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Leipzig, den 22. Januar 2017

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(Odette Sylvia Yaptieu Djeungue)

“*A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.*”

Tolstoy

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Dedication

To

my parents Djeungue Hubert & Jacqueline,

my older brother Pefeli Djeungue Serge who trained me in Mathematics,

my friend Dr. Abdelrhman Elkasapy who passed away on 13.01.2017 ($R.I.P$).

Abstract

The aim of this work is to generalize Forman's discrete Morse theory, on one end to discrete Morse-Bott theory, motivated by Morse-Bott theory in the smooth setting, see [9], [8]. On the other, motivated by J-N. Corvellec's Morse theory for continuous functionals, see [13], we generalize Forman's discrete Morse-floer theory by considering a vector field more general than the one extracted from a discrete Morse function.

A discrete Morse function, see [20], is defined on a CW complex such that, it locally increases in dimension except possibly in one direction. The extracted vector field from this discrete Morse function has the properties that each cell has either one incoming or outgoing arrow but not both and there are no closed orbits. A boundary operator is constructed from this vector field configuration, see [21], by considering the chain complex generated only by the critical cells, these are the cells without arrows. This yields the Betti numbers of the cell complex. One obtains the Morse inequalities which are inequalities between the numbers of critical cells of fixed indices and the Betti numbers.

Our first generalization of the above theory is our discrete analogue of Morse-Bott theory, on CW complexes. Where, a function assumes the same value on systematically defined collections of cells. This function will then admit critical collection of cells instead of just critical cells. We define a discrete Morse-Bott function by requiring some specific conditions on the collections of cells. The reduced collections represent our critical collections. We obtain some discrete Morse-Bott inequalities, that is, the Poincaré polynomial of the CW complex is expressed in terms of those of the reduced collections, excluding the noncritical pairs. The vector field originating from this discrete Morse-Bott function is such that, inside each collection a cell can have as many incoming and/or outgoing arrows, but between the collections there are no closed orbits. We also do some Conley theory analysis, see [12], [32], [23], by using the reduced collections, also excluding the noncritical pairs, as the isolated invariant sets. We systematically define their

respective isolating neighborhoods and exit sets, and these two constitute the index pairs. The Poincaré polynomial of the CW complex is then expressed in terms of those of the index pairs.

Next, we generalize Forman's discrete Morse-Floer theory. We take a finite CW complex, in which every cell is given an orientation. This CW complex has a vector field configuration which is such that a cell can have as many outgoing or incoming arrows as possible but there are no closed orbits. This in particular tells us that this vector field originates from some function. Using a systematic definition for our critical cells, we then define a boundary operator, using all the given arrows, by means of a probabilistic and averaging technique. From this boundary operator, the Betti numbers of the CW complex are extracted, and we also get an analogue of the Morse inequalities for the CW complex under consideration. After appropriately defining the isolated invariant sets using this arrow configuration, we also obtain the result in Conley theory analysis.

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Introduction

Morse theory is a very important tool for the study of the topology of differentiable manifolds. It was first introduced by Morse in 1925, in [40]. It recovers the homology groups of the manifold from the critical points of a Morse function and the relations between them. The Morse inequalities are inequalities between the Betti numbers (these are the dimensions of the homology groups) of the manifold and the numbers of critical points of fixed indices of the function. To get the homology groups, one attaches a k-dimensional cell for each critical point of index k, and gluing relations between those cells then yield the homological boundary operator. Floer, in [19], discovered a more direct way to achieve this. He directly constructed the boundary operator from the critical points by counting the gradient lines between critical points with index difference one. Floer's direct construction of the boundary operator in terms of critical points and gradient lines, without having to invoke the local geometry of the manifold in question, made spectacular applications to symplectic geometry possible. In fact, Floer's theory needs only index differences, but no absolute indices, and it therefore also applies in certain infinite dimensional situations, with functionals like the Dirac functional where each critical point would have an infinite index. Floer homology was fully developed in [44]. For a presentation in the context of Riemannian geometry, see also [31].

Morse-Bott theory, introduced by Bott in [8], being a generalization of Morse theory, was developed to treat the cases where instead of having critical points, one has critical submanifolds. To each critical submanifold, one associates a certain index that is determined by looking at the Hessian restricted to the normal part of the submanifold, since on the tangential part it vanishes. The Poincaré polynomial (and hence the Euler number) of the manifold is then obtained from the Morse-Bott inequalities in the sense that it is expressed in terms of the Poincaré polynomials of the critical submanifolds, taking their respective indices into account.

Conley theory on the other hand, introduced by Conley in [12], and being more dynamics related, focuses on the study of the topological invariants of a given manifold. Using the (negative) gradient flow lines generated by some (Morse) function, one obtains for each isolated invariant set its isolating neighborhood and exit set for the flow, and these two constitute an index pair. One obtains the Poincaré polynomial of the manifold by summing up those of the index pairs up to some correction term. In particular, the Euler number of the manifold is then obtained by summing for all isolating invariant sets the alternating sums of the dimensions of the homology groups of the index pairs.

1.1 Discrete Morse theory

In a rather different direction, Forman in 1998, in [20], developed a discrete version of Morse theory for CW complexes. A CW complex, introduced by J. C. Whitehead in [48], is a decomposition of a space into cells each of which is homeomorphic to an open disc. The dimension of the disc specifies the dimension of the cell. The CW construction uses a specific gluing procedure via the characteristic maps. This space is endowed with the weak topology and satisfies the condition that the closure of each cell intersects only a finite number of cells. We write $\sigma^{(k)}$ to emphasize that σ is a cell of dimension k. The topological boundary elements of a cell are called its faces. If a cell $\sigma^{(k)}$ is a face of another cell τ, we write $\sigma < \tau$ to indicate that $\dim\sigma = \dim\tau - 1$, in which case σ is called a facet of τ. We say $\sigma^{(k)} < \tau$ is a regular facet of τ if, for φ_τ, the characteristic map of τ, we have: the map $\varphi_\tau \colon \varphi_\tau^{-1}(\sigma^{(k)}) \to \sigma^{(k)}$ is a homeomorphism and $\overline{\varphi_\tau^{-1}(\sigma^{(k)})}$ is a closed k-ball.

We denote the cardinality of a set A by $\sharp A$.

A discrete Morse function, according to Forman, is a real-valued function defined on the set of cells such that it locally increases in dimension, except possibly in one direction. More formally, Forman's definition of a discrete Morse function f on a CW complex requires that for all cells $\sigma^{(k)}$,

$$\begin{cases} \text{for all } \tau \text{ s.t. } \sigma \text{ is an irregular face of } \tau, \quad f(\sigma) < f(\tau); \\ Un(\sigma) := \sharp\{\tau^{(k+1)} \,|\, \sigma \text{ is a regular facet of } \tau \text{ and } f(\tau) \leq f(\sigma)\} \leq 1; \end{cases}$$

and

$$\begin{cases} \text{for all } \nu \text{ s.t. } \nu \text{ is an irregular face of } \sigma, \quad f(\nu) < f(\sigma); \\ Dn(\sigma) := \sharp\{\nu^{(k-1)} \,|\, \nu \text{ is a regular facet of } \sigma \text{ and } f(\nu) \geq f(\sigma)\} \leq 1. \end{cases}$$

When σ is not a regular facet of some other cell, it is automatically critical; in the regular case, σ is critical if both $Dn(\sigma)$ and $Un(\sigma)$ are 0. In fact, one easily sees that at most one of them can be 1; the other then has to be 0.

A CW complex is regular if all the faces are regular. As examples one has polyhedral, cubical and simplicial complexes.

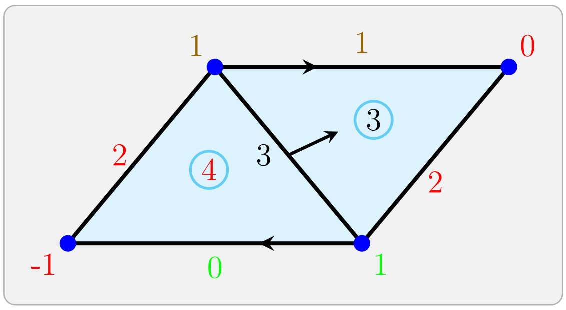


Figure 1.1: Vector field of a Morse function

A pair $\{\sigma, \tau\}$ with $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$ is called a noncritical pair. If we draw an arrow from σ to τ whenever $\sigma < \tau$ but $f(\sigma) \geq f(\tau)$, then we get a vector field associated to this function, and each noncritical cell has precisely one arrow which is either incoming or outgoing. Therefore, for the Euler number, we only need to count the critical cells with appropriate signs according to their dimensions, since the noncritical cells cancel in pairs. This is illustrated in Figure 1.1, where the 0-cells are the blue nodes, the 1-cells are the black edges and the 2-cells are the triangles in cyan. Moreover the noncritical pairs have their values with the same color (different from the color red) while the critical cells have their values in red.

We recall that, see [23], a combinatorial vector field on a CW complex $\mathbb{K}$ is a map $V \colon \mathbb{K} \to \mathbb{K} \cup \{0\}$ that satisfies: if the image of a cell is non zero, then the dimension of the image is the dimension of the cell plus one; the image cells have zero images; for each cell σ, either $V(\sigma) = 0$ or σ is a regular face of $V(\sigma)$; and the pre-image of a given cell contains at most one element. One thinks of the function V as assigning an arrow from σ to τ whenever $V(\sigma) = \tau$. In this way, each cell can either have one incoming or outgoing arrow but not both. We write $\sigma \to \tau$ whenever there is an arrow from σ to τ.

It is an important consequence of Forman's definition that the vector field admits no closed orbits, where by a closed orbit we mean a path of the form

$$\sigma_0 \to \tau_0 > \sigma_1 \to \tau_1 > \dots \sigma_m \to \tau_m > \sigma_0.$$

Since a combinatorial vector field can also admit closed orbits, the vector field constructed from a discrete Morse function is a combinatorial vector field that has no closed orbits. Conversely, one can always construct a discrete Morse function from a combinatorial vector field that admits no closed orbits.

The main purpose of this work is to relax Forman's assumptions, by considering a vector field originating from some function. We want for such a vector field, in contrast to Forman's assumption, that a cell can have arbitrarily many incoming or outgoing arrows. In addition we exclude closed orbits.

Before moving any further, let us first describe some motivating results in the contexts of discrete Morse-Floer and Conley theories.

Forman's definition leads to the natural question whether in that setting also an analogue of Floer's theory is possible, which was answered still by Forman, in [21], in the case of CW complexes, where he uses the notion of gradient path to define the boundary operator in the discrete setting. That is, he defined a boundary operator using the vector field generated from this discrete Morse function. See Definition 5.2.3 for the reformulation for CW complexes. We provide a proof for the fact that, the square of this boundary operator is zero, see Theorem 5.2.1. Our proof is based on an inductive argument: first, we show that any discrete Morse function can be transformed into one where all the cells are critical; then, moving from a situation in which the square of the boundary is zero, arrows are added one by one in the complex, and at each step, it is checked that the square of the boundary computed is still zero.

Forman answered the question of a discrete analogue of Conley theory for CW complexes, see [23]. He first uses a combinatorial vector field, and as isolated invariant sets he considers the rest points (which are the critical cells) and the closed orbits. The isolating neighborhoods here are the unions of all the cells in the isolated invariant sets together with the ones in their boundaries; the exit set is just the collections of cells in the isolating neighborhood that are not in the isolated invariant sets.

1.2 The project and the results

Now going back to this definition of a discrete Morse function according to Forman, and that of a boundary operator on a Morse complex, a natural question is whether any generalization of these concepts is possible. That is, if any one of those conditions is omitted is there a possibility of developing a similar concept like the one by Forman in discrete Morse theory? This is our main concern. We are interested in answering this question because we would like to derive the topological properties of a cell complex from a vector field more general than the one extracted from a discrete Morse function. This idea comes from the fact that Conley theory in general uses arbitrary gradient flow lines to extract the topological invariants of the manifold under consideration. In the smooth setting, Morse-Bott theory is a generalization of Morse theory. Also, the notion of Morse theory for continuous functionals by Corvellec [13] tells us that Morse theory can be broadened. That is, considering continuous functions that are not necessarily differentiable having isolated critical points, some generalized Morse inequalities can still be obtained.

We recall that for the vector field of a discrete Morse function, the number of incoming arrows or outgoing arrows of a given cell cannot exceed one. Also, each cell can have either one incoming or one outgoing arrow but never both. Finally, the vector field does not admit any closed orbit.

We now present the results we obtained.

Discrete Morse-Bott theory

Our first goal is to construct a discrete analogue of Morse-Bott theory. This involves generalizing discrete Morse functions to include the case where a discrete function can have larger critical collections of cells, instead of only simple critical cells. We define a discrete Morse-Bott function by requiring some conditions on specific collections of cells, where the function assumes the same value for all the cells in the collection. Using the reduced collections, excluding the noncritical pairs, we obtain some discrete Morse-Bott inequalities. That is, the Poincaré polynomial of the CW complex is expressed in terms of those of the reduced collections. The vector field originating from this discrete Morse-Bott function is such that inside each collection a cell can have possibly more than one incoming and/or outgoing arrow, but between the collections there are no closed orbits. We also do some Conley theory analysis by using the reduced collections (excluding the noncritical pairs) as the isolated invariant sets, and define their respective isolating neighborhoods and exit sets. These two constitute the index pairs. The Poincaré polynomial of the CW complex is then expressed in terms of those of the index pairs.

It should be noted that, when considering our discrete Morse-Bott function, the extracted discrete vector field is not a combinatorial vector field, see Figure 5.26. Thus Forman's discrete Conley theory and our approach are complementary.

However, we could not achieve a direct Floer theory approach to our discrete Morse-Bott framework. That is, we could not define a boundary operator between the reduced collections. The only way was perturbing the discrete Morse-Bott function to get a discrete Morse function and then applying Forman's discrete Morse-Floer theory.

A generalized discrete Morse-Floer theory

The second point is a generalization of Forman's discrete Morse-Floer theory. We consider a vector field on a cell complex which is more general than the one extracted from a discrete Morse function. In particular, a cell can have an arbitrary number of incoming or outgoing arrows. However, we need the vector field not to have closed orbits. We define a boundary operator from which we can derive the Betti numbers of the CW complex under consideration. We also derive some

Morse-related inequalities as well.

More specifically, on a finite CW complex $\mathbb{K}$, in which each cell is given an orientation, we consider the following arrow configuration.

Definition 1.2.1 (Arrow configuration). *An arrow configuration assigns to each k-cell σ a collection of $(k+1)$-cells that have σ as a facet. We draw an arrow from σ to each cell in that collection. The cardinality of that collection is denoted by $n_{ou}(\sigma)$. Conversely, for each k-cell σ, we let $n_{in}(\sigma)$ be the number of arrows that it receives from its facets. Thus $n_{ou}(\sigma)$ is the number of outgoing arrows of σ while $n_{in}(\sigma)$ is the number of incoming arrows of σ.*

We require that at most one of $n_{ou}(\sigma)$ and $n_{in}(\sigma)$ be different from zero and that there should not be any closed orbit.

When $n_{in}(\sigma) \geq 2$ (resp. $n_{ou}(\sigma) \geq 2$), we say the corresponding cell σ is abnormally downward (resp. abnormally upward) noncritical.

On one hand, the Euler number of the finite CW complex $\mathbb{K}$ at hand can be retrieved using the following idea:
$\forall\, \sigma^{(k)} \in \mathbb{K}$, the contribution of $\sigma^{(k)}$ is the function $C\colon \mathbb{K} \to \mathbb{Z}$ defined by:

$$C(\sigma^{(k)}) \;=\; (-1)^k \;+\; n_{in}(-1)^{k-1} \;+\; n_{ou}(-1)^{k+1}.$$

We have the following:

$$\chi(\mathbb{K}) = \sum_{\sigma\in\mathbb{K}} C(\sigma), \tag{1.1}$$

where $\chi(\mathbb{K})$ is the Euler number of the CW complex $\mathbb{K}$.
Observe that if a cell has only one arrow (either incoming or outgoing) then its contribution is zero. See Proposition 6.1.2 for the precise statement and proof of (1.1).

We recall that if $\mathbb{K}$ is a CW complex (in which every cell is endowed with an orientation called initial orientation), and R is any principal ideal domain, $C_k(\mathbb{K}; R)$ is the free R-module generated by the (oriented) k-cells of $\mathbb{K}$. The cellular boundary operator $\partial^c\colon C_{k+1}(\mathbb{K}; R) \to C_k(\mathbb{K}; R)$, is given by

$$\partial^c(\tau^{(k+1)}) = \sum_{\sigma<\tau} [\tau^{(k+1)} : \sigma^{(k)}]\sigma^{(k)},$$

where $[\tau : \sigma]$ is the incidence number of τ and σ. That is, the number of times that τ (along its boundary) is wrapped around σ. (Taking the induced orientation from τ onto σ into account: for σ a regular facet of τ, if the induced orientation on σ coincides with the initial orientation of σ, $[\tau : \sigma] = +1$; if not then it is -1).

In order to develop Floer's theory in this setup, we start with a finite CW complex, in which each cell is given an orientation and whose Betti numbers can be computed using cellular homology. We define a boundary operator, using all the arrows, which is based on some probabilistic and averaging technique. This boundary operator is the composition of some systematically well-defined "flow map" with the cellular boundary operator. The critical cells are of the following types: the cells with no incoming and outgoing arrow; the abnormally downward noncritical cells; the cells having an outgoing arrow pointing to an abnormally downward noncritical cell; the abnormally upward noncritical cells; the cells having an incoming arrow from an abnormally upward noncritical cell. The most important facts about this boundary operator is that the extracted Betti numbers are exactly the topological Betti numbers.

1.3 Organization of the thesis

This thesis consists of six chapters:

In Chapter 2, we define the basic concepts about cell complexes and the possible operations on them. We focus on the deformation retraction in CW complexes in general. This is because Forman's boundary operator preserves the homotopy type as a result of applying some deformation retraction. Most importantly, we provide our definition of a symmetric subdivision of a cell, see Definition 2.3.3, which is the logical geometric interpretation of the many incoming arrows of a given cell. Since for irregular CW complexes it is not very clear what it means to subdivide in a symmetric way, we subdivide cells having incoming arrows from their regular facets. In this way, after mapping the cell with its arrows to a disc of appropriate dimension (under the inverse image of the characteristic map), we subdivide this ball symmetrically, and project back this subdivision to the cell, see Figure 2.15. We provide along some proofs of the fact that applying the above mentioned operations to CW complexes still yields CW complexes.

Chapter 3 is about the concept of cellular topology, in which we recall the notions of homotopy equivalence of two cell complexes, as well as that of cellular homology. We also emphasize that the operations of subdivision and deformation retraction do not change the homotopy type of the complex under consideration. We then also define the Betti numbers which are just the dimensions of the corresponding homology groups of the complex, the Poincaré polynomial as well as the Euler number of the given complex which is just the alternating sum of the Betti numbers.

Chapter 4 focuses on the different notions of Morse, Morse-Floer, Morse-Bott and Conley theories, and the emphasis by which all these methods yield the Poincaré polynomial and hence the Euler number of the object under consideration. We give the definition of a Morse function and the statement of the Morse inequalities. In Morse-Floer theory, it is shown how the Betti numbers are obtained from Floer's boundary operator whose definition is also given. After giving the definition of the Morse-Bott function which generalizes that of a Morse function, the Morse-Bott inequalities are stated. We also recall what Conley theory analysis is all about.

In Chapter 5, after briefly reviewing discrete Morse theory and discrete Morse-Floer theory according to Forman, we provide our proof of the fact that the square of Forman's boundary operator is zero, see Theorem 5.2.1.

We present a solution to the question of an analogue of Morse-Bott theory for CW complexes, see Section 5.3. We consider a function assuming the same value on maximal collections of cells, where the union of the closure of the cells in each collection should be connected. The idea is, such a function has to be discrete Morse on the complex except possibly in these collections. That is, we define our discrete Morse-Bott function by requiring that for each such collection, the discrete Morse conditions have to be valid for those cells that have faces or are faces of cells not contained in the collection. See Definition 5.3.2 for the precise formulation. A cell σ in a collection C is said to be upward noncritical (resp. downward noncritical) w.r.t. C, if there exists a $\tau \notin C$, $\tau > \sigma$ (resp. $\tau < \sigma$) such that, $f(\tau) < f(\sigma)$ (resp. $f(\tau) > f(\sigma)$). Also of importance in this discrete setting is the analogue of a critical submanifold, which we call a reduced collection. For a collection C, the reduced collection C^{red} is obtained by taking out of C all the upward or downward noncritical cells w.r.t. C. See Definition 5.3.5. We also derive an analogue of the Morse-Bott inequalities, excluding all those reduce collections that are noncritical pairs, since the contribution of any noncritical pair always cancels out in the computation of the Euler number. Surprisingly, it turns out that the reduced collections will always have a positive contribution in terms of their Poincaré polynomial, when the computation of the Poincaré polynomial of the entire complex is carried out. This justifies the fact that our analogue of a Morse-Bott index is just zero. For more insight, see Theorem 5.3.7 for the formulation and proof of our result. The proof of the desired discrete Morse-Bott inequalities is just based on some perturbation technique, that is the discrete Morse-Bott function is perturbed to get a discrete Morse function.

A discrete analogue of Conley theory using a discrete Morse function is quite simple. This is why, using our discrete Morse-Bott function on a CW complex, we present our brief approach on discrete Conley theory, see Section 5.4. The isolated invariant sets are the reduced collections, excluding the noncritical pairs. As index pair for each isolated invariant set, we have: the isolating neighborhood is the union of the closure of all the cells in the reduced collection; the exit set is just the collection of those cells in the isolating neighborhood that are not in the isolated invariant set, such that either their value is less than the value assumed in the reduced collection, or they are upward noncritical w.r.t. the collection. Thus the same technique as in the smooth case is applied. Our result is formulated in Theorem 5.4.2. The proof however is based on our analogue of discrete Morse-Bott theory, by making the observation that, the homology groups of a reduced collection are the well-defined relative homology groups of its respective index pair.

Our solution to the question of a generalized boundary operator is then given in Chapter 6, where, our approach to answering this question is mainly based on some probabilistic method using averaging techniques. It consists in finding a boundary operator that uses all the arrows, despite the difficulties we encounter which are: abnormally downward noncritical cells and abnormally upward noncritical cells. The definition of the boundary operator is given by Definition 6.4.1. To prove that the square of this boundary operator is zero, which is the statement of Theorem 6.4.1, moving from a situation without any arrows (where the square of the cellular boundary operator is zero), we create abnormally upward/downward noncritical cells by adding arrows and at each step we show that the square of the newly defined boundary operator is also zero.

Our result, the Morse-related inequalities, is expressed in Theorem 6.4.3. For the proof, we suppose we are in a situation where all the cells in the CW complex belong to either one of the following: the cells with no incoming and outgoing arrow; the abnormally downward noncritical cells; the cells having an outgoing arrow pointing to an abnormally downward noncritical cell; the abnormally upward noncritical cells; the cells having an incoming arrow from an abnormally upward noncritical cell. This gives us an equality between the chain complex generated in our framework with the one generated using cellular homology. This is because we already know that the Forman-type noncritical cells can always be collapsed, preserving the homotopy type of the CW complex in the process. Also, this boundary operator is defined in such a way that it maps the kernels of the cellular chain complex to kernels in the new chain complex and vice versa. Hence the newly computed Betti numbers coincide with the topological ones.

After constructing collections consisting of either: the cells with no arrows; the abnormally downward noncritical cells together with their facets from which their arrows come; the abnormally upward noncritical cells as well as those cells to which their arrows point. The isolated invariant sets consist of these collections with the exception that if any two collections intersect then you take their union. Similarly, the isolating neighborhood is the union of the closure of each cell in the isolated invariant set, and the exit set is the difference of both. We then also do some Conley theory analysis using this arrow configuration and the result is stated in Theorem 6.5.5.

2

Cell Complexes

The focal point of this chapter is on cell complexes, which provide a natural way of describing spaces combinatorially, preserving their homotopy types. We shall mainly be interested in CW complexes which are also the most general of all. First introduced by J. H. C. Whitehead in [48], when they are not regular, CW complexes can provide the smallest number of cells needed in describing the topology of spaces.

In this chapter, we briefly recall the different examples of cell complexes, putting a little emphasis on the regular CW complexes, in particular the polyhedral complexes. We also recall some of the operations to be applied on CW complexes.

Section 2.1 focuses on CW complexes since our work is within the category of CW complexes. An introduction to category theory can be found in [45], and also [6]. Section 2.2 is about the regular cell complexes whereas Section 2.3 emphasizes the given operations that we have on CW complexes, in particular our symmetric subdivision. Given that this chapter is mainly a review of what is already in the literature, among the very large number of those, we refer the reader to [35], [14],[25], [48],[27],[24], and [34].

2.1 CW complexes

Let $n > 0$. An n-cell (a cell of dimension n) is a (topological) space which is homeomorphic to the open n-disc $\{x \in \mathbb{R}^n \,|\, ||x|| < 1\}$. The closed n-disc is given by $D^n = \{x \in \mathbb{R}^n \,|\, ||x|| \leq 1\}$. A closed n-cell is a space homeomorphic to D^n. The

0-cells are points.

One defines a cell complex as a decomposition of a space into cells, where the highest dimension of the cells is that of the space. It should also be noted that there is a specific gluing method that is used in order to properly construct the space (from its cell decomposition) from gluing the cells along their boundaries. This gluing procedure is by means of some maps called the characteristic maps and it starts from the cells of lowest dimension to the cells of highest dimension.

We first describe what it means to build a new space from a given one, by gluing along boundaries.

Definition 2.1.1. *Let X and Y be topological spaces, $A \subseteq X$ be closed, and let $f\colon A \to Y$ be continuous.*

We denote by $Y \cup_f X$ the quotient space $X \sqcup Y/\sim$, where the equivalence relation $\sim$ is given by $a \sim f(a)$ (identifying every $a \in A$ with $f(a) \in Y$).

We say the space $Y \cup_f X$ is obtained from Y by attaching X along f, and it is equipped with the quotient topology in the sense that, if $q\colon X \sqcup Y \to Y \cup_f X$ is the quotient map, then a set $S \subseteq Y \cup_f X$ is open if and only if $q^{-1}(S)$ is open in $X \sqcup Y$.

A cell-decomposition of a space X is a family $\Xi = \{e_\alpha |\, \alpha \in \mathcal{A}\}$ of subspaces of X such that each e_α is a cell and

$$X = \amalg_{\alpha\in\mathcal{A}} e_\alpha \quad \text{(disjoint union of the cells).}$$

The n-skeleton of X, is given by

$$X^{(n)} = \amalg_{\alpha\in\mathcal{A}, \dim e_\alpha \leq n} e_\alpha.$$

Example 2.1.1.
(i) Every set with the discrete topology has a cell-decomposition consisting of only 0-cells.

(ii) A cell-decomposition of $\mathbb{S}^1$ is $\mathbb{S}^1 = \{e_1,\, e_2\}$, where e_1 is a 0-cell also understood as a point, and e_2 is a 1-cell also understood as $\mathbb{S}^1\backslash\text{pt}$, where the 1-cell is attached to the 0-cell along its boundary which is a disjoint union of two 0-cells. See Figure 2.1.

From now on, we denote by $e^{(n)}$ a cell e of dimension n. Let ∂^{top} be the topological boundary operator given by $\partial^{top} A = \bar{A} \setminus \mathring{A}$, where $\bar{A}$, also denoted $Cl(A)$, is the closure of A, and $\mathring{A}$, also denoted $int(A)$, is the interior of A.

The topological boundary elements of a cell are called its faces. If a cell σ is a face of another cell τ, we write $\sigma < \tau$ to indicate that $\dim \sigma = \dim \tau - 1$, in which case σ is called a facet of τ.

A finite cell-decomposition is one consisting of finitely many cells.

We now give the precise definition of a CW complex according to Whitehead.

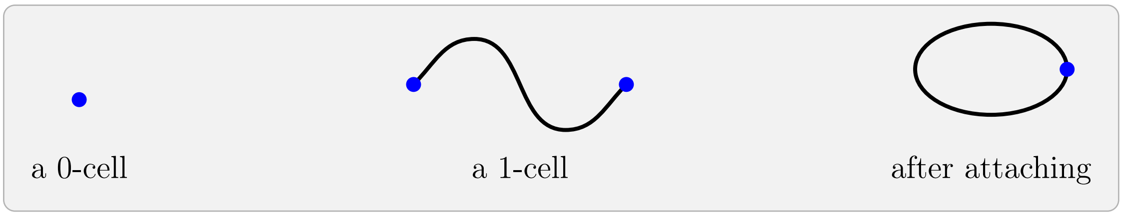


Figure 2.1: A CW construction of $\mathbb{S}^1$.

Definition 2.1.2 (CW complex, Whitehead). *A pair (X, Ξ), consisting of a Hausdorff space X and a cell-decomposition Ξ of X, is called a* ***CW complex*** *if the following are satisfied:*

(1) *('Characteristic Maps') For each $e_\alpha^{(n)} \in \Xi$, there is a map $\varphi_\alpha \colon D^n \to X$ continuous, restricting to a homeomorphism $\varphi_{\alpha|int(D^n)} \colon int(D^n) \to e_\alpha^{(n)}$, and taking $\mathbb{S}^{n-1}$ into $X^{(n-1)}$, that is, $\varphi_\alpha(\mathbb{S}^{n-1})$ is a subset of a union of finitely many cells of dimension less than n.*

(2) *('Closure Finiteness') For any cell $e_\alpha \in \Xi$ the closure $\bar{e}_\alpha$ intersects only a finite number of other cells in Ξ.*

(3) *('Weak Topology') $A \subseteq X$ is closed (in X) if and only if $A \cap \bar{e}_\alpha$ is closed (in $\bar{e}_\alpha$) for all $e_\alpha \in \Xi$.*

The name 'CW' comes from (2) and (3) where C is for closure and W is for weak.

We denote by $\mathcal{A}^n$ the index set of the n-cells.

Definition 2.1.3 (Construction of a CW complex). *A CW complex is obtained by the following inductive construction of the skeletons:*

(i) *The 0-skeleton, $X^{(0)}$, is a discrete set consisting of the 0-cells.*

(ii) *Inductively construct $X^{(n)}$, by simultaneously attaching the n-cells $e_\alpha^{(n)}$ along their boundaries via maps $\varphi_\alpha \colon \mathbb{S}^{n-1} \to X^{(n-1)}$. This in particular tells us that $X^{(n)}$ is the quotient space of $X^{(n-1)} \amalg_{\alpha \in \mathcal{A}^n} D^n$ under the identification $x \sim \varphi_\alpha(x)$, for $x \in \mathbb{S}^{n-1} = \partial^{top} D^n$.*

(iii) *The space $X = \bigcup_n X^{(n)}$ is given the weak topology that is: $A \subseteq X$ is open if and only if $A \cap X^{(n)}$ is open for all n.*

From now on we will abuse notation and denote a CW complex (X, Ξ) by the space X.

Remark 2.1.1. (*i*) $A \subseteq X$ is open if and only if $\varphi_\alpha^{-1}(A)$ is open for any cell $e_\alpha^{(n)}$ with $\varphi_\alpha \colon D^n \to X$ being the characteristic map.

(*ii*) $X^{(n)}/X^{(n-1)}$ is homeomorphic to a wedge of n-dimensional spheres, one for each n-cell of X.

Definition 2.1.4. *A CW complex is*

a) finite if it has finitely many cells;

b) locally finite if every cell meets only finitely many closed cells;

c) of finite type if every skeleton is a finite CW complex;

d) countable if it has countably many cells.

The characteristic map is the attaching map, that is, in the process of constructing the CW complex, it tells us how to attach the n-cell (along its boundary) to the $(n-1)$-skeleton. In the finite case there exists $n \in \mathbb{N}$ s.t. $X^{(n)} = X$, whereas in the infinite case, $X = \cup_{i=0}^{\infty} X^{(i)}$.

In this work we will only consider finite CW complexes.

By CW structure of a space we mean its cell decomposition.

Example 2.1.2. (*a*) The n-dimensional sphere $\mathbb{S}^n$ has the CW structure with one 0-cell and one n-cell. Since the n-cell is attached via the map $\mathbb{S}^{n-1} \to e^{(0)}$, $\mathbb{S}^n$ can be seen as the quotient space $D^n/\partial^{top} D^n$.

(*b*) Recall that the real projective n-space, $\mathbb{R}\mathrm{P}^n$, is defined as the space of all lines through the origin in $\mathbb{R}^{n+1}$ which is also the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence $v \sim \lambda v$, with v a vector and $\lambda \neq 0$ a scalar. This is then the same as considering $\mathbb{S}^n/\sim$ with $v \sim -v$. It has the CW structure with one cell in each dimension. Thus, since $\mathbb{R}\mathrm{P}^{n-1}$ is just ∂D^n with antipodal points identified, $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ via the attaching map (quotient projection) $\mathbb{S}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$.
For $\mathbb{C}\mathrm{P}^n$, one takes one cell in each even dimension and the attaching map to be $\mathbb{S}^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$.

(*c*) The torus $\mathbb{T}^2$ also given by $\mathbb{S}^1 \times \mathbb{S}^1$, has a CW structure consisting of one 0-cell, two 1-cells, and one 2-cell.

The following are examples of spaces that are not CW complexes.

a) Every infinite dimensional Hilbert space. This is because, it is a Baire space. A Baire space is a topological space in which the union of every countable collection of closed sets with empty interior has empty interior. Therefore, one cannot write an infinite dimensional Hilbert space as a countable union of n-skeletons, since each skeleton is a closed set with empty interior.

b) The Hawaiian earring is given by the infinite union of circles with centers $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$. Hence, it is not locally contractible whereas CW complexes are locally contractible (see [27, p. 522]).

A subcomplex of a CW complex X is a closed subspace of X which is such that if it contains any cell, then it also contains the closure of that cell.

We call the pair (X, A) a CW pair whenever A is a subcomplex of the CW complex X.

Let A be a subcomplex of X. A CW decomposition of the pair (X, A) consists of a sequence of subspaces $A \subset X^{(0)} \subset X^{(1)} \subset \cdots \subset X$, where $X = \bigcup_n X^{(n)}$ and

(1) $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells,

(2) X has the weak topology w.r.t. all $X^{(n)}$, $n \geq 0$.

Set $A = X^{(-1)}$, then we say a pair (X, A) with the decomposition $(X^{(n)}, n \geq -1)$ is a relative CW complex.

Proposition 2.1.1. (*i*) *If φ is the characteristic map for the cell $e^{(n)}$, then $\varphi(D^n) = \bar{e}^{(n)}$, in particular, $\bar{e}^{(n)}$ is compact.*

(*ii*) *A compact subspace of a CW complex is contained in a finite subcomplex.*

(*iii*) *CW complexes are normal and in particular Hausdorff.*

(*iv*) *A function on a CW complex is continuous if and only if its restriction on each cell is continuous.*

Proof. (*i*) Indeed, $\bar{e}^{(n)} = \overline{\varphi(int(D^n))} \subseteq \overline{\varphi(D^n)} = \varphi(D^n)$, since $\varphi(D^n)$ is compact in a Hausdorff space and hence closed. On the other hand, $\varphi(D^n) = \varphi\big(\overline{int(D^n)}\big) \subseteq \overline{\varphi(int(D^n))} = \bar{e}^{(n)}$.

(*ii*) See [14, p. 200].

(*iii*) See [27, p. 522].

(*iv*) See [35, p. 42].

□

Remark 2.1.2. In Definition 2.1.2, condition (2) always holds. Indeed, as seen above, for φ the characteristic map of $e^{(n)}$, $\bar{e}^{(n)} = \varphi(D^n)$ is compact and hence contained in a finite subcomplex. The idea then follows from the fact that $\varphi(\mathbb{S}^{n-1})$ is a subset of a union of finitely many cells of dimension less than n. Condition (3) is only needed in the case of an infinite cell-decomposition. Thus, since we are only interested in the finite CW complexes, we only need to check condition (1).

The following theorem gives the conditions under which a given CW complex can be embedded in some Euclidean space. We refer the reader to [24, p. 46] for the proof.

Theorem 2.1.2. *Every locally finite and countable CW complex of dimension k can be embedded in $\mathbb{R}^{2k+1}$.*

We now move to the regular CW complexes.

2.2 Regular CW complexes

We present some of the regular CW complexes, putting a little emphasis on the polyhedral complexes. Roughly speaking, a CW complex is said to be regular if the characteristic maps are homeomorphisms for each cell.

Definition 2.2.1. *A cell $\sigma^{(k)}$ is a **regular face** of another cell $\tau^{(k+1)}$ if, for the characteristic map φ_τ of τ, we have:*

(i) $\varphi_\tau \colon \varphi_\tau^{-1}(\sigma) \to \sigma$ *is a homeomorphism,*

(ii) $\overline{\varphi_\tau^{-1}(\sigma)}$ *is a closed k-ball.*

The construction in Figure 2.1 is an example of a non-regular CW complex for $\mathbb{S}^1$ and Figure 2.2 shows an example of a regular CW complex for $\mathbb{S}^1$.

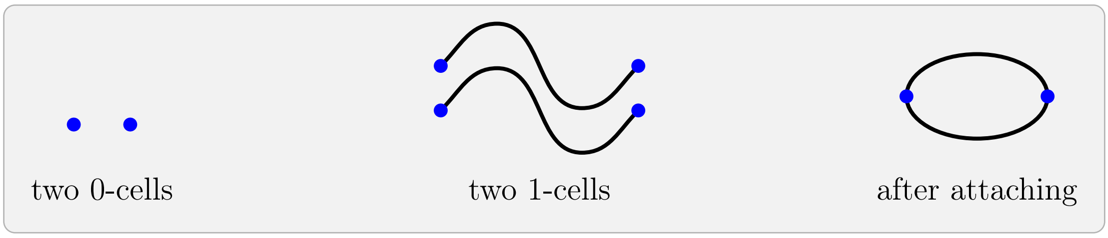


Figure 2.2: Regular CW construction of $\mathbb{S}^1$.

Definition 2.2.2. *A CW complex X is regular if for each n-cell $e_\alpha^{(n)}$, the characteristic map $\varphi_\alpha \colon \mathbb{S}^{n-1} \to X^{(n-1)}$ is a homeomorphism.*

Observe that in a regular CW complex, all the faces are regular. As examples we have polyhedral, cubical and simplicial complexes, where the cells are polytopes, cubes and simplices respectively.

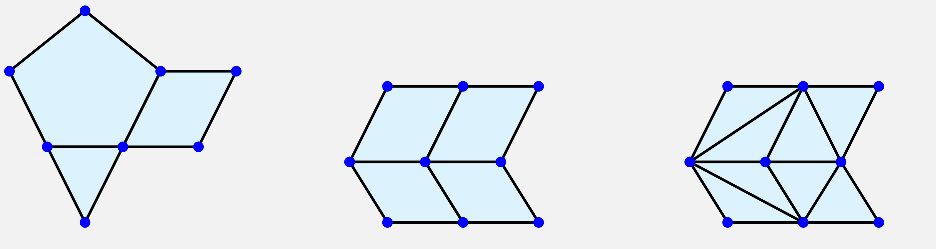


Figure 2.3: Examples of regular CW complexes.

Polyhedral complexes

Following [34], [43] and mostly [36], we give a brief review on polyhedral complexes.

Let v_0, $v_1, \cdots, v_n$ be points in $\mathbb{R}^d$. An *affine combination* of the v_i*'s* is a point $x = \sum_{i=0}^{n} \lambda_i v_i$, where $\sum_{i=0}^{n} \lambda_i = 1$. An affine combination is a *convex combination* if the λ_i*'s* are all nonnegative. The affine (resp. convex) hull is the set of affine (resp. convex) combinations. It is an n-plane if the $n+1$ points are affinely independent, that is, the n vectors $v_i - v_0$, for $i = 1, \cdots, n$, are linearly independent.

A *convex d-polytope* $\mathbb{P}$ is the convex hull of a finite collection of points $\{v_0, \cdots, v_n\}$, denoted $\mathrm{conv}(v_0, \cdots, v_n)$, that affinely spans $\mathbb{R}^d$. Its faces are also polytopes. It can alternatively be defined as a bounded subset of $\mathbb{R}^d$, that is the solution of a finite number of linear inequalities and equalities, in which case, $S \subseteq \mathbb{P}$ is called a face of $\mathbb{P}$ if there exists a linear functional f on $\mathbb{R}^d$ such that $f(s) = 0$ for all $s \in S$ and $f(p) \geq 0$ for all $p \in \mathbb{P}$.

Definition 2.2.3. *A geometric polyhedral complex* $\mathbb{P}$ *is a collection of convex polytopes in* $\mathbb{R}^d$ *such that:*

(*i*) *Every face of a polytope in* $\mathbb{P}$ *is itself a polytope in* $\mathbb{P}$.

(*ii*) *The intersection of any two polytopes in* $\mathbb{P}$ *is a face of each of them.*

The problem with geometric polytopes is that we are restricted when it comes to the dimension. This is the reason why, in general, it is better to work with abstract ones. We now give the definition for an abstract polyhedral complex.

Let $(P, \leq)$ be a partially ordered set (poset), we have the following:

Two elements F and G of P are said to be *incident* if $F \leq G$ or $G \leq F$.

The least element, denoted by F_{-1}, is such that $F_{-1} \leq F$ for all $F \in P$, and it has rank -1. If P has rank n, the greatest element, if it exists, denoted by F_n, is such that $F_n \geq F$ for all $F \in P$, and it also has rank n.

A *chain* of P is a totally ordered subset of P, and it has length l if it contains exactly $l + 1$ elements. The maximal chains are called *flags* and they contain the least and greatest elements. Denote by $\mathcal{F}(P)$ the set of all flags of P. Two flags are said to be *adjacent* if they differ in exactly one element. If this element has rank k, we say they are k-adjacent. P is *flag-connected* if any two distinct flags f and $g \in \mathcal{F}(P)$ can be joined by a sequence of flags $f = f_0, f_1, \cdots, f_{s-1}, f_s = g$, such that, f_i and f_{i-1} are adjacent for $i = 1, \cdots, s$.

For any two elements F, G such that $F \leq G$, the set $\{A \mid A \in P, F \leq A \leq G\}$ is called a *section* of P, denoted by G/F, its rank is given by $\operatorname{rank} G - \operatorname{rank} F - 1$. It then follows that, if the rank of G/F_{-1} is i, then G also has rank i. The rank of P is the maximal rank of its elements. A k-section is a section of rank k.

P is *strongly flag-connected* if each section of P is flag-connected, equivalently, if any two distinct flags f and $g \in \mathcal{F}(P)$ can be joined by a sequence of flags $f = f_0, f_1, \cdots, f_{s-1}, f_s = g$, all containing $f \cap g$, such that, f_i and f_{i-1} are adjacent for $i = 1, \cdots, s$.

A poset P of rank n is said to be connected if either $n \leq 1$ or $n \geq 2$, and for any two proper elements G and F of P, there exists a finite sequence $F = A_0, A_1, \cdots, A_k = G$ of proper elements of P, such that, A_{i-1} and A_i are incident for $i = 1, \cdots, k$.

P is said to be *strongly connected* if each section of P including P itself is connected.

We are now ready to give the formal definition of an abstract polytope. The elements of P are called its faces.

Definition 2.2.4. *An abstract n-polytope P is a poset satisfying the following:*

(1) *P has a least and greatest face,*

(2) *each flag of P has length $n + 1$,*

(3) *P is strongly connected,*

(4) *for each $i = 0, \cdots, n - 1$, if F of rank $i - 1$ and G of rank $i + 1$ are incident faces of P, then there are precisely two i-faces A_1 and A_2 of P such that $F < A_i < G$, for $i = 1, 2$.*

Note that the last condition is crucial since it is the reason why the square of the natural boundary operator applied to a given polytope will be zero, as we will see later. We shall refer to it as the *incidence property of polytopes*.

In Figure 2.4, the first and last complexes are not polyhedral complexes.

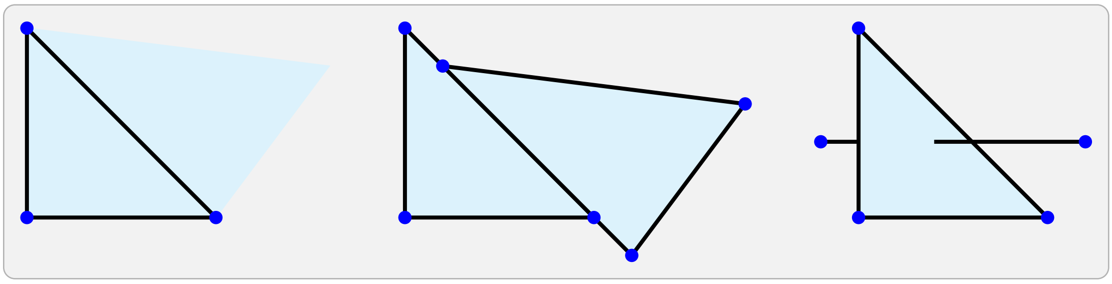


Figure 2.4: Examples of complexes.

Proposition 2.2.1. *If P satisfies conditions* (1) *and* (2) *above, then it is strongly connected if and only if it is strongly flag-connected.*

For the proof see [36, p. 24].

We can now give the definition of a polyhedral complex.

Definition 2.2.5 (Abstract polyhedral complex). *A polyhedral complex is obtained by the gluing procedure of a CW complex, where each cell to be glued is a polytope, and the intersection of any two polytopes is a face of both.*
Its dimension is the maximum dimension (or rank) of its polytopes.

Geometric realization

In this part we follow [36] and [43].

A realization of an abstract polytope is a geometric polytope in some Euclidean space which is in correspondence with the abstract one in the sense that the one-to-one image of the vertex set of the abstract polytope is the vertex set of the geometric one. Also, the vertex map should induce maps on the respective faces in such a way that there is a one-to-one correspondence between the faces of the abstract polytopes with those of the geometric one. More importantly, the faces of the abstract and geometric polytope are isomorphic and the partial ordering of the abstract polytope is inherited by the geometric one. This then means that the realization commutes with intersections.

Denote by P_j the set of j-faces of P and 2^X be the family of subsets of X. More formally, we have the following definition.

Definition 2.2.6. *A realization of an abstract polytope P is a mapping f_0 of the vertex set P_0 of P into some Euclidean space. The image V_0 of the vertex set of P is the vertex set of the realization.*

This mapping f_0 induces maps on the faces of rank greater than 0, this is the content of the following theorem (see also [36, p. 122]).

Theorem 2.2.2. *Let f_0 be a realization of an abstract n-polytope, then f_0 induces surjections $f_j\colon P_j \to V_j$, and for $j = 1, \cdots, n$, with $V_j \subseteq 2^{V_{j-1}}$, consisting of the elements*

$$f_j F := \{f_{j-1} G \,|\, G \in P_{j-1} \quad \text{and} \quad G \leq F\} \quad \text{for } F \in P_j.$$

Also, f_{-1} is given by $f_{-1}\emptyset = \emptyset$.

A realization is given by all the mappings f_j. If all the f_j's are bijective we say that the realization is *faithful*. We say that a realization of an abstract polytope P is *symmetric* if each automorphism of P induces an isometric permutation of the vertex set of the realization. Faithful and symmetric realizations always exist in the case of finite abstract polytopes. In this way each j-face of P is uniquely determined by the $(j-1)$-faces that belong to it.

A realization of a given abstract polytope is unique up to an affine transformation.

The reverse procedure, i.e. moving from a geometric polytope to an abstract one, is also possible. This is achieved by mapping all the faces of the geometric polytope to the faces of the abstract polytope bijectively, and of course preserving the inclusion. In this way, the mappings also commute with the intersections and we get an abstract polytope equivalent to the geometric one.

Polyhedral complexes are the most general among all the regular cell complexes.

Hasse diagram: The Hasse diagram provides a good description for posets and polytopes in particular. It is drawn from the lowest rank elements to the highest rank elements, from the bottom to the top in the sense that, any elements on the same horizontal line have the same rank, and if F is a face that has as subface A (i.e. $A < F$), draw a line from A to F. Figure 2.5 shows the Hasse diagram of a (regular) pentagon.

A regular polytope is one for which all the faces "look" the same, that is, the faces of the same dimension are all isomorphic, in the sense that their Hasse diagrams are isomorphic (in terms of posets). In Figure 2.3, the middle and right polyhedral complexes are both regular, while the complex on the left is not.

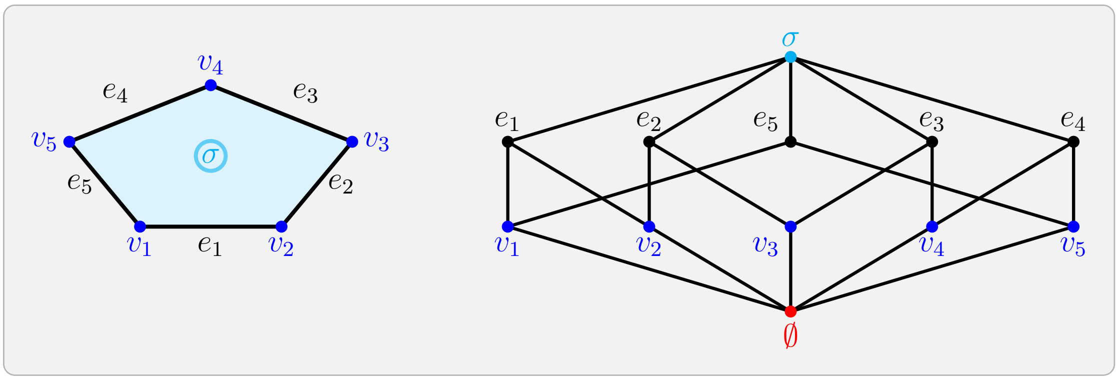


Figure 2.5: A polytope and its Hasse diagram.

Definition 2.2.7. *Let C be a collection of polytopes (it might as well be a subcomplex) of a polyhedral complex $\mathbb{K}$.*

(i) *The Closure of C, denoted $Cl(C)$ or $\overline{C}$, is the smallest subcomplex that contains C.*

(ii) *The **Star** of C, denoted $St(C)$, is an open neighborhood of C containing the set of all the polytopes of $\mathbb{K}$ which have a face in C. It need not be a subcomplex.*

(iii) *The **Link** of C, denoted $Lk(C)$, is the topological boundary of the Star, that is, $Cl(St(C)) - St(Cl(C))$. It is also a subcomplex.*

In Figure 2.6, C is given by the two red vertices and the red edge. The Star of C is highlighted in green in the middle subfigure and the Link is the green subcomplex in the last subfigure.

A **cubical complex** is a polyhedral complex for which all the cells are cubes. It is mostly used in the area of image processing, but it will not be relevant for our purpose.

Simplicial complexes

A simplicial complex is a polyhedral complex for which all the cells are simplices.

A k**-simplex** (simplex of dimension k) is the convex hull of $k + 1$ affinely independent points. We use special names for the first few dimensions, vertex for 0-simplex, edge for 1-simplex, triangle for 2-simplex, and tetrahedron for 3-simplex.

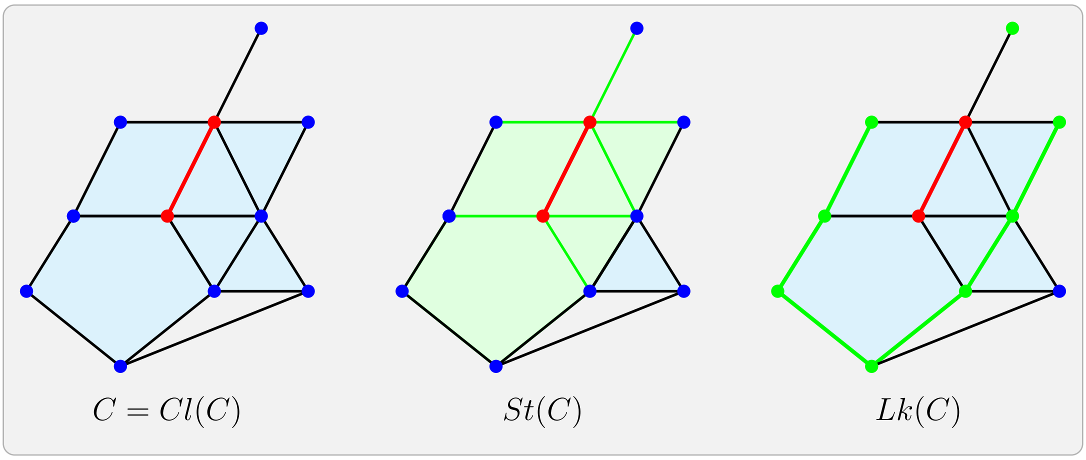


Figure 2.6: The closure, star and link of a collection of polytopes.

Definition 2.2.8. *A* ***geometric simplicial complex*** $\mathbb{K}$ *is a finite collection of simplices such that:*

1) *if* $\tau \in \mathbb{K}$ *and* $\sigma \leq \tau$ *then* $\sigma \in \mathbb{K}$,

2) *if* $\tau_1, \tau_2 \in \mathbb{K}$ *then* $\tau_1 \cap \tau_2 \in \mathbb{K}$ *and is either empty or a face of both.*

The dimension of $\mathbb{K}$ *is the maximum dimension of its simplices.*

A subcomplex of a simplicial complex is a subset which is itself a simplicial complex.

Figure 2.4 shows how the conditions of the definition of a simplicial complex may be violated.

If we denote by $\| \mathbb{K} \|$ the union of all the simplices of $\mathbb{K}$ with the topology inherited from $\mathbb{R}^d$, then we define a triangulation of a given topological space X to be a simplicial complex $\mathbb{K}$ for which there is a homeomorphism between X and $\| \mathbb{K} \|$.

The fact that $\mathbb{K}$ should lie in $\mathbb{R}^d$ puts a restriction not only on the cardinality of $\mathbb{K}$ but also on the dimension of its simplices.

In general replacing each simplex in $\mathbb{K}$ by its set of vertices yields a collection of sets $\mathcal{A}$ such that if $V \in \mathcal{A}$ and $W \subseteq V$ then $W \in \mathcal{A}$; this follows from the first condition for having a simplicial complex $\mathbb{K}$. The second condition tells us that if $W = V_1 \cap V_2$, then $W \subseteq V_1$ and $W \subseteq V_2$, and this clearly is true. One remark is that, the set of all the vertices of $\mathcal{A}$ is equal to the union of all the elements of $\mathcal{A}$. $\mathcal{A}$ is called the abstraction of $\mathbb{K}$ or the **vertex scheme** of $\mathbb{K}$. Thus we have the following definition.

Definition 2.2.9 (Abstract simplicial complex). *An abstract simplicial complex is a finite collection of sets $\mathcal{A}$ such that if $\alpha \in \mathcal{A}$ and $\beta \subseteq \alpha$, then $\beta \in \mathcal{A}$.*

The sets in $\mathcal{A}$ are the simplices and the dimension of each $\alpha \in \mathcal{A}$ is its cardinality minus one, that is, $\dim \alpha = \sharp(\alpha) - 1$. Let us denote by $V(\mathcal{A})$ and $V(\mathbb{K})$ the sets of vertices of $\mathcal{A}$ and $\mathbb{K}$ respectively.

In what follows, if $\alpha = \{v_1, \cdots, v_l\} \in \mathcal{A}$ by $f(\alpha)$ we mean $\{f(v_1), \cdots, f(v_l)\}$.

Two abstract simplicial complexes $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be isomorphic if there is a bijection f between $V(\mathcal{A}_1)$ and $V(\mathcal{A}_2)$ s.t. $\alpha \in \mathcal{A}_1$ if and only if $f(\alpha) \in \mathcal{A}_2$.

Out of all cell complexes, it is usually easiest to deal with simplicial complexes due to their easy combinatorial structure.

Definition 2.2.10 (Geometric realization). *A geometric realization of an abstract simplicial complex $\mathcal{A}$ is a simplicial complex $\mathbb{K}$ for which there exists a bijection $f\colon V(\mathcal{A}) \to V(\mathbb{K})$ s.t. $\alpha \in \mathcal{A}$ if and only if $\operatorname{conv}(f(\alpha)) \in \mathbb{K}$.*

The geometric realization is unique up to an affine isomorphism. In general it makes more sense to just map any abstract simplex to the standard geometric simplex.

We have the following theorem whose proof is found in [42, p. 15].

Theorem 2.2.3. *Every abstract simplicial complex A is (isomorphic to) the vertex scheme of some simplicial complex $\mathbb{K}$.*

Better still, we have the following theorem, see [17, p. 64] for its proof.

Theorem 2.2.4. *Every abstract simplicial complex of dimension k has a geometric realization in $\mathbb{R}^{2k+1}$.*

We have seen that every finite CW complex can be embedded it into some Euclidean space. This brings us to the second part of this chapter dedicated to the operations one can apply on a CW complex.

2.3 Some operations on CW complexes.

Here we briefly mention some of the operations on CW complexes that we come across, following [27].

Product: Let X and Y be two CW complexes, then their product $X \times Y$ is also a CW complex with the cell structure given by: the cells of $X \times Y$ are of the form $e_\alpha^{(n)} \times e_\beta^{(m)}$, where the $e_\alpha^{(n)}$ are the cells in X and the $e_\beta^{(m)}$ are the cells in Y. One should however bear in mind that the product topology on $X \times Y$ is coarser that the topology on $X \times Y$ as a cell complex, but they both coincide if either one of them has finitely many cells or if both have countably many cells.

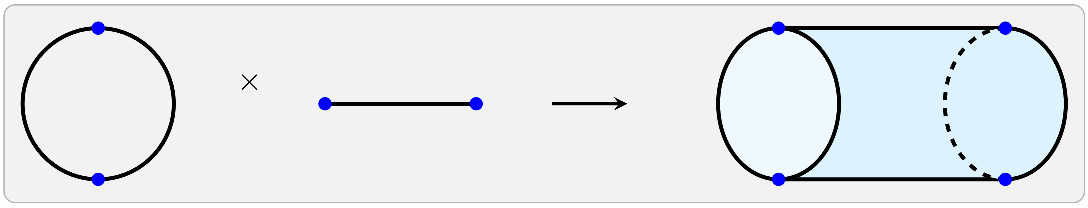


Figure 2.7: The product of $\mathbb{S}^1$ and $I = [0, 1]$.

Quotient: Let (X, A) be a CW pair , then the quotient X/A is a CW complex with the cell structure given by all the cells in $X \setminus A$ plus one 0-cell which comes from the set A being contracted into a point. Indeed the characteristic maps of X/A are just the compositions of the characteristic maps of X with the quotient map $X \to X/A$. Figure 2.8 illustrates the quotient space of the torus $\mathbb{T}^2$ (or $\mathbb{S}^1 \times \mathbb{S}^1$) with $\mathbb{S}^1$ i.e. $\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1$.

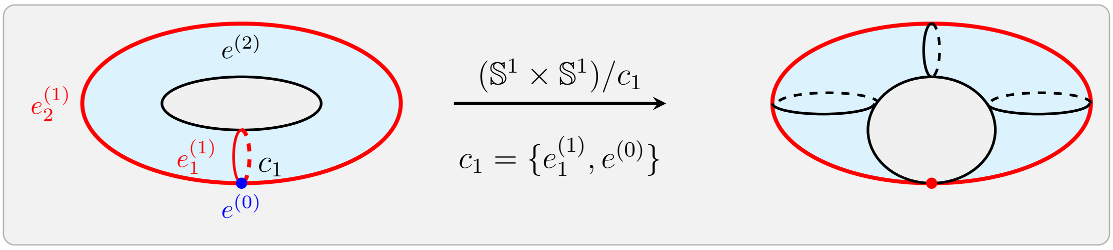


Figure 2.8: The quotient of the torus with $\mathbb{S}^1$.

