\partial_i \partial_j f(x+\theta y)| \\
&\leq c_4(n,s) \sup_{z\in\mathbb{R}^n} (1+|z|)^{n+2s} \max_{i,j=1,\dots,n} |\partial_i \partial_j f(z)| \\
&\leq c_4(n,s) \max_{i,j=1,\dots,n} \sup_{z\in\mathbb{R}^n} (1+|z|)^{n+2} |\partial_i \partial_j f(z)| \\
&\overset{(2)}{\leq} c_4(n,s) \max_{i,j=1,\dots,n} C_{\alpha_{ij},n+2}(f) \\
&= const.
\end{aligned}
$$

In (1) we used inequality 2.1.14 again but now in the form

$$(1+|x|)^{n+2s} \leq (1+|\theta y|)^{n+2s}(1+|x+\theta y|)^{n+2s}.$$

In (2) $\alpha_{ij} \in \mathbb{N}_0^n$ with $|\alpha_{ij}| = 2$ and it can be understood as $e_i + e_j$ where the e_k are the unit vectors that are zero in every entry except in the kth position where they are 1.
To summarize we showed that

$$(1+|x|)^{n+2s} \int_{\mathbb{R}^n} \frac{2|f(x)| + |f(x+y)| + |f(x-y)|}{|y|^{n+2s}} dy < const.$$

for every $x \in \mathbb{R}^n$ and therefore we proved that

$$\sup_{x\in\mathbb{R}^n} (1+|x|^{n+2s})|(-\Delta)^s f(x)| < \infty.$$

□

2.1.16 Remark. *The last proposition allows us to extend the definition of $(-\Delta)^s$ to a larger class of functions. With $L_{1,s}$ we denote the space of all functions u for which $\|u|L_{1,s}\| < \infty$, where*

$$\|u|L_{1,s}\| = \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx.$$

Now we define $(-\Delta)^s u$ as a tempered distribution then for every $f \in \mathcal{S}$ we have

$$\langle (-\Delta)^s u, f\rangle = \langle u, (-\Delta)^s f\rangle.$$

The expression on the right side exists because

$$
\begin{aligned}
|\langle u, (-\Delta)^s f\rangle| &\leq \int_{\mathbb{R}^n} |u(x)||((-\Delta)^s f)(x)| dx \\
&= \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} (1+|x|^{n+2s})|((-\Delta)^s f)(x)| dx \\
&\leq C \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx \\
&= C\|u|L_{1,s}\|.
\end{aligned}
$$

It should be noted that this coincides with our old definition in the case that $u \in \mathcal{S}$. We will show that in 2.1.19.

So now we want to show that

$$\langle (-\Delta)^s u, f\rangle = \langle u, (-\Delta)^s f\rangle$$

is always true when $f, u \in \mathcal{S}$.

2.1.17 Lemma. *Let $f \in \mathcal{S}$ and $s \in (0,1)$, then $|\xi|^{2s} f(\xi) \in L_2(\mathbb{R}^n)$.*

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{4s} |f(\xi)|^2 d\xi &\leq C_{0,n+2} \int_{\mathbb{R}^n} |\xi|^{4s} (1+|\xi|)^{-2n-4} d\xi \\ &= \Xi(n) C_{0,n} \int_0^\infty \frac{\rho^{4s+n-1}}{(1+\rho)^{2n+4}} d\rho \\ &\leq \Xi(n) C_{0,n} \int_0^\infty (1+\rho)^{4s+n-1-2n-4} d\rho \\ &= \Xi(n) C_{0,n} \int_1^\infty \rho^{4(s-1)-1-n} d\rho.\end{aligned}$$

The integral exists if the exponent is smaller than -1, which it obviously is, since $(s-1) < 0$. □

2.1.18 Conclusion. *For $s \in (0,1)$*

$$(-\Delta)^s : \mathcal{S} \to L_2.$$

Proof. We know that $\mathcal{F}[f] \in \mathcal{S}$, when $f \in \mathcal{S}$, see A.2.6. This means $|\xi|^{2s}\mathcal{F}[f](\xi) \in L_2(\mathbb{R}^n)$, by lemma 2.1.17. Lastly, when we use conclusion A.2.12 we get that $(-\Delta)^s f = \mathcal{F}^{-1}\left[|\xi|^{2s}\mathcal{F}[f](\xi)\right] \in L_2(\mathbb{R}^n)$. □

2.1.19 Proposition. *For any $f, g \in \mathcal{S}$ we have*

$$\langle (-\Delta)^s f, g\rangle = \langle f, (-\Delta)^s g\rangle.$$

Proof. First we will check whether the expressions $\langle (-\Delta)^s f, g\rangle$ and $\langle f, (-\Delta)^s g\rangle$ are well-defined. $f, g \in L_2$ by proposition A.1.2 and $(-\Delta)^s f, (-\Delta)^s g \in L_2$ by conclusion 2.1.18. So the L_2 scalar products are well-defined.
We will use the formulas from Proposition A.2.9.

$$\begin{aligned}\langle (-\Delta)^s f, g\rangle &= \langle \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}[f](\xi)], g\rangle \\ &= \langle |\xi|^{2s}\mathcal{F}[f](\xi), \mathcal{F}[g](\xi)\rangle \\ &= \int_{\mathbb{R}^n} \left(|\xi|^{2s}\mathcal{F}[f](\xi)\right) \overline{\mathcal{F}[g](\xi)} d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{F}[f](\xi)\overline{(|\xi|^{2s}\mathcal{F}[g](\xi))} d\xi \\ &= \langle \mathcal{F}[f](\xi), |\xi|^{2s}\mathcal{F}[g](\xi)\rangle \\ &= \langle f, \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}[g](\xi)]\rangle \\ &= \langle f, (-\Delta)^s g\rangle.\end{aligned}$$

□

Next we will show that $(-\Delta)^s$ commutes with the translation operator $\rho_y(u)(x) = u(x+y)$.

2.1.20 Proposition. *For $y \in \mathbb{R}^n$ fixed, we have*

$$(-\Delta)^s[u(x+y)] = [(-\Delta)^s u](x+y)$$

i.e. the operator commutes with rigid motions. To clarify, x is variable of the function and y is the rigid motion.

Proof. To prove this we use the actual definition of $(-\Delta)^s$. First we note that

$$\begin{aligned}\mathcal{F}[u(\cdot + y)](\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x+y)dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-iz\xi} e^{iy\xi} u(z)dz \\ &= e^{iy\xi}\mathcal{F}[u](\xi).\end{aligned}$$

Here we used the transformation $z = x + y$ with $dz = dx$. Now we get

$$\begin{aligned}(-\Delta)^s[u(x+y)] &= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}|\xi|^{2s}\mathcal{F}[u(\cdot + y)](\xi)d\xi \\ &= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}|\xi|^{2s} e^{iy\xi}\mathcal{F}[u](\xi)d\xi \\ &= (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x+y)\xi}|\xi|^{2s}\mathcal{F}[u](\xi)d\xi \\ &= [(-\Delta)^s u](x+y)\end{aligned}$$

□

2.2. Removal of the singularity

We want to work with the fundamental solution of the operator, but the singularity in $x = 0$ hinders us. Because of that, we are going to define a function Γ that coincides with the fundamental solution Ψ for $|x| > 1$ and if $|x| \leq 1$ it should be a paraboloid φ. The function we get should be at least continuously differentiable, but we will even get Lipschitz-continuity of the first derivatives. See the picture below to get an idea what it should look like.

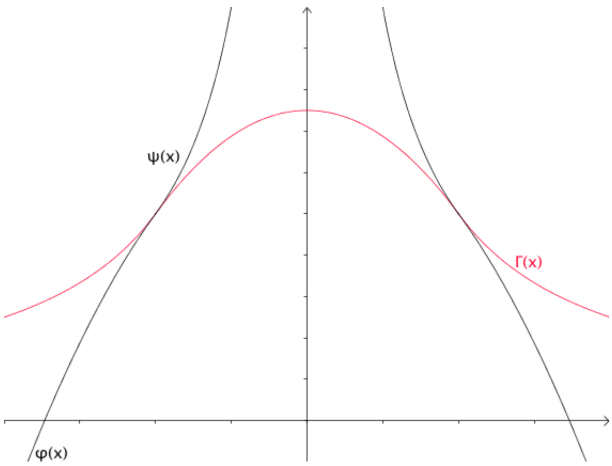

Figure 2.1.: The modified fundamental solution.

2.2.1 Proposition. *Let $c \in \mathbb{R}$ be arbitrary. Then*

$$\Gamma(x) = \begin{cases} \frac{c(n-2s)}{2}(1-|x|^2) + c & \text{für } |x| \leq 1 \\ \frac{c}{|x|^{n-2s}} & \text{für } |x| > 1 \end{cases}$$

is a continuously differentiable function.

Proof. The functions $\varphi(x) = \frac{c(n-2s)}{2}(1-|x|^2)+c$ and $\psi(x) = \frac{c}{|x|^{n-2s}}$ are at least C^2 in their respective domains. Therefore, we focus on what happens if $|x| = 1$. In order to do that, we choose $x^0 \in \mathbb{R}^n$ with $|x^0| = 1$, we get:

$$\lim_{x\to x^0, |x|\le 1} \varphi(x) = \frac{c(n-2s)}{2}(1-1)+c = c. \qquad \lim_{x\to x^0, |x|>1} \psi(x) = c$$

This proves the continuity, let's look at the first derivatives:

$$\partial_{x_j}\varphi(x) = -c(n-2s)x_j \qquad \partial_{x_j}\psi(x) = -\frac{c(n-2s)x_j}{|x|^{n-2s+2}}.$$

Again we choose $x^0 \in \mathbb{R}^n$ mit $|x^0| = 1$ and obtain:

$$\lim_{x\to x^0, |x|\le 1} \partial_{x_j}\varphi(x) = -c(n-2s)x_j^0 \qquad \lim_{x\to x^0, |x|>1} \partial_{x_j}\psi(x) = -c(n-2s)x_j^0.$$

This is true for $j = 1, \ldots, n$ and so all partial derivatives of Γ are continuous. □

2.2.2 Remark. 1. *But the second derivatives of Γ are not continuous along the unit sphere. For $i \neq j$, we have*

$$\partial_{x_i x_j}\varphi(x) \equiv 0 \qquad \partial_{x_i x_j}\psi(x) = \frac{c(n-2s)(n-2s+2)x_i x_j}{|x|^{n-2s+4}}$$

However $\partial_{x_i x_j}\psi(x)$ is not identical 0.
We will see that we at least have Lipschitz continuity of the first partial derivaties of Γ, so altogether $\Gamma \in C^{1,1}$.

2. *Landkof had a similar idea when he studied superharmonic functions of fractional order, see* [7, p. 112]. *But he was a bit more drastic by cutting off the fundamental solution on a neighbourhood of the origin in order to avoid the singularity.*

We need the following Lemma to prove that $\Gamma \in C^{1,1}$.

2.2.3 Lemma. *We have:*

(i) $|\partial_{x_i x_j}\varphi(x)| < C_1$ *for all x with $|x| \le 1$ if $1 \le i, j \le n$.*

(ii) $|\partial_{x_i x_j}\psi(x)| < C_2$ *for all x with $|x| > \frac{1}{2}$ if $1 \le i, j \le n$.*

Proof. (i) First we note that:

$$\partial_{x_i x_j}\varphi(x) = -c(n-2s)\delta_{ij}$$

and so apparently

$$|\partial_{x_i x_j}\varphi(x)| \le |c|(n-2s).$$

(ii) Now note:

$$\partial_{x_i x_j}\psi(x) = \frac{c(n-2s)(n-2s+2)x_i x_j}{|x|^{n-2s+4}} - \frac{c(n-2s)\delta_{ij}}{|x|^{n-2s+2}}.$$

Thus

$$\begin{aligned}
|\partial_{x_i,x_j}\psi(x)| &\leq \frac{|c|(n-2s)(n-2s+2)}{|x|^{n-2s+2}}\frac{|x_i|}{|x|}\frac{|x_j|}{|x|} + \frac{|c|(n-2s)}{|x|^{n-2s+2}} \\
&\leq 2^{n-2s+2}|c|(n-2s)(n-2s+2) + 2^{n-2s+2}|c|(n-2s) \\
&= 2^{n-2s+2}|c|(n-2s)(n-2s+3).
\end{aligned}$$

Keep in mind that $|x_i| \leq |x|$ is always true and that $\frac{1}{|x|} < 2$.

□

2.2.4 Proposition. *Γ as defined above is in $C^{1,1}$.*

Proof. Let's review the definition of Γ

$$\Gamma(x) = \begin{cases} \frac{c(n-2s)}{2}(1-|x|^2) + c = \varphi(x) & \text{für } |x| \leq 1 \\ \frac{c}{|x|^{n-2s}} = \psi(x) & \text{für } |x| > 1. \end{cases}$$

Our goal is to use propostion A.3.3 on the first derivatives of φ and ψ.

Lemma 2.2.3.(i) shows us that the second derivatives of φ are bounded on $B_1(0)$ and since this domain is convex we can apply proposition A.3.1 to the first derivatives of φ and get

$$|\partial_{x_i}\varphi(x) - \partial_{x_i}\varphi(y)| \leq m_1|x-y|$$

for all $i = 1,\ldots,n$. Next we use lemma 2.2.3.(ii) to see that the second derivatives of ψ are bounded for $|x| \geq \frac{1}{2}$ so by proposition A.3.2 we have Lipschitz-continuity of $\partial_{x_i}\psi$ for $i = 1,\ldots,n$ as long as $|x| \geq 1$.

At last all the requirements of proposition A.3.3 are fulfilled and we can conclude the Lipschitz-continuity of $\partial_{x_i}\Gamma$ for all $i = 1,\ldots,n$. □

2.2.5 Remark. *From now on we denote*

$$\Gamma(x) = \begin{cases} \frac{c_{n,-s}(n-2s)}{2}(1-|x|^2) + c_{n,-s} = \varphi(x) & \textit{für } |x| \leq 1 \\ \frac{c_{n,-s}}{|x|^{n-2s}} = \Psi(x) & \textit{für } |x| > 1 \end{cases}$$

where $c_{n,-s}$ is the constant mentioned in proposition 2.1.9.

2.3. Supersolutions and Comparison

We start this section with a closer investigation of our newly found function Γ but we will quickly focus on the function $(-\Delta)^s\Gamma$ because it will give us an approximation of the identity and thus play a vital role in our characterization of supersolutions.

2.3.1 Proposition. *For $\lambda > 0$ we define: $\Gamma_\lambda(x) = \frac{\Gamma(\frac{x}{\lambda})}{\lambda^{n-2s}}$. Then:*

1. *$\Gamma_\lambda \in C^{1,1}$.*
2. *Γ_λ coincides with the fundamental solution $\Psi(x)$ outside of the ball with radius λ around zero.*
3. *Γ_λ is bounded and*

$$\sup_{x\in\mathbb{R}^n} \Gamma_\lambda(x) = \frac{c_{n,-s}(n+2(1-s))}{2\lambda^{n-2s}}.$$

Proof. 1. First we note that $\Gamma \in C^{1,1}$. Then we compute $\partial_j \Gamma_\lambda(x) = \partial_j \Gamma(\frac{x}{\lambda})\frac{1}{\lambda^{n+1-2s}}$ and since $\partial_j \Gamma \in C^0$ for any j, also $\partial_j \Gamma_\lambda \in C^0$ for any j.
Since also $\partial_j \Gamma \in C^{0,1}$, i.e. for every j there exists a constant L_j such that $|\Gamma(x)-\Gamma(y)| \leq L_j|x-y|$, we have

$$\begin{aligned}
|\partial_j \Gamma_\lambda(x) - \partial_j \Gamma_\lambda(y)| &= \frac{1}{\lambda^{n+1-2s}} \left|\partial_j \left[\Gamma\left(\frac{x}{\lambda}\right)\right] - \partial_j \left[\Gamma\left(\frac{y}{\lambda}\right)\right]\right| \\
&\leq \frac{L_j}{\lambda^{n+2-2s}}|x-y|.
\end{aligned}$$

So also $\partial_j \Gamma \in C^{0,1}$. In summary we have $\Gamma_\lambda \in C^{1,1}$.

2. We note that $\Gamma(x)$ coincides with the fundamental solution Ψ outside of the ball with radius 1 around zero. So if $|x| > \lambda$ we get $\frac{|x|}{\lambda} > 1$ and

$$\begin{aligned}
\Gamma_\lambda(x) &= \Gamma\left(\frac{x}{\lambda}\right)\frac{1}{\lambda^{n-2s}} \\
&= \Psi\left(\frac{x}{\lambda}\right)\frac{1}{\lambda^{n-2s}} \\
&= \frac{c_{n,-s}}{|x|^{n-2s}}\frac{\lambda^{n-2s}}{\lambda^{n-2s}} \\
&= \Psi(x).
\end{aligned}$$

3. Outside $B_1(0)$ Γ_λ is a decreasing function and inside $B_1(0)$ it is continuous and thus bounded. By taking a look at the picture above one quickly realizes that the maximum must be attained in 0 and the value in 0 is

$$\Gamma_\lambda(0) = \frac{c_{n,-s}(n+2(1-s))}{2\lambda^{n-2s}}.$$

□

2.3.2 Proposition. *If $\lambda_1 \leq \lambda_2$, then for every $x \in \mathbb{R}^n$ we have $\Gamma_{\lambda_1}(x) \geq \Gamma_{\lambda_2}(x)$.*

Proof. The inequality is fulfilled if $|x| > \lambda_2$, since both functions coincide with $\Psi(x)$ there.
First the case $\lambda_1 \leq |x| \leq \lambda_2$. Here Γ_{λ_1} already coincides with Ψ but Γ_{λ_2} is still a paraboloid. So we just show that $\Psi(x) \geq \varphi_\lambda(x)$ for $|x| \leq \lambda$ with $\varphi_\lambda(x) = \frac{1}{\lambda^{n-2s}}\varphi\left(\frac{x}{\lambda}\right)$.
Because $\Psi(x) = \Psi(|x|e_1)$ and $\varphi(x) = \varphi(|x|e_1)$, we can further simplify the problem by studying the functions $\Psi(t) = \Psi(te_1)$ and $\varphi_\lambda(t) = \varphi_\lambda(te_1)$, where e_1 is the first unit vector.
By definition of the functions we can immediately conclude that $\Psi(\lambda) = \varphi_\lambda(\lambda)$. Next we check the first derivatives:

$$\Psi'(t) = -c(n-2s)\frac{1}{t^{n-2s+1}} \qquad \varphi'_\lambda(t) = -c(n-2s)\frac{t}{\lambda^{n+2-2s}}.$$

From $t \leq \lambda$ follows

$$\frac{1}{t^{n-2s+1}} \geq \frac{1}{\lambda^{n-2s+1}} \geq \frac{t}{\lambda^{n-2s+2}}.$$

So we conclude that $\Psi'(t) \leq \varphi'_\lambda(t)$ for $t \leq \lambda$. Together with the fact that $\Psi(\lambda) = \varphi_\lambda(\lambda)$, proposition A.3.5 yields that $\Psi(t) \geq \varphi_\lambda(t)$ if $t \leq \lambda$ and thus $\Psi(x) \geq \varphi_\lambda(x)$ if $|x| \leq \lambda$.
Next is the case $|x| < \lambda_1 \leq \lambda_2$. Here both Γ_{λ_1} and Γ_{λ_2} still coincide with paraboloids φ_{λ_1} and φ_{λ_2}. We again reduce this to a one-dimensional problem by defining $\varphi_{\lambda_j}(t) = \varphi_{\lambda_j}(te_1)$ for $j = 1, 2$. We just showed that $\varphi_{\lambda_1}(\lambda_1) = \Psi(\lambda_1) \geq \varphi_{\lambda_2}(\lambda_1)$. Since

$$\frac{t}{\lambda_1^{n-2s+2}} \geq \frac{t}{\lambda_2^{n-2s+2}}$$

we have $\varphi'_{\lambda_1}(t) \leq \varphi'_{\lambda_2}(t)$ and again, by using proposition A.3.5, $\varphi_{\lambda_1}(t) \geq \varphi_{\lambda_2}(t)$ for $t < \lambda_1$ and thus $\varphi_{\lambda_1}(x) \geq \varphi_{\lambda_2}(x)$ if $|x| < \lambda_1$. □

2.3.3 Proposition. *The function* $(-\Delta)^s\Gamma$ *is positive. See* [14, Proposition 2.11]

Proof. In propostion 2.2.4 we showed that $\Gamma \in C^{1,1}$ and because of conclusion 2.1.12 and proposition 2.3.1 we can use the formula in proposition 2.1.11 to compute $(-\Delta)^s\Gamma$.

(i) For starters, we take a look at the case $x_0 = 0$. Γ attains its maximum in x_0, this means of course $\Gamma(0) \geq \Gamma(y)$ for every $y \in \mathbb{R}^n$. So $\Gamma(0) - \Gamma(y)$ is positive for every $y \in \mathbb{R}^n$ which gives us

$$\begin{aligned}(-\Delta)^s\Gamma(0) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{\Gamma(0) - \Gamma(y) + \Gamma(0) - \Gamma(-y)}{|y|^{n+2s}} dy \\ &> 0.\end{aligned}$$

(ii) Now $x_0 \notin B_1$. Obviously, $\Gamma(x_0) = \Psi(x_0)$, where $\Psi(x) = \frac{c_{n,-s}}{|x|^{n-2s}}$ was the fundamental solution of $(-\Delta)^s$. For every other x we have $\Gamma(x) \leq \Psi(x)$ and especially $\Gamma(x) < \Psi(x)$ for $x \in B_1(0)$ so we get:

$$\begin{aligned}(-\Delta)^s\Gamma(x_0) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Gamma(x_0) - \Gamma(x_0+y) - \Gamma(x_0-y)}{|y|^{n+2s}} dy \\ &> \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Psi(x_0) - \Psi(x_0+y) - \Psi(x_0-y)}{|y|^{n+2s}} dy \\ &= (-\Delta)^s\Psi(x_0).\end{aligned}$$

(iii) At last we consider $x_0 \in B_1\backslash\{0\}$. We use the function $F(y)$ as defined in proposition A.3.4 and we know that $F(x_0) = \Gamma(x_0)$ and $F(y) > \Gamma(y)$ for every $y \in \mathbb{R}^n\backslash\{x_0\}$ because of conclusion A.3.11.

$$\begin{aligned}(-\Delta)^s\Gamma(x_0) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Gamma(x_0) - \Gamma(x_0+y) - \Gamma(x_0-y)}{|y|^{n+2s}} dy \\ &> \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2F(x_0) - F(x_0+y) - F(x_0-y)}{|y|^{n+2s}} dy \\ &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Psi(\lambda x_0) - \Psi(\lambda x_0+y) - \Psi(\lambda x_0-y)}{|y|^{n+2s}} dy \\ &= (-\Delta)^s\Psi(\lambda x_0) \\ &= (-\Delta)^s\Psi\left(\frac{x_0}{|x_0|^{\frac{n-2s+2}{n-2s+1}}}\right).\end{aligned}$$

(iv) Now we will apply the results from (ii) and (iii), where we set $\alpha = \frac{n-2s+2}{n-2s+1} > 1$. We choose a positive function $\varphi \in C_c^\infty(\mathbb{R}^n - \{0\})$ and have

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s\Gamma(y)\varphi(y)dy &= \int_{B_1(0)} (-\Delta)^s\Gamma(y)\varphi(y)dy + \int_{B_1^c(0)} (-\Delta)^s\Gamma(y)\varphi(y)dy \\ &> \int_{B_1(0)} (-\Delta)^s\Psi(y)\varphi(y)dy + \int_{B_1^c(0)} (-\Delta)^s\Psi\left(\frac{y}{|y|^\alpha}\right)\varphi(y)dy \\ &> \int_{B_1(0)} (-\Delta)^s\Psi(y)\varphi(y)dy + \int_{B_1^c(0)} (-\Delta)^s\Psi(z)\varphi\left(\frac{z}{|z|^{\frac{\alpha}{\alpha-1}}}\right)\frac{dz}{|z|^{\frac{\alpha n}{\alpha-1}}} \\ &= \delta(\varphi\chi_{B_1(0)}) + \delta(\beta\chi_{B_1^c(0)}) \\ &= 0.\end{aligned}$$

Where $\beta(z) = \varphi\left(\frac{z}{|z|^{\frac{\alpha}{\alpha-1}}}\right)\frac{1}{|z|^{\frac{\alpha n}{\alpha-1}}}$. We used the substitution $z = \frac{y}{|y|^\alpha}$, then $|z| = \frac{1}{|y|^{\alpha-1}}$ or $|y| = \frac{1}{|z|^{\frac{1}{\alpha-1}}}$ thus $y = z|y|^\alpha = \frac{z}{|z|^{\frac{\alpha}{\alpha-1}}}$. This substitution gives us $dz = \frac{dy}{|y|^{\alpha n}}$ or, by the beforementioned,

$dy = \frac{dz}{|z|^{\frac{\alpha n}{\alpha - 1}}}$.
To summarize we have seen that $\int_{\mathbb{R}^n} (-\Delta)^s \Gamma(y) \varphi(y) dy > 0$ for every positive $\varphi \in C_c^\infty(\mathbb{R}^n - \{0\})$, this means that $(-\Delta)^s\Gamma(x) > 0$ for every $x \in \mathbb{R}^n - \{0\}$ because it is continuous. This will be shown in the next proposition.