We recall that a space is normal if every two disjoint closed sets can be separated by disjoint open neighborhoods, and every normal space is Hausdorff in the sense that we take, instead of closed sets, just two different points (understood as two singletons).

The following proposition is found in [35, p. 59], [27], [24] and also [34].

Proposition 2.3.1. *If X is normal and the quotient map $X \to X/\sim$ is a closed map, then the quotient space $X/\sim$ is normal.*

Proof. Let us denote by $\widetilde{X}$ the space $X/\sim$. Let $\widetilde{C_1}$ and $\widetilde{C_2}$ be two disjoint closed sets in $\widetilde{X}$. By the quotient topology, both $q^{-1}(\widetilde{C_1})$ and $q^{-1}(\widetilde{C_2})$ are closed in X and disjoint since $q^{-1}(\widetilde{C_1}) \cap q^{-1}(\widetilde{C_2}) \subseteq q^{-1}(\widetilde{C_1} \cap \widetilde{C_2}) = \emptyset$. Thus, normality of X gives the existence of two disjoint open neighborhoods $U_1 := \mathcal{N}(q^{-1}(\widetilde{C_1}))$ and $U_2 := \mathcal{N}(q^{-1}(\widetilde{C_2}))$ of $q^{-1}(\widetilde{C_1})$ and $q^{-1}(\widetilde{C_2})$ respectively. Now consider their respective complements with respect to X i.e. $X \setminus U_1$ and $X \setminus U_2$, they are both closed so that their respective images with respect to our closed map q are also closed in $\widetilde{X}$. So take $\widetilde{U_1} := \widetilde{X} \setminus (q(X \setminus U_1))$ and $\widetilde{U_2} := \widetilde{X} \setminus (q(X \setminus U_2))$ to be the desired open neighborhoods of $\widetilde{C_1}$ and $\widetilde{C_2}$ respectively. We have $\widetilde{U_1} \cap \widetilde{U_2} = \emptyset$. Indeed:

$$x \in \widetilde{U_1} \Rightarrow x \notin q(X \setminus U_1) \Rightarrow q^{-1}(x) \notin X \setminus U_1 \Rightarrow q^{-1}(x) \in U_1,$$

so $q^{-1}(\widetilde{U_1}) \subset U_1$. Thus

$$\emptyset = U_1 \cap U_2 \supset q^{-1}(\widetilde{U_1}) \cap q^{-1}(\widetilde{U_2}) \;\Rightarrow\; \widetilde{U_1} \cap \widetilde{U_2} = \emptyset.$$

□

Join: If X and Y are two CW complexes, the join of X and Y, denoted $X * Y$, is defined by the quotient space $(X \times Y \times I)/\sim$, where $\sim$ is given by $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. That is $X \times Y \times \{0\}$ is collapsed into X while $X \times Y \times \{1\}$ is collapsed to Y. Another way of seeing this is by considering $X * Y$ to be the collection of line segments joining points in X to points in Y. Note that $X * Y$ has the cell structure consisting of the product cells of $X \times Y \times (0, 1)$ and the cells of X and Y. If $Y = \{y\}$ then $X * Y = CX$, the cone of X. See Figure 2.9 for an illustration.

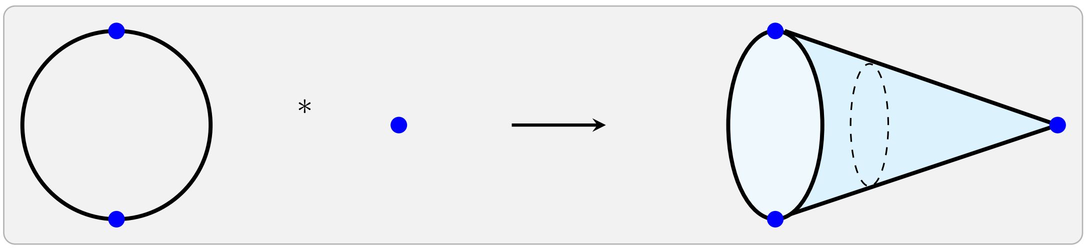


Figure 2.9: The join of $\mathbb{S}^1$ and $I = [0, 1]$.

Wedge sum: Let $a \in X$ and $b \in Y$, the wedge sum of X and Y, denoted $X \wedge Y$, is the quotient of their disjoint union $X \bigsqcup Y$ obtained by identifying

a and b to a point. For 0-cells a and b, $X \wedge Y$ is also a CW complex since it can be regarded as $X \bigsqcup Y/(a \sim b)$. If we consider two spheres then their wedge sum is the new space obtained from gluing the two spheres at two given points. See Figure 2.10, where the first $\mathbb{S}^2$ has the CW structure of one 0-cell and one 2-cell, while the second has the structure of one 0-cell, one 1-cell and two 2-cells.

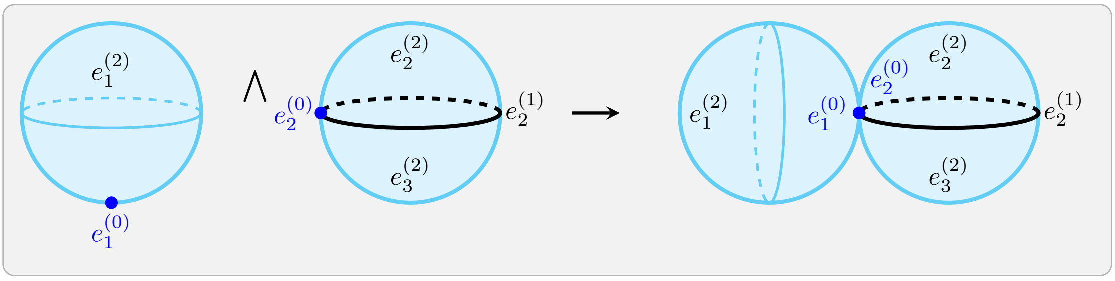


Figure 2.10: The wedge product of $\mathbb{S}^2$ and $\mathbb{S}^2$.

Smash product: The smash product of X and Y, denoted $X \vee Y$, is defined by $X \times Y/X \wedge Y$. Consider for example the torus $\mathbb{T}^2$ also given by $\mathbb{S}^1 \times \mathbb{S}^1$: if we consider its two generators say c_1 and c_2, they intersect at a given point, take this as $\mathbb{S}^1 \wedge \mathbb{S}^1$ then collapsing c_1 and c_2 into a point yields $\mathbb{S}^2$, this is illustrated in Figure 2.11.

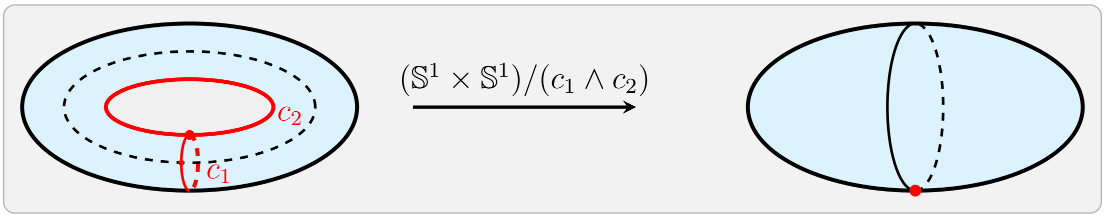


Figure 2.11: The smash product of $\mathbb{S}^1$ and $\mathbb{S}^1$.

Some of the operations above will be used in the following subsection which is about symmetric subdivision.

Symmetric subdivision

We give here our definition of the symmetric subdivision of a cell, since our theory starts with CW complexes. It should be noted that the resulting complex is still a CW complex.

We consider arrows coming from one cell to another. We associate to a pair $\{\sigma, \tau \mid \sigma < \tau\}$ an arrow from σ to τ. Then σ has an outgoing arrow while τ has an incoming arrow.

We consider a given cell that has incoming arrows from some of its facets and we want to isolate the facets having the outgoing arrows in a symmetric way.

We write $\sigma \to \tau$ if there is an arrow from σ to τ.

We recall that if φ is the characteristic map of a cell $\tau^{(n)}$, then $\varphi(D^n) = \bar{\tau}^{(n)}$, where $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$, and φ maps $int(D^n)$ homeomorphically into $\tau^{(n)}$. The idea behind the symmetric subdivision is, we assume that the closure of each cell is mapped to D^n under the inverse image of the characteristic map, we symmetrically subdivide D^n and then map the symmetric subdivisions of D^n back to the original cell under the characteristic map. First we define what we mean by the "expansion" of incoming arrows of a disc.

Definition 2.3.1 (Expansion of incoming arrows). *Let τ be a cell with facets $\sigma_1, \cdots, \sigma_k$, and incoming arrows from some of its regular facets. Let D^τ be the corresponding closed disc which is the inverse image of $\bar{\tau}$ under the characteristic map φ of τ. That is $D^\tau := \varphi^{-1}(\bar{\tau})$. Let $I \subset \{1, \cdots, k\}$ be the index set of the cells having the incoming arrows of τ, that is, for each $j \in I$, there is $\sigma_j \to \tau$. This also means that there is an arrow $\varphi^{-1}(\sigma_j) \to D^\tau$. Let b be the center of the disc D^τ, define $D^{\tau,i}$ to be*

$$D^{\tau,i} := b * \varphi^{-1}(\sigma_i), \quad \textit{for } i = 1, \cdots, k.$$

For each $j \in I$, expand the incoming arrow of each $D^{\tau,j}$ step by step in the following way:

Step 1 *For each $j \in I$, define I_j to be s.t. $I_j \cap I = \emptyset$ and any $w < D^{\tau,j}, w \neq \varphi^{-1}(\sigma_j)$ satisfies $w < D^{\tau,s}$ for $s \in I_j$. If $I_j \neq \emptyset$, draw an arrow $w \to D^{\tau,s}$ for every $s \in I_j$. Let $I^1 := \cup_j I_j$.*

Step 2 *For each $j \in I^1$, define I_j^1 to be s.t. $I_j^1 \cap (I \cup I^1) = \emptyset$ and any $w^1 < D^{\tau,j}, w^1 \neq w$ satisfies $w^1 < D^{\tau,s}$ for $s \in I_{j^1}$. If $I_j^1 \neq \emptyset$, draw an arrow $w^1 \to D^{\tau,s}$ for each $s \in I_{j^1}$. Let $I^2 := \cup_j I_j^1$.*

Continue as before until you get to step l.

Step l *For each $j \in I^{l-1}$, define I_j^{l-1} to be s.t. $I_j^{l-1} \cap (I \cup_{m=1}^{l-1} I^m) = \emptyset$ and any $w^{l-1} < D^{\tau,j}, w^{l-1} \neq w^{l-2}$ satisfies $w^{l-1} < D^{\tau,s}$ for $s \in I_j^{l-1}$. If $I_j^{l-1} \neq \emptyset$, draw an arrow $w^{l-1} \to D^{\tau,s}$ for each $s \in I_j^{l-1}$. Let $I^l := \cup_j I_j^{l-1}$.*
The process stops when

$$\{1, \cdots, k\} \setminus (I \cup_{m=1}^{l-1} I^m)) = \emptyset.$$

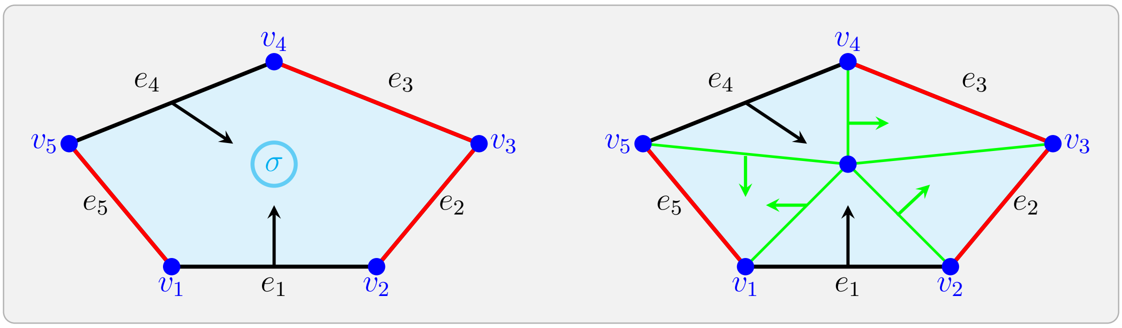


Figure 2.12: Expansion of incoming arrows.

We call the result of this procedure the expansion of the incoming arrows of τ.

First we state what we mean by a given regular facet of a cell inherits a projected arrow from the cell.

Definition 2.3.2 (Arrow projection). *Suppose a given cell τ has incoming arrows from its regular facets $\sigma_1, \cdots, \sigma_l$ then we project its incoming arrows onto its regular facets having no arrows in the following way:*
If τ has no incoming arrows from its regular facet w, and w intersects one or more of the σ_i's for $i = 1, \cdots, l$ we then project the incoming arrows of τ onto w by "supposing" there is an arrow from $w \cap \sigma_i$ to w.

In Figure 2.13, σ has two incoming arrows $e_2 \to \sigma$ and $e_3 \to \sigma$. We illustrate the arrow projection by highlighting the face e_1 of τ that has no arrow in red. The second subfigure shows how e_1 inherits two incoming arrows, one from $v_2 = e_2 \cap e_1$ and the other from $v_1 = e_3 \cap e_1$.

The arrows generated by the two definitions above are "fictive" since they will be eliminated in the next step. They are only useful for the subdivision of our cell. The subdivision is achieved from the lower dimensional cells to the higher dimensional ones and a subdivision of a k-cell will use the subdivisions of some of its $(k-1)$-faces.

The next definition provides a way to symmetrically subdivide an arbitrary cell from some CW complex that need not be regular, provided its incoming arrows come from its regular facets.

Definition 2.3.3 (Symmetric subdivision of a cell). *We proceed from the cells of lowest dimension to the highest and assume that the incoming arrows come from regular facets.*

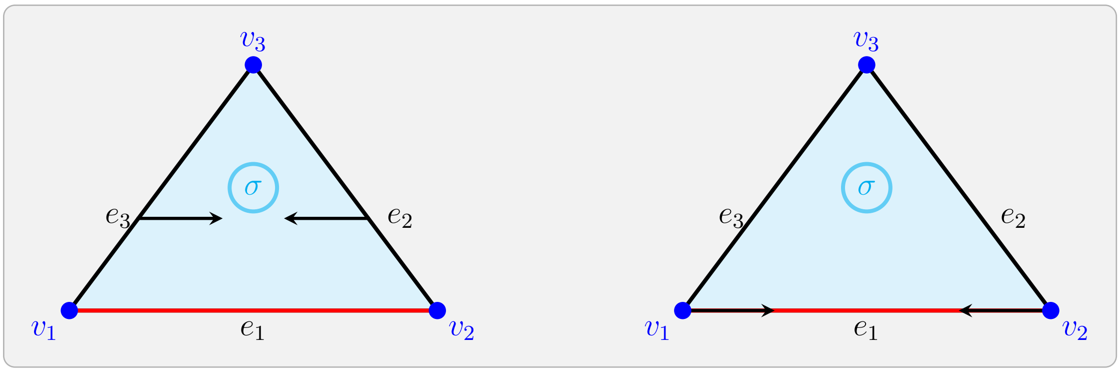


Figure 2.13: Arrow projection.

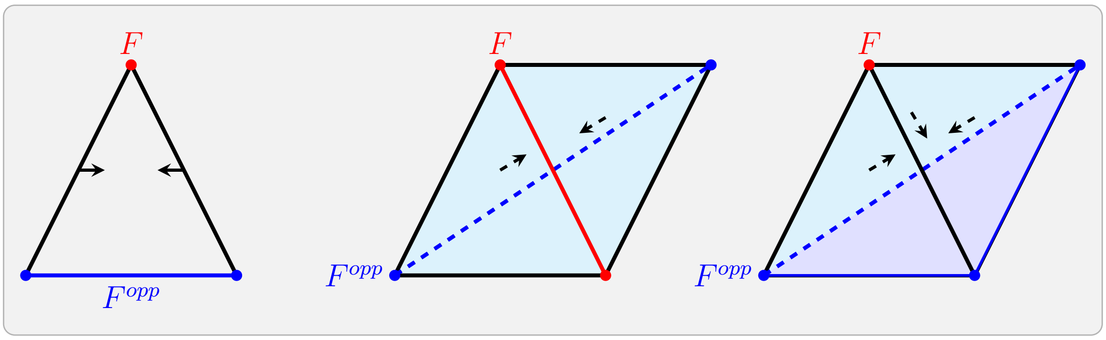


Figure 2.14: The examples of F and F^{opp}.

(1) *Let S^1 be a 1-cell having two incoming arrows from its regular facets v_1 and v_2. Let D^1 s.t. $\varphi(D^1) = \bar{S}^1$ be the inverse image of $\bar{S}^1$ under its characteristic map φ. Let b_{D^1} be the center of D^1. Then we define a symmetric subdivision for S^1 by:*

$$S^{1,1} = \varphi(b_{D^1} * \varphi^{-1}(v_1)), \quad S^{1,2} = \varphi(b_{D^1} * \varphi^{-1}(v_2))$$

and both are 1-cells.

(2) *Let S^k, $k \geq 2$ be a k-cell having arrows from $\tau_1, \cdots, \tau_p$, all regular facets, and let D^k be its inverse image under its characteristic map φ.*
Let $F = \bigcap_{i=1}^{p} \bar{\tau}_i$, and denote by I_F its vertex set. Define F^{opp} to be the largest subcomplex of S^k that does not contain F i.e. $F^{opp} = \mathrm{conv}\{v_i \,|v_i \notin I_F\}$. See Figure 2.14.

(a) *If $Cl(\partial^{top}(S^k) \backslash \cup_{i=1}^{p} \bar{\tau}_i) = \emptyset$, set $S^{k,l} = \varphi(b_{D^k} * \varphi^{-1}(\tau_l))$ for $l = 1, \cdots, p$.*

(*b*) *Else,*

If $F \neq \emptyset$ and F^{opp} is a face of S^k, "project the arrows" and map $\bar{S}^k$ to D^k. Then projecting the arrows will generate $(k-1)$-dimensional subsets of the disc D^k with incoming arrows, subdivide them, and get subdivisions $D^{k-1,1}, \cdots, D^{k-1,m}$ of dimension $k-1$ of the corresponding $(k-1)$-dimensional subsets of the disc D^k. The k-dimensional subdivisions are given by:

$$S^{k,l} = \varphi\big(\varphi^{-1}(\tau_l) * N(\varphi^{-1}(\tau_l))\big) \quad \textit{for } l = 1, \cdots, p,$$

where

$$N(\varphi^{-1}(\tau_l)) := \{D^{k-1,i} \,|\, D^{k-1,i} \cap \varphi^{-1}(\tau_l) \quad \textit{is a } (k-2)\textit{-cell}\}.$$

This is illustrated in Figure 2.16.

Else do the "expansion of the incoming arrows first", see Definition 2.3.1. If, as a result of expanding the arrows, any part of D^k has more than one "expanded" arrow, then subdivide it.
The subdivisions of D^k are obtained by collapsing each of the cells having an "expanded" incoming arrow, and to get the subdivisions of S^k we take the image of the subdivisions of D^k. This is shown in Figure 2.17.

In the process of collapsing the cells having the incoming "expanded" arrows, if two cells σ_1 and σ_2 are such that: there is a v satisfying $\sigma_1 > v < \sigma_2$; the incoming arrows of σ_1 and σ_2 (from some other cells) come from the expansion of the same initial incoming arrow, that is from the expansion of the same $b_{D^k} * \tau$; then also remove this cell v together with its faces in $\sigma_1 \cap \sigma_2$ that were created by the subdivision.

Remark 2.3.1. Observe that the symmetric subdivision generated by Definition 2.3.3 is unique and does not depend on the geometry of the cells. The reason we do the subdivision on the disc is to avoid any possible problem due to convexity.

Note that, the resulting subdivisions need no longer be convex, but this is not a problem since we do not require any convexity, given that we are interested only in the abstract cells.

Figure 2.15 illustrates the construction when the cell has some irregular facets: after taking the inverse image of the cell under the characteristic map, the green edge is unglued and this results in a disc with two incoming arrows. We then apply Definition 2.3.3 to symmetrically subdivide this disc to get the subfigure in the middle. The green edges of the subdivided disc are then glued back together, and this gives a subdivision on the initial cell.

Remark 2.3.2. Since the characteristic map maps the interior of the disc homeomorphically to the cell, this in particular tells us that: whenever we subdivide our abstract cell by drawing a straight line, we actually mean some curve having no self intersections in the interior of the cell, and whose end points are those of the straight line.

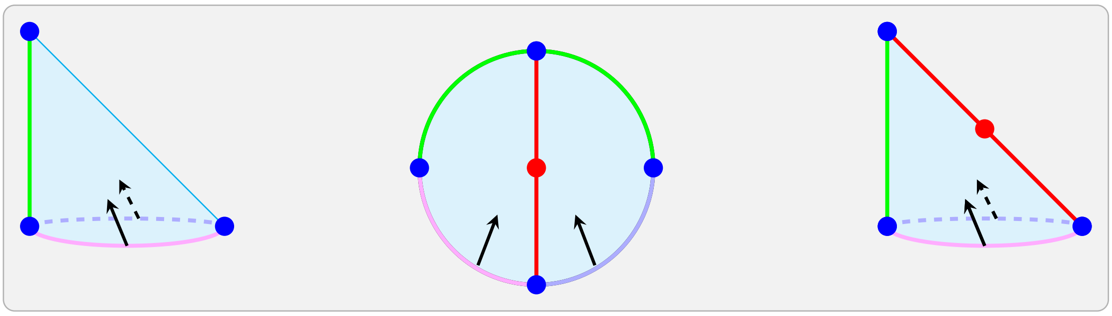


Figure 2.15: Projecting a symmetric subdivision.

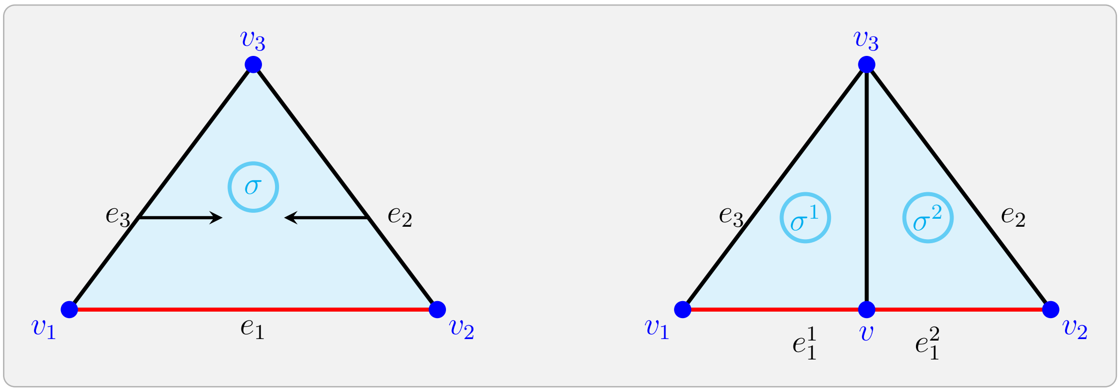


Figure 2.16: Symmetric subdivision of a 2-simplex.

Example 2.3.1. In this example, we work on a disc of dimension 2.

- In Figure 2.16, the 2-cell σ has two incoming arrows, after projecting its incoming arrows, we get the illustration in Figure 2.13, meaning that we have to subdivide the edge e_1, and then get $\sigma^1 := e_1^1 * e_3$ and $\sigma^2 := e_1^2 * e_2$.
- In the top left subfigure in Figure 2.17, our σ is such that $F^{opp} = \emptyset$, so we first expand its incoming arrows, these are the arrows in green, at the top right subfigure. In the process, the 2-cell $w = (b, v_1, v_5)$ now gets two incoming

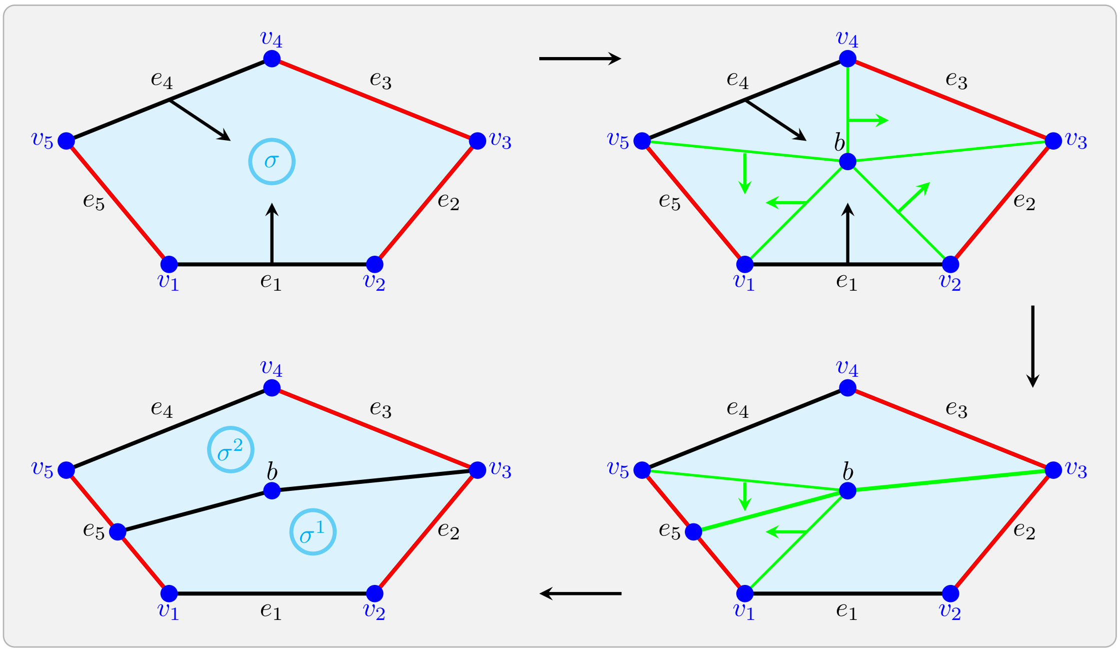


Figure 2.17: Symmetric subdivision of a pentagon.

"expanded" arrows, so we have to subdivide it using (b) of Definition 2.3.3 above, refer to Figure 2.16. After this subdivision, we collapse all the 2-cells having a green incoming arrow, and the resulting subfigure is the one at the bottom left.

Example 2.3.2. In this example, we work on a disc of dimension 3. Let us suppose that S^3 is given by the first subfigure in Figure 2.18, and it has incoming arrows from $\tau_1 := (v_6, b_4, v_5)$ the triangle in blue, $\tau_2 := (v_4, b_6, v_3)$ the triangle in pink and $\tau_3 := (v_2, b_2, v_7)$ the triangle in green. We join the barycenter b of S^3 with each triangle in its boundary, we get a total of 24 cells. Indeed, we have

$$(b, b_1, v_1, v_6), (b, b_1, v_6, v_7), (b, b_1, v_7, v_2), (b, b_1, v_2, v_1)$$

$$(b, b_2, v_2, v_7), (b, b_2, v_7, v_8), (b, b_2, v_8, v_3), (b, b_2, v_3, v_2)$$

$$(b, b_3, v_3, v_8), (b, b_3, v_8, v_5), (b, b_3, v_5, v_4), (b, b_3, v_4, v_3)$$

$$(b, b_4, v_5, v_4), (b, b_4, v_4, v_1), (b, b_4, v_1, v_6), (b, b_4, v_6, v_5)$$

$$(b, b_5, v_6, v_5), (b, b_5, v_5, v_8), (b, b_5, v_8, v_7), (b, b_5, v_7, v_6)$$

$$(b, b_6, v_4, v_1), (b, b_6, v_1, v_2), (b, b_6, v_2, v_3), (b, b_6, v_3, v_4).$$

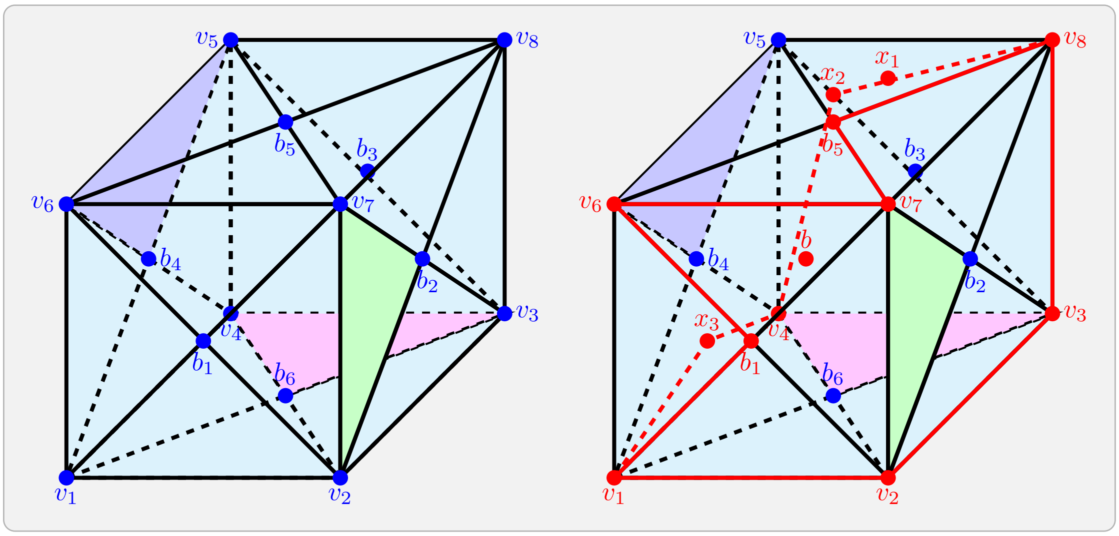


Figure 2.18: Example of a symmetric subdivision in dimension 3.

Figure 2.19 demonstrates the expansion at each step. If there is an arrow from an edge (w_1, w_2) to a cell (w, w_1, w_2), this is understood as the expanded arrow from (b, w_1, w_2) to the cell (b, w, w_1, w_2). If the arrow carries a number k and has the same color with τ_i, then this arrow is the expanded arrow of $b * \tau_i$ at step k. At step 2, (b, b_4, v_4, v_1) has 3 arrows, two blue and one pink, so we subdivide it symmetrically into 3 parts (b, x_3, v_4, v_1),(b, x_3, b_4, v_4) and (b, x_3, b_4, v_1). We have to delete the cell (b, x_3, b_4) along with (x_3, b_4) because (b, x_3, b_4, v_4) and (b, x_3, b_4, v_1) have incoming arrows of the same color, that is, they both belong to the expansion of the same $b * \tau_i$. Similarly, the cells (b, b_3, v_5, v_4) and (b, b_3, v_8, v_5) also get subdivided accordingly.

We represented the intersection of the subdivision in red, meaning, after doing all the necessary expansions and collapses, we end up with the following:

the cell $S^{3,1}$ is given by the expansion of τ_1, that is, it is the 3-cell bounded by the following cells:
(b_4, v_6, v_5), (b_4, v_5, v_4), (b_4, v_6, v_1), (b_4, v_4, x_3, v_1), (v_6, v_1, b_1), (b_5, v_7, v_6),
(v_6, b_5, v_5), (b_5, v_8, v_5), (v_4, x_2, v_5), $(v_5, x_2, x_1, v_8, v_5)$, (b, v_1, b_1),
(b, b_1, v_6), (b, v_6, v_7), (b, v_7, b_5), (b, b_5, v_8), (b, v_8, x_1), (b, x_1, x_2),
(b, x_2, v_4), (b, v_4, x_3), (b, x_3, v_1);

the cell $S^{3,2}$ is given by the expansion of τ_2, that is, it is the 3-cell bounded by the following cells:
(b_6, v_4, v_1), (b_6, v_1, v_2), (b_6, v_2, v_3), (b_6, v_3, v_4), (v_8, b_3, v_3), (b_3, v_3, v_4),

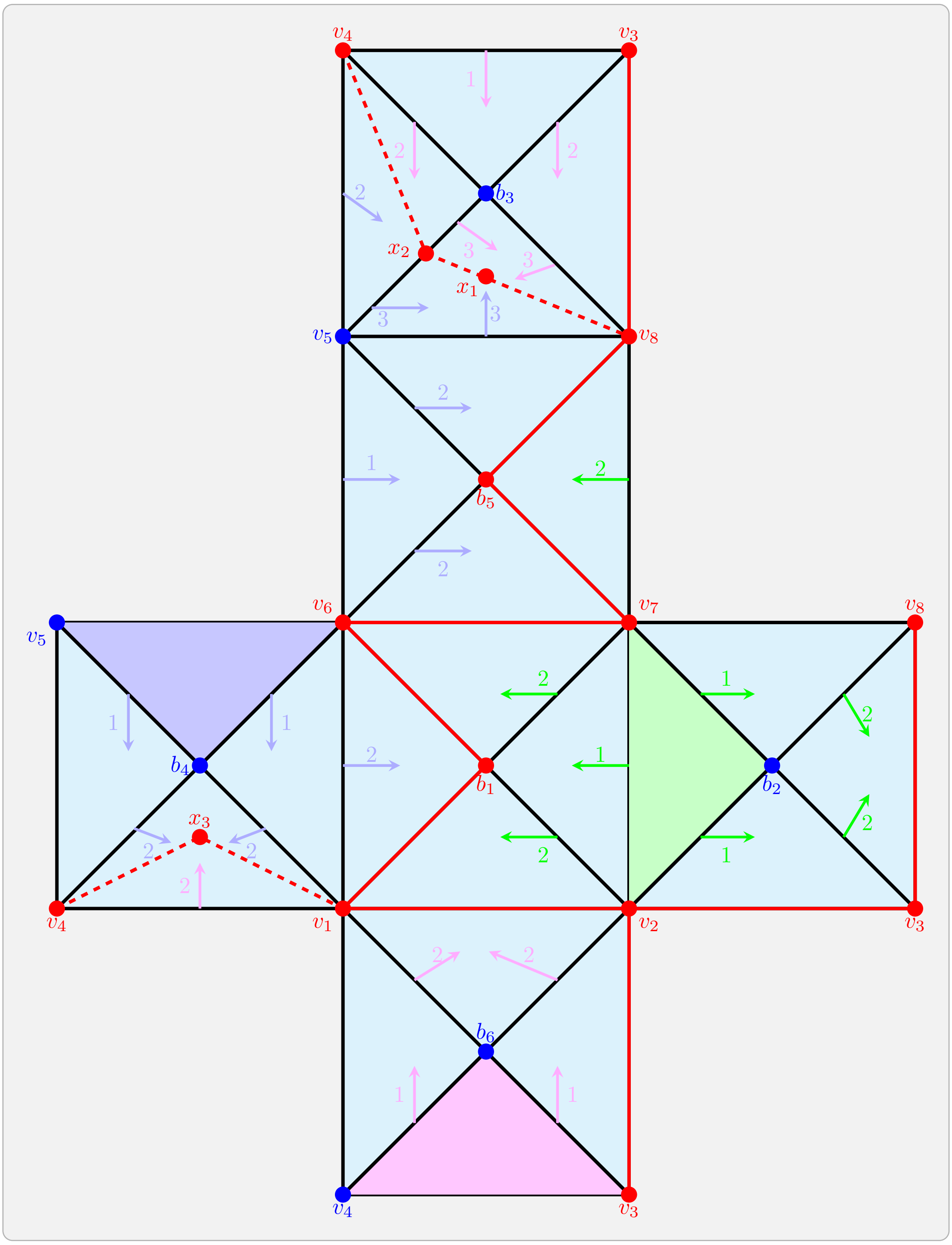


Figure 2.19: Arrow expansion in dimension 3.

$(v_4, x_2, b_3), (b_3, x_2, x_1, v_8), (v_4, x_3, v_1), (b, v_4, x_3), (b, x_3, v_1), (b, v_8, x_1),$
$(b, x_1, x_2), (b, x_2, v_4), (b, v_1, v_2), (b, v_2, v_3), (b, v_3, v_8);$

the cell $S^{3,3}$ is given by the expansion of τ_3, that is, it is the 3-cell bounded by the following cells:
$(b_2, v_2, v_7), (b_2, v_7, v_8), (b_2, v_8, v_3), (b_2, v_3, v_2), (b_1, v_6, v_7), (b_1, v_7, v_2),$
$(b_1, v_2, v_1), (b_5, v_7, v_8)(b, v_2, v_3), (b, v_3, v_8), (b, v_1, v_2), (b, v_1, b_1),$
$(b, b_1, v_6), (b, v_6, v_7), (b, v_7, b_5), (b, b_5, v_8).$

Thus this is obtained from the right subfigure in Figure 2.18 by joining the barycenter b to all the red edges.

The most important fact here is the preservation of the category of CW complexes whenever a given cell of a CW complex is subdivided.

Proposition 2.3.2. *Suppose we have a CW complex X and we subdivide a cell of X to get a complex X'. Then X' is also a CW complex.*

Proof. Let $e_\alpha^{(n)}$ be the cell that we subdivide into cells of dimension equal and less than n.

Let φ_α be its corresponding characteristic map, then φ_α maps the interior of an n-disc homeomorphically into $e_\alpha^{(n)}$. Subdividing $e_\alpha^{(n)}$ therefore is equivalent to subdividing the open n-disc, and each subdivision of the disc can in turn be mapped homeomorphically into an open disc of the appropriate dimension. That is, if $e_\alpha^{(n)} = \bigcup_{\beta\in\vartheta} w_\beta$, where ϑ indexes all the cells in the subdivision of $e_\alpha^{(n)}$, then there exist continuous functions $\tilde{\varphi}_\beta \colon D \to e_\alpha^{(n)}$ s.t.

(*i*) ${\tilde{\varphi}_\beta}_{|int(D^s)} \to w_\beta^{(s)}$ is a homeomorphism where $0 \leq s \leq n$. Indeed: for each cell $w_\beta^{(s)} \in e_\alpha^{(n)}$, $0 \leq s \leq n$, $\tilde{\varphi} = \varphi_\alpha \circ h_\beta^{-1}$, where h_β is the homeomorphism between $\varphi_\alpha^{-1}(w_\beta^{(s)})$ and $int(D^s)$. Note that this h_β is also a homeomorphism between $\overline{\varphi_\alpha^{-1}(w_\beta^{(s)})}$ and D^s. We illustrated this in Figure 2.20.

(*ii*) $\tilde{\varphi}_\beta(\mathbb{S}^{s-1})$ is the union of a finite number of cells of dimension less than s. To get this, use the fact that we have a finite subdivision on our open disc and it is true for φ_α.

(*iii*) X' has the weak topology and is Hausdorff. Indeed, $A \cap \bar{\sigma}$ is closed in $\bar{\sigma}$ for all $\sigma \in X'$ if and only if $A \cap \bar{e}_\alpha^{(n)}$ is closed in $\bar{e}_\alpha^{(n)}$, if and only if $A \cap \bar{w}_\beta^{(s)}$ is closed in $\bar{w}_\beta^{(s)}$ for all w_β in the subdivision, since $\bar{w}_\beta^{(s)} = \tilde{\varphi}_\beta(D^s)$ is compact (continuous image of compact) in Hausdorff $\bar{e}_\alpha^{(n)}$ (the Hausdorff property is hereditary and X is Hausdorff) hence closed in $\bar{e}_\alpha^{(n)}$. Note that X' is Hausdorff from its construction (since $\bar{e}_\alpha^{(n)}$ is Hausdorff).

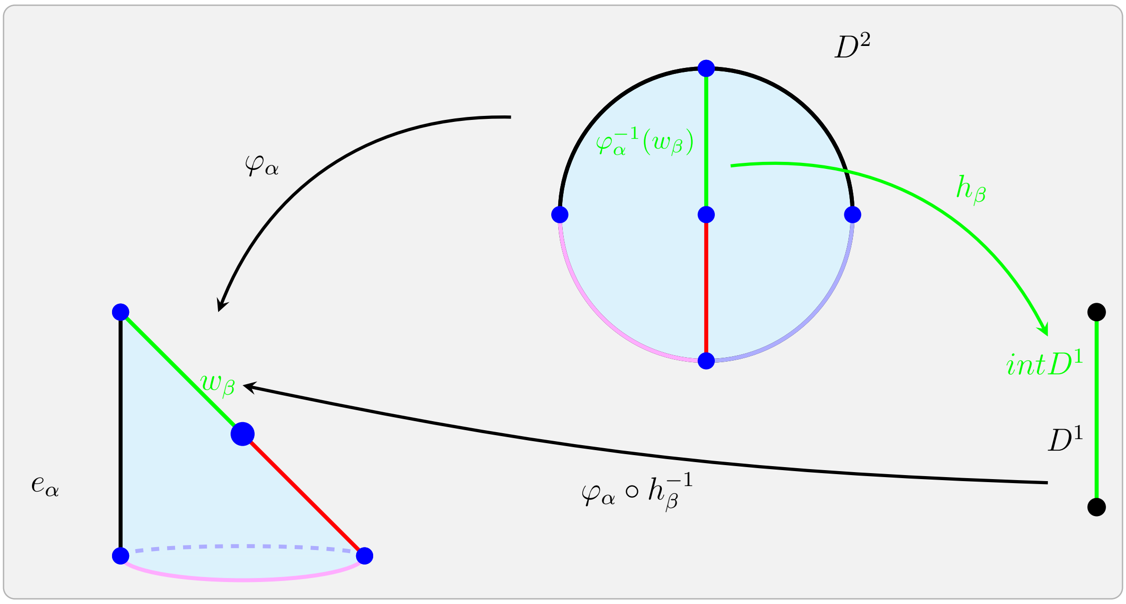


Figure 2.20: Characteristic map of a cell in the subdivision.

□

We now move to the next subsection which is about deformation retraction in CW complexes.

Deformation retraction

In this subsection, we prove that the operation of deformation retracting a given subcomplex of a CW complex still leaves us within the category of CW complexes.

Definition 2.3.4. *A subspace $A \subset X$ is a retract of X if there is a retraction $r\colon X \to A$, such that, $r \circ i = id_A$ where, $i\colon A \hookrightarrow X$ is the inclusion map.*

A is a deformation retract if there exists a homotopy $F\colon X \times [0\,,1] \to X$, such that, $F(x,0) = id_X$ and $F(x,1) = i \circ r(x)$.

A is called a strong deformation retract if in addition, $F(a,t) = a$ for all $a \in A$ and for all $t \in [0,1]$.

Proposition 2.3.3. *Suppose that a (closed) subcomplex C of a CW complex X is deformation retracted/collapsed into its (closed) subcomplex A, then the resulting complex is also a CW complex.*

Proof. Let $r\colon C \to A$ be the deformation retraction, and let f be a (continuous) extension on the entire complex given by: for $x \in X$, $f(x) = r(x)$ if $x \in C$, and

$f(x) = x$ else. Observe that our deformation retraction of C into A means we collapse C into A and this then means that the resulting complex is the quotient $X/\sim$, where $\sim$ is given by $x \sim f(x)$. Thus the map $q\colon X \to X/\sim$ is a quotient map, and satisfies: for all $x \in C$, $q(x) \in A$; and $q_{|A} = id_A$.

The cells in $X' := X/\sim$ are the cells in $X \setminus C$ plus the cells in A. Any cell e_α in X' has as characteristic map $q \circ \varphi_\alpha$, where φ_α is the characteristic map for X.

We only need to show that X' with the quotient topology is Hausdorff, and that the weak and quotient topologies on X' coincide. Since X is normal, from Proposition 2.3.1, we need to show that our quotient map q is closed in order to deduce that X' is also normal and hence Hausdorff.

Since C is a subcomplex (in a finite CW complex), it is the continuous image (under the characteristic maps) of a finite union of compacts (the closed discs), thus C is actually compact. Let $S \subseteq C$ be closed then S is compact which implies that $q(S)$ is also compact in $q(A) = A$. But A is Hausdorff (as a subset of the Hausdorff space X) thus, $q(S)$ is closed in A and A is closed in X' which implies that $q(S)$ is closed in X'. Therefore, $q^{-1}(q(S))$ is closed in X (by definition of quotient topology) hence $q_{|C}$ is a closed map.

Now, for S closed in X, we have

$$S = (S \cap C^c) \cup (S \cap C) \quad \text{and} \quad q^{-1}(q(S)) = (S \cap \bar{C}^c) \cup \big(q^{-1}(q(S \cap C))\big),$$

thus, $q^{-1}(q(S))$ is the union of two closed sets in X, hence it is closed in X. This implies that q is a closed map. Hence, X' is Hausdorff. The proof ends with the following lemma, see also [35, p. 59].

Lemma 2.3.4. *The weak and quotient topologies on X' coincide. That is, for C closed in X', $q^{-1}(C)$ is closed in X if and only if $C \cap \bar{\sigma}$ is closed in $\bar{\sigma}$, for all $\sigma \in X'$.*

Proof: '$\Rightarrow$' Let C closed in X' and suppose $q^{-1}(C)$ is closed in X. Take $\sigma \in X'$, then $\bar{\sigma}$ is compact (in a Hausdorff space) and hence closed. The set $q^{-1}(C) \cap q^{-1}(\bar{\sigma})$ is also closed in X (as a finite intersection of two closed sets), this implies that $q^{-1}_{|q^{-1}\bar{\sigma}}(C)$ is closed in $q^{-1}(\bar{\sigma})$. Thus, $C \cap \bar{\sigma} = q(q^{-1}_{|q^{-1}\bar{\sigma}}(C))$ is closed in $\bar{\sigma}$, since q is a closed map.

'$\Leftarrow$' Let $\sigma \in X'$ and suppose $C \cap \bar{\sigma}$ is closed in $\bar{\sigma}$. Take $\tau \in X$ s.t. $q(\tau) = \sigma$. Recalling that every continuous map from a compact space to a Hausdorff space is a quotient map, we have that the map $q_{|\bar{\tau}}\colon \bar{\tau} \to \bar{\sigma}$ is a quotient map, thus $q^{-1}_{|\bar{\tau}}(C \cap \bar{\sigma})$ is also closed in $\bar{\tau}$ which implies that $q^{-1}(C) \cap \bar{\tau} = q^{-1}(C \cap \bar{\sigma}) \cap \bar{\tau}$ which is closed in $\bar{\tau}$ and $\bar{\tau}$ is closed in X, hence $q^{-1}(C)$ is closed in X.

$\square$

Subdivisions and deformation retractions do not change the category of a CW complex. On this basis, we shall now move to the topology of CW complexes in the next chapter.

3

The topology of CW complexes

A CW complex, as mentioned before, provides a decomposition of a space into cells together with some gluing method. The most important property about it is the fact that any CW construction of a given space has the same homotopy type with the space. In this chapter, we study the topology of CW complexes thereby first having an insight on their homotopy types and the operations under which they might or might not change. We also recall the definitions of cellular homology that we shall need later on in order to find the Betti numbers of the complex under consideration. For a general review on topology, we refer the reader to [16] and [41].

Section 3.1 is about the homotopy types of CW complexes whereas Section 3.2 is about cellular homology.

In the sequel, by maps we mean continuous maps and by spaces we mean topological spaces.

3.1 The homotopy types of CW complexes

In this section, following [25] and [34], we present those properties of spaces that make them to have the same homotopy types, and properties of functions on spaces that make them homotopy preserving.

Given two topological spaces X and Y, let f_0 and f_1 be continuous maps from X to Y.

Definition 3.1.1. *We say that f_0 is homotopic to f_1, and write $f_0 \simeq f_1$, if there exists a continuous map $F\colon X \times [0\,,1] \to Y$ s.t. $F(x,0) = f_0$ and $F(x,1) = f_1$.*

The relation of being homotopic is an equivalence relation. This is the content of the following proposition.

Proposition 3.1.1. *"$\simeq$" is an equivalence relation.*

Proof. a) Reflexivity: $f \simeq f$. Indeed, define $F \times [0\,,1] \to X$ by $F(x,t) = f(x)$. F is well defined and continuous since it is the composition of f and the projection of $X \times [0\,,1]$ onto X.

b) Symmetry: if $f \simeq g$ then $g \simeq f$. Indeed, if $F\colon X \times [0\,,1] \to Y$ is continuous s.t. $F(x,0) = f$ and $F(f,1) = g$ then define $G\colon X \times [0,1] \to Y$ s.t. $G(x,t) = F(x, 1-t)$, then $G(x,0) = g$ and $G(x,1) = f$.

c) Transitivity: $f \simeq g$ and $g \simeq h \Rightarrow f \simeq h$. Indeed, let F be a homotopy from f to g and G a homotopy from g to h. A homotopy $H\colon X \times [0,1] \to Y$ from f to h is given by:

$$H(x,t) \;\; = \begin{cases} F(x,2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ G(x,2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

□

The next definition gives the conditions under which two given spaces are homotopy equivalent.

Definition 3.1.2. *A map $f\colon X \to Y$ is said to be a homotopy equivalence if there exists a map $g\colon Y \to X$ s.t. $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We call g a homotopy inverse of f and we say that the spaces X and Y are homotopy equivalent or have the same homotopy type.*

We then have the following basic properties about homotopy equivalence of spaces and homotopy inverses of functions, and their proofs are found in [28], [3] and [2].

Proposition 3.1.2. *We have the following:*

i) if g is a homotopy inverse of some function, it is also a homotopy equivalence;

ii) a map homotopic to a homotopy equivalence is also a homotopy equivalence;

iii) a map homotopic to a homotopy inverse is also a homotopy inverse;

iv) any two homotopy inverses of the same map are homotopic.

Remark 3.1.1. From the above proposition, we make the following observations:

1) Any homeomorphism is a homotopy equivalence.

2) The identity map is a homotopy equivalence.

From the statements in Proposition 3.1.2, one sees that a homotopy inverse is unique up to homotopy, and the composition of two homotopy equivalences is also a homotopy equivalence. It should be noted that the same definitions and facts also hold for maps of pairs.

Proposition 3.1.3. *Homotopy equivalence is an equivalence relation.*

Proof. Let X, Y and Z be topological spaces.

i) Reflexivity: X is homotopy equivalent to X by using the identity map.

ii) Symmetry: if X is homotopy equivalent to Y by means of a homotopy equivalence f with inverse g, then Y is homotopy equivalent to X since if $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$ then this means that f is a homotopy inverse for g.

iii) Transitivity: if X is homotopy equivalent to Y and Y homotopy equivalent to Z, then we have homotopy equivalences $f\colon X \to Y$ and $g\colon Y \to X$; $h\colon Y \to Z$ and $k\colon Z \to Y$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$; $k \circ h \simeq id_Y$ and $h \circ k \simeq id_Z$. Then $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq id_Z$ and $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq id_X$ which means that X is homotopy equivalent to Z.

□

The next proposition is important for this work since it establishes the fact that any strong deformation retraction, see Definition 2.3.4, of some space into its subspace preserves its homotopy type.

Proposition 3.1.4. *If A is a strong deformation retract of X, then the inclusion map, $i\colon A \hookrightarrow X$, is a homotopy equivalence.*

Proof. The proof follows from the fact that the strong deformation retraction $r\colon X \to X$ is a homotopy inverse for the inclusion map. □

In the next subsection we present two important notions in homotopy theory, although they are not used in this thesis.

Mapping cone and mapping cylinder

The mapping cone and mapping cylinder are two very important notions in homotopy theory as they also provide a necessary and sufficient condition for a given function to be a homotopy equivalence. One can express the mapping cone in terms of the mapping cylinder. We will however not put too much emphasis on this since it is not directly related to our work.

Definition 3.1.3. *Let $f\colon X \to Y$ be a continuous map and $I = [0,1]$. The mapping cylinder of f is the quotient space*

$$M(f) = ((X \times I) \sqcup Y)/\sim,$$

where $\sim$ is given by $(x,1) \sim f(x)$ for all $x \in X$.

Definition 3.1.4. *The mapping cone of f is*

$$C(f) = ((X \times I) \sqcup Y)/\sim,$$

where $\sim$ is given by $(x,1) \sim f(x)$ and $(x,0) \sim (y,0)$ for all $x, y \in X$.

One immediately sees that $C(f) = M(f)/(X \times \{0\})$.

See Figure 3.1 for an illustration where $X = \mathbb{S}^1$ and $Y = \{pt\}$, pt denotes a point. Then the function f is the constant map $\mathbb{S}^1 \to \{pt\}$, and the mapping cylinder of f is a cone whereas the mapping cone of f is the sphere.

Lemma 3.1.5. *Let $f\colon X \to Y$ be a continuous map. Then Y is a strong deformation retract of $M(f)$.*

Proof. See [35, P. 117]. □

The following theorem establishes the condition for which a given map $f\colon X \to Y$ is a homotopy equivalence.

Theorem 3.1.6. *A map $f\colon X \to Y$ is a homotopy equivalence if and only if X is a deformation retract of $M(f)$.*

Proof. See [35, P. 119]. □

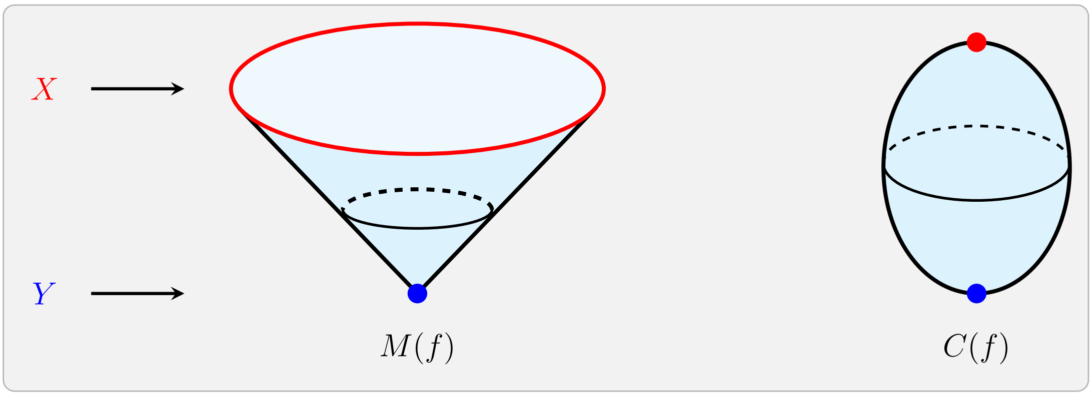


Figure 3.1: The mapping cone and mapping cylinder for the constant map on $\mathbb{S}^1$.

We recall that A pair (X, A) consisting of a CW complex X and a subcomplex A of X is called a CW pair.

Now we move to the homotopy extension property of a given CW pair. For more insight, we refer the reader to [27] or [25].

Definition 3.1.5 (Homotopy Extension Property). *A CW pair (X, A) has the homotopy extension property if, for a homotopy $f_t\colon A \to Y$ and a map $F_0\colon X \to Y$ s.t. $F_0|_A = f_0$, there exists an extension of F_0 to a homotopy $F_t\colon X \to Y$ s.t. $F_t|_A = f_t$.*

The following proposition gives a necessary and sufficient condition for a CW pair to have the homotopy extension property.

Proposition 3.1.7. *A CW pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

Proof. See [27, P. 532]. □

Lemma 3.1.8. *The subspace $(B^n \times \{0\}) \cup (S^{n-1} \times [0, 1])$ is a strong deformation retract of $B^n \times [0, 1]$.*

Proof. See also [25, P. 25]. □

The following theorem also known as the Homotopy Extension Theorem together with Proposition 3.1.7 tells us that every CW pair has the homotopy extension property.

Theorem 3.1.9 (Homotopy Extension Theorem). *If (X, A) is a CW pair, then $(X \times \{0\}) \cup (A \times I)$ is a strong deformation retract of $X \times I$.*

Proof. See [25, P. 27]. □

The next proposition is also of importance since it tells us that any quotient space obtained by collapsing a contractible subspace has the same homotopy type as the original space.

Proposition 3.1.10. *If (X, A) is a CW pair and A is contractible, then the quotient map $X \xrightarrow{q} X/A$ is a homotopy equivalence.*

Proof. We need to find a function $f\colon X/A \to X$ s.t. $q \circ f \simeq id_{X/A}$ and $f \circ q \simeq id_X$. A being contractible, we can find a homotopy, $F_t\colon X \to X$, extending the contraction of A, such that

$$F_0 = id_X, \quad F_t(A) \subset A \quad \text{and} \quad F_1(A) = pt,$$

where $pt \in X$ denotes the point to which A is contracted. We then have the following facts:

$$\begin{tikzcd} X \arrow[r, "F_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{F}_t"] & X/A \end{tikzcd}$$

$q \circ F_t(A) = pt$, it therefore factors into $X \xrightarrow{q} X/A \xrightarrow{\bar{F}_t} X/A$ satisfying $q \circ F_t = \bar{F}_t \circ q$.

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{F}_1"] \arrow[ur, "f"] & X/A \end{tikzcd}$$

Similarly, since $F_1(A) = pt$, it induces a map $f\colon X/A \to X$ satisfying $f \circ q = F_1$. Thus, for all $[x] \in X/A$,
$q \circ f[x] = q \circ f \circ q(x) = q \circ F_1(x) = \bar{F}_1 \circ q(x) = \bar{F}_1[x]$.
We have therefore found the desired function f since,

$$f \circ q = F_1 \simeq F_0 = id_X \quad \text{and} \quad q \circ f = \bar{F}_1 \simeq \bar{F}_0 = id_{X/A}.$$

□

The fact that the homotopy type of a CW complex does not change if we subdivide a given cell or apply a deformation retraction is of importance. This is the content of the following proposition.

Proposition 3.1.11. *The homotopy types of CW complexes are left unchanged under the operations of subdivisions and deformation retraction.*

Proof. It follows from the facts that: any subdivision of a cell is the composition of homeomorphisms with the characteristic map of this cell, and hence preserves the homotopy type; and a deformation retraction is a homotopy equivalence. □

The homotopy of CW complexes is understood, we now move to the next section of this chapter which is about the homology of CW complexes.

3.2 The homology groups of CW complexes

The computation of the homology groups of the CW complex is of great importance since it gives the Betti numbers of the complex, these being the dimensions of the homology groups. The Betti numbers denoted b_i's also give a topological insight since each b_i is understood as the number of i-dimensional "holes" that the complex has. Just as in the theory of simplicial complexes, there are two ways to compute the homology groups of CW complexes, one way is using singular homology while the other is cellular homology. We recall that singular homology on a given topological space X, see [47], is based on the fact that, the standard simplex is mapped to the space by some continuous function, but it will not be of our interest, we will only recall what is more appropriate to our work, which is cellular homology.

We will start with an insight about how orientations of CW complexes are carried out, then we give the definition of cellular homology. Here we mostly follow [25] and [27] where the reader can find all the proofs of the statements made.

Orientations of CW complexes

A very important notion in orienting a CW complex is that of the degree of a map. We talk of orientation of CW complexes for simplicity, but in fact we look at local orientations, that is, each individual cell is oriented. Thus, before we show how the orientation of a cell is carried out, we first recall the notion of the degree of a map defined from $\mathbb{S}^n$ to $\mathbb{S}^n$ along with some of its properties.

Degree of a map $\mathbb{S}^n \to \mathbb{S}^n$

For a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$, the degree of f is the number of times that f wraps $\mathbb{S}^n$ around itself. For each such map f, we have that $f_*\colon H_*^S(\mathbb{S}^n) \to H_*^S(\mathbb{S}^n)$, where H^S denotes the singular homology, is a homomorphism from an infinite cyclic group to itself. In this case, the degree of f, denoted deg f, is the integer d s.t. $f_*[x] = d[x]$. We have the following facts for $n \geq 1$.

Proposition 3.2.1. *(i) The degree of the identity map is* 1*, since* $Id_{\mathbb{S}^n *} = Id_{H_*^S(\mathbb{S}^n)}$.

(ii) If f is not surjective, deg f =0.

(iii) If $f \simeq g$ then deg f = deg g.

(iv) *deg* $f \circ g$= *deg* f *deg* g.

(v) *If* f *is a homotopy equivalence, deg* $f = \pm 1$.

(vi) *If* f *is a reflection of* $\mathbb{S}^n$, *def* $f = -1$.

(vii) *The antipodal map* $\mathbb{S}^n \to \mathbb{S}^n$, $x \mapsto -x$, *has degree* $(-1)^{n+1}$.

(viii) *If* f *has no fixed points, it has degree* $(-1)^{n+1}$.

Proof. See [27, P. 134]. □

In the next theorem, we see that for two maps from $\mathbb{S}^n$ to $\mathbb{S}^n$ to be homotopic, they must have the same degree and vice versa.

Theorem 3.2.2 (Brouwer-Hopf). *Two maps from* $\mathbb{S}^n$ *to* $\mathbb{S}^n$ *are homotopic if and only if they have the same degree.*

We recall that all the topological boundary elements of a cell are called its faces and the co-dimension one faces are called facets.

Let $e_\alpha^{(n)}$ be a cell, an orientation on this cell is determined by its characteristic map, say φ_α. This means that an orientation is chosen on the respective disc D^n for which $\varphi_\alpha \colon D^n \to e^{(n)}$. This orientation is then mapped to $e^{(n)}$ via φ. Recall that a characteristic map for $e_\alpha^{(n)}$ can also be understood as a map $\varphi_\alpha \colon (D^n, \mathbb{S}^{n-1}) \to (\bar{e}_\alpha^{(n)}, \partial^{top}\bar{e}_\alpha^{(n)})$. This then induces a homeomorphism $\varphi' \colon D^n/\mathbb{S}^{n-1} \to \bar{e}^{(n)}/\partial^{top}\bar{e}^{(n)}$. We also know that the map $h_n \colon D^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$ is a homeomorphism. Thus, if φ_1 and φ_2 are two characteristic maps for the cell $e^{(n)}$, the homeomorphism $H := h_n \circ (\varphi_2')^{-1} \circ \varphi_1' \circ h_n^{-1} \colon \mathbb{S}^n \to \mathbb{S}^n$ has degree ± 1. This follows from the fact that any homeomorphism is a homotopy equivalence and any homotopy equivalence has degree ± 1. The following is found in [25, P. 52].

Definition 3.2.1. *Two characteristic maps* φ_1 *and* φ_2 *for a given cell are equivalent if and only if the corresponding map* H *has degree* 1.

The equivalence of characteristic maps is an equivalence relation. Since the homeomorphism H can only have degree ± 1. The content of the following proposition is about the orientations of cells.

Proposition 3.2.3. *Each cell has exactly two orientations, one denoted by* $+$ *and the other by* $-$. *If a given characteristic map* φ *determines one orientation, then* $\varphi \circ r$ *determines the other, where,* $r \colon (D^n, \mathbb{S}^{n-1}) \to (D^n, \mathbb{S}^{n-1})$ *is such that* $r(x_1, \cdots, x_n) = (-x_1, x_2, \cdots, x_n)$.

Proof. See [25, P. 53]. □

Thus, two characteristic maps belong to the same equivalence class if and only if they yield the same orientation on the cell. This means that there are only 2 equivalence classes, one for which $\deg H = +1$ and the other for which $\deg H = -1$.

Let us consider each of the cells given in Figure 3.2 from which we want to construct a ball of dimension 3. Each of them is given an orientation, and we would like to show how these orientations will induce orientations on each of their faces. This is crucial when determining the incidence number, see Definition 3.2.3 below, between two cells σ and τ with $\sigma < \tau$. This incidence number is roughly speaking the number of times that τ (along its boundary) is wrapped around σ, taking the induced orientation from τ onto σ into consideration.

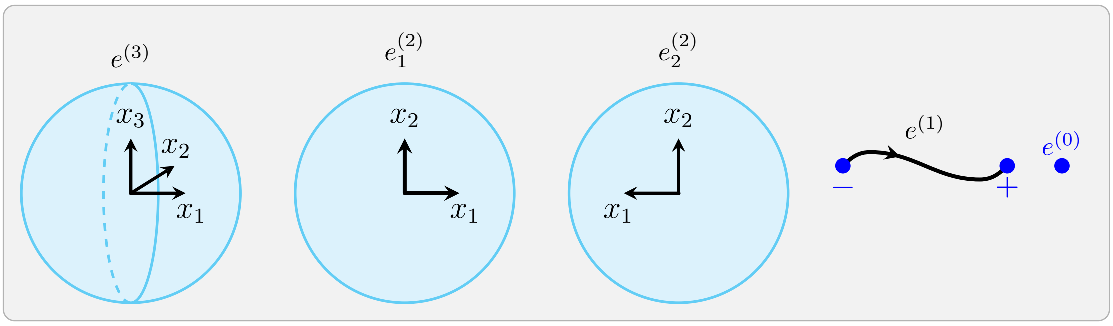


Figure 3.2: Initial orientation for each cell.

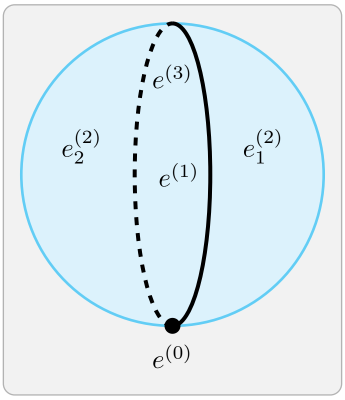


Figure 3.3: Constructed CW complex.

It should be noted that if we have an orientation on a given n-cell determined by coordinates $x_1, \cdots, x_n$, the induced orientation on its facets is determined in the following way: we consider the first coordinate x_1 to be an outer normal to the tangent space of the inverse image of this facet under the characteristic map of the n-cell. Then we take the remaining coordinates, that is $y_1 := x_2, \cdots, y_{n-1} := x_n$, to generate the induced orientation on the inverse image of that facet (under the characteristic map of the n-cell). Mapping this orientation back the to facet using the characteristic map of the n-cell gives us an induced orientation on the facet. This is always possible since the image of the closed disc under the characteristic map is the closed cell, which is the cell together with all its faces.

The case of an edge is already shown in the Figure 3.2, in the sense that we use the convention that if a vertex has the arrow of the orientation coming towards it, we give it a $+$ sign; if not then a $-$ sign.

One then easily sees that after gluing everything together, see Figure 3.3, $e^{(0)}$ will have two induced orientations from $e^{(1)}$: one being $+$ and the other $-$. So that the incidence number of $e^{(1)}$ and $e^{(0)}$, denoted $[e^{(1)} : e^{(0)}]$, will be $[e^{(1)} : e^{(0)}] = +1 - 1 = 0$. The other incidence numbers are illustrated in Figure 3.4, where, the induced orientation from $e^{(3)}$ on $e_1^{(2)}$ and the initial orientation of $e_1^{(2)}$ coincide and the same applies to $e_2^{(2)}$. This yields $[e^{(3)} : e_1^{(2)}] = 1 = [e^{(3)} : e_2^{(2)}]$. The induced orientation from $e_2^{(2)}$ on $e^{(1)}$ coincides with the initial orientation on $e^{(1)}$, whereas the one induced from $e_1^{(2)}$ does not, this gives $[e_2^{(2)} : e^{(1)}] = 1$ and $[e_1^{(2)} : e^{(1)}] = -1$.

We shall see later on, when we define cellular homology, that with these incidence numbers the square of the boundary operator is zero as expected.

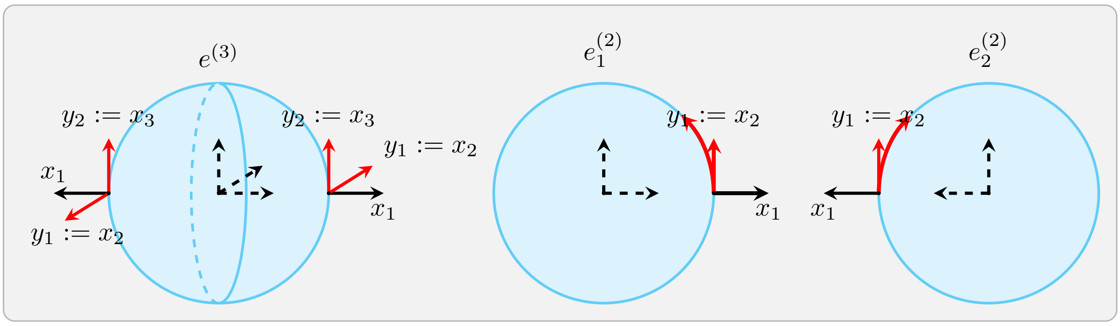


Figure 3.4: The orientation of a cell induces an orientation on each of its faces.

Remark 3.2.1. For each cell $e^{(n)}$, the above procedure is applied to the n-disc and then mapped to $e^{(n)}$ via the its characteristic map φ. To get the induced orientation of a facet of $e^{(n)}$, one uses the fact that $\varphi(D^n) = \bar{e}^{(n)}$.

In the case of simplicial complexes, the orientation is easily determined by a given order of the vertices, this is explained in the following remark.

Remark 3.2.2. Orientation of simplicial complexes:
Let $\sigma = \{v_0, v_1, \cdots, v_d\}$ be a d-simplex, there are $(d+1)!$ ways of ordering its vertices, and an orientation of a simplex is determined by ordering its vertices.
A permutation is said to be even if it consists of an even number of transpositions.
We will say that two orderings are equivalent if they differ by an even permutation.
This defines an equivalence relation on the set of orderings of σ. There are exactly two equivalence classes for each σ, each of which is called an orientation of σ, one corresponds to $+$ and the other to $-$.

The consistency of the orientation of a simplicial complex is such that, if for $i = 1, 2$, $\sigma_i < \tau$ and $v < \sigma_i$, then the orientations on v induced by the induced orientation of τ on σ_1 and σ_2 are different.

We also know from the definition of a polytope (since a simplex is also a polytope) that if $v^{(k-2)}$ and $\tau^{(k)}$ are incident there will always exist σ_1 and σ_2, $\sigma_1 \neq \sigma_2$ such that $v < \sigma_i < \tau$, for $i = 1, 2$.

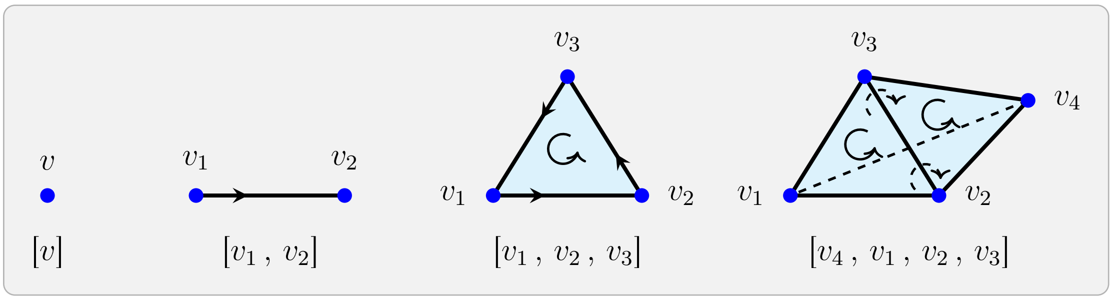


Figure 3.5: The orientation of a simplex induces an orientation on each of its faces.

From Figure 3.5, we see that the edge $\{v_1, v_2\}$ has two different induced orientations, $[v_1, v_2]$ and $[v_2, v_1]$, with respect to the oriented triangles $[v_1, v_2, v_3]$ and $[v_4, v_2, v_1]$ respectively. Note that the orientations in this case are given by the natural boundary operator on simplicial complexes (see Definition 3.2.5 below). On this note we move to the next subsection which is about cellular homology.

Cellular homology

We can now give the definition of cellular homology but first, we need to recall some facts about singular homology, see [47] for more details.

We recall that for a subspace A of a topological space X, one has the short exact sequence:

$$0 \to C_*(A) \to C_*(X) \to C_*(X)/C_*(A) \to 0,$$

where $C_*(A)$ denotes the singular chains on A and $C_*(X)$ denote the singular chains on X. From this, one defines the relative singular homology groups over a principal ideal domain R (or a commutative Ring)

$$H_n^S(X, A; R) = H_n^S(C_*(X)/C_*(A); R), \quad \text{for } n = 0, 1, \cdots .$$

In the sequel, when we write $H^S(X)$ we mean $H^S(X; R)$.

Recall that if we have a CW complex X, $X^{(n)}/X^{(n-1)}$ is homotopy equivalent to $\bigwedge_{\alpha \in \mathcal{A}^n} \mathbb{S}^n$, where $\mathcal{A}^n$ is the index set of the n-cells of our CW complex X. This is equivalent to saying that $H_n^S(X^{(n)}, X^{(n-1)})$ is isomorphic to $\bigoplus_{\alpha\in\mathcal{A}^n} H_n^S(D^n, \mathbb{S}^{n-1})$. Some properties of singular homology are given in the following proposition.

In what follows, R is a principal ideal domain.

Proposition 3.2.4. *(i)* $H_n^S\Big(\coprod_{\alpha\in\mathcal{A}} X_\alpha; R\Big) \cong \bigoplus_{\alpha\in\mathcal{A}} H_n^S(X_\alpha; R)$.

(ii) For $2 \leq k \leq n$,

$$\begin{aligned} H_k^S(D^n, \mathbb{S}^{n-1}; R) &\cong H_{k-1}^S(\mathbb{S}^{n-1}; R) \\ &\cong \begin{cases} R & \text{for } k = n \\ 0 & \text{for } k \neq n. \end{cases} \end{aligned}$$

(iii) The map $i\colon X^{(n)} \hookrightarrow X$ *induces an isomorphism* $i_*\colon H_k(X^{(n)}) \to H_k(X)$.

(iv)

$$H_k^S(X^{(n)}, X^{(n-1)}; R) \cong \begin{cases} 0 & \text{for } k \neq n \\ \bigoplus_{\alpha\in\mathcal{A}^n} R & \text{for } k = n. \end{cases}$$

We can now give the definitions of cellular homology both from the abstract point of view and the geometric one. We are mostly following [25]. It should be noted that for both versions of the boundary operator, we only provide the proof in the abstract case .

Abstract cellular homology

The abstract theory of cellular homology uses that of singular homology. By abstract, we mean that no effect of orientation whatsoever is of importance to this way of defining cellular homology.

Define the chain complex $C_n(X; R) := H_n^S(X^{(n)}, X^{(n-1)}; R)$. From the previous proposition it is a free R-module and its elements are called the cellular chains in X over R.

Proposition 3.2.5. *(i)* $C_n(X; R) \cong \bigoplus_{\alpha\in\mathcal{A}^n} R$.

(ii) $C_n(X; R) \cong \bigoplus_{\alpha\in\mathcal{A}^n} H_n^S(\bar{e}_\alpha^{(n)}, \partial^{top}\bar{e}_\alpha^{(n)}; R)$.

Proof. It follows from the previous proposition and the previous assertions in the sense that,
$H_n^S(X^{(n)}, X^{(n-1)}) \cong \bigoplus_{\alpha\in\mathcal{A}^n} H_n^S(D^n, \mathbb{S}^{n-1}) \cong \bigoplus_{\alpha\in\mathcal{A}^n} H^S(\bar{e}_\alpha^{(n)}, \partial^{top}\bar{e}_\alpha^{(n)})$. □

We recall that in general for a pair (X, A) we have the long exact sequence

$$\cdots \to H_k^S(A) \xrightarrow{i_*} H_k^S(X) \xrightarrow{j_*} H_k^S(X,A) \xrightarrow{\partial_*} H_{k-1}^S(A) \xrightarrow{i_*} \cdots$$
$$\cdots \to H_0^S(A) \xrightarrow{i_*} H_0^S(X) \xrightarrow{j_*} H_0^S(X,A) \to 0 \tag{3.1}$$

where,
∂_* is the "connecting homomorphism" defined by: it maps a relative cycle $[z] \in H_k^S(X, A) = H_k^S(C(X)/C(A))$ to its boundary which is a cycle in A, that is,

$$\partial_*([z]) = [\partial z].$$

j_* maps a cycle $[\gamma] \in H_k^S(X)$ to a cycle whose boundary is in A, thus,

$$j_*[\gamma] = [\gamma].$$

Now if we consider the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$, we can extract the following:

$$H_n^S(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial_*} H_{n-1}^S(X^{(n-1)}) \xrightarrow{j_*} H_{n-1}^S(X^{(n-1)}, X^{(n-2)}) \xrightarrow{\partial_*} H_{n-2}^S(X^{(n-2)}) \tag{3.2}$$

The cellular boundary operator is defined as the composition $j_* \circ \partial_*$, more precisely we have the following definition.

Definition 3.2.2. *The cellular boundary operator, denoted ∂_n^c, is defined to be the composition of the two following maps:*

$$H_n^S(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial_*} H_{n-1}^S(X^{(n-1)}) \xrightarrow{j_*} H_{n-1}^S(X^{(n-1)}, X^{(n-2)}).$$

That is, $j_* \circ \partial_* := \partial_n^c \colon C_n(X; R) \to C_{n-1}(X; R)$.

The next proposition establishes the fact that the square of the cellular boundary operator is zero.

Proposition 3.2.6. $\partial^c \circ \partial^c = 0$.

Proof. Using (3.2), we have

$$\partial_{n-1}^c \circ \partial_n^c = (j_{*,n-2} \circ \partial_{*,n-1}) \circ (j_{*,n-1} \circ \partial_{*,n}),$$

but $\partial_{*,n-1} \circ j_{*,n-1} = 0$, indeed: for $[\gamma] \in H_{n-1}^S(X^{(n-1)})$, $j_*[\gamma] = [\gamma]$, where, $\partial\gamma \in X^{(n-2)}$. Thus,

$$\partial_{*,n-1} \circ j_{*,n-1}[\gamma] = \partial_{*,n-1}[\gamma] = [\partial\gamma] = 0.$$

□

Proposition 3.2.6 tells us that the family $\{C_n(X;R), \partial_n^c\}$ is a free chain complex over R. We now show how the same result can be achieved when dealing with the geometric viewpoint of cellular homology.

Geometric cellular homology

The geometric viewpoint of cellular homology is based on the fact that all the cells are oriented and the free R-modules are spanned by the oriented cells of the appropriate dimensions. Thus, let X be a CW complex in which each cell is endowed with an orientation (called initial orientation), and $C_n(X;R)$ the free R-module generated by the oriented n-cells of X.

As a formal definition for $[\tau^{(n)} : \sigma^{(n-1)}]$ we have the following: it is the degree of the compositions of the attaching map of τ and the quotient map that collapses $X^{(n-1)} \setminus \sigma$ into a point. Indeed, we have the following diagram:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[d, "\varphi_\tau"] & X^{(n-1)} \arrow[d, "q"] & & \mathbb{S}^{n-1} \\
\partial^{top}\bar{\tau} \arrow[ur, "i"] \arrow[r, "i\circ q"'] & X^{(n-1)}/(X^{(n-1)} - \sigma) & \bar{\sigma}/(\partial^{top}\bar{\sigma}) \arrow[l, "h_\sigma"'] & \mathbb{D}^{n-1}/\mathbb{S}^{n-2} \arrow[l, "\varphi'_\sigma"'] \arrow[u, "h_{n-1}"]
\end{tikzcd}$$

Definition 3.2.3. *The incidence number $[\tau : \sigma]$ is equal to the degree of the map*

$$h_{n-1} \circ {\varphi'_\sigma}^{-1} \circ h_\sigma^{-1} \circ (i \circ q) \circ \varphi_\tau \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}.$$

Note that all the above maps, with the exception of φ_τ and $i \circ q$, are obviously homeomorphisms. Also, φ'_σ is the homeomorphism induced by the characteristic map of σ.

Remark 3.2.3. If a given cell $\tilde{\sigma}^{(n-1)}$ is not in the topological boundary of τ, then from Definition 3.2.3, the corresponding map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ for $\tilde{\sigma}$ will not be surjective. This will then imply that it will be a map homotopic to a constant map, and will therefore have degree 0.

Definition 3.2.4 (Cellular boundary operator). *The cellular boundary operator, $\partial_n^c : C_n(X;R) \to C_{n-1}(X;R)$, is given by:*

$$\partial_n^c(\tau^{(n)}) = \sum_{\sigma<\tau} [\tau^{(n)} : \sigma^{(n-1)}]\sigma^{(n-1)}. \tag{3.3}$$

Roughly speaking, the definition given by (3.3) means that the cellular boundary operator of a given cell is the sum of its co-dimension one topological boundary elements (also called facets) counted with their multiplicity and also taking their orientations into account. This sum only contains finitely many elements since the attaching map of τ has compact image hence it can only intersect a finite number of cells of dimension $n-1$.

Remark 3.2.4. We recall that the CW complex is endowed with an orientation in the sense that, each of its cells is given an orientation called initial orientation. In the process of finding the cellular boundary of a given cell, its orientation will induce some orientations on its facets. If for a given cell this induced orientation coincides with its initial orientation, this cell appears with the sign $+$; if not, then it appears with the sign $-$.

Thus a chain $c \in C_n(X;R)$ can be written as

$$c := \sum_{\alpha \in \mathcal{A}^n} c_\alpha e_\alpha^{(n)},$$

where $\mathcal{A}^n$ indexes the n-cells, and only finitely many of the c_α's are non zero.

Similarly, $c \times I = \sum_\alpha c_\alpha (e_\alpha^{(n)} \times I) \in C_{n+1}(X \times I; R)$, see [25, P. 63] or [10].

Now we emphasize on the fact that in the case of a simplicial complex $\mathbb{K}$, the definition of the natural boundary operator of an oriented simplex is given in a simple form.

Let $C_d(\mathbb{K}; R)$ be the free R-module generated by the oriented d-simplices of $\mathbb{K}$.

Definition 3.2.5. *The natural boundary operator is the linear function defined for each oriented d-simplex σ by: $\partial_d : C_d(\mathbb{K}; R) \to C_{d-1}(\mathbb{K}; R)$,*

$$\partial_d(\sigma) = \partial_d[v_0, v_1, \cdots, v_d] = \sum_{i=0}^{d} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_d],$$

where the $\hat{v}_i$ means that the vertex v_i is omitted.

Note: The boundary operator allows the oriented simplex to induce an orientation on its faces as shown in Figure 3.5.

The square of this boundary operator is also zero as seen in the following theorem.

Theorem 3.2.7.

$$\partial \circ \partial = 0.$$

Proof.

$$\begin{aligned}
\partial_{d-1} \circ \partial_d(\sigma) &= \partial_{d-1}\Big(\sum_{i=0}^{d}(-1)^i[v_0, v_1, \cdots, \hat{v}_i, \cdots, v_d]\Big) \\
&= \sum_{0 \le i < j \le d-1} (-1)^j(-1)^i\,[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_d] \\
&+ \sum_{0 \le j < i \le d} (-1)^j(-1)^i\,[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_d] \\
&= \sum_{j=0}^{i-1} (-1)^{i+j}\,[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_d] \\
&+ \sum_{j=i+1}^{d} (-1)^{i+j-1}\,[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_d] \\
&= 0.
\end{aligned}$$

□

In the sequel, when we write $H(X)$ we mean $H(X; R)$.

We have a similar result as in singular homology about the fact that every cellular map has an induced chain map. It turns out that unlike for simplicial maps (which map simplices to simplices), the cellular maps just need to map skeletons to skeletons of the same dimension, rather than mapping cells to cells.

Definition 3.2.6. *A map f from two CW complexes X and Y is a cellular map if $f(X^n) \subset Y^n$, for all $n \geq 0$.*

The following definition then applies to both the geometric and abstract viewpoint of cellular homology and it is shown later that cellular homology coincides with singular homology (see also [27]).

Definition 3.2.7. *The cellular homology groups are given by:*

$$H_k(X) := \ker \partial_k^c / \operatorname{im} \partial_{k+1}^c, \ \textit{for all } k = 0, 1, \cdots n.$$

Theorem 3.2.8.

$$H_k(X) \approx H_k^S(X), \quad \textit{for all} \ \ k = 0, 1, \cdots n.$$

Proof. Using (3.2), the fact that $j_{*,n-1} \circ \partial_{*,n} = \partial_n^c$ and j_* is injective, we have:

$$\ker \partial_* = \ker \partial^c.$$

The First Isomorphism Theorem [26] and [15] yields

$$\operatorname{im} \partial_{*,n+1} / \ker j_{*,n} \cong \operatorname{im}(j_{*,n} \circ \partial_{*,n+1}) \quad \text{and} \quad H_n^S(X^{(n)}) / \ker j_{*,n} \cong \operatorname{im} j_{*,n},$$

but

$$\ker j_* = 0, \quad \operatorname{im}(j_{*,n} \circ \partial_{*,n+1}) = \operatorname{im} \partial^c_{n+1}, \quad \text{and} \quad \operatorname{im} j_{*,n} = \ker \partial_{*,n}.$$

Thus,

$$H_n^S(X^{(n)}) / \operatorname{im} \partial_{*,n+1} \cong \ker \partial^c_n / \operatorname{im} \partial^c_{n+1},$$

and the result follows since $H_n^S(X)$ can be identified with
$H_n^S(X^{(n)}) / \operatorname{im} \partial_{*,n+1}$. □

Proposition 3.2.9. *If $\{C_n(X), \partial^c_n\}$ and $\{C_n(Y), \partial'^c_n\}$ are the free chain complexes for the cell complexes X and Y respectively, then every cellular map $f : X \to Y$ induces a chain map $f_\sharp : C_n(X) \to C_n(Y)$, that is, $\partial'^c_n \circ f_\sharp = f_\sharp \circ \partial^c_n$.*

Let us denote an element $e^{(n)}_\alpha$ in X by $e^{(n)}_{x,\alpha}$, and an element $e^{(n)}_\beta$ in Y by $e^{(n)}_{y,\beta}$. In fact, the chain map $f_\sharp$ is given by:

$$f_\sharp(e^{(n)}_{x,\alpha}) = \sum_{e^{(n)}_{y,\beta} < f(e^{(n)}_{x,\alpha})} [f : e^{(n)}_{x,\alpha} : e^{(n)}_{y,\beta}] e^{(n)}_{y,\beta},$$

where $[f : e^{(n)}_{x,\alpha} : e^{(n)}_{y,\beta}]$, called the mapping degree of f with respect to $e^{(n)}_{x,\alpha}$ and $e^{(n)}_{y,\alpha}$, is the number of times that f maps $e^{(n)}_{x,\alpha}$ onto $e^{(n)}_{y,\alpha}$, and is formally defined as the following:

$$\begin{tikzcd}
 & e^{(n)}_\alpha \arrow[r, "f_{|e^{(n)}_\alpha}"] \arrow[d] & Y^n \arrow[d] & \\
\mathbb{D}^n/\mathbb{S}^{n-1} \arrow[r, "\varphi'_{x,\alpha}"] \arrow[d, "h_n"] & \bar{e}^{(n)}_{x,\alpha}/(\partial^{top}\bar{e}^{(n)}_{x,\alpha}) \arrow[r, "f'_{\alpha,\beta}"] & Y^n/(Y^n - e^{(n)}_{y,\beta}) & \bar{e}^{(n)}_{y,\beta}/(\partial^{top}\bar{e}^{(n)}_{y,\beta}) \arrow[l, "h_{y,\beta}"'] \\
\mathbb{S}^n & & \mathbb{S}^n & \mathbb{D}^n/\mathbb{S}^{n-1} \arrow[l, "h_n"] \arrow[u, "\varphi'_{y,\beta}"']
\end{tikzcd}$$

Definition 3.2.8. *The number $[f : e^{(n)}_{x,\alpha} : e^{(n)}_{y,\beta}]$ is the degree of the map*

$$h_n \circ {\varphi'_{y,\beta}}^{-1} \circ h^{-1}_{y,\beta} \circ f'_{\alpha,\beta} \circ \varphi'_{x,\alpha} \circ h_n^{-1} : \mathbb{S}^n \to \mathbb{S}^n.$$

Note that $f'_{\alpha,\beta}$ is the map induced by the cellular map f.
We now give some properties of cellular homology in the next subsection.

Some properties of cellular homology

Here, we just present some of the properties of cellular homology among those that are known from singular homology. For the proofs, we refer the reader to [47], [42], [25] and [27]. We mostly present the proofs for those related to the question of homotopy equivalence.

Let X be a CW complex, and R a commutative Ring.

Proposition 3.2.10. *$H_n(X;R) = 0$ for $n > \dim X$.*

Proof. Follows from the fact that if $k = \dim X$, we have $C_{k+l}(X,R) = 0$ for $l \geq 1$. □

Lemma 3.2.11. *$H_k(X^{(n)}, X^{(n-1)})$ is zero for $k \neq n$ and is free abelian for $k = n$ with a basis in one to one correspondence with the n-cells of X.*

Proof. See [27, P. 173]. □

Proposition 3.2.12. *If X is path connected, then $H_0(X;R) = R$.*

Proposition 3.2.13. *If X is of finite type, and R is a Principal Ideal Domain, then $H_n(X;R)$ is finitely generated for each n.*

We give some examples of homology groups for the most commonly used spaces.

Example 3.2.1.

$$H_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, n \\ 0 & \text{else} \end{cases}, \qquad H_k(\mathbb{T}^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } k = 1 \end{cases},$$

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd},\ 0 < k < n \\ 0 & \text{else} \end{cases},$$

$$H_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, 4, \cdots, 2n \\ 0 & \text{else.} \end{cases}$$

The following proposition establishes the relation between the homology groups of a pair (X, A) and those of the quotient X/A. For the proof, see [25, P. 66].

Proposition 3.2.14.

$$H_n(X/A;R) \cong \begin{cases} H_0(X,A;R)\oplus R & \textit{if } n=0 \\ H_n(X,A;R) & \textit{if } n\neq 0. \end{cases}$$

Example 3.2.2.

$$H_k(\mathbb{S}^n, pt;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=n \\ 0 & \text{else} \end{cases}, \qquad H_k(\mathbb{S}^n/pt;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=0,n \\ 0 & \text{else.} \end{cases}$$

$$H_k(\mathbb{T}^2, \mathbb{S}^1;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=1,2 \\ 0 & \text{else} \end{cases}, \qquad H_k(\mathbb{T}^2/\mathbb{S}^1;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=0,1,2 \\ 0 & \text{else.} \end{cases}$$

Proposition 3.2.15. *If $\{C_n(X), \partial_n^c\}$ and $\{C_n(Y), \partial_n'^c\}$ are the free chain complexes for the cell complexes X and Y respectively, then every cellular map $f: X \to Y$ induces an R-linear map $f_*: H_n(X) \to H_n(Y)$, for all $n \geq 0$.*

The next theorems provide the conditions under which the homology groups of spaces are isomorphic.

Theorem 3.2.16 (Homotopy Invariance). *Given two CW complexes X and Y, and two homotopic cellular maps, $f, g: X \to Y$, the induced maps on the homology groups are the same, that is $f_* = g_*$.*

Proof. The proof uses the following fact, see [25, P. 63], [10, P. 221]:

$$\partial(c^{(n)} \times [0,1]) = (-1)^n(c \times \{1\} - c \times \{0\}) + \partial(c) \times [0,1].$$

$f_* = g_*$ if and only if $f_*[x] = g_*[x]$ for every $[x] \in H_n(X)$, that is, $[f_\sharp(x)] = [g_\sharp(x)]$.
Thus, we need to show that $f_\sharp - g_\sharp = 0_{H_n(X)}$, that is, we can find a y s.t. $(f_\sharp - g_\sharp)(x) = \partial y$.
$f \simeq g$ implies there exists $F: X \times [0,1] \to Y$ s.t. $F(x,0) = f$ and $F(x,1) = g$. F also extends to a chain map $F_\sharp$.

$$\begin{aligned} \partial F_\sharp(x \times [0,1]) &= F_\sharp \partial(x \times [0,1]) \\ &= F_\sharp\Big((-1)^n(x \times \{1\} - x \times \{0\}) + (\partial x) \times [0,1]\Big) \\ &= (-1)^n(g_\sharp(x) - f_\sharp(x)), \\ &\quad \text{since } [x] \in H_n(X) \Rightarrow x \in Z_n(X) \text{ that is } \partial x = 0. \end{aligned}$$

So, $y = (-1)^{n+1}(F_\sharp(x \times I))$. □

Theorem 3.2.17. *The Homology groups are homotopy invariants that is, if X is homotopy equivalent to Y, then $H_n(X) \cong H_n(Y)$, for all $n \geq 0$.*

Proof. It follows from Theorem 3.2.16 in the sense that, X homotopy equivalent to Y means that, there exist functions $f : X \to Y$ and $g : Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. This then implies that $f_* \circ g_* = id_{H_*(Y)}$ and $g_* \circ f_* = id_{H_*(X)}$. Thus f_* and g_* are inverses of each other, therefore $H_*(X) \cong H_*(Y)$. □

Just as it is the case for simplicial complexes, there is also a Euler-Poincaré formula for CW complexes. Before its statement we recall that the Betti numbers denoted b_k, $k \geq 0$, of a CW complex are given by the dimensions of the homology groups, that is,

$$b_k := \dim H_k(X, R).$$

Theorem 3.2.18 (Euler-Poincaré formula). *Let X be a CW complex of dimension n, and let $C_k(X; R)$ be the free R-module generated by the (oriented) cells of dimension k, where R is a principal ideal domain. Then*

$$\sum_{k=0}^{n} (-1)^k \dim C_k(X) = \sum_{k=0}^{n} (-1)^k b_k.$$

The proof needs the following lemma:

Lemma 3.2.19. *If A, B, C are finitely generated R-modules (R a principal ideal domain), and the short sequence*

$$0 \xrightarrow{h_1} A \xrightarrow{h_2} B \xrightarrow{h_3} C \xrightarrow{h_4} 0$$

is "exact", that is, $\operatorname{im} h_i = \ker h_{i+1}$, *for* $i = 1, 2, 3$, *then* $\dim A + \dim C = \dim B$.

Proof. The exactness of the short sequence gives: h_2 injective, that is $\ker h_2 = 0$; h_3 surjective, that is $\operatorname{im} h_3 = C$. Also, $\operatorname{im} h_2 = \ker h_3$ and the First Isomorphism Theorem, see [26] and [15], yields

$$\operatorname{im} h_2 \cong A/\ker h_2 \quad \text{and} \quad \operatorname{im} h_3 \cong B/\ker h_3.$$

This then gives $C \cong B/A$ and the result follows. □

Proof of Euler-Poincaré formula. From the chain complex and the definition of the homology groups, we have the following two short exact sequences:

$$0 \to \ker \partial_k^c \hookrightarrow C_k \xrightarrow{\partial_k^c} \operatorname{im} \partial_k^c \to 0,$$

$$0 \to \operatorname{im} \partial_{k+1}^c \hookrightarrow \ker \partial_k^c \to H_k \to 0.$$

Applying Lemma 3.2.19, we have

$$\dim C_k = \dim \ker \partial_k^c + \dim \operatorname{im} \partial_k^c,$$

$$\dim \ker \partial_k^c = \dim \operatorname{im} \partial_k^c + \dim H_k.$$

Adding the two equations, multiplying by $(-1)^k$ and taking the sum over all k from 0 to n yields

$$\sum_{k=0}^{n} (-1)^k \dim C_k = \sum_{k=0}^{n} (-1)^k \big(\dim H_k + (\dim \operatorname{im} \partial_k^c + \dim \operatorname{im} \partial_{k+1}^c)\big).$$

And the result follows using the fact that, when the dimension of the CW complex is n, $\operatorname{im} \partial_{n+1}^c = 0$. □

We now have all the necessary preliminary tools needed to understand both the operations on CW complexes as well as their topology. Thus, not only do we know how cellular homology is computed, we also know exactly what type of operations preserve or not the homotopy types of CW complexes. We can now recall the different Morse-related theories in the smooth setting. This is the content of the next chapter.

4

Morse, Morse-Floer, Morse-Bott and Conley theories

This chapter is about the smooth theories of Morse and the Morse-related ones, and how the Poincaré polynomial of a given cell complex can be retrieved using any one of them which include, Morse theory, Morse-Floer theory, Morse-Bott theory and (Morse-)Conley theory. In Morse-Floer theory, finding the Betti numbers is achieved through finding a boundary operator associated to the negative gradient flow lines of a Morse function. While in Morse theory the critical points have to be isolated, the scope of Morse-Bott theory is beyond that of Morse theory in the sense that, it allows functions that admit critical submanifolds and not only critical points. (Morse-)Conley theory focuses on the isolated critical points together with their isolating neighborhood as well as their exit set for a given flow (not necessarily originating from a Morse function), thus it uses the dynamics of flow lines.

Section 4.1 will focus on Morse theory while Section 4.2 is about Morse-Floer theory and and how the boundary operator is defined counting the negative gradient flow lines of a Morse function. Finally, in Section 4.3 and Section 4.4, we give the respective ideas on smooth Morse-Bott theory and smooth (Morse-)Conley theory respectively.

In this chapter, M denotes a compact smooth finite-dimensional manifold without boundary.

4.1 Morse theory

In this section we briefly recall the notion of Morse theory, we refer the reader to [40], [44], and [37].

Whenever we have a Morse function on a given smooth manifold, we can always construct a CW complex from the critical points of the function, in such a way that the manifold and the CW complex are homotopy equivalent. Also, the cellular homology of the CW complex and the singular homology of the manifold are isomorphic. The Morse inequalities not only provide some upper bounds for the Betti numbers of the manifold but also ensure that one can always retrieve the Euler number of the manifold just from the Morse numbers. Let us first recall the following facts about smooth functions defined on smooth manifolds, see also [31] and [11].

Let M be a smooth, m-dimensional manifold and $f\colon M \to \mathbb{R}$ a smooth function.

A point p is said to be a *critical* point of f if $df_p \equiv 0$, where $df_p\colon T_pM \to \mathbb{R}$ is the differential of f at p, and T_pM is the tangent space to the manifold at p. Note that $T_{f(p)}\mathbb{R} = \mathbb{R}$.

Thus, if we take a local chart around p, $\varphi\colon U \to \mathbb{R}^m$, where U is an open neighborhood of p, with $\varphi(x) = (x_1, \cdots, x_m)$, we have:

$$\frac{\partial}{\partial x_j}(f \circ \varphi^{-1})(\varphi(p)) = 0 \quad \text{for all} \quad j = 1, \cdots, m.$$

The *Hessian* of f at the critical point p is a symmetric bilinear map $H_p(f)\colon T_pM \times T_pM \to \mathbb{R}$.

Using φ, the coordinate chart around p, with $\frac{\partial}{\partial x_1}|_p, \cdots, \frac{\partial}{\partial x_m}|_p$ as a basis for T_pM, we can write the matrix of $H_p(f)$ with respect to this basis (with an abuse of notation) as

$$H_p(f) = \Big(\frac{\partial^2(f \circ \varphi^{-1})}{\partial x_i \partial x_j}(\varphi(p))\Big).$$

The following definition gives the conditions under which a given smooth function is said to be Morse.

Definition 4.1.1 (Morse function). *A smooth function $f\colon M \to \mathbb{R}$ is said to be* ***Morse*** *if all its critical points are nondegenerate, that is, $H(f)$ is nondegenerate at every critical point.*

An important notion needed for the Morse inequalities is the index of a given critical point of the Morse function.

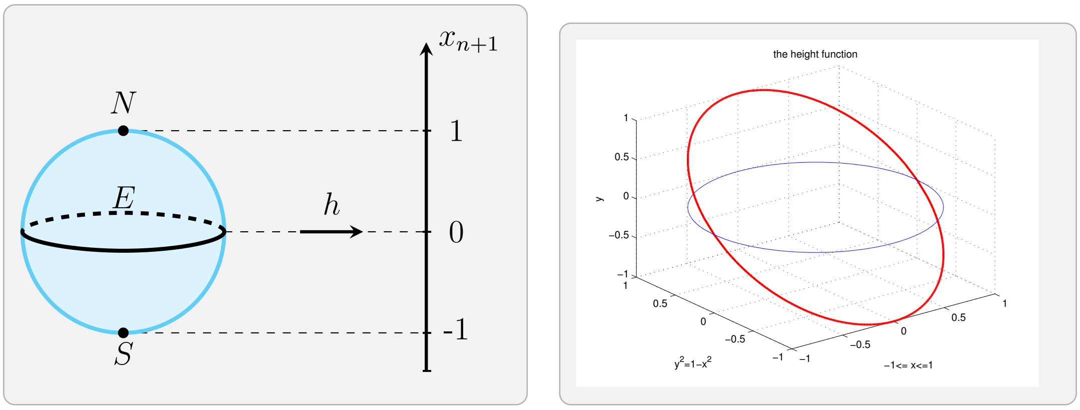
Definition 4.1.2 (Index). *If p is critical, we define the index of p, denoted $ind(p)$, to be the dimension of the largest subspace of T_pM on which $H_p(f)$ is negative definite that is the number of negative eigenvalues of the corresponding matrix.*

One of the most important properties about the critical points of a Morse function is that:

The nondegenerate critical points of a Morse function are isolated.

Example 4.1.1. Figure 4.1a provides an example of a Morse function, the height function on the sphere. The critical points here are the North and South pole, and they are nondegenerate. One can check that: the Hessian at N is nondegenerate and has 2 negative eigenvalues, so N has index 2; at S the Hessian has no negative eigenvalues, so S has index 0.

An example of a function which is not Morse would be the one obtained by taking the square of the height function, see Section 4.3.



(a) The height function on $\mathbb{S}^2$.

(b) Graphical representation on $\mathbb{S}^1$.

Figure 4.1: The height function and a graphical representation.

The following theorem establishes the fact that the homotopy types of the sublevel sets, $M^t := \{x \in M \mid f(x) \leq t\}$, changes whenever one crosses a critical point of the Morse function.

Theorem 4.1.1. *Let M be a smooth, finite-dimensional manifold without boundary, and let $f\colon M \to \mathbb{R}$ be a smooth Morse function. Suppose for $a < b$, $\{x \in M \,|\, a \leq f(x) \leq b\} := M_a^b$ is compact and contains no critical point, then for $a \leq t < b$, the sublevel set M^t is diffeomorphic to M^b, and M^a is a deformation retract of M^b.*

If M_a^b contains one nondegenerate critical point of index k, then M^b has the homotopy type of M^a with one k-cell attached, in fact M^b deformation retracts into $M^a \cup e^k$.

Proof. See [7, P. 64]. □

We have the following important theorem for the homotopy type of a CW complex associated to a Morse function:

Theorem 4.1.2. *Let M be a smooth, finite-dimensional manifold without boundary, and let $f\colon M \to \mathbb{R}$ be a smooth Morse function. Then M is homotopy equivalent to a CW complex with a k-cell for each critical point of f of index k.*

Proof. See [7, P. 69]. □

We illustrate this by considering the height function on the torus $\mathbb{T}^2$ lying vertically, see Figure 4.2. The critical points are: s of index 0, r of index 1, q of index 1 and p of index 2.

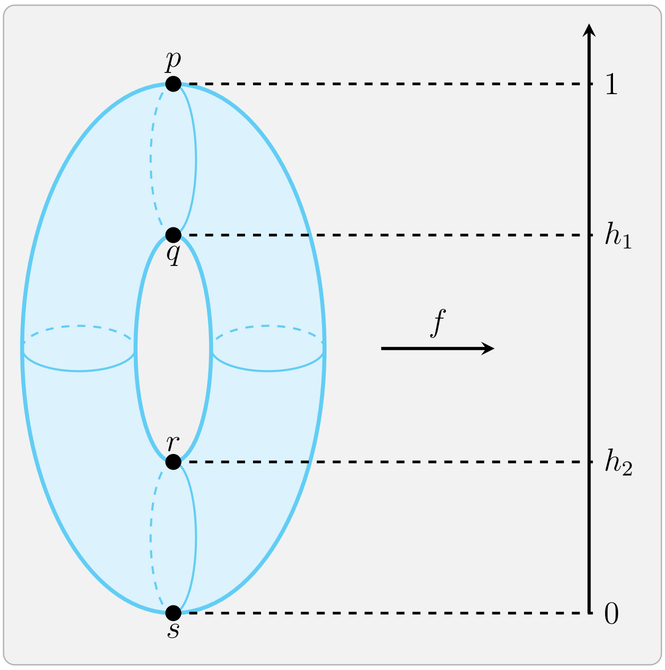


Figure 4.2: A Morse function.

The critical point s has index 0, and $M^0 = \{s\}$. Thus, for $0 \leq t_0 < h_2$, M^{t_0} is homeomorphic to a disc and is homotopy equivalent to a point, that is, M^{t_0} deformation retracts into M^0, see Figure 4.3.

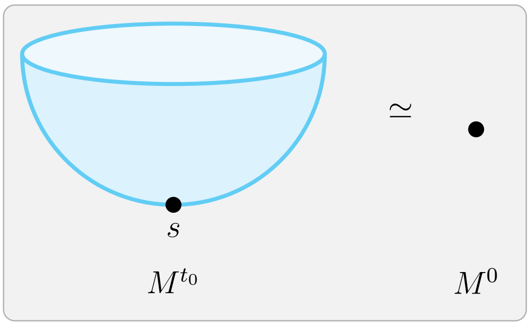


Figure 4.3: Homotopy equivalence between M^0 and M^{t_0}.

The critical point r has index 1, and for $h_2 \leq t_1 < h_1$, M^{t_1} is homotopy equivalent to $\mathbb{S}^1$, in fact, M^{t_1} is homotopy equivalent to M^{t_0} with a 1-cell attached and it deformations retracts into $M^0 \cup e_1^{(1)}$, where the 1-cell is denoted by $e_1^{(1)}$. This is illustrated in Figure 4.4.

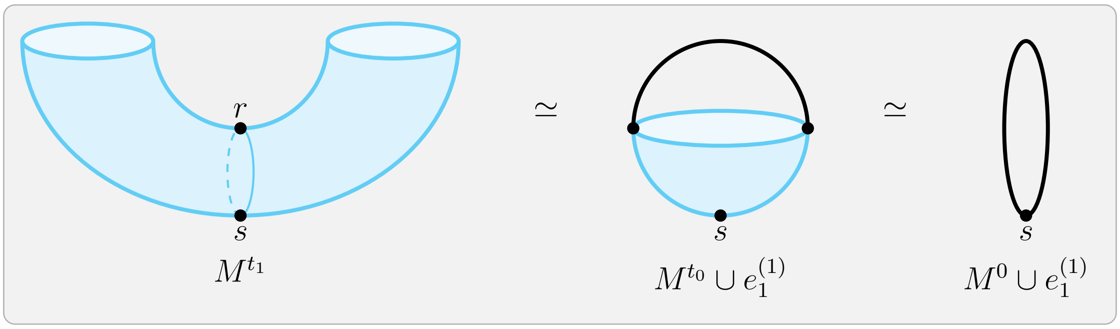


Figure 4.4: Homotopy equivalence between M^{t_1} and $M^0 \cup e_1^{(1)}$.

For the critical point q also of index 1, we have that, for $h_1 \leq t_2 < 1$, M^{t_2} is homotopy equivalent to $\mathbb{S}^1 \wedge \mathbb{S}^1$, it is in fact homotopy equivalent to M^{t_1} with a 1-cell, $e_2^{(1)}$, attached and it deformation retracts into $M^{h_2} \cup e_2^{(1)}$, see Figure 4.5.

The point p is critical of index 2, and, for $t_3 = 1$, M^{t_3} is homotopy equivalent to $\mathbb{T}^2$, in fact it is homotopy equivalent to M^{t_2} with a 2-cell, $e^{(2)}$, or $M^{h_1} \cup e^{(2)}$. This is shown in Figure 4.6.

Before going to the next important fact about Morse theory, namely the Morse inequalities, we give an important lemma which states that, at a given nondegenerate critical point, a Morse function assumes in a local coordinates chart a certain canonical form which is helpful in determining the index of that critical point. For the proof or more insight see [40].

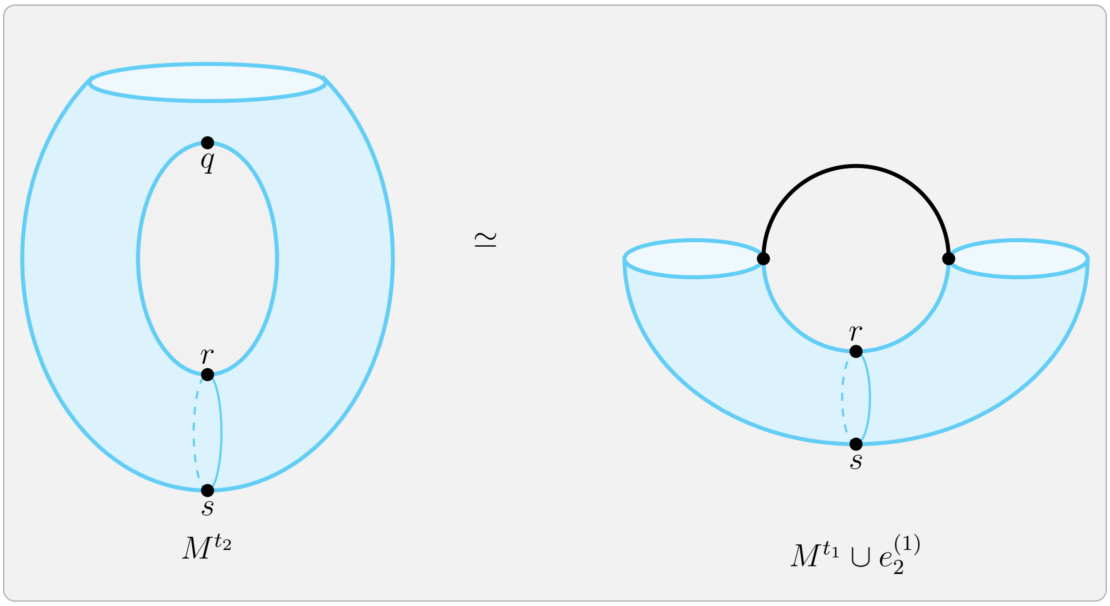


Figure 4.5: Homotopy equivalence between M^{t_2} and $M^{h_2} \cup e_2^{(1)}$.

Lemma 4.1.3 (Morse Lemma). *Let $f\colon M \to \mathbb{R}$ be a smooth Morse function and p a nondegenerate critical point of index k. There exists a smooth chart $\varphi\colon U \to \mathbb{R}^m$ around p with $\varphi(p) = 0$ s.t.*

$$(f \circ \varphi^{-1})(x_1, \cdots, x_m) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2.$$

We now state the most important theorem of Morse theory namely the Morse inequalities, see also [38].

Let m_i be the number of critical points of index i, and b_i the i^{th} Betti number.

Note that to determine the Betti numbers of the manifold, we may triangulate the manifold to obtain a simplicial complex and compute the homology groups of this simplicial complex, but one has to prove that these homology groups are independent of the triangulation. Thus, to avoid the problem of equivalence of triangulations, one looks at singular homology or cellular homology.

Theorem 4.1.4 (Morse inequalities). *For $k = 0, 1, \cdots, n$,*

(i) (Weak): $m_k \geq b_k$,

$$m_0 - m_1 + m_2 - ... \pm m_n = b_0 - b_1 + b_2 - ... \pm b_n = \chi(M).$$

(ii) (Strong):

$$m_k - m_{k-1} + m_{k-2} - \ldots \pm m_0 \geq b_k - b_{k-1} + b_{k-2} - \ldots \pm b_0.$$

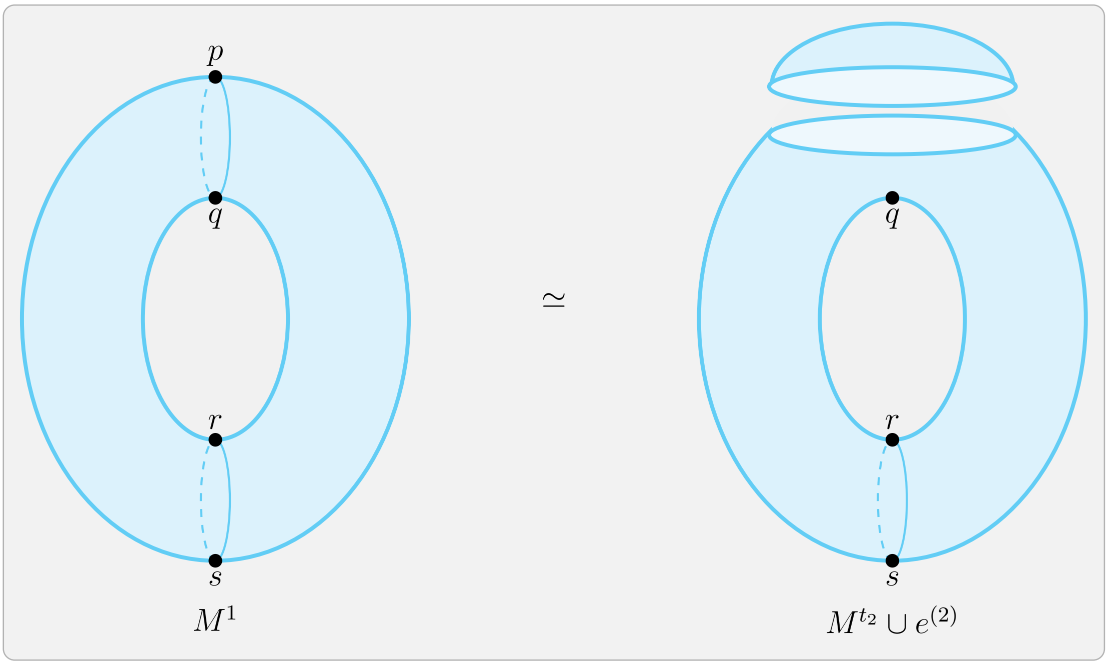


Figure 4.6: Homotopy equivalence between $\mathbb{T}^2$ and $M^{h_1} \cup e^{(2)}$.

Since the idea of the proof for the above theorem is the same both in the smooth and discrete settings, a proof is given in Section 5.2.

We now move to the next part of this chapter which is about Morse-Floer theory, where, we will see how Floer's boundary operator is defined.

4.2 Morse-Floer theory

Floer homology is based on the fact that, a boundary operator is defined by counting the negative gradient flow lines of a Morse function, moving from one critical point of a given index, going downwards to another one of index difference one. For more insight see [19], [44], [7] and [31].

We recall, see [31] or [11], that the gradient vector field of a smooth function $f\colon M \to \mathbb{R}$, where (M, g) is a Riemannian manifold with metric g, denoted ∇f, is the vector field dual to df given by:

$$g(\nabla f, V) = df(V) = V(f), \quad \text{for all vector fields } V.$$

The negative gradient flow lines are given by $\gamma_x(t) = \phi_t(x)$, where the local 1-parameter family of diffeomorphisms $\phi_t\colon M \to M$ is generated by the negative gradient vector field of f. That is,

$$\begin{aligned}\frac{d}{dt}\phi_t(x) &= -(\nabla f)(\phi_t(x)),\\ \phi_0(x) &= x.\end{aligned}$$

That is, the negative gradient flow lines between two critical points p and q are given by:

$$\frac{d}{dt}\gamma(t) = (-\nabla f)(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = p, \quad \lim_{t\to+\infty}\gamma(t) = q.$$

The unstable manifold of p denoted $W^u(p)$ and the stable manifold of q denoted $W^s(q)$ are given by:

$$W^u(p) = \{x \in M \text{ s.t. } \lim_{t\to-\infty}\phi_t(x) = p\} \quad \text{and} \quad W^s(q) = \{x \in M \text{ s.t. } \lim_{t\to\infty}\phi_t(x) = q\}.$$

The space of flow lines between the points p and q is

$$M(p,q) := W^u(p) \cap W^s(q).$$

The important condition in Floer theory is, the given function should be Morse-Smale. This is the content of the following definition.

Definition 4.2.1. *A Morse function on a finite-dimensional smooth Riemannian manifold M is said to be* ***Morse-Smale*** *if and only if, for every two critical points p, of index k, and q of index $k-1$, the unstable manifold of p and the stable manifold of q intersect* ***transversally****, that is for all $x \in M(p,q)$,*

$$T_xM = T_xW^u(p) \oplus T_xW^s(q).$$

We have the following, assuming f is Morse-Smale on M with critical points p and q.

Proposition 4.2.1. *If $W^u(p) \cap W^s(q) \neq \emptyset$ then $M(p,q)$ is an embedded submanifold of M of dimension* $\mathrm{ind}(p) - \mathrm{ind}(q)$.

Corollary 4.2.2. *The index of the critical points of f is strictly decreasing along the flow lines, that is, if $W^u(p) \cap W^s(q) \neq \emptyset$ then* $\mathrm{ind}(p) > \mathrm{ind}(q)$.

Thus, whenever $f\colon M \to \mathbb{R}$ is a Morse-Smale function, the space of negative gradient flow lines from a critical point p to another critical point q is of dimension $\mathrm{ind}(p) - \mathrm{ind}(q)$. Also, if $\mathrm{ind}(p)$-$\mathrm{ind}(q)$=1, we have that $M(p,q)$ is a submanifold of dimension 1. Note that $\mathbb{R}$ acts on the set of flow lines $M(p,q)$ by translations, so that the space

$$\mathcal{M}(x,y) = M(x,y)/\mathbb{R}$$

is a compact manifold of dimension $\mathrm{ind}(p) - \mathrm{ind}(q) - 1$, hence zero-dimensional, that is, it consists of a finite number of elements which are just the flow lines from p to q. Floer's boundary operator counts the number of flow lines between two critical points of index difference 1 (there are finitely many of them).

We can now give the definition of Floer's boundary operator. Let $C_k(M;R)$ be the free R-module generated by the critical points of index k, over a principal ideal domain R.

Definition 4.2.2 (Floer). *The boundary operator $\partial_k \colon C_k(M;R) \to C_{k-1}(M;R)$ is given by:*

$$\partial(p) = \sum_{ind(q)=ind(p)-1} n(p,q)q,$$

where $n(p,q)$ is the number of flow lines between p and q.

This sum can either be taken modulo 2 (in $\mathbb{Z}_2$) that is $n(p,q)$ is either 0 or 1, or in $\mathbb{Z}$. For Floer's theory in $\mathbb{Z}$, one needs to consider orientable manifolds and the notion of orientation of the flow lines, for more details on this refer to [44] or [7].

We have the following theorem establishing the fact that the square of the boundary operator defined above is zero.

Theorem 4.2.3 (Floer).

$$\partial \circ \partial = 0.$$

The main idea behind this theorem is the fact that (see [32], [44], [19], [31]),

the broken trajectories, that is, the trajectories from the critical points of index k to those of index $k-2$ via the critical points of index $k-1$, always occur in pairs. And each of these broken trajectories is the limit of an infinite number of unbroken ones.