□

Our next goal is to show the continuity of $(-\Delta)^s\Gamma$.

2.3.4 Proposition. $(-\Delta)^s\Gamma$ *is continuous.*

Proof. To prove this, we take the representation given in 2.1.11 and apply Lebesgue's Theorem on the continuity of parameter integrals, see [1, Theorem A.3], note that Γ is $C^{1,1}$ and bounded, therefore we can use that representation.
We consider the function

$$f(x,y) = \begin{cases} \frac{2\Gamma(x) - \Gamma(x+y) - \Gamma(x-y)}{|y|^{n+2s}} & y \neq 0 \\ 0 & y = 0. \end{cases}$$

Then we still have

$$(-\Delta)^s\Gamma(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} f(x,y) dy,$$

because we changed the integrand only on a set of measure zero.
f is continuous in x for every y, because Γ is continuous. It is also integrable for every y, as we have seen in 2.1.12. In the proof of that proposition we also found a integrable majorant that is independent of y, it was given by

$$F(x) = 2\left(\sum_{j=1}^n M_j^2\right) \frac{1}{|y|^{n+2s-2}} \chi_{B_1(0)} + 4M \frac{1}{|y|^{n+2s}} \chi_{\mathbb{R}^n \setminus B_1(0)}.$$

Where the M_j are the Lipschitz constants of the first derivatives of Γ and $M = \sup_{x \in \mathbb{R}^n} |\Gamma(x)|$. This concludes the proof. □

We denote $\gamma_\lambda = (-\Delta)^s\Gamma_\lambda$.

2.3.5 Proposition. *The function* γ_1 *is also symmetric, i.e. if* $|x_1| = |x_2|$ *then* $\gamma_1(x_1) = \gamma_1(x_2)$.

Proof. Let A be a rotation that maps x_1 onto x_2, i.e. $Ax_1 = x_2$ or $x_1 = A^{-1}x_2$. We will use the variable transformation $z = Ay$, then we have $dz = dy$, because $\det(A) = 1$ and we get

$$\begin{aligned} \gamma_1(x_1) &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2\Gamma(x_1) - \Gamma(x_1 + y) - \Gamma(x_1 - y)}{|y|^{n+2s}} dy \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2\Gamma(x_1) - \Gamma(x_1 + A^{-1}z) - \Gamma(x_1 - A^{-1}z)}{|z|^{n+2s}} dz \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2\Gamma(x_1) - \Gamma\left(A^{-1}(x_2 + z)\right) - \Gamma\left(A^{-1}(x_2 - z)\right)}{|z|^{n+2s}} dz \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2\Gamma(x_2) - \Gamma(x_2 + z) - \Gamma(x_2 - z)}{|z|^{n+2s}} dz \\ &= \gamma_1(x_2). \end{aligned}$$

We used the symmetry of Γ, i.e. $|x_1| = |x_2| \Rightarrow \Gamma(x_1) = \Gamma(x_2)$ and the fact that $|A^{-1}z| = |z|$. □

The next proposition tells us something about the asymptotic behaviour of γ_λ.

2.3.6 Proposition. *For every $\lambda > 0$ constants $0 < c_1 = c_1(n,s,\lambda)$ and $0 < c_2 = c_2(n,s,\lambda)$ exist such that*
$$\frac{c_1}{|x|^{n+2s}} \leq \gamma_\lambda(x) \leq \frac{c_2}{|x|^{n+2s}}$$
when $|x| > 2\lambda$. See [14, Proposition 2.12].

Proof. We start with the upper approximation. And we choose $x \in \mathbb{R}^n$ such that $|x| \geq 2\lambda$.
$$\begin{aligned}
\gamma_\lambda(x) &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2\Gamma_\lambda(x) - \Gamma_\lambda(x+y) - \Gamma_\lambda(x-y)}{|y|^{n+2s}} dy \\
&= \frac{c_{n,s}}{2}\left\{\int_{\mathbb{R}^n} \frac{2\Gamma_\lambda(x) - \Psi(x+y) - \Psi(x-y)}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{\Psi(x+y) - \Gamma_\lambda(x+y)}{y^{n+2s}} dy \right.\\
&\qquad \left. + \int_{\mathbb{R}^n} \frac{\Psi(x-y) - \Gamma_\lambda(x-y)}{y^{n+2s}} dy\right\} \\
&= \frac{c_{n,s}}{2}\left\{\int_{\mathbb{R}^n} \frac{2\Psi(x) - \Psi(x+y) - \Psi(x-y)}{|y|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{\Psi(z) - \Gamma_\lambda(z)}{|x-z|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{\Psi(z) - \Gamma_\lambda(z)}{|z-x|^{n+2s}} dy\right\} \\
&= \frac{c_{n,s}}{2}\left\{(-\Delta)^s\Psi(x) + 2\int_{\mathbb{R}^n} \frac{\Psi(z) - \Gamma_\lambda(z)}{|x-z|^{n+2s}} dy\right\} \\
&= c_{n,s}\int_{\mathbb{R}^n} \frac{\Psi(z) - \Gamma_\lambda(z)}{|x-z|^{n+2s}} dy.
\end{aligned}$$

Here $(-\Delta)^s\Psi(x) = 0$ by an similar argument like the one in step (iv) of the proof of 2.3.3.
The function $\Psi(y) - \Gamma_\lambda(y)$ has a compact support, because of proposition 2.3.1.2. Its support is the ball around zero with radius λ. Therefore
$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\Psi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2s}} dy &= \int_{B_\lambda(0)} \frac{\Psi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2s}} dy \\
&\leq \left(\int_{B_\lambda(0)} [\Psi(y) - \Gamma_\lambda(y)]\, dy\right)\left(\sup_{y \in B_\lambda(0)} \frac{1}{|x-y|^{n+2s}}\right).
\end{aligned}$$

Because Γ_λ is bounded, we can estimate
$$\begin{aligned}
\left|\int_{B_\lambda(0)} [\Psi(y) - \Gamma_\lambda(y)]\, dy\right| &\leq \int_{B_\lambda(0)} [\Psi(y) + |\Gamma_\lambda(y)|]\, dy \\
&\leq \int_{B_\lambda(0)} \frac{c_{n,-s}}{|y|^{n-2s}} dy + \sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y)|B_\lambda| \\
&= c_{n,-s}\Xi(n)\int_0^\lambda \frac{1}{\rho^{1-2s}} d\rho + \sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y)|B_\lambda| \\
&= \frac{c_{n,-s}\Xi(n)}{2s}\lambda^{2s} + \sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y)|B_\lambda| < \infty.
\end{aligned}$$

Here $|B_\lambda(0)|$ denotes the volume of $B_\lambda(0)$.
Next we will find an estimation for
$$\sup_{y \in B_\lambda(0)} \frac{1}{|x-y|^{n+2s}}.$$
We choose the $y \in \overline{B_\lambda(0)}$ that is the closest to x, which is $y = \frac{x\lambda}{|x|}$. Then we get
$$\sup_{y \in B_\lambda(0)} \frac{1}{|x-y|^{n+2s}} \leq \frac{1}{|x - \frac{x\lambda}{|x|}|^{n+2s}}$$

$$\begin{aligned}&\stackrel{(1)}{\leq} \frac{1}{(|x|-\lambda)^{n+2s}}\\&\stackrel{(2)}{\leq} \frac{2^{n+2s}}{|x|^{n+2s}}.\end{aligned}$$

(1) is correct by left side of the triangle inequality, i.e. $|a-b| \geq ||a|-|b||$, and the fact that $|x| > \lambda$.
(2) is true because $\lambda \leq \frac{|x|}{2}$ implies that $|x| - \lambda \geq \frac{|x|}{2}$.
In summary we have seen that

$$\begin{aligned}\gamma_\lambda(x) &= c_{n,s} \int_{\mathbb{R}^n} \frac{\Psi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2s}} dy\\&\leq c_{n,s} \left(\frac{c_{n,-s}\Xi(n)}{2s} \lambda^{2s} + \sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y) |B_\lambda| \right) \frac{2^{n+2s}}{|x|^{n+2s}}\\&= c_2(n,s,\lambda) \frac{1}{|x|^{n+2s}}.\end{aligned}$$

Now to the lower estimation. We first want to explain the basic idea, see the picture 2.2.

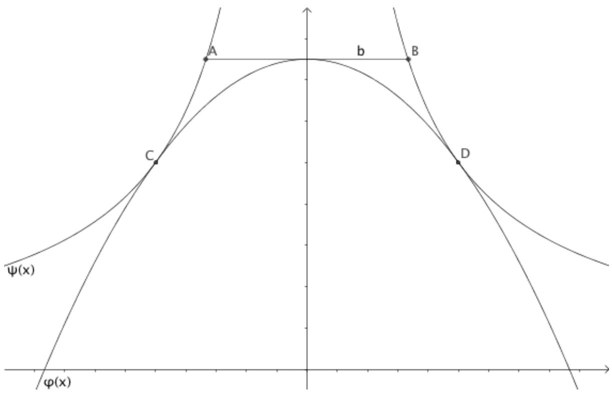

Figure 2.2.: The relevant set.

We want an estimation of the form

$$\Psi(x) - \Gamma_\lambda(x) \geq \Psi(x) - \sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y)$$

but the right hand side does not have to be positive for all $x \in B_\lambda(0)$. In figure 2.2 that would be the interval between C and D. So we want the set of all points, where it is always positive, which would be the interval between A and B. As the first step we have to determine the value of Γ_λ in its maximum. Because Γ_λ coincides with a paraboloid inside $B_\lambda(0)$, it attains its maximum in $x = 0$. By definition of Γ_λ we have

$$\sup_{y \in \mathbb{R}^n} \Gamma_\lambda(y) = c_{n,-s} \left(1 + \frac{n-2s}{2}\right) \frac{1}{\lambda^{n-2s}}.$$

Next we have to find the points x_0 such that

$$\begin{aligned}\Psi(x_0) &= c_{n,-s} \left(1 + \frac{n-2s}{2}\right) \frac{1}{\lambda^{n-2s}}\\&= c_{n,-s} \left(1 + \frac{n-2s}{2}\right)^{\frac{n-2s}{n-2s}} \frac{1}{\lambda^{n-2s}}\end{aligned}$$

$$= c_{n,-s} \left(\frac{1}{\left(\frac{2}{n+2(1-s)}\right)^{\frac{1}{n-2s}} \lambda} \right)^{n-2s}.$$

This is true for those x_0 with

$$|x_0| = \left(\frac{2}{n+2(1-s)}\right)^{\frac{1}{n-2s}} \lambda.$$

We denote

$$\alpha = \left(\frac{2}{n+2(1-s)}\right)^{\frac{1}{n-2s}},$$

note that $\alpha < 1$ because $n \geq 2$.

Now to the actual estimation:

$$\int_{\mathbb{R}^n} \frac{\Psi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2s}} dy \geq \left(\int_{B_\lambda(0)} [\Psi(y) - \Gamma_\lambda(y)]\, dy \right) \left(\inf_{y \in B_\lambda(0)} \frac{1}{|x-y|^{n+2s}} \right).$$

With

$$\begin{aligned}
\int_{B_\lambda(0)} [\Psi(y) - \Gamma_\lambda(y)]\, dy &\geq \int_{B_{\alpha\lambda}(0)} [\Psi(y) - \Gamma_\lambda(y)]\, dy \\
&\geq \int_{B_{\alpha\lambda}(0)} \left[\Psi(y) - \sup_{z \in \mathbb{R}^n} \Gamma_\lambda(z) \right] dy \\
&= \frac{c_{n,-s} \Xi(n) \alpha^{2s}}{2s} \lambda^{2s} - \sup_{z \in \mathbb{R}^n} \Gamma_\lambda(z) |B_{\alpha\lambda}| > 0-
\end{aligned}$$

The last expression must be positive because the function we are integrating is positive. Now

$$\begin{aligned}
\inf_{y \in B_\lambda(0)} \frac{1}{|x-y|^{n+2s}} &\geq \frac{1}{|x + \frac{x\lambda}{|x|}|^{n+2s}} \\
&\geq \frac{1}{(|x| + \lambda)^{n+2s}} \\
&\geq \frac{1}{2^{n+2s} |x|^{n+2s}},
\end{aligned}$$

because $y = -\frac{x\lambda}{|x|}$ has the greatest distance to x and $\lambda < |x|$. So in the end:

$$\begin{aligned}
\gamma_\lambda(x) &= c_{n,s} \int_{\mathbb{R}^n} \frac{\Psi(y) - \Gamma_\lambda(y)}{|x-y|^{n+2s}} dy \\
&\geq c_{n,s} \left(\frac{c_{n,-s} \Xi(n) \alpha^{2s}}{2s} \lambda^{2s} - \sup_{z \in \mathbb{R}^n} \Gamma_\lambda(z) |B_{\alpha\lambda}| \right) \frac{1}{2^{n+2s} |x|^{n+2s}} \\
&= c_1(n,s,\lambda) \frac{1}{|x|^{n+2s}}.
\end{aligned}$$

□

2.3.7 Conclusion. *The positive function γ_λ is bounded for every $\lambda > 0$, i.e. there exists a constant $c = c(\lambda, n, s)$ such that $\sup_{x\in\mathbb{R}^n} \gamma_\lambda(x) \le c$.*

Proof. We just saw that $\gamma_\lambda(x) \le \frac{c_2}{|x|^{n+2s}}$ for $|x| > 2\lambda$ so we conclude that $\gamma_\lambda(x) \le \frac{c_2}{(2\lambda)^{n+2s}}$ when $|x| > 2\lambda$.
In proposition 2.3.4 we have seen that γ_1 is continuous and so is γ_λ because we obtain it from γ_1 through scaling.
Thus γ_λ must be bounded over the compact set $B_{2\lambda}(0)$. So we set

$$c = \max\left(\sup_{y\in B_{2\lambda}(0)} \gamma_\lambda(y), \frac{c_2}{(2\lambda)^{n+2s}}\right).$$

□

2.3.8 Remark. *We further want to analyze the appearing constants. Our main goal is to see their dependency of λ. For instance*

$$c_2(n, s, \lambda) = c_{n,s}\left(\frac{c_{n,-s}\Xi(n)}{2s}\lambda^{2s} + \sup_{y\in\mathbb{R}^n} \Gamma_\lambda(y)|B_\lambda|\right)2^{n+2s}.$$

We have already seen that

$$\sup_{y\in\mathbb{R}^n} \Gamma_\lambda(y) = c_{n,-s}\left(1 + \frac{n-2s}{2}\right)\frac{1}{\lambda^{n-2s}}.$$

The volume of a n-dimensional ball with radius r is given by

$$|B_r| = r^n \frac{\pi^{\frac{n}{2}}}{\Gamma_f(\frac{n}{2}+1)}.$$

Here Γ_f is the actual Gamma-function.
Now we can conclude that

$$\sup_{y\in\mathbb{R}^n} \Gamma_\lambda(y)|B_\lambda| = c(n, s)\lambda^{2s}$$

and finally

$$c_2(n, s, \lambda) = c_2(n, s)\lambda^{2s}$$

with

$$c_2(n, s) = c_{n,s}\left(\frac{c_{n,-s}\Xi(n)}{2s} + c_{n,-s}\left(1 + \frac{n-2s}{2}\right)\frac{\pi^{\frac{n}{2}}}{\Gamma_f(\frac{n}{2}+1)}\right)2^{n+2s}.$$

And we get analogously

$$c_1(n, s, \lambda) = c_1(n, s)\lambda^{2s}$$

with

$$c_1(n, s) = c_{n,s}\left(\frac{c_{n,-s}\Xi(n)\alpha^{2s}}{2s} - c_{n,-s}\left(1 + \frac{n-2s}{2}\right)\frac{\alpha^n\pi^{\frac{n}{2}}}{\Gamma_f(\frac{n}{2}+1)}\right)\frac{1}{2^{n+2s}}.$$

2.3.9 Proposition. *For all $\lambda > 0$ $\gamma_\lambda \in L_1(\mathbb{R}^n)$.*

Proof. In proposition 2.3.6 we have seen that for $|x| > 2\lambda$, we have

$$0 < \gamma_\lambda(x) \le \frac{c_2}{|x|^{n+2s}}.$$

This allows the estimation

$$\int_{\mathbb{R}^n - B_{2\lambda}(0)} |\gamma_\lambda(x)|dx \le \int_{\mathbb{R}^n - B_{2\lambda}(0)} \frac{c_2}{|x|^{n+2s}}dx < \infty.$$

Because γ_λ is continuous, it is bounded on $B_{2\lambda}(0)$ and therefore

$$\int_{B_{2\lambda}(0)} |\gamma_\lambda(x)| dx \leq \sup_{z \in B_{2\lambda}(0)} \gamma_\lambda(z) |B_{2\lambda}(0)| < \infty.$$

This concludes the proof. □

2.3.10 Remark. *We have* $\Gamma_\lambda, \Psi \in L_{1,s}$.

Proof. We note that $\Psi \in L_{1,s}$ because

$$\begin{aligned}
\|\Psi|L_{1,s}\| &= \int_{B_1(0)} \frac{\Psi(x)}{1+|x|^{n+2s}} dx + \int_{B_1^c(0)} \frac{\Psi(x)}{1+|x|^{n+2s}} dx \\
&\leq \int_{B_1(0)} \Psi(x) dx + \int_{B_1^c(0)} \frac{\Psi(x)}{|x|^{n+2s}} dx \\
&\leq \int_{B_1(0)} \frac{c_{n,-s}}{|x|^{n-2s}} dx + \int_{B_1^c(0)} \frac{c_{n,-s}}{|x|^{2n}} dx \\
&= c_{n,-s} \Xi(n) \left\{ \left[\frac{\rho^{2s}}{2s} \right]_0^1 - \left[\frac{\rho^{-n}}{n} \right]_1^\infty \right\} \\
&< \infty.
\end{aligned}$$

Since $\Gamma(x) \leq \Psi(x)$ for every $x \in \mathbb{R}^n$ we also have

$$\|\Gamma|L_{1,s}\| \leq \|\Psi|L_{1,s}\| < \infty.$$

□

2.3.11 Conclusion. $(-\Delta)^s \Gamma \in L_1(\mathbb{R}^n)$ *and additionally*

$$\int_{\mathbb{R}^n} (-\Delta)^s \Gamma(x) dx = 1$$

Proof. We consider a smooth cut off function η such that $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\eta(x) = 1$ for all $x \in B_1(0)$ and $\Theta\eta \subset B_2(0)$. Then we denote $\eta_R(x) = \eta(\frac{x}{R})$. We start with $\langle (-\Delta)^s \Gamma, \eta_R \rangle$, this converges to $\int_{\mathbb{R}^n} (-\Delta)^s \Gamma(x) dx$ when $R \to \infty$ by the theorem of dominated convergence. Remember that $(-\Delta)^s \Gamma \in L_1$ as seen in 2.3.9 and that $|((-\Delta)^s \Gamma)\eta| \leq |(-\Delta)^s \Gamma|$. We also have $\langle (-\Delta)^s \Psi, \eta_R \rangle = 1$ for every $R > 0$. So we get

$$\begin{aligned}
\int_{\mathbb{R}^n} (-\Delta)^s \Gamma(x) dx - 1 &= \lim_{R\to\infty} \langle (-\Delta)^s \Gamma, \eta_R \rangle - \lim_{R\to\infty} \langle (-\Delta)^s \Psi, \eta_R \rangle \\
&= \lim_{R\to\infty} \langle (-\Delta)^s \Gamma - (-\Delta)^s \Psi, \eta_R \rangle \\
&= \lim_{R\to\infty} \langle \Gamma - \Psi, (-\Delta)^s \eta_R \rangle.
\end{aligned}$$

We used the the fact that $\Gamma, \Psi \in L_{1,s}$, see 2.3.10, and the definition of $(-\Delta)^s$ for these functions in the last line.
The function $\Gamma - \Psi$ is in L_1 and has compact support so we will study the behaviour of $(-\Delta)^s \eta_R$ on compact sets when R goes to ∞.
Let K be a compact set, then there exists a constant k such that $|x| < k$ for every $x \in K$, choose $R > k$. So for every $x \in K$ we have $\eta_R(x) = 1$.

$$0 \leq (-\Delta)^s \eta_R(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} dy$$

$$
\begin{aligned}
&= c_{n,s} PV \int_{|y|\geq R} \frac{1-\eta_R(y)}{|x-y|^{n+2s}} dy \\
&\leq c_{n,s} PV \int_{|y|\geq R} \frac{1}{|x-y|^{n+2s}} dy \\
&\overset{(1)}{\leq} c_{n,s} \int_{|y|\geq R} \frac{1}{(|y|-k)^{n+2s}} dy \\
&= c_{n,s} \Xi(n) \int_R^\infty \frac{\rho^{n-1}}{(\rho-k)^{n+2s}} d\rho \\
&= c_{n,s} \Xi(n) \int_R^\infty \frac{1}{(\rho-k)^{1+2s}} \frac{1}{\left(1-\frac{k}{\rho}\right)^{n-1}} d\rho \\
&\overset{(2)}{\leq} c_{n,s} \frac{\Xi(n)}{\left(1-\frac{k}{R}\right)^{n-1}} \left[-\frac{1}{2s}(\rho-k)^{-2s}\right]_{\rho=R}^\infty \\
&= c_{n,s} \frac{\Xi(n)}{2s} \frac{R^{n-1}}{(R-k)^{n+2s-1}}.
\end{aligned}
$$

$0 \leq (-\Delta)^s \eta_R(x)$ is true because $\eta_R(x) - \eta_R(y) \geq 0$ for $x \in K$ and y arbitrary. In (1) we see that the integral is not singular anymore and used that $|y-x| \geq ||y|-|x|| \geq ||y|-k|$.
(2) is true because

$$
\begin{aligned}
& \rho \geq R \\
\iff \quad & 1-\frac{k}{\rho} \geq 1-\frac{k}{R} \\
\iff & \left(1-\frac{k}{\rho}\right)^{-1} \leq \left(1-\frac{k}{R}\right)^{-1}.
\end{aligned}
$$

We see that

$$
0 \leq (-\Delta)^s \eta_R(x) \leq \frac{\Xi(n)}{2s} \frac{R^{n-1}}{(R-k)^{n+2s-1}}
$$

for every $x \in K$. This means

$$
\sup_{x\in K} (-\Delta)^s \eta_R(x) \leq \frac{\Xi(n)}{2s} \frac{R^{n-1}}{(R-k)^{n+2s-1}}.
$$

This enables us to finally show

$$
\begin{aligned}
\lim_{R\to\infty} \langle \Gamma - \Psi, (-\Delta)^s \eta_R \rangle &= \lim_{R\to\infty} \int_{\mathbb{R}^n} (\Gamma(x)-\Psi(x))(-\Delta)^s \eta_R(x) dx \\
&\overset{(3)}{=} \lim_{R\to\infty} \int_{B_1(0)} (\Gamma(x)-\Psi(x))(-\Delta)^s \eta_R(x) dx \\
&\leq \lim_{R\to\infty} \left(\sup_{y\in B_1(0)} |(-\Delta)^s \eta_R(y)| \right) \int_{B_1(0)} |\Gamma(x)-\Psi(x)| \, dx \\
&\leq \lim_{R\to\infty} \left(\frac{\Xi(n)}{2s} \frac{R^{n-1}}{(R-1)^{n+2s-1}} \right) \|\Gamma - \Psi | L_1(B_1(0))\| \\
&= 0.
\end{aligned}
$$

(3) is true because $\Theta(\Gamma - \Psi) = B_1(0)$ by definition of Γ. The integral

$$
\int_{B_1(0)} (\Gamma(x)-\Psi(x)) dx
$$

exists, because Γ is continuous and Ψ is integrable over $B_1(0)$. At last we see that

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s\Gamma(x)dx - 1 &= \lim_{R\to\infty} \langle \Gamma - \Psi, (-\Delta)^s \eta_R \rangle \\ &= 0.\end{aligned}$$

□

2.3.12 Conclusion. *We also have* $\int_{\mathbb{R}^n} \gamma_\lambda(x)dx = 1$.

Proof. We will check the rescaling properties of $\gamma_\lambda = (-\Delta)^s\Gamma_\lambda$.

$$\begin{aligned}\gamma_\lambda(x) = [(-\Delta)^s\Gamma_\lambda](x) &= (-\Delta)^s\left(\frac{1}{\lambda^{n-2s}}\Gamma\left(\frac{\cdot}{\lambda}\right)\right)(x) \\ &= \frac{1}{\lambda^{n-2s}}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{2\Gamma\left(\frac{x}{\lambda}\right) - \Gamma\left(\frac{x+y}{\lambda}\right) - \Gamma\left(\frac{x-y}{\lambda}\right)}{|y|^{n+2s}}dy \\ &= \frac{1}{\lambda^{n-2s}}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{2\Gamma\left(\frac{x}{\lambda}\right) - \Gamma\left(\frac{x}{\lambda}+z\right) - \Gamma\left(\frac{x}{\lambda}-z\right)}{|\lambda z|^{n+2s}}\frac{\lambda^{n+2s}}{\lambda^{2s}}dz \\ &= \frac{1}{\lambda^{n}}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\frac{2\Gamma\left(\frac{x}{\lambda}\right) - \Gamma\left(\frac{x}{\lambda}+z\right) - \Gamma\left(\frac{x}{\lambda}-z\right)}{|z|^{n+2s}}dz \\ &= \frac{1}{\lambda^n}((-\Delta)^s\Gamma)\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^n}\gamma_1\left(\frac{x}{\lambda}\right).\end{aligned}$$

Where we used the transformation $z = \frac{y}{\lambda}$ with $dy = \lambda^n dz$.
Thus we have

$$\begin{aligned}\int_{\mathbb{R}^n}\gamma_\lambda(x)dx &= \int_{\mathbb{R}^n}\gamma_1\left(\frac{x}{\lambda}\right)\frac{dx}{\lambda^n} \\ &= \int_{\mathbb{R}^n}\gamma_1(t)dt = 1.\end{aligned}$$

We again used the transformation $t = \frac{x}{\lambda}$ with $dt = \frac{dx}{\lambda^n}$. □

The properties up to now were more of a general nature. From now on we will have our later application in mind.