Define the Betti numbers to be

$$b_i := \dim\left(\ker \partial_i / \operatorname{im} \partial_{i+1}\right).$$

We illustrate how to compute ∂ in $\mathbb{Z}_2$, using Figure 4.7. Where, we consider a Morse function (the height function) on a deformed sphere, and have that N_1 and

N_2 both have index 2, q has index 1 and S has index 0 since it is the minimum. The action of the boundary operator is given by:

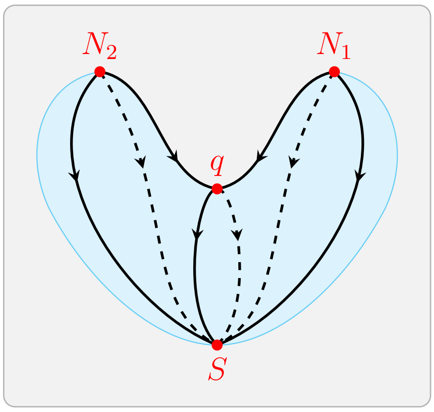


Figure 4.7: Flow lines on a deformed sphere.

$\partial N_1 = q = \partial N_2$;
$\partial q = S + S = 2S = 0$;
$\partial S = 0$;
thus,
$\ker \partial_0 = \langle\, S \,\rangle, \quad \operatorname{im} \partial_1 = 0$;
$\ker \partial_1 = \langle\, q \,\rangle = \operatorname{im} \partial_2$;
$\ker \partial_2 = \langle\, N_1 + N_2 \,\rangle, \quad \operatorname{im} \partial_3 = 0$.
One checks easily that $\partial \circ \partial = 0$, and also, the corresponding Betti numbers, $b_0 = 1$; $b_1 = 0$; and $b_2 = 1$, are the right ones.

Now, if instead of having isolated critical points a given function admits critical manifolds, this brings us to the generalization of Morse theory that is Morse-Bott theory.

4.3 Morse-Bott theory

We now consider functions whose set of critical points are not discrete set of points but rather smooth manifolds. This is the idea behind Morse-Bott theory, see [8], [9] and [7]. The Poincaré polynomial of the manifold is retrieved by adding those of the critical submanifolds taking their respective indices into account, up to some correction term.

Consider for example a torus lying horizontally on a plane, and the height function defined on it, see Figure 4.8. The critical sets are the two circles C_1 and C_2, one of which is mapped to the maximum and the other to the minimum.

Now, let M be a differentiable manifold and suppose f is a smooth function on M whose critical set contains a submanifold C of dimension greater than zero. Pick a Riemannian metric g on M (this is possible since M is a smooth manifold, see [31]) and use it to split $T_*M_{|C}$ as

$$T_*M_{|C} = T_*C \oplus \nu_*C,$$

where T_*C is the tangent space of C and ν_*C is the normal bundle of C.

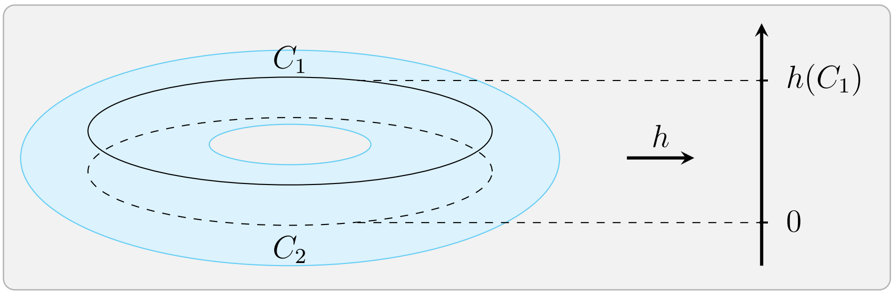


Figure 4.8: A Morse-Bott function.

Let $p \in C$, $V \in T_pC$, and $W \in T_pM$, and let $H_p(f)$ be the Hessian of f at p. We have:

$$H_p(f)(V,W) = \widetilde{V}(\widetilde{W}(f))(p) = V_p(\widetilde{W}(f)) = 0.$$

Since $V_p \in T_pC$ and any vector field $\widetilde{W}$ which is an extension of W will satisfy $df(\widetilde{W})_{|C} = 0$. Indeed, p critical implies that $df_p(V) = (V(f))(p) = g(\nabla f, V)(p) = 0$ for every $V \in T_pM$. Thus, $\widetilde{W}(f_{|C}) = 0$, which implies $H_p(f) = 0$ on T_pC.

Since the Hessian is symmetric in V and W, $H_p(f)$ induces a symmetric bilinear form, $H_p^\nu(f)$, on ν_pC. Because the Hessian vanishes on the tangential component, we define a Morse-Bott function to be one for which the Hessian restricted to the normal component satisfies the Morse conditions. This is the content of the following definition.

Definition 4.3.1 (Morse-Bott function). *Let M be a smooth manifold. A smooth function $f\colon M \to \mathbb{R}$ is said to be a Morse-Bott function if and only if, the set of critical points of f, $Cr(f)$, is a disjoint union of connected submanifolds, and for each connected submanifold $C \subseteq Cr(f)$, the bilinear form $H_p^\nu(f)$ is nondegenerate for all $p \in C$.*

Example 4.3.1.

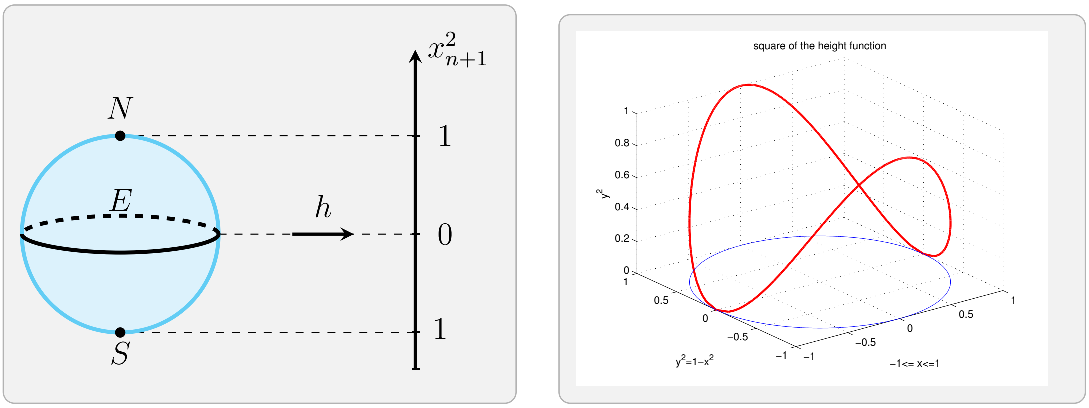
1) Every Morse function $f\colon M \to \mathbb{R}$ is Morse-Bott since, $C_i = p_i$, has dimension 0 so that, $T_*M_{|C_i} = \nu_*C_i$ and $H_{p_i}(f) = {H^\nu}_{p_i}(f)$ is nondegenerate for all $p_i \in Cr(f)$.

2) The example given in Figure 4.8 is a Morse-Bott function.

3) Every constant function $f\colon M \to \mathbb{R}$, $f \equiv c$ is a Morse-Bott function, here $Cr(f) = M$ and $\nu_*M = \{0\}$.

4) The square of the height function defined on a sphere is also a Morse-Bott function, see Figure 4.9a. The critical points are the North pole N and South pole S, and the critical submanifold is the Equator E. To be able to visualize

what really happens in this case, we have a look at the graph of this function, the simplest situation is when we consider $\mathbb{S}^1$, see Figure 4.9b.



(a) A Morse-Bott function on $\mathbb{S}^2$. (b) Graphical representation on $\mathbb{S}^1$.

Figure 4.9: The square of the height function and a graphical representation.

Theorem 4.3.1 (Morse-Bott Lemma). *Let f be a Morse-Bott function, and $C \subset Cr(f)$ a connected component. For any $p \in C$, there is a local chart around p and a local splitting $\nu_* C = \nu_*^+ C \oplus \nu_*^- C$, identifying a point $x \in M$ in its domain to (u, v, w), where $u \in C, v \in \nu_*^- C$ and $w \in \nu_*^+ C$, such that within this chart f assumes the form*

$$f(x) = f(u, v, w) = f(C) - |v|^2 + |w|^2.$$

Theorem 4.3.1 tells us that there is a well defined Morse-Bott index $\lambda_C \in \mathbb{Z}_+$ for each C, this is the content of the following definition.

Definition 4.3.2. *Let M be a smooth, finite-dimensional manifold and let $f\colon M \to \mathbb{R}$ be Morse-Bott. Let $C \subset Cr(f)$. For any $p \in C$, let λ_p be the index of p, that is, the number of negative eigenvalues of $H_p^\nu(f)$. The Morse-Bott Lemma, see also [7], establishes the fact that λ_p is locally constant. The connectedness of C tells us that λ_p is constant in C. Define the index of C to be $\lambda_C = \lambda_p$.*

Note that since the Hessian on C vanishes on the tangential component, the integer λ_p is the dimension of the normal bundle $\nu_p C$.

We now state the analogue in this setting of the Morse inequalities, namely the Morse-Bott inequalities.

Theorem 4.3.2 (Morse-Bott inequalities). *Let M be a smooth, finite-dimensional manifold and $f\colon M \to \mathbb{R}$ be Morse-Bott. Assume that all critical* *submanifolds of f are orientable. Then there exists a polynomial $R(t)$ with nonnegative integer coefficients such that*

$$MB_t(f) = P_t(M) + (1+t)R(t),$$

where,
$MB_t(f) = \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j}$ *is the Morse-Bott polynomial of f, and*
$P_t(M) = \sum_i b_i t^i$ *is the Poincaré polynomial of M.*

The version involving non-orientable submanifolds can be found in [4].

Example 4.3.2. 1) If f is the constant function, then $Cr(f) = M$, meaning that $MB_t(f) = P_t(M)$ and therefore $R(t) = 0$.

2) If f is the square of the height function given in Figure 4.9a, then the critical submanifolds of f are given by S, N and E. The points S and N have index n while E has index 0, since it is the minimum. So, $MB_t(f) = P_t(E)+t^n+t^n = P_t(\mathbb{S}^{n-1}) + 2t^n = 1 + t^{n-1} + 2t^n$; $P_t(S^n) = 1 + t^n$. Thus, $R(t) = t^{n-1}$.

3) Consider the height function on the torus $\mathbb{T}^2$ lying horizontally on a plane, (see Figure 4.8) then the circles C_1 and C_2 are the critical submanifolds, C_1 has Morse-Bott index 0, while C_2 has Morse-Bott index 1. We then have $P_t(C_1) = P_t(C_2) = 1 + t$; $P_t(\mathbb{T}^2) = 1 + 2t + t^2$; $MB_t(f) = 1 + t + (1 + t)t$. Hence, $R(t) \equiv 0$.

Morse theory and Morse-Bott theory need the constraint of having some specific functions defined on a given manifold in order to study its topology. A less constrained dynamics-related theory, more general than the above two, and which also studies the topology of manifolds is Conley theory.

4.4 Conley theory

Conley theory uses the idea of exit sets and isolating neighborhoods of isolated invariant sets to derive the Euler number of the manifold at hand. Thus we consider our manifold with a given flow on it, this flow need not originate from some Morse function. The idea behind Conley theory is that the topology of the given space and the qualitative properties of the dynamical system that defines the flow lines are encoded in the isolated invariant sets. One obtains the Poincaré polynomial of the manifold by summing up those of the index pairs (the pair isolating neighborhood and exit set) of the isolated invariant sets up to some correction term. In particular, the Euler number of the manifold is then obtained by summing for all isolated invariant sets, the alternating sums of the dimensions of the homology groups of the index pairs. For more insight see also [32], [39] and [12].

We recall that here we also use the negative gradient flow lines generated by a given function, and let us denote by ϕ_t the generated flow.

For a given set $N \subset M$, we define

$$I(N) := \{x \in N \mid \phi(\mathbb{R}, x) \subset N\},$$

that is, the set of points in N that remain in N at all times under the flow.

N is said to be invariant if $I(N) = N$. Thus a set N is said to be an *invariant* set if the set of all points in N that remain in N for all positive and negative times under the flow, is equal to N.

A set I is an **isolated invariant set** if it is invariant (w.r.t. the flow) and has a compact neighborhood that contains no other invariant sets besides I. Such a neighborhood will be called an **isolating neighborhood** for I. We have the following:

Definition 4.4.1. *A pair (N, E) of compact sets is called an* ***index pair*** *for an isolated invariant set I if:*

i) $\overline{N \setminus E}$ is an isolating neighborhood of I with $E \cap I = \emptyset$;

ii) if $x \in E$ and $\phi([0, t], x) \subset N$, then $\phi([0, t], x) \subset E$;

iii) if $x \in N$ and $\phi_t(x) \notin N$ for some $t > 0$, then there exists $t_0 \in [0, t)$ with $\phi([0, t_0], x) \subset N$ and $\phi_{t_0}(x) \in E$.

Thus E is the exit set for the flow from I. That is, the flow cannot return to N from E and the flow leaving N has to pass through E.

Definition 4.4.2. *The* ***topological Conley index*** *of an isolated invariant set I is the homotopy type of N/E and the* ***homological Conley*** *index of I is the polynomial*

$$C_t(I) := \sum_k \dim H_k(N, E; \mathbb{Z}) t^k.$$

A *Morse decomposition* of a compact invariant set S is a finite collection of disjoint compact invariant subsets $S_1, \cdots, S_n$ of S (called Morse sets) that permit an admissible ordering, that is, for $y \in S \setminus \bigcup_{i=1}^n S_i$, there exist i, j, $i < j$ with $\alpha(y) \subset S_i$, $\omega(y) \subset S_j$, where

$$\alpha(y) := \bigcap_{t \in \mathbb{R}} \overline{y((-\infty, t))} \quad \text{and} \quad \omega(y) := \bigcap_{t \in \mathbb{R}} \overline{y((t, +\infty))}$$

are the assymptotic limit sets, see [32] for a detailed definition. The smallest Morse decomposition of a given manifold (compact invariant set) M is given by the collection of critical points of the Morse function on M. The ordering is such that the initial point of a flow line has a smaller index than the final point.

Theorem 4.4.1. *Let $I_1, \cdots, I_l$ be isolated invariant sets that form a Morse decomposition for a given flow on a manifold M, then there exists $R(t)$, a polynomial in t with nonnegative coefficients, such that*

$$\sum_{j=1}^{l} C_t(I_j) = P_t(M) + (1+t)R(t). \tag{4.1}$$

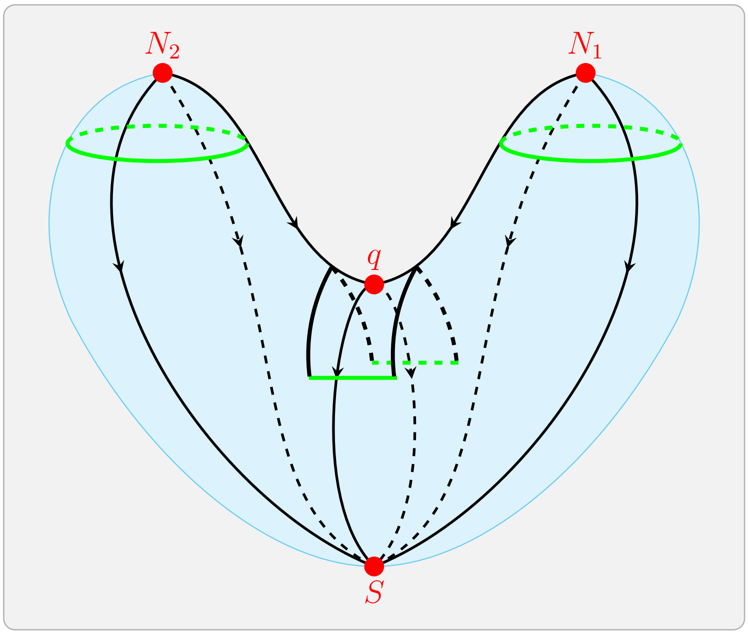


Figure 4.10: Critical points with their respective exit sets.

Example 4.4.1. We illustrate this using Figure 4.7, where for each critical point of the given (Morse) function (height function), we determine its isolating neighborhood and exit set. Note that we use the negative gradient flow lines. The resulting figure is given by Figure 4.10, where the exit sets for the different isolated invariant sets are highlighted in green.

The index pair for N_1 is given by: the isolating neighborhood is any disc around N_1 and the exit set is the boundary of the chosen disc represented by the green circle in the figure. So we have as index pair $(D, \partial^{top} D)$ also equivalent to $(\mathbb{S}^2, pt)$. So its contribution in (4.1) is t^2. We get the same contribution from N_2.

If we consider q, as isolating neighborhood we have a rectangle, and as exit set we have two opposite sides of the rectangle. This is equivalent to having an interval where its sides are the exit sets, that is $(\mathbb{S}^1, pt)$ therefore contributes with t.

The minimum point S has any disc around it to be its isolating neighborhood but an empty exit set, so its index pair is of the form $(D, \emptyset)$ which is equivalent to $(pt, \emptyset)$, thus contributes with 1.

We then have that $2t^2 + t + 1$ is the polynomial of $M \simeq \mathbb{S}^2$, $P_t(\mathbb{S}^2) = t^2 + 1$, so that $R(t) = t$. Setting $t = -1$ yields $2 - 1 + 1 = 2 = \chi(M)$.

After recalling all of the theories needed in studying the topology of manifolds, we now move to the next chapter which is about their discrete analogues.

5

Discrete Morse, Morse-Floer, Morse-Bott and Conley theories

This chapter focuses on the discrete analogues of the theories of Morse and the Morse-related ones. Analogously as in the smooth case, the Poincaré polynomial of a given cell complex can be retrieved using any one of the discrete analogues of Morse theory, Morse-Floer theory, Morse-Bott theory and (Morse-)Conley theory. A discrete Morse function is one that locally increases in dimension except possibly in one direction. In discrete Morse-Floer theory, finding the Betti numbers is achieved through finding a boundary operator associated to the combinatorial vector field of a discrete Morse function. In discrete Morse theory at most two connected cells can have the same value, and one has critical cells. The scope of our discrete Morse-Bott theory is beyond that of discrete Morse theory in the sense that, the discrete Morse-Bott function is discrete Morse except on some maximal collection of cells, where each cell has the same value and for which the union of the closure of the cells is connected. Also, one has critical collection of cells, that we call reduced collections. The discrete Morse-Bott inequalities are expressed in terms of the Poincaré polynomial of some of the reduced collections and that of the cell complex under consideration. Our discrete Conley theory uses some of the reduced collections of the discrete Morse-Bott function as isolated invariant sets.

Section 5.1 will focus on recalling discrete Morse theory. Section 5.2 focuses on the analogue of Morse-Floer theory in the discrete setting. Finally in Section 5.3 and Section 5.4, we give our versions of discrete Morse-Bott theory and discrete Morse-Bott-Conley theory, respectively.

5.1 Discrete Morse theory

The focal point of this section is Morse theory in the discrete setting. That is, we give the definition of a discrete Morse function, discuss its critical points as well as their indices, and the desired Morse inequalities. We also refer the reader to [20] and [21].

Roughly speaking, a discrete Morse function is a function defined on a cell complex, assigning a real value to each cell, with the condition that it should be locally increasing in dimension except possibly in one direction. That is, each cell should have a function value greater than the function values of its facets, except for at most one of them. Also, using the discrete analogue of sublevel sets of a function, which are the level subcomplexes, we have a homotopy equivalence between two level subcomplexes not containing any critical cell; and if there is a critical cell, we need to attach a cell of the same dimension.

Let $\mathbb{K}$ be a finite CW complex. We recall that all the topological boundary elements of a cell are called its faces and the co-dimension one faces are called facets. For a set A, we denote the cardinality of A by $\sharp A$.

Definition 5.1.1 (Discrete Morse function). *A function $f\colon \mathbb{K} \to \mathbb{R}$ is a discrete Morse function if, for all $\sigma^{(k)} \in \mathbb{K}$,*

$$\begin{cases} \text{for all } \tau \text{ s.t. } \sigma \text{ is an irregular face of } \tau, \quad f(\sigma) < f(\tau); \\ Un(\sigma) := \sharp\{\tau^{(k+1)} \,|\, \sigma \text{ is a regular facet of } \tau \text{ and } f(\tau) \le f(\sigma)\} \le 1, \end{cases}$$

and

$$\begin{cases} \text{for all } \nu \text{ s.t. } \nu \text{ is an irregular face of } \sigma, \quad f(\nu) < f(\sigma); \\ Dn(\sigma) := \sharp\{\nu^{(k-1)} \,|\, \nu \text{ is a regular facet of } \sigma \text{ and } f(\nu) \ge f(\sigma)\} \le 1. \end{cases}$$

Equivalently,

(i) if $\tau_1 > \sigma$ and $\tau_2 > \sigma$, $\tau_1 \neq \tau_2$ or $\tau_1 = \tau_2$ and σ is an irregular facet of τ_1, then $f(\sigma) < \max\{f(\tau_1), f(\tau_2)\}$,

(ii) if $\nu_1 < \sigma$ and $\nu_2 < \sigma$, $\nu_1 \neq \nu_2$ or $\nu_1 = \nu_2$ and ν_1 is an irregular facet of σ, then $f(\sigma) > \min\{f(\nu_1), f(\nu_2)\}$.

Definition 5.1.2 (Discrete Witten-Morse function). *A function $f\colon \mathbb{K} \to \mathbb{R}$ is a discrete Witten-Morse function if for all $\sigma^{(k)} \in \mathbb{K}$,*

(i) whenever $\tau_1 > \sigma$ and $\tau_2 > \sigma$ satisfy $\tau_1 \neq \tau_2$, or $\tau_1 = \tau_2$ and σ is an irregular facet of τ_1, then $f(\sigma) < \operatorname{avg}\{f(\tau_1), f(\tau_2)\}$,

(ii) whenever $\nu_1 < \sigma$ and $\nu_2 < \sigma$ satisfy $\nu_1 \neq \nu_2$, or $\nu_1 = \nu_2$ and ν_1 is an irregular facet of σ, then $f(\sigma) > \operatorname{avg}\{f(\nu_1), f(\nu_2)\}$.

Definition 5.1.3 (Flat Witten-Morse function). *A discrete Witten-Morse function $f\colon \mathbb{K} \to \mathbb{R}$ is said to be flat if for all $\sigma^{(k)} \in \mathbb{K}$,*

(i) whenever $\tau_1 > \sigma$ and $\tau_2 > \sigma$ satisfy $\tau_1 \neq \tau_2$ then $f(\sigma) \leq \min\{f(\tau_1), f(\tau_2)\}$,

(ii) whenever $\nu_1 < \sigma$ and $\nu_2 < \sigma$ satisfy $\nu_1 \neq \nu_2$ then $f(\sigma) \geq \max\{f(\nu_1), f(\nu_2)\}$.

Example 5.1.1. Using Figure 5.1, we have:

1) the subfigure at the top left shows an example of a function that is not discrete Morse, since the edge with value 1 has two facets (vertices), one with value 1 and the other with value 2.

2) The function given by the example in the top right subfigure is discrete Morse but not discrete Witten-Morse, since the 2-cell with value 3 has two facets (edges) whose average value is not less than 3, indeed $3 < (7 + 1)/2 = 4$.

3) The function given by the example in the subfigure at the bottom left is discrete Witten-Morse but not flat Witten-Morse, since the edge with value 2 has a smaller value than the maximum value of its two vertices.

4) The bottom right subfigure depicts a flat Witten-Morse function.

We observe that discrete Witten-Morse implies discrete Morse, but the converse is not true as shown in the examples in Figure 5.1. We shall only be focusing on the discrete Morse function.

Definition 5.1.4. *A cell $\sigma^{(k)} \in \mathbb{K}$ is said to be **critical** if*

$$\sharp\{\tau \,|\, \tau > \sigma,\ f(\tau) \leq f(\sigma)\} = 0 \quad \textit{and} \quad \sharp\{\nu \,|\, \nu < \sigma,\ f(\nu) \geq f(\sigma)\} = 0.$$

Example 5.1.2. In Figure 5.1, the critical cells are the ones whose values are highlighted in red.

Definition 5.1.5 (Index). *If $\sigma^{(k)}$ is a critical cell of dimension k, then the index of $\sigma^{(k)}$, denoted by $ind(\sigma^{(k)})$, is k.*

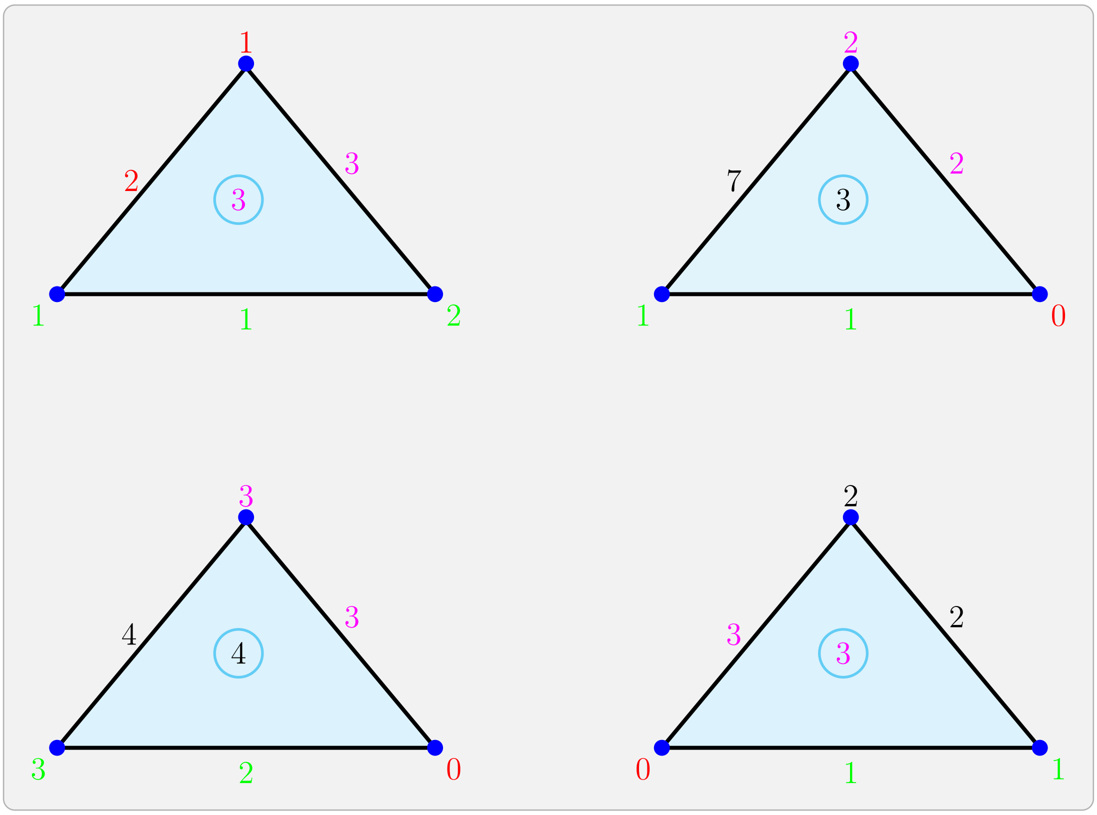


Figure 5.1: Examples of discrete functions on a cell complex.

Remark 5.1.1. The reason why the index is defined as in Definition 5.1.5 above is the following: in smooth Morse theory, the (Morse) index of a critical point p is the number of independent directions at p in which the function decreases. Thus the Morse index of the minimum is 0 and that of the maximum is the dimension of the space. So if we use the fact that if $\sigma^{(k)}$ is a critical cell, then all the $(k-1)$-dimensional cells in the topological boundary of σ have values less than the value of σ. Thus, if we consider the function value of each cell to be assigned to its barycenter, a smooth interpolation of a neighborhood of the barycenter b_σ of σ will then yield that b_σ is a critical point of index k, since the function decreases in all directions at b_σ and there are exactly k independent ones.

Definition 5.1.6 (Level subcomplex). *Let $\mathbb{K}$ be a regular CW complex and let $f\colon \mathbb{K} \to \mathbb{R}$ be a discrete Morse function. For $c \in \mathbb{R}$, the level subcomplex of c, denoted by $M(c)$, is the union of all the cells with values less than or equal to c*

together with all of their faces. That is,

$$M(c) = \bigcup_{f(\tau)\le c} \bigcup_{\sigma \le \tau} \sigma.$$

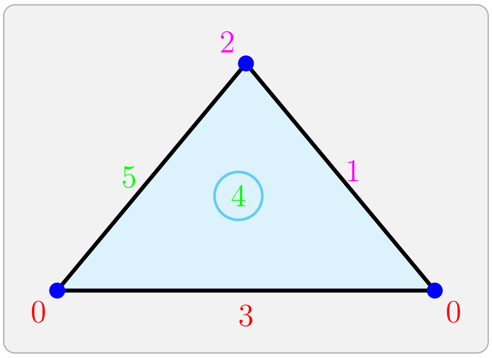


Figure 5.2: A discrete Morse function.

Analogous to the smooth case, the level subcomplex of a given value is the collection of all cells whose values are less than the specific value, together with their faces. Also, an observation about the noncritical cells is they always occur in pairs. That is, if a cell is not critical, then there is another (unique) cell of one dimension lower or higher that is also not critical. In Figure 5.2, the critical cells are in red, while the noncritical ones occur in pairs, with their values having the same color. The different level subcomplexes are shown in Figure 5.3. One can also observe that Theorem 4.1.1 applies in this discrete setting as well, (see [20, p. 104] for the formulation in the discrete setting) that is, for $a < b$, the homotopy type of a level subcomplex $M(b)$ only changes from that of $M(a)$ if there is a critical cell whose value is in $(a, b]$; if $(a, b]$ contains no critical value, then $M(b)$ deformation retracts into $M(a)$. Theorem 4.1.2 also applies here, (see [20, p. 107] for the formulation in this setting): the (regular) CW complex is homotopy equivalent to a CW complex with one k-cell attached for each critical k-cell of the (regular) CW complex.

Indeed, moving from $M(0)$ to $M(1) = M(2)$, we do not encounter any critical cell, this is why $M(1) = M(2)$ deformation retracts into $M(0)$. We come across a critical edge, moving from $M(2)$ to $M(3)$, and this explains why $M(3)$ is homotopy equivalent to $M(2)$ with an edge attached. Finally, moving from $M(3)$ to $M(4) = M(5)$, no critical cell is added, thus, $M(5)$ indeed deformation retracts into $M(3)$. In total we have two critical vertices and one critical edge, and a triangle is indeed homotopy equivalent to a CW complex, having two 0-cells and one 1-cell.

The following definition is taken from [23].

Definition 5.1.7. *A **combinatorial vector field** on a CW complex* $\mathbb{K}$ *is a map*

$$V : \mathbb{K} \to \mathbb{K} \cup \{0\}$$

satisfying:

(*i*) *if* $V(\sigma) \neq 0$, *then* $\dim V(\sigma) = \dim(\sigma) + 1$ *and* $\sigma < V(\sigma)$;

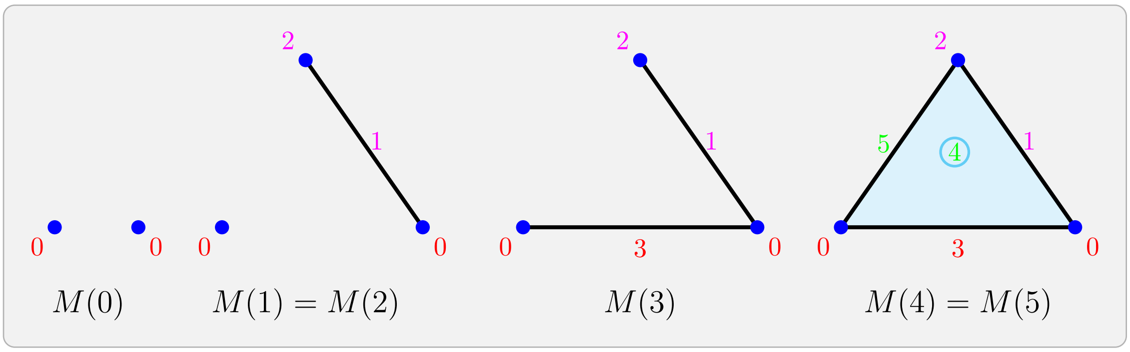


Figure 5.3: Different level subcomplexes.

(ii) *if* $V(\sigma) = \tau \neq 0$, *then* $V(\tau) = 0$;

(iii) *for any* τ, *there is at most one* σ *s.t.* $V(\sigma) = \tau$;

(iv) *for each* σ, *either* $V(\sigma) = 0$ *or* σ *is a regular face of* $V(\sigma)$.

Thus, if we draw an arrow from σ to τ whenever $\tau = V(\sigma)$ (since σ is in the boundary of τ), condition (iii) tells us that, at most one arrow should point at τ, that is, τ cannot have more than one incoming arrow. Condition (ii) tells us that, if an arrow points at τ, there should not be any arrow starting at τ. Thus, one already sees that a cell cannot be at the same time the head and the tail of an arrow, and each cell has a unique incoming or outgoing arrow but never both.

If σ is not contained in the image of V and $V(\sigma) = 0$, then σ is called a **rest point** and its index is given by its dimension.

Another important observation is the fact that, the points that are not rest points occur in pairs, and therefore the Euler number of the complex is equal to the alternating sum of the rest points that is.

$$\chi(\mathbb{K}) = \sum_{i=0}^{\dim K} (-1)^i n_i$$

where n_i is the number of rest points of index i.

A V-orbit (V-path) of index k is a finite sequence

$$\sigma_0^{(k)}, \tau_0^{(k+1)}, \sigma_1^{(k)}, \tau_1^{(k+1)}, \ldots, \sigma_m^{(k)}, \tau_m^{(k+1)}, \quad \text{for some } m \in \mathbb{N}$$

where, for $j = 0, \ldots, m-1$,

$$\tau_j^{(k+1)} = V(\sigma_j^{(k)}),$$

$$\sigma_j^{(k)} \neq \sigma_{j+1}^{(k)} < \tau_j^{(k+1)}.$$

The V-orbit is closed if $\sigma_0^{(k)} = \sigma_m^{(k)}$. We will sometimes refer to a closed V-orbit as a closed orbit.

If we consider a discrete Morse function on the cell complex $\mathbb{K}$, a combinatorial vector field is constructed in the following manner:

If $\tau > \sigma$ with $f(\tau) \leq f(\sigma)$, draw an arrow from σ to τ.

This then yields that the corresponding orbit cannot be closed. Conversely, for every combinatorial vector field without closed orbits, there exists a discrete Morse function associated to it. In fact, an orbit of a discrete Morse function will be of the form

$$f(\sigma_0) \geq f(\tau_0) > f(\sigma_1) \geq f(\tau_1) > \cdots > f(\sigma_{m+1}) \cdots$$

We now discuss discrete Morse-Floer theory, where we will see how the vector field constructed using the method above is used to define a boundary operator.

5.2 Discrete Morse-Floer theory

A boundary operator, using a similar idea as the one described in Section 4.2 in the smooth setting, can also be computed using a combinatorial vector field extracted from a discrete Morse function. This is the purpose of this section, see also [21].

The following remarks will be useful for both this chapter and the next one.

Remark 5.2.1 (Incidence property). The incidence property of regular CW complexes states that if $\nu < \sigma < \tau$ in a regular CW complex, then there exists $\widetilde{\sigma} \neq \sigma$ s.t. $\nu < \widetilde{\sigma} < \tau$. The same holds for CW complexes provided ν is a regular facet of σ and σ is a regular facet of τ.

We let f be a discrete Morse function on a finite CW complex $\mathbb{K}$.

It can be shown that for a noncritical cell σ, either
$\sharp\{\tau > \sigma \,|\, f(\tau) \leq f(\sigma)\} = 1$, in which case we say that σ is **upward noncritical**
or
$\sharp\{\nu < \sigma \,|\, f(\nu) \geq f(\sigma)\} = 1$, in which case σ is said to be **downward noncritical**.

Indeed, if σ is such that $\nu < \sigma < \tau$ and $f(\nu) \geq f(\sigma) \geq f(\tau)$, then in particular, we must have that ν is a regular facet of σ and σ is a regular facet of τ. Then there exists a cell $\widetilde{\sigma} \neq \sigma$, $\widetilde{\sigma} < \tau$ s.t. $f(\widetilde{\sigma}) < f(\tau)$. Choose $\widetilde{\sigma}$ s.t. $\nu < \widetilde{\sigma}$. This is always possible from the incidence of ν and τ (see Remark 5.2.1). Then $f(\nu) < f(\widetilde{\sigma})$ for the discrete Morse conditions not to be violated. Hence,

$$f(\nu) \geq f(\sigma) \geq f(\tau) > f(\widetilde{\sigma}) > f(\nu),$$

which is a contradiction. Thus, a cell cannot be downward and upward noncritical at the same time.

Definition 5.2.1 (Trivial discrete Morse function). *A discrete Morse function $f\colon \mathbb{K} \to \mathbb{R}$ is said to be trivial if all the cells in $\mathbb{K}$ are critical.*

We recall that whenever we have a discrete Morse function f, we draw an arrow from σ to τ if $\sigma < \tau$ but $f(\sigma) \geq f(\tau)$. In this way, we get a vector field from which we can define the boundary operator.

Note that there can only be an arrow from σ to τ if σ is a regular facet (codimension one face) of τ.

Since not every CW complex is orientable, we look at local orientations: we consider a CW complex in which every cell is endowed with an orientation called initial orientation. The orientations on the higher dimensional cells will induce some orientations on the lower dimensional ones, see Subsection 3.2.1. If the induced orientation, on a cell coincides with its initial one, the cell will be counted with a + sign; if not then a − sign. We recall that the incidence number between two critical cells $\tau^{(k+1)}$ and $\sigma^{(k)}$, denoted $[\tau : \sigma]$, is the number of times that τ is wrapped (along its boundary) around σ. When orientations are taken into account and σ is a regular facet of τ, $[\tau : \sigma]$ is equal to $+1$ if the induced orientation from τ to σ coincides with the initial orientation of σ, and is -1 otherwise. See Definition 3.2.3 for the precise formulation.

Definition 5.2.2. *Two cells $\sigma^{(k)}$ and $\omega^{(k+s)}$ are said to be path-connected if there exists a sequence of cells connecting them. That is there is a path between them of the form*

$$\sigma^{(k)} < \tau_1^{(k+1)} > \sigma_2^{(k)} < \tau_2^{(k+1)} > \cdots > \sigma_m^{(k)} < \tau_{m_1}^{(k+1)}$$

$$\tau_{m_1}^{(k+1)} < \omega_1^{(k+2)} > \tau_{m_1+1}^{(k+1)} < \omega_2^{(k+2)} > \cdots > \tau_{m_1+m_2}^{(k+1)} < \omega_{m_2}^{(k+2)}$$

$$\omega_{m_2}^{(k+2)} < \omega_1^{(k+3)} > \omega_{m_2+1}^{(k+2)} < \omega_2^{(k+3)} > \cdots > \omega_{m_2+m_3}^{(k+2)} < \omega_{m_3}^{(k+3)}$$

$$\vdots$$

$$\omega_{m_{s-1}}^{(k+s-1)} < \omega_1^{(k+s)} > \omega_{m_{s-1}+1}^{(k+s-1)} < \omega_2^{(k+s)} > \cdots > \omega_{m_{s-1}+m_s}^{(k+s-1)} < \omega^{(k+s)}.$$

Remark 5.2.2. For a regular CW complex, if a cell $\sigma^{(k)}$ is a face of another cell $\omega^{(k+2)}$ then there exist $\tau_1^{(k+1)}$ and $\tau_2^{(k+1)}$ s.t. $\sigma < \tau_i < \omega$. Looking at the orientations, see [25], ω will induce some orientations on τ_1 and τ_2. The orientation on τ_1 (induced from ω) will induce an orientation on σ that will be different from the one induced from τ_2. However, when the CW complex is not regular, and σ is an irregular facet of $\tau < \omega$), then we cannot induce a consistent orientation on σ from τ. In particular, if $\tau_1 = \tau_2$ (meaning that the CW complex is no longer regular, and σ is an irregular facet of τ_1), then, on one side τ_1 will induce a positive orientation on σ but on another side it will induce a negative orientation on σ. More generally, even if σ is not a face of ω, in the regular case, the orientation on ω will induce an orientation on σ along each path from ω to σ.

Let $C_k(\mathbb{K};\mathbb{Z})$ be the free $\mathbb{Z}$-module (or $\mathbb{Z}_2$-module) generated by the critical oriented k-cells of $\mathbb{K}$. We denote Forman's boundary operator by ∂^F, and $\partial_k^F : C_k \to C_{k-1}$ is roughly given by the following:

for τ critical, to find $\partial^F\tau$ proceed as follows: among all $\sigma < \tau$,

(i) if σ critical then keep it,

(ii) if σ is downward noncritical, ignore it,

(iii) if it is upward noncritical then there exists a unique $\tau' \neq \tau$, $\tau' > \sigma$ with $f(\tau') \leq f(\sigma)$; take all the critical facets of this τ'; if it has a downward noncritical facet ignore it; if it has an upward noncritical facet, start over again.

More formally we have the following definition, see also [20] and [21].

Let

$$U := \{\sigma \in \mathbb{K} \mid \exists!\, \tau \text{ s.t } \ \sigma \to \tau\}.$$

We define $v^F : \mathbb{K} \cup \{0\} \to \mathbb{K} \cup \{0\}$ by:

$$v^F(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is critical,} \\ \displaystyle\sum_{\widetilde{\sigma} < \tau,\, \widetilde{\sigma} \neq \sigma} v^F(\widetilde{\sigma}) & \text{if } \sigma \in U, \text{ that is. } \exists! \text{ arrow } \sigma \to \tau, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

This definition is recursive so we have to argue that it terminates. Indeed, for a discrete Morse function f, the cells σ and $\widetilde{\sigma}$ in (5.1) satisfy: $f(\sigma) > f(\widetilde{\sigma})$. Also, the finiteness of the CW complex ensures that we will stop at some point.

Recall that ∂^c denotes the cellular boundary operator, see Definition 3.2.4.

Definition 5.2.3 (Definition of the boundary operator ∂^F). *The boundary operator $\partial^F : C_k \to C_{k-1}$ is given by:*

$$\partial^F(\tau) := \upsilon^F \circ \partial^c \tau = \sum_{\sigma < \tau} [\tau : \sigma] \upsilon^F \sigma.$$

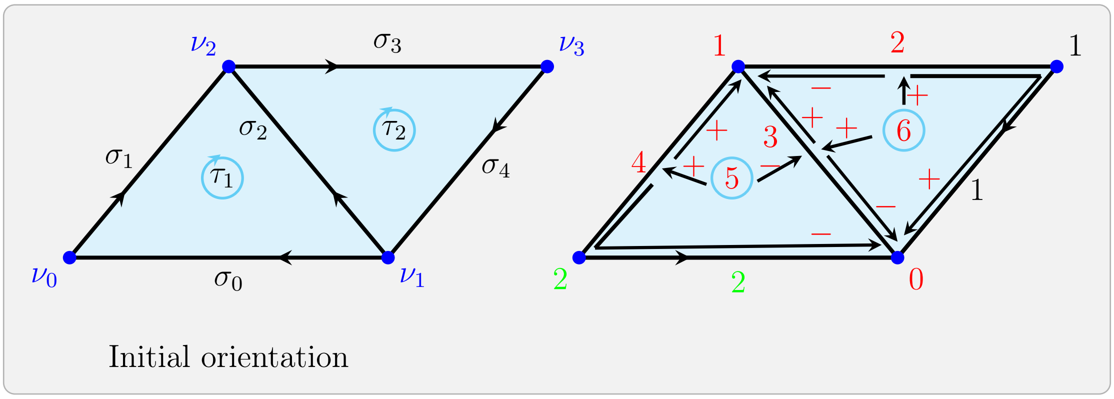


Figure 5.4: Definition of the boundary operator.

Example 5.2.1. Using Figure 5.4, the arrows are drawn between the critical cells of index difference one (going downward), and the sign of each arrow between two critical cells represents the incidence number. The left subfigure specifies the initial orientations of the cells. That is,
$\sigma_0 = [\nu_1, \nu_0]$, $\sigma_1 = [\nu_0, \nu_2]$, $\sigma_2 = [\nu_1, \nu_2]$, $\sigma_3 = [\nu_2, \nu_3]$, $\sigma_4 = [\nu_3, \nu_1]$,
$\tau_1 = [\nu_0, \nu_2, \nu_1]$, $\tau_2 = [\nu_1, \nu_2, \nu_3]$.
Now, using the initial orientations of each cell, we have:
$\partial_2^F(\tau_2) = \sigma_2 + \sigma_3$; the edge σ_4 is downward noncritical so we ignore it.
$\partial_2^F(\tau_1) = -\sigma_2 + \sigma_1$; the edge σ_0 is downward noncritical so we ignore it.
$\partial_1^F(\sigma_3) = \nu_1 - \nu_2$, since the other vertex ν_3 is upward noncritical with the edge σ_4 which in turn has the vertex ν_1 as critical.
$\partial_1^F(\sigma_1) = \nu_2 - \nu_1$, since the other vertex ν_0 is upward noncritical with the edge σ_0 which in turn has the vertex ν_1 as critical.
$\partial_1^F(\sigma_2) = \nu_2 - \nu_1$.
$\partial_0^F(\nu_i) = 0$ for all $i = 0, \cdots, 3$.
Then one can easily check that $\partial_{k-1}^F \circ \partial_k^F = 0$ for all $k = 1, 2$, and for $b_k := \ker \partial_k^F / \operatorname{im} \partial_{k+1}^F$, we get the desired Betti numbers, that is $b_0 = 1$, $b_1 = 0$, $b_2 = 0$.

Remark 5.2.3. Forman, in [20], using a combinatorial vector field V of the discrete Morse function, first defined a discrete gradient flow ϕ on the CW complex under consideration given by $\phi = 1+\partial^c V + V\partial^c$, where ∂^c is the cellular boundary operator. The vector field V satisfies $V \circ V = 0$ and this flow ϕ commutes with ∂^c. He then showed that the ϕ-invariant chains form a differential complex. He constructed a canonical isomorphism between the homology groups of this differential complex and those of the CW complex and then showed that the space of ϕ-invariant chains is isomorphic to the span of the critical cells.

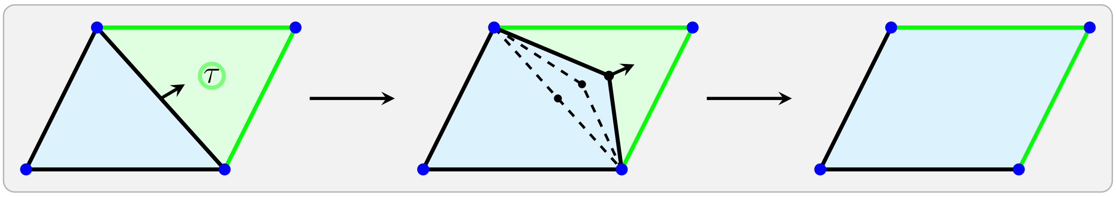


Figure 5.5: A collapse/deformation retraction.

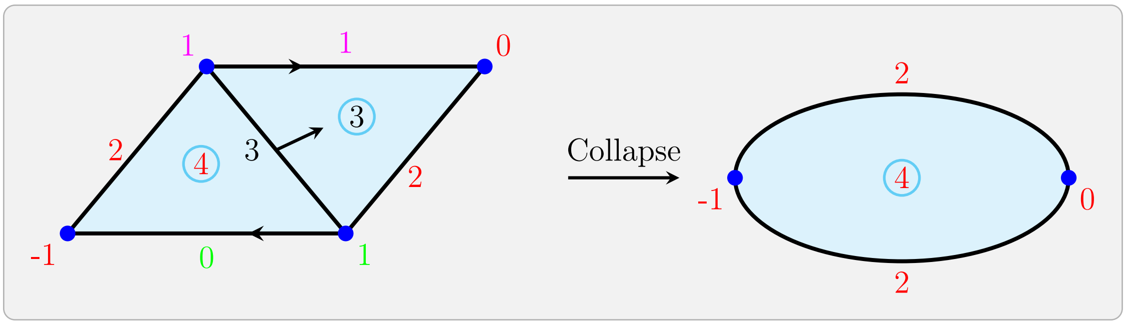


Figure 5.6: Equivalent definition of the boundary operator.

Remark 5.2.4. Definition 5.2.3 is equivalent to applying some collapses whenever there is an outgoing/incoming arrow, and then taking the cellular boundary operator of the new complex obtained after all the collapses have been carried out. Also, each collapse is exactly a strong deformation retraction, which is also homotopy preserving. See Figure 5.5 for an illustration. For example, after applying all the collapses in the left subfigure of Figure 5.6, we get the right subfigure to which, applying the cellular boundary operator gives exactly the same result as applying Definition 5.2.3 to the left subfigure.

The following theorem establishes the fact that the square of the boundary operator given by Definition 5.2.3 is zero, and thus the Betti numbers will be well defined.

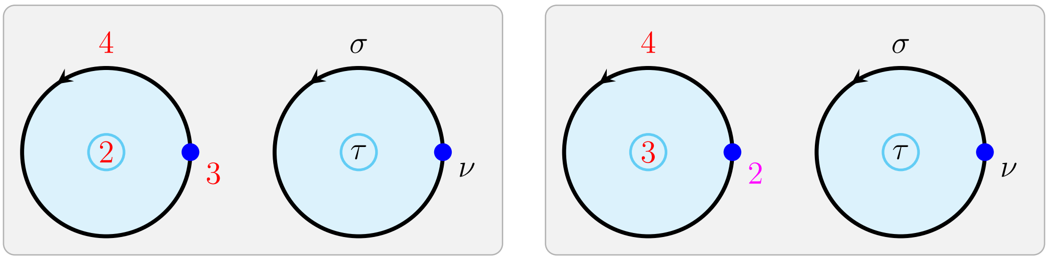
Theorem 5.2.1.

$$\partial^F_{k-1} \circ \partial^F_k = 0 \quad \textit{for } k = 1, \cdots, n.$$

The idea behind the proof is as follows: a given cell σ having an outgoing arrow pointing to some τ is replaced by the sum of all the critical facets of τ in such a way that its boundary and the boundary of the sum of all these critical facets are equal (up to an orientation) and so on. This is in fact an internal collapse (strong deformation retraction) of $\bar{\tau}$ into some part of its topological boundary, and we know that any such collapse does not change the homotopy type. The orientation idea is such that, between any two incident cells of index difference two, there are exactly two cells of the intermediate dimension being facets of the higher dimensional cell and having as facet the lowest dimensional one. This guarantees the cancellation, since the induced orientations from the highest dimensional cell to the lowest dimensional one will be opposite.

To prove Theorem 5.2.1 we argue as follows:

(1) First we show that any arbitrary discrete Morse function on a CW complex can be transformed into a trivial discrete Morse function, where by **trivial** we mean that all the cells are critical. This can always happen. In particular, when the CW complex is not regular, we need the following remark.



(a) Not acceptable (b) Acceptable

Figure 5.7: Framework.

Remark 5.2.5. In each subfigure of Figure 5.7, the cell ν is an irregular facet of σ and σ is a regular facet of τ. In Figure 5.7a, $f(\sigma) > f(\nu) > f(\tau)$ whereas, in Figure 5.7b, $f(\sigma) > f(\tau) > f(\nu)$. We would like the discrete Morse function to be such that: instead of having a situation like the one described in Figure 5.7a, we would rather have the situation in Figure 5.7b. Therefore we consider the framework given by Figure 5.7b. That is,

if ν is an irregular facet of σ but σ is a regular facet of τ, one requires $f(\nu) < f(\tau)$.

Similarly,

if ν is a regular facet of σ but σ is an irregular facet of τ, we require that $f(\nu) < f(\tau)$.

(2) Since for a trivial Morse function we have $\partial^F \circ \partial^F = 0$, we now show that if we transform this trivial discrete Morse function into a non-trivial one, step by step, by creating a noncritical pair, that is, a pair $\{\sigma, \tau\}$ where $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$, then we will still have $\partial^F \circ \partial^F = 0$.

Proof. (1) To show this, we proceed from lower dimensional noncritical pairs to higher dimensional ones. We denote by s_k the number of upward noncritical cells of dimension k.

Let $\{v_*, e\}$ be a noncritical pair with v_* a vertex and e an edge. Then $f(v_*) \geq f(e)$. Define $f_0^0 \colon \mathbb{K} \to \mathbb{R}$ by

$$f_0^0(\sigma) = \begin{cases} f(\sigma) - \varepsilon_0 & \text{if } \sigma = v_*, \\ f(\sigma) & \text{else,} \end{cases}$$

where,

$$\varepsilon_0 > f(v_*) - f(e). \tag{5.2}$$

It follows that f_0^0 is discrete Morse and both v_* and e are critical. Indeed,

$$f_0^0(v_*) = f(v_*) - \varepsilon_0 < f(e) \quad \text{from} \quad (5.2),$$

also,

$$f_0^0(v_*) < f(v_*) < \min_{\widetilde{e} > v_*, \widetilde{e} \neq e} f(\widetilde{e}) = \min_{\widetilde{e} > v_*, \widetilde{e} \neq e} f_0^0(\widetilde{e}).$$

We apply this definition to all the upward noncritical vertices. Set $f_0 := f_0^{s_0}$.

Now we proceed by induction on the dimension of the upward noncritical cell. Suppose $\{\sigma_*^{(k)}, \tau^{(k+1)}\}$ is a noncritical pair, and let $f_{k-1} := f_{k-1}^{s_{k-1}}$, then we have

$$f_{k-1}(\sigma_*) \geq f_{k-1}(\tau) \quad \text{and} \quad f_{k-1}(\sigma_*) > \max_{\sigma < \tau, \sigma \neq \sigma_*} f_{k-1}(\sigma).$$

We then modify the value of f_{k-1} at σ as follows:

$f_k^0 \colon \mathbb{K} \to \mathbb{R}$ is given by

$$f_k^0(\sigma) = \begin{cases} f_{k-1}(\sigma) - \varepsilon_k & \text{if } \sigma = \sigma_*, \\ f_{k-1}(\sigma) & \text{else,} \end{cases}$$

where ε_k is chosen such that,

$$f_{k-1}(\sigma_*) - f_{k-1}(\tau) < \varepsilon_k < f_{k-1}(\sigma_*) - \max_{\nu<\sigma_*} f_{k-1}(\nu). \tag{5.3}$$

Claim:

$$f_{k-1}(\tau) > \max_{\nu<\sigma_*} f_{k-1}(\nu).$$

Proof of the claim. Since we proceed from lower dimensional pairs to the higher dimensional ones, all the $\nu^{(k-1)} < \sigma_*^{(k)}$ are critical. Indeed, if we suppose there is a $\nu^\star < \sigma_*$ with $f_{k-1}(\nu^\star) \geq f_{k-1}(\tau)$, then if $\nu^\star$ is an irregular facet of σ_*. And we get a contradiction since $f_{k-1}(\nu^\star)$ has to be less than $f_{k-1}(\tau)$, from Remark 5.2.5 above. If $\nu^\star$ is a regular facet of σ and σ is a regular facet of τ, by the incidence property, Remark 5.2.1, there exists some $\widetilde{\sigma} \neq \sigma$ s.t. $\tau > \widetilde{\sigma} > \nu^\star$ with $f_{k-1}(\tau) \leq f_{k-1}(\nu^\star) < f_{k-1}(\widetilde{\sigma})$, and this will contradict the discrete Morse condition at τ. Hence,

$$f_{k-1}(\tau) > f_{k-1}(\nu) \text{ for all } \nu < \sigma_*,$$

and this implies that

$$f_{k-1}(\tau^{(k+1)}) > \max_{\nu^{(k-1)}<\sigma_*^{(k)}} \{ f_{k-1}(\nu^{(k-1)})\}.$$

□

Now, using the definition of f_k^0 and the condition for ε_k, one check that it is discrete Morse and that $\sigma_*^{(k)}$ and $\tau^{(k+1)}$ are critical.

To check this it suffices to check at the two cells σ_* and τ.

Indeed, the LHS of (5.3) implies

$$f_k^0(\sigma_*) = f_{k-1}(\sigma_*) - \varepsilon_k \; < \; f_{k-1}(\tau) = f_k^0(\tau).$$

Also, the RHS of (5.3) implies

$$f_k^0(\sigma_*) = f_{k-1}(\sigma_*) - \varepsilon_k \; > \; \max_{\nu<\sigma_*} f_{k-1}(\nu) = \max_{\nu<\sigma_*} f_k^0(\nu).$$

The inequality

$$f_{k-1}(\sigma_*) < \min_{\widetilde{\tau}>\sigma_*,\widetilde{\tau}\neq\tau} f_{k-1}(\widetilde{\tau})$$

implies that the same inequality holds for f_k^0. We then set $f_k := f_k^{s_k}$ and proceed as before.

The desired trivial discrete Morse function is $f_T = f_n$.

(2) To show that $\partial^F \circ \partial^F = 0$, it is enough to only consider the cells which can reach either σ or τ by means of some v^F-paths.
Assume that to move from step t to step $t+1$, the noncritical pair $\{\sigma\,,\,\tau\}$ was created. We denote by $\partial^{F,t}$, the boundary operator at step t. Then $\partial^{F,t} \circ \partial^{F,t} = 0$.

Let $\varsigma^{(k+1)}$ be a critical cell that can reach $\sigma^{(k)}$ by means of some v^F-paths. We denote by b_ς the part of $\partial^{F,t}\varsigma$ that is not connected (by any v^F-path) to σ. At step t, before the noncritical pair was created, that is before the arrow $\sigma \to \tau$ was added, assuming that σ is negatively oriented w.r.t τ and positively oriented w.r.t. ς, we have, for $c \in \mathbb{Z}$,

$$\partial^{F,t}\varsigma = b_\varsigma + c\sigma \quad \text{and} \quad \partial^{F,t}\tau = b_\tau - \sigma.$$

Observe that when $c = 0$, the result follows trivially. The case $c = 1$ includes the situation where $\varsigma > \sigma$.

Now, $\partial^{F,t} \circ \partial^{F,t} = 0 \quad \Rightarrow$

$$\partial^{F,t}(b_\varsigma) + c\partial^{F,t}\sigma = 0, \tag{5.4}$$

$$\partial^{F,t}(b_\tau) - \partial^{F,t}\sigma = 0. \tag{5.5}$$

Note that the orientation on ς will induce some orientation on each element in b_ς and b_τ. If this induced orientation coincides with the initial orientation of a given cell then this cell will appear in $\partial^{F,t+1}\varsigma$ with positive sign, if not then with a negative sign. Also, the elements $\nu < \sigma$ are incident to ς and to τ, so that the induced orientations from b_ς (induced by the orientation from ς) to the critical facets of σ will be different from those induced by the elements in b_τ. Thus, the signs of the elements in $\partial^{F,t}\sigma$ in (5.4) and (5.5) indeed have to be different.

To get $\partial^{F,t+1}(\varsigma^{(k+1)})$, we take all the critical cells that were part of its boundary operator at step t. Since σ is upward noncritical with τ, we will also take into account the critical facets of τ. Thus

$$\partial^{F,t+1}(\varsigma^{(k+1)}) = b_\varsigma + c\, b_\tau, \tag{5.6}$$

where b_ς is the sum of all the critical cells that are in $\partial^{F,t}\varsigma$ excluding σ and b_τ is the sum of all the critical cells in $\partial^{F,t}\tau$ excluding σ.