2.3.13 Lemma. *For functions* $u \in L_{\frac{2n}{n-2s}}(\mathbb{R}^n)$ *the integral*

$$\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}dx$$

does exist.

Proof. The idea is to apply the Hölder inequality for $p = \frac{2n}{n-2s}$ and thus $q = \frac{2n}{n+2s}$.

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}dx &\le \left(\int_{\mathbb{R}^n}|u|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}\frac{1}{(1+|x|^{n+2s})^q}dx\right)^{\frac{1}{q}} \\ &\le \|u|L^p\| C\left(\int_{\mathbb{R}^n}\frac{1}{(1+|x|^{2n})}dx\right)^{\frac{1}{q}}.\end{aligned}$$

Where we used the fact that there exists a constant C such that

$$(1+|x|^{n+2s})^q \ge C(1+|x|^{(n+2s)q}) = C(1+|x|^{2n}).$$

Since

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^{2n})} dx &= \int_{B_1(0)} \frac{1}{(1+|x|^{2n})} dx + \int_{\mathbb{R}^n \setminus B_1(0)} \frac{1}{(1+|x|^{2n})} dx \\ &\leq |B_1(0)| + \Xi(n) \int_1^\infty \frac{1}{\rho^{n+1}} d\rho \\ &< \infty\end{aligned}$$

we can conclude that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty.$$

□

2.3.14 Remark. *The choice of* $p = \frac{2n}{n-2s}$ *seems arbitrary but this* L_p *space will turn out to be the important one for our application.*
The fact that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty$$

will interact quite nicely with the asymptotic behaviour of γ_λ *we derived in proposition 2.3.6, which we will see in the proposition 2.3.20.*

2.3.15 Remark. *By what we have seen in remark 2.1.16, we are now able to define* $(-\Delta)^s u$ *for* $u \in L_{\frac{2n}{n-2s}}(\mathbb{R}^n)$.
$(-\Delta)^s u$ *is a tempered distribution and for every* $f \in \mathcal{S}$ *we have*

$$\langle (-\Delta)^s u, f\rangle = \langle u, (-\Delta)^s f\rangle.$$

The existence of the expression on the right hand side follows from the beforementioned remark since $L_{\frac{2n}{n-2s}}(\mathbb{R}^n) \subset L_{1,s}(\mathbb{R}^n)$.

Now we can sensibly define what it means for a function $u \in L_{\frac{2n}{n-2s}}(\mathbb{R}^n)$ to be a supersolution of $(-\Delta)^s$ in an open set Ω.

2.3.16 Definition. We say that $u \in L_{\frac{2n}{n-2s}}$ satisfies $(-\Delta)^s u \geq 0$ in an open set Ω if for every nonnegative test function φ with $\Theta\varphi \Subset \Omega$,

$$\langle u, (-\Delta)^s \varphi\rangle \geq 0.$$

This definition is hard to work with since one would have to test against all possible test functions, so our next goal will be to find a characterization of supersolutions that is a bit easier to deal with. We would like to derive something similar to the mean value formula for superharmonic functions, see the following remark.

2.3.17 Remark. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $u \in C^2(\Omega)$ *be a superharmonic function in* Ω*, i.e.* $(-\Delta)u \geq 0$ *in* Ω*,* $x \in \Omega$ *and* $r > 0$ *such that* $\overline{B(x,r)} \Subset \Omega$*, then*

$$u(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy.$$

The next proposition shows us that the functions γ_λ plays an important role for us.

2.3.18 Proposition. *Let* $u \in L_p(\mathbb{R}^n)$ *for any* $p \geq 1$ *then the* γ_λ *approximate the identity in the following sense:*

$$\lim_{\lambda\to 0} (u * \gamma_\lambda)(x) = \lim_{\lambda \to 0} \int_{\mathbb{R}^n} u(y)\gamma_\lambda(x-y) dy = u(x)$$

for almost every $x \in \mathbb{R}^n$. *See* [14, Proposition 2.13].

Proof. The existence of the convolution will be proved in 2.3.20. We have already seen in 2.3.12 that

$$\gamma_\lambda(x) = \frac{1}{\lambda^n}\gamma_1\left(\frac{x}{\lambda}\right)$$

and

$$\int_{\mathbb{R}^n} \gamma_\lambda(x)dx = 1.$$

The rest follows from [16, 6.1.(16)] □

2.3.19 Proposition. *Let $u \in L_{\frac{2n}{n-2s}}$. If $(-\Delta)^s u$ is continuous at a point $x \in \mathbb{R}^n$, then*

$$(-\Delta)^s u(x) = \lim_{\lambda\to 0} \frac{C}{\lambda^{2s}}(u(x) - (u * \gamma_\lambda)(x))$$

with $C = C(s,n)$.

Proof. We choose a $g \in L_1(\mathbb{R}^n)$ with compact support, i.e. there exists a ball $B_R(0)$ with $\Theta g \Subset B_R(0)$ and Θg is always closed. We want to show that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^n} (-\Delta)^s u(x-y)\frac{1}{\lambda^n}g\left(\frac{y}{\lambda}\right)dy = (-\Delta)^s u(x)\int_{\mathbb{R}^n} g(y)dy. \tag{1}$$

This means that for every $\epsilon > 0$ we can find a λ such that

$$\int_{\mathbb{R}^n} |(-\Delta)^s u(x-y) - (-\Delta)^s u(x)|\frac{1}{\lambda^n}|g\left(\frac{y}{\lambda}\right)|dy < \epsilon.$$

Since $(-\Delta)^s u$ is continuous in x, there exists a $\delta > 0$ such that $|x-y| < \delta$ implies

$$|(-\Delta)^s u(x) - (-\Delta)^s u(y)| < \frac{\epsilon}{\|g|L_1\|}.$$

Again we use the transformation $z = \frac{y}{\lambda}$ with $dy = \lambda^n dz$ and get

$$\int_{\mathbb{R}^n} |(-\Delta)^s u(x-y) - (-\Delta)^s u(x)|\frac{1}{\lambda^n}|g\left(\frac{y}{\lambda}\right)|dy = \int_{\mathbb{R}^n} |(-\Delta)^s u(x-\lambda z) - (-\Delta)^s u(x)||g(z)|dz.$$

We only have to integrate over Θg

$$\int_{\mathbb{R}^n} |(-\Delta)^s u(x-\lambda z) - (-\Delta)^s u(x)||g(z)|dz = \int_{\Theta g} |(-\Delta)^s u(x-\lambda z) - (-\Delta)^s u(x)||g(z)|dz.$$

We know that $|z| < R$ for every $z \in \Theta g$, so if we choose $\lambda < \frac{\delta}{R}$ we obtain

$$|x - \lambda z - x| = |\lambda z| \le \frac{\delta R}{R} = \delta,$$

i.e. $x - \lambda z \in B_\delta(x)$ and so we conclude that there exists a $\lambda > 0$ with

$$\int_{\Theta g} |(-\Delta)^s u(x-\lambda z) - (-\Delta)^s u(x)||g(z)|dz \le \frac{\epsilon}{\|g|L_1\|}\int_{\mathbb{R}^n} |g(z)|dz = \epsilon$$

for a given $\epsilon > 0$. So we proved (1).
Now we want to apply this to $g(y) = \Psi(y) - \Gamma(y)$, then

$$\frac{1}{\lambda^n}g\left(\frac{y}{\lambda}\right) = \frac{1}{\lambda^n}\left[\Psi\left(\frac{y}{\lambda}\right) - \Gamma\left(\frac{y}{\lambda}\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{\lambda^n}\left[\frac{\lambda^n}{\lambda^{2s}}\frac{c_{n,-s}}{|y|^{n-2s}} - \frac{\lambda^n}{\lambda^{2s}}\Gamma_\lambda(y)\right] \\
&= \frac{1}{\lambda^{2s}}\left[\Psi(y) - \Gamma_\lambda(y)\right].
\end{aligned}
$$

We insert this into (1) and get

$$
\begin{aligned}
(-\Delta)^s u(x) &= C \lim_{\lambda\to 0}\int_{\mathbb{R}^n}(-\Delta)^s u(x-y)\frac{1}{\lambda^{2s}}\left[\Psi(y)-\Gamma_\lambda(y)\right]dy \\
&\overset{(2)}{=} \lim_{\lambda\to 0}\frac{C}{\lambda^{2s}}\int_{\mathbb{R}^n} u(x-y)(-\Delta)^s\left[\Psi(y)-\Gamma_\lambda(y)\right]dy \\
&\overset{(3)}{=} \lim_{\lambda\to 0}\frac{C}{\lambda^{2s}}\left(u(x) - \int_{\mathbb{R}^n} u(x-y)(-\Delta)^s\Gamma_\lambda(y)dy\right) \\
&= \lim_{\lambda\to 0}\frac{C}{\lambda^{2s}}\left(u(x) - (u*\gamma_\lambda)(x)\right).
\end{aligned}
$$

In (2) we used the definition of $(-\Delta)^s u$ for $u \in L_{\frac{2n}{n-2s}}$ and in (3) the fact that Ψ is the fundamental solution of $(-\Delta)^s$. The constant C is given through $C = \left(\int_{\mathbb{R}^n}\Psi(y)-\Gamma(y)dy\right)^{-1}$. □

2.3.20 Proposition. *For $u \in L_{\frac{2n}{n-2s}}(\mathbb{R}^n)$ we have*

$$
\int_{\mathbb{R}^n} u(y)\gamma_\lambda(x-y)dy < \infty
$$

$\forall x \in \mathbb{R}^n$.

Proof. We start off by summarizing what we know about the functions. We saw in proposition 2.3.13 that

$$
\int_{\mathbb{R}^n}\frac{|u(y)|}{1+|y|^{n+2s}}dy < \infty
$$

and the positive function $\gamma_\lambda(x-y)(1+|x-y|^{n+2s})$ is bounded for $|x-y| \geq 2\lambda$ by Proposition 2.3.6. So we will deal with

$$
\int_{\mathbb{R}^n}\frac{u(y)}{1+|y|^{n+2s}}\frac{1+|y|^{n+2s}}{1+|x-y|^{n+2s}}(1+|x-y|^{n+2s})\gamma_\lambda(x-y)dy.
$$

To continue, we will investigate the behaviour of

$$
\iota_x(y) = \frac{1+|y|^{n+2s}}{1+|x-y|^{n+2s}}.
$$

It is obviously a nonnegative, continuous function and

$$
\iota_x(y) = \frac{1+\frac{1}{|y|^{n+2s}}}{\frac{1}{|y|^{n-2s}} + \left|\frac{x}{|y|}-\frac{y}{|y|}\right|^{n+2s}}.
$$

This implies

$$
\lim_{|y|\to\infty}\iota_x(y) = 1.
$$

So there must exist a $R > 0$ such that $\iota_x(y) \leq 2$ for all y with $|y| > R$. Without loss of generality, we assume that $B_{2\lambda}(x) \subset B_R(0)$.
Now we integrate over $B_R(0)$:

$$\begin{aligned}\left|\int_{B_R(0)} u(y)\gamma_\lambda(x-y)dy\right| &\leq \int_{B_R(0)} \frac{|u(y)|}{1+|y|^{n+2s}}(1+|y|^{n+2s})\gamma_\lambda(x-y)dy \\ &\leq (1+R^{n+2s}) \sup_{z\in\mathbb{R}^n} \gamma_\lambda(z) \int_{B_R(0)} \frac{|u(y)|}{1+|y|^{n+2s}}dy \\ &\leq (1+R^{n+2s}) \sup_{z\in\mathbb{R}^n} \gamma_\lambda(z)\|u|L_{1,s}\|.\end{aligned}$$

Where we used the fact that γ_λ is bounded, see Conclusion 2.3.7 .
Next will be rest of the integral

$$\begin{aligned}\left|\int_{\mathbb{R}^n-B_R(0)} u(y)\gamma_\lambda(x-y)dy\right| &= \int_{\mathbb{R}^n-B_R(0)} \frac{|u(y)|}{1+|x-y|^{n+2s}}(1+|x-y|^{n+2s})\gamma_\lambda(x-y)dy \\ &\overset{(1)}{\leq} c_2(n,s,\lambda) \int_{\mathbb{R}^n-B_R(0)} \frac{|u(y)|}{1+|x-y|^{n+2s}}dy \\ &= c_2(n,s,\lambda) \int_{\mathbb{R}^n-B_R(0)} \frac{|u(y)|}{1+|y|^{n+2s}} \frac{1+|y|^{n+2s}}{1+|x-y|^{n+2s}}dy.\end{aligned}$$

(1) is true by Proposition 2.3.6.

$$\begin{aligned}\int_{\mathbb{R}^n-B_R(0)} \frac{|u(y)|}{1+|y|^{n+2s}} \frac{1+|y|^{n+2s}}{1+|x-y|^{n+2s}}dy &\leq 2\int_{\mathbb{R}^n-B_R(0)} \frac{|u(y)|}{1+|y|^{n+2s}}dy \\ &\leq 2\|u|L_{1,s}\|.\end{aligned}$$

All in all

$$\begin{aligned}\int_{\mathbb{R}^n-B_R(0)} |u(y)\gamma_\lambda(x-y)|\,dy &\leq c_2(n,s,\lambda)\int_{\mathbb{R}^n-B_{2\lambda}(x)} \frac{|u(y)|}{1+|y|^{n+2s}} \frac{1+|y|^{n+2s}}{1+|x-y|^{n+2s}}dy \\ &\leq 2c_2(n,s,\lambda)\|u|L_{1,s}\|\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |u(y)\gamma_\lambda(x-y)|\,dy &= \int_{\mathbb{R}^n-B_R(0)} |u(y)\gamma_\lambda(x-y)|\,dy + \int_{B_R(0)} |u(y)\gamma_\lambda(x-y)|\,dy \\ &\leq \left[2c_2(n,s,\lambda) + (1+R^{n+2s}) \sup_{z\in\mathbb{R}^n}\gamma_\lambda(z)\right]\|u|L_{1,s}\|.\end{aligned}$$

□

2.3.21 Remark. *The way we proved that $u * \gamma_\lambda$ is finite seems cumbersome. Using Hölder inequality would be the natural ansatz. The advantage of our approach is the fact that we found an integrable majorant that is independent of x, which we will need in the next lemma.*

2.3.22 Lemma. *For every $u \in L_{\frac{2n}{n-2s}}(\mathbb{R}^n)$ the function $u * \gamma_\lambda$ is continuous for $\lambda > 0$.*

Proof. We want to use Lebesgue's Theorem on continuity of parameter integrals, see [1, Theorem A.3.]. We have to show that

$$(u * \gamma_\lambda)(y) = \int_{\mathbb{R}^n} u(x)\gamma_\lambda(y-x)dx$$

is continuous in y. We denote $f(x,y) = u(x)\gamma_\lambda(y-x)$.
We saw in Proposition 2.3.20 that f is integrable for every $x \in \mathbb{R}^n$. It is continuous in y, because y only appears in the argument of γ_λ, which is a continuous function.
Lastly we need some kind of integrable majorant that is independent of x. But we have already seen in the proof of Proposition 2.3.20 that such a majorant does exist. We choose:

$$F(y) = \frac{u(y)}{1+|y|^{n+2s}} \left((1+R^{n+2s}) \sup_{z\in\mathbb{R}^n} \gamma_\lambda(z)\chi_{B_R(0)}(y) + 2c_2(n,s,\lambda)\chi_{\mathbb{R}^n\setminus B_R(0)}(y) \right).$$

R was defined in Proposition 2.3.20 and we refer to the proof of that proposition for the details.
Finally we can conclude that the function

$$g(y) = \int_{\mathbb{R}^n} f(x,y)dy = (u * \gamma_\lambda)(y)$$

is continuous. □

Now we are ready to derive the characterization of supersolutions. But we first want to gather some facts about lower-semicontinuous functions because these will play an important role.

2.3.23 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower-semicontinuous if

$$\{x \in \mathbb{R}^n | f(x) \le a\} \text{ is closed for every } a \in \mathbb{R}$$

or

$$f(x) \le \liminf_{y\to x} f(y) \text{ for any } x \in \mathbb{R}^n.$$

Upper-semicontinuity is defined analogously.

2.3.24 Remark. *The first definition says that $\{x \in \mathbb{R}^n | f(x) > a\}$ is open. This means especially that, for a fixed $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, the set $\{x \in \mathbb{R}^n | f(x) > f(x_0) - \epsilon\}$ is an open neighbourhood of x_0.*
So we derive a ϵ-δ-definition for lower-semicontinuity in x_0:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow f(x) > f(x_0) - \epsilon$$

The next proposition will contain some facts about lower-semicontinuous functions that we will use later on.

2.3.25 Proposition. *Let u, v be two lower-semicontinuous functions then*

(i) $\min(u,v)$ *is a lower-semicontinuous function.*

(ii) $u+v$ *is a lower-semicontinuous function.*

Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence of lower-semicontinuous functions then

(iii) $u(x) = \lim_{n\to\infty} u_n(x)$ *is a lower-semicontinuous function.*

Proof. (i) and (ii) can be found directly in [5, Lemma 4.2.1], so we only have to prove (iii). By [5, Lemma 4.2.1] we get that $w(x) = \sup_{n\in\mathbb{N}} u_n(x)$ is lower-semicontinuous. But in the case of an increasing sequence, the limit and the supremum coincide. Because $\forall \epsilon > 0$ there exists an n_0 such that $u(x) - u_{n_0}(x) < \epsilon$ and this is equivalent to $u(x) - \epsilon < u_{n_0}(x)$, which is the definition of the supremum of the u_n. □

2.3.26 Proposition. *For $u \in L_{\frac{2n}{n-2s}}$, we have $(-\Delta)^s u \ge 0$ in an open set Ω iff u is lower-semi-continuous in Ω and*

$$u(x_0) \ge \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x-x_0)dx$$

for any $x_0 \in \Omega$ and $\lambda < dist(x_0, \partial\Omega)$.

Proof. Let $(-\Delta)^s u \geq 0$. For $r > \lambda_1 > \lambda_2$, we form $\Gamma_{\lambda_2} - \Gamma_{\lambda_1}$. This is a nonnegative function by Proposition 2.3.2, beyond that it is also a $C^{1,1}$ function with support in B_r. If $(-\Delta)^s u \geq 0$ in $B_r(x_0)$, then

$$\langle (-\Delta)^s u(x), \Gamma_{\lambda_2}(x - x_0) - \Gamma_{\lambda_1}(x - x_0)\rangle \geq 0.$$

This is true because we interpret $(-\Delta)^s u$ as a nonnegative Radon measure. Next we use the definition of the operator for functions in $L_{1,s}$ to get

$$\langle u(x), (-\Delta)^s \Gamma_{\lambda_2}(x - x_0) - (-\Delta)^s \Gamma_{\lambda_1}(x - x_0)\rangle \geq 0.$$

One should note proposition 2.1.20.
Now we rewrite this as

$$\begin{aligned} \langle u, \gamma_{\lambda_2}(x - x_0)\rangle &\geq \langle u, \gamma_{\lambda_1}(x - x_0)\rangle \\ (u * \gamma_{\lambda_2})(x_0) &\geq (u * \gamma_{\lambda_1})(x_0). \end{aligned}$$

Next we take $\Omega_0 \Subset \Omega$ with $(-\Delta)^s u \geq 0$ in Ω. Let $r = \text{dist}(\Omega_0, \partial\Omega)$. Then if $r > \lambda_1 > \lambda_2 > 0$, we have

$$(u * \gamma_{\lambda_2}) \geq (u * \gamma_{\lambda_1}) \tag{1}$$

everywhere in Ω_0.
We know that γ_λ is an approximate identity, so $u * \gamma_\lambda \to u$ a.e. in Ω_0 as $\lambda \to 0$. We saw in Lemma 2.3.22, that $u * \gamma_\lambda$ is continuous for every λ. So u is the limit of an increasing sequence of continuous functions. This means that u is lower-semicontinuous, see proposition 2.3.25.(iii).
Lastly we take $\lambda_2 \to 0$ in (1) to get:

$$u(x_0) \geq (u * \gamma_\lambda)(x_0)$$

for a.e. $x_0 \in \Omega$ and λ small enough. Keep in mind that we can modify u in a set of measure zero, we can set $u(x_0) = (u * \gamma_\lambda)(x_0)$ in those x_0 where $u * \gamma_\lambda$ does not converge. So we finally get that

$$u(x_0) \geq (u * \gamma_\lambda)(x_0)$$

is true for every $x_0 \in \mathbb{R}^n$.
The other implication follows from [7, Lemma 1.10]. □

2.3.27 Proposition. *Let C be a constant, then we have that*

$$u(x) \geq u * \gamma_\lambda(x) - C\lambda^{2s}$$

for every $x \in \Omega$ and $\lambda <$ dist$(x, \partial\Omega)$ iff $(-\Delta)^s u \geq -C$ in Ω, i.e. $(-\Delta)^s u + C \geq 0$ in the sense of definition 2.3.16.