When $c = 1$, one quick way of understanding this is the fact that, ς is expanded through σ while τ is collapsed. Thus we have a new cell ς that has as boundary elements b_ς and $c\, b_\tau$, and any $\nu < \sigma$ will be incident to ς so that the induced orientations (induced by the orientation from ς) from $\sigma_1 \in b_\varsigma$ and $\sigma_2 \in b_\tau$, $\sigma_1 > \nu < \sigma_2$, to ν must indeed be different. Since the elements in b_ς and b_τ were not affected by the creation of the noncritical pair, we have

$$\partial^{F,t+1}(b_\varsigma) = \partial^{F,t}(b_\varsigma) \quad \text{and} \quad \partial^{F,t+1}(b_\tau) = \partial^{t}(b_\tau).$$

This yields

$$\partial^{F,t+1} \circ \partial^{F,t+1}(\,\varsigma^{(k+1)}\,) = \partial^{F,t+1}(b_\varsigma) + c\,\partial^{F,t+1}(b_\tau) = \partial^{F,t}(b_\varsigma) + c\,\partial^{F,t}(b_\tau) = 0,$$

by adding (5.4) and $c \cdot$ (5.5).

Now let us consider $\omega^{(k+2)}$ which can reach $\tau^{(k+1)}$ by means of some v^F-paths. The argument we will use here is an extension of the idea in the following remark.

Remark 5.2.6. If $\sigma^{(k)}$ is a face of a cell $\omega^{(k+2)}$, then we have the possibilities:

1) There does not exist a cell τ s.t. $\sigma < \tau < \omega$. There is nothing to show here.
2) There is a cell τ s.t. $\sigma < \tau < \omega$. This gives the following situations:
   σ is an irregular facet of τ and τ is an irregular facet of ω;
   σ is a regular facet of τ and τ is an irregular facet of ω;
   σ is an irregular facet of τ and τ is a regular facet of ω;
   σ is a regular facet of τ and τ is a regular facet of ω. In this case there must also exist a $\tau' \neq \tau$ s.t. $\sigma < \tau' < \omega$, and the induced orientation of ω onto τ will induce an opposite orientation on σ as compared to the induced orientation of ω onto τ'.

In the case where τ is an irregular facet of ω, then τ need not appear in $\bar{\partial}\omega$ (this can happen when the incidence number of ω and τ is zero). This situation will also not be interesting. We then summarize all the situations above into the following:

If $\sigma^{(k)}$ is a face of a cell $\omega^{(k+2)}$, there exist cells τ and τ' s.t. $\sigma < \tau < \omega$ and $\sigma < \tau' < \omega$.

Observe that we do not specify the regularity of facets. We also not not specify that τ' should be different from τ since they can also be equal (as in the case where σ is an irregular facet of τ).

At step t there exist some ς_i's such that $\varsigma_i > \sigma$ and for which

$$\partial^{F,t}\omega = b_\omega + \sum_i \alpha_i\varsigma_i + c\tau, \quad \text{for } \alpha_i,\, c \in \mathbb{Z},$$

where, for

$$\partial^{F,t}\tau = b_\tau - \sigma, \tag{5.7}$$

$\varsigma := \sum_i \alpha_i\varsigma_i$ must be such that

$$\partial^{F,t}\varsigma = b_\varsigma + c\sigma. \tag{5.8}$$

At step t the square of the boundary operator is zero, and we get

$$\partial^{F,t} \circ \partial^{F,t}(\,\omega^{(k+2)}\,) \;=\; 0 \;\Rightarrow\; \partial^{F,t}\varsigma \;+\; c\,\partial^{F,t}(\,\tau\,) \;+\; \partial^{F,t}(b_\omega) = 0, \tag{5.9}$$

Note that the boundary of $\varsigma^{(k+1)}$ at step $t+1$ is the sum of its critical facets plus the sum of the critical facets of $\tau^{(k+1)}$ multiplied by c, which is the sum of all facets of both except σ (the noncritical one).

It should be noted that because there is a v^F-path from ω to τ, ω and σ are path-connected. Following Remark 5.2.2, we get that the induced orientations from ω to $\varsigma^{(k+1)}$ and $\tau^{(k+1)}$ will induce opposite orientations on $\sigma^{(k)}$. Thus, if the orientation on $\sigma^{(k)}$ induced by $\varsigma^{(k+1)}$ is $+$ then the one induced by τ has to be $-$, that is, σ must indeed appear in (5.8) and (5.7) with opposite orientations. This implies that

$$c\,\partial^{F,t}\tau + \partial^{F,t}\varsigma = b_\varsigma + c\,b_\tau \tag{5.10}$$

and,

$$\begin{aligned}
\partial^{F,t+1} \circ \partial^{F,t+1}(\,\omega^{(k+2)}\,) \;&=\; \partial^{F,t+1}(\,\varsigma^{(k+1)}\,) + \partial^{F,t+1}(b_\omega) \\
&=\; b_\varsigma + c\,b_\tau \; + \partial^{F,t}(b_\omega) \quad \text{using (5.6)} \\
&=\; 0,
\end{aligned}$$

from (5.9), (5.10) and the fact that $\partial^{F,t+1}(b_\omega) = \partial^{F,t}(b_\omega)$.

If $\omega^{(k+2)}$ can reach $c\sigma$, $c \in \mathbb{Z}$, by means of some v^F-paths, but not τ, then there exist critical cells ς_i and $\widetilde{\varsigma}_j$ s.t., for $\varsigma := \sum_i \alpha_i\varsigma_i$ and $\widetilde{\varsigma} := \sum_j \gamma_j\widetilde{\varsigma}_j$, $\alpha_i, \gamma_j \in \mathbb{Z}$, we have

$$\bar{\partial}^t\omega = b_\omega + \varsigma + \widetilde{\varsigma} = \bar{\partial}^{t+1}\omega,$$

where,

$$\bar{\partial}^t \varsigma = b_\varsigma + c\sigma, \quad \text{whereas} \quad \bar{\partial}^t \widetilde{\varsigma} = b_{\widetilde{\varsigma}} - c\sigma.$$

Then,

$$\bar{\partial}^{t+1} \varsigma = b_\varsigma + c\, v^F \sigma, \quad \text{and} \quad \bar{\partial}^{t+1} \widetilde{\varsigma} = b_{\widetilde{\varsigma}} - c\, v^F \sigma.$$

Hence

$$\bar{\partial}^{t+1} \circ \bar{\partial}^{t+1} \omega = \bar{\partial}^t b_\omega + b_\varsigma + b_{\widetilde{\varsigma}} = \bar{\partial}^t \circ \bar{\partial}^t \omega = 0.$$

□

We can now give a proof in the discrete setting for the Morse inequalities, where the Betti numbers are now obtained from the discrete boundary operator using the combinatorial vector field of a discrete Morse function.

From the previous proof, we know that $\partial_i^F \circ \partial_{i+1}^F = 0$, that is, $\mathrm{im}(\partial_{i+1}^F) \subseteq \ker(\partial_i^F)$, so the homology groups are well-defined.

The **Betti numbers** are given by

$$b_i \,:=\, \dim\, H_i = \dim\, \left(\ker \partial_i^F / \operatorname{im} \partial_{i+1}^F \right)$$

and the **Morse numbers**

$$m_i \,:=\, \sharp\{ \text{critical cells of dimension } i \} \,=\, \dim\, C_i.$$

We then also have the Morse inequalities, see also [20].

Theorem 5.2.2 (Discrete Morse inequalities). *Let f be a discrete Morse function on a finite n-dimensional CW complex $\mathbb{K}$. For $k = 0, 1, \cdots, n$,*

(i) *(Weak): $m_k \geq b_k$,*

$$m_0 - m_1 + m_2 - \cdots \pm\, m_n = b_0 - b_1 + b_2 - \cdots \pm\, b_n =: \chi(\mathbb{K}),$$

where, $\chi(\mathbb{K})$ is the Euler number of the n-dimensional CW complex $\mathbb{K}$.

(ii) *(Strong):*

$$m_k \,-\, m_{k-1} \,+\, m_{k-2} \,-\, \cdots \,\pm\, m_0 \,\geq\, b_k \,-\, b_{k-1} \,+\, b_{k-2} \,-\, \cdots \,\pm\, b_0$$

with equality when $k = n$.

Proof. (*i*) We prove (*i*) using (*ii*): adding (*ii*) for k and (*ii*) for $k-1$ yields $m_k \geq b_k$. One can also observe this from the fact that $\ker \partial_k \subseteq C_k$.

Using (*ii*) for $k = n+1$, we get that $m_{n+1} = 0$ (since there are no critical points of dimension greater than n) similarly $b_{n+1} = 0$. Thus we have

$$-m_n + m_{n-1} - \cdots \pm m_0 \geq -b_n + b_{n-1} - \cdots \pm b_0,$$

comparing this with (*ii*) for $k = n$ yields

$$m_0 - m_1 + m_2 - \cdots \pm m_n = b_0 - b_1 + b_2 - \cdots \pm b_n = \chi(\mathbb{K}).$$

(*ii*) To prove this we will need the following fact from Linear Algebra:

If $L : X \to Y$ is a linear map on finitely generated R-modules, and R a principal ideal domain, then

$$\dim X = \dim \ker L + \dim \operatorname{im} L.$$

Thus, we have a chain complex of finite-dimensional free $\mathbb{Z}$-modules:

$$0 \xrightarrow{\partial^F_{n+1}} C_n \xrightarrow{\partial^F_n} C_{n-1} \xrightarrow{\partial^F_{n-1}} \cdots \xrightarrow{\partial^F_2} C_1 \xrightarrow{\partial^F_1} C_0 \xrightarrow{\partial^F_0} 0 \quad (\partial^F \circ \partial^F = 0)$$

with

$$\dim C_k = m_k = \dim \ker \partial^F_k + \dim \operatorname{im} \partial^F_k,$$

and

$$b_k = \dim H_k = \dim \ker \partial^F_k / \operatorname{im} \partial^F_{k+1} = \dim \ker \partial^F_k - \dim \operatorname{im} \partial^F_{k+1}$$

for $k = 0, 1, \cdots, n$, follows from the short exact sequence

$$0 \to \operatorname{im} \partial^F_{k+1} \hookrightarrow \ker \partial^F_k \to H_k \to 0.$$

So,

$$\begin{aligned}
& m_k - m_{k-1} + m_{k-2} \cdots \pm m_0 \\
= \; & \dim \ker \partial^F_k + \dim \operatorname{im} \partial^F_k - \dim \ker \partial^F_{k-1} - \dim \operatorname{im} \partial^F_{k-1} + \dim \ker \partial^F_{k-2} \\
& + \dim \operatorname{im} \partial^F_{k-2} - \cdots \pm \dim \ker \partial^F_0 \pm \underbrace{\dim \operatorname{im} \partial^F_0}_{0} \\
= \; & [\dim \ker \partial^F_k - \dim \operatorname{im} \partial^F_{k+1} + \dim \operatorname{im} \partial^F_{k+1}] + [\dim \operatorname{im} \partial^F_k - \dim \ker \partial^F_{k-1}] \\
& [-\dim \operatorname{im} \partial^F_{k-1} + \dim \ker \partial^F_{k-2}] + \cdots \pm [-\dim \operatorname{im} \partial^F_1 + \dim \ker \partial^F_0]
\end{aligned}$$

$$
\begin{aligned}
&= \dim\, \mathrm{im}\, \partial^F_{k+1} + b_k - b_{k-1} + b_{k-2} - \cdots \pm b_0 \\
&\geq b_k - b_{k-1} + b_{k-2} - \cdots \pm b_0 \\
&\quad \text{(with equality when } k = n \text{ since } \mathrm{im}\, \partial^F_{n+1} = 0).
\end{aligned}
$$

□

It should be noted that one also has the polynomial version of the Morse inequalities which is more general than the above Morse inequalities and whose proof is similar to the one above. The **Poincaré polynomial** of an n-dimensional CW complex $\mathbb{K}$ is given by

$$P_t(\mathbb{K}) := \sum_{k=0}^{n} b_k t^k.$$

Proposition 5.2.3 (Polynomial Morse inequalities). *Let f be a discrete Morse function on a finite n-dimensional CW complex $\mathbb{K}$. Then there exists $R(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_{k=0}^{n} m_k t^k = P_t(\mathbb{K}) + (1+t)R(t).$$

Proof.

$$
\begin{aligned}
\sum_{k=0}^{n} m_k t^k - \sum_{k=0}^{n} b_k t^k &= \sum_{k=0}^{n} (\dim \ker \partial^F_k + \dim \mathrm{im}\, \partial^F_k) t^k \\
&\quad - \sum_{k=0}^{n} (\dim \ker \partial^F_k - \dim \mathrm{im}\, \partial^F_{k+1}) t^k \\
&= \sum_{k=0}^{n} (\dim \mathrm{im}\, \partial^F_k + \dim \mathrm{im}\, \partial^F_{k+1}) t^k \\
&= \sum_{k=0}^{n} (m_k - \dim \ker \partial^F_k) t^k + \sum_{k=0}^{n} (m_{k+1} - \dim \ker \partial^F_{k+1}) t^k \\
&= (t+1) \sum_{k=1}^{n} (m_k - \dim \ker \partial^F_k) t^{k-1},
\end{aligned}
$$

since $m_0 = \dim \ker \partial^F_0$ and $m_{n+1} = 0 = \dim \ker \partial^F_{n+1}$.

The proof ends by using the fact that $\dim \ker \partial^F_k \leq m_k$ for all $k = 1, 2, \cdots, n$.

□

Observe that the Euler number of a CW complex $\mathbb{K}$ is given by

$$\chi(\mathbb{K}) = P_{-1}(\mathbb{K}).$$

Trajectory configurations

Restricting ourselves to regular CW complexes, we now define a trajectory to be an arrow between two critical cells of index difference one, going downward in indices. We show that just as in the smooth setting, the square of the boundary operator is zero because

the broken trajectories, that is, the trajectories between two critical cells of index difference two via a critical cell of intermediate index, occur in pairs.

Let $a_{n,n-1}$ denote a trajectory, that is an arrow, from an n-cell to an $(n-1)$-cell, assuming of course that they are both critical. We observe that just as in the smooth setting, the square of the boundary operator is zero means that the broken arrows, that is, the arrows between two critical cells of index difference two, passing through a critical cell of intermediate index, occur in pairs. So that if we take orientations into account or if we are working in $\mathbb{Z}_2$, they will both cancel.

Indeed, suppose that there is a broken trajectory from $\tau^{(n)}$ to $\nu^{(n-2)}$ through a given cell $\sigma_1^{(n-1)}$, then this means that $\sigma_1^{(n-1)}$ is either in the topological boundary of $\tau^{(n)}$ or it can be attained from it by using some upward noncritical cell in the topological boundary of τ, and the same holds for both σ_1 and ν. Thus we have $a_{n,n-1}$, an arrow from τ to σ_1, and $a_{n-1,n-2}$, an arrow from σ_1 to ν.
We show that there exists arrows $a'_{n,n-1}$, from τ to some σ' and $a'_{n-1,n-2}$, from σ' to ν. We have the following situations:

a) if $\nu < \sigma < \tau$ then there exists $\widetilde{\sigma} < \tau$ s.t. $\nu < \widetilde{\sigma}$, if $\widetilde{\sigma}$ is critical then take $\sigma' = \widetilde{\sigma}$ as seen in Figure 5.8.

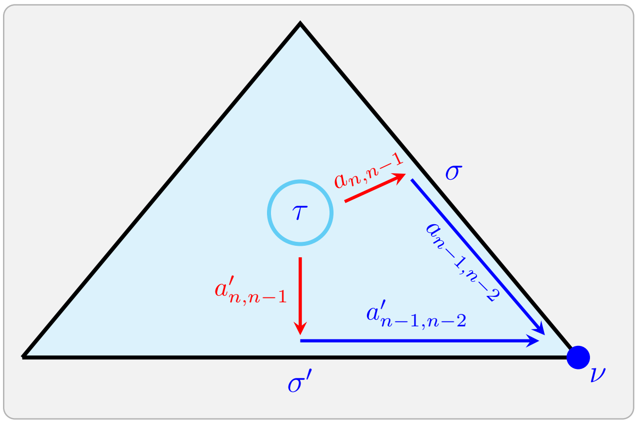


Figure 5.8: Broken trajectories with $\sigma' > \nu$.

If $\widetilde{\sigma}$ is not critical, we have:

if $\widetilde{\sigma}$ is downward noncritical then there is an $(n-1)$-cell $\nu_1 < \widetilde{\sigma}$ s.t. $f(\nu_1) \geq f(\widetilde{\sigma})$, then take σ' to be the critical facet of τ which is neither $\widetilde{\sigma}$ nor σ but which has ν_1 as a facet. Here $a'_{n-1,n-2}$ is the trajectory from σ' to ν passing through ν_1 and $\widetilde{\sigma}$. See Figure 5.9.

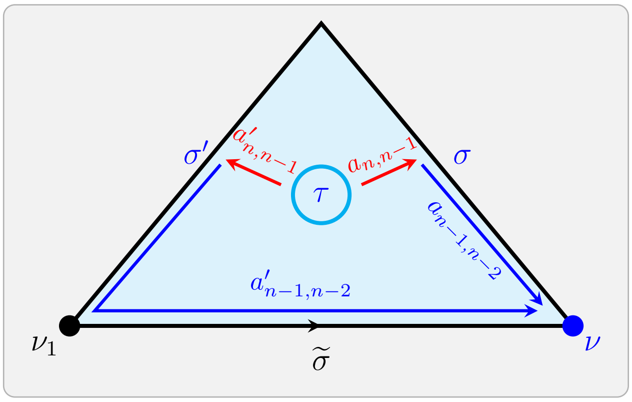


Figure 5.9: Broken trajectories when $\sigma' > \nu_1 \to \widetilde{\sigma} > \nu$.

If $\widetilde{\sigma}$ is upward noncritical with a τ_1 such that a $\nu_1 < \widetilde{\sigma}_1$ is upward noncritical with $\widetilde{\sigma}_1$ and then replaced by ν, then σ' is such that $\nu_1 < \sigma' < \tau_1$. In this case, $a'_{n-1,n-2}$ is the trajectory from σ' to ν passing through ν_1. This is illustrated in Figure 5.10.

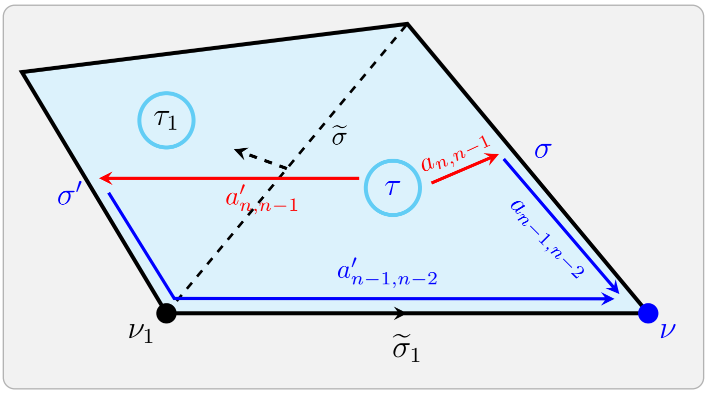


Figure 5.10: Broken trajectories for $\widetilde{\sigma} \to \tau_1 > \sigma'$.

b) If ν is not a face of τ, then there is a ν_1 face of τ such that ν_1 is upward noncritical, and then replaced by ν. Let $\widetilde{\sigma} < \tau$ be such that $\nu_1 < \widetilde{\sigma}$.

If $\widetilde{\sigma}$ is critical then take $\sigma' = \widetilde{\sigma}$ in which case $a'_{n-1,n-2}$ is the trajectory from σ' to ν; if not do as in a).

If $\widetilde{\sigma} < \tau$ is upward noncritical, then there is a τ_1 with $f(\tau_1) \leq f(\widetilde{\sigma})$ and $\widetilde{\sigma}$ is replaced by the topological boundary elements of τ_1 of the same dimension. Among them, pick the one, say $\widetilde{\sigma}_1$, which is such that $\widetilde{\sigma}_1 > \nu$. If it is critical then set $\sigma' = \widetilde{\sigma}_1$, so that $a'_{n,n-1}$ is the trajectory from τ to σ' passing through $\widetilde{\sigma}$ and τ_1; if not then continue as before. See Figure 5.11.

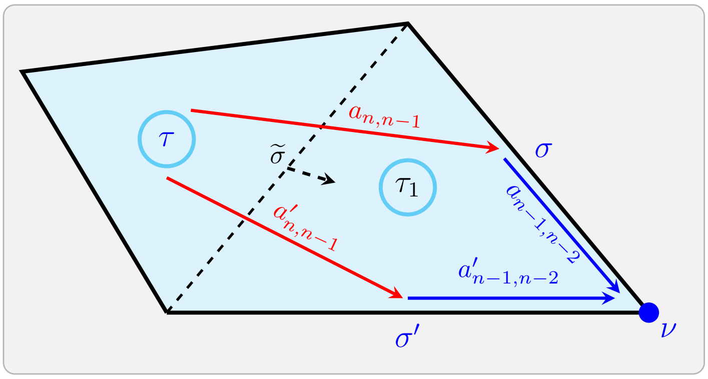


Figure 5.11: Broken trajectories when ν is not a face of τ.

Given $f\colon V \to \mathbb{R}$, where V is the vertex set of some CW complex $\mathbb{K}$, when can it be extended to a Forman's discrete Morse function on a CW complex?

It turns out that an extension $F\colon \mathbb{K} \to \mathbb{R}$ should be such that: For each $k = 0, 1, \cdots, \dim \mathbb{K}$,

$$F(\sigma^{(k)}) > \operatorname{avg}\{F(\nu^{(k-1)}) \,|\, \nu^{(k-1)} < \sigma^{(k)}\}$$

and

$$F(\sigma^{(k)}) < \operatorname{avg}\{F(\tau^{(k+1)}) \,|\, \tau^{(k+1)} > \sigma^{(k)}\}.$$

Observe that these are exactly the conditions given in the definition of a discrete Witten-Morse function, see Definition 5.1.2.

Example 5.2.2. Consider the extension given by

$$F(\sigma^{(k)}) = \max\{F(\nu^{(k-1)})|\nu^{(k-1)} < \sigma^{(k)}\} + 1,$$

if f is the constant 0 function on the set of vertices, we see that F will be the function that assigns to every cell its dimension (which is trivially a discrete Morse function).

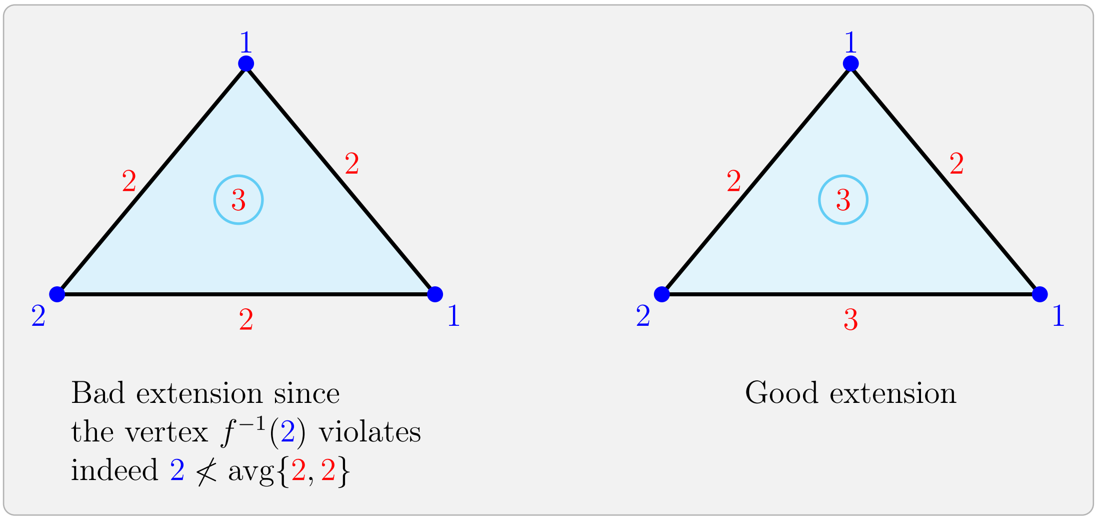


Figure 5.12: Extension of a function defined on vertices.

What happens if f is extended by taking averages? This will give rise to the notion of discrete Morse-Bott theory which is the object of the next section.

5.3 Discrete Morse-Bott Theory

The content in this section has been published in [50].

As mentioned earlier, an analogue of Morse-Bott theory in the discrete case might be of importance, since extending a function defined on the set of vertices might not always result in a discrete Morse function everywhere on the cell complex. The idea is, we consider a function that is discrete Morse on a cell complex except possibly at some maximal collection of cells, where every cell in the collection has the same value. We derive the analogue of the smooth Morse-Bott inequalities, that is, inequalities involving the Poincaré polynomial (and hence the Betti numbers) of the cell complex and those of the reduced collections. First we shall need an analogue of the Morse-Bott function in the discrete setting.

Definition 5.3.1 (Collection). *Let f be a discrete function on some CW complex. A collection C for f is a maximal set of cells such that:*

(*i*) *all the cells in* C *have the same function value,*

(*ii*) $\bigcup_{\sigma \in C} \bar{\sigma}$ *is (path-) connected.*

We recall that a pair $\{\sigma \,,\, \tau\}$ is said to be noncritical if $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$.

Definition 5.3.2 (Discrete Morse-Bott function). *Let* f *be a function defined on a CW complex* $\mathbb{K}$ *and let* $C^1, \cdots, C^l$ *be the collections for* f*. We say* f *is discrete Morse-Bott if, for each* i *and for all* $\sigma^{(k)} \in C^i$,

if σ *is an irregular face of some* τ, $f(\sigma) < f(\tau)$, *else*

$$U_n^c(\sigma) \;:=\; \sharp\{\tau^{(k+1)} \notin C^i \,|\, \sigma \text{ a regular facet of } \tau,\; f(\tau) < f(\sigma)\} \leq 1; \quad (5.11)$$

if $\nu < \sigma$ *is an irregular face of* σ, $f(\nu) < f(\sigma)$, *else*

$$D_n^c(\sigma) \;:=\; \sharp\{\nu^{(k-1)} \notin C^i \,|\, \nu \text{ a regular facet of } \sigma,\; f(\nu) > f(\sigma)\} \leq 1. \quad (5.12)$$

Remark 5.3.1. (*i*) If for all i, $C^i = \{\sigma_i\}$ and f is a discrete Morse-Bott function, then f is also a discrete Morse function.

(*ii*) Every discrete Morse function is a discrete Morse-Bott function in which each collection has at most two elements.

Remark 5.3.2. From the definition above, we get that in any collection there cannot be cells σ and τ with σ an irregular face of τ. This is useful in the situation where, if a collection reduces to only two cells of adjacent dimension, then one should not be an irregular facet of the other. This is important because in discrete Morse theory, the noncritical pairs always have no contribution.

It can be shown that either $U_n^c(\sigma) = 1$ or $D_n^c(\sigma) = 1$, but not both. The argument used is the same as in the discrete Morse case, by considering a discrete Morse-Bott function for which all the collections are singletons.

Figure 5.13 shows an example of a discrete function that is not a discrete Morse-Bott function. Take $C = \{\nu, \nu_2, \nu_1, \sigma_1, \sigma_2, \sigma_3, \tau\}$, then the vertex ν violates (5.12), indeed there are w_1 and w_2 not in C, with $w_1 > \nu < w_2$ but $f(w_2) < f(\nu) > f(w_1)$.

Definition 5.3.3. *Let* f *be a discrete Morse-Bott function and* C *a collection. We say that* $\sigma \in C$ *is* ***upward noncritical with respect to*** C *if there exists a cell* $w(\sigma) \notin C$, $w(\sigma) > \sigma$ *s.t.* $f(\sigma) > f(w(\sigma))$.

Definition 5.3.4. *Let* f *be a discrete Morse-Bott function and* C *a collection. We say that* $\sigma \in C$ *is* ***downward noncritical with respect to*** C *if there exists a cell* $w(\sigma) \notin C$, $w(\sigma) < \sigma$ *s.t.* $f(\sigma) < f(w(\sigma))$.

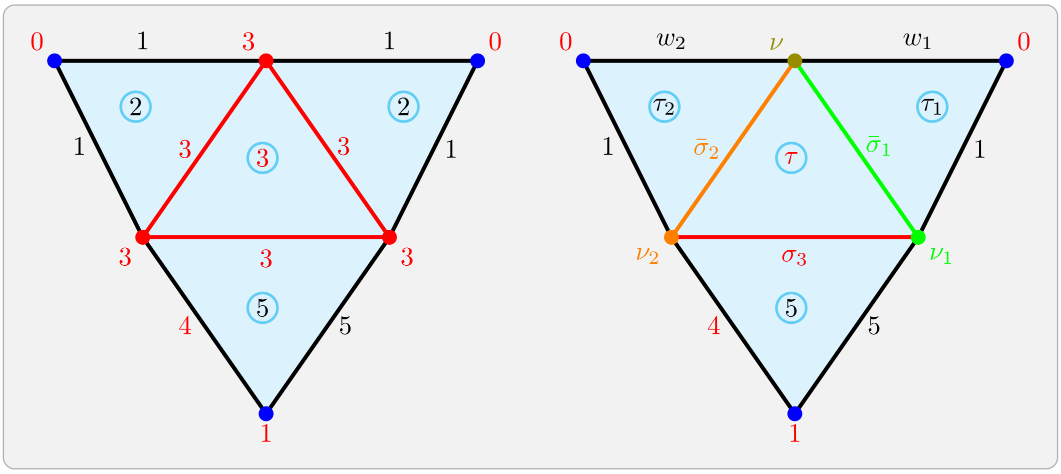


Figure 5.13: A discrete function that is not Morse-Bott.

Remark 5.3.3. If C is a singleton in the two definitions above, then we have the usual (upward and downward) noncriticalities in Forman's framework.

Observe that a cell cannot be at the same time upward and downward noncritical with respect to the same collection.

The following is the analogue of a critical submanifold.

Definition 5.3.5 (Reduced collection). *For each collection C, we define the reduced collection C^{red} by taking out of C all the cells that are upward or downward noncritical with respect to C.*

Remark 5.3.4. 1) If $C = \{\sigma\}$ with σ neither upward nor downward noncritical, then σ is critical in the discrete Morse sense for the function f and $C = C^{red}$.

2) If $C = \{\sigma\}$ where σ is either upward or downward noncritical, then $C^{red} = \emptyset$.

3) Observe that if C^{red} consists of only one element, then this element is not necessarily critical in the usual sense for the function f. To see this just consider a given function and a $C = \{\sigma_1, \nu, \sigma_2\}$, with $\sigma_1 > \nu < \sigma_2$ but both σ_1 and σ_2 being downward noncritical. Then $C^{red} = \{\nu\}$, but ν is not critical in the usual sense for this function since it has the same value with σ_1 and σ_2.

The following definition helps to distinguish between a cell that we will call critical and a reduced collection that is a singleton.

Definition 5.3.6 (Critical cell). *A cell σ is said to be critical if*

$$\sharp\{\tau > \sigma | f(\tau) \leq f(\sigma)\} = 0 \quad \textit{and} \quad \sharp\{\nu < \sigma | f(\nu) \geq f(\sigma)\} = 0.$$

The definition above tells us that if $C = C^{red} = \{\sigma\}$ then σ is critical.

The following lemma establishes the fact that the faces of upward noncritical cells w.r.t. a collection are also upward noncritical.

Lemma 5.3.1. *Let C be a collection for a discrete Morse-Bott function. If $\sigma \in C$ is upward noncritical with respect to C, then every face of σ in C is also upward noncritical w.r.t. C.*
In particular, if C is a subcomplex and $\sigma \in C$ is upward noncritical with respect to C, then so is $\bar{\sigma}$.

Proof. Let $\nu < \sigma$, and $\nu \in C$. The fact that σ is upward noncritical w.r.t. C implies there is an $w(\sigma) \notin C$, $w(\sigma) \in C'$ with $w(\sigma) > \sigma$, s.t. $f(\sigma) > f(w(\sigma))$. Since there are no irregular faces in a collection we get that ν is a regular face of σ, and the fact that there are no closed orbits tells us that σ is a regular face of $w(\sigma)$. We have $\nu < \sigma < w(\sigma)$, that is ν and $w(\sigma)$ are incident. Let $\tau \notin C$ be such that $w(\sigma) > \tau > \nu < \sigma$, such a τ exists by the incidence property. Then either $\tau \in C'$ or $\tau \notin C'$: if $\tau \in C'$ then $f(\tau) = f(w(\sigma))$; if $\tau \notin C'$, the definition of a discrete Morse-Bott function yields $f(w(\sigma)) > f(\tau)$. Thus, $f(\nu) = f(\sigma) > f(w(\sigma)) \geq f(\tau)$, therefore ν is also upward noncritical w.r.t. C, hence all the faces of σ in C are also upward noncritical with respect to C. $\square$

The most important fact about our definition of a discrete Morse-Bott function is the following.

Lemma 5.3.2. *If a collection C is such that $C \neq C^{red}$, then there is a one-to-one correspondence between the upward (resp. downward) noncritical cells σ w.r.t. C and the cells $w(\sigma) \notin C$, $w(\sigma) > \sigma$ with $f(\sigma) > f(w(\sigma))$ (resp. $w(\sigma) < \sigma$ with $f(\sigma) < f(w(\sigma))$).*

Proof. We already know from Lemma 5.3.1 that if $\sigma \in C$ is upward noncritical with respect to C, then every (regular) face of σ in C is also upward noncritical w.r.t. C. If C is a subcomplex, the interesting case would be to consider a situation where $\sigma_1, \sigma_2 \in C$ are upward noncritical w.r.t. C satisfying $\bar{\sigma}_1 \nsubseteq \bar{\sigma}_2$ and $\bar{\sigma}_2 \nsubseteq \bar{\sigma}_1$. Then, if we take a $\nu \in C$ such that $\sigma_1 > \nu < \sigma_2$, we have the existence of other cells $w(\sigma_i) \notin C$ for $i = 1, 2$ s.t. $w(\sigma_i) > \sigma_i$. Thus ν is a regular face of the σ_i*'s* and this also implies (by the incidence property), there exist $\tau_i \notin C$ for $i = 1, 2$ with $\nu < \tau_i < w(\sigma_i)$, and the following holds:

$$f(\nu) = f(\sigma_1) > f(w(\sigma_1)) \geq f(\tau_1) \quad \text{and} \quad f(\nu) = f(\sigma_2) > f(w(\sigma_2)) \geq f(\tau_2).$$

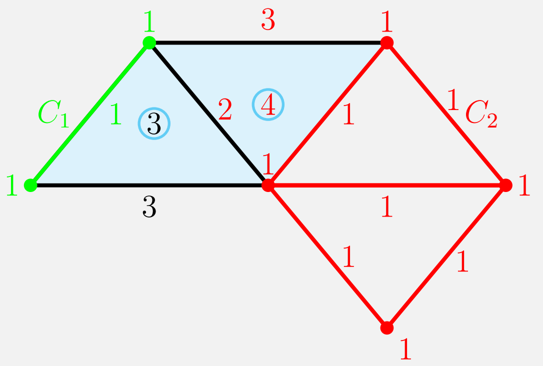
Thus, if $\tau_1 \neq \tau_2$, f at ν will have a value greater than that of more than one cell in the complement of C, and this will contradict the definition of a discrete Morse-Bott function.

If however $\tau_1 = \tau_2$, then there is a unique $w(\nu) = \tau \notin C$ for which $f(\nu) > f(w(\nu))$.

If $\sigma \in C$ is downward noncritical, then it follows directly from the definition of f. $\square$

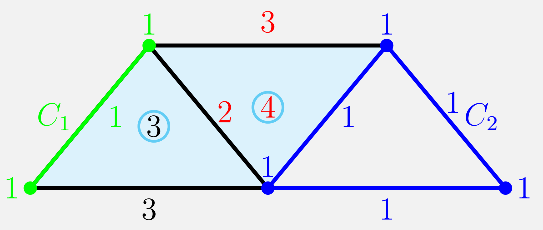
Figure 5.13 also shows that the result in Lemma 5.3.2 does not hold whenever we have a function that is not discrete Morse-Bott.

The notion of the index here however is ambiguous, but it will be shown in subsequent examples that the Euler number will always be counted with positive sign. See for instance Figure 5.14, Figure 5.15, and Figure 5.16; each of them involves a simplicial complex $\mathbb{K}$ with negative, zero and positive Euler number, and also reduced collections with negative, zero and positive Euler numbers. Recall that the contribution of a cell of index k, in the computation of the Euler number of a CW complex, is given by $(-1)^k$.



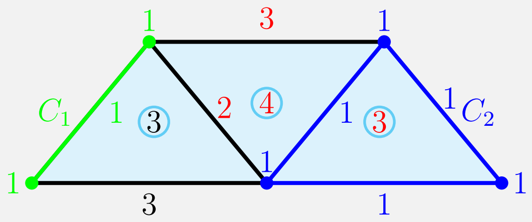
there are 2 critical edges $\rightsquigarrow$ -2
there is 1 critical 2-simplex $\rightsquigarrow$ 1
$-2 + 1 + \chi(C_1) + \chi(C_2)$
$= -1 + 1 + -1 = -1 = \chi(\mathbb{K})$.

Figure 5.14: A discrete Morse-Bott function on $\mathbb{K}$ with $\chi(\mathbb{K}) = -1$.



there are 2 critical edges $\rightsquigarrow$ -2
there is 1 critical 2-simplex $\rightsquigarrow$ 1
$-2 + 1 + \chi(C_1) + \chi(C_2)$
$= -1 + 1 + 0 = 0 = \chi(\mathbb{K})$.

Figure 5.15: A discrete Morse-Bott function on $\mathbb{K}$ with $\chi(\mathbb{K}) = 0$.



there are 2 critical edges $\rightsquigarrow\ -2$
there are 2 critical 2-simplices $\rightsquigarrow\ 2$
$-2+2+\chi(C_1)+\chi(C_2)$
$=0+1+0=1=\chi(\mathbb{K})$.

Figure 5.16: A discrete Morse-Bott function on $\mathbb{K}$ with $\chi(\mathbb{K})=1$.

Given that the reduced collections are not necessary subcomplexes, we need to make precise how their Betti numbers are obtained. First we prove the following:

Lemma 5.3.3. *If $\sigma \in \overline{C^{red}} \setminus C^{red}$ and $f(\sigma)=f(\tau)$ for $\tau \in C^{red}$, then σ cannot be downward noncritical.*

Proof. If $\sigma \in \overline{C^{red}} \setminus C^{red}$, then $\sigma \notin C^{red}$ but σ is a face of an element in C^{red}, say $\sigma < \tau$ for some $\tau \in C^{red}$.
If σ is downward noncritical, then there is a $\nu^\star < \sigma$ s.t. $f(\nu^\star) > f(\sigma)$ and all the other ($\nu \notin C^{red}$) $\nu < \sigma$ are such that $f(\sigma) > f(\nu)$. Note that $\nu^\star$ and τ are incident. Thus there is a $\sigma^\star > \tau$ s.t. $\nu^\star < \sigma^\star \neq \sigma$.
If $\sigma^\star \in C^{red}$ we have a contradiction since $f(\nu^\star) > f(\sigma) = f(\sigma^\star)$ that is, $f(\nu^\star)$ will be greater than $f(\sigma)$ and $f(\sigma^\star)$. Thus $\sigma^\star \notin C^{red}$, and $f(\sigma^\star) > f(\nu^\star)$ because $\nu^\star$ is already upward noncritical with σ. This automatically means that $f(\sigma^\star) > f(\nu^\star) > f(\sigma) = f(\tau)$, which also contradicts the fact that $\tau \in C^{red}$. $\square$

Lemma 5.3.4. *For any reduced collection C^{red}, $\overline{C^{red}} \setminus C^{red}$ is a subcomplex.*

Proof. Let $\sigma \in \overline{C^{red}} \setminus C^{red}$, and $\nu < \sigma$, then by definition of $\overline{C^{red}}$, ν is also in $\overline{C^{red}}$. Thus, to show that $\nu \in \overline{C^{red}} \setminus C^{red}$, we only need to show that $\nu \notin C^{red}$.

For $\sigma \in \overline{C^{red}} \setminus C^{red}$, σ is a face of an element in C^{red}, say $\sigma < \tau$ for some $\tau \in C^{red}$. The fact that $\tau \in C^{red}$ and $\sigma < \tau$ implies that $f(\sigma) \leq f(\tau)$.
If $f(\sigma)=f(\tau)$, this means that σ is either downward or upward noncritical w.r.t. C. From Lemma 5.3.3, such a σ cannot be downward noncritical. If σ is upward noncritical, then since σ cannot be downward noncritical, we have $f(\nu) \leq f(\sigma)$. If $f(\nu)=f(\sigma)$, then ν has to be in C, and by Lemma 5.3.1, ν also has to be upward noncritical w.r.t. C, which implies that $\nu \notin C^{red}$. If $f(\nu) < f(\sigma)$ then $\nu \notin C^{red}$ by definition of C^{red}.
If $f(\sigma) < f(\tau)$ then either $f(\nu) \leq f(\sigma) < f(\tau)$, $f(\sigma) < f(\nu) < f(\tau)$, or $f(\nu) = f(\tau) > f(\sigma)$. In any case ν cannot be in C^{red}. $\square$

Remark 5.3.5. For a given collection C, $\overline{C^{red}}$ is not always equal to C. Figure 5.18 illustrates this: C is the collection of the 2-cell with value 3, the two red edges, the red vertex and the two green vertices. After removing the upward noncritical vertices w.r.t. C, that is the two green vertices, and the downward noncritical 2-cell w.r.t. C, we end up with the two red edges and the red vertex. Thus, $\overline{C^{red}}$ consists of the two red edges, the red vertex and the two green vertices, which is different from C.

Let $[\tau : \sigma]$ denote the incidence number of τ and σ, which is the number of times that τ is wrapped around σ, taking the induced orientation from τ onto σ into account.

Let $C_k(C^{red}; \mathbb{Z})$ be the free $\mathbb{Z}$-module generated by the oriented cells of C^{red} of dimension k. One can also take $C_k(C^{red}; \mathbb{Z}_2)$.

Definition 5.3.7. *The boundary operator* $\partial_k^{red} \colon C_k(C^{red}, \mathbb{Z}) \to C_{k-1}(C^{red}, \mathbb{Z})$ *is given by:*

$$\partial_k^{red} \tau^{(k)} = \sum_{\sigma \in C^{red}, \sigma < \tau} [\tau : \sigma] \sigma^{(k-1)}.$$

Proposition 5.3.5.

$$\partial_{k-1}^{red} \circ \partial_k^{red} = 0.$$

Proof. If C^{red} is a subcomplex then it follows from the classical theory.
If C^{red} is not a subcomplex, the result will follow if we show that the above boundary operator is a relative boundary operator, that is, a boundary operator for relative homology. Indeed

$\overline{C^{red}} := \bigcup_{\sigma \in C^{red}} \bar{\sigma}$ is a subcomplex by definition.

Let $X := \overline{C^{red}}$ and $A := \overline{C^{red}} \setminus C^{red}$ then X is a subcomplex, and A is a subcomplex of X by Lemma 5.3.4, and the result follows since $C^{red} = X \setminus A$. □

Let $b_k^{red} := \dim \left(\ker \partial_k^{red} / \operatorname{im} \partial_{k+1}^{red} \right)$ then we define the Poincaré polynomial of C^{red} by

$$P_t(C^{red}) = \sum_k b_k^{red} t^k.$$

Example 5.3.1. Now we can change the function in Figure 5.13 to make it discrete Morse-Bott and this is illustrated in Figure 5.17. Where, C is the (subcomplex) collection of all the simplices with value 3. C has three upward noncritical vertices and one upward noncritical edge, highlighted in green. Then $C^{red} = \{\sigma_2, \sigma_3, \tau\}$. Observe that one can retrieve the Euler number of the complex just by adding that of C^{red} to those of the critical simplices. The contribution for C^{red} is -1. There are three critical vertices and one critical edge which all together contribute for 2. Thus one has $\chi(C^{red}) + 2 = -1 + 2 = 1 = \chi(\mathbb{K})$, where $\chi(\mathbb{K})$ is the Euler number of the cell complex.

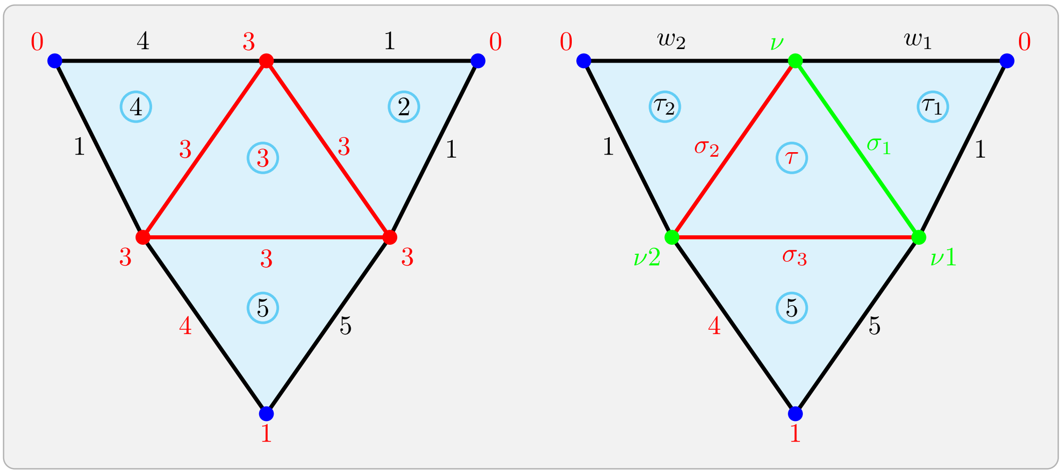


Figure 5.17: A discrete Morse-Bott function.

Before we state our analogue of the Morse-Bott inequalities which generalizes the idea above, we prove the following proposition.

Let $n_k := \sharp\{\sigma \in C^{red} | \dim \sigma = k\}$, then $n_k = \dim C_k(C^{red}; \mathbb{Z}) := \dim C_k^{red}$, and let $s := \dim C^{red}$.

Proposition 5.3.6. *Let C^{red} be a reduced collection for a discrete Morse-Bott function. Then there exists $r(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_{k=0}^{s} n_k t^k = P_t(C^{red}) + (1+t)r(t).$$

Proof. The idea of the proof is the same as in the proof of Proposition 5.2.3. □

Remark 5.3.6. In general, the contribution of each noncritical pair cancels out whenever one looks at the Euler number, thus it saves a lot of time to ignore the noncritical pairs.

The idea in Example 5.3.1 is the content of the following analogue of the Morse-Bott inequalities.

Theorem 5.3.7 (Discrete Morse-Bott inequalities). *Let f be a discrete Morse-Bott function on an n-dimensional CW complex $\mathbb{K}$, and let $C^{i,red}$, $i = 1, \dots, l$ be its nonempty disjoint reduced collections that are not noncritical pairs. Then there exists $R(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_{i=1}^{l} P_t(C^{i,red}) = P_t(\mathbb{K}) + (1+t)R(t). \tag{5.13}$$

The result (5.13) implies that, setting $t = -1$ above, the Euler number of each reduced collection should always be counted with positive sign.

If we want to have the formula

$$\sum_{i=1}^{l} P_t(C^{i,red}) t^{ind(C^{i,red})} = P_t(\mathbb{K}) + (1+t)R(t),$$

then we have to put $\operatorname{ind}(C^{i,red}) = 0$ for all $i = 1, \cdots, l$.

Remark 5.3.7. One of the reasons for defining the index as above is the fact that the Poincaré polynomial of a point in the smooth setting is given by 1 while in the discrete setting, the Poincaré polynomial of a cell of dimension k is given by t^k, as we shall see later on.

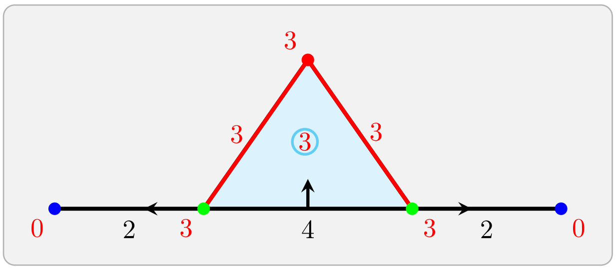


Figure 5.18: A discrete Morse-Bott-Conley method.

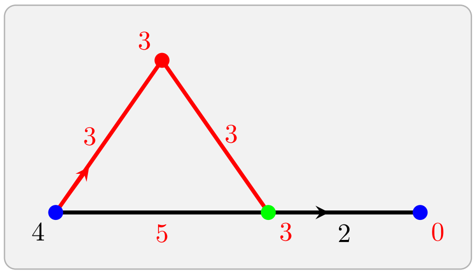


Figure 5.19: A reduced collection that is a noncritical pair.

Example 5.3.2. 1) In Figure 5.18, C consists of the red vertex, the two green vertices, the two red edges and the 2-simplex with value 3. Removing all the noncritical cells w.r.t. C yields that C^{red} consists of the two red edges and the red vertex. Thus, $P_t(C^{red}) = t$; in addition, there are two critical vertices, thus their overall contribution is 2. Hence the discrete Morse-Bott inequalities are satisfied since $\sum_i P_t(C^{i,red}) = t + 2$ and $P_t(\mathbb{K}) = 1$, that is, $R(t) \equiv 1$.

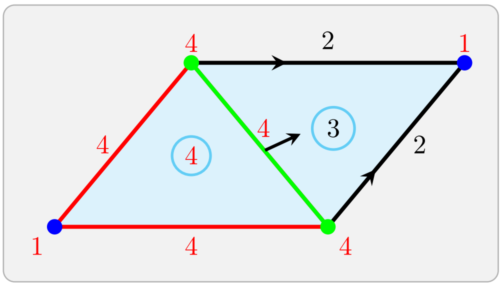


Figure 5.20: Example of a discrete Morse-Bott-Conley method.

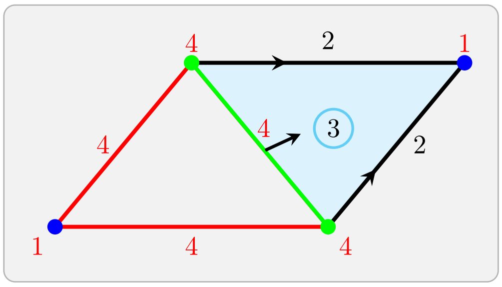


Figure 5.21: Another example of a discrete Morse-Bott-Conley method.

2) In Figure 5.19, C consists of the red vertex, the two red edges and the green vertex. After removing the upward or downward noncritical cells w.r.t. C, we obtain a reduced collection that is a noncritical pair, so we do not take it into consideration. Thus we only take into account the critical edge and the critical vertex, and adding their contributions yield $t+1$ which is the Poincaré polynomial of the complex, thus, $R(t) \equiv 0$.

3) In Figure 5.20, C is the collection of the two red edges, the green edge, the two green vertices and the 2-simplex with value 2. C^{red} consists of the two red edges and the 2-simplex. $P_t(C^{red}) = t$, to this we add the contributions for the two critical vertices. We get $\sum_i P_t(C^{i,red}) = t + 2$ and $P_t(\mathbb{K}) = 1$. Here, $R(t) \equiv 1$.

3) In Figure 5.21, C is the collection of the two red edges, the green edge and the two green vertices. C^{red} consists of the two red edges. $P_t(C^{red}) = 2t$, to this we add the contributions for the two critical vertices, and we get, $\sum_i P_t(C^{i,red}) = 2t + 2$, $P_t(\mathbb{K}) = 1 + t$. Thus, $R(t) \equiv 1$.

The following lemma is useful for the proof of Theorem 5.3.7.

Lemma 5.3.8. $\sum_i \dim\ker \partial_k^{C^{i,red}} - \dim\ker \partial_k^F \geq 0$ *for each* $k \geq 1$.

Proof. Suppose that the reduced collections $C^{i,red}$, $i = 1, \cdots, l$ are indexed such that $f(C^{i,red}) \leq f(C^{j,red})$ for $i \leq j$. We know that if $\sigma \in \partial^F \tau$, where ∂^F denotes Forman's boundary operator, then $f(\sigma) < f(\tau)$. Using the reduced collections, if $\tau \in C^{i,red}$, and $\sigma \in \partial^F \tau$, then either $\sigma \in C^{i,red}$, in which case $f(\sigma) = f(\tau)$, or there is a $C^{j,red}$, s.t. $\sigma \in C^{j,red}$, in which case one immediately sees that $f(C^{j,red}) < f(C^{i,red})$, so $j \leq i$.
Also, we know that
$$C_k(\mathbb{K}, R) = \oplus_{i=1}^{l} C_k(C^{i,red}, R),$$

where ($R = \mathbb{Z}$ or $\mathbb{Z}_2$).

Let $\partial_k^i := \partial_k^{C^{i,red}}$, then $\sigma \in C_k(\mathbb{K}, R) \Rightarrow \sigma = \sigma^1 + \sigma^2 + \cdots + \sigma^l$ with $\sigma^i \in C_k(C^{i,red}, R)$ where,

$$\begin{aligned} \partial^F \sigma^i &= \mathrm{Proj}_{C^{i,red}} \partial^F \sigma^i + \sum_{j=1}^{i-1} \mathrm{Proj}_{C^{j,red}} \partial^F \sigma^i \\ &= \partial^i \sigma^i + \sum_{j=1}^{i-1} \mathrm{Proj}_{C^{j,red}} \partial^F \sigma^i, \quad \text{for} \quad i = 1, \cdots, l, \end{aligned}$$

and Proj denotes the projection map.

If $\sigma \neq 0$ and $\sigma \in \ker \partial^F$, then $\sum_i \partial^F \sigma^i = 0$.
If $\sigma = \sigma^1$ then $\partial^F \sigma = 0 \Leftrightarrow \partial^1 \sigma^1 = 0$ and we are done.
If at least one of the σ^i's is different from zero, since $\mathrm{Proj}_{C^{j,red}} \partial^F \sigma^i = 0$ if $j > i$, we get immediately that $\partial^l \sigma^l = \mathrm{Proj}_{C^{l,red}} \partial^F \sigma = 0$.
If $\sigma^l = 0$, do the same for σ^{l-1} and so on, until you get to σ^1, which we know will have to be different from zero in order not to contradict the fact that $\sigma \neq 0$. Thus at least one of the σ^i's must be different from zero. Hence $\sum_i \dim\ker \partial_k^i \geq \dim\ker \partial_k^F$.

Now let us suppose $\dim\ker \partial_k^F = 2$. Let $\sigma_i \neq 0$, for $i = 1, 2$ and $\sigma_1 \neq \sigma_2$. Suppose $\sigma_i \in \ker \partial_k^F$, for $i = 1, 2$ then $\sigma_1 = \sigma_1^1 + \sigma_1^2 + \cdots + \sigma_1^l$ and $\sigma_2 = \sigma_2^1 + \sigma_2^2 + \cdots + \sigma_2^l$. As before, we will have $\partial^l \sigma_1^l = 0$ and $\partial^l \sigma_2^l = 0$. If $\sigma_1^l = \sigma_2^l \neq 0$, then $0 \neq \bar{\sigma} := \sigma_1 + \sigma_2 = \sigma_1^1 - \sigma_2^1 + \cdots + \sigma_1^{l-1} - \sigma_2^{l-1}$ satisfies $\partial^F \bar{\sigma} = 0$. From the previous step it follows that $\sum_{i=1}^{l-1} \dim\ker \partial_k^i \geq 1$. Using the fact that $\sigma_1^l \in \ker \partial_k^l$, we then get that $\sum_i^l \dim\ker \partial_k^i \geq 2$. If $\sigma_1^l = 0 \,\&\, \sigma_2^l \neq 0$ or $\sigma_1^l \neq 0 \,\&\, \sigma_2^l = 0$, we are done. If $\sigma_1^l = 0 \,\&\, \sigma_2^l = 0$, then do the same for $\sigma_1 = \sigma_1^1 + \cdots + \sigma_1^{l-1}$ and $\sigma_2 = \sigma_2^1 + \cdots + \sigma_2^{l-1}$. Hence $\sum_i^l \dim\ker \partial_k^i \geq \dim\ker \partial_k^F$.

We assume that the statement is true for $\dim\ker \partial_k^F \leq m - 1$ and we show it is true for m. Suppose $\sigma_i \in \ker \partial_k^F$ and are all linearly independent for $i = 1, \cdots, m$, where $\sigma_i = \sigma_i^1 + \cdots + \sigma_i^l$.
Proceeding as before, we get $\sigma_i^l \in \ker \partial_k^l$ for $i = 1, \cdots, m$. If $\sigma_i^l \neq 0$ for all i and $\sigma_i^l \neq \sigma_j^l$ for all $i \neq j$ we are done.
If $\sigma_i^l \neq 0$ for all i but $\sigma_1^l = \sigma_2^l = \cdots = \sigma_m^l$, then we get $\bar{\sigma}_i := \sigma_1 - \sigma_{i+1} \in \ker \partial_k^F$ for $i = 1, \cdots, m-1$. The linear independence of the $\bar{\sigma}_i$'s follows from that of the σ_i's, and by induction hypothesis, $\sum_{i=1}^{l-1} \dim\ker \partial_k^i \geq m - 1$. Hence, $\sum_{i=1}^{l} \dim\ker \partial_k^i \geq m = \dim\ker \partial_k^F$, also taking $\sigma_i^l \in \ker \partial_k^l$.
If $\sigma_i^l \neq 0$ for all i but $\sigma_1^l \neq \sigma_2^l = \cdots = \sigma_m^l$, then we get $\bar{\sigma}_i := \sigma_2 - \sigma_{i+2} \in \ker \partial_k^F$ for $i = 1, \cdots, m-2$. By induction hypothesis, $\sum_{i=1}^{l-1} \dim\ker \partial_k^i \geq m - 2$. Hence, $\sum_{i=1}^{l} \dim\ker \partial_k^i \geq m = \dim\ker \partial_k^F$, also adding σ_1^l and σ_2^l in $\ker \partial_k^l$.
The same idea is used if there are subsets $A_1, \cdots, A_s$ of $\{1, \cdots, m\}$ s.t. $\sigma_i^l = \sigma_j^l$

for all $i \neq j$, $i, j \in A_p$, but $\sigma_i^l \neq \sigma_j^l$ for $i \in A_p$ and $j \in A_q$, $p \neq q$.
Indeed: $|A_1| + \cdots + |A_s| = m$, and for each A_i, we define $\bar{\sigma}_j^{A_i} := \sigma_1^{A_i} - \sigma_{j+1}^{A_i}$ for $j = 1, \cdots, |A_i| - 1$. From the $\bar{\sigma}_j^{A_i}$ for $i = 1, \cdots, s$ and $j = 1, \cdots, |A_i| - 1$, and the induction hypothesis, we have: $\sum_{i=1}^{l-1} \dim \ker \partial_k^i \geq \sum_{i=1}^{s}(|A_i| - 1) = m - s$. Taking into consideration the fact that in each A_i we have $\sigma_j^l \in \ker \partial_k^l$, we get $\sum_{i=1}^{l} \dim \ker \partial_k^i \geq m$.
If $\sigma_i^l = 0$, we do the same for σ_i^{l-1} and so on.

□

We now have all the necessary tools to prove Theorem 5.3.7.

Proof of Theorem 5.3.7. Let $f_\varepsilon \colon \mathbb{K} \to \mathbb{R}$ given by

$$f_\varepsilon(\sigma) = f(\sigma) - \frac{\varepsilon}{\dim \sigma + 1},$$

then $f_\varepsilon \to f$ as $\varepsilon \to 0$.

Let $n_k^i := \sharp\{\sigma \in C^{i,red} \mid \dim \sigma = k\}$ and $s^i = \dim C^{i,red}$.

Claim:
For sufficiently small ε, f_ε is discrete Morse and
$\{\sigma^{(k)} \text{ critical for } f_\varepsilon\} = \bigcup_{i=1}^{l}\{\sigma^{(k)} \in C^{i,red}\}$, that is

$$m_k^{f_\varepsilon} = \sum_{i=1}^{l} n_k^i, \tag{5.14}$$

where $m_k^{f_\varepsilon}$ is the number of critical points of f_ε of dimension k.