Proof. We assume that Ω is bounded, because $f \geq -C$ locally in Ω is the same as $f \geq -C$ in all of Ω for every distribution.
We define $v = C\Psi * \chi_\Omega$, then $v(x)$ is lower semicontinuous. $(-\Delta)^s v = C\chi_\Omega$ which means that $(-\Delta)^s v$ is continuous in Ω and we can apply Proposition 2.3.19 and get for $x \in \Omega$

$$C = \lim_{\lambda \to 0} \frac{1}{\lambda^{2s}}(v(x) - v * \gamma_\lambda(x)).$$

We will study the right hand side more precisely.

$$\begin{aligned} \frac{1}{\lambda^{2s}}(v(x) - v * \gamma_\lambda(x)) &= \frac{1}{\lambda^{2s}}((-\Delta)^s v * (\Psi - \Gamma_\lambda)(x)) \\ &\overset{(1)}{=} \frac{1}{\lambda^{2s}} \int_{B_\lambda(x)} (-\Delta)^s v(y) \left(\Psi - \Gamma_\lambda\right)(x - y) dy. \end{aligned}$$

(1) is true because $\Theta(\Psi - \Gamma_\lambda) = B_\lambda(0)$. The function $(-\Delta)^s v$ is constant on $B_\lambda(x)$ iff $\lambda < \operatorname{dist}(x, \partial\Omega)$ and then:

$$\begin{aligned}
\frac{1}{\lambda^{2s}}(v(x) - v * \gamma_\lambda(x)) &= C\int_{B_\lambda(x)} \frac{1}{\lambda^{2s}}\left(\Psi - \Gamma_\lambda\right)(x-y)dy \\
&\overset{(2)}{=} C\int_{B_\lambda(x)} \frac{1}{\lambda^{n}}\left(\Psi - \Gamma\right)\left(\frac{x-y}{\lambda}\right)dy \\
&= C\int_{B_1(0)} (\Psi - \Gamma)(z)dz.
\end{aligned}$$

We showed (2) at the end of the proof of Proposition 2.3.19. As we can see, the right hand side is independent of λ, so we conclude

$$C = \frac{1}{\lambda^{2s}}(v(x) - v * \gamma_\lambda(x))$$

iff $\lambda < \operatorname{dist}(x, \partial\Omega)$.
Now if we use that, we see

$$\begin{aligned}
& & u(x) &\geq (u * \gamma_\lambda)(x) - C\lambda^{2s} \\
&\iff & u(x) + C\lambda^{2s} + (v * \gamma_\lambda)(x) &\geq ((u+v) * \gamma_\lambda)(x) \\
&\iff & (u+v)(x) &\geq ((u+v) * \gamma_\lambda)(x).
\end{aligned}$$

Proposition 2.3.26 tells us that the last line is equivalent to $(-\Delta)^s(u+v)(x) \geq 0$ in Ω, i.e. $(-\Delta)^s u \geq -C$ in Ω. That is what we wanted to show. □

The next proposition will be our variant of a maximum principle.

2.3.28 Proposition. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and u be a lower-semicontinuous function in $\overline{\Omega}$ such that $(-\Delta)^s u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^n \backslash \Omega$. Then we have $u \geq 0$ in all of $\mathbb{R}^n$ and if $u(x) = 0$ for one $x \in \Omega$, then $u \equiv 0$ in all $\mathbb{R}^n$. See* [14, Proposition 2.17]

Proof. We assume that $u(x)$ has negative values in some points. These points must then lie inside Ω, since u is nonnegative outside of Ω. Because u is lower semicontinuous on a compact set in $\mathbb{R}^n$ it attains its minimum in that set (we needed to require semicontinuity in $\overline{\Omega}$ since we can't assure it). So this minimum must be negative and we suppose it is attained in $x_0 \in \Omega$. By Proposition 2.3.26 there must exist a λ such that

$$u(x_0) \geq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx.$$

Since γ_λ has integral 1, this is equivalent to

$$0 \geq \int_{\mathbb{R}^n} (u(x) - u(x_0))\gamma_\lambda(x - x_0)dx.$$

This is impossible because $u(x) - u(x_0)$ is positive outside of Ω and γ_λ is always nonnegative. So our assumption must be false, this means, that $u(x)$ is always nonnegative.
Now we assume that $u(x_0) = 0$, again by using Proposition 2.3.26 we get

$$0 \geq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx.$$

But, since $u(x)$ is nonnegative, we also have

$$0 \leq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx.$$

So in total

$$0 = \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx.$$

By positivity of γ_λ we obtain $u(x) \equiv 0$. □

The next proposition shows that the minimum of two supersolutions is again a supersolution. This fact will be needed in the 3rd chapter.

2.3.29 Proposition. *If $u_1, u_2 \in L_{\frac{2n}{n-2s}}$ fulfill $(-\Delta)^s u_1 \geq 0$ and $(-\Delta)^s u_2 \geq 0$ in Ω then so does $u(x) = min(u_1(x), u_2(x))$, i.e. $(-\Delta)^s u(x) \geq 0$ in Ω. See* [14, Proposition 2.18]

Proof. We choose an arbitrary $x_0 \in \Omega$. By Proposition 2.3.26 we get

$$u_j(x_0) \geq \int_{\mathbb{R}^n} u_j(x)\gamma_\lambda(x - x_0)dx$$

for $j = 1, 2$ and λ small enough and also that u_1, u_2 are lower-semicontinuous. $u(x_0) = u_j(x_0)$ for $j = 1$ or $j = 2$ by definition and $u(x) \leq u_j(x)$ for every other $x \in \mathbb{R}^n$, so

$$u(x_0) \geq \int_{\mathbb{R}^n} u_j(x)\gamma_\lambda(x - x_0)dx \geq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx$$

and u is lower-semicontinuous by lemma 2.3.25, thus $(-\Delta)^s u \geq 0$ by Proposition 2.3.26. □

We get similar properties for functions u such that $(-\Delta)^s u \leq 0$.

2.3.30 Proposition. *For $u \in L_{\frac{2n}{n-2s}}$, we have $(-\Delta)^s u \leq 0$ in an open set Ω iff u is upper-semi-continuous in Ω and*

$$u(x_0) \leq \int_{\mathbb{R}^n} u(x)\gamma_\lambda(x - x_0)dx$$

for any $x_0 \in \Omega$ and $\lambda < dist(x_0, \partial\Omega)$. See [14, Proposition 2.19]

Also the analogue to Proposition 2.3.27 can be obtained.

2.3.31 Proposition. *Let C be a constant, then*

$$u(x) \leq u * \gamma_\lambda(x) + C\lambda^{2s}$$

for every $x \in \Omega$ and $\lambda < dist(x, \partial\Omega)$ iff $(-\Delta)^s u \leq C$ in Ω.

Proof. The proof works mostly like in 2.3.27. Now we define $v(x) = -C\Psi * \chi_\Omega$, then $v(x)$ is upper-semicontinuous and $(-\Delta)^s v(x) = -C\chi_\Omega$. Here we get

$$-C = \frac{1}{\lambda^{2s}}(v(x) - v * \gamma_\lambda(x))$$

iff $\lambda < \text{dist}(x, \partial\Omega)$. Then

$$\begin{aligned} & u(x) \leq (u * \gamma_\lambda)(x) + C\lambda^{2s} \\ \iff \quad & u(x) - C\lambda^{2s} + (v * \gamma_\lambda)(x) \leq ((u + v) * \gamma_\lambda)(x) \\ \iff \quad & (u + v)(x) \leq ((u + v) * \gamma_\lambda)(x). \end{aligned}$$

This is again equivalent to $(-\Delta)^s(u + v)(x) \leq 0$ in Ω. □

Now we formulate our comparison principle.

2.3.32 Proposition. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. For two functions $u, v \in L_{\frac{2n}{n-2s}}$ such that $(-\Delta)^s u \geq 0$ and $(-\Delta)^s v \leq 0$ in Ω, $u \geq v$ in $\mathbb{R}^n \backslash \Omega$ and $u - v$ is a lower-semicontinuous function in $\overline{\Omega}$, then we have $u \geq v$ in all of $\mathbb{R}^n$ and if $u(x) = v(x)$ for one $x \in \Omega$, then $u \equiv v$ in all $\mathbb{R}^n$. See* [14, Proposition 2.21].

Proof. $u - v$ is lower-semicontinuous in $\overline{\Omega}$. $(-\Delta)^s(u - v) = (-\Delta)^s u - (-\Delta)^s v \geq 0$ in Ω and $u - v \geq 0$ in $\mathbb{R}^n \backslash \Omega$. Now we can apply Proposition 2.3.28 to $u - v$ and get $u - v \geq 0$, i.e. $u \geq v$ in $\mathbb{R}^n$ and if $(u - v)(x) = 0$, i.e. $u(x) = v(x)$, for one $x \in \Omega$, then $(u - v)(x) = 0$, i.e. $u(x) = v(x)$ in all of $\mathbb{R}^n$. □

We finally get a similar property for functions u such that $(-\Delta)^s u = 0$ in Ω.

2.3.33 Proposition. *For $u \in L_{\frac{2n}{n-2s}}$, we have $(-\Delta)^s u = 0$ in an open set Ω iff u is continuous in Ω and*

$$u(x_0) = \int_{\mathbb{R}^n} u(x) \gamma_\lambda (x - x_0) dx$$

for any $x_0 \in \Omega$ and $\lambda < dist(x_0, \partial\Omega)$. See [14, Proposition 2.22]

Proof. We use Proposition 2.3.26 and 2.3.30 simultaneously to see that this Proposition is true. □

Chapter 3.

Basic Properties of the Free Boundary Problem

In the first section of this chapter we will turn to the actual obstacle problem. First we will prove the existence of the solution u in the homogeneous Sobolev space $\overline{H}^s$ in case that the obstacle φ is continuous and has compact support. We will furthermore be able to show that u is also continuous. In the second section we will derive further regularity results.

3.1. Construction of the Solution

First, we will state the problem again that we are going to study.
We take a continuous function with compact support $\varphi : \mathbb{R}^n \to \mathbb{R}$, which will be our obstacle. We will look for a function u that satisfies:

- $u \geq \varphi$ in $\mathbb{R}^n$
- $(-\Delta)^s u \geq 0$ in $\mathbb{R}^n$
- $(-\Delta)^s u(x)(u(x) - \varphi(x)) = 0$ for all $x \in \mathbb{R}^n$
- $\lim_{|x|\to\infty} u(x) = 0$.

To construct the solution, we first need to define the space $\overline{H}^s$. This is the completion of $\mathcal{S}$ in the norm $\| \cdot |\overline{H}^s\|$. Which is given by

$$\|f|\overline{H}^s\| = \sqrt{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} dxdy}$$

for any $f \in \mathcal{S}$. Note that $\| \cdot |\overline{H}^s\|$ is usually a semi-norm, but functions $f \in \mathcal{S}$ go to zero, when $|x|$ goes to ∞. That's why $\| \cdot |\overline{H}^s\|$ is a norm on S.

3.1.1 Remark. *We briefly want to discuss the connections to the fractional Sobolev space $H^s(\mathbb{R}^n)$. One could define it as the set of all $f \in L_2(\mathbb{R}^n)$ such that $\|f|\overline{H}^s\| < \infty$. In fact $H^s(\mathbb{R}^n)$ is the set of all functions in $\overline{H}^s(\mathbb{R}^n)$ that are in $L_2(\mathbb{R}^n)$, see* [17, Remark 5.2.3.3].
Furthermore we have the continuous embedding $\overline{H}^s \subset L_{\frac{2n}{n-2s}}$, see [2, Theorem 1.38].

The space $\overline{H}^s$ even is a Hilbert space, see [2, Proposition 1.34], with the inner product

$$\langle f, g\rangle_{\overline{H}^s} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))\overline{(g(x)-g(y))}}{|x-y|^{n+2s}} dxdy.$$

3.1.2 Proposition. *For $f, g \in \overline{H}^s$ there exists a constant $d = d(n,s) > 0$ such that*

$$\langle f, g\rangle_{\overline{H}^s} = d(n,s) \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)} d\xi$$

$$= d(n,s)\int_{\mathbb{R}^n}((-\Delta)^{\frac{s}{2}}f)(x)\overline{((-\Delta)^{\frac{s}{2}}g)(x)}dx.$$

Proof. In [2, Proposition 1.37] it is shown that for every $f \in \overline{H}^s$ there exists a constant $d = d(n,s) > 0$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}dxdy = d(n,s)\int_{\mathbb{R}^n}|\xi|^{2s}|\mathcal{F}[f](\xi)|^2 d\xi.$$

The left hand side is $\|f|\overline{H}^s\|^2$ and the right hand side is $\||\xi|^s\mathcal{F}[f]|L_2\|^2$. So we can immediately conclude that $\||\xi|^s\mathcal{F}[f]|L_2\| < \infty$. So for all $f \in \overline{H}^s$ $\||\xi|^s\mathcal{F}[f]|L_2\|$ is finite.
Next we note the Polarization identity for the complex case, see [20, p. 175],

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

which basically says that a scalar product can be expressed through the norm. Thus we conclude for $f,g \in \overline{H}^s$

$$\begin{aligned}
\langle f,g\rangle_{\overline{H}^s} &= \frac{1}{4}\Big(\|f+g|\overline{H}^s\|^2 - \|f-g|\overline{H}^s\|^2 + i\|f+ig|\overline{H}^s\|^2 - i\|f-ig|\overline{H}^s\|^2\Big)\\
&= \frac{d(n,s)}{4}\big(\||\xi|^s\mathcal{F}[f+g]|L_2\|^2 - \||\xi|^s\mathcal{F}[f-g]|L_2\|^2\\
&\qquad + i\||\xi|^s\mathcal{F}[f+ig]|L_2\|^2 - i\||\xi|^s\mathcal{F}[f-ig]|L_2\|^2\big)\\
&= d(n,s)\langle|\xi|^s\mathcal{F}[f],|\xi|^s\mathcal{F}[g]\rangle_{L_2}\\
&= d(n,s)\int_{\mathbb{R}^n}|\xi|^{2s}\mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)}d\xi.
\end{aligned}$$

This proves the first equality. For the second equality, we simply note that $|\xi|^s\mathcal{F}[f], |\xi|^s\mathcal{F}[g] \in L_2$ and apply Remark A.2.9 to get

$$\begin{aligned}
\langle|\xi|^s\mathcal{F}[f],|\xi|^s\mathcal{F}[g]\rangle_{L_2} &= \langle\mathcal{F}^{-1}\left[|\xi|^s\mathcal{F}[f]\right],\mathcal{F}^{-1}\left[|\xi|^s\mathcal{F}[g]\right]\rangle_{L_2}\\
&= \langle(-\Delta)^{\frac{s}{2}}f,(-\Delta)^{\frac{s}{2}}g\rangle_{L_2}.
\end{aligned}$$

□

3.1.3 Remark. *We wish for a representation of the scalar product on $\overline{H}^s$ of the form*

$$\langle f,g\rangle_{\overline{H}^s} = \langle f,(-\Delta)^s g\rangle_{L_2},$$

but in this general setup we cannot guarantee that $f,(-\Delta)^s g \in L_2$, so we have to understand it as the dual paring.
We start with the case $f,g \in \mathcal{S}$. We have just seen that then

$$\langle f,g\rangle_{\overline{H}^s} = \int_{\mathbb{R}^n}|\xi|^{2s}\mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)}d\xi = \langle(-\Delta)^{\frac{s}{2}}f,(-\Delta)^{\frac{s}{2}}g\rangle_{L_2}.$$

Furthermore we know that $|\xi|^{2s}\mathcal{F}[g] \in L_2(\mathbb{R}^n)$ by lemma 2.1.17 and $\mathcal{F}[f] \in L_2(\mathbb{R}^n)$ by A.1.2. So we get

$$\begin{aligned}
\langle f,g\rangle_{\overline{H}^s} &= \int_{\mathbb{R}^n}|\xi|^{2s}\mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)}d\xi\\
&= \langle f,(-\Delta)^s g\rangle.
\end{aligned}$$

Now we take $f \in \overline{H}^s$, *then there exists a sequence* $f_n \in \mathcal{S}$ *such that* $f_n \to f$ *in* $\overline{H}^s$. *Using* [17, Theorem 5.2.3.1] *we get that* $(-\Delta)^{\frac{s}{2}} f_n \to (-\Delta)^{\frac{s}{2}} f$ *in* $L_2(\mathbb{R}^n)$. *Thus we get*

$$\langle (-\Delta)^{\frac{s}{2}} f_n, (-\Delta)^{\frac{s}{2}} g \rangle_{L_2} \to \langle (-\Delta)^{\frac{s}{2}} f, (-\Delta)^{\frac{s}{2}} g \rangle_{L_2}.$$

Because of [17, Theorem 5.2.3.1] *we have that* $(-\Delta)^s g \in \overline{H}^{-s}$, *which is the dual space to* $\overline{H}^s$, *since* $\mathcal{S} \subset \overline{H}^s$. *So we can understand* $\langle f_n, (-\Delta)^s g \rangle$ *as the dual pairing and since* $f_n \to f$ *in* $\overline{H}^s$, *we have*

$$\langle f_n, (-\Delta)^s g \rangle \to \langle f, (-\Delta)^s g \rangle.$$

Remember that we think of $(-\Delta)^s g$ *as a continuous functional on* $\overline{H}^s$.
So in total, from

$$\langle (-\Delta)^{\frac{s}{2}} f_n, (-\Delta)^{\frac{s}{2}} g \rangle_{L_2} = \langle f_n, (-\Delta)^s g \rangle$$

it follows that

$$\langle (-\Delta)^{\frac{s}{2}} f, (-\Delta)^{\frac{s}{2}} g \rangle_{L_2} = \langle f, (-\Delta)^s g \rangle.$$

And finally

$$\begin{aligned} \langle f, g \rangle_{\overline{H}^s} &= \langle (-\Delta)^{\frac{s}{2}} f, (-\Delta)^{\frac{s}{2}} g \rangle_{L_2} \\ &= \langle f, (-\Delta)^s g \rangle. \end{aligned}$$

We will find u as the function that minimizes

$$J(u) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy = \|u|\overline{H}^s\|^2$$

over all functions u in $\overline{H}^s$ that satisfy $\varphi \leq u$ almost everywhere.

3.1.4 Proposition. *The functional* $J(u)$ *attains a unique minimum in the set* $U = \{u \in \overline{H}^s : \varphi \leq u \text{ a.e. }\}$.

Proof. We start with the properties of U. First we see that it is convex. For $u, v \in U$ we have

$$\lambda u + (1-\lambda) v \geq \lambda \min(u,v) + (1-\lambda)\min(u,v) = \min(u,v) \geq \varphi \text{ a.e..}$$

It is also closed. We take a sequence $(u_n) \in U$ that converges to a $u \in \overline{H}^s$, remember that $\overline{H}^s$ is embedded into $L_p(\mathbb{R}^n)$ with $p = \frac{2n}{n-2s}$. Then we can extract a subsequence (u_{n_k}) such that $u_{n_k}(x) \to u(x)$ for almost every $x \in \mathbb{R}^n$. Since $u_{n_k}(x) \geq \varphi(x)$ for almost every $x \in \mathbb{R}^n$ this also has to be true for the limit u.
Lastly we will ensure that it is not empty. We recap that φ is bounded and has compact support. There must exist an $R > 0$ such that $\Theta\varphi \subset B_R(0)$. By [21, Folgerung 1.2], there exists a function $\alpha \in C^\infty$ such that $0 \leq \alpha(x) \leq 1$ for every $x \in \mathbb{R}^n$, $\alpha(x) = 1$ for $x \in \Theta\varphi$ and $\Theta\alpha \subset B_R(0)$. Now we define $\psi(x) = \alpha(x) \sup_{y \in \Theta\varphi} \varphi(y)$, remember that φ is bounded. This function fulfills $\psi(x) \geq \varphi(x)$ for every $x \in \mathbb{R}^n$ and $\psi \in C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \overline{H}^s$, thus $\psi \in U$.
Next we check the properties of J. It is continuous. We take a convergent sequence $(u_n) \subset \overline{H}^s$ that converges to $u \in \overline{H}^s$. This means $\|u_n - u|\overline{H}^s\| \to 0$ for $n \to \infty$. Thus we get

$$\begin{aligned} |J(u_n) - J(u)| &= |\|u_n|\overline{H}^s\|^2 - \|u|\overline{H}^s\|^2| \\ &= |\|u_n|\overline{H}^s\| - \|u|\overline{H}^s\||(\|u_n|\overline{H}^s\| + \|u|\overline{H}^s\|) \\ &\leq \|u_n - u|\overline{H}^s\|(\|u_n|\overline{H}^s\| + \|u|\overline{H}^s\|) \\ &\to 0. \end{aligned}$$

Note that the second factor is bounded since convergent sequences are bounded.
J is strictly convex. It is sufficient to show that $\beta(t) = J(u + t(v-u))$ is strictly convex on $[0,1]$ for every $u, v \in \overline{H}^s$ with $u \neq v$, see [24, Proposition 42.4]. So we fix arbitrary $u, v \in \overline{H}^s$

$$\beta(t) = J(u + t(v-u))$$

$$
\begin{aligned}
&= \langle u + t(v-u), u + t(v-u) \rangle_{\overline{H}^s} \\
&= J(u) + t^2 J(v-u) + 2t \langle u, (v-u) \rangle_{\overline{H}^s}.
\end{aligned}
$$

We see that β is a polynomial of second degree and therefore twice differentiable. So it is strictly convex iff $\varphi'' > 0$ on $[0,1]$, see [24, Proposition 42.5.(d)]. Since

$$
\varphi''(t) = 2J(v-u) > 0
$$

for $v \neq u$, we conclude that J is strictly convex.
Lastly we note that $J(u) = \|u|\overline{H}^s\|^2 \to \infty$ when $\|u|\overline{H}^s\| \to \infty$. Now we are able to apply [24, Proposition 38.15], which says that a convex and continuous function, that fulfills the beforementioned property, possesses a minimum on a closed, convex and nonempty set. [24, Theorem 38.C] ensures that this minimum is unique since U is convex and J is strictly convex. □

From here on we denote this unique minimum as u. In the rest of this section we will show that u is the solution to our obstacle problem. We already know that $u \geq \varphi$ because $u \in U$. Next we will show that it is a supersolution of $(-\Delta)^s$.

3.1.5 Lemma. *The function u is bounded and* $\sup u \leq \sup \varphi$.

Proof. Let us assume that there is a x_0 such that $u(x_0) > \sup \varphi$, then there exists a $\epsilon > 0$ such that $u(x_0) - \epsilon > \sup \varphi$. Thus we get the existence of a $\delta > 0$ such that $u(x) > u(x_0) - \epsilon > \sup \varphi$ for every $x \in B_\delta(x_0)$, by the $\epsilon - \delta$ definition of semicontinuity[1] , see 2.3.24.
Now we define a function

$$
v(x) = \begin{cases} u(x) & x \notin B_\delta(x_0) \\ \sup \varphi & x \in B_\delta(x_0). \end{cases}
$$

This function is also always above φ, we will show that it has a smaller norm than u. For that sake we make the following estimations.
First of all, we notice that $v(x) - v(y) = 0$ for $x, y \in B_\delta(x_0)$. Then that $v(x) - v(y) = u(x) - u(y)$ for $x, y \in B_\delta(x_0)^c$ and lastly for $x \in B_\delta(x_0)$ and $y \notin B_\delta(x_0)$ we have

$$
\begin{aligned}
|v(x) - v(y)| &= |\sup \varphi - u(y)| \\
&< |u(x) - u(y)|.
\end{aligned}
$$

Now we insert all of this in the norm and get

$$
\begin{aligned}
\|v|\overline{H}^s\|^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy \\
&= 2 \int_{B_\delta(x_0)} \int_{B_\delta(x_0)^c} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy + \int_{B_\delta(x_0)^c} \int_{B_\delta(x_0)^c} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy \\
&< 2 \int_{B_\delta(x_0)} \int_{B_\delta(x_0)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy + \int_{B_\delta(x_0)^c} \int_{B_\delta(x_0)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \\
&\leq \|u|\overline{H}^s\|^2.
\end{aligned}
$$

The only thing we have left to show is that $v \in U$. We just saw that its norm is finite, but we have to make sure that it even is in the space $\overline{H}^s$. For that we use a characterization of $\overline{H}^s$ given by [11, (2.3)]. He characterizes it as all functions f in $L_{\frac{2n}{n-2s}}$ with $\|f|\overline{H}^s\| < \infty$. Thus it is enough to show that $v \in L_{\frac{2n}{n-2s}}$. Which simply follows from the fact that $u \in L_{\frac{2n}{n-2s}}$ and that $v = u + (\sup \varphi - u)\chi_{B_\delta(x_0)}$.
To summarize: We have found a function $v \in U$ such that $J(v) < J(u)$, which is a contradiction to 3.1.4. Thus our assumptions must be false. Therefore u must be bounded by $\sup \varphi$. □

[1] The semicontinuity of u will be shown in 3.1.7.

3.1.6 Proposition. *The function u is a supersolution of $(-\Delta)^s$, i.e. $(-\Delta)^s u \geq 0$. See* [14, Proposition 3.1].

Proof. Let h be an arbitrary smooth and nonnegative function with compact support and $t > 0$. The function $u + th$ is then still above the obstacle and so we can conclude that $\|u + th|\overline{H}^s\| \geq \|u|\overline{H}^s\|$, since u was the minimum of J. Therefore

$$J(u) = \langle u, u\rangle_{\overline{H}^s} \leq \langle u + th, u + th\rangle_{\overline{H}^s} = J(u + th).$$

Now we define the function $f(t) = J(u + th)$ for $t > 0$, then f has a righthanded minimum in $t = 0$. This means that $f'_+(0) \geq 0$ and since

$$f(t) = J(u + th) = \langle u, u\rangle_{\overline{H}^s} + 2t\langle u, h\rangle_{\overline{H}^s} + t^2\langle h, h\rangle_{\overline{H}^s}$$

we have

$$f'_+(0) = 2\langle u, h\rangle_{\overline{H}^s} \geq 0.$$

Because of remark 3.1.3, this means

$$\langle u, (-\Delta)^s h\rangle_{L_2} \geq 0$$

which, by definition, means that $(-\Delta)^s u \geq 0$. □

In the next two propositions we will show that $(-\Delta)^s u(x) = 0$ for those x with $u(x) > \varphi(x)$.

3.1.7 Proposition. *The function u is lower-semicontinuous and the set $\{u > \varphi\}$ is open. See* [14, Corollary 3.2].

Proof. We know that u is a supersolution of $(-\Delta)^s$, i.e. $(-\Delta)^s u \geq 0$ and we have seen in Lemma 2.3.16 that $u \in L_{1,s}$. This means we can apply Proposition 2.3.26 and get the lower-semicontinuity of u.
The set $\{u > \varphi\}$ consists of all $x \in \mathbb{R}^n$ such that $u(x) > \varphi(x)$, this can also be written as $u(x) - \varphi(x) > 0$. $u - \varphi$ is also lower-semicontinuous, because φ is continuous[2]. The set $\{x \in \mathbb{R}^n | u(x) - \varphi(x) > 0\}$ is open by definition of lower-semicontinuity (see Definition 2.3.23). This is exactly the set $\{u > \varphi\}$. □

3.1.8 Proposition. *We choose $x_0 \in \mathbb{R}^n$ such that $u(x_0) > \varphi(x_0)$ and $r > 0$ such that $u > \varphi$ in $B_r(x_0)$, then $(-\Delta)^s u(x) = 0$ in $B_r(x_0)$. See* [14, Proposition 3.3].

Proof. We will argue analogously to proposition 3.1.6. Because we already know that u is a supersolution, we only have to show that

$$\langle u, (-\Delta)^s h\rangle \geq 0$$

for every nonpositive testfunction h with compact support in $B_r(x_0)$, because then $(-\Delta)^s u \leq 0$. Its support Θh has a positive distance to $\partial B_r(x_0)$, so $u(x) > \varphi(x)$ for every $x \in \Theta h$. We just saw that u is lower-semicontinuous and because φ is upper-semicontinuous, we get that $u - \varphi$ is lower-semicontinuous. A lower-semicontinuous function attains its minimum over a compact set. So there exists a $z \in \Theta h$ such that $u(z) - \varphi(z) = \inf_{y \in \Theta h}(u(y) - \varphi(y)) > 0$. Thus, there must exist an $\epsilon > 0$ such that

$$u(x) > \varphi(x) + \epsilon \text{ in all of } \Theta h.$$

Now $u + th$ is still going to be above φ as long as $t \geq 0$ is small enough.
Like before, we see that $f(t) = J(u + th)$ has a minimum in $t = 0$, thus $f'_+(0) \geq 0$ and we get that

$$\langle u, (-\Delta)^s h\rangle_{L_2} \geq 0.$$

We can do that for every nonpositive test function and thus $(-\Delta)^s u = 0$. □

3.1.9 Remark. *The last Proposition means that $\Theta[(-\Delta)^s u] \subset \{u = \varphi\}$.*

[2]A function is continuous if it is upper- and lower-semicontinuous

3.1.10 Remark. *Finally we see that u solves the obstacle problem we stated at the beginning. We know it is a supersolution of $(-\Delta)^s$ that is above φ. The fact that $\Theta[(-\Delta)^s u] \subset \{u = \varphi\}$ means that*

$$(-\Delta)^s u(x)(u(x) - \varphi(x)) = 0$$

for every $x \in \mathbb{R}^n$. The decay at infinity follows from general fractional potential theory, see [15].

We use the rest of this section to show that u is continuous.

3.1.11 Proposition. *Let $v \in L_{\frac{2n}{n-2s}}$ be a bounded function in $\mathbb{R}^n$ such that $(-\Delta)^s v \geq 0$ and v is continuous on $E = \Theta[(-\Delta)^s v]$. Then v is continuous in $\mathbb{R}^n$. See* [14, Proposition 3.4].

Proof. We first note that $(-\Delta)^s v = 0$ in $\mathbb{R}^n \backslash E$ and thus v is continuous there by proposition 2.3.33 and since v is continuous on E we only have to check whether it is continuous on ∂E.
We take a point $x_0 \in \partial E$ and a sequence $x_k \to x_0$ as $k \to \infty$. We know that $\liminf_{k\to\infty} v(x_k) \geq v(x_0)$ by lower-semicontinuity of v. We only have to show that $\limsup_{k\to\infty} v(x_k) \leq v(x_0)$.
Suppose the contrary is true, i.e. we assume there exists a subsequence, which we call (x_k) again, such that

$$\lim_{k\to\infty} v(x_k) = v(x_0) + a$$

where $a > 0$. To clarify, we assume that the biggest accumulation point is bigger than $v(x_0)$ and (x_k) is now a sequence that converges to that accumulation point. v is continuous in E by hypothesis, so x_k is not in E from some k on. We drop the first elements in our sequence and can thus assume that $x_k \notin E$ for every $k \in \mathbb{N}$.
We call y_k one of the points in E, that are closest to x_k. So also $y_k \to x_0$ for $k \to \infty$ and since v is continuous in E, we also have

$$\lim_{k\to\infty} v(y_k) = v(x_0).$$

Let $\lambda_k = \frac{|x_k - y_k|}{2} = \frac{1}{2}\mathrm{dist}(x_k, E)$, so $\lambda_k \to 0$ as $k \to \infty$.
Next, we take a look at the function

$$f_e(x) = \frac{\gamma_1(x+e)}{\gamma_1(x)}$$

with an arbitrary unit vector e. f is continuous and always positive because γ_1 is continuous and always positive. The translation by the vector e becomes neglectable when $|x|$ becomes big because γ_1 behaves asymptotically like $\frac{1}{|x|^{n+2s}}$, see 2.3.6, this means

$$\lim_{|x|\to\infty} f_e(x) = 1.$$

This means there exists an $R > 0$ such that $f_e(x) \geq \frac{1}{2}$ for $x \notin B_R(0)$ and there must exist a $z \in \overline{B_R(0)}$ such that $0 < f_e(z) \leq f_e(x)$ for every $x \in \overline{B_R(0)}$, because f_e is continuous. These last two facts allow us to conclude that

$$c_0 = \inf_{x\in\mathbb{R}^n} f_e(x) \geq \min\left(\frac{1}{2}; f_e(z)\right) > 0.$$

Next we state that c_0 is independent of the choice of e. For given unit vectors e_1, e_2 we show that for every $x \in \mathbb{R}^n$ there exists an $y \in \mathbb{R}^n$, such that $f_{e_1}(x) = f_{e_2}(y)$. We simply take a rotation A that maps e_1 onto e_2, i.e. $Ae_1 = e_2$ and define $y = Ax$. Then we have $|y| = |Ax| = |x|$ and $|x + e_1| = |A(x + e_1)| = |y + e_2|$, so by spherical symmetrie of γ_1, see 2.3.5, we get $f_{e_1}(x) = f_{e_2}(y)$.
We take $e = \frac{x_k - y_k}{\lambda_k}$ and get

$$\begin{aligned}
\gamma_{\lambda_k}(x - y_k) - c_0\gamma_{\lambda_k}(x - x_k) &= \frac{1}{\lambda_k^n}\left(\gamma_1\left(\frac{x - y_k}{\lambda_k}\right) - c_0\gamma_1\left(\frac{x - x_k}{\lambda_k}\right)\right) \\
&= \frac{1}{\lambda_k^n}\left(\gamma_1\left(\frac{x - x_k}{\lambda_k} + \frac{x_k - y_k}{\lambda_k}\right) - c_0\gamma_1\left(\frac{x - x_k}{\lambda_k}\right)\right)
\end{aligned}$$

$$= \frac{1}{\lambda_k^n}\gamma_1\left(\frac{x-x_k}{\lambda_k}\right)\left(\frac{\gamma_1\left(\frac{x-x_k}{\lambda_k}+e\right)}{\gamma_1\left(\frac{x-x_k}{\lambda_k}\right)} - c_0\right)$$
$$\geq 0.$$

Because the difference is nonnegative by definition of c_0.
Next we will use proposition 2.3.26 on $v(y_k)$ to get

$$\begin{aligned} v(y_k) &\geq \int_{\mathbb{R}^n} \gamma_{\lambda_k}(x-y_k)v(x)dx \\ &= \int_{\mathbb{R}^n} c_0\gamma_{\lambda_k}(x-x_k)v(x)dx + \int_{\mathbb{R}^n} \left(\gamma_{\lambda_k}(x-y_k) - c_0\gamma_{\lambda_k}(x-x_k)\right)v(x)dx \\ &= c_0 v(x_k) + I_1 + I_2. \end{aligned} \tag{1}$$

In (1) we used proposition 2.3.33 on $v(x_k)$ because $x_k \notin E$ and therefore $(-\Delta)^s v = 0$ in $B_{\lambda_k}(x_k)$. Here we defined

$$I_1 = \int_{B_{\sqrt{\lambda_k}}(y_k)} \left(\gamma_{\lambda_k}(x-y_k) - c_0\gamma_{\lambda_k}(x-x_k)\right)v(x)dx$$

and

$$I_2 = \int_{\mathbb{R}^n \setminus B_{\sqrt{\lambda_k}}(y_k)} \left(\gamma_{\lambda_k}(x-y_k) - c_0\gamma_{\lambda_k}(x-x_k)\right)v(x)dx.$$

Now we take a nullsequence ϵ_l. For every $l \in \mathbb{N}$, there exists a $\delta_l > 0$ such that $|x - x_0| < \delta_l$ implies $v(x) > v(x_0) - \epsilon_l$, because v is lower-semicontinuous (compare remark 2.3.24). For every $l \in \mathbb{N}$ there exists an $k_l \in \mathbb{N}$ such that $B_{\sqrt{\lambda_{k_l}}}(y_{k_l}) \subset B_{\delta_l}(x_0)$, since $y_k \to x_0$ and $\lambda_k \to 0$. To summarize, there exists a subsequence (y_{k_l}) such that $v(x) > v(x_0) - \epsilon_l$ for every $x \in B_{\sqrt{\lambda_{k_l}}}(y_{k_l})$.
Our next goal is to find proper estimations for I_1 and I_2 in case of the subsequence (y_{k_l}). We will start with an estimation for the following integral while using the notation $\alpha_k = \frac{1}{\sqrt{\lambda_k}}$.

$$\begin{aligned} \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k}(0)} \left(\gamma_1(z) - c_0\gamma_1(z+e)\right)dz &= \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k}(0)} \gamma_1(z)dz - c_0 \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k}(0)} \gamma_1(z+e)dz \\ &\leq c_2(n,s) \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k}(0)} \frac{1}{|z|^{n+2s}}dz - c_0c_1(n,s) \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k}(0)} \frac{1}{|z+e|^{n+2s}}dz & (1) \\ &\leq c_2(n,s)\Xi(n)\int\limits_{\alpha_k}^{\infty} \frac{\rho^{n-1}}{\rho^{n+2s}}d\rho - c_0c_1(n,s) \int\limits_{\mathbb{R}^n\setminus B_{\alpha_k+1}(0)} \frac{1}{|z|^{n+2s}}dz & (2) \\ &\leq c_2(n,s)\Xi(n)\left[\frac{-1}{2s\rho^{2s}}\right]_{\alpha_k}^{\infty} - c_0c_1(n,s)\Xi(n)\left[\frac{-1}{2s\rho^{2s}}\right]_{\alpha_k+1}^{\infty} \\ &\leq \frac{c_2(n,s)\Xi(n)}{2s}\lambda_k^s - \frac{c_0c_1(n,s)\Xi(n)}{2s}\frac{\lambda_k^s}{(1+\sqrt{\lambda_k})^{2s}} \\ &=: \beta_k. \end{aligned}$$

Note that β_k converges to 0 when $k \to \infty$. We used the upper and lower approximation found in proposition 2.3.6 in (1). In (2) we used that $B_{\alpha_k}(e) \subset B_{\alpha_k+1}(0)$, the positivity of γ_1 and the fact that we substract the second integral.

Now moving to the approximation of I_2, we will use the transformation $z = \frac{x-y_{k_l}}{\lambda_{k_l}}$, denote $e = \frac{y_{k_l}-x_{k_l}}{\lambda_{k_l}}$ and use that $\gamma_\lambda(x) = \frac{1}{\lambda^n}\gamma_1\left(\frac{x}{\lambda}\right)$

$$\begin{aligned}
I_2 &\geq -\|v|L_\infty\| \int_{\mathbb{R}^n\setminus B_{\sqrt{\lambda_{k_l}}}(y_{k_l})} \Big(\gamma_{\lambda_{k_l}}(x-y_{k_l}) - c_0\gamma_{\lambda_{k_l}}(x-x_k)\Big)\, dx \qquad (3)\\
&= -\|v|L_\infty\| \int_{\mathbb{R}^n\setminus B_{\sqrt{\lambda_{k_l}}}(y_{k_l})} \left(\gamma_1\left(\frac{x-y_{k_l}}{\lambda_{k_l}}\right) - c_0\gamma_1\left(\frac{x-y_{k_l}+y_{k_l}-x_{k_l}}{\lambda_{k_l}}\right)\right)\frac{dx}{\lambda_{k_l}^n}\\
&= -\|v|L_\infty\| \int_{\mathbb{R}^n\setminus B_{\alpha_{k_l}}(0)} (\gamma_1(z) - c_0\gamma_1(z+e))\, dz\\
&\geq -\|v|L_\infty\|\beta_{k_l}.
\end{aligned}$$

In (3), we used that v is a bounded function.
The approximation for I_1 is very similar. Here we will use that $v(x) > v(x_0) - \epsilon_l$ for every $x \in B_{\sqrt{\lambda_{k_l}}}(y_{k_l})$.

$$\begin{aligned}
I_1 &\geq (v(x_0)-\epsilon_l)\int_{B_{\sqrt{\lambda_{k_l}}}(y_{k_l})} \Big(\gamma_{\lambda_{k_l}}(x-y_{k_l}) - c_0\gamma_{\lambda_{k_l}}(x-x_k)\Big)\, dx\\
&= (v(x_0)-\epsilon_l)\int_{B_{\alpha_{k_l}}(0)} (\gamma_1(z) - c_0\gamma_1(z+e))\, dz\\
&= (v(x_0)-\epsilon_l)\left(\int_{\mathbb{R}^n} (\gamma_1(z) - c_0\gamma_1(z+e))\, dz - \int_{\mathbb{R}^n\setminus B_{\alpha_{k_l}}(0)} (\gamma_1(z) - c_0\gamma_1(z+e))\, dz\right)\\
&\geq (v(x_0)-\epsilon_l)(1-c_0-\beta_{k_l}).
\end{aligned}$$

The fact that $\int_{\mathbb{R}^n}\gamma_1(x)dx = 1$ was used here. Now we continue where we left off in (1).

$$\begin{aligned}
v(y_{k_l}) &\geq c_0 v(x_{k_l}) + I_1 + I_2\\
&\geq c_0 v(x_{k_l}) + (v(x_0)-\epsilon_l)(1-c_0-\beta_{k_l}) - \|v|L_\infty\|\beta_{k_l}\\
&= c_0 v(x_{k_l}) + v(x_0)(1-c_0) - \epsilon_l(1-c_0-\beta_{k_l}) - \|v|L_\infty\|\beta_{k_l} - v(x_0)\beta_{k_l}\\
&\geq c_0 v(x_{k_l}) + v(x_0)(1-c_0) - \epsilon_l(1-c_0-\beta_{k_l}) - 2\|v|L_\infty\|\beta_{k_l}.
\end{aligned}$$

We remember that $\epsilon_{k_l}, \beta_{k_l} \to 0, v(y_{k_l}) \to v(x_0)$ and $v(x_{k_l}) \to v(x_0) + a$ as $l \to \infty$. If we take $l \to \infty$ in the inequality we finally get

$$\begin{aligned}
v(x_0) &\geq c_0(v(x_0)+a) + v(x_0)(1-c_0) + 0\\
&= v(x_0) + c_0 a.
\end{aligned}$$

But this is a contradiction, because $ac_0 > 0$. This concludes our proof. □

3.1.12 Conclusion. *The function u is continuous.*

Proof. We already know that $u \in L_{\frac{2n}{n-2s}}$. It is continuous on $\Theta[(-\Delta)^s u]$ because it coincides with the continuous function φ there, see remark 3.1.9. We also know that it is bounded, see 3.1.5. Lastly it is a supersolution of $(-\Delta)^s$, see 3.1.6. Thus we can apply proposition 3.1.11 and conclude that u is continuous. □

3.2. Further Results

3.2.1 Proposition. *The function u is the least supersolution of $(-\Delta)^s$ that is above φ, i.e. $u \geq \varphi$ a.e. and is nonnegative at infinity, i.e. $\liminf_{|x|\to\infty} u(x) \geq 0$. See* [14, Proposition 3.6].

Proof. We choose another supersolution v, i.e. $(-\Delta)^s v \geq 0$ such that $v \geq \varphi$ and $\liminf_{|x|\to\infty} v(x) \geq 0$. Let $m = \min(u, v)$. We have to show that $u = m$. By definition we already know that $m \leq u$, so we only need to show that $m \geq u$.
We want to use our comparison principle 2.3.32. We first note that m is again a supersolution because of proposition 2.3.29. It also is above φ because u and v are always above φ.
We choose $\Omega = \{u > \varphi\}$, this is an open set, as seen in 3.1.7. In Ω we have $(-\Delta)^s u = 0$ because of proposition 3.1.8. Since $\varphi \leq m \leq u$, we can conclude $u(x) = m(x)$ for $x \in \mathbb{R}^n \setminus \Omega$. We know from conclusion 3.1.12 that u is continuous and therefore that $m - u$ is lower-semicontinuous. Remember that m is lower-semicontinuous, because it is a supersolution.
Finally, we can apply proposition 2.3.32[3] and get that $u(x) = m(x)$ for every $x \in \mathbb{R}^n$. □

Now we want to show that the solution u is Lipschitz iff the obstacle φ is Lipschitz. For that we prove a proposition, that is a bit more general. But first we need the following definition.

3.2.2 Definition. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ admits a function $\omega : [0, \infty] \to [0, \infty]$ as a (global) modulus of continuity iff

$$|f(x) - f(y)| \leq \omega(|x - y|)$$

for every $x, y \in \mathbb{R}^n$.

Now we can state the following proposition.

3.2.3 Proposition. *If φ admits ω as a modulus of continuity, then so does the function u. See* [14, Theorem 3.8].

Proof. By definition of a modulus of continuity, we have for any $h \in \mathbb{R}^n$

$$\varphi(x + h) + \omega(|h|) \geq \varphi(x)$$

for every $x \in \mathbb{R}^n$. Since $u(x) \geq \varphi(x)$ is always true, we conclude that

$$u(x + h) + \omega(|h|) \geq \varphi(x).$$

The function $u(x + h) + \omega(|h|)$ is again a supersolution of $(-\Delta)^s$, note proposition 2.1.20, that is above the obstacle φ. From proposition 3.2.1 it follows that

$$u(x + h) + \omega(|h|) \geq u(x)$$

for every $x \in \mathbb{R}^n$. Thus we can immediately conclude that $u(x + h) - u(x) \geq -\omega(|h|)$. If we define $y = x + h$ and $k = -h$, we get from this inequality: $u(y) - u(y + k) \geq -\omega(|k|)$ which is equivalent to $u(y + k) - u(y) \leq \omega(|k|)$. So in total

$$|u(x + h) - u(x)| \leq \omega(|h|)$$

for every $x, h \in \mathbb{R}^n$. This means u admits ω as a modulus of continuity. □

3.2.4 Conclusion. *The function u is Lipschitz and its Lipschitz constant is not larger than the one of φ. See* [14, Corollary 3.9].

[3] Our u takes on the role of v in that proposition and m the role of u.

Proof. φ is Lipschitz by hypothesis, i.e. $\forall x, y \in \mathbb{R}^n$

$$|\varphi(x) - \varphi(y)| \leq L|x - y|.$$

But this means that φ admits $\omega(r) = Lr$ as a modulus of continuity. Thus proposition 3.2.3 tells us that

$$|u(x) - u(y)| \leq L|x - y|$$

$\forall x, y \in \mathbb{R}^n$. So we get that also u is Lipschitz, but there could be a smaller Lipschitz constant. □

3.2.5 Proposition. *Take $\varphi \in C^{1,1}$. Assume that the second order incremental quotients of φ are bounded below, i.e. there exists a constant $C > 0$ such that*

$$\frac{\varphi(x + te) + \varphi(x - te) - 2\varphi(x)}{2t^2} \geq -C$$

for every $x, e \in \mathbb{R}^n$ and $t > 0$. Then the same is true for u with the same constant C.

Proof. We reformulate the inequality for φ so it looks like this

$$\frac{\varphi(x + te) + \varphi(x - te)}{2} + Ct^2 \geq \varphi(x).$$

Again, we conclude that

$$\frac{u(x + te) + u(x - te)}{2} + Ct^2 \geq \varphi(x).$$

The function

$$v(x) = \frac{u(x + te) + u(x - te)}{2} + Ct^2$$

is a supersolution of $(-\Delta)^s$, because

$$\begin{aligned} (-\Delta)^s v(x) &= (-\Delta)^s \left[\frac{u(x + te) + u(x - te)}{2} + Ct^2 \right] \\ &= \frac{(-\Delta)^s [u(x + te)] + (-\Delta)^s [u(x - te)]}{2} \\ &= \frac{(-\Delta)^s [u] (x + te) + (-\Delta)^s [u] (x - te)}{2} \\ &\geq 0. \end{aligned}$$

Compare proposition 2.1.20 and note that $(-\Delta)^s$ "differentiates" with respect to x.
Thus far we learned that v is a supersolution, that it is above φ, and by proposition 3.2.1 it must be above u. For the sake of completeness, it should be mentioned that v is nonnegative at infinity because u is and the constant C as well as t is positive.
So

$$v(x) = \frac{u(x + te) + u(x - te)}{2} + Ct^2 \geq u(x)$$

and this is equivalent to

$$\frac{u(x + te) + u(x - te) - 2u(x)}{2t^2} \geq -C.$$

This means that the second order incremental quotients of u are also bounded below. □

3.2.6 Proposition. *If $(-\Delta)^s \varphi \leq C$ for some nonnegative constant C and some $s \in (0, 1)$, then also $(-\Delta)^s u \leq C$. See* [14, Proposition 3.11].