Proof of the claim. Let C^i be a collection and $\sigma \in C^i$. We recall that if $\sigma < \tau$ is an irregular facet of τ, then $f(\sigma) < f(\tau)$ and the same holds for f_ε. So, it is enough to do the following at the regular facets.

$$\begin{aligned}
\sharp\{\nu < \sigma | f_\varepsilon(\nu) \geq f_\varepsilon(\sigma)\} &= \sharp\{\nu \in C^i, \nu < \sigma | f_\varepsilon(\nu) \geq f_\varepsilon(\sigma)\} \\
&+ \sharp\{\nu \notin C^i, \nu < \sigma | f_\varepsilon(\nu) > f_\varepsilon(\sigma)\} \\
&=: A_1 + B_1
\end{aligned}$$

$$\begin{aligned}
\sharp\{\tau > \sigma | f_\varepsilon(\tau) \leq f_\varepsilon(\sigma)\} &= \sharp\{\tau \in C^i, \tau > \sigma | f_\varepsilon(\tau) \leq f_\varepsilon(\sigma)\} \\
&+ \sharp\{\tau \notin C^i, \tau > \sigma | f_\varepsilon(\tau) < f_\varepsilon(\sigma)\}
\end{aligned}$$

$$=: \quad A_2 + B_2$$

We have, $A_1 = 0$ (resp. $A_2 = 0$) since, if $\nu < \sigma$, meaning that $\dim \nu = \dim \sigma - 1$ (resp. $\tau > \sigma$ meaning that $\dim \tau = \dim \sigma + 1$), and both are in the same collection that is $f(\sigma) = f(\nu)$ (resp. $f(\sigma) = f(\tau)$), we get that $f_\varepsilon(\sigma) > f_\varepsilon(\nu)$ (resp. $f_\varepsilon(\sigma) < f_\varepsilon(\tau)$).

For sufficiently small ε, that is, as $\varepsilon \to 0$, we have $B_1 = \sharp\{\nu \notin C^i, \nu < \sigma | f(\nu) > f(\sigma)\}$ and $B_2 = \sharp\{\tau \notin C^i, \tau > \sigma | f(\tau) < f(\sigma)\}$, so that, $B_1 \leq 1$ and $B_2 \leq 1$ follow from the discrete Morse-Bott condition for f. Hence $A_1 + B_1 \leq 1$ and $A_2 + B_2 \leq 1$ which implies that f_ε is discrete Morse.

We show that

$$\{\sigma^{(k)} \text{ critical for } f_\varepsilon\} = \bigcup_i \{\sigma^{(k)} \ \in \ C^{i,red}\}. \tag{5.15}$$

'$\Rightarrow$ ' Let σ be critical for f_ε, this implies that $B_1 = 0$ and $B_2 = 0$. For ε small enough, $B_1 = 0$ means that x is not downward noncritical w.r.t C^i, and $B_2 = 0$ means that σ is not upward noncritical w.r.t C^i. Therefore σ should be in $C^{i,red}$.
'$\Leftarrow$ ' If $\sigma \in C^{i,red}$, then σ was neither upward noncritical nor downward noncritical w.r.t. C^i for f. So for sufficiently small ε, $B_1 = 0 = B_2$ and this implies that σ is critical for f_ε. □

Now to end the proof of the theorem: from Proposition 5.3.6, we have for each i,

$$\sum_{k=0}^{s^i} n_k^i t^k = P_t(C^{i,red}) + (1+t) r_i(t), \tag{5.16}$$

where each $r_i(t)$ is a polynomial in t with nonnegative integer coefficients.
The function f_ε is discrete Morse on $\mathbb{K}$ for sufficiently small ε, so from the Morse inequalities we have:

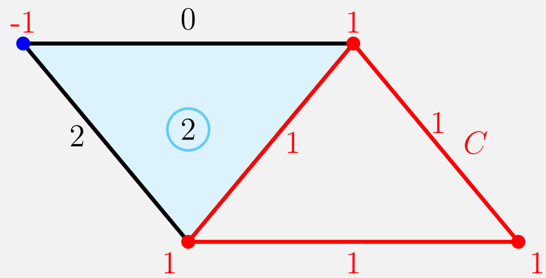
$$\begin{aligned}
& \sum_{k=0}^{n} m_k^{f_\varepsilon} t^k = \sum_{k=0}^{n} b_k t^k + (1+t) r^1(t) \\
\Rightarrow \quad & \sum_{k=0}^{n} \Big(\sum_{i=1}^{l} n_k^i\Big) t^k = \sum_{k=0}^{n} b_k t^k + (1+t) r^1(t) \quad \text{from (5.14)} \\
\Rightarrow \quad & \sum_{i=1}^{l} P_t(C^{i,red}) + (1+t) r_i(t) = \sum_{k=0}^{n} b_k t^k + (1+t) r^1(t) \quad \text{from (5.16)} \\
\Rightarrow \quad & \sum_{i=1}^{l} P_t(C^{i,red}) = P_t(\mathbb{K}) + (1+t) R(t).
\end{aligned}$$

Note that

$$r^1(t) = \sum_{k=1}^{n} (m_k^{f_\varepsilon} - \dim\ker\partial_k^F)t^{k-1} \quad \text{and } r_i(t) = \sum\nolimits_{k=1}^{s^i} (n_k^i - \dim\ker\partial_k^{C^{i,red}})t^{k-1},$$

where $\partial_k^{C^{i,red}} := \partial^c_{k|C^{i,red}}$, . Thus $R(t) = \sum_{k=1}^n (\sum_{i=1}^l \dim\ker\partial_k^{C^{i,red}} - \dim\ker\partial_k^F)t^{k-1}$. and the result follows from Lemma 5.3.8. □

Figure 5.22 illustrates the fact that the discrete Morse-Bott inequalities are not satisfied when we do not reduce the collections.



there is 1 critical vertex $\rightsquigarrow$ 1
$1 + \chi(C) = 1 + 0 = 1 \neq \chi(S) = 0$.
Thus the Morse-Bott inequalities are not satisfied. This is not surprising since C is not a reduced collection.

Figure 5.22: A counter example when C is not a reduced collection.

Remark 5.3.8. 1) Let f be a discrete Morse on $\mathbb{K}$ except on some subcomplex C^i where it is constant. To approximate f to get a discrete Morse function f_ε on the entire complex $\mathbb{K}$, one needs the condition that:

While approximating $f^i := f_{|C^i}$ into a discrete Morse function, any cell $\sigma \in C^i$ noncritical w.r.t. C^i for f should not be noncritical for f^i_ε.

Figure 5.23 illustrates why we need this condition. The green vertex was initially not critical for the function f, and, after approximating we also made it noncritical for the function $f_{\varepsilon|C}$. This yields a function on the entire complex that is not discrete Morse.

2) Observe that if C^i is a subcomplex of some CW complex $\mathbb{K}$, $C^{i,red} \neq C^i$, and f_ε an approximating discrete Morse function on $\mathbb{K}$, a critical point for $f^i_\varepsilon := f_{\varepsilon|C^i}$ need not be critical for f_ε. This will then mean that in the proof above, (5.15) need not hold if we replace $C^{i,red}$ by C^i. This is shown in Figure 5.24, where C is the collection of all the simplices having the value 1, and C^{red} is the collection of all the simplices of C except the green vertex. The function f_ε is a trivial discrete Morse function on C. After approximating, the green vertex is critical inside C but not critical on $\mathbb{K}$.

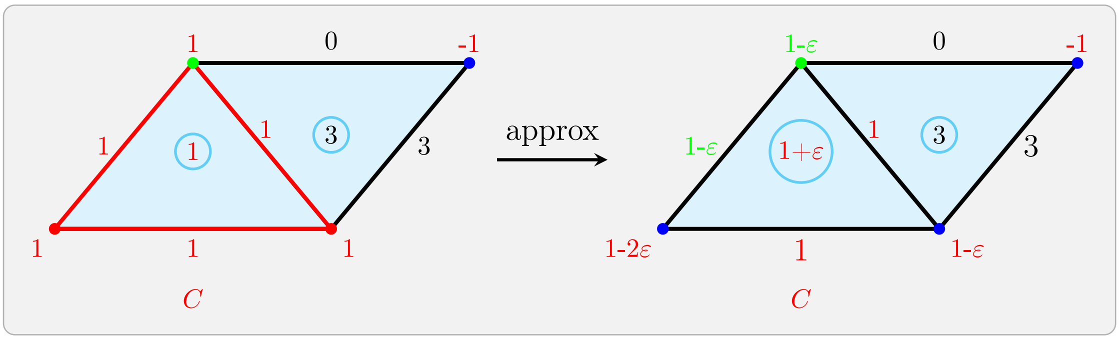


Figure 5.23: A noncritical cell violates the discrete Morse condition after approximating.

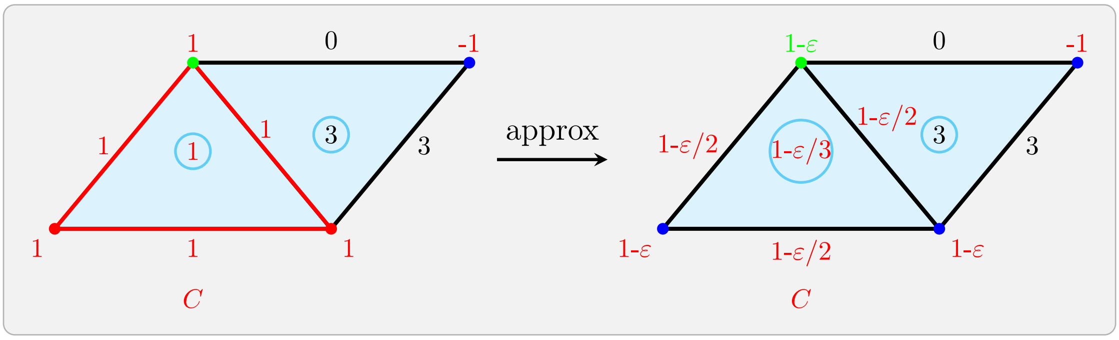


Figure 5.24: A noncritical cell stays noncritical after approximating.

3) One should also observe that our proof, being based on some perturbation idea of the discrete Morse-Bott function to get a discrete Morse function, will not work if we use a function that does not necessarily assume the same value on the collections.

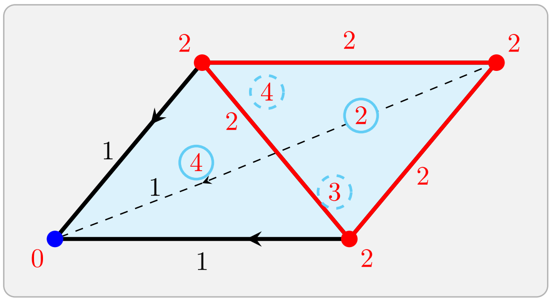
After realizing that an approach of Floer's in the case of a discrete Morse function is possible one would like to ask the question if something similar can be done using a discrete Morse-Bott function.

Question: *is it possible to do some kind of Floer-related theory on a complex having a discrete Morse-Bott function?*

It turns out that,

we cannot consider the whole reduced collection as one critical object.

Indeed, using Figure 5.25, let the reduced collection C^{red} to be $C^{red} = \{\sigma_1, \sigma_2, \sigma_3, \tau_1\}$, that is, from the collection with values 1, we have removed the upward noncritical vertices. If we suppose that:



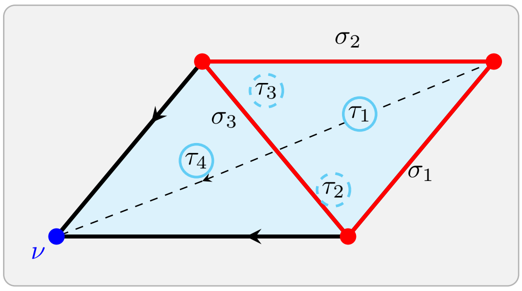


Figure 5.25: A reduced collection cannot be one critical object.

$\partial_0(\nu) = 0, \quad \partial_1(C^{red}) = (\nu - \nu) + (\nu - \nu) + (\nu - \nu) = 0;$
$\partial_2(\tau_4) = \partial_2(\tau_3) = \partial_2(\tau_2) = C^{red}, \quad \partial_2(C^{red}) = 0,$ then:
$\ker \partial_0 = \langle \nu \rangle, \quad \operatorname{im} \partial_1 = 0;$
$\ker \partial_1 = \langle C^{red} \rangle, \quad \operatorname{im} \partial_2 = \langle C^{red} \rangle;$
$\ker \partial_2 = \langle C^{red}, \tau_3 + \tau_2, \tau_3 + \tau_4 \rangle;$
this will then yield, $b_0 = 1, \ b_1 = 0, \ b_2 = 3$ which is not correct.
This problematic result is mostly because we have

$$\partial_2(\tau_4) = \partial_2(\tau_3) = \partial_2(\tau_2) = C^{red}$$

which would not be the case if C^{red} was considered as a collection of critical cells.

Thus, a plausible idea would be the following: if C^{red} has dimension k, then we have to consider it as a critical element in all dimensions $0, 1, 2, \cdots k$, whenever it has an element of dimension $0, 1, \cdots, k$, and to be more precise, the only solution to the question above is the following:

Solution: *we approximate the discrete Morse-Bott function on each collection to get a discrete Morse function; this is always possible as seen before, so that any upward or downward noncritical cell w.r.t. a given collection will not be critical after the approximation. In this way, all the cells in the reduced collections will be critical just by making the approximating function to be a trivial discrete Morse function.*

The above solution shows that the only way we can have a boundary operator on some complex on which is defined our discrete Morse-Bott function is by approximating to get a discrete Morse function. We now ask: having a discrete Morse-Bott function, is it possible to develop any other theory using a more dynamics-related method to get some insight on the Euler number? This is what we will be investigating in the last part of this chapter. But before we get to that, a good question to ask will be the following:

Why should the Euler number of the reduced collection always be counted with positive sign whereas this is not always the case in the smooth setting when critical submanifolds are taken into account?

Now we use the discrete Morse-Bott function to derive a similar notion to smooth Conley theory, this is our version of discrete Conley theory.

5.4 Discrete Morse-Bott-Conley theory

The content in this section has also been published in [50].

The scope of Conley theory is more general than that of Morse theory, because if we consider the discrete vector field originating from some discrete Morse function, then this combinatorial vector field does not admit any closed orbit. This makes things easy in the sense that, the only critical objects under consideration are the critical cells of the discrete Morse function, and we will show below that for a critical cell of dimension k, the homological Conley index is just given by t^k, or $(-1)^k$ for $t = -1$.

Forman, in [23], made things more interesting by considering a combinatorial vector field in general (it need not originate from some discrete Morse function and can therefore admit closed orbits). In general, a combinatorial vector field on a CW complex yields a disjoint collection C_r of rest points or closed orbits, where the rest points are exactly the critical cells. For each C_r, let $\overline{C_r}$ be the union of all the cells in C_r with the ones in their boundaries, and let $\widetilde{C_r} := \overline{C_r} \setminus C_r$. The pair $(\overline{C_r}, \widetilde{C_r})$ is taken to be an index pair for the corresponding C_r. Forman showed that with

$$m_i := \sum_{C_r} \dim H^i(\overline{C_r}, \widetilde{C_r}; \mathbb{Z}),$$

the Morse inequalities are satisfied.

Now we want to go beyond a combinatorial vector field and we do this by considering our discrete Morse-Bott function on a CW complex $\mathbb{K}$. The vector field originating from this function inside each collection need not be a combinatorial vector field as a cell in the collection can have more than one incoming and/or outgoing arrow, see Figure 5.26, but between the collections there should not be any closed orbit.

We shall consider as isolated invariant sets the reduced collections (excluding the noncritical pairs), and their isolating neighborhood will be the subcomplex

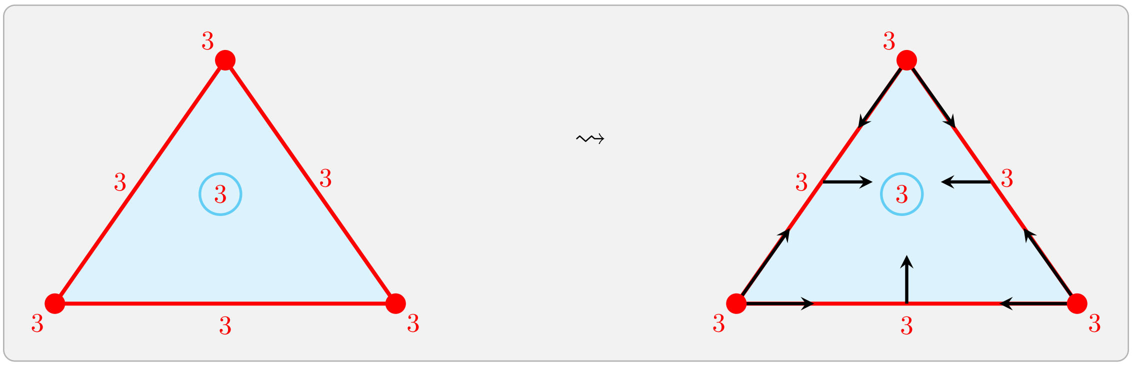


Figure 5.26: Vector field inside a collection.

generated by the isolated invariant set. The exit set will be the part of $\partial^{top} N$ where the values of the function are smaller than on the isolated invariant set. This is the content of the following definition.

Definition 5.4.1. 1) *I is said to be an **isolated invariant** set if it is a reduced collection that is not a noncritical pair.*

2) *An **isolating neighborhood** N for I is the union of all the cells in I together with all the cells in their boundaries. That is,*

$$N = \bigcup_{\sigma \in I} \bar{\sigma}.$$

3) *The **exit set** for the flow from I is just given by the part of N not in I where the values of f are smaller than or equal to the value on I. In other words*

$$E = \{\sigma \in N \setminus I \mid f(\sigma) \leq f(\tau) \text{ for } \tau \in I\}.$$

4) *We call (N, E) an index pair for I.*

Remark 5.4.1. 1) The reduced collections I_i^{red} are such that there is no path (following the arrows) that moves from a cell in I_i^{red} to another cell outside of I_i^{red}, meaning that each of them is invariant. Since they constitute the building block for computing the Poincaré polynomial of the CW complex, we consider them to be isolated. Hence each collection I_i^{red} is an isolated invariant set.

2) In the definition of the exit set above, taking $I = C^{red}$, the cells in E that have the same value with the ones in I are exactly those cells that are upward

noncritical w.r.t. C. Indeed, we know from Lemma 5.3.3 that any $\sigma \in N \setminus I$ satisfying $f(\sigma) = f(\tau)$ for $\tau \in I$ cannot be downward noncritical. Thus,

$$E = \left\{ \begin{array}{l} \sigma \in N \setminus I \quad \text{s.t.} \quad \text{either } f(\sigma) < f(\tau) \text{ for } \tau \in I, \quad \text{or} \\ f(\sigma) = f(\tau) \ \text{ and } \ \sigma \text{ is upward noncritical w.r.t. } C \end{array} \right\}.$$

Later on, see the proof of Theorem 5.4.2, we will show that $E = N \setminus I$.

Definition 5.4.2. *The **topological Conley index** of an isolated invariant set I is the homotopy type of N/E and the **homological Conley** index of I is the polynomial*

$$C_t(I) := \sum_k \dim H_k(N, E; \mathbb{Z}) t^k.$$

We then have the following lemma.

Lemma 5.4.1. *If $I = C^{red}$ with $\sharp C^{red} = 1$ of dimension k, then its homological Conley index is t^k.*

Proof. Let $I = \sigma^{(k)}$, then $N = \bar{\sigma}$, and $E = \partial^{top}\bar{\sigma}$, where ∂^{top} denotes the topological boundary. Thus we obtain an index pair $(\bar{\sigma}, \partial^{top}(\sigma))$ for σ, and the homotopy type of this index pair is just that of a k-dimensional closed disc with its boundary identified, which is that of the pair (S^k, pt). Thus the Conley index is given by

$$C_t(\sigma^{(k)}) = \sum_i \mathit{dim}\, H_i(\mathbb{S}^k, pt) t^i = t^k,$$

since $H_i(\mathbb{S}^k, pt) = 1$ for $i = k$ and $H_i(\mathbb{S}^k, pt) = 0$ for $i \neq k$. □

In general, if $I = C^{red}$ is such that $C^{red} \neq C$, then there are elements that are upward noncritical w.r.t. C, and (in order not to contradict the definition of a discrete Morse-Bott function) there is a one to one correspondence between those elements and those out of C making them upward noncritical, this is the content of Lemma 5.3.2. Thus we have disjoint noncritical pairs $\{\sigma < \tau\}$, $\sigma \in C$ and $w(\sigma) = \tau \notin C$ with $f(\sigma) \geq f(\tau)$. We call such a σ an **exit cell**.

Remark 5.4.2. If $I = C^{red}$, then the exit set of I denoted E is the union of all the cells in N whose values are smaller than the value in C^{red}, together with all the exit cells of C that are contained in N.

We have the following analogue of the result in the smooth setting:

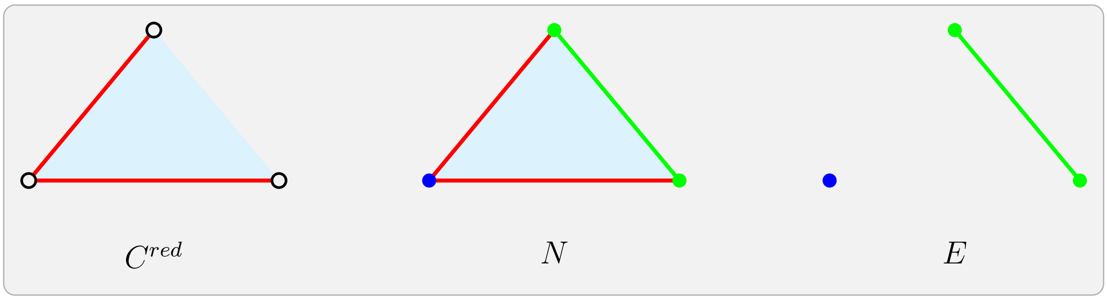


Figure 5.27: A reduced collection and its index pair.

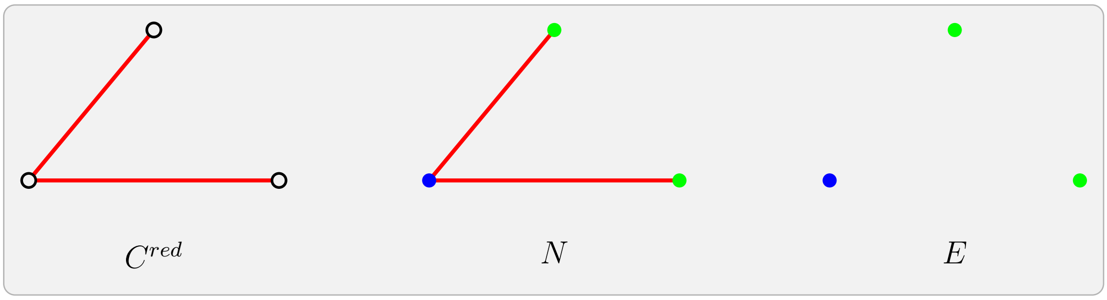


Figure 5.28: Another reduced collection and its index pair.

Theorem 5.4.2. *Let f be a discrete Morse-Bott function having isolated invariant sets $I_1, \cdots, I_l$, then there exists $R(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$P_t(\mathbb{K}) + (1+t)R(t) = \sum_{j=1}^{l} C_t(I_j). \tag{5.17}$$

Recall that $P_t(N,E) = P_t(N/E) - 1$.

Example 5.4.1. 1) For the discrete Morse-Bott function in Figure 5.20, we get the respective index pair in Figure 5.27. Thus $P_t(N,E) = t$. Observe that this can be achieved geometrically by performing N/E and using the fact that $P_t(N,E) = P_t(N/E) - 1$. Hence, $P_t(\mathbb{K}) = 1$ and $\sum_j C_t(I_j) = t+2$, this implies that $R(t) \equiv 1$.

2) Using Figure 5.21, the respective index pair for the discrete Morse-Bott function is given by Figure 5.28. In this case, we get $P_t(N,E) = 2t$. Thus $P_t(\mathbb{K}) = 1+t$ and $\sum_j C_t(I_j) = 2t+2$, this implies that $R(t) \equiv 1$.

Proof of Theorem 5.4.2.

- We have shown above, see Lemma 5.4.1, that whenever I is a singleton, a cell of dimension k,

$$C_t(I) = t^k.$$

- If $I = C^{red}$ where $\sharp C^{red} > 1$, we only need to show that $C^{red} = N \setminus E$ (which holds for all C^{red}).
The fact that $C^{red} \subseteq N \setminus E$ follows from the definitions of those sets.
We show that $N \setminus E \subseteq C^{red}$.
Let $\sigma \in N \setminus E$, then σ is either in C^{red} or it is a face of an element in C^{red}. If $\sigma \in C^{red}$, we are done. If there is a $\tau \in C^{red}$ s.t. $\sigma < \tau$, then from the definition of C^{red} and the fact that σ is not in E, we must have $f(\sigma) = f(\tau)$ and σ is not upward noncritical. Lemma 5.3.3 also establishes the fact that σ cannot be downward noncritical. Thus, $f(\sigma) = f(\tau)$ and σ is neither upward nor downward noncritical, this implies σ is in C^{red}.
Thus $N := N(I)$ is a subcomplex by definition, and $E := E(I) = N \setminus I := \overline{C^{red}} \setminus C^{red}$ is a subcomplex by Lemma 5.3.4. Hence, the relative homology is well defined. Thus,

$$C_t(I) = \sum_k \dim H_k(N(I), E(I); \mathbb{Z}) t^k = P_t(I),$$

and the second equality follows from Theorem 5.3.7 since $C^{red} = N \setminus E$ and both $N := N(I_j)$ and $E := E(I_j)$ are subcomplexes, indeed

$$\begin{aligned}\sum_{j=1}^{l} C_t(I_j) &= \sum_{j=1}^{l} \sum_k \dim H_k(N(I_j), E(I_j); \mathbb{Z}) t^k \\ &= \sum_{j=1}^{l} P_t(I_j) = P_t(\mathbb{K}) + (1+t)R(t).\end{aligned}$$

$\square$

A consequence of the proof above is the following:

Lemma 5.4.3. *If (N, E) is an index pair for $C^{i,red}$, then*

$$\chi(C^{i,red}) = \chi(N, E) = \chi(N) - \chi(E).$$

Remark 5.4.3. The lemma above is not true in general if we consider the Poincaré polynomials instead of the Euler numbers.

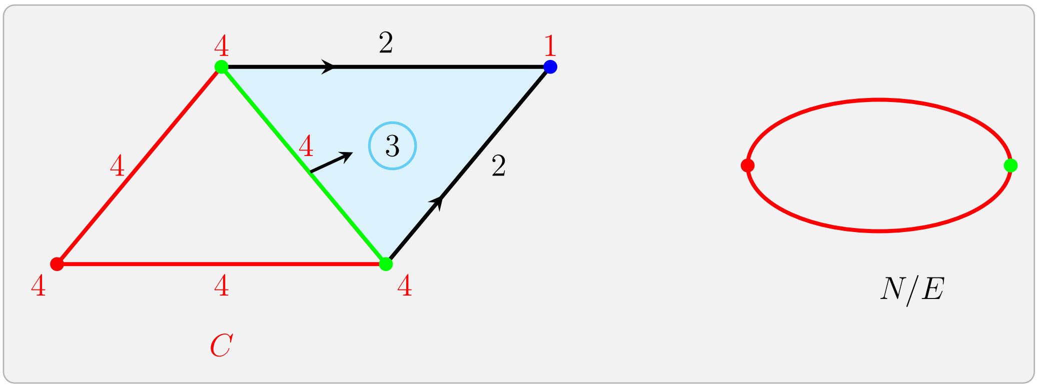


Figure 5.29: The corresponding quotient space to an index pair.

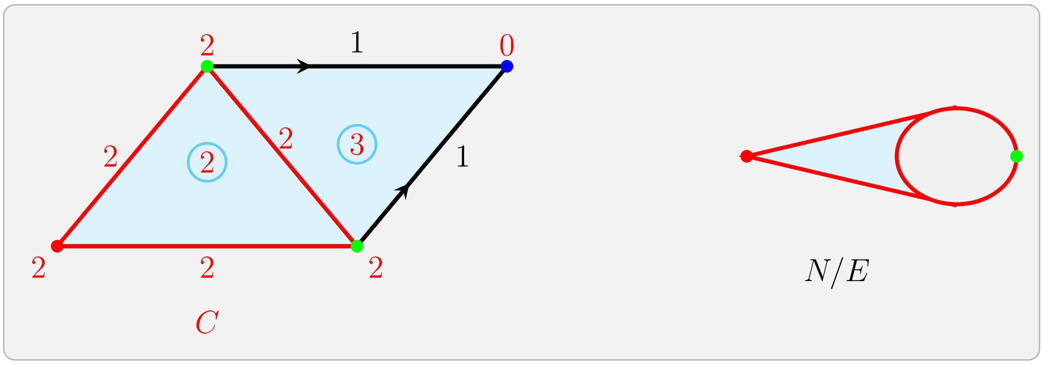


Figure 5.30: The quotient space corresponding to an index pair.

Example 5.4.2. 1) Using Figure 5.29, C^{red} consists of the two red edges and the red vertex, N is the subcomplex generated by the two red edges, so that E is the collection of the two green vertices. $P_t(N, E) = P_t(N/E) - 1 = t$. The critical vertex contributes 1. Thus $\sum_j C_t(I_j) = t + 1 = P_t(\mathbb{K})$ that is $R(t) \equiv 0$.

2) In Figure 5.30, C^{red} takes all the elements of C except the two green vertices. $N = C$ and E is the two green vertices. $P_t(N, E) = t$, there is one critical 2-simplex whose contribution is t^2, and one critical vertex whose contribution is 1. Hence $\sum_j C_t(I_j) = t^2 + t + 1$, but $P_t(\mathbb{K}) = 1$ this implies that $R(t) = t$.

Now, we need to consider the situation where we go beyond a discrete Morse-Bott function, as discussed above, Forman already considered combinatorial vector fields that may have closed orbits. An important observation is the fact that our discrete Conley theory and that of Forman's complement each other in the sense that, in a collection where the function is constant, the vector field is not a combinatorial vector field, see Figure 5.26 (indeed Definition 5.1.7, tells us that for a combinatorial vector field we cannot have situations where a cell has more than one outgoing arrow or more than one incoming arrow), but between the collections we cannot have a closed orbit.

Using this idea we wonder if we can extend our theory to a function that is discrete Morse except on some subcomplexes where it is not. This will then mean that a situation like in Figure 5.31 is allowed. But the problem would be to check if any Conley theory is possible. In fact if we take C to be the triangle in red including

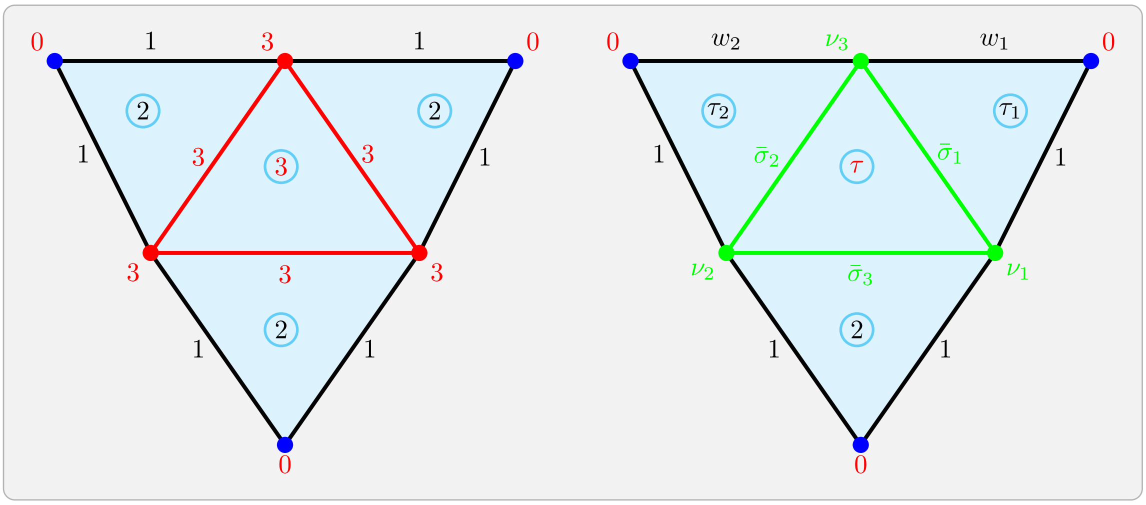


Figure 5.31: A reduced collection with its exit set for a function not discrete Morse-Bott.

the 2-simplex, then its exit set is the subcomplex highlighted in green, so that $H_k(C, E; \mathbb{Z}) = H_k(\mathbb{S}^2, pt; \mathbb{Z}) = \mathbb{Z}$ for $k = 2$ and $H_k(C, E; \mathbb{Z}) = H_k(\mathbb{S}^2, pt; \mathbb{Z}) = 0$ for $k \neq 2$. But adding to this the contributions for the three critical vertices, we get a total of 4 which is clearly not the Euler number of our complex.

To avoid all such ambiguities, we take it upon ourselves to develop a nicer theory needed to derive the Betti numbers, and thus the Euler number, of a given CW complex, and which is more general than Forman's. This is the reason for our attempt in solving a possible generalization of discrete Morse-Floer theory, by considering on a finite CW complex a vector field that allows forking and merging. In particular we provide a way to deal with the ambiguous example given by Figure 5.31. Our solution to this problem is the object of our final chapter.

6

A generalized boundary operator

In this chapter, we present our generalization of Forman's notion of discrete Morse-Floer theory. By considering an arrow configuration more general than the one extracted from a discrete Morse function on a finite CW complex, we provide a definition for a boundary operator depending on the given arrow pattern. This boundary operator will then be used for the computation of the Betti numbers of the complex under consideration. Our construction of the boundary operator is mainly based on some probabilistic method using averaging techniques, using all the arrows, despite the difficulties arising from cells with more than one incoming or outgoing arrow. We construct this boundary operator step by step, first dealing with each case individually, in order for the general boundary operator to be more understandable. The critical cells are of the following types: the cells with no arrows; the cells with more than one incoming arrow; the cells having an outgoing arrow pointing to a cell with more than one incoming arrow; the cells with more than one outgoing arrow; the cells having an incoming arrow coming from a cell with more than one outgoing arrow.

In Section 6.1, we state our assumptions and make precise what type of cells we encounter in our framework that were not present in Forman's framework. We show that the arrow configuration that we consider is generated by some discrete function. We also give the result about computing the Euler number of a CW complex using the arrow configuration but without the use of any boundary operator.

In Section 6.2, we define the boundary operator just in the abnormally upward noncritical case. The proof of the fact that the square of this boundary operator is zero is a step by step procedure, moving from a situation with no arrows (where the square of the boundary operator is zero), creating abnormally upward noncritical cells by adding arrows, and then showing at each step that the square of the boundary operator is still zero. We also show that the Betti numbers in this case coincide with the topological ones.

A boundary operator in the abnormally downward noncritical case is defined in Section 6.3. Using the same idea as in the abnormally upward noncritical case, we show that its square is zero and that the extracted Betti numbers are exactly the topological ones.

Section 6.4 focuses on the definition of the boundary operator in the general case and the statements and proofs of the main theorems, that is, the proof of the fact that the square of this boundary operator is zero and that we can recover the Betti numbers of the CW complex from it. We obtain some Morse-type inequalities as well.

In Section 6.5, we do some Conley theory analysis on the arrow configuration under consideration in Section 6.4.

The content in this chapter will be published in [33].

6.1 Notations

Let $\mathbb{K}$ be a finite CW complex in which each cell is endowed with an orientation (called initial orientation). We recall that the topological boundary elements of a cell are called its faces and the co-dimension one faces are called facets. Suppose there is an arrow configuration on $\mathbb{K}$, for which each cell has finitely many outgoing or incoming arrows but not both. We shall however require that the arrow configuration should not have closed orbits.

We write $\sigma \to \tau$ if there is an arrow from σ to τ.

Definition 6.1.1 (Closed orbit). *A closed orbit (of dimension k) is defined to be a closed path, that is,*

$$\sigma_1^{(k)} \to \tau_1^{(k+1)} > \sigma_2^{(k)} \to \tau_2^{(k+1)} \cdots \tau_{l-1}^{(k+1)} > \sigma_l^{(k)} \to \tau_l^{(k+1)} > \sigma_1^{(k)} \quad (P).$$

Figure 6.1 shows an example of a closed orbit.

More precisely we have the following definition for our arrow configuration.

Definition 6.1.2 (Arrow configuration). *An arrow configuration assigns to each k-cell σ a collection of $(k+1)$-cells that have σ as a facet. We draw an arrow* *from σ to each cell in that collection. The cardinality of that collection is defined by $n_{ou}(\sigma)$. Conversely, for each k-cell σ, we let $n_{in}(\sigma)$ be the number of arrows that it receives from its facets. Thus $n_{ou}(\sigma)$ is the number of outgoing arrows while $n_{in}(\sigma)$ is the number of incoming arrows of σ.*

We require that at most one of $n_{ou}(\sigma)$ and $n_{in}(\sigma)$ be different from zero and that there should not be any closed orbit.

Remark 6.1.1. For all σ, $n_{in}(\sigma)$ and $n_{ou}(\sigma)$ are both finite since the CW complex is finite. Also, $n_{in}(\sigma)$ cannot exceed the number of facets of σ.

We let

$$ADn(\sigma) := \{\nu \in \mathbb{K} \,|\, \nu < \sigma,\, \exists\, \nu \to \sigma\}, \quad AUn(\sigma) := \{\tau \in \mathbb{K} \,|\, \tau > \sigma,\, \exists\, \sigma \to \tau\}.$$

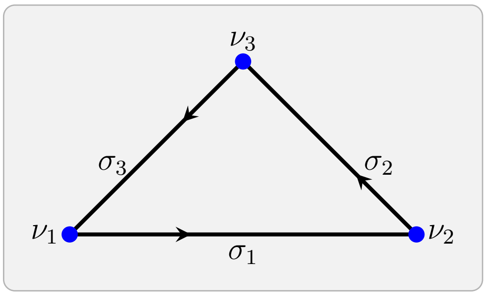


Figure 6.1: A vector field not originating from a discrete function.

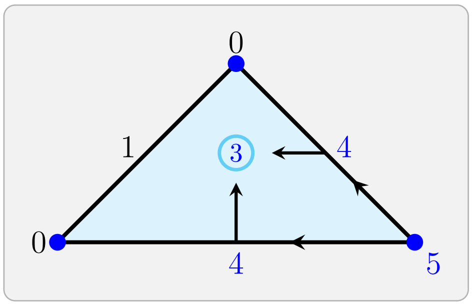


Figure 6.2: A discrete function whose vector field is not our arrow configuration.

The reason for such an arrow configuration instead of one including closed orbits is that we want the arrow configuration to come from some discrete function. The following lemma shows that every such arrow configuration comes from some discrete function.

Lemma 6.1.1. *The arrow configuration given by Definition 6.1.2 is generated by some discrete function.*

Proof. We recall that we draw an arrow from σ to τ if and only if $\sigma < \tau$ and $f(\sigma) \geq f(\tau)$.

(*a*) If we suppose that a given arrow configuration has closed orbits, then consider the arrow configuration given by Figure 6.1. If there is a function f generating the arrow pattern, then f should satisfy the following inequalities: $f(\nu_1) \geq f(\sigma_1) > f(\nu_2) \geq f(\sigma_2) > f(\nu_3) \geq f(\sigma_3) > f(\nu_1)$, which is a contradiction.

(*b*) Also, if in the arrow configuration a cell can have at the same time an incoming and an outgoing arrow, then if there are arrows $\nu \to \sigma$ and $\sigma \to \tau$, the following situation might occur: $f(\nu) \geq f(\sigma) \geq f(\tau) > f(\rho) > f(\nu)$ for some $\rho \neq \sigma,\ \nu < \rho < \tau$. This also would yield a contradiction.

We construct the desired function $f\colon \mathbb{K} \to \mathbb{R}$ in the following way. For every $\sigma \in \mathbb{K}$,

$$f(\sigma) \leq \min\{f(\nu) \,|\, \nu \in ADn(\sigma)\} \quad \text{if} \quad n_{in}(\sigma) \geq 1,$$

$$f(\sigma) \geq \max\{f(\tau) \,|\, \tau \in AUn(\sigma)\} \quad \text{if} \quad n_{ou}(\sigma) \geq 1,$$

$$f(\sigma) > \max\{f(\nu) \,|\, \nu < \sigma,\, \nu \notin ADn(\sigma)\},$$

$$f(\sigma) < \min\{f(\tau) \,|\, \tau > \sigma,\, \tau \notin AUn(\sigma)\}.$$

Then f is indeed well defined.

$\square$

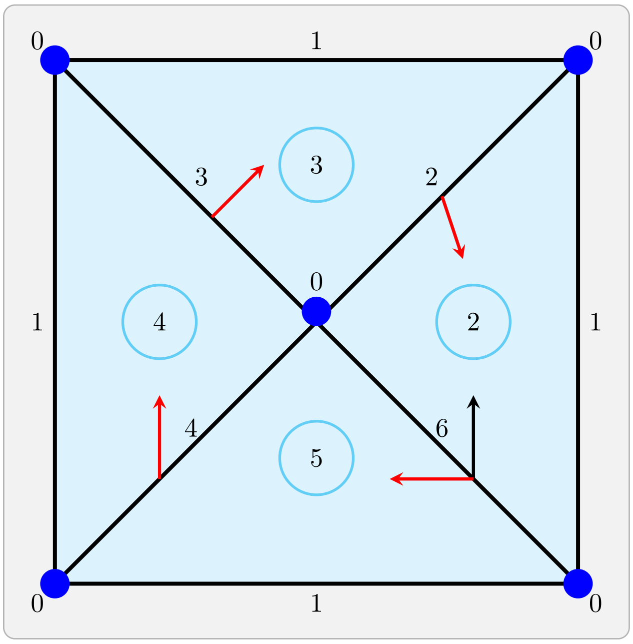


Figure 6.3: A discrete function whose vector field has a closed orbit.

The converse of Lemma 6.1.1 is not entirely true. The extracted vector field from an arbitrary discrete function defined on a CW complex $\mathbb{K}$ need not be the arrow configuration given by Definition 6.1.2. Indeed Figure 6.2 shows an example of a function whose extracted vector field allows for a cell to have at the same time an incoming and an outgoing arrow. Also, in Figure 6.3 we have a discrete function whose extracted vector field has a closed orbit highlighted with the red arrows. In this case the edge with value 6 has more than one outgoing arrow, one of which points to the 2-cell with value 2 which has more than one incoming arrow. To get our desired arrow configuration from a discrete function f, it is not very clear what conditions f should satisfy to avoid closed orbits. However, one condition f should satisfy is the following:

$$\text{For each pair } \{\nu, \sigma \,|\, \nu < \sigma\}, \text{ if } f(\nu) \geq f(\sigma) \quad \text{then} \quad f(\sigma) < \min_{\tau > \sigma} f(\tau). \tag{6.1}$$

The condition given by (6.1) tells us that a cell cannot have at the same time an incoming and an outgoing arrow.

We recall that the cells in Forman's framework are the downward noncritical cells which are those with only one incoming arrow, the upward noncritical cells which are those with only one outgoing arrow and the critical cells which are those without arrows.

Definition 6.1.3 (Forman-type noncritical cell). *Let $\mathbb{K}$ be a CW complex together with the arrow configuration as given in Definition 6.1.2. A cell $\sigma \in \mathbb{K}$ is said to be a Forman-type noncritical cell if it satisfies exactly one of the following:*

- *it has a single incoming arrow, from some cell ν, and ν satisfies $n_{ou}(\nu) = 1$,*
- *it has a single outgoing arrow, to some cell τ, and τ satisfies $n_{in}(\tau) = 1$.*

In addition to the Forman-type cells, we introduce another two types of cells.

Definition 6.1.4 (Abnormally downward noncritical cell). *A cell τ is said to be abnormally downward noncritical if $n_{in}(\tau) > 1$, that is, the number of incoming arrows of τ is greater than 1.*

Definition 6.1.5 (Abnormally upward noncritical cell). *A cell τ is said to be abnormally upward noncritical if $n_{ou}(\tau) > 1$, that is, the number of outgoing arrows of τ is greater than 1.*

For our examples, in which we mostly use simplicial complexes, we write $[\nu_1, \nu_2, \cdots, \nu_k]$ to denote the oriented cell with vertices $\nu_1, \cdots, \nu_k$.

Example 6.1.1. *a*) In Figure 6.4a, the vertex ν_2 is abnormally upward noncritical.

b) In Figure 6.4b, the 2-cell $\tau = [\nu_1, \nu_2, \nu_3]$ is abnormally downward noncritical.

	Notations
CW complex dimension of the CW complex	$\mathbb{K}$ n
cells with the exception that in some concrete cases, vertex edge	$\nu,\ \sigma,\ \rho,\ \tau,\ \varsigma,\ \omega,\ \varpi$ v e
the dimension of ν	$k-1$
the dimension of σ, ρ	k
the dimension of τ, ς	$k+1$.
the dimension of ω	$k+2$
the closure of a cell σ	$\bar{\sigma}$
the coefficients in a linear combination of cells	α
the number of outgoing arrows in a forking situation	m,
the number of incoming arrows in a merging situation	l,
indices the indices that run through other indices	i, j, h, p, q, r
the induction step	t
the coefficients that arise as a result of our computations	capital letters.
Forman's boundary operator	∂^F
the cellular boundary operator	∂^c
an arrow from σ to τ	$\sigma \to \tau$.

Table 6.1: Notations for different types of objects in this chapter.

In our framework, the critical cells are: the cells with no incoming and outgoing arrow; the abnormally downward noncritical cells; the cells having an outgoing arrow pointing to an abnormally downward noncritical cell; the abnormally upward noncritical cells; the cells having an incoming arrow from an abnormally upward noncritical cell. The following is the definition of a critical cell in this framework.

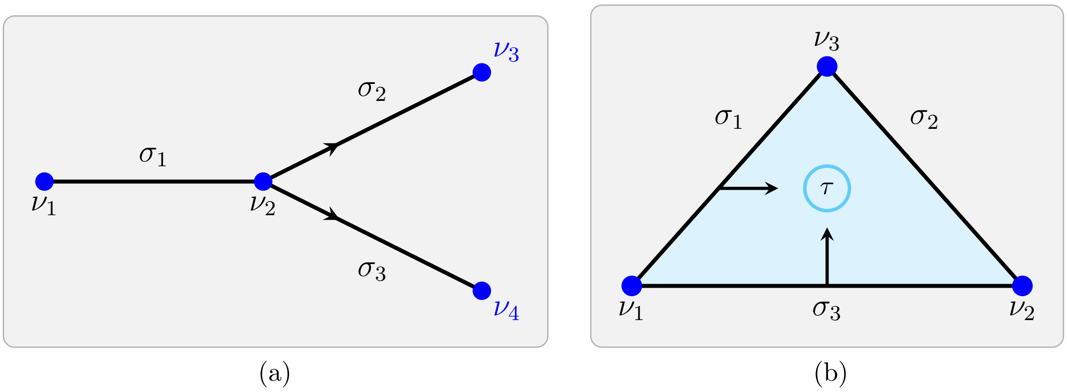


Figure 6.4: Different types of cells.

Definition 6.1.6 (Critical cell). *A cell σ with its arrow pattern is said to be critical if it satisfies any one of the following:*

a) $n_{in}(\sigma) = 0$ *and* $n_{ou}(\sigma) = 0$;

b) $n_{in}(\sigma) > 1$;

c) $\sigma \in ADn(\tau)$ *for some* τ *satisfying* $n_{in}(\tau) > 1$;

d) $n_{ou}(\sigma) > 1$;

e) $\sigma \in AUn(\nu)$ *for some* ν *satisfying* $n_{ou}(\nu) > 1$.

We state a result useful in determining the Euler number of the CW complex using the arrow configuration without any notion whatsoever of a boundary operator.

Definition 6.1.7 (Contribution function). *We define the contribution function of a cell* $\sigma^{(k)} \in \mathbb{K}$, $C\colon \mathbb{K} \to \mathbb{Z}$, *by:*

$$C(\sigma^{(k)}) = (-1)^k + (-1)^{k-1} n_{in}(\sigma^{(k)}) + (-1)^{k+1} n_{ou}(\sigma^{(k)}),$$

where $n_{in}(\sigma)$ *and* $n_{ou}(\sigma)$ *denote the number of incoming and outgoing arrows of* σ *respectively.*

In particular, $C(\sigma) = 0$ if σ has only a single (incoming or outgoing) arrow.

The next proposition shows how the Euler number can be computed just using the contribution function of each cell.

Proposition 6.1.2. *The Euler number of the cell complex* $\mathbb{K}$ *is given by:*

$$\chi(\mathbb{K}) = \sum_{\sigma \in \mathbb{K}} C(\sigma). \tag{6.2}$$

Proof. Essentially this follows because an outgoing (resp. incoming) arrow of a k-cell is an incoming arrow of a $(k+1)$-cell (resp. outgoing arrow of a $(k-1)$-cell). Therefore, the contributions cancel in (6.2).

□

Now we need to find a way to compute the Euler number using the notion of boundary operator that will depend on the arrow configuration that we have. Note that we need to construct a boundary operator whose square is zero and also whose derived Betti numbers coincide with the topological Betti numbers of the CW complex under consideration.

From now on we will assume that each cell is oriented, of course also implicitly taking the induced orientations into account.

We now move to the next section which is about defining a boundary operator in the forking case that is in the situation where cells can have more than one outgoing arrow.

6.2 The forking case

In this section, we define a boundary operator in the case where a cell can have many outgoing arrows or at most one incoming arrow, and there are no closed orbits. We use a probabilistic idea combined with an averaging technique to define this boundary operator.

Let $\mathbb{K}$ be a finite CW complex in which each cell is endowed with an orientation, and let ∂^c being the cellular boundary operator. Suppose that on $\mathbb{K}$ we have the arrow configuration given by Definition 6.1.2 with the assumption that for each cell σ,

$$n_{in}(\sigma) \leq 1.$$

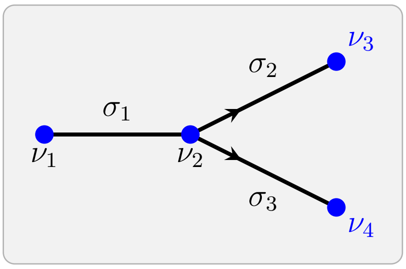


Figure 6.5: A forking case.

In Figure 6.5, the vertex ν_2 is abnormally upward noncritical since $n_{ou}(\nu_2) = 2$.

Let the set of cells having outgoing arrows be denoted by

$$A^u = \{\, \sigma \in \mathbb{K} \mid n_{ou}(\sigma) \geq 1 \,\}.$$

Let us define the following:

(1) the set of cells that are critical in Forman's framework is given by

$$\bar{C}_o^{(k)} := \{\sigma^{(k)} \in \mathbb{K} \,|\, n_{in}(\sigma) = 0 \;\&\; n_{ou}(\sigma) = 0\};$$

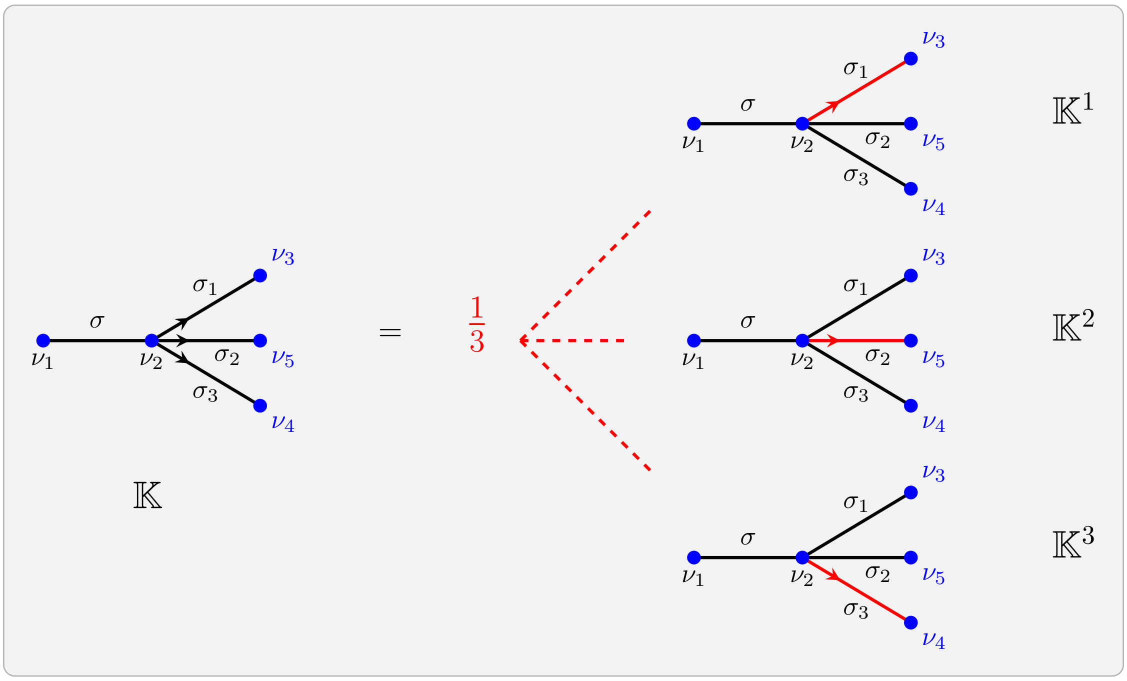


Figure 6.6: Illustration in the forking case.

(2) the set of abnormally upward noncritical cells is

$$\bar{C}_{ou}^{(k)} := \{\sigma^{(k)} \in \mathbb{K} \,|\, n_{ou}(\sigma) > 1 \;\&\; n_{in}(\sigma) = 0\};$$

(3) the set of all cells having an incoming arrow from an abnormally upward noncritical cell is

$$\bar{C}_{sou}^{(k)} := \{\sigma^{(k)} \in \mathbb{K} \,|\, \sigma \in AUn(\nu) \;\&\; n_{ou}(\nu) > 1 \,\text{for some}\;\; \nu\}.$$

Let the set of all critical k-cells be given by

$$\bar{C}^{(k)} := \bar{C}_o^{(k)} \cup \bar{C}_{ou}^{(k)} \cup \bar{C}_{sou}^{(k)},$$

and $\bar{C}_k$ be the free $\mathbb{R}$-module generated by the oriented cells in $\bar{C}^{(k)}$.
Let $\mathbb{K}_k$ denote the free $\mathbb{R}$-module generated by the oriented k-cells of $\mathbb{K}$.

We define the "flow" map v^{up}.

The definition is recursive in nature. Roughly speaking, v^{up} of a cell σ is a linear combination of the cells of the same dimension that are in the cellular boundary of the cells to which the arrows of σ point, and so on. If σ is in $\bar{C}_o \cup \bar{C}_{sou}$ then v^{up} of σ is equal to σ.

The map $v^{up}\colon \mathbb{K}_k \to \mathbb{K}_k$ is given by:

$$v^{up}(\sigma) \quad = \quad \begin{cases} \sigma & \text{if} \quad \sigma \in \bar{C}_o \cup \bar{C}_{sou}, \\ V^{up}(\sigma) & \text{if} \quad \sigma \in A^u, \\ 0 & \text{else}, \end{cases}$$

where, if $AUn(\sigma) = \{\tau_1, \tau_2, \cdots, \tau_m\}$,

$$V^{up}(\sigma) = \beta(m)\,\sigma + (1 - \beta(m))\frac{1}{m}\sum_{i=1}^{m} v^{up}(\sigma \to \tau_i),$$

with

$$v^{up}(\sigma \to \tau_i) = \sum_{\rho < \tau_i, \rho \neq \sigma} v^{up}(\rho), \tag{6.3}$$

and

$$\beta(m) \in [0,1) \ \text{ is such that } \beta(m) = 0 \ \text{ if } m = 1 \ \text{ and } \beta(m) > 0 \ \text{ if } m > 1.$$

We have to argue that the recursive definition above terminates after finitely many steps.
Recall that by Lemma 6.1.1 there is a discrete function f that generates the given arrow configuration. Note that the argument of $v^{up}(\rho)$ in (6.3) has strictly smaller value for the function f than the value $f(\sigma)$. Indeed, from the proof of Lemma 6.1.1 $f(\sigma) \geq f(\tau_i)$ for each i. We are in the forking case so each τ_i has only one arrow coming from σ. Thus, $f(\tau_i) > f(\rho)$, since there is no arrow from ρ to τ_i. Hence $f(\sigma) > f(\rho)$. This tells us that the flow map v^{up} cannot return to σ. Since $\mathbb{K}$ is finite, it implies we stop at some point.

Definition 6.2.1. *We define* $\bar{C}_k \xrightarrow{\bar{\partial}_k} \bar{C}_{k-1}$ *as follows:*

$$\bar{\partial}\tau = v^{up} \circ \partial^c \tau = \sum_{\sigma < \tau} v^{up}(\sigma).$$

Remark 6.2.1. The crucial fact about the definition above is that the coefficient $\beta(m)$ is not zero whenever $m > 1$. Consider for example Figure 6.5 with the orientations:

$$\sigma_1 = [\nu_2, \nu_1],\ \sigma_2 = [\nu_2, \nu_3],\ \sigma_3 = [\nu_4, \nu_3].$$

If we suppose that $\beta(m) = 0$ for all $m > 1$, we obtain
$\bar{\partial}\sigma_1 = \nu_1 - \frac{1}{2}(\nu_3 + \nu_4)$, $\bar{\partial}\sigma_2 = \nu_3 - \frac{1}{2}(\nu_3 + \nu_4) = \frac{1}{2}(\nu_3 - \nu_4)$, and
$\bar{\partial}\sigma_3 = \nu_4 - \frac{1}{2}(\nu_4 + \nu_3) = \frac{1}{2}(\nu_4 - \nu_3)$.
Then one immediately sees that
$\ker \bar{\partial}_0 = \langle \nu_1, \nu_2, \nu_3, \nu_4 \rangle$, $\operatorname{im} \bar{\partial}_1 = \langle \bar{\partial}\sigma_1, \bar{\partial}\sigma_3 \rangle$, $\ker \bar{\partial}_1 = \langle \sigma_3 - \sigma_4 \rangle$, $\operatorname{im} \bar{\partial}_2 = 0$.
This does not give the right Betti numbers since we obtain
$\bar{b}_0 = \dim\left(\ker \bar{\partial}_0 / \operatorname{im} \bar{\partial}_1\right) = 2 \neq 1 = b_0$, and $\bar{b}_1 = \dim\left(\ker \bar{\partial}_1 / \operatorname{im} \bar{\partial}_2\right) = 1 \neq 0 = b_1$.