Proof. If $x \notin \{u = \varphi\}$, then $(-\Delta)^s u = 0$ by proposition 3.1.8.
If $x \in \{u = \varphi\}$, we use the representation of $(-\Delta)^s$ given in proposition 2.1.11 to get

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{-c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy \\ &\leq \frac{-c_{n,s}}{2}\int_{\mathbb{R}^n} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{n+2s}}dy \\ &= (-\Delta)^s \varphi(x) \leq C.\end{aligned}$$

We used that $u(x \pm y) \geq \varphi(x \pm y)$ and that $u(x) = \varphi(x)$. □

3.2.7 Proposition. *If $(-\Delta)^s\varphi \in L^\infty(\mathbb{R}^n)$, then $(-\Delta)^s u \in L^\infty(\mathbb{R}^n)$. See* [14, Proposition 3.12].

Proof. From proposition 3.2.6 it follows that $(-\Delta)^s u$ is bounded above, because $(-\Delta)^s\varphi$ is and $(-\Delta)^s u$ is naturally bounded below because it is a supersolution. □

From now on, we will investigate the case of weaker requirements on φ. To be precise, we study the case $\varphi \in C^{1,\alpha}$. We will start with a general property of $C^{1,\alpha}$ functions.

3.2.8 Lemma. *For $\varphi \in C^{1,\alpha}$ we can find a constant $C > 0$ such that*

$$\varphi(x+h) \geq \varphi(x) + \nabla\varphi(x)\cdot h - C|h|^{1+\alpha}$$

for $x, h \in \mathbb{R}^n$. This constant is independent of the choice of h.

Proof. The proof uses the mean value theorem. With it we get

$$\varphi(x+h) - \varphi(x) = \nabla\varphi(x+\theta h)\cdot h$$

with $\theta \in (0,1)$. Now if we additionally substract $\nabla\varphi(x)\cdot h$ on both sides we get

$$\begin{aligned}|\varphi(x+h) - \varphi(x) - \nabla\varphi(x)\cdot h| &= |\left(\nabla\varphi(x+\theta h) - \nabla\varphi(x)\right)\cdot h| \\ &\leq |\left(\nabla\varphi(x+\theta h) - \nabla\varphi(x)\right)||h| \qquad (1) \\ &\leq \left(\sum_{j=1}^n |\partial_j\varphi(x+\theta h) - \partial_j\varphi(x)|^2\right)^{\frac{1}{2}} |h| \\ &\leq \left(\sum_{j=1}^n C_j^2(\theta|h|)^{2\alpha}\right)^{\frac{1}{2}} |h| \qquad (2) \\ &= \left(\sum_{j=1}^n C_j^2\theta^{2\alpha}\right)^{\frac{1}{2}} |h|^{1+\alpha} \\ &\leq \left(\sum_{j=1}^n C_j^2\right)^{\frac{1}{2}} |h|^{1+\alpha} \\ &= C|h|^{1+\alpha}.\end{aligned}$$

We used the Cauchy-Schwarz inequality in (1) and the Hölder-continuity of the partial derivatives, i.e. the fact that $|\partial_j\varphi(y) - \partial_j\varphi(x)| \leq C_j|x-y|^\alpha$, in (2). We see that the constant C does only depend on the Lipschitz constants of the partial derivatives. The inequality especially means that

$$-C|h|^{1+\alpha} \leq \varphi(x+h) - \varphi(x) - \nabla\varphi(x)\cdot h$$

or

$$\varphi(x) + \nabla\varphi(x)\cdot h - C|h|^{1+\alpha} \leq \varphi(x+h).$$

□

Our last goal will be to show that a similar statement is also true for u, even though u is only Lipschitz continuous. To do that, we first need a couple of lemmatas.

3.2.9 Lemma. *Suppose $\varphi \in C^{1,\alpha}$. If we have $h_j \in \mathbb{R}^n$ and $\lambda_j \in [0,1]$ for $j = 1, \ldots, k$ such that $0 = \sum_{j=1}^k \lambda_j h_j$ and $\sum_{j=1}^k \lambda_j = 1$ then there exists a constant ζ such that*

$$u(x) \leq \sum_{j=1}^k \lambda_j u(x+h_j) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha}$$

for any $x \in \mathbb{R}^n$.

Proof. We first apply lemma 3.2.8 for every h_j to find constants $C_j = C(h_j)$ such that

$$\varphi(x+h_j) \geq \varphi(x) + \nabla\varphi(x) \cdot h_j - C_j |h_j|^{1+\alpha}$$

for every $j = 1, \ldots, k$. We define $\zeta = \max_{j=1,\ldots,k} C_j$ and get

$$\varphi(x+h_j) \geq \varphi(x) + \nabla\varphi(x) \cdot h_j - \zeta |h_j|^{1+\alpha}$$

for every $j = 1, \ldots, k$. So

$$\begin{aligned}
\sum_{j=1}^k \lambda_j \varphi(x+h_j) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} &\geq \sum_{j=1}^k \lambda_j \left(\varphi(x) + \nabla\varphi(x) \cdot h_j - \zeta |h_j|^{1+\alpha} \right) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} \\
&= \varphi(x) \sum_{j=1}^k \lambda_j + \nabla\varphi(x) \cdot \sum_{j=1}^k \lambda_j h_j - \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} \\
&= \varphi(x).
\end{aligned}$$

Again we conclude that the function

$$v(x) = \sum_{j=1}^k \lambda_j u(x+h_j) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha}$$

lies above the obstacle and

$$\begin{aligned}
(-\Delta)^s v(x) = (-\Delta)^s &\left[\sum_{j=1}^k \lambda_j u(x+h_j) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} \right] \\
&= \sum_{j=1}^k \lambda_j (-\Delta)^s [u(x+h_j)] + (-\Delta)^s [\zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha}] \\
&= \sum_{j=1}^k \lambda_j (-\Delta)^s [u](x+h_j) \\
&\geq 0.
\end{aligned}$$

Where we used proposition 2.1.20.
To summarize, v is a supersolution that is always above φ and nonnegative at infinity because u is nonnegative at infinity and $\zeta, \lambda_j \geq 0$. We can apply proposition 3.2.1 and get

$$v(x) = \sum_{j=1}^k \lambda_j u(x+h_j) + \zeta \sum_{j=1}^k \lambda_j |h_j|^{1+\alpha} \geq u(x).$$

That is what we wanted to show. □

3.2.10 Lemma. *For $0 < k < h$ and $\alpha \in (0,1)$ we have*

$$\frac{h-k}{h}(k)^{1+\alpha} + \frac{k}{h}(h-k)^{1+\alpha} \leq 2kh^{\alpha}.$$

Proof. We have

$$\begin{aligned} & \frac{h-k}{h}(k)^{1+\alpha} + \frac{k}{h}(h-k)^{1+\alpha} \leq 2kh^{\alpha} \\ \iff & \frac{h}{k}\left(\frac{k}{h}\right)^{1+\alpha} - \left(\frac{k}{h}\right)^{1+\alpha} + \frac{(h-k)^{1+\alpha}}{h^{1+\alpha}} \leq 2 \\ \iff & \left(\frac{k}{h}\right)^{\alpha} - \left(\frac{k}{h}\right)^{1+\alpha} + \left(1-\frac{k}{h}\right)^{1+\alpha} \leq 2 \\ \iff & \left(1-\frac{k}{h}\right)\left(\left(\frac{k}{h}\right)^{\alpha} + \left(1-\frac{k}{h}\right)^{\alpha}\right) \leq 2. \end{aligned}$$

We only have to verify the last inequality. To do that, we will study the function

$$v_{\alpha}(x) = (1-x)\left(x^{\alpha} + (1-x)^{\alpha}\right)$$

on the interval $[0,1]$ since $0 < x = \frac{k}{h} < 1$. First we note that $v_{\alpha}(x) \leq (x^{\alpha} + (1-x)^{\alpha}) = w_{\alpha}(x)$ since $0 < (1-x) < 1$. Next we note that $w_{\alpha}(x)$ is monotonically decreasing in α, when $x \in [0,1]$, so we will attain an upper bound if $\alpha = 0$. Then $w_{\alpha}(x) < 1+1 = 2$. In total we get $v_{\alpha}(x) \leq 2$ and this is what we wanted to show. ☐

3.2.11 Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function that satisfies*

$$\frac{f(x_0+k) - f(x_0)}{k} \leq \frac{f(x_0+h) - f(x_0)}{h} + Ch^{\alpha} \tag{1}$$

for $0 < k < h \leq 1$, $\alpha \in (0,1)$ and a positive constant C. Then the right derivative $f'_+(x_0)$ of f in x_0 exists.

Proof. We first define $k_n = \frac{1}{n}$, choose $h = 1$ and note that the incremental quotients of f must be bounded since it is a Lipschitz function, i.e. $\frac{|f(x)-f(y)|}{|x-y|} \leq D$. In this case our inequality looks like this

$$-D \leq \left[f\left(x_0 + \frac{1}{n}\right) - f(x_0)\right] n \leq f(x_0+1) - f(x_0) + C \leq D + C.$$

Thus, the bounded sequence $a_n = \left[f\left(x_0 + \frac{1}{n}\right) - f(x_0)\right] n$ must have a convergent subsequence $(a_{n_j}) = \left(\left[f\left(x_0 + \frac{1}{n_j}\right) - f(x_0)\right] n_j\right)$ by the theorem of Bolzano and Weierstraß. We denote the limit of this subsequence with a, i.e. $a_{n_j} \to a$ when $j \to \infty$.
Next we choose an arbitrary nullsequence (h_l) such that $0 < h_1 < \frac{1}{n_1} \leq 1$. Without loss of generality we can assume that this sequence is monotonically decreasing, [4] then for every $l \in \mathbb{N}$ there exist $n_l^1, n_l^2 \in \mathbb{N}$, that are elements of the subsequence (n_j) we used before, such that $\frac{1}{n_l^1} < h_l < \frac{1}{n_l^2}$. First we apply inequality (1) to $\frac{1}{n_l^1} < h_l$ to get

$$\left[f\left(x_0 + \frac{1}{n_l^1}\right) - f(x_0)\right] n_l^1 \leq \frac{f(x_0+h_l) - f(x_0)}{h_l} + Ch_l^{\alpha} \qquad \iff$$

[4] If it were not decreasing we would define $a_l = \frac{f(x_0+h_l)-f(x_0)}{h_l}$ and then choose an arbitrary subsequence (a_{l_j}). From the corresponding sequence (h_{l_j}) we can again extract a monotonically decreasing subsequence (h'_l). All sequences $a'_l = \frac{f(x_0+h'_l)-f(x_0)}{h'_l}$ will converge to the same value following the rest of the proof. Then we can simply apply [23, Proposition 10.13] to get the convergence of the initial sequence.

$$\left[f\left(x_0+\frac{1}{n_l^1}\right)-f(x_0)\right]n_l^1 - Ch_l^\alpha \le \frac{f(x_0+h_l)-f(x_0)}{h_l}$$

now we apply it to $h_l < \frac{1}{n_l^2}$ and get

$$\frac{f(x_0+h_l)-f(x_0)}{h_l} \le \left[f\left(x_0+\frac{1}{n_l^2}\right)-f(x_0)\right]n_l^2 + C\left(\frac{1}{n_l^2}\right)^\alpha.$$

So, when we combine these two inequalities, we get for every $l \in \mathbb{N}$

$$\left[f\left(x_0+\frac{1}{n_l^1}\right)-f(x_0)\right]n_l^1 - Ch_l^\alpha \le \frac{f(x_0+h_l)-f(x_0)}{h_l} \le \left[f\left(x_0+\frac{1}{n_l^2}\right)-f(x_0)\right]n_j^2 + C\left(\frac{1}{n_l^2}\right)^\alpha.$$

The left and the right side of the inequality converge to a when l goes to infinity so this means

$$a \le \lim_{l\to\infty}\frac{f(x_0+h_l)-f(x_0)}{h_l} \le a$$

and since this is true for an arbitrary nullsequence we even have

$$a \le \lim_{h\to+0}\frac{f(x_0+h)-f(x_0)}{h} \le a.$$

We see that the limit exists and is equal to a. This concludes the proof.

□

3.2.12 Remark. *Of course a similar statement is true for the left derivative $f'_-(x_0)$. In that case, we need the inequality to look like*

$$\frac{f(x_0)-f(x_0-k)}{k} \ge \frac{f(x_0)-f(x_0-h)}{h} - Ch^\alpha$$

for $0 < k < h \le 1$ and a positive constant C. The proof goes analogously.

3.2.13 Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function that satisfies an inequality like in lemma 3.2.9. That means, if we find $h_j \in \mathbb{R}$ and $\lambda_j \in [0,1]$ for $j = 1, \dots, m$ such that $\sum_{j=1}^m \lambda_j = 1$ and $\sum_{j=1}^m \lambda_j h_j = 0$, then there exists a constant ζ such that*

$$f(x) \le \sum_{j=1}^m \lambda_j f(x+h_j) + \zeta\sum_{j=1}^m \lambda_j |h_j|^{1+\alpha}$$

for every $x \in \mathbb{R}$. Then for every $x_0 \in \mathbb{R}$

(i) f has a right derivative $f'_+(x_0)$ and a left derivative $f'_-(x_0)$ at x_0.

(ii) the right derivative is greater than the left derivative.

(iii) for any number a in the closed interval $[f'_-(x_0), f'_+(x_0)]$ and every $h \in \mathbb{R}$, there exists a $\eta \in \mathbb{R}$ such that

$$f(x_0+h) \ge f(x_0) + ah - \eta|h|^{1+\alpha}$$

where η only depends on the constant ζ in the beforementioned inequality. See [14, Lemma 3.15].

Proof. (i) Choose $x_0 \in \mathbb{R}$ and $0 < k < h$. Let $x = x_0 + k$, $h_1 = -k$, $h_2 = h - k$ and $\lambda_1 = \frac{h-k}{h}$ and $\lambda_2 = \frac{k}{h}$. Then

$$\lambda_1 + \lambda_2 = \frac{h-k+k}{h} = 1$$

and

$$\lambda_1 h_1 + \lambda_2 h_2 = \frac{-k(h-k)}{h} + \frac{(h-k)k}{h} = 0.$$

By assumption there exists a ζ such that

$$\begin{aligned} f(x) = f(x_0+k) &\le \frac{h-k}{h} f(x_0) + \frac{k}{h} f(x_0+h) + \zeta \left(\frac{h-k}{h} |k|^{1+\alpha} + \frac{k}{h} |h-k|^{1+\alpha} \right) \\ &\le f(x_0) + \frac{k}{h} \left(f(x_0+h) - f(x_0) \right) + 2\zeta k h^{\alpha}. \end{aligned}$$

We used lemma 3.2.10 in the second inequality.
In total, this is equivalent to

$$\frac{f(x_0+k) - f(x_0)}{k} \le \frac{f(x_0+h) - f(x_0)}{h} + 2\zeta h^{\alpha}. \tag{2}$$

Lemma 3.2.11 ensures the existence of the right derivative $f'_+(x_0)$ when $h \to 0$.
For the left derivative, we set $x = x_0 - k$, $h_1 = k$, $h_2 = k - h$ and λ_1, λ_2 remain unaltered. Then we get

$$\begin{aligned} f(x) = f(x_0-k) &\le \frac{h-k}{h} f(x_0) + \frac{k}{h} f(x_0-h) + \zeta \left(\frac{h-k}{h} |k|^{1+\alpha} + \frac{k}{h} |h-k|^{1+\alpha} \right) \\ &\le f(x_0) + \frac{k}{h} \left(f(x_0-h) - f(x_0) \right) + 2\zeta k h^{\alpha}. \end{aligned}$$

That is equivalent to

$$\frac{f(x_0-k) - f(x_0)}{k} \le \frac{f(x_0-h) - f(x_0)}{h} + 2\zeta h^{\alpha}$$

or

$$\frac{f(x_0) - f(x_0-k)}{k} \ge \frac{f(x_0) - f(x_0-h)}{h} - 2\zeta h^{\alpha} \tag{3}$$

and thus remark 3.2.12 ensures the existence of the left derivative.

(ii) We consider $h_1 = h$ and $h_2 = -h$ together with $\lambda_1 = \lambda_2 = \frac{1}{2}$. By assumption there exists a constant ζ_1, so that we get

$$f(x) \le \frac{1}{2} f(x+h) + \frac{1}{2} f(x-h) + \zeta_1 h^{1+\alpha}.$$

Therefore,

$$\frac{f(x) - f(x-h)}{h} \le \frac{f(x+h) - f(x)}{h} + 2\zeta_1 h^{\alpha}.$$

Thus, if we take $h \to 0$

$$f'_-(x) \le f'_+(x).$$

(iii) We choose $a \in [f'_-(x_0), f'_+(x_0)]$ and start off with positive $h \in \mathbb{R}$. We let k go to 0 in (2) to get

$$a \le f'_+(x_0) \le \frac{f(x_0+h) - f(x_0)}{h} + 2\zeta h^{\alpha}$$

which is equivalent to

$$f(x_0+h) \ge f(x_0) + ah - 2\zeta h^{1+\alpha}.$$

Next, we want to show the same for negative $l \in \mathbb{R}$. We let k go to 0 in (3) to get

$$a \geq f'_-(x_0) \geq \frac{f(x_0) - f(x_0 - h)}{h} - 2\zeta h^\alpha$$

remember that h was positive here. From this we get

$$\begin{aligned} & f(x_0 - h) \geq f(x_0) - ah - 2\zeta h^{1+\alpha} \\ \Longleftrightarrow & f(x_0 + (-h)) \geq f(x_0) + a(-h) - 2\zeta|-h|^{1+\alpha} \\ \Longleftrightarrow & \quad f(x_0 + l) \geq f(x_0) + al - 2\zeta|l|^{1+\alpha}. \end{aligned}$$

When we define $l = -h$, i.e. l is negative. So in total we have seen that

$$f(x_0 + h) \geq f(x_0) + ah - \eta|h|^{1+\alpha}$$

for all $h \in \mathbb{R}$, where η only depends on ζ.

□

3.2.14 Proposition. *If $\varphi \in C^{1,\alpha}$, then for every $x_0 \in \mathbb{R}^n$, there exists an $a \in \mathbb{R}^n$ such that*

$$u(x_0 + h) \geq u(x_0) + a \cdot h - C|h|^{1+\alpha} \text{ for every } h \in \mathbb{R}^n.$$

The geometric interpretation is, that the function u has a supporting plane in each point with an error of order $1 + \alpha$.

Proof. We choose $t_j \in \mathbb{R}$, $\lambda_j \in [0, 1]$ for $j = 1, \dots, m$ and $e \in \mathbb{R}^n$ such that $\sum_{j=1}^m \lambda_j = 1$, $\sum_{j=1}^m \lambda_j t_j = 0$ and $|e| = 1$. First we note that for $h_j = t_j e$ also $\sum_{j=1}^m \lambda_j h_j = e\left(\sum_{j=1}^m \lambda_j t_j\right) = 0$. We can apply lemma 3.2.9 for $x = x_0 + se$ with $s \in \mathbb{R}$ arbitrary and get

$$u(x) \leq \sum_{j=1}^m \lambda_j u(x + h_j) + \zeta \sum_{j=1}^m \lambda_j |h_j|^{1+\alpha}. \tag{1}$$

After defining $f(r) = u(x_0 + re)$ with $r \in \mathbb{R}$, this inequality reads as

$$f(s) \leq \sum_{j=1}^m \lambda_j f(s + t_j) + \zeta \sum_{j=1}^m \lambda_j |t_j|^{1+\alpha}$$

since $x + h_j = x_0 + (s + t_j)e$. This enables us to use 3.2.13 and get that

$$f(s + t) \geq f(s) + f'_+(s)t - \eta|t|^{1+\alpha} \text{ for every } s, t \in \mathbb{R}.$$

The case $s = 0$ is enough for us, so

$$f(t) \geq f(0) + f'_+(0)t - \eta|t|^{1+\alpha} \text{ for every } t \in \mathbb{R}.$$

This means for the function u

$$u(x_0 + te) \geq u(x_0) + f'_+(0)t - \eta|t|^{1+\alpha}. \tag{2}$$

Let us take a closer look at $f'_+(0)$.

$$f'_+(0) = \lim_{t \to +0} \frac{f(t) - f(0)}{t}$$

$$= \lim_{t\to+0} \frac{u(x_0+te)-u(x_0)}{t}.$$

We see that $f'_+(0)$ is the directional derivative of u in direction of e in the point x_0. From now on we denote it as $a_{x_0}(e)$.
Next we will deal with the function

$$c(x) = a_{x_0}\left(\frac{x}{|x|}\right)|x| \text{ for every } x \in \mathbb{R}^n\backslash\{0\} \text{ and } c(0) = 0$$

which is basically the directional derivative of u in direction of the vector x in x_0, now we allow $|x| \neq 1$. First we note that for every $h \in \mathbb{R}^n$ c is bounded on a neighborhood of h. Choose $k \in B_\epsilon(h)$, then $|c(k)| \leq L(|h|+\epsilon)$, where L is the Lipschitz constant of u. We can reformulate inequality (2) using c and get

$$u(x_0+h) \geq u(x_0) + c(h) - \eta|h|^{1+\alpha} \text{ for every } h \in \mathbb{R}^n. \tag{3}$$

Next, we want to show that c is convex, i.e.

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y) \text{ for every } x, y \in \mathbb{R}^n.$$

For that purpose we define $\lambda_1 = \lambda$, $\lambda_2 = (1-\lambda)$ with $\lambda \in [0,1]$, $h_1 = (1-\lambda)t(x-y)$ and $h_2 = \lambda t(y-x)$ where $t > 0$. Then $\lambda_1 + \lambda_2 = 1$ and

$$\lambda_1 h_1 + \lambda_2 h_2 = \lambda(1-\lambda)t(x-y) + \lambda(1-\lambda)t(y-x) = 0.$$

Thus, if we insert that in (1) for $x = x_0 + \lambda tx + (1-\lambda)ty$ we get

$$u(x_0+\lambda tx+(1-\lambda)ty) \leq \lambda u(x_0+tx) + (1-\lambda)u(x_0+ty) + \zeta\left[\lambda(1-\lambda)\left((1-\lambda)^\alpha + \lambda^\alpha\right)\right](t|x-y|)^{1+\alpha}.$$

We substract $u(x_0)$ on both sides and divide by $t > 0$.

$$\frac{u(x_0+\lambda tx+(1-\lambda)ty)-u(x_0)}{t} \leq \lambda\frac{u(x_0+tx)-u(x_0)}{t} + (1-\lambda)\frac{u(x_0+ty)-u(x_0)}{t} + Ct^\alpha|x-y|^{1+\alpha}.$$

Now we let $t \to +0$ and we see that the quotients are the directional derivatives of u in x_0 in direction of arbitrary vectors, so we insert the function c.

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y).$$

Thus c is convex and therefore we can conclude that c is continuous and finite in every point, because it is bounded in a neighborhood of any point (see [24, Proposition 47.5]). We can conclude that there exists a vector $b \in \mathbb{R}^n$ such that

$$c(h) \geq c(0) + b\cdot h = b\cdot h \text{ for every } h \in \mathbb{R}^n.$$

(Because the subdifferential is nonempty in every point, see [24, Theorem 47.A].) Finally, inserting this into (3) yields

$$u(x_0+h) \geq u(x_0) + c(h) - \eta|h|^{1+\alpha} \geq u(x_0) + b\cdot h - \eta|h|^{1+\alpha}.$$

□

Appendix A.