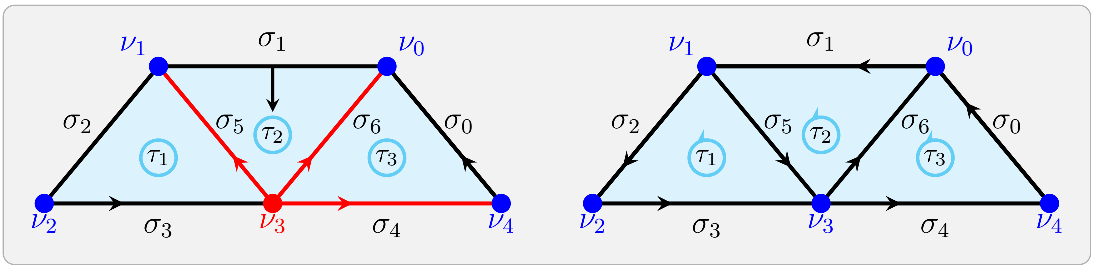


Figure 6.7: Example in the forking case.

We give some examples to illustrate the definition of $\bar{\partial}$.

Example 6.2.1. In Figure 6.7, the initial orientation of each cell is given in the right subfigure, that is:
$\tau_1 = [\nu_1, \nu_2, \nu_3]$; $\tau_2 = [\nu_1, \nu_3, \nu_0]$; $\tau_3 = [\nu_0, \nu_3, \nu_4]$; $\sigma_0 = [\nu_4, \nu_0]$; $\sigma_1 = [\nu_0, \nu_1]$; $\sigma_2 = [\nu_1, \nu_2]$; $\sigma_3 = [\nu_2, \nu_3]$; $\sigma_4 = [\nu_3, \nu_4]$; $\sigma_5 = [\nu_1, \nu_3]$; $\sigma_6 = [\nu_3, \nu_0]$.
The cell ν_3 is abnormally upward noncritical with the cells σ_4, σ_5 and σ_6. We have the following:

$$\bar{C}_o^{(0)} = \{\nu_0\,,\, \nu_1\}, \quad \bar{C}_o^{(1)} = \{\sigma_2\}, \quad \bar{C}_o^{(2)} = \{\tau_1, \tau_3\};$$

$$\bar{C}_{ou}^{(0)} = \{\nu_3\}, \quad \bar{C}_{sou}^{(1)} = AUn(\nu_3) = \{\sigma_4\,,\, \sigma_5\,,\, \sigma_6\};$$

$$\begin{aligned} v^{up}(\nu_3) &= \beta(3)\nu_3 + (1-\beta(3))\frac{1}{3}\Big(v^{up}(\nu_3 \to \sigma_5) + v^{up}(\nu_3 \to \sigma_6) + v^{up}(\nu_3 \to \sigma_4)\Big) \\ &= \beta(3)\nu_3 + (1-\beta(3))\frac{1}{3}(\nu_1 + 2\nu_0); \end{aligned}$$

$\bar{\partial}\tau_1 = -\sigma_5 + \sigma_2, \quad \bar{\partial}\tau_3 = (-\sigma_6 + \sigma_4),$
since the induced orientation from τ_1 (resp. τ_3) onto σ_5 (resp. σ_6) does not coincide with the initial orientation of σ_5 (resp. σ_6);

$\bar{\partial}\sigma_2 = -\nu_1 + v^{up}(\nu_3),$
since the induced orientation from σ_2 (also σ_5) onto ν_3 coincides with the initial orientation of ν_3, whereas the one induced by σ_4 (or σ_6) does not coincide with the initial orientation. Also, the initial orientation of ν_1 does not coincide with its induced orientation from σ_2. We therefore have:

$$\bar{\partial}\sigma_2 = -\nu_1 + v^{up}(\nu_3), \quad \bar{\partial}\sigma_5 = -\nu_1 + v^{up}(\nu_3), \quad \bar{\partial}\sigma_4 = \nu_0 - v^{up}(\nu_3),$$

$$\bar{\partial}\sigma_6 = \nu_0 - v^{up}(\nu_3).$$

One easily checks that $\bar{\partial} \circ \bar{\partial} = 0$.

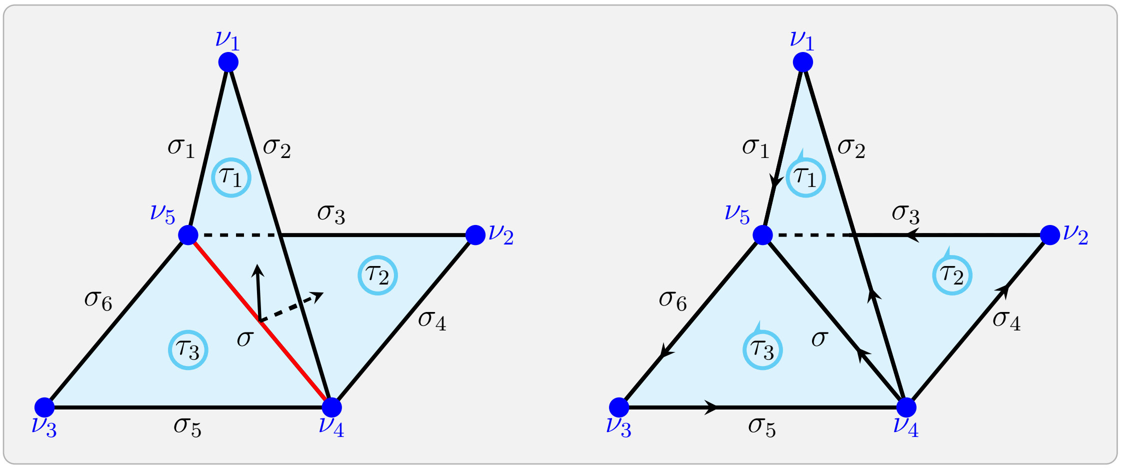


Figure 6.8: Another example in the forking case.

Example 6.2.2. In Figure 6.8, the labels and initial orientations given by the right subfigure are as follows:
$\tau_1 = [\nu_1, \nu_5, \nu_4]$; $\tau_2 = [\nu_2, \nu_5, \nu_4]$; $\tau_3 = [\nu_4, \nu_5, \nu_3]$; $\sigma = [\nu_4, \nu_5]$; $\sigma_1 = [\nu_1, \nu_5]$; $\sigma_2 = [\nu_4, \nu_1]$; $\sigma_3 = [\nu_2, \nu_5]$; $\sigma_4 = [\nu_4, \nu_2]$; $\sigma_5 = [\nu_3, \nu_4]$; $\sigma_6 = [\nu_5, \nu_3]$.
The cell σ is abnormally upward noncritical with the cells τ_1 and τ_2, and we have:

$$\begin{aligned}
v^{up}(\sigma) &= \beta(2)\sigma + (1-\beta(2))\frac{1}{2}\big(v^{up}(\sigma \to \tau_1) + v^{up}(\sigma \to \tau_2)\big) \\
&= \beta(2)\sigma + (1-\beta(2))\frac{1}{2}\big(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4\big),
\end{aligned}$$

since the induced orientation from σ onto each σ_i coincides with the initial orientation of σ_i. Now, the induced orientation from τ_3 onto σ coincides with the initial orientation of σ, but the one from τ_1 (or τ_2) does not. Hence,

$$\begin{aligned}
\bar{\partial}\tau_3 &= \sigma_6 + \sigma_5 + v^{up}(\sigma) \\
&= \beta(2)(\sigma_6 + \sigma_5 + \sigma) \\
&\quad + (1-\beta(2))\Big(\frac{1}{2}(\sigma_6 + \sigma_5 + \sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_6 + \sigma_5 + \sigma_3 + \sigma_4)\Big),
\end{aligned}$$

$$\bar{\partial}\tau_1 = \sigma_1 + \sigma_2 - v^{up}(\sigma) = \beta(2)(\sigma_1 + \sigma_2 - \sigma) + (1-\beta(2))\big(\tfrac{1}{2}(\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4)\big),$$

$$\bar{\partial}\tau_2 = \sigma_3 + \sigma_4 - v^{up}(\sigma) = \beta(2)(\sigma_3 + \sigma_4 - \sigma) + (1-\beta(2))\frac{1}{2}(\sigma_3 + \sigma_4 - \sigma_1 - \sigma_2).$$

Observe that each term in brackets is the cellular boundary of some linear combination of cells. Hence, $\bar{\partial} \circ \bar{\partial} = 0$.

The next proposition establishes the fact that the square of the boundary operator defined above is zero.

Proposition 6.2.1. $\bar{\partial} \circ \bar{\partial} = 0$.

The proof of the fact that the square of this boundary operator is zero is a step by step procedure, moving from a situation where there are no arrows (where the square of the boundary is zero), creating abnormally upward noncritical cells by adding arrows, and then showing at each step that the square of the boundary operator is still zero. The idea behind this proof is the same as the one used to prove Theorem 5.2.1. Observe that in the proof of Theorem 5.2.1, the number of outgoing arrows is at most 1, so we move to the situation where we have $m > 1$ outgoing arrows. Note that the definition of the boundary operator when there are $m > 3$ arrows cannot be expressed in terms of the definition when there are $m - 1$ arrows. This is the reason why our induction step moves directly from a situation with no arrows to one with m arrows. We will also frequently make use of Remark 5.2.2.

We shall proceed in two steps.

(1) Supposing that our arrow configuration comes from some discrete function f, we show that for every such function f, there exists a trivial discrete Morse function f_T such that we can move from f_T to f by a series of operations (consisting in creating (abnormally) upward noncritical cells). When the CW complex is not regular, one may need to put an emphasis on Remark 5.2.5.

(2) When the CW complex has no arrows, $\bar{\partial} = \partial^c$, where ∂^c is the cellular boundary operator, hence $\bar{\partial}^2 = 0$. We show that this latter relation is preserved under the operations of creating (abnormally) upward noncritical cells at each step.

Proof. (1) We proceed in increasing dimensions of the cells. We denote by s_i the number of (abnormally) upward noncritical cells of dimension i. Let f be a function whose extracted vector field coincides with our arrow configuration. Let v_*, be an (abnormally) upward noncritical vertex, then $f(v_*) \geq \max_{e \in AUn(v_*)} f(e)$, and we define $f_0^0 \colon \mathbb{K} \to \mathbb{R}$ by

$$f_0^0(\sigma) \quad = \quad \begin{cases} f(\sigma) - \varepsilon_0 & \text{if} \quad \sigma = v_*, \\ f(\sigma) & \text{else}, \end{cases}$$

where,

$$\varepsilon_0 \; > \; f(v_*) - \min_{e \in AUn(v_*)} f(e). \tag{6.4}$$

We can check that f_0^0 satisfies the discrete Morse conditions at v_* and all the e_i's $\in AUn(v_*)$. Indeed,

$$f_0^0(v*) = f(v_*) - \varepsilon_0 < \min_{e \in AUn(v_*)} f(e) = \min_{e \in AUn(v_*)} f_0^0(e) \quad \text{by} \quad (6.4).$$

Also,

$$f(v_*) < \min_{\widetilde{e} > v_*, \widetilde{e} \notin AUn(v_*)} f(\widetilde{e})$$

implies that the same inequality holds for f_0^0.

This definition is applied to all (abnormally) upward noncritical cells of dimension 0. If we let s_0 be the number of (abnormally) upward noncritical cells of dimension 0, the we will have functions $f_0^0, f_0^1, \cdots, f_0^{s_0}$. Set $f_0 := f_0^{s_0}$.

Now we proceed by induction on the dimension of the (abnormally) upward noncritical cell. Let s_{k-1} be the number of (abnormally) upward noncritical cells of dimension $k-1$, and $f_{k-1} := f_{k-1}^{s_{k-1}}$.

Let σ_* be a k-dimensional (abnormally) upward noncritical cell. We then have:

$$f_{k-1}(\sigma_*) \geq \max_{\tau \in AUn(\sigma_*)} f_{k-1}(\tau) \quad \text{and} \quad f_{k-1}(\sigma_*) > \max_{\substack{\sigma < \tau, \tau \in AUn(\sigma_*) \\ \sigma \neq \sigma_*}} f_{k-1}(\sigma).$$

Also,

$$\forall\ \rho \text{ s.t. } 0 \leq \dim \rho \leq k-1,\ \max_{\nu < \rho} f_{k-1}(\nu) < f_{k-1}(\rho) < \min_{\tau > \rho} f_{k-1}(\tau). \tag{6.5}$$

Define $f_k^0 \colon \mathbb{K} \to \mathbb{R}$ by

$$f_k^0(\sigma) \quad = \quad \begin{cases} f_{k-1}(\sigma) - \varepsilon_k & \text{if} \quad \sigma = \sigma_*, \\ f_{k-1}(\sigma) & \text{else}, \end{cases}$$

where ε_k is chosen so that

$$f_{k-1}(\sigma_*) - \min_{\tau \in AUn(\sigma_*)} f_{k-1}(\tau) < \varepsilon_k < f_{k-1}(\sigma_*) - \max_{\nu < \sigma_*} f_{k-1}(\nu). \tag{6.6}$$

Claim:

$$\min_{\tau \in AUn(\sigma_*)} f_{k-1}(\tau) > \max_{\nu < \sigma_*} f_{k-1}(\nu).$$

Proof of the claim. Let $\tau \in AUn(\sigma_*)$.
If ν is an irregular facet of σ_*, then assuming also the conditions in Remark 5.2.5 one requires $f_{k-1}(\nu) < f_{k-1}(\tau)$.
If $\nu < \sigma_*$ is a regular face of σ_*, then from the incidence property, Remark 5.2.1, there exists $\sigma' \neq \sigma_*$ s.t. $\nu < \sigma' < \tau$, which implies that $f_{k-1}(\tau) > f_{k-1}(\sigma') > f_{k-1}(\nu)$. This is because ν satisfies (6.5) and τ satisfies $n_{in}(\tau) \leq 1$. Since τ is already downward noncritical with σ_* it cannot be with σ'. Thus $f_{k-1}(\tau) > \max_{\nu < \sigma_*} f_{k-1}(\nu)$, and the same inequality holds for the minimum since it holds for all $\tau \in AUn(\sigma_*)$. $\square$

It is easy to check that f_k^0 satisfies the discrete Morse conditions at σ_* and all the τ_i's $\in AUn(\sigma_*)$. Indeed, the left inequality of (6.6) implies

$$f_k^0(\sigma_*) = f_{k-1}(\sigma_*) - \varepsilon_k < \min_{\tau \in AUn(\sigma_*)} f_{k-1}(\tau) = \min_{\tau \in AUn(\sigma_*)} f_k^0(\tau).$$

The right inequality of (6.6) implies

$$f_k^0(\sigma_*) = f_{k-1}(\sigma_*) - \varepsilon_k > \max_{\nu < \sigma_*} f_{k-1}(\nu) = \max_{\nu < \sigma_*} f_k^0(\nu).$$

The inequality

$$f_{k-1}(\sigma_*) < \min_{\widetilde{\tau} > \sigma_*, \widetilde{\tau} \notin AUn(\sigma_*)} f_{k-1}(\widetilde{\tau})$$

implies that the same inequality holds for f_k^0.

We apply the same definition to all the (abnormally) upward noncritical k-dimensional cells. If there are s_k of them, then after step k we have the function $f_k := f_k^{s_k}$.

We point out that in modifying the initial function f, at each step we only modify the value of the function at the (abnormally) upward noncritical cell. Also, at any intermediate step, the function satisfies the discrete Morse property. Continuing this way, after finitely many steps we arrive at the desired trivial discrete Morse function $f_T = f_n$.

(2) We now reverse the preceding steps, that is, we move from the situation where the CW complex has no arrows, then $\bar{\partial} \circ \bar{\partial} = 0$, as we have observed above. We shall show that this relation is preserved at any step. Therefore, ultimately, it has to hold for the given arrow configuration.
Thus, consider a step that transforms a cell $\sigma^{(k)}$ into an (abnormally) upward noncritical cell with $|AUn(\sigma^{(k)})| = l,\ l \geq 1$. We let $\bar{\partial}^{t-1}$ be the boundary operator before the arrows were added. After adding the arrows, we are at step t. By induction, we assume that $\bar{\partial}^{t-1} \circ \bar{\partial}^{t-1} = 0$, and we set out to show that also $\bar{\partial}^t \circ \bar{\partial}^t = 0$.

Let $\tau_i^{(k+1)} \in AUn(\sigma^{(k)})$ for $i = 1, \cdots, m$, and $\tau^{(k+1)} \notin AUn(\sigma)$ be a critical cell that can reach σ by some v^{up}-paths. We denote by b_τ the part of $\bar{\partial}^t \tau$ not connected (by means of any v^{up}-path) to σ.

Suppose

$$\bar{\partial}^{t-1} \tau_i = \sigma^{(k)} + b_{\tau_i}, \quad \bar{\partial}^{t-1} \tau = -\, c\sigma^{(k)} + b_\tau, \quad \text{for some } c \in \mathbb{R},$$

that is, the induced orientation on $\sigma^{(k)}$ from τ_i coincides with its initial orientation. Then

$$v^{up}(\sigma \to \tau_i) = -b_{\tau_i} \quad \text{for all } i = 1, \cdots, m.$$

Let us denote by $\mathbb{K}^{t-1}$ the cell complex (with the arrow configuration) at step $t-1$, where the cell σ has no arrows. The cell complex $\mathbb{K}^t$ is the one for which σ has all the outgoing arrows $\sigma \to \tau_1, \cdots, \sigma \to \tau_l$. Then referring to Figure 6.6, we get that $\mathbb{K}^t$ can be decomposed into $\mathbb{K}^{t,1}, \cdots, \mathbb{K}^{t,m}$ where in $\mathbb{K}^{t,i}$, the cell σ has the unique outgoing arrow $\sigma \to \tau_i$.

$$
\begin{tikzcd}
\cdots \arrow[r, "\bar\partial^{t,i}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{t,i}_{k+2}"] \arrow[d, hook, "\imath_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{j,i}_{k+1}"] \arrow[d, hook, "\imath_{k+1}"] & \bar C_{k}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{j,i}_{k}"] \arrow[d, hook, "\imath_{k}"] & \cdots \\
\cdots \arrow[r, "\bar\partial^{t}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k+2}"] \arrow[d, "h_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k+1}"] \arrow[d, "h_{k+1}"] & \bar C_{k}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k}"] \arrow[d, "h_{k}"] & \cdots \\
\cdots \arrow[r, "\bar\partial^{t-1}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k+1}"] & \bar C_{k}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k}"] & \cdots
\end{tikzcd}
$$

Note that the diagram above is not a commutative diagram. Indeed, using Figure 6.6, and for $\imath$ the inclusion map and h the identity map,

$$\imath_0 \circ \bar\partial^{t,1}_1(\sigma) = \nu_1 - \nu_3 \neq \nu_1 - \nu_2 = \bar\partial^{t-1} \circ \imath_1(\sigma).$$

Also,

$$\bar\partial^t_1 \circ h_1(\sigma) = \bar\partial^t\sigma = \nu_1 - \beta(3)\nu_2 - \frac{1-\beta(3)}{3}(\nu_3+\nu_4+\nu_5)$$

whereas

$$h_0 \circ \bar\partial^{t-1}_1(\sigma) = h_0(\nu_1 - \nu_2) = \nu_1 - \nu_2.$$

One should look at it as three different chain complexes $\big(\bar C_*(\mathbb{K}^{t,i},\mathbb{R}), \bar\partial^{t,i}\big)$, $\big(\bar C_*(\mathbb{K}^{t-1},\mathbb{R}), \bar\partial^{t-1}\big)$ and $\big(\bar C_*(\mathbb{K}^{t},\mathbb{R}), \bar\partial^{t}\big)$, where, $\bar C_k(\mathbb{K}^{t,i},\mathbb{R}) \subseteq \bar C_k(\mathbb{K}^{t-1},\mathbb{R}) = \bar C_k(\mathbb{K}^{t},\mathbb{R})$. The diagram is only needed to understand how we can rewrite the boundary operator $\bar\partial^t$ in terms of $\bar\partial^{t-1}$ and $\bar\partial^{t,i}$ for $i = 1, \cdots, m$, as shown below.

Now, after putting the m outgoing arrows of σ, we are at step t and get:

$$
\begin{aligned}
\bar\partial^t_{k+1}\tau_i &= b_{\tau_i} + \beta(m)\sigma + (1-\beta(m))\frac{1}{m}\sum_{j=1}^{m} v^{up}(\sigma \to \tau_j) \\
&= \beta(m)\big(\sigma + b_{\tau_i}\big) + (1-\beta(m))\frac{1}{m}\sum_{j=1}^{m}\big(b_{\tau_i} - b_{\tau_j}\big)
\end{aligned}
$$

$$=: \quad \beta(m)\bar{\partial}^{t-1}\tau_i + \frac{1-\beta(m)}{m}\sum_{j\neq i}\bar{\partial}^{t,j}\tau_i \tag{6.7}$$

$$= \quad \Big(1 - \frac{1-\beta(m)}{m}\Big)\bar{\partial}^{t-1}_{k+1}\tau_i - \frac{1-\beta(m)}{m}\sum_{j\neq i}\bar{\partial}^{t-1}_{k+1}\tau_j. \tag{6.8}$$

Similarly one gets

$$\begin{aligned}
\bar{\partial}^t_{k+1}\tau \quad &= \quad b_\tau - c\Big(\beta(m)\sigma + \frac{1-\beta(m)}{m}\sum_{i=1}^m v(\sigma\to\tau_i)\Big)\\
&= \quad \beta(m)\big(-c\sigma + b_\tau\big) + \frac{1-\beta(m)}{m}\sum_{i=1}^m \big(b_\tau + c\, b_{\tau_i}\big)\\
&=: \quad \beta(m)\bar{\partial}^{t-1}\tau + \frac{1-\beta(m)}{m}\sum_{i=1}^m \bar{\partial}^{t,i}\tau \qquad (6.9)\\
&= \quad \bar{\partial}^{t-1}_{k+1}\tau + c\frac{1-\beta(m)}{m}\sum_{i=1}^m \bar{\partial}^{t-1}_{k+1}\tau_i. \qquad (6.10)
\end{aligned}$$

We also know that

$$\bar{\partial}^{t-1}_i = \bar{\partial}^t_i \quad \text{for } i\leq k \quad \text{and} \quad \bar{\partial}^{t-1}_j = \bar{\partial}^t_j \quad \text{for } j\geq k+2. \tag{6.11}$$

Thus to show that $\bar{\partial}^t \circ \bar{\partial}^t$ we only need to show that

$$\bar{\partial}^t_{k+1}\circ\bar{\partial}^t_{k+2} = 0, \tag{6.12}$$

and

$$\bar{\partial}^t_k\circ\bar{\partial}^t_{k+1} = 0. \tag{6.13}$$

Using the fact that if a cell does not have σ or any of the τ_i's as a face, more generally if there is no v^{up}-path (orbit) connecting this cell to σ or any of the τ_i's, then there is nothing to prove. The equality (6.13) follows immediately by applying $\bar{\partial}^t_k$ to (6.10) and (6.8), and using (6.11).

We proceed to prove (6.12).
Suppose that $c_1\tau_1, \cdots, c_m\tau_m,\ c_i\in\mathbb{R}$ for $i=1,\ldots,m$, can be reached from some critical $(k+2)$-dimensional cell ω. Then there exist $\varsigma := \sum_i \gamma_i\varsigma_i$ for

$\gamma_i \in \mathbb{R}$ and $(k+1)$-dimensional critical cells ς_i such that the boundary of w can be written as

$$\bar{\partial}^{t-1}\omega = b_\omega + \varsigma + \sum_{i=1}^{m} c_i \tau_i = \bar{\partial}^t \omega.$$

Furthermore for

$$\bar{\partial}^{t-1}\tau_i = b_{\tau_i} + \sigma,$$

ς must be such that

$$\bar{\partial}^{t-1}(\varsigma) = b_\varsigma - \sum_{i=1}^{m} c_i \sigma.$$

The fact that the square of the boundary operator at step $t-1$ is zero yields

$$\bar{\partial}^{t-1} \circ \bar{\partial}^{t-1}\omega = 0 \;\Rightarrow\; \bar{\partial}^{t-1} b_\omega + b_\varsigma + \sum_{i=1}^{m} c_i b_{\tau_i} = 0. \tag{6.14}$$

After adding the outgoing arrows at σ, that is at step t, we get

$$\bar{\partial}^t \tau_i = b_{\tau_i} + v^{up}(\sigma), \tag{6.15}$$

similarly,

$$\bar{\partial}^t \varsigma = b_\varsigma - \sum_{i=1}^{m} c_i v^{up}(\sigma). \tag{6.16}$$

This yields

$$\begin{aligned} \bar{\partial}^t \circ \bar{\partial}^t \omega &= \bar{\partial}^t b_\omega + b_\varsigma + \sum_{i=1}^{m} c_i b_{\tau_i} \quad \text{from (6.16) and (6.15)} \\ &= \bar{\partial}^{t-1} b_\omega + b_\varsigma + \sum_{i=1}^{m} c_i b_{\tau_i} = 0, \end{aligned}$$

from (6.14) and the fact that $\bar{\partial}^{t-1} b_\omega = \bar{\partial}^t b_\omega$.

Suppose $\sigma^{(k)}$ can be reached (by means of some v^{up}-paths) from some critical cell $\omega^{(k+2)} \in \bar{C}$. Then there exist $\varsigma := \sum_i \alpha_i \varsigma_i$ and $\widetilde{\varsigma} := \sum_j \gamma_j \widetilde{\varsigma}_j$ with $\alpha_i, \gamma_j \in \mathbb{R}$, s.t.

$$\bar{\partial}^{t-1}\omega = b_\omega + \varsigma + \widetilde{\varsigma} = \bar{\partial}^t \omega,$$

where, for $c \in \mathbb{R}$,

$$\bar{\partial}^{t-1}\varsigma = b_\varsigma + c\sigma \quad \text{and} \quad \bar{\partial}^{j-1}\widetilde{\varsigma} = b_{\widetilde{\varsigma}} - c\sigma.$$

One important fact is the following: the induced orientation (induced from ω) from ς and $\widetilde{\varsigma}$ onto σ must be opposite, thus suppose σ is positively oriented w.r.t. ς then it has to be negatively w.r.t. $\widetilde{\varsigma}$. In any case we get $\bar{\partial}^t_{k+1}\varsigma = b_\varsigma + c\, v^{up}(\sigma)$ and $\bar{\partial}^t_{k+1}\widetilde{\varsigma} = b_{\widetilde{\varsigma}} - c\, v^{up}(\sigma)$. It then follows immediately that $\bar{\partial}^t_{k+1}(\varsigma + \widetilde{\varsigma}) = \bar{\partial}^{t-1}_{k+1}(\varsigma + \widetilde{\varsigma})$. Using the fact that $\bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega$, this then tells us that $\bar{\partial}^t_{k+1} \circ \bar{\partial}^t_{k+2}\omega = \bar{\partial}^{t-1}_{k+1} \circ \bar{\partial}^{t-1}_{k+2}\omega = 0$.

Hence (6.12) also holds.

□

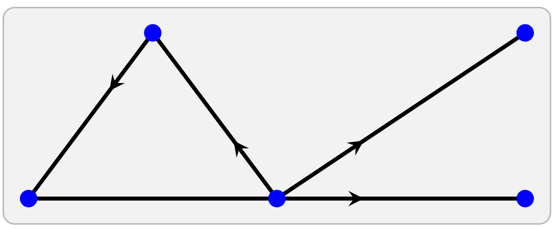
We recall that the topological Betti numbers of the CW complex are defined by $b_i := \dim\left(\ker \partial^c_i / \operatorname{im} \partial^c_{i+1}\right)$, where ∂^c is the cellular boundary operator.
Let $\bar{C}_k := \bar{C}_k(\mathbb{K}, \mathbb{R})$, and $\bar{\partial}_k \colon \bar{C}_k \to \bar{C}_{k-1}$, define

$$\bar{b}_i := \dim\left(\ker \bar{\partial}_i / \operatorname{im} \bar{\partial}_{i+1}\right).$$

The next proposition establishes the fact that the Betti numbers of the CW complex are obtained from this boundary operator, that is $b_i = \bar{b}_i$, for all i.

Proposition 6.2.2. *$b_i = \bar{b}_i$, for all i.*

To prove this proposition, we assume for simplicity that the CW complex $\mathbb{K}$ has no noncritical cells that belong to Forman's framework, meaning that the only cells with outgoing arrows are the abnormally upward noncritical cells. The reason is that we already know that Forman's framework preserves the homotopy type and hence the Betti numbers, since the Forman-type noncritical cells can be collapsed, preserving the homotopy type of the CW complex in the process. This means that instead of working with the example presented in Figure 6.9a, we rather work with the one in Figure 6.9b.


(a)

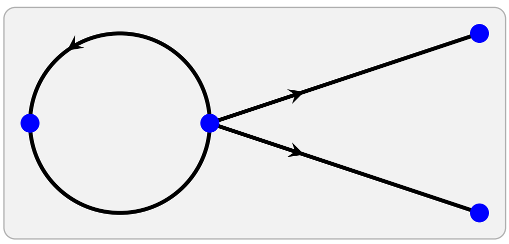

(b)

Figure 6.9: Framework.

Proof. Let $\bar{C}^{(k)} = \bar{C}_o^{(k)} \cup \bar{C}_{ou}^{(k)} \cup \bar{C}_{sou}^{(k)}$. Consider the CW complex $\mathbb{K}$ with the chain complex (C_*, ∂^c) where, ∂^c is the cellular boundary operator. Then, using $(\bar{C}_*, \bar{\partial})$ as the chain complex obtained from $\bar{C}^{(k)}$, we get $\bar{C}_k = C_k$ (since we collapse the Forman-type noncritical cells) which already means that $\dim C_k = \dim \bar{C}_k$.

We know from the definition of v^{up} that it is linear.
Indeed, if $\tau = 0$, then $v^{up}(\tau) = 0$; also, $v^{up}(\sum_{\sigma<\tau} \sigma) = v^{up} \circ \partial\tau = \sum_{\sigma<\tau} v^{up}(\sigma)$, which implies that $v^{up}(\alpha_1\tau_1 + \alpha_2\tau_2) = \alpha_1 v^{up}(\tau_1) + \alpha_2 v^{up}(\tau_2)$.
Also, if $\partial^c\tau = 0$, then $0 = v^{up}(\sum_{\sigma<\tau} \sigma) = \sum_{\sigma<\tau} v^{up}(\sigma) = \bar{\partial}\tau$.
Thus, $\bar{\partial}$ satisfies: for $\varpi \in C_k$, $\partial^c(\varpi) = 0 \Rightarrow \bar{\partial}(\varpi) = 0$. This tells us that $\ker \partial^c \subseteq \ker \bar{\partial}$.

We now show that $\ker \bar{\partial} \subseteq \ker \partial$.

For $\varpi \in \bar{C}_k$ such that $\bar{\partial}\varpi = 0$, we show that $\partial^c\varpi = 0$. Proceeding by induction, we suppose that $\bar{\partial}^t\varpi = 0$ and we show that $\bar{\partial}^{t-1}\varpi = 0$. Because of (6.11), it is enough to show this when ϖ is a linear combination of $(k+1)$-dimensional cells. For any $\varpi = \sum_i \alpha_i\varpi_i$, for $\alpha_i \in \mathbb{R}^*$ for all i, the interesting ones among all the ϖ_i's are those connected to σ. Say

$$\varpi = \sum_{i=1}^{m} \alpha_i\tau_i + \sum_{h=m+1}^{q} \alpha_h\tau^h + \sum_{p\geq q+1} \alpha_p\varpi_p,$$

where,

- for $p \geq q+1$, ϖ_p is not connected to σ by any v^{up}-path using the arrows,
- each τ^h reaches (algebraically) $c^h\sigma$ by means of some v^{up}-paths, $\tau^h \notin AUn(\sigma)$,
- each $\tau_i \in AUn(\sigma)$.

Then

$$\begin{aligned}
0 &= \bar{\partial}^t\varpi = \sum_{i=1}^{m} \alpha_i\bar{\partial}^t\tau_i + \sum_{h=m+1}^{q} \alpha_h\bar{\partial}^t\tau^h + \sum_{p\geq q} \alpha_p\bar{\partial}^t\varpi_p \\
&= \sum_{i=1}^{m} \Big(\alpha_i\big(1 - \frac{1-\beta(m)}{m}\big) - \frac{1-\beta(m)}{m} \sum_{j\neq i, j=1}^{m} \alpha_j\Big)\bar{\partial}^{t-1}\tau_i \\
&\quad + \sum_{h=m+1}^{q} \alpha_h\Big(\bar{\partial}^{t-1}\tau^h + c^h\frac{1-\beta(m)}{m}\sum_{i=1}^{l}\bar{\partial}^{t-1}\tau_i\Big) + \sum_{p\geq q} \alpha_p\bar{\partial}^{t-1}\varpi_p \\
&= \sum_{i=1}^{m} \alpha_i\bar{\partial}^{t-1}\tau_i + \sum_{h=m+1}^{q} \alpha_h\bar{\partial}^{t-1}\tau^h + \sum_{p\geq q} \alpha_p\bar{\partial}^{t-1}\varpi_p =: \bar{\partial}^{t-1}\varpi.
\end{aligned}$$

The last equality follows from the following fact: evaluating the coefficients of σ in (6.7) and (6.9), one immediately gets:

$$\operatorname{Proj}_\sigma \bar{\partial}^t \tau_i = \beta(m), \quad \operatorname{Proj}_\sigma \bar{\partial}^t \tau^h = -\, c^h \beta(m), \quad \text{and} \quad \operatorname{Proj}_\sigma \bar{\partial}^t \varpi_h = 0.$$

Hence, for $\bar{\partial}^t \varpi$ to be zero, one must have

$$0 = \operatorname{Proj}_\sigma \bar{\partial}^t \varpi = \beta(m)\Big(\sum_{i=1}^{m} \alpha_i - \sum_{h=m+1}^{q} c^h \alpha_h\Big).$$

The assumption that there is no Forman-type noncritical cell gives us $m > 1$, so that $\beta(m) > 0$ and we must have

$$\sum_{i=1}^{m} \alpha_i - \sum_{h=m+1}^{q} c^h \alpha_h = 0.$$

Thus $\varpi \in \ker \partial^c \Leftrightarrow \varpi \in \ker \bar{\partial}$.
The result is concluded from the fact that

$$\dim \ker \bar{\partial}_k + \dim \operatorname{im} \bar{\partial}_k = \dim \bar{C}_k = \dim C_k = \dim \ker \partial^c_k + \dim \operatorname{im} \partial^c_k.$$

Also, $\dim \ker \partial^c_k = \dim \ker \bar{\partial}_k \Rightarrow \dim \operatorname{im} \partial^c_k = \dim \operatorname{im} \bar{\partial}_k$. Hence,

$$\bar{b}_k = \dim \ker \bar{\partial}_k - \dim \operatorname{im} \bar{\partial}_{k+1} = \dim \ker \partial^c_k - \dim \operatorname{im} \partial^c_{k+1} = b_k.$$

□

Example 6.2.3. Using Example 6.2.1, we get:
$\ker \bar{\partial}_0 = \langle \nu_0\,, \nu_1\,, \nu_3 \rangle$, $\quad \operatorname{im} \bar{\partial}_1 = \langle \upsilon^{up}(\nu_3) - \nu_1\,, \nu_0 - \upsilon^{up}(\nu_3) \rangle$;
$\ker \bar{\partial}_1 = \langle \sigma_2 - \sigma_5\,, \sigma_4 - \sigma_6 \rangle = \operatorname{im} \bar{\partial}_2$.
Hence one gets $\bar{b}_0 = 1 = b_0$, $\bar{b}_1 = 0 = b_1$, $\bar{b}_2 = 0 = b_2$.

The next section is about defining the boundary operator in the merging case, that is in the case where cells have more than one incoming arrows.

6.3 The merging case

In this section, we define a boundary operator just in the abnormally downward noncritical case.

Let $\mathbb{K}$ be a finite CW complex in which each cell is endowed with an orientation, together with the arrow configuration given by Definition 6.1.2. Suppose that each cell can have at most one outgoing arrow, that is for each cell σ,

$$n_{ou}(\sigma) \le 1.$$

In addition to the sets $\bar{C}_o^{(k)}$ and A^u given in Section 6.2, we define the following:

(1) the set of all the abnormally downward noncritical cells is

$$\bar{C}_{in}^{(k)} := \{\sigma^{(k)} \in \mathbb{K} \,|\, n_{ou}(\sigma) = 0 \;\&\; n_{in}(\sigma) > 1\};$$

(2) the set of all the facets from which the arrows of an abnormally downward noncritical cell come is

$$\bar{C}_{sin}^{(k)} := \{\sigma^{(k)} \in \mathbb{K} \,|\, \sigma \in ADn(\tau) \text{ with } n_{in}(\tau) > 1 \,\text{for some}\, \tau\}.$$

Set

$$\bar{C}^{(k)} = \bar{C}_{o}^{(k)} \cup \bar{C}_{in}^{(k)} \cup \bar{C}_{sin}^{(k)},$$

and $\bar{C}_k$ the free $\mathbb{R}$-module generated by the oriented cells in $\bar{C}^{(k)}$.

We define the "flow" map $v^{do}\colon \mathbb{K}_k \to \mathbb{K}_k$.

The definition is also recursive. Roughly speaking, if a cell σ is in $\bar{C}_o \cup \bar{C}_{in}$, then v^{do} of σ is equal to σ. If σ has an outgoing arrow, then v^{do} of σ is a linear combination of the cells of the same dimension that are in the cellular boundary of the cell to which the arrow of σ points, and so on.

We proceed as follows:

$$v^{do}(\sigma) \;\; = \begin{cases} \sigma & \text{if} \quad \sigma \in \bar{C}_o \cup \bar{C}_{in}, \\ V^{do}(\sigma) & \text{if} \quad \sigma \in A^u, \\ 0 & \text{else}, \end{cases}$$

where, for $\sigma \in A^u$, there exists a τ s.t. $ADn(\tau) = \{\sigma_1, \cdots, \sigma_l\}$ with $\sigma = \sigma_1$, and $V^{do}(\sigma)$ is given by

$$V^{do}(\sigma) := \beta(l)\,\sigma + \frac{1-\beta(l)}{l}\Big((l-1)\sigma + \sum_{j\neq 1}\sigma_j + \sum_{\widetilde{\sigma}<\tau, \widetilde{\sigma}\notin ADn(\tau)} v^{do}(\widetilde{\sigma})\Big), \qquad (6.17)$$

and as before,

$$\beta(l) \in [0,1) \;\text{ is such that } \beta(l) = 0 \;\text{ if } l = 1 \;\text{ and } \beta(l) > 0 \;\text{ if } l > 1.$$

We also have to argue that the recursive definition above terminates after finitely many steps.

Recall that by Lemma 6.1.1 there is a discrete function f that generates the given arrow configuration. For such a function f, (see the proof of Lemma 6.1.1) because there is an arrow from σ to τ we have $f(\sigma) \geq f(\tau)$. In turn $f(\tau) > f(\widetilde{\sigma})$ since there is no arrow from $\widetilde{\sigma}$ to τ. Hence $f(\sigma) > f(\widetilde{\sigma})$. That is, the argument of $v^{do}(\widetilde{\sigma})$ in (6.17) has strictly smaller value, for the function f, than the value $f(\sigma)$. Hence the flow map v^{do} cannot return to σ. Since $\mathbb{K}$ is finite, it implies we stop at some point.

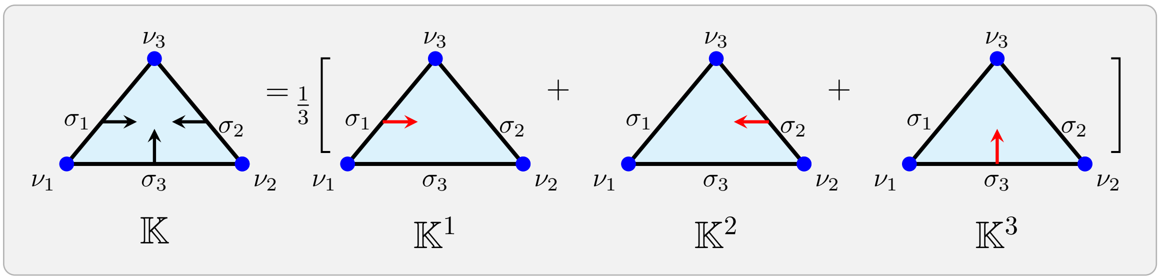


Figure 6.10: Illustration in the merging case.

Definition 6.3.1. *We define* $\bar{C}_k \xrightarrow{\bar{\partial}_k} \bar{C}_{k-1}$ *as follows:*

$$\bar{\partial}\tau = v^{do} \circ \partial^c \tau = \sum_{\sigma<\tau} v^{do}(\sigma).$$

Remark 6.3.1. What is crucial about the definition above is the fact that $\beta(l) \neq 0$ for $l > 1$. Consider for example Figure 6.11, with the initial orientations given by the right subfigure, we have:

$$\tau = [\nu_1, \nu_2, \nu_3], \quad \sigma_1 = [\nu_1, \nu_3], \quad \sigma_2 = [\nu_3, \nu_2], \quad \sigma_3 = [\nu_1, \nu_2].$$

Assuming $\beta(l) = 0$ for all l, we get:
$\bar{\partial}\sigma_1 = \nu_3 - \frac{1}{2}(\nu_1 + \nu_2)$, $\bar{\partial}\sigma_2 = -\nu_3 + \frac{1}{2}(\nu_1 + \nu_2)$, $\bar{\partial}\sigma_3 = 0$, and $\bar{\partial}\tau = -\sigma_1 - \sigma_2 + \sigma_3$.
Then one immediately sees that σ_3 adds an additional element in $\ker \bar{\partial}_1$. Indeed,
$\ker \bar{\partial}_0 = \langle \nu_1, \nu_2, \nu_3 \rangle, \quad \operatorname{im} \bar{\partial}_1 = \langle \bar{\partial}\sigma_1 \rangle$,
$\ker \bar{\partial}_1 = \langle \sigma_1 + \sigma_2, \sigma_3 \rangle, \quad \operatorname{im} \bar{\partial}_2 = \langle -\sigma_1 - \sigma_2 + \sigma_3 \rangle$.
This then does not give the right Betti numbers since we get
$\bar{b}_0 = \dim\left(\ker \bar{\partial}_0 / \operatorname{im} \bar{\partial}_1\right) = 2 \neq b_0 = 1$, and $\bar{b}_1 = \dim\left(\ker \bar{\partial}_1 / \operatorname{im} \bar{\partial}_2\right) = 1 \neq 0 = b_1$.

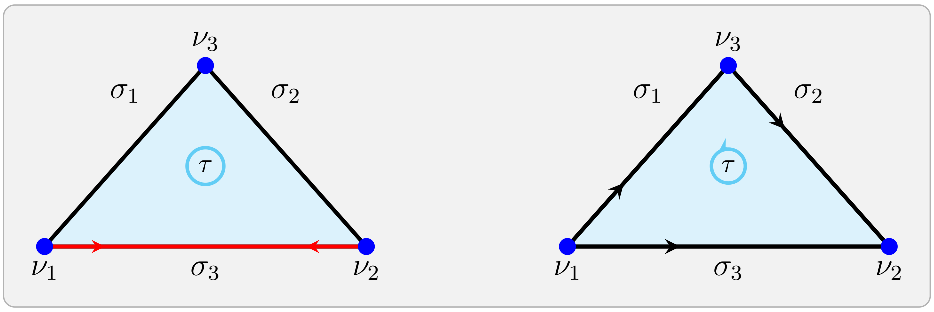


Figure 6.11: Example of a merging case.

We now give some examples to illustrate the definition of $\bar{\partial}$ in this case.

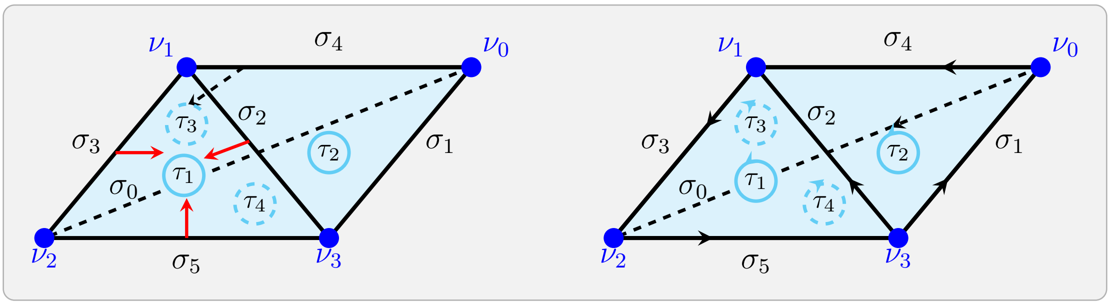


Figure 6.12: Another example of a merging case.

Example 6.3.1. Using Figure 6.11, with the initial orientations given by the right subfigure, we have:

$$\tau = [\nu_1, \nu_2, \nu_3], \quad \sigma_1 = [\nu_1, \nu_3], \quad \sigma_2 = [\nu_3, \nu_2], \quad \sigma_3 = [\nu_1, \nu_2].$$

The cell σ_3 is abnormally downward noncritical with the vertices ν_1 and ν_2, and we have
$\bar{C}_o^{(2)} = \{\tau\}, \quad \bar{C}_o^{(1)} = \{\sigma_1\, , \sigma_1\}, \quad \bar{C}_o^{(0)} = \{\nu_3\}, \quad \bar{C}_{in}^{(1)} = \{\sigma_3\},$
$\bar{C}_{sin}^{(0)} = \{\nu_1\, , \nu_2\}.$
Then

$$v^{do}(\nu_1) = \beta(2)\nu_1 + \tfrac{1-\beta(2)}{2}(\nu_1 + \nu_2), \quad v^{do}(\nu_2) = \beta(2)\nu_2 + \tfrac{1-\beta(2)}{2}(\nu_2 + \nu_1),$$

$$\bar{\partial}\tau = \sigma_3 - \sigma_2 - \sigma_1, \quad \bar{\partial}\sigma_2 = -\nu_3 + v^{do}(\nu_2), \quad \bar{\partial}\sigma_1 = \nu_3 - v^{do}(\nu_1),$$

$$\bar{\partial}\sigma_3 = -v^{do}(\nu_1) + v^{do}(\nu_2) = \beta(2)(\nu_2 - \nu_1).$$

It is easy to check that $\bar{\partial} \circ \bar{\partial} = 0$.

Example 6.3.2. Using Figure 6.12 above, the initial orientations given by the right subfigure are such that:
$\tau_1 = [\nu_1, \nu_2, \nu_3]; \quad \tau_2 = [\nu_0, \nu_1, \nu_3]; \quad \tau_3 = [\nu_1, \nu_0, \nu_2]; \quad \tau_4 = [\nu_2, \nu_0, \nu_3];$
$\sigma_0 = [\nu_0, \nu_2]; \quad \sigma_1 = [\nu_3, \nu_0]; \quad \sigma_2 = [\nu_3, \nu_1]; \quad \sigma_3 = [\nu_1, \nu_2]; \quad \sigma_4 = [\nu_0, \nu_1];$
$\sigma_5 = [\nu_2, \nu_3].$
The cell σ_4 is upward noncritical with the cell τ_3, and the cell τ_1 is abnormally downward noncritical with the cells σ_2, σ_3 and σ_5. We then have:

$$v^{do}(\sigma_3) = \beta(3)\sigma_3 + \frac{1-\beta(3)}{3}(2\sigma_3 - \sigma_2 - \sigma_5);$$

$$v^{do}(\sigma_2) = \beta(3)\sigma_2 + \frac{1-\beta(3)}{3}(2\sigma_2 - \sigma_3 - \sigma_5);$$

$$v^{do}(\sigma_5) = \beta(3)\sigma_5 + \frac{1-\beta(3)}{3}(2\sigma_5 - \sigma_2 - \sigma_3).$$

$$\begin{aligned}
\bar{\partial}\tau_4 &= -\sigma_0 - \sigma_1 - v^{do}(\sigma_5) \\
&= -\beta(3)(\sigma_0 + \sigma_1 + \sigma_5) \\
&\quad - \frac{(1-\beta(3))}{3}\Big(2(\sigma_0 + \sigma_1 + \sigma_5) + (\sigma_0 + \sigma_1 - \sigma_2 - \sigma_3)\Big).
\end{aligned}$$

Also,

$$\begin{aligned}
\bar{\partial}\tau_2 &= \sigma_1 - v^{do}(\sigma_2) + \sigma_0 - v^{do}(\sigma_3) \\
&= \sigma_1 + \sigma_0 + 2\frac{(1-\beta(3))}{3}\sigma_5 - \frac{1+2\beta(3)}{3}(\sigma_2 + \sigma_3). \\
&= 2\frac{(1-\beta(3))}{3}(\sigma_1 + \sigma_0 + \sigma_5) + \frac{1+2\beta(3)}{3}(\sigma_1 + \sigma_0 - \sigma_2 - \sigma_3).
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
\bar{\partial}\tau_1 &= v^{do}(\sigma_3) + v^{do}(\sigma_2) + v^{do}(\sigma_5) \\
&= \beta(3)\sigma_3 + \frac{1-\beta(3)}{3}(2\sigma_3 - \sigma_2 - \sigma_5) + \beta(3)\sigma_2 + \frac{1-\beta(3)}{3}(2\sigma_2 - \sigma_3 - \sigma_5) \\
&\quad + \beta(3)\sigma_5 + \frac{1-\beta(3)}{3}(2\sigma_5 - \sigma_2 - \sigma_3) = \beta(3)(\sigma_5 + \sigma_3 + \sigma_2).
\end{aligned}$$

Each term in brackets is the cellular boundary of some linear combination of cells. One then checks by direct computation that $\bar{\partial} \circ \bar{\partial} = 0$.

The next proposition establishes the fact that the square of this boundary operator is zero.

Proposition 6.3.1. $\bar{\partial} \circ \bar{\partial} = 0$.

The proof of the fact that the square of this boundary operator is zero is a step by step procedure, moving from a situation with no arrows (where the square of the boundary is zero), creating abnormally downward noncritical cells by adding arrows, and then showing at each step that the square is still zero. We also use the same techniques and ideas as in the proof of Theorem 5.2.1.

We shall proceed in two steps, using the same idea as in the abnormally upward noncritical situation.

(1) Supposing that the arrow configuration comes from some discrete function f, we show that for every such function f, there exists a trivial discrete Morse function f_T such that we can move from f_T to f by a series of operations (consisting in creating (abnormally) downward noncritical cells). When the CW complex is not regular, one may need to take note of Remark 5.2.5.

(2) When the CW complex has no arrows, $\bar{\partial} = \partial^c$, where ∂^c is the cellular boundary operator, hence $\bar{\partial}^2 = 0$. We show that this latter relation is preserved under the operations of adding arrows at each step.

Proof. (1) We proceed in decreasing dimension of the (abnormally) downward noncritical cells. For $i = 0. \cdots, n$, let s_i be the number of (abnormally) downward noncritical cells of dimension i.

Let f be the discrete function whose extracted vector field coincides with our arrow configuration. Let $\sigma_*^{(n)}$ be an (abnormally) downward noncritical cell, then $f(\sigma_*) \leq \min\limits_{\nu \in ADn(\sigma_*)} f(\nu)$.

Define $f_n^0 \colon \mathbb{K} \to \mathbb{R}$ by

$$f_n^0(\sigma) \;=\; \begin{cases} f(\sigma) + \varepsilon_n & \text{if} \quad \sigma = \sigma_*, \\ f(\sigma) & \text{else,} \end{cases}$$

where

$$\varepsilon_n \;>\; -f(\sigma_*) + \max_{\nu \in ADn(\sigma_*)} f(\nu).$$

This definition is applied to all (abnormally) downward noncritical cells of dimension n and we can easily check that f_n^0 satisfies the discrete Morse conditions at σ_* and all the ν_i's $\in ADn(\sigma_*)$. In fact we have

$$f_n^0(\sigma_*) > \max_{\nu \in ADn(\sigma_*)} f(\nu).$$

We set $f_n := f_n^{s_n}$.

Now we proceed by induction on the decreasing dimension of the (abnormally) downward noncritical cells. Considering a k-dimensional (abnormally) downward noncritical cell $\sigma_*^{(k)}$. For $f_{k+1} := f_{k+1}^{s_{k+1}}$, we then have

$$f_{k+1}(\sigma_*) \leq \min_{\nu \in ADn(\sigma_*)} f_{k+1}(\nu) \quad \text{and} \quad f_{k+1}(\sigma_*) < \min_{\tau > \sigma_*} f_{k+1}(\tau).$$

Also,

$$\forall\ \rho \text{ s.t. } k+1 \leq \dim \rho \leq n,\ \max_{\nu < \rho} f_{k+1}(\nu) < f_{k+1}(\rho) < \min_{\tau > \rho} f_{k+1}(\tau). \quad (6.18)$$

Define $f_k^0 \colon \mathbb{K} \to \mathbb{R}$ by

$$f_k^0(\sigma) \;=\; \begin{cases} f_{k+1}(\sigma) + \varepsilon_k & \text{if} \quad \sigma = \sigma_* \\ f_{k+1}(\sigma) & \text{else,} \end{cases}$$

where ε_k is chosen such that

$$-f_{k+1}(\sigma_*) + \max_{\nu \in ADn(\sigma_*)} f_{k+1}(\nu) < \varepsilon_k < -f_{k+1}(\sigma_*) + \min_{\tau > \sigma_*} f_{k+1}(\tau).$$

Claim:

$$\max_{\nu \in ADn(\sigma_*)} f_{k+1}(\nu) < \min_{\tau > \sigma_*} f_{k+1}(\tau).$$

Proof of the claim. let $\nu \in ADn(\sigma_*)$ and $\tau > \sigma_*$.
If the cell τ is such that σ_* is an irregular facet of τ. Then assuming also the conditions in Remark 5.2.5, one requires $f_{k+1}(\nu) < f_{k+1}(\tau)$.
For σ_* a regular facet of τ, we get the existence of some cell $\sigma' \neq \sigma_*$ s.t. $\nu < \sigma' < \tau$, and $f_{k+1}(\nu) < f_{k+1}(\sigma') < f_{k+1}(\tau)$. The inequality on the left follows from the fact that $n_{ou}(\nu) \leq 1$. The one on the right follows from (6.18). □

It is easy to check that f_k^0 satisfies the discrete Morse conditions at σ_* and all the ν_i's $\in ADn(\sigma_*)$. More importantly we have:

$$\max_{\nu \in ADn(\sigma_*)} f_k^0(\nu) < f_k^0(\sigma_*) < \min_{\tau > \sigma_*} f_k^0(\tau).$$

We set $f_k := f_k^{s_k}$ and continue in this way.

The desired trivial discrete Morse function in this case is $f_0 := f_0^{s_0}$.

(2) We now reverse the preceding steps, that is, we move from the situation where the CW complex has no arrows, then $\bar{\partial} \circ \bar{\partial} = 0$, as we have observed above. We shall now show that this relation is preserved at any step. Therefore, ultimately, it has to hold for the given arrow configuration.
Thus, consider a step that transforms a cell $\tau^{(k+1)}$ into an (abnormally) downward noncritical one. We let $\bar{\partial}^{t-1}$ be the boundary operator at step $t-1$, that is before the arrows where added. After adding the arrows, we are at step t. By induction, we assume that $\bar{\partial}^{t-1} \circ \bar{\partial}^{t-1} = 0$, and we set out to show that also $\bar{\partial}^t \circ \bar{\partial}^t = 0$.

Let $ADn(\tau^{(k+1)}) = \{\sigma_1^{(k)}, \sigma_2^{(k)}, \cdots, \sigma_l^{(k)}\}$ and $\tau_i^{(k+1)}$ be critical cells that reach $\sigma_i^{(k)}$ by means of some v^{do}-path for $i = 1, \cdots, l$. Assume that each σ_i is positively oriented w.r.t. τ but negatively w.r.t. τ_i. Then for $c_i \in \mathbb{R}$ for all $i = 1, \cdots, l$,

$$\bar{\partial}^{t-1}\tau = b_\tau + \sum_i \sigma_i, \quad \bar{\partial}^{t-1}\tau_i = b_{\tau_i} - c_i\sigma_i.$$

Let us denote by $\mathbb{K}^{t-1}$ the cell complex together with the arrow configuration at step $t-1$, where the cell τ has no arrows. The cell complex $\mathbb{K}^t$ is the one for which τ has all the incoming arrows $\sigma_1 \to \tau, \cdots, \sigma_l \to \tau$. Then referring to Figure 6.10, we get that $\mathbb{K}^t$ can be decomposed into $\mathbb{K}^{t,1}, \cdots, \mathbb{K}^{t,l}$ where in $\mathbb{K}^{t,i}$, the cell τ has the unique incoming arrow $\sigma_i \to \tau$.

$$\begin{tikzcd}
\dots \arrow[r, "\bar\partial^{t,i}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{t,i}_{k+2}"] \arrow[d, hook, "\imath_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{t,i}_{k+1}"] \arrow[d, hook, "\imath_{k+1}"] & \bar C_{k}(\mathbb{K}^{t,i},\mathbb{R}) \arrow[r, "\bar\partial^{t,i}_{k}"] \arrow[d, hook, "\imath_{k}"] & \dots \\
\dots \arrow[r, "\bar\partial^{t}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k+2}"] \arrow[d, "h_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k+1}"] \arrow[d, "h_{k+1}"] & \bar C_{k}(\mathbb{K}^{t},\mathbb{R}) \arrow[r, "\bar\partial^{t}_{k}"] \arrow[d, "h_{k}"] & \dots \\
\dots \arrow[r, "\bar\partial^{t-1}_{k+3}"] & \bar C_{k+2}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k+2}"] & \bar C_{k+1}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k+1}"] & \bar C_{k}(\mathbb{K}^{t-1},\mathbb{R}) \arrow[r, "\bar\partial^{t-1}_{k}"] & \dots
\end{tikzcd}$$

Similarly as in the proof of Proposition 6.2.1, this diagram is not a commutative diagram. It is only needed to understand what follows.
We have for $i = 1, \cdots, l$,

$$\begin{aligned}
\bar\partial^t \tau_i &= b_{\tau_i} - c_i\Big(\beta(l)\sigma_i + \frac{1-\beta(l)}{l}\Big((l-1)\sigma_i - \sum_{j\neq i}\sigma_j - b_\tau\Big)\Big) \\
&= \beta(l)\bar\partial^{t-1}\tau_i + \frac{(1-\beta(l))}{l}\sum_{j\neq i}\bar\partial^{t,j}\tau_i + \frac{1-\beta(l)}{l}\bar\partial^{t,i}\tau_i && (6.19) \\
&= \bar\partial^{t-1}\tau_i + c_i\frac{1-\beta(l)}{l}\bar\partial^{t-1}\tau. && (6.20)
\end{aligned}$$

Observe that for $j \neq i$, $\bar\partial^{t,j}\tau_i = \bar\partial^{t-1}\tau_i$.

$$\begin{aligned}
\bar\partial^t \tau &= b_\tau + \sum_{i=1}^{l}\Big(\beta(l)\sigma_i + \frac{1-\beta(l)}{l}((l-1)\sigma_i - \sum_{j\neq i}\sigma_j - b_\tau)\Big) && (6.21) \\
&= \beta(l)\bar\partial^{t-1}\tau.
\end{aligned}$$

Using the fact that

$$\bar\partial^t_i = \bar\partial^{t-1}_i = 0 \quad \text{for } i \leq k \quad \text{and} \quad \bar\partial^t_j = \bar\partial^{t-1}_j = 0 \quad \text{for } j \geq k+2, \qquad (6.22)$$

to prove that $\bar\partial^t \circ \bar\partial^t = 0$, it is enough to show

$$\bar\partial^t_k \circ \bar\partial^t_{k+1} = 0, \qquad (6.23)$$

and

$$\bar{\partial}^t_{k+1} \circ \bar{\partial}^t_{k+2} = 0. \tag{6.24}$$

The equality (6.23) follows by applying $\bar{\partial}^t_k$ to (6.20) and (6.21), using (6.22).