The next two sections in this appendix are just an amalgamation of facts about rapidly decreasing functions and the Fourier transformation.

A.1. Rapidly decreasing functions

The rapidly decreasing functions will play a big role for us:

A.1.1 Definition. A function $f \in C^\infty(\mathbb{R}^n)$ is called **rapidly decreasing** iff there exists a constant $C_{\alpha,j}(f)$ for every multiindex $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}_0$ such that:

$$|\partial^\alpha f(x)| \leq C_{\alpha,j}(f)(1+|x|)^{-j}$$

for all $x \in \mathbb{R}^n$. We use $\mathcal{S}$ as the symbol for the set of all these functions.
We can define a family of seminorms by

$$|f|_{j,k;\mathcal{S}} = \sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} (1+|x|)^j |\partial^\alpha f(x)|.$$

A.1.2 Proposition. *(i) Let $f \in \mathcal{S}$, then for every multiindex $\alpha \in \mathbb{N}_0^n$ we have*

$$\partial^\alpha f \in L_1(\mathbb{R}^n).$$

(ii) Let $f \in \mathcal{S}$, then for every $k \in \mathbb{N}_0$ we have

$$(1+|x|)^k f(x) \in L_1(\mathbb{R}^n).$$

(iii) Let $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in L_2(\mathbb{R}^n)$.

Proof. (i)

$$\begin{aligned}
\|\partial^\alpha f | L_1(\mathbb{R}^n)\| &= \int_{\mathbb{R}^n} |\partial^\alpha f(x)| dx \\
&\leq C_{\alpha,n+1}(f) \int_{\mathbb{R}^n} (1+|x|)^{-(n+1)} dx && (*) \\
&= \Xi(n) C_{\alpha,n+1}(f) \int_0^\infty (1+\rho)^{-(n+1)} \rho^{n-1} d\rho && (**) \\
&\leq \Xi(n) C_{\alpha,n+1}(f) \int_0^\infty (1+\rho)^{-(n+1)} (\rho+1)^{n-1} d\rho \\
&= \Xi(n) C_{\alpha,n+1}(f) \int_1^\infty \rho^{-(n+1)} \rho^{n-1} d\rho \\
&= \Xi(n) C_{\alpha,n+1}(f) \int_1^\infty \rho^{-2} d\rho \\
&< \infty.
\end{aligned}$$

In (*) we use the definition of rapidly decreasing functions and in (**) we go to spherical coordinates, $\Xi(n)$ is the constant that arises when doing this.

(ii)

$$
\begin{aligned}
\|(1+|x|)^k f(x)|L_1(\mathbb{R}^n)\| &= \int_{\mathbb{R}^n} (1+|x|)^k |f(x)| dx \\
&\leq C_{0,k+n+1}(f) \int_{\mathbb{R}^n} \frac{(1+|x|)^k}{(1+|x|)^{k+n+1}} dx \\
&\leq C_{0,k+n+1}(f) \int_{\mathbb{R}^n} (1+|x|)^{-(n+1)} dx.
\end{aligned}
$$

The rest follows analogously to (i).

(iii)

$$
\begin{aligned}
\|f|L_2(\mathbb{R}^n)\| &= \int_{\mathbb{R}^n} |f(x)|^2 dx \\
&\leq C_{0,n+1}(f) \int_{\mathbb{R}^n} (1+|x|)^{-2(n+1)} dx \\
&\leq C_{0,n+1}(f) \int_{\mathbb{R}^n} (1+|x|)^{-(n+1)} dx.
\end{aligned}
$$

The rest follows analogously to (i).

□

The dual of $\mathcal{S}$ will also be important for us.

A.1.3 Definition. The space of all linear and bounded functionals $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ on $\mathcal{S}$ is called the space of **tempered distributions**. We denote it with $\mathcal{S}'(\mathbb{R}^n) = (\mathcal{S}(\mathbb{R}^n))'$

A.1.4 Remark. *A linear mapping $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is bounded iff there exists a constant C and $j, k \in \mathbb{N}_0$ such that*

$$|\langle f, \varphi \rangle| \leq C |\varphi|_{j,k;\mathcal{S}}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here $\langle f, \varphi \rangle := f(\varphi)$ denotes the duality product.

A.1.5 Proposition. *We have that $L^1(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.*

Proof. Choose $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then.

$$
\begin{aligned}
|\langle f, \varphi \rangle| &\leq \int_{\mathbb{R}^n} |f(x)||\varphi(x)| dx \\
&\leq |\varphi|_{0,0;\mathcal{S}} \int_{\mathbb{R}^n} |f(x)| dx \\
&= \|f|L^1(\mathbb{R}^n)\| |\varphi|_{0,0;\mathcal{S}}.
\end{aligned}
$$

□

A.2. Fourier transformation

The Fourier transformation will be an important tool for us, therefore we will summarize some useful facts. Let us start with:

A.2.1 Definition. Let $f \in L^1(\mathbb{R}^n)$, we call

$$\mathcal{F}[f](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

the **Fourier transformation** of f and

$$\mathcal{F}^{-1}[g](x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)d\xi$$

is called the **inverse Fourier transformation** of g. It is easy to see that

$$\mathcal{F}^{-1}[g](x) = \mathcal{F}[g](-x).$$

Here $x \cdot \xi$ stands for the scalar product in $\mathbb{R}^n$ so $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$.

A.2.2 Proposition. *The mapping $\mathcal{F} : L^1(\mathbb{R}^n) \mapsto C_b^0(\mathbb{R}^n)$ is linear and*

$$\sup_{\xi\in\mathbb{R}^n} |\mathcal{F}[f](\xi)| \leq \|f|L^1(\mathbb{R}^n)\|.$$

C_b^0 is the set of bounded, continuous functions. The same is true for the inverse transformation.

Proof. See [1, Theorem 2.1.1]. □

A.2.3 Proposition. *Let $f \in \mathcal{S}$, then*

$$\mathcal{F}[\partial_{x_j} f] = i\xi_j \mathcal{F}[f]$$

and

$$\partial_{\xi_j} \mathcal{F}[f] = \mathcal{F}[(-ix_j)f].$$

Proof. See [4, Satz 43.1]. □

A.2.4 Remark. *We now consider a continuously differentiable function that has the property $f, \partial_{x_j} f \in L^1(\mathbb{R}^n)$ for $j = 1, \ldots, n$. Proposition A.2.2 tells us that $\mathcal{F}[f]$ and $\xi_j\mathcal{F}[f]$ for $j = 1, \ldots, n$ are bounded that means for every $j = 0, \ldots, n$ there exists a constant C_j such that $|\mathcal{F}[f]| < C_0$ and $|\xi_j\mathcal{F}[f]| < C_j$. It follows:*

$$\begin{aligned}(1 + |\xi|)|\mathcal{F}[f](\xi)| &= |\mathcal{F}[f](\xi)| + \left(\sum_{j=1}^n \xi_j^2 |\mathcal{F}[f](\xi)|^2\right)^{\frac{1}{2}} \\ &\leq C_0 + \left(\sum_{j=1}^n C_j^2\right)^{\frac{1}{2}} \leq C.\end{aligned}$$

This implies:

$$|\mathcal{F}[f](\xi)| \leq \frac{C}{1 + |\xi|}.$$

We can generalize this statement the following way:
Let $f \in C^k(\mathbb{R}^n)$ such that $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq k$, then:

$$|\mathcal{F}[f](\xi)| \leq \frac{C}{(1 + |\xi|)^k}.$$

We summarize:

Differentiability of f implies polynomial decay of $\mathcal{F}[f](\xi)$ as $|\xi| \longrightarrow \infty$.

This also works the other way round. If $(1+|x|)^k f(x) \in L^1(\mathbb{R}^n)$, then the second equation of a modified version of proposition A.2.3 - see [1, Theorem 2.1.3] *- implies that $\mathcal{F}[f] \in C^k(\mathbb{R}^n)$. We summarize that the following way:*

Quicker decay of $f(x)$ as $|x| \longrightarrow \infty$, means higher differentiability of $\mathcal{F}[f]$.

Compare [1, Remark 2.2.].

This remark shows us that the Fourier transformation operates in a natural way on the set of rapidly decreasing functions - note proposition A.1.2.

A.2.5 Proposition. *The mapping $\mathcal{F} : \mathcal{S} \mapsto \mathcal{S}$ is linear and sequential continuous.*

Proof. See [4, Satz 43.2]. □

A.2.6 Proposition. *The Fourier transformation $\mathcal{F} : \mathcal{S} \mapsto \mathcal{S}$ is an isomorphism, i.e. $\mathcal{F}^{-1}[\mathcal{F}[\varphi]] = \mathcal{F}[\mathcal{F}^{-1}[\varphi]] = \varphi$, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are sequential continuous.*

Proof. See [4, Satz 43.3 and 43.4]. □

A.2.7 Proposition. *For every $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \mathcal{F}[f](\xi)\overline{\mathcal{F}[g](\xi)}d\xi.$$

Proof. See [1, Theorem 2.11]. □

A.2.8 Proposition. *For every $f, g \in \mathcal{S}$ we have*

$$\langle \mathcal{F}[f], g\rangle = \langle f, \mathcal{F}^{-1}[g]\rangle \text{ and } \langle \mathcal{F}^{-1}[f], g\rangle = \langle f, \mathcal{F}[g]\rangle.$$

Proof. This follows from the formula in proposition A.2.7. For example if we insert $h(x) = \mathcal{F}^{-1}[g](x)$ into the right sight of the formula, while using proposition A.2.6, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{F}[f](\xi)\overline{\mathcal{F}[h](\xi)}d\xi &= \int_{\mathbb{R}^n} \mathcal{F}[f](\xi)\overline{g(\xi)}d\xi = \langle \mathcal{F}[f], g\rangle \\ = \int_{\mathbb{R}^n} f(x)\overline{h(x)}dx &= \int_{\mathbb{R}^n} f(x)\overline{\mathcal{F}^{-1}[g](x)}dx = \langle f, \mathcal{F}^{-1}[g]\rangle.\end{aligned}$$

□

A.2.9 Remark. *The last two propositions are also true for $f, g \in L_2(\mathbb{R}^n)$.*

Proof. See [21, Satz 1.24]. □

We can extend the definition of the Fourier transformation to the class of tempered distributions.

A.2.10 Definition. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the Fourier transformation $\mathcal{F}[f]$ and the inverse Fourier transformation as

$$\langle \mathcal{F}[f], \varphi\rangle := \langle f, \mathcal{F}[\varphi]\rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$
$$\langle \mathcal{F}^{-1}[f], \varphi\rangle := \langle f, \mathcal{F}^{-1}[\varphi]\rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

A.2.11 Proposition. *The Fourier transformation $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is a linear isomorphism with inverse $\mathcal{F}^{-1}$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are sequentially continuous.*

Proof. See [4, Satz 44.3]. □

A.2.12 Conclusion. *The Fourier transformation $\mathcal{F} : L_2 \to L_2$ is a linear isomorphism, i.e.*

$$\|f|L_2\| = \|\mathcal{F}[f]|L_2\|.$$

Proof. See [1, Theorem 2.11]. □

A.3. Some propositions about functions

A.3.1. Needed for proposition 2.2.4

In this section we derive sufficient criteria for Lipschitz continuity for functions we will be using. First will be the case of a function with bounded derivatives on a convex domain, next up will be the case of a radial symmetric function on $K_{1,\infty}$, lastly we take a look at a function that is a mixture of both.

A.3.1 Proposition. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $f : \Omega \to \mathbb{R}$ is differentiable in Ω, furthermore $\sup_{x\in\Omega} |\partial_{x_j} f(x)| = M_j < \infty$ for $1 \leq j \leq n$, then f satisfies a Lipschitz condition*

$$|f(x) - f(y)| \leq m|x - y|.$$

Here we have $m = |M|$ and $M = (M_1, \ldots, M_n)$.

Proof. The mean value theorem gives us the existence of $\theta \in (0, 1)$, such that

$$f(x) - f(y) = \nabla f(x + \theta(y - x)) \cdot (x - y).$$

We conclude

$$\begin{aligned} |f(x) - f(y)| &= |\nabla f(x + \theta(y - x)) \cdot (x - y)| \\ &\leq \sum_{j=1}^{n} |\partial_{x_j} f(x + \theta(y - x))||(x_j - y_j)| \\ &\leq \left(\sum_{j=1}^{n} |\partial_{x_j} f(x + \theta(y - x))|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |x_j - y_j|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^{n} M_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |x_j - y_j|^2\right)^{\frac{1}{2}} = m|x - y|. \end{aligned}$$

□

A.3.2 Proposition. *Let $g : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be a continuously differentiable function, $\sup\limits_{|x|\geq\frac{1}{2}} |\partial_j g(x)| = M_j < \infty$ for $1 \leq j \leq n$ and g is rotational symmetric, i.e. $|x| = |y| \Rightarrow g(x) = g(y)$. $g(x)$ is then Lipschitz continuous in $\mathbb{R}^n \backslash B_1(0)$.*

Proof. The basic idea is to reduce the problem to the situation of Proposition A.3.1.
Let us start with the case where x and y lie on one line, so $|x| \geq 1$ and $y = xt$, we can assume $t \geq 1$ or else we simply switch the roles of x and y.
We define $B = B_r(\frac{x+y}{2})$, where $r = \frac{|x-y|}{2} + \frac{1}{4}$. By construction, $x, y \in B$ and $B \cap B_{\frac{1}{2}}(0) = \emptyset$. B is a convex domain and the partial derivatives of g are bounded in B, by proposition A.3.1 we get the Lipschitz-continuity of g on B. Therefore, $|g(x) - g(y)| \leq m|x - y|$.

Let $x, y \in \mathbb{R}^n \backslash B_1(0)$ be arbitrary. Again we assume $|x| \leq |y|$. We define $z = \frac{x|y|}{|x|}$, and so $|z| = |y|$. Using the rotational symmetry of g gives us $g(z) = g(y)$. We now have $z = xt$ with $t \geq 1$, so we reduced it to the first case and get $|g(x) - g(z)| \leq m|x - z|$.
Beyond that:

$$\begin{aligned} |x - z| &= \left|x - \frac{x|y|}{|x|}\right| \\ &= \left|\frac{x|x| - x|y|}{|x|}\right| \end{aligned}$$

$$= \frac{|x|}{|x|}||x| - |y||$$
$$\leq |x - y|.$$

We can conclude:

$$|g(x) - g(y)| = |g(x) - g(z)| \leq m|x - z| \leq m|x - y|.$$

This shows the Lipschitz-continuity of g. □

A.3.3 Proposition. *Let $f : B_1(0) \subset \mathbb{R}^n \to \mathbb{R}$ be Lipschitz-continuous i.e. $|f(x)-f(y)| \leq M_1|x-y|$ for $|x|, |y| \leq 1$ and $g : \mathbb{R}^n \backslash B_1(0) \to \mathbb{R}$ also Lipschitz-continuous, so $|g(x)-g(y)| \leq M_2|x-y|$ for $|x|, |y| \geq 1$, where $f(x) = g(x)$ for all $|x| = 1$. Then we get that*

$$h(x) = \begin{cases} f(x) & \text{for } |x| \leq 1 \\ g(x) & \text{for } |x| > 1 \end{cases}$$

is Lipschitz-continuous on $\mathbb{R}^n$.

Proof. The Lipschitz-continuity of h, when $|x|, |y| \leq 1$ or $|x|, |y| \geq 1$, simply follows from the Lipschitz-continuity of f or g.
Now we have $|x| < 1$ and $|y| > 1$. We look at the line from x to y and label the point on that line with absolute vaule 1 z. For this point we have:

$$\begin{aligned} |h(x) - h(y)| &= |h(x) - h(z) + h(z) - h(y)| \\ &\leq |h(x) - h(z)| + |h(z) - h(y)| \\ &= |f(x) - f(z)| + |g(z) - g(y)| \\ &\leq M_1|x - z| + M_2|z - y| \\ &\leq max(M_1, M_2)[|x - z| + |z - y|] \\ &= M_3|x - y|. \end{aligned}$$

Here $|x - z| + |z - y| = |x - y|$ only is true because all those points are on a line. □

A.3.2. Needed for proposition 2.3.3

In this section, we will validate a construction Silvestre is using in the proof of proposition 2.3.3. We use the same notations as in proposition 2.2.1.
He uses that for an arbitrary $x_0 \in B_1(0)\backslash\{0\}$ one can shift Ψ in such a way that it touches Γ in x_0 and is bigger than Γ in every other point, see the picture below.

Silvestre is using that without any explanation on how to shift Ψ or why it is always bigger, but we will prove it in detail.
In the first proposition, we define the shift and prove that it touches Γ in x_0.

A.3.4 Proposition. *Let $x_0 \in B_1\backslash\{0\}$ and define*

$$F(x) = \psi(x - (1 - \lambda)x_0) + \delta$$

with $\delta = \Gamma(x_0) - \psi(\lambda x_0)$ and $\lambda = |x_0|^{-\frac{n-2s+2}{n-2s+1}} > 1$. $F(x)$ then touches Γ in x_0.

Proof. First we will verify that $F(x_0) = \Gamma(x_0)$.

$$\begin{aligned} F(x_0) &= \psi(x_0 - (1 - \lambda)x_0) + \delta \\ &= \psi(\lambda x_0) + \Gamma(x_0) - \psi(\lambda x_0) \qquad = \Gamma(x_0). \end{aligned}$$

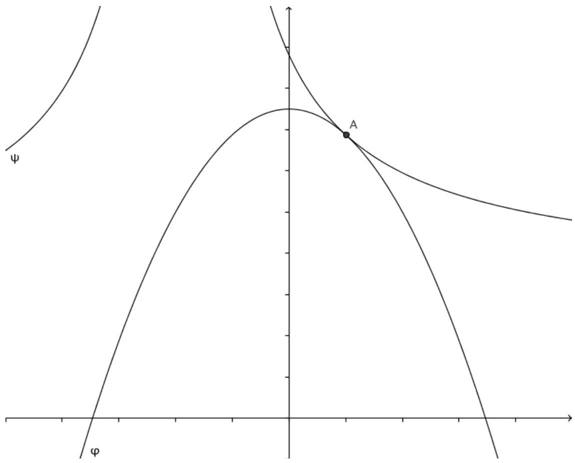

Figure A.1.: Ψ after the shift.

What remains is to show that $\partial_i F(x_0) = \partial_i \Gamma(x_0)$ for all $i = 1, \dots, n$. By definition of Γ this is equivalent to $\partial_i F(x_0) = \partial_i \varphi(x_0)$. We have

$$\partial_i F(x) = \frac{-c(n-2s)(x_i - (1-\lambda)x_0^i)}{|x-(1-\lambda)x_0|^{n-2s+2}}.$$

So we get

$$\begin{aligned}
\partial_i F(x_0) &= \frac{-c(n-2s)(\lambda x_0^i)}{|\lambda x_0|^{n-2s+2}} \\
&= \frac{-c(n-2s)x_0^i}{|x_0|^{n-2s+2}} \frac{\lambda}{\lambda^{n-2s+2}} \\
&= \frac{-c(n-2s)x_0^i}{|x_0|^{n-2s+2}} \frac{1}{\lambda^{n-2s+1}} \\
&= \frac{-c(n-2s)x_0^i}{|x_0|^{n-2s+2}} \left(|x_0|^{\frac{n-2s+2}{n-2s+1}}\right)^{n-2s+1} \\
&= -c(n-2s)x_0^i = \partial_i \varphi(x_0).
\end{aligned}$$

And this is true for every $i = 1, \dots, n$. □

Next we want to show that $F(y) > \Gamma(y)$ for every $y \in \mathbb{R}^n - \{x_0\}$. To do what, we first reduce it to a one dimensional problem by showing that $F(tx_0) > \Gamma(tx_0)$ for $t > 0$.
We will break that down into showing that

(i) $F(tx_0) > \varphi(tx_0)$ for $t \leq \frac{1}{|x_0|}$.

(ii) $F(tx_0) > \psi(tx_0)$ for $t \geq \frac{1}{|x_0|}$.

To prove that, we first need the following general lemma.

A.3.5 Lemma. *Let f and g be two continuously differentiable functions such that $f(a) \leq g(a)$ for $a \in \mathbb{R}_{>0}$.*

(i) If $f'(t) > g'(t)$ for all $0 < t < a$, then $g(t) > f(t)$ for all $0 < t < a$.

(ii) If $f'(t) < g'(t)$ for all $t > a$, then $g(t) > f(t)$ for all $t > a$.

Proof. The proof uses the mean value theorem. We will apply it to the difference $h(t) = g(t) - f(t)$. Then we have $h(a) \geq 0$.

(i) Let $0 < t < a$ then there exists a $\theta \in (0,1)$ such that

$$\begin{aligned} h(a) - h(t) &= h'((1-\theta)a + t\theta)(a-t) && \iff \\ -h(t) &= -h(a) + (g' - f')((1-\theta)a + t\theta)(a-t) && \iff \\ h(t) &= h(a) + (f' - g')((1-\theta)a + t\theta)(a-t) > h(a). \end{aligned}$$

Because $(1-\theta)a + t\theta \in (t,a)$ and by assumption $f'(t) > g'(t)$ for $t \in (0,a)$.

(ii) Let $t > a$ then there exists a $\theta \in (0,1)$ such that

$$\begin{aligned} h(t) - h(a) &= h'((1-\theta)a + t\theta)(t-a) && \iff \\ h(t) &= h(a) + (g' - f')((1-\theta)a + t\theta)(t-a) && \iff \\ h(t) &= h(a) + (g' - f')((1-\theta)a + t\theta)(t-a) > h(a). \end{aligned}$$

Because $(1-\theta)a + t\theta > a$ and by assumption $g'(t) > f'(t)$ for $t > a$.

$\square$

Now we will begin with (i).

A.3.6 Lemma. *We define $g(t) = F(tx_0) = \psi(tx_0 - (1-\lambda)x_0) = \psi((t-1+\lambda)x_0)$ and $f(t) = \varphi(tx_0)$. Then $g'(t) < f'(t)$ for $0 < t < 1$ and $g'(t) > f'(t)$ for $t > 1$.*

Proof. We have:

$$\begin{aligned} f'(t) &= \sum_{j=1}^{n} \partial_{x_j}\varphi(tx_0)\frac{d}{dt}(tx_0^j) \\ &= \sum_{j=1}^{n}(-c(n-2s)tx_0^j)x_0^j \\ &= (-c(n-2s))t|x_0|^2 \end{aligned}$$

and

$$\begin{aligned} g'(t) &= \sum_{j=1}^{n} \partial_{x_j}\psi((t-1+\lambda)x_0)\frac{d}{dt}((t-1+\lambda)x_0^j) \\ &= \sum_{j=1}^{n} -\frac{c(n-2s)((t-1+\lambda)x_0^j)}{|(t-1+\lambda)x_0|^{n-2s+2}}x_0^j \\ &= -\frac{c(n-2s)(t-1+\lambda)}{|(t-1+\lambda)|^{n-2s+2}|x_0|^{n-2s+2}}\sum_{j=1}^{n}(x_0^j)^2 \\ &= -\frac{c(n-2s)}{|(t-1+\lambda)|^{n-2s+1}|x_0|^{n-2s+2}}|x_0|^2 && (*) \\ &= -\frac{c(n-2s)}{|(t-1+\lambda)|x_0|^{\frac{n-2s+2}{n-2s+1}}|^{n-2s+1}}|x_0|^2 \\ &= -\frac{c(n-2s)}{|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}}+1|^{n-2s+1}}|x_0|^2. \end{aligned}$$

At (*) we used that since $\lambda > 1$ we have $(t - 1 + \lambda) > 0$ even when t is close to zero.

Let us start with the case $0 < t < 1$

$$\begin{aligned} g'(t) &< f'(t) && \iff \\ -\frac{c(n-2s)}{|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{n-2s+1}}|x_0|^2 &< (-c(n-2s))t|x_0|^2 && \iff \\ \frac{1}{|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{n-2s+1}} &> t. \end{aligned}$$

This is true because $-1 < (t-1) < 0$ and with this $-1 < (t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} < 0$ since $|x_0| < 1$. Adding 1 gives us $0 < (t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1 < 1$, so we can also take the absolute value $0 < |(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1| < 1$. Taking the inverse finally gives us

$$|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{-1} > 1$$

and so

$$|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{-\frac{n-2s+2}{n-2s+1}} > 1 > t.$$

Now let us take $t > 1$. Then we have to show

$$\begin{aligned} g'(t) &> f'(t) && \iff \\ \frac{1}{|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{n-2s+1}} &< t. \end{aligned}$$

Now $(t-1)$ is positive and thereby $|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1| > 1$. Again taking the inverse leads to $|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{-1} < 1$ and finally

$$|(t-1)|x_0|^{\frac{n-2s+2}{n-2s+1}} + 1|^{-\frac{n-2s+2}{n-2s+1}} < 1 < t.$$

□

A.3.7 Conclusion. *$F(tx_0) \geq \varphi(tx_0)$ for $t \leq \frac{1}{|x_0|}$ and especially $F(tx_0) > \varphi(tx_0)$ for $t \neq 1$.*

Proof. In A.3.4 we showed that $F(x_0) = \varphi(x_0)$. When we denote $g(t) = F(tx_0)$ and $f(t) = \varphi(tx_0)$ this means that $f(1) = g(1)$. In lemma A.3.6 we showed that $g'(t) < f'(t)$ for $0 < t < 1$ and $g'(t) > f'(t)$ for $t > 1$. Finally applying lemma A.3.5 gives us that $g(t) > f(t)$ for all $t > 0$ with $t \neq 1$ and this is what we wanted to show. □

Now we prove (ii). We take a similar approach.

A.3.8 Lemma. *We define $g(t) = F(tx_0) = \psi(tx_0 - (1-\lambda)x_0) = \psi((t-1+\lambda)x_0)$ and $k(t) = \psi(tx_0)$. Then $g'(t) > k'(t)$ for $t \geq \frac{1}{|x_0|}$.*

Proof. We have:

$$\begin{aligned} k'(t) &= \sum_{j=1}^{n} \partial_{x_j}\psi(tx_0)\frac{d}{dt}(tx_0^j) \\ &= \sum_{j=1}^{n} -\frac{c(n-2s)tx_0^j}{|tx_0|^{n-2s+2}}x_0^j \\ &= -\frac{c(n-2s)t}{|t|^{n-2s+2}|x_0|^{n-2s+2}}|x_0|^2. \end{aligned}$$

In lemma A.3.6 we have seen that

$$g'(t) = -\frac{c(n-2s)(t-1+\lambda)}{|(t-1+\lambda)|^{n-2s+2}|x_0|^{n-2s+2}}|x_0|^2$$

$$\begin{aligned}
g'(t) &> k'(t) && \iff \\
-\frac{c(n-2s)(t-1+\lambda)}{|(t-1+\lambda)|^{n-2s+2}|x_0|^{n-2s+2}}|x_0|^2 &> -\frac{c(n-2s)t}{|t|^{n-2s+2}|x_0|^{n-2s+2}}|x_0|^2 && \iff \\
\frac{(t-1+\lambda)}{|(t-1+\lambda)|^{n-2s+2}} &< \frac{t}{|t|^{n-2s+2}} && \iff \\
\frac{1}{(t-1+\lambda)^{n-2s+1}} &< \frac{1}{t^{n-2s+1}} && \iff \\
t &< (t-1+\lambda) && \iff \\
1 &< \lambda.
\end{aligned}$$

The last inequality is true by definition of lambda. We should note that t and $(t-1+\lambda)$ are bigger than zero. □

A.3.9 Conclusion. *$F(tx_0) > \psi(tx_0)$ for $t \geq \frac{1}{|x_0|}$.*

Proof. We saw in conclusion A.3.7 that $F\left(\frac{x_0}{|x_0|}\right) > \varphi\left(\frac{x_0}{|x_0|}\right)$. Since $\left|\frac{x_0}{|x_0|}\right| = 1$ we also get that $\varphi\left(\frac{x_0}{|x_0|}\right) = \psi\left(\frac{x_0}{|x_0|}\right)$ because both coincide on the unit sphere. We conclude:

$$F\left(\frac{x_0}{|x_0|}\right) > \psi\left(\frac{x_0}{|x_0|}\right).$$

Additionally lemma A.3.8 tells us that $g'(t) > k'(t)$ for $t \geq \frac{1}{|x_0|}$ with the notation $g(t) = F(tx_0)$ and $k(t) = \psi(tx_0)$.
We use (ii) from lemma A.3.5 to get that $F(tx_0) > \psi(tx_0)$ for $t \geq \frac{1}{|x_0|}$. □

To summarize, we now know that $F(tx_0) > \Gamma(tx_0)$ for $t > 0$ and $t \neq 1$. The last step is to conclude that the inequality is true for every $y \in \mathbb{R}^n$. The idea is to use the radial symmetry to extend the inequality $F \geq \Gamma$ from the line $\{tx_0 : t > 0\}$ onto the whole $\mathbb{R}^n$. We need the next lemma to show that.

A.3.10 Lemma. *Let f and g be 2 C^1 functions such that for $x_0 \in B_1(0)\backslash\{0\}$:*

(i) *$g(tx_0) \geq f(tx_0)$ for $t > 0$.*

(ii) *f is rotational symmetric to the origin, i.e. $|x| = |y| \Rightarrow f(x) = f(y)$.*

(iii) *g is rotational symmetric to the point $-\beta x_0$ with $\beta > 0$, i.e. $|x + \beta x_0| = |y + \beta x_0| \Rightarrow g(x) = g(y)$ and monotonically decreasing, i.e. $|x + \beta x_0| > |y + \beta x_0| \Rightarrow g(x) < g(y)$.*

Then we have $g(y) > f(y)$ for every $y \in \mathbb{R}^n\backslash\{tx_0 : t > 0\}$.

Proof. We choose $y \in \mathbb{R}^n\backslash\{tx_0 : t > 0\}$.
We define $z = \frac{x_0|y|}{|x_0|}$, so $|z| = |y|$ and $z \in \{tx_0 : t > 0\}$. This gives us

$$f(y) = f(z) \leq g(z) \tag{1}$$

by (ii) and (i). Then we denote w the point along the line $\{yt : t \geq 0\}$ with

$$|w + \beta x_0| = |z + \beta x_0|$$

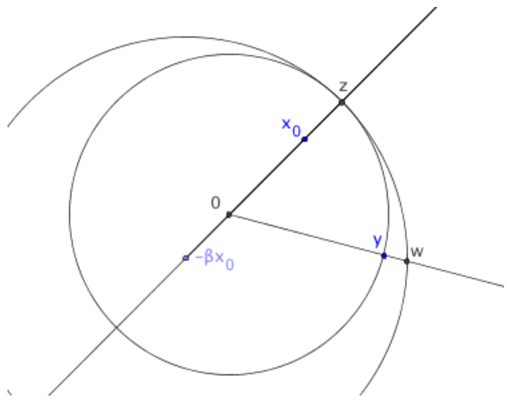

Figure A.2.: Relation of the points.

(iii) then gives us that

$$g(w) = g(z). \tag{2}$$

Because $|z + \beta x_0| = |z| + \beta|x_0| = |y| + \beta|x_0|$, remember that z and x_0 are on one line, we have

$$|y| + \beta|x_0| = |w + \beta x_0| \leq |w| + \beta|x_0|$$

and therefore

$$|y| \leq |w|. \tag{3}$$

This means that $w = yt$ with $t \geq 1$. The last step will be to show that

$$\begin{aligned} &|y + \beta x_0|^2 \leq |w + \beta x_0|^2 \\ \iff &|y|^2 + 2\beta y \cdot x_0 + \beta^2|x_0|^2 \leq |w|^2 + 2\beta w \cdot x_0 + \beta^2|x_0|^2 \\ \iff &\qquad |y|^2 + 2\beta y \cdot x_0 \leq |w|^2 + 2\beta w \cdot x_0. \end{aligned}$$

The last inequality is true because of (3) and the fact that

$$2\beta y \cdot x_0 = 2\beta\frac{1}{t}(ty) \cdot x_0 = 2\frac{\beta}{t}w \cdot x_0 \leq 2\beta w \cdot x_0$$

since $t \geq 1$. Lastly, using (iii), this yields

$$g(w) < g(y). \tag{4}$$

In the end, using (1), (2) and (4) we get

$$f(y) = f(z) \leq g(z) = g(w) < g(y).$$

This is exactly what we wanted to show. □

The last lemma was specifically tailored to our needs, so we can immediatey draw the following

A.3.11 Conclusion. *$F(y) > \Gamma(y)$ for every $y \in \mathbb{R}^n \backslash \{x_0\}$.*

Proof. We start off by noting that $(1 - \lambda)x_0$ is the singularity of F and, since Γ is always bounded, the inequality will be true there.
Because of proposition A.3.4 we have $F(x_0) = \Gamma(x_0)$ and because of conclusion A.3.9 and A.3.7, we have $F(tx_0) > \Gamma(tx_0)$ for $t > 0$ and $t \neq 1$.
Γ is rotational symmetric to the origin, F is rotational symmetric to the point $(1 - \lambda)x_0 = -(\lambda - 1)x_0$ and decreasing in the sense of lemma A.3.10. Therefore, we can apply lemma A.3.10 and get that $F(y) > \Gamma(y)$ for every $y \in \mathbb{R}^n \backslash \{tx_0 : t > 0\}$. □

A.4. Pseudodifferential Operators

The goal in this section is to find a way to avoid the problems with the singularity of the multiplier. We will take a smooth cut off function μ with $\Theta\mu \subset B_2(0)$ such that $\mu(x) = 1$ for $|x| \leq 1$, then we consider the multiplier $(1 - \mu(\xi))|\xi|^{2s}$. It will be in the symbol class $S^{2s}_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$, see below for the definiton, and therefore we will be able to use the theory of pseudodifferential operators.

A.4.1 Definition. Let $m \in \mathbb{R}, n, N \in \mathbb{N}$. Then $S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$ is the vector-space of all smooth functions $p : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$

holds for all $\alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^N$. The functions p are called pseudodifferential symbols. This is the general definition, we are only interested in the case $n = N$. See [1, Definition 3.1].

A.4.2 Definition. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then for $f \in \mathcal{S}$

$$p(x, D_x)f(x) = \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\mathcal{F}[f](\xi)d\xi$$

defines the associated pseudodifferential operator to the symbol p. See [1, Remark 3.2].

In order to show that $(1 - \mu(\xi))|\xi|^{2s} \in S^m_{1,0}$, we need a way to estimate $|\partial_\xi^\alpha |\xi|^{2s}|$. The next definition will prove useful for that purpose.

A.4.3 Definition. A function $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ is called homogeneous of degree $d \in \mathbb{R}$ if $f(rx) = r^d f(x)$ for all $r > 0$ and $x \neq 0$.

A.4.4 Proposition. *Let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ be a continuous function that is homogeneous of degree $d \in \mathbb{R}$. Then*

$$|f(x)| \leq \left(\sup_{|y|=1} |f(y)| \right) |x|^d$$

for all $x \in \mathbb{R}^n$. Note that the supremum over the unit sphere is attained, since f is continuous.

Proof. Because f is homogeneous we have

$$\frac{f(x)}{|x|^d} = f\left(\frac{x}{|x|}\right)$$

and thus

$$\left|\frac{f(x)}{|x|^d}\right| = \left| f\left(\frac{x}{|x|}\right)\right| \leq \sup_{|y|=1} |f(y)|.$$

This concludes the proof. ∎

A.4.5 Proposition. *Let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ be k-times continuously differentiable and homogeneous of degree $d \in \mathbb{R}$. Then there exists a constant $C_\alpha > 0$ such that*

$$|\partial^\alpha f(x)| \leq C_\alpha |x|^{d-|\alpha|} \text{ for all } x \in \mathbb{R}^n \backslash \{0\},$$

for all $|\alpha| \leq k$.

Proof. We have

$$\begin{aligned} r^d \partial_j f(x) &= \partial_j [f(rx)] \\ &= r [\partial_j f](rx). \end{aligned}$$

In total

$$[\partial_j f](rx) = r^{d-1}\partial_j f(x).$$

This means that $\partial_j f$ is homogeneous of degree $d-1$ and thus by proposition A.4.4

$$|\partial_j f(x)| \leq \left(\sup_{|y|=1} |\partial_j f(y)|\right) |x|^{d-1}.$$

The rest follows by mathematical induction. □

A.4.6 Conclusion. *We have*

$$\left|\partial^\alpha |\xi|^{2s}\right| \leq C_\alpha |\xi|^{2s-|\alpha|}.$$

For a positive constant C_α.

Proof. Since

$$|r\xi|^{2s} = r^{2s}|\xi|^{2s},$$

$|\xi|^{2s}$ is homogeneous of degree $2s$ and we can apply proposition A.4.5. □

A.4.7 Proposition. *Let $\mu \in C_c^\infty(B_2(0))$ such that $0 \leq \mu(\xi) \leq 1$ and especially $\mu(\xi) = 1$ if $|\xi| \leq 1$. We define $g(\xi) = (1-\mu(\xi))|\xi|^{2s}$. Then $g \in S^{2s}_{1,0}$.*

Proof. First off, we note that g is independent of x and therefore we only have to show that

$$|\partial^\alpha g(\xi)| \leq D_\alpha (1+|\xi|)^{2s-|\alpha|}.$$

Where D_α is a constant, that is independent of ξ. We will apply the product rule of Leibniz and get

$$\partial^\alpha \left[(1-\mu(\xi))|\xi|^{2s}\right] = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(1-\mu(\xi))\partial^\beta |\xi|^{2s}.$$

1st case: $|\xi| \leq 1$. Here $1-\mu$ is the constant zero function, so itself and all partial derivatives are zero. This means

$$\begin{aligned}\left|\partial^\alpha \left[(1-\mu(\xi))|\xi|^{2s}\right]\right| &= \left|\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(1-\mu(\xi))\partial^\beta |\xi|^{2s}\right| \\ &= 0 \\ &\leq C_\alpha (1+|\xi|)^{2s-|\alpha|}.\end{aligned}$$

2nd case: $|\xi| > 2$. Here μ is constant zero, so all partial derivatives are also zero, i.e. $\partial^\alpha(1-\mu(\xi)) = 0$ if $\alpha \neq 0$, but now $1-\mu = 1$, therefore

$$\begin{aligned}\left|\partial^\alpha \left[(1-\mu(\xi))|\xi|^{2s}\right]\right| &= \left|\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(1-\mu(\xi))\partial^\beta |\xi|^{2s}\right| \\ &= \left|\partial^\alpha \left[|\xi|^{2s}\right]\right| \\ &\leq C_\alpha (1+|\xi|)^{2s-|\alpha|}.\end{aligned}$$

Here we applied conclusion A.4.6.

3rd case: $1 < |\xi| \leq 2$. We will use the fact that all appearing partial derivatives of μ and $|\xi|^{2s}$ are continuous on the closure of the annulus $K_{1,2}$ and thus are also bounded on $K_{1,2}$. We will denote $\sup_{\xi\in K_{1,2}} \left|\partial^{\alpha-\beta}(1-\mu(\xi))\right| = \mu_{\alpha,\beta} < \infty$ and $\zeta_\beta = \sup_{\xi\in K_{1,2}} |\xi|^{2s-|\beta|} < 4$.

$$\begin{aligned}\left|\partial^\alpha \left[(1-\mu(\xi))|\xi|^{2s}\right]\right| &= \left|\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}(1-\mu(\xi))\partial^\beta|\xi|^{2s}\right| \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\mu_{\alpha,\beta}\left|\partial^\beta|\xi|^{2s}\right| \\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\mu_{\alpha,\beta}C_\beta|\xi|^{2s-|\beta|} \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\mu_{\alpha,\beta}C_\beta\zeta_\beta \\ &\leq 3^{|\alpha|-2s}\left(\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\mu_{\alpha,\beta}C_\beta\zeta_\beta\right)(1+|\xi|)^{2s-|\alpha|}.\end{aligned}$$

The last inequality is true because

$$\min_{\zeta\in K_{1,2}}\frac{1}{(1+|\zeta|)^{|\alpha|-2s}} = \frac{1}{3^{|\alpha|-2s}} < 1$$

in the case where $|\alpha| > 2s$ and

$$\min_{\zeta\in K_{1,2}}(1+|\zeta|)^{2s-|\alpha|} = 2^{2s-|\alpha|} > 1$$

in the case where $|\alpha| \leq 2s$.

Now we define

$$D_\alpha = \max\left(C_\alpha, 3^{|\alpha|-2s}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\mu_{\alpha,\beta}C_\beta\zeta_\beta\right).$$

Then

$$|\partial^\alpha g(\xi)| \leq D_\alpha(1+|\xi|)^{2s-|\alpha|}.$$

Thus $g \in S^{2s}_{1,0}$. □

A.4.8 Lemma. *Let μ be like above and $f \in L_1$, then*

$$H(x) = \int_{\mathbb{R}^n} e^{ix\xi}|\xi|^{2s}\mu(\xi)\mathcal{F}[f](\xi)d\xi$$

is a function in C^∞.

Proof. The point here is that we only have to integrate over $B_2(0)$, because μ is zero outside.
First we define

$$h(\xi,x) = e^{ix\xi}|\xi|^{2s}\mu(\xi)\mathcal{F}[f](\xi)\chi_{B_2(0)}.$$

Where the characteristic funciton $\chi_{B_2(0)}$ is a function in ξ. Now h is differentiable in x infinitely often, because only $e^{ix\xi}$ depends on x.
Next we choose an arbitrary $\alpha \in \mathbb{N}_0^n$. Then

$$\begin{aligned}|\partial_x^\alpha h(\xi,x)| &= |\xi^\alpha||\xi|^{2s}|\mu(\xi)||\mathcal{F}[f](\xi)|\chi_{B_2(0)} \\ &\leq \|f|L_1\|(f)|\xi|^{2s+|\alpha|}\chi_{B_2(0)}.\end{aligned}$$

Where we used proposition A.2.2. The last expression is an integrable function for every α because we are only integrating over $B_2(0)$.
With that we can apply Lebesgue's Theorem about the differentiability of parameter integrals, see [1, Theorem A.4], to every partial derivative of H and get therefore that $H \in C^\infty$. □

The next proposition is a standard result about the mapping properties of pseudodifferential operators on the set of rapidly decreasing functions.

A.4.9 Proposition. *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n), m \in \mathbb{R}$ a pseudodifferential symbol. Then*

$$p(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a bounded and linear operator.

Proof. See [1, Theorem 3.6]. □

A.4.10 Conclusion. *Let $f \in \mathcal{S}$ and $s \in (0, 1)$ then there exist functions $g \in \mathcal{S}$ and $h \in C^\infty$ such that*

$$(-\Delta)^s f = g + h.$$

This especially means that $(-\Delta)^s f \in C^\infty$.

Proof.

$$\begin{aligned}
(-\Delta)^s f &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} \mathcal{F}[f](\xi) d\xi \\
&= (2\pi)^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} \mu(\xi) \mathcal{F}[f](\xi) d\xi + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2s} (1 - \mu(\xi)) \mathcal{F}[f](\xi) d\xi \right) \\
&= (2\pi)^{-\frac{n}{2}} H(x) + (2\pi)^{-\frac{n}{2}} g(D_x) f(x).
\end{aligned}$$

Here $H(x)$ is the C^∞ function defined in A.4.8 and $g(D_x)$ is the associated pseudodifferential operator to the symbol g given in A.4.7 and by Proposition A.4.9 $g(D_x)f(x) \in \mathcal{S}$. This concludes the proof. □

A.4.11 Remark. *We showed that our operator $(-\Delta)^s$ can be written as the sum of a pseudodifferential operator $g(D_x)$ and a smoothing operator.*

List of Symbols

$\mathbb{N}_0$	The set of natural numbers including zero.
$\mathbb{N}_0^n$	$= \{(a_1, \ldots, a_n) \mid a_j \in \mathbb{N}_0\}$
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
Θf	The support of f, i.e. $\overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$
$B_r(x_0)$	Ball of radius r around the point x_0.
$K_{r,R}$	$= \{x \in \mathbb{R}^n : r < \lvert x\rvert < R\}$
$K_{r,\infty}$	$= \mathbb{R}^n - B_r(0)$
$A \Subset B$	$A \subset B$ such that $dist(a, \partial B) > 0$ for every $a \in A$.
$\Xi(n)$	Surface area of the n-dimensional unit sphere.
x^α	$= x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$
$\lvert\beta\rvert$	$= \sum_{j=1}^n \beta_j$ for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$
δ_{ij}	This is the Kronecker symbol.
$\beta \leq \alpha$	for $\alpha, \beta \in \mathbb{N}_0^n$ means $\beta_j \leq \alpha_j$ for $j = 1, \ldots, n$.
$\binom{\alpha}{\beta}$	$= \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_n}{\beta_n}$ for $\alpha, \beta \in \mathbb{N}_0^n$.
$\Psi(x)$	The fundamental solution of $(-\Delta)^s$. See 2.1.13
$\Gamma(x)$	See 2.2.5
$\Gamma_\lambda(x)$	$= \frac{\Gamma(\frac{x}{\lambda})}{\lambda^{n-2s}}$
χ_A	The characteristic function of the set A.
$\mathcal{S}$	The set of rapidly decreasing functions.
$\mathcal{S}'$	The set of tempered distributions.
$L_1(\mathbb{R}^n)$	Set of all functions f such that $\int_{\mathbb{R}^n} \lvert f(x)\rvert dx < \infty$
$L_{1,s}(\mathbb{R}^n)$	Set of all functions f such that $\int_{\mathbb{R}^n} \frac{\lvert f(x)\rvert}{1+\lvert x\rvert^{n+2s}} < \infty$
$L_p(\mathbb{R}^n)$	Set of all functions f such that $\int_{\mathbb{R}^n} \lvert f(x)\rvert^p dx < \infty$
$L_\infty(\mathbb{R}^n)$	Set of all functions f such that $\sup_{x\in\mathbb{R}^n} f(x) < \infty$.
$C^d(\Omega)$	All functions that are d times continuously differentiable on Ω.
$C_b^0(\Omega)$	The set of all bounded and continuous functions on Ω.
C^α	$= \{u \in C^0 : \sup_{x\neq y} \frac{\lvert u(x)-u(y)\rvert}{\lvert x-y\rvert^\alpha} < \infty\}$ for $0 < \alpha < 1$
$C^{d,\alpha}$	$= \{u \in C^d : \partial^\beta u \in C^\alpha$ for all $\beta \in \mathbb{N}_0^n$ with $\lvert\beta\rvert \leq d\}$
$C_c^\infty(\Omega)$	Set of all functions that are infinitely often differentiable and have a compact support that is contained in Ω.
$S_{1,0}^m(\mathbb{R}^N \times \mathbb{R}^n)$	See definition A.4.1.
$(-\Delta)^s f(x)$	The fractional Laplacian, see 2.1.1.
$\mathcal{F}[f](\xi)$	Fourier Transformation of f in ξ
$\partial_j f$	Partial derivative of f in direction of the j-th unit vector.
$\partial^\alpha f$	$= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$
$\lVert f \mid L_1(\mathbb{R}^n)\rVert$	$= \int_{\mathbb{R}^n} \lvert f(x)\rvert dx$
$\lVert u \mid L_{1,s}\rVert$	$= \int_{\mathbb{R}^n} \frac{\lvert u(x)\rvert}{1+\lvert x\rvert^{n+2s}} dx$
$\langle f, g\rangle$	$= \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$, the L_2 scalar product.
$(f * g)(x)$	$= \int_{\mathbb{R}^n} f(y)g(x-y)dy$.

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Erklärung der Urheberschaft

Ich erkläre hiermit an Eides statt, dass ich die vorliegende Arbeit ohne Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher in gleicher oder ähnlicher Form in keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

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