We now proceed to show (6.24).

a) Suppose $\omega^{(k+2)}$ can reach $c\tau$, $c \in \mathbb{R}$, then there exist $\varsigma_1, \cdots, \varsigma_l$ such that

$$\bar{\partial}^t \omega = b_\omega + \sum_{i=1}^{l} \varsigma_i + c\tau = \bar{\partial}^{t-1}\omega,$$

where each ς_i is such that

$$\bar{\partial}^{t-1}\varsigma_i = b_{\varsigma_i} - c\sigma_i.$$

Note that as argued before, each ς_i is actually a linear combination of cells. Since the fact that σ_i and ω are path-connected (and there are paths through τ and others through ς_i) tells us that the induced orientation (induced from ω) from ς_i onto σ_i has to be different from the one induced from τ.

$$0 = \bar{\partial}^{t-1} \circ \bar{\partial}^{t-1}\omega = \bar{\partial}^{t-1} b_\omega + \sum_i b_{\varsigma_i} + c b_\tau. \tag{6.25}$$

Then using (6.21) and (6.20), and the fact that $\bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega$, we get

$$\begin{aligned}
\bar{\partial}^t \circ \bar{\partial}^t \omega &= \bar{\partial}^t b_\omega + \sum_{i=1}^{l} b_{\varsigma_i} + c b_\tau \\
&= \bar{\partial}^{t-1} b_\omega + \sum_{i=1}^{l} b_{\varsigma_i} + c b_\tau, \\
&= 0 \quad \text{from (6.25).}
\end{aligned}$$

b) Suppose τ cannot be reached from ω, but instead $c_i\sigma_i$, $c_i \in \mathbb{R}$, can. In this case there exist ς and $\widetilde{\varsigma}$, both linear combination of critical cells, such that

$$\bar{\partial}^{t-1}\omega = b_\omega + \varsigma + \widetilde{\varsigma} = \bar{\partial}^t \omega,$$

where

$$\bar{\partial}^{t-1}\varsigma = b_\varsigma + \sum_i c_i\sigma_i, \quad \text{and } \bar{\partial}^{t-1}\widetilde{\varsigma} = b_{\widetilde{\varsigma}} - \sum_i c_i\sigma_i.$$

We also use the fact that the induced orientations (induced from ω) from ς onto σ_i has to be different from the one induced by $\widetilde{\varsigma}$. It follows immediately that

$$\bar{\partial}^t(\bar{\partial}^t\omega) = \bar{\partial}^{t-1}(\bar{\partial}^{t-1}\omega) = 0, \quad \text{since} \quad \bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega.$$

□

The next proposition, just as in the abnormally upward noncritical case, establishes the fact that the Betti numbers of the CW complex are obtained from this boundary operator, that is $b_i = \bar{b}_i$, for all i.

Proposition 6.3.2. *$b_i = \bar{b}_i$, for all i.*

Using the same idea as in the proof for the abnormally upward noncritical case, we assume that the CW complex $\mathbb{K}$ has no Forman-type noncritical cell. Then $\dim C_k = \dim \bar{C}_k$.

Proof. Similarly as in the abnormally upward noncritical case, we know from the definition of v^{do} that it is linear. This then implies that, if $\partial^c\tau = 0$, then $0 = v^{do}(\sum_{\sigma<\tau}\sigma) = \sum_{\sigma<\tau} v^{do}(\sigma) = \bar{\partial}\tau$. Thus, for $\varpi \in C_k = \bar{C}_k$, $\partial^c(\varpi) = 0 \Rightarrow \bar{\partial}(\varpi) = 0$. This tells us that $\ker\partial^c \subseteq \ker\bar{\partial}$.

We show the opposite inclusion. We suppose $\bar{\partial}\varpi = 0$ and we have to show that $\partial^c\varpi = 0$. We also achieve this by induction, meaning that we assume that $\bar{\partial}^t\varpi = 0$ and we show that $\bar{\partial}^{t-1}\varpi = 0$. Because of (6.22), it is enough to show this when ϖ is a linear combination of $(k+1)$-dimensional cells.

Suppose that $\varpi = \sum_{i=1}^{l}\sum_{q_i=1}^{n_i}\alpha_i^{q_i}\tau_i^{q_i} + \alpha\tau + \sum_j \alpha_j\varpi_j$, for $\alpha_i^{q_i}, \alpha, \alpha_j \in \mathbb{R}$, where,

- the ϖ_j'*s* are not connected by means of some v^{do}-paths to any of the σ_i'*s*,
- each $\tau_i^{q_i}$ reaches $c_i^{q_i}\sigma_i$, $c_i^{q_i} \in \mathbb{R}$.

Proceeding by induction, this means that we suppose that $\bar{\partial}^t\varpi = 0$ and we show that $\bar{\partial}^{t-1}\varpi = 0$.

$$\begin{aligned}
0 &= \bar{\partial}^t\varpi = \sum_{i=1}^{l}\sum_{q_i}\alpha_i^{q_i}\left(\bar{\partial}^{t-1}\tau_i^{q_i} + c_i^{q_i}\frac{1-\beta(l)}{l}\bar{\partial}^{t-1}\tau\right) + \beta(l)\alpha\bar{\partial}^{t-1}\tau + \sum_j \alpha_j\bar{\partial}^t\varpi_j \\
&= \sum_i\sum_{q_i}\alpha_i^{q_i}\bar{\partial}^{t-1}\tau_i^{q_i} + \left(\frac{1-\beta(l)}{l}\sum_i\sum_{q_i}c_i^{q_i}\alpha_i^{q_i} + \beta(l)\alpha\right)\bar{\partial}^{t-1}\tau + \sum_j \alpha_j\bar{\partial}^{t-1}\varpi_j \\
&= \sum_i\sum_{q_i}\alpha_i^{q_i}\bar{\partial}^{t-1}\tau_i^{q_i} + \alpha\bar{\partial}^{t-1}\tau + \sum_j \alpha_j\bar{\partial}^{t-1}\varpi_j =: \bar{\partial}^{t-1}\varpi.
\end{aligned}$$

The last equality follows from the fact that, looking at the coefficients of the σ_i's, that is looking at the first equalities of (6.20) and (6.21), we get

$$\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau_i^{q_i} = -c_i^{q_i}\big(\beta(l)+(l-1)\tfrac{(1-\beta(l))}{l}\big), \quad \text{for } h\neq i,\ \mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau_h^{q_h} = c_h^{q_h}\tfrac{1-\beta(l)}{l},$$

$$\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau = \beta(l) \qquad \text{and}\ \mathrm{Proj}_{\sigma_i}\bar{\partial}^t\varpi_j = 0.$$

Thus, for $\bar{\partial}^t\varpi$ to be zero, one must have, for $i = 1,\cdots,l$,

$$\begin{aligned} 0 &= \mathrm{Proj}_{\sigma_i}\bar{\partial}^t\varpi \\ &= \frac{1-\beta(l)}{l}\sum_{h\neq i}\sum_{q_h} c_h^{q_h}\alpha_h^{q_h} - \big(\beta(l)+(1-\beta(l))\frac{l-1}{l}\big)\sum_{q_i}\alpha_i^{q_i}c_i^{q_i} + \beta(l)\alpha, \end{aligned}$$

and summing this for all i yields $\beta(l)\big(-\sum_i\sum_{q_i}c_i^{q_i}\alpha_i^{q_i}+l\alpha\big) = 0$. The absence of the Forman-type noncritical cells gives $l > 1$. We also know that for $l > 1$, $\beta(l) > 0$, so the term inside brackets should be zero. That is

$$-\sum_i\sum_{q_i}c_i^{q_i}\alpha_i^{q_i} + l\alpha = 0.$$

Thus, $\varpi \in \ker\partial^c \Leftrightarrow \varpi \in \ker\bar{\partial}$.

As before, the result is concluded from the fact that

$$\dim\ker\bar{\partial}_k + \dim\mathrm{im}\,\bar{\partial}_k = \dim\bar{C}_k = \dim C_k = \dim\ker\partial_k^c + \dim\mathrm{im}\,\partial_k^c.$$

Also, $\dim\ker\partial_k^c = \dim\ker\bar{\partial}_k \Rightarrow \dim\mathrm{im}\,\partial_k^c = \dim\mathrm{im}\,\bar{\partial}_k$. Hence

$$\bar{b}_k = \dim\ker\bar{\partial}_k - \dim\mathrm{im}\,\bar{\partial}_{k+1} = \dim\ker\partial_k^c - \dim\mathrm{im}\,\partial_{k+1}^c = b_k.$$

$\square$

Example 6.3.3. Using the computations of Example 6.3.2, we get:
$\ker\bar{\partial}_0 = \langle\nu_0\,,\,\nu_1\,,\,\nu_2\,,\,\nu_3\rangle$, $\quad \mathrm{im}\,\bar{\partial}_1 = \langle-\nu_0+\nu_1\,,\,\nu_2-\nu_1\,,\,\nu_3-\nu_2\rangle$;
$\ker\bar{\partial}_1 = \langle\sigma_3+\sigma_5+\sigma_2\,,\,-(\sigma_0+\sigma_5+\sigma_1)\,,\,\sigma_1-\sigma_2+\sigma_0-\sigma_3\rangle = \mathrm{im}\,\bar{\partial}_2$;
$\ker\bar{\partial}_2 = \langle\tau_1+\tau_4+\tau_2\rangle$, $\quad \mathrm{im}\,\bar{\partial}_2 = 0$.
We then get the desired Betti numbers: $\bar{b}_0 = 1 = b_0$, $\bar{b}_1 = 0 = b_1$, $\bar{b}_2 = 1 = b_2$.

Now that we have established how the boundary operator is defined in each isolated situation, we combine the two approaches to define the boundary operator in the general setting.

6.4 The mixed case

We present our approach to answering the question of a generalization of Forman's discrete Morse-Floer theory. The data for our construction consists of a finite CW complex, where each cell is given an orientation, and an arrow configuration such that a cell can have as many outgoing or incoming arrows but not both, and there are no closed orbits. The definition we provide in this part is based on some probabilistic idea and averaging technique in the sense that, when looking for the boundary of a cell, if any one of its facets has a certain number of outgoing arrows, then the average over this is taken, up to a certain factor, and so on, and if a facet has an arrow that gets into a cell that has many incoming arrows, some kind of average is also taken into account. This definition is a combination of the two definitions that we provided in the forking and the merging cases. For this reason, in the proof provided in this section we will not give all the details that are already contained in the proofs in Section 6.2 and Section 6.3.

Let $\mathbb{K}$ be a finite CW complex in which each cell is endowed with an orientation, with ∂^c be the cellular boundary operator on $\mathbb{K}$. We assume on $\mathbb{K}$ an arrow configuration satisfying the conditions in Definition 6.1.2. Let $\sigma \in \mathbb{K}$ be a cell, recall that $n_{in}(\sigma)$ (resp. $n_{ou}(\sigma)$) denotes the number of incoming arrows (resp. outgoing arrows) of σ.

Definition of the boundary operator

Here we give our definition of the boundary operator, using the above arrow configuration. Before defining the boundary operator we first define some related notions.

We let the sets A^u, $AUn(\sigma)$, $ADn(\tau)$, $\bar{C}_o^{(k)}$, $\bar{C}_{in}^{(k)}$, $\bar{C}_{ou}^{(k)}$, $\bar{C}_{sin}^{(k)}$, $\bar{C}_{sou}^{(k)}$, be defined as before, but instead take

$$\bar{C}^{(k)} = \bar{C}_o^{(k)} \cup \bar{C}_{ou}^{(k)} \cup \bar{C}_{in}^{(k)} \cup \bar{C}_{sin}^{(k)} \cup \bar{C}_{sou}^{(k)},$$

and $\bar{C}_k$ the free $\mathbb{R}$-module generated by the (oriented) cells in $\bar{C}^{(k)}$.

Let $\beta(l) \in [0, 1)$ be such that $\beta(l) = 0$ for $l = 1$ and $\beta(l) > 0$ for $l > 1$.

Below we define the generalized "flow" map v^G.

The definition is recursive in nature. Roughly speaking, when a cell σ has some outgoing arrows, v^G of σ is a linear combination of the cells of the same dimension that are in the cellular boundary operator of the cells to which the arrows of σ point, and so on. When σ is in $\bar{C}_o \cup \bar{C}_{in} \cup \bar{C}_{sou}$, v^G of σ is equal to σ.

Now we provide a detailed definition.

We define the map $v^G\colon \mathbb{K}_k \to \mathbb{K}_k$ in the following way:

$$v^G(\sigma) \;=\; \begin{cases} \sigma & \text{if} \quad \sigma \in \bar{C}_o \cup \bar{C}_{in} \cup \bar{C}_{sou}, \\ V^G(\sigma) & \text{if} \quad \sigma \;\in\; A^u, \\ 0 & \text{else.} \end{cases}$$

To define V^G for a cell $\sigma \in A^u$, we consider the set $AUn(\sigma) = \{\tau_1^\sigma, \cdots, \tau_m^\sigma\}$, of the target cells of arrows coming from σ and define

$$V^G(\sigma) := \beta(m)\,\sigma + \frac{1-\beta(m)}{m}\sum_{r=1}^{m} v^G(\sigma \to \tau_r^\sigma).$$

For an arrow $\sigma \to \tau$, the value $v^G(\sigma \to \tau)$ is defined as follows.
Let τ be s.t. $ADn(\tau) = \{\sigma_1, \sigma_2, \cdots, \sigma_l\}$, with $\sigma = \sigma_1$, and for each $i = 1, \cdots, l$,

$AUn(\sigma_i) = \{\, \tau_1^{\sigma_i},\, \tau_2^{\sigma_i}, \cdots,\, \tau_{m_i}^{\sigma_i}\}$ with $\tau_1^{\sigma_i} = \tau$. Then

$$A^{\sigma_1} = A^\sigma := \{\tau\} \times AUn(\sigma_2) \times \cdots \times AUn(\sigma_l).$$

Note that the outgoing cells of σ itself are not included into the product above.
For an element $E \in A^\sigma$, define

$$P_\tau(E) = \{\sigma_i \in ADn(\tau) \mid \mathrm{Proj}_i(E) = \tau,\ \text{for some } i\} \quad \text{and set} \quad \eta_E := |P_\tau(E)|.$$

To illustrate the definition of P_τ, let's say $E = (\tau, \tau, \tau, E_4, \cdots, E_l)$ then for $i = 1, 2, 3,\ \mathrm{Proj}_i(E) = \tau$, this implies that $P_\tau(E) = \{\sigma, \sigma_2, \sigma_3\}$.

Now we come back to the definition of $v^G(\sigma \to \tau)$. We define

$$v^G(\sigma \to \tau) := \frac{1}{|A^\sigma|}\sum_{E \in A^\sigma} v(\sigma, E) \qquad (|A^\sigma| = \prod_{j \neq 1} m_j).$$

Finally, we define for $E \in A^\sigma$,

$$\begin{aligned} v(\sigma, E) \;:=\; & \beta(\eta_E)\,\sigma + \frac{1-\beta(\eta_E)}{\eta_E}\Bigg((\eta_E - 1)\sigma + \sum_{\substack{h \neq 1 \\ \sigma_h \in P_\tau(E)}} \sigma_h \\ & + \sum_{\substack{\sigma' < \tau \\ \sigma' \notin ADn(\tau)}} v^G(\sigma') + \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} v^G(\sigma_j \to \tau^{\sigma_j})\Bigg). \end{aligned} \tag{6.26}$$

Note: $v^G(-\sigma \to \tau) = -v^G(\sigma \to \tau)$ where $-\sigma$ is the cell σ with the opposite orientation.

We have to argue that the recursive definition above terminates after finitely many steps.
Recall that by Lemma 6.1.1 there is a discrete function f that generates the given arrow configuration. From the proof of Lemma 6.1.1, $f(\sigma) \geq f(\tau) > f(\sigma')$. That is, the arguments of $v^G(\sigma')$ in (6.26) has strictly smaller value for the function f than the value $f(\sigma)$. However, $f(\sigma)$ need not be greater than $f(\sigma_j)$. But the absence of closed orbits in our arrow configuration ensures that the flow map v^G cannot return to σ. So, the absence of closed orbits and the finiteness of $\mathbb{K}$ both imply we stop at some point.

Remark 6.4.1. The absence of closed orbits is very crucial to the fact that the definition above will terminate. In the forking and merging cases, the existence of a function for the given arrow configuration already implies that there will not be any closed orbits. However, in the mixed case this is not always true. Figure 6.3 shows an example of a discrete function whose extracted vector field has a closed orbit. This is only possible because the edge with the value 6 is abnormally upward noncritical and one of its outgoing arrows points to an abnormally downward noncritical cell, the 2-cell with the value 2.

Definition 6.4.1 (A boundary operator). *We define* $\bar{C}_k \xrightarrow{\bar{\partial}_k} \bar{C}_{k-1}$ *as follows:*

$$\bar{\partial}\tau = v^G \circ \partial^c \tau = \sum_{\sigma < \tau} v^G(\sigma).$$

Below we provide some examples of CW complexes where we evaluate the boundary operator $\bar{\partial}$.

Example 6.4.1. Using Figure 6.14 and Figure 6.15, we have the following:
$\omega_1^{(3)} = [\nu_1, \nu_2, \nu_3, \nu_4]$; $\omega_2^{(3)} = [\nu_0, \nu_1, \nu_3, \nu_2]$;
$\tau_0 = [\nu_0, \nu_1, \nu_2]$; $\tau_1 = [\nu_0, \nu_2, \nu_3]$; $\tau_2 = [\nu_0, \nu_3, \nu_1]$; $\tau_4 = [\nu_1, \nu_3, \nu_2]$;
$\tau_3 = [\nu_1, \nu_2, \nu_4]$; $\tau_5 = [\nu_1, \nu_4, \nu_3]$; $\tau_6 = [\nu_2, \nu_3, \nu_4]$;
$\sigma_0 = [\nu_0, \nu_2]$; $\sigma_1 = [\nu_1, \nu_0]$; $\sigma_2 = [\nu_3, \nu_0]$; $\sigma_3 = [\nu_1, \nu_2]$; $\sigma_4 = [\nu_4, \nu_2]$;
$\sigma_5 = [\nu_2, \nu_3]$; $\sigma_6 = [\nu_3, \nu_1]$; $\sigma_7 = [\nu_3, \nu_4]$; $\sigma_8 = [\nu_4, \nu_1]$;
The cell $w_2^{(3)}$ is abnormally downward noncritical with the cells τ_0 and τ_4.
The cell τ_5 is abnormally downward noncritical with the cells σ_8 and σ_6.
The edge σ_6 is abnormally upward noncritical with the cells τ_5 and τ_2.
The vertex ν_4 is abnormally upward noncritical with the edges σ_7 and σ_8.
We then get the following sets:

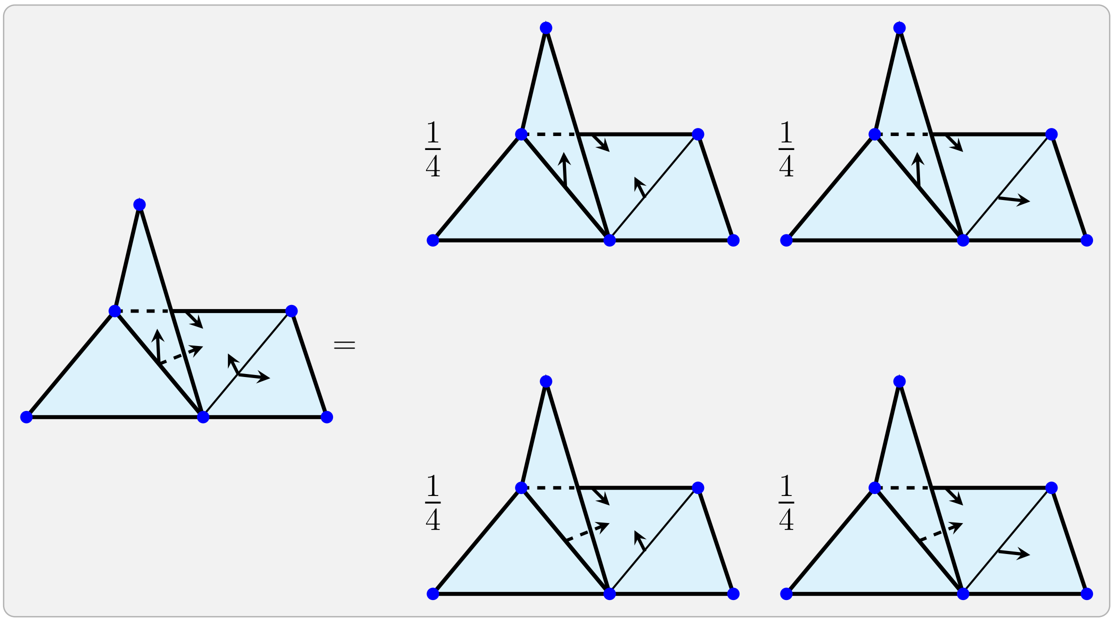


Figure 6.13: Illustration in the mixed case.

$\bar{C}_o^{(0)} = \{\nu_2, \nu_3\},\ \bar{C}_o^{(1)} = \{\sigma_5, \sigma_2, \sigma_3\},\ \bar{C}_o^{(2)} = \{\tau_6, \tau_3, \tau_1\},\ \bar{C}_o^{(3)} = \{\omega_1^{(3)}\};$

$\bar{C}_{in}^{(2)} = \{\tau_5\},\ \bar{C}_{in}^{(3)} = \{\omega_2^{(3)}\},\ \bar{C}_{ou}^{(0)} = \{\nu_4\},\ \bar{C}_{ou}^{(1)} = \{\sigma_6\};$

$\bar{C}_{sin}^{(1)} = \{\sigma_8, \sigma_6\},\ \bar{C}_{sin}^{(2)} = \{\tau_4, \tau_0\},\ \bar{C}_{sou}^{(1)} = \{\sigma_4, \sigma_7\},\ \bar{C}_{sou}^{(2)} = \{\tau_5, \tau_2\};$

We obtain

$$\bar{\partial}\omega_1^{(3)} = \tau_6 + \tau_3 + \tau_5 + V^G(\tau_4),$$

since we take all the cells in $\bar{C}_o \cup \bar{C}_{in} \cup \bar{C}_{sou}$ and $\tau_4 \in \bar{C}_{sin}$. We also take induced orientations into account. Following the outgoing arrow from τ_4, the cell $\omega_2^{(3)}$ has τ_0 *s.t.* $AUn(\tau_0) = \{\omega_2^{(3)}\}$, thus,

$$A = \{(\omega_2^{(3)}, \omega_2^{(3)})\} \text{ and } P_{\omega_2^{(3)}}((\omega_2^{(3)}, \omega_2^{(3)})) = \{\tau_4, \tau_0\}.$$

We then have

$$v^G(\tau_4 \to \omega_2^{(3)}) = v(\tau_4, (\omega_2^{(3)}, \omega_2^{(3)})) = \beta(2)\tau_4 + \frac{1-\beta(2)}{2}(\tau_4 - \tau_0 - \tau_1 - \tau_2),$$

since the arrow from τ_4 meets the incoming one from τ_0. Hence,

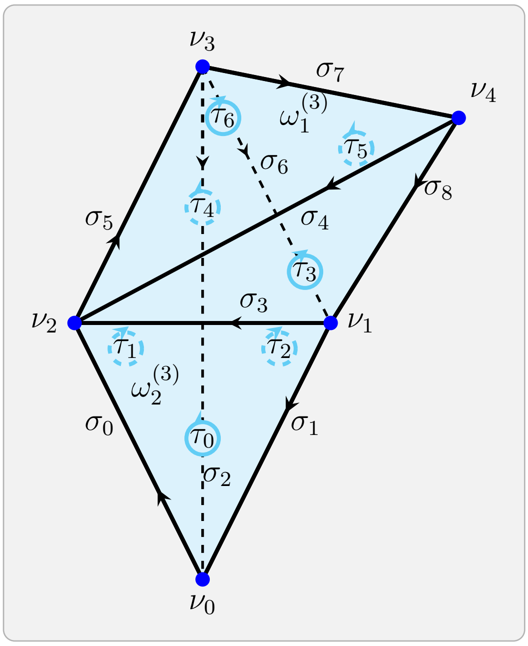


Figure 6.14: Initial orientation for Figure 6.15.

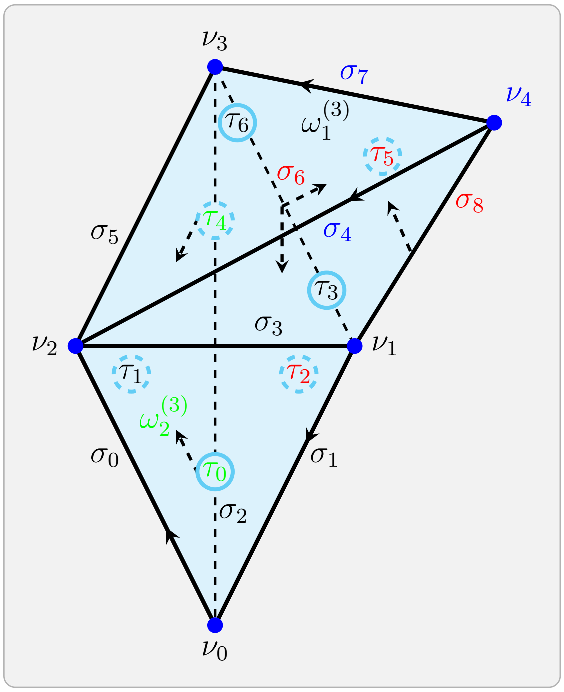


Figure 6.15: A general example.

$$
\begin{aligned}
\bar{\partial}\omega_1^{(3)} &= \tau_6 + \tau_3 + \tau_5 + \beta(2)\tau_4 + \frac{1-\beta(2)}{2}(\tau_4 - \tau_0 - \tau_1 - \tau_2) \\
&= \Big(\beta(2) + \frac{1-\beta(2)}{2}\Big)(\tau_6 + \tau_3 + \tau_5 + \tau_4) \\
&\quad + \frac{1-\beta(2)}{2}(\tau_6 + \tau_3 + \tau_5 - \tau_0 - \tau_1 - \tau_2);
\end{aligned}
$$

$$
\begin{aligned}
\bar{\partial}\omega_2^{(3)} &= \tau_1 + \tau_2 + \beta(2)\tau_4 + \frac{1-\beta(2)}{2}(\tau_4 - \tau_0 - \tau_1 - \tau_2) \\
&\quad + \beta(2)\tau_0 + \frac{1-\beta(2)}{2}(\tau_0 - \tau_4 - \tau_1 - \tau_2) \\
&= \beta(2)(\tau_1 + \tau_2 + \tau_0 + \tau_4);
\end{aligned}
$$

$\bar{\partial}\tau_3 = \sigma_3 - \sigma_4 + V^G(\sigma_8)$,

and following the arrow $\sigma_8 \to \tau_5$, the cell τ_5 has σ_6 *s.t.* $AUn(\sigma_6) = \{\tau_5, \tau_2\}$. Thus, $A^{\sigma_8} = \{(\tau_5, \tau_5), (\tau_5, \tau_2)\}$, and we need to take the average over the possibilities that we have at the edge σ_6. We have

$v^G(\sigma_8 \to \tau_5) = \frac{1}{2}(v(\sigma_8,(\tau_5,\tau_5)) + v(\sigma_8,(\tau_5,\tau_2)))$.
Also,

$$P_{\tau_5}((\tau_5,\tau_5)) = \{\sigma_8,\sigma_6\} \Rightarrow v(\sigma_8,(\tau_5,\tau_5)) = \beta(2)\sigma_8 + \tfrac{1-\beta(2)}{2}(\sigma_8+\sigma_6-\sigma_7),$$

$$P_{\tau_5}((\tau_5,\tau_2)) = \{\sigma_8\} \Rightarrow v(\sigma_8,(\tau_5,\tau_2)) = -\sigma_7 + v^G(\sigma_6 \to \tau_2) = -\sigma_7 + \sigma_2,$$

where $\tau_2 \in AUn(\sigma_6) \cap E$, and $\tau_2 \neq \tau_5$. This yields

$$\begin{aligned}
\bar{\partial}\tau_3 &= \sigma_3 - \sigma_4 + \frac{1}{2}\Big(\beta(2)\sigma_8 + \frac{1-\beta(2)}{2}\big(\sigma_8 - \sigma_7 + \sigma_6\big) + (-\sigma_7 + \sigma_2)\Big) \\
&= \big(\frac{\beta(2)}{2} + \frac{1-\beta(2)}{4}\big)(\sigma_3 - \sigma_4 + \sigma_8) + \frac{1-\beta(2)}{4}(\sigma_3 - \sigma_4 - \sigma_7 + \sigma_6) \\
&\quad + \frac{1}{2}(\sigma_3 - \sigma_4 - \sigma_7 + \sigma_2);
\end{aligned}$$

Similarly one gets

$$\begin{aligned}
\bar{\partial}\tau_4 &= -\sigma_5 - \sigma_3 - V^G(\sigma_6) \\
&= -\sigma_5 - \sigma_3 - \beta(2)\sigma_6 - \frac{1-\beta(2)}{2}\big(v^G(\sigma_6 \to \tau_5) + v^G(\sigma_6 \to \tau_2)\big) \\
&= -\sigma_5 - \sigma_3 - \beta(2)\sigma_6 - \frac{1-\beta(2)}{2}\Big(\beta(2)\sigma_6 + \frac{1-\beta(2)}{2}(\sigma_6 + \sigma_8 + \sigma_7) + \sigma_2\Big) \\
&= -\big(\beta(2)(1 + \frac{1-\beta(2)}{2}) + \frac{(1-\beta(2))^2}{4}\big)(\sigma_5 + \sigma_3 + \sigma_6) \\
&\quad - \frac{(1-\beta(2))^2}{4}(\sigma_5 + \sigma_3 + \sigma_8 + \sigma_7) - \frac{1-\beta(2)}{2}(\sigma_5 + \sigma_3 + \sigma_2);
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}\tau_2 &= -\sigma_2 + V^G(\sigma_6) \\
&= -\sigma_2 + \beta(2)\sigma_6 + \frac{1-\beta(2)}{2}\Big(\beta(2)\sigma_6 + \frac{1-\beta(2)}{2}(\sigma_6 + \sigma_8 + \sigma_7) + \sigma_2\Big) \\
&= -\Big(\beta(2)\big(1 + \frac{1-\beta(2)}{2}\big) + \frac{(1-\beta(2))^2}{4}\Big)\big(\sigma_2 + \sigma_6\big) \\
&\quad - \frac{(1-\beta(2))^2}{4}(\sigma_2 + \sigma_8 + \sigma_7);
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}\tau_5 &= -\sigma_7 + V^G(\sigma_6) - V^G(\sigma_8) \\
&= -\sigma_7 + \beta(2)\sigma_6 + \frac{1-\beta(2)}{2}\Big(\beta(2)\sigma_6 + \frac{1-\beta(2)}{2}(\sigma_6 + \sigma_8 + \sigma_7) + \sigma_2\Big) \\
&\quad - \frac{1}{2}\Big(\beta(2)\sigma_8 + \frac{1-\beta(2)}{2}\big(\sigma_8 - \sigma_7 + \sigma_6\big) + (-\sigma_7 + \sigma_2)\Big) \\
&= \big(\beta(2)(1 + \frac{1-\beta(2)}{4})\big)(\sigma_6 - \sigma_8 - \sigma_7) - \frac{\beta(2)}{2}(\sigma_2 - \sigma_8 - \sigma_7);
\end{aligned}$$

$$\bar{\partial}\tau_0 = \sigma_3; \quad \bar{\partial}\tau_6 = (\sigma_7 + \sigma_4) + \sigma_5; \quad \bar{\partial}\tau_1 = \sigma_2 + \sigma_5;$$

$$\bar{\partial}\sigma_2 = \nu_2 - \nu_3 = \bar{\partial}\sigma_6; \quad \bar{\partial}\sigma_5 = \nu_3 - \nu_2; \quad \bar{\partial}\sigma_3 = 0;$$

$$\bar{\partial}\sigma_8 = \nu_2 - \beta(2)\nu_4 - \tfrac{1-\beta(2)}{2}(\nu_3 + \nu_2); \quad \bar{\partial}\sigma_4 = \nu_2 - \beta(2)\nu_4 - \tfrac{1-\beta(2)}{2}(\nu_3 + \nu_2);$$

$$\bar{\partial}\sigma_7 = -\nu_3 + \beta(2)\nu_4 + \tfrac{1-\beta(2)}{2}(\nu_3 + \nu_2).$$

One checks by direct computation that $\bar{\partial} \circ \bar{\partial} = 0$.

Generalized Morse inequalities

We now state and prove the mains theorems of this chapter, one of which establishes the fact that the square of the boundary operator $\bar{\partial}$ is zero.

Theorem 6.4.1. $\bar{\partial} \circ \bar{\partial} = 0$.

The proof of the fact that the square of this boundary operator is zero is a step by step procedure, moving from a situation with no arrows (where the square of the boundary is zero), creating abnormally upward/downward noncritical cells by adding arrows, and then showing at each step that the square is still zero.

Proof. We move from the situation where there are no arrows. In that case, the boundary operator is just the cellular boundary operator and therefore $\bar{\partial} \circ \bar{\partial} = 0$. After adding arrows, we show that this relation is preserved at any step. Therefore, ultimately, it has to hold for our arrow configuration as desired.

We have shown in Section 6.2 and Section 6.3 that, adding a forking or merging cell preserves the equation $\bar{\partial} \circ \bar{\partial} = 0$. We are left to address the mixed case.

We let $\bar{\partial}^{t-1}$ be the boundary operator at step $t-1$ satisfying $\bar{\partial}^{t-1} \circ \bar{\partial}^{t-1} = 0$. After creating the abnormally upward/downward noncritical cell, we are at step t and we set out to show that $\bar{\partial}^t \circ \bar{\partial}^t = 0$.
In order to understand what is happening, we start with the simplest case where a cell has only two incoming arrows from abnormally upward noncritical cells.

Suppose that τ **is such that** $ADn(\tau) = \{\sigma_1, \sigma_2\}$ **with** $AUn(\sigma_i) = \{\tau_1^{\sigma_i} = \tau, \tau_2^{\sigma_i}, \cdots, \tau_{m_i}^{\sigma_i}\}$ **for** $i = 1, 2$**, and w.l.o.g.** $n_{in}(\tau_p^{\sigma_i}) = 1$ **for** $p \neq 1$.

At step t, τ has no incoming arrows and the σ_i's have no outgoing arrows. Suppose ς is a critical cell and that one could reach either σ_1 or σ_2 from ς by following the v^G-path. Without loss of generality assume that for $\varsigma = \varsigma^{\sigma_1}$ and

$c_1 \in \mathbb{R}$,

$$\bar{\partial}^{t-1}\varsigma^{\sigma_1} = b_{\varsigma^{\sigma_1}} + c_1\sigma_1.$$

We choose the orientations of σ_1, σ_2, τ, $\tau_p^{\sigma_i}$ in such a way that:

$$\bar{\partial}^{t-1}\tau = b_\tau + \sigma_1 + \sigma_2, \quad \text{and } \bar{\partial}^{t-1}\tau_q^{\sigma_i} = b_{\tau_q^{\sigma_i}} - \sigma_i, \text{ for all } i = 1,2 \text{ and } q \neq 1.$$

Now we show how the value of the boundary operator $\bar{\partial}^t$ on ς, τ and $\tau_q^{\sigma_i}$ can be expressed in terms of the operator $\bar{\partial}^{t-1}$.

$$\begin{aligned}
\bar{\partial}^t\varsigma^{\sigma_1} &= b_{\varsigma^{\sigma_1}} + c_1\Bigg(\beta(m_1)\sigma_1 + \frac{1-\beta(m_1)}{m_1}\Big[\sum_{p=2}^{m_1} v^G(\sigma_1 \to \tau_p^{\sigma_1}) \\
&\quad + \frac{1}{m_2}\Big(\beta\sigma_1 + \frac{1-\beta}{2}(\sigma_1 - \sigma_2 - b_\tau) - (m_2-1)b_\tau - \sum_{q=2}^{m_2} v^G(\sigma_2 \to \tau_q^{\sigma_2})\Big)\Big]\Bigg) \\
&= \beta(m_1)(b_{\varsigma^{\sigma_1}} + c_1\sigma_1) + \frac{1-\beta(m_1)}{m_1}\Big[\sum_{p=2}^{m_1}(b_{\varsigma^{\sigma_1}} + c_1 v^G(\sigma_1 \to \tau_p^{\sigma_1})) \\
&\quad + \frac{1}{m_2}\Big(b_{\varsigma^{\sigma_1}} + c_1\Big(\beta\sigma_1 + \frac{1-\beta(2)}{2}(\sigma_1 - \sigma_2 - b_\tau)\Big) \\
&\quad - (m_2-1)(-b_{\varsigma^{\sigma_1}} + c_1 b_\tau) - \; c_1\sum_{q=2}^{m_2} v^G(\sigma_2 \to \tau_q^{\sigma_2})\Big)\Big] \\
&= \frac{1-\beta(m_1)}{m_1}\sum_{p=2}^{m_1}\Big((b_{\varsigma^{\sigma_1}} + c_1\sigma_1) + c_1(-\sigma_1 + v^G(\sigma_1 \to \tau_p^{\sigma_1}))\Big) \\
&\quad + \beta(m_1)(b_{\varsigma^{\sigma_1}} + c_1\sigma_1) + \frac{1-\beta(m_1)}{m_1 m_2}\Big[\beta(2)(b_{\varsigma^{\sigma_1}} + c_1\sigma_1) + \\
&\quad + \frac{1-\beta(2)}{2}\Big((b_{\varsigma^{\sigma_1}} + c_1\sigma_1) + (b_{\varsigma^{\sigma_1}} - \; c_1\sigma_2 - \; c_1 b_\tau)\Big) \\
&\quad + (m_2-1)\Big((b_{\varsigma^{\sigma_1}} + c_1\sigma_1) - c_1(\sigma_1 + \sigma_2 + b_\tau)\Big) \\
&\quad - c_1\sum_{q=2}^{m_2}(-\sigma_2 + v^G(\sigma_2 \to \tau_q^{\sigma_2}))\Big] \\
&= \bar{\partial}^{t-1}\varsigma - c_1\frac{1-\beta(m_1)}{m_1 m_2}\big((m_2-1) + \frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau \qquad (6.27) \\
&\quad + c_1\frac{1-\beta(m_1)}{m_1}\Big(\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1} - \frac{1}{m_2}\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2}\Big).
\end{aligned}$$

Similarly for ς^{σ_2} that can reach $c_2\sigma_2,\ c_2 \in \mathbb{R}$, we get

$$\begin{aligned}\bar{\partial}^t\varsigma^{\sigma_2} &= \bar{\partial}^{t-1}\varsigma^{\sigma_2} - c_2\frac{1-\beta(m_2)}{m_1m_2}\big((m_1-1)+\frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau \\ &\quad + c_2\frac{1-\beta(m_2)}{m_2}\Big(\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2} - \frac{1}{m_1}\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1}\Big).\end{aligned} \tag{6.28}$$

For $p = 2, \cdots, m_1$,

$$\begin{aligned}\bar{\partial}^t\tau_p^{\sigma_1} &= b_{\tau_p^{\sigma_1}} - \beta(m_1)\sigma_1 - \frac{1-\beta(m_1)}{m_1}\Big[\sum_{r=2}^{m_1} v^G(\sigma_1 \to \tau_r^{\sigma_1}) \\ &\quad + \frac{1}{m_2}\Big(\beta(2)\sigma_1 + \frac{1-\beta(2)}{2}(\sigma_1-\sigma_2-b_\tau) - (m_2-1)b_\tau \\ &\quad - \sum_{q=2}^{m_2} v^G(\sigma_2 \to \tau_q^{\sigma_2})\Big)\Big] \\ &= (1-\frac{1-\beta(m_1)}{m_1})\bar{\partial}^{t-1}\tau_p^{\sigma_1} - \frac{1-\beta(m_1)}{m_1}\sum_{\substack{r\neq p\\ r=2}}^{m_1}\bar{\partial}^{t-1}\tau_r^{\sigma_1} \\ &\quad + \frac{1-\beta(m_1)}{m_1m_2}\big((m_2-1)+\frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau + \frac{1-\beta(m_1)}{m_1m_2}\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2}.\end{aligned} \tag{6.29}$$

Similarly, for $q = 2, \cdots, m_2$,

$$\begin{aligned}\bar{\partial}^t\tau_q^{\sigma_2} &= (1-\frac{1-\beta(m_2)}{m_2})\bar{\partial}^{t-1}\tau_q^{\sigma_2} - \frac{1-\beta(m_2)}{m_2}\sum_{\substack{r\neq q\\ r=2}}^{m_2}\bar{\partial}^{t-1}\tau_r^{\sigma_2} \\ &\quad + \frac{1-\beta(m_2)}{m_1m_2}\big((m_1-1)+\frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau + \frac{1-\beta(m_2)}{m_1m_2}\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1}.\end{aligned} \tag{6.30}$$

To get the final result, we also need to know explicitly what $-\sigma_i + v^G(\sigma_i \to \tau)$ is for $i = 1, 2$.

$$\begin{aligned}-\sigma_1 + v^G(\sigma_1 \to \tau) &= -\sigma_1 + \frac{1}{m_2}\Big(\beta(2)\sigma_1 + \frac{1-\beta(2)}{2}(\sigma_1-\sigma_2-b_\tau) - (m_2-1)b_\tau \\ &\quad - \sum_{q=2}^{m_2} v^G(\sigma_2 \to \tau_q^{\sigma_2})\Big) \\ &= \frac{1}{m_2}\Big(-\frac{1-\beta(2)}{2}(\sigma_1+\sigma_2+b_\tau) - (m_2-1)(\sigma_1+\sigma_2+b_\tau)\Big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{m_2}\sum_{q=2}^{m_2}\big(-\sigma_2+v^G(\sigma_2\to\tau_q^{\sigma_2})\big)\\
=&\quad -\frac{1}{m_2}\big((m_2-1)+\frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau-\frac{1}{m_2}\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2},
\end{aligned}
$$

and it similarly follows that

$$
-\sigma_2+v^G(\sigma_2\to\tau)=-\frac{1}{m_1}\big((m_1-1)+\frac{1-\beta(2)}{2}\big)\bar{\partial}^{t-1}\tau-\frac{1}{m_1}\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1}.
$$

Now,

$$
\begin{aligned}
\bar{\partial}^t\tau &= b_\tau+\beta(m_1)\sigma_1+\frac{1-\beta(m_1)}{m_1}\sum_{p=1}^{m_1}v^G(\sigma_1\to\tau_p^{\sigma_1})\\
&\quad+\beta(m_2)\sigma_2+\frac{1-\beta(m_2)}{m_2}\sum_{q=1}^{m_2}v^G(\sigma_2\to\tau_q^{\sigma_2})\\
&= (b_\tau+\sigma_1+\sigma_2)+\frac{1-\beta(m_2)}{m_2}\sum_{p=1}^{m_2}\big(v^G(\sigma_2\to\tau_p^{\sigma_2})-\sigma_2\big)\\
&\quad+\frac{1-\beta(m_1)}{m_1}\sum_{q=1}^{m_1}\big(v^G(\sigma_1\to\tau_q^{\sigma_1})-\sigma_1\big)\\
&= \bar{\partial}^{t-1}\tau+\frac{1-\beta(m_2)}{m_2}\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2}+\frac{1-\beta(m_1)}{m_1}\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1}\\
&\quad+\frac{1-\beta(m_1)}{m_1}v^G(\sigma_1\to\tau)+\frac{1-\beta(m_2)}{m_2}v^G(\sigma_2\to\tau)\\
&= \Bigg(1-\frac{1-\beta(m_1)}{m_1m_2}\Big((m_2-1)+\frac{1-\beta(2)}{2}\Big) \qquad (6.31)\\
&\quad-\frac{1-\beta(m_2)}{m_1m_2}\Big((m_1-1)+\frac{1-\beta(2)}{2}\Big)\Bigg)\bar{\partial}^{t-1}\tau\\
&\quad+\Big(\frac{1-\beta(m_1)}{m_1}-\frac{1-\beta(m_2)}{m_1m_2}\Big)\sum_{p=2}^{m_1}\bar{\partial}^{t-1}\tau_p^{\sigma_1}\\
&\quad+\Big(\frac{1-\beta(m_2)}{m_2}-\frac{1-\beta(m_1)}{m_1m_2}\Big)\sum_{q=2}^{m_2}\bar{\partial}^{t-1}\tau_q^{\sigma_2}.
\end{aligned}
$$

Showing that the square of the boundary operator $\bar{\partial}$ is zero in this case is similar to the case when $l>2$. This is why we only provide a proof in the general case.

Suppose τ **is such that** $ADn(\tau) = \{\sigma_1, \sigma_2, \cdots, \sigma_l\}$ **with** $AUn(\sigma_i) = \{\tau_1^{\sigma_i} = \tau, \tau_2^{\sigma_i}, \cdots, \tau_{m_i}^{\sigma_i}\}$ **for** $i = 1, 2, \cdots, l$**, and w.l.o.g.** $n_{in}(\tau_p^{\sigma_i}) = 1$ **for** $p \neq 1$. In particular this means that at least one of $l, m_1, \cdots, m_l$ is in $\mathbb{N}^* \setminus \{1\}$ where $\mathbb{N}^* := \{1, 2, \cdots\}$.

Let ς^{σ_i} be critical and reaches $c_i\sigma_i$, $c_i \in \mathbb{R}$. Suppose we have the following:

$$\bar{\partial}^{t-1}\varsigma^{\sigma_i} = b_{\varsigma^{\sigma_i}} + c_i\sigma_i, \quad \text{for all } i = 1, \cdots, l,$$

$$\bar{\partial}^{t-1}\tau = b_\tau + \textstyle\sum_i \sigma_i, \quad \bar{\partial}^{t-1}\tau_r^{\sigma_i} = b_{\tau_r^{\sigma_i}} - \sigma_i, \quad \text{for all } \ r = 2, \cdots, m_i.$$

Let $\quad A^{\sigma_i} = \{\tau\} \times AUn(\sigma_1) \times \cdots \times AUn(\sigma_{i-1}) \times AUn(\sigma_{i+1}) \times \cdots \times AUn(\sigma_l)$,

$$v^G(\sigma_i) = \beta(m_i)\sigma_i + \frac{1-\beta(m_i)}{m_i}\sum_{r=1}^{m_i} v^G(\sigma_i \to \tau_r^{\sigma_i}).$$

Set $|P_\tau(E)| = \eta_E$, in general,

$$-\sigma_i + v^G(\sigma_i \to \tau) = -\sigma_i + \frac{1}{|A^{\sigma_i}|}\sum_{E \in A^{\sigma_i}} v(\sigma_i, E) = \frac{1}{|A^{\sigma_i}|}\sum_{E \in A^{\sigma_i}} (v(\sigma_i, E) - \sigma_i),$$

where,

$$v(\sigma_i, E) = \beta(\eta_E)\sigma_i + \frac{1-\beta(\eta_E)}{\eta_E}\Big[(\eta_E - 1)\sigma_i - b_\tau - \sum_{\substack{h \neq i \\ \sigma_h \in P_\tau(E)}} \sigma_h - \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} v(\sigma_j \to \tau^{\sigma_j})\Big].$$

Then

$$\begin{aligned} v(\sigma_i, E) - \sigma_i &= \frac{1-\beta(\eta_E)}{\eta_E}\Big[-\sigma_i - b_\tau - \sum_{h \neq i, \sigma_h \in P_\tau(E)} \sigma_h - \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} v^G(\sigma_j \to \tau^{\sigma_j})\Big] \\ &= -\frac{1-\beta(\eta_E)}{\eta_E}\Big[\big(\sum_{i=1}^{l} \sigma_i + b_\tau\big)\big) + \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} (-\sigma_j + v^G(\sigma_j \to \tau^{\sigma_j}))\Big] \\ &= -\frac{1-\beta(\eta_E)}{\eta_E}\Big[\bar{\partial}^{t-1}\tau + \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} \bar{\partial}^{t-1}\tau^{\sigma_j}\Big]; \end{aligned}$$

For $i = 1, \cdots, l$,

$$\bar{\partial}^t\varsigma^{\sigma_i} \ = \ b_{\varsigma^{\sigma_i}} + c_i\Big(\beta(m_i)\sigma_i + \frac{1-\beta(m_i)}{m_i}\sum_{r=1}^{m_i} v^G(\sigma_i \to \tau_r^{\sigma_i})\Big)$$

$$
\begin{aligned}
&= \beta(m_i)(b_{\varsigma^{\sigma_i}} + c_i\sigma_i) + \frac{1-\beta(m_i)}{m_i}\Big(\sum_{r=2}^{m_i}\big(b_{\varsigma^{\sigma_i}} + c_i v^G(\sigma_i \to \tau_r^{\sigma_i})\big) + b_{\varsigma^{\sigma_i}} \\
&\quad + c_i\frac{1}{|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}} v(\sigma_i, E)\Big) \\
&= \beta(m_i)(b_{\varsigma^{\sigma_i}} + c_i\sigma_i) \\
&\quad + \frac{1-\beta(m_i)}{m_i}\left(\sum_{r=2}^{m_i}\Big((b_{\varsigma^{\sigma_i}} + c_i\sigma_i) + c_i(-\sigma_i + v^G(\sigma_i \to \tau_r^{\sigma_i}))\Big)\right. \\
&\quad \left. + \frac{1}{|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\Big((b_{\varsigma^{\sigma_i}} + c_i\sigma_i) + c_i(-\sigma_i + v(\sigma_i, E))\Big)\right) \\
&= \bar{\partial}^{t-1}\varsigma^{\sigma_i} + c_i\frac{1-\beta(m_i)}{m_i}\left(\sum_{r=2}^{m_i}\bar{\partial}^{t-1}\tau_r^{\sigma_i}\right. \\
&\quad \left. - c_i\frac{1}{|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\Big(\bar{\partial}^{t-1}\tau + \sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}}\bar{\partial}^{t-1}\tau^{\sigma_j}\Big)\right) \\
&= \bar{\partial}^{t-1}\varsigma^{\sigma_i} + c_i\frac{1-\beta(m_i)}{m_i}\sum_{r=2}^{m_i}\bar{\partial}^{t-1}\tau_r^{\sigma_i} - c_i\frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\bar{\partial}^{t-1}\tau \\
&\quad - c_i\frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}}\bar{\partial}^{t-1}\tau^{\sigma_j};
\end{aligned}
$$

For $p = 2, \cdots, m_i$,

$$
\begin{aligned}
\partial^t\tau_p^{\sigma_i} &= b_{\tau_p^{\sigma_i}} - \beta(m_i)\sigma_i - \frac{1-\beta(m_i)}{m_i}\sum_{r=1}^{m_i} v^G(\sigma_i \to \tau_r^{\sigma_i}) \\
&= \bar{\partial}^{t-1}\tau_p^{\sigma_i} - \frac{1-\beta(m_i)}{m_i}\left(\sum_{r=2}^{m_i}\bar{\partial}^{t-1}\tau_r^{\sigma_i}\right. \\
&\quad \left. - \frac{1}{|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\Big(\bar{\partial}^{t-1}\tau + \sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}}\bar{\partial}^{t-1}\tau^{\sigma_j}\Big)\right) \\
&= \Big(1 - \frac{1-\beta(m_i)}{m_i}\Big)\bar{\partial}^{t-1}\tau_p^{\sigma_i} - \frac{1-\beta(m_i)}{m_i}\sum_{\substack{r\neq p \\ s=2}}^{m_i}\bar{\partial}^{t-1}\tau_r^{\sigma_i} \\
&\quad + \frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\bar{\partial}^{t-1}\tau \\
&\quad + \frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|}\sum_{E\in A^{\sigma_i}}\frac{1-\beta(\eta_E)}{\eta_E}\sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}}\bar{\partial}^{t-1}\tau^{\sigma_j};
\end{aligned}
$$

$$\begin{aligned}
\bar{\partial}^t \tau &= b_\tau + \sum_{p=1}^{l} \Big(\beta(m_p)\sigma_p + \frac{1-\beta(m_p)}{m_p} \sum_j v^G(\sigma_p \to \tau_j^{\sigma_p}) \Big) \\
&= \big(b_\tau + \sum_{i=1}^{l} \sigma_p\big) + \sum_{p=1}^{l} \Big(\frac{1-\beta(m_p)}{m_p} \sum_j (v^G(\sigma_p \to \tau_j^{\sigma_p}) - \sigma_p) \Big) \\
&= \bar{\partial}^{t-1}\tau + \sum_{k=1}^{p} \frac{1-\beta(m_p)}{m_p} \Big((v^G(\sigma_p \to \tau) - \sigma_p) + \sum_{j=2}^{m_p} (v^G(\sigma_p \to \tau_j^{\sigma_p}) - \sigma_p) \Big) \\
&= \bar{\partial}^{t-1}\tau + \sum_{i=1}^{l} \frac{1-\beta(m_i)}{m_i} \sum_{r=2}^{m_i} \bar{\partial}^{t-1}\tau_r^{\sigma_i} - \sum_{i=1}^{l} \frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|} \sum_{E\in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E} \bar{\partial}^{t-1}\tau \\
&\quad - \sum_{i=1}^{l} \frac{1-\beta(m_i)}{m_i|A^{\sigma_i}|} \sum_{E\in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E} \sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}} \bar{\partial}^{t-1}\tau^{\sigma_j}.
\end{aligned}$$

Let $|A| = |m_i A^{\sigma_i}| = \prod_{i=1}^{l} m_i$,

$$P^{\sigma_i} := \frac{1-\beta(m_i)}{|A|} \sum_{E\in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E},$$

and

$$Q := \frac{1-\beta(m_i)}{|A|} \sum_{E\in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E} \sum_{\substack{\tau^{\sigma_j}\neq\tau \\ \tau^{\sigma_j}\in E\cap AUn(\sigma_j)}} \bar{\partial}^{t}\tau^{\sigma_j}.$$

We now proceed to find P^{σ_i} and Q.

For $i \in \{1,\cdots,l\}$ and $h = 1,\cdots,l$, let us denote by $C^h_{\{1,\cdots,l\}}$ the set of all the different combinations of h elements in $\{1,\cdots,l\}$. That is, the set of all subsets of $\{1,\cdots,l\}$ of cardinality h. For example,

$$C^1_{\{1,\cdots,l\}} = \{\{1\},\{2\},\cdots,\{l\}\}, \quad C^1_{\{1,\cdots,l\}\setminus\{1\}} = \{\{2\},\{3\},\cdots,\{l\}\}$$
$$C^2_{\{1,\cdots,l\}\setminus\{1\}} = \{\{2,3\},\cdots,\{2,l\},\{3,4\},\cdots,\{3,l\},\cdots,\{l-1,l\}\}.$$

Let

$$S_h = \sum_{e\in\Gamma^h} \prod_{r\in e} (m_r - 1), \quad \Gamma^h := C^h_{\{1,\cdots,l\}},$$

and

$$\text{for } h = 1,\cdots,l-1, \quad S_{i,h} = \sum_{e\in\Gamma_i^h} \prod_{r\in e} (m_r - 1), \quad \Gamma_i^h := C^h_{\{1,\cdots,l\}\setminus\{i\}},$$

then

$$S_h = S_{i,h} + (m_i - 1)S_{i,h-1} \quad \text{for} \quad h = 1, \cdots, l,$$

where, $S_{i,0} = 1$, $S_{i,l} = 0$, and for $l = 1$, $S_{i,1} = 0$.

Lemma 6.4.2.

$$1 + \sum_{i=1}^{l} S_i = |A| \quad \textit{and} \quad 1 + \sum_{h=1}^{l-1} S_{i,h} = \frac{|A|}{m_i}.$$

Proof. We show both equalities at the same time using induction on l. Suppose that for $\{1, \cdots, l\}$ we have $m_1, \cdots, m_l$.

- For $l = 2$ we have $\{1, 2\}$ and $\{m_1, m_2\}$,

  $$1 + S_1 + S_2 = 1 + \big((m_1 - 1) + (m_2 - 1)\big) + (m_1 - 1)(m_2 - 1) = m_1 m_2 = |A|.$$

  For $i = 1$, $1 + S_{1,1} = 1 + (m_2 - 1) = m_2 = \frac{|A|}{m_1}$.

- We assume that the statements are true for l and we show that they are true for $l + 1$. That is we consider $\{1, \cdots, l, l + 1\}$ and $\{m_1, \cdots, m_l, m_{l+1}\}$. Set

  $$|A^l| := \prod_{r=1}^{l} m_r \quad \text{and} \quad |A^{l+1}| := \prod_{r=1}^{l+1} m_r.$$

  For $h = 1, \cdots, l$, $S_h^l := \sum_{e \in \Gamma^h} \prod_{r \in e}(m_r - 1)$, $\quad \Gamma^h := C^h_{\{1, \cdots, l\}}$.

  For $h = 1, \cdots, l - 1$, $S_{i,h}^l := \sum_{e \in \Gamma_i^h} \prod_{r \in e}(m_r - 1)$, $\quad \Gamma_i^h := C^h_{\{1, \cdots, l\} \setminus \{i\}}$.

  For $h = 1, \cdots, l + 1$, $S_h^{l+1} := \sum_{e \in \Gamma^h} \prod_{r \in e}(m_r - 1)$, $\quad \Gamma^h := C^h_{\{1, \cdots, l, l+1\}}$.

  For $h = 1, \cdots, l$, $S_{i,h}^{l+1} := \sum_{e \in \Gamma_i^h} \prod_{r \in e}(m_r - 1)$, $\quad \Gamma_i^h := C^h_{\{1, \cdots, l, l+1\} \setminus \{i\}}$.
  The induction hypothesis yields

  $$1 + \sum_{h=1}^{l} S_h^l = |A^l|, \quad 1 + \sum_{h=1}^{l-1} S_{i,h}^l = \frac{|A^l|}{m_i}.$$

  Then

  $$S_h^{l+1} = S_h^l + (m_{l+1} - 1)S_{h-1}^l \text{ for } h = 1, \cdots, l + 1,$$

  where, $S_0^l = 1$, $S_{l+1}^l = 0$.
  Also,

  $$S_{i,h}^{l+1} = S_{i,h}^l + (m_{l+1} - 1)S_{i,h-1}^l \text{ for } h = 1, \cdots, l,$$

where, $S^l_{i,0} = 1, \quad S^l_{i,l} = 0$, and for $l = 1,\ S^l_{i,1} = 0$.
This implies that

$$\begin{aligned} 1 + \sum_{h=1}^{l+1} S^{l+1}_h &= 1 + \sum_{h=1}^{l} S^l_h + (m_{l+1} - 1)\big(1 + \sum_{h=1}^{l} S^l_h\big) \\ &= m_{l+1}|A^l| = |A^{l+1}|. \end{aligned}$$

We also get

$$\begin{aligned} 1 + \sum_{h=1}^{l} S^{l+1}_{i,h} &= 1 + \sum_{h=1}^{l-1} S^l_{i,h} + (m_{l+1} - 1)\big(1 + \sum_{h=1}^{l-1} S^l_{i,h}\big) \\ &= m_{l+1}\frac{|A^l|}{m_i} = \frac{|A^{l+1}|}{m_i}. \end{aligned}$$

□

Consider the expression $\sum_{E\in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E}$. Then there are the following possibilities for the set $E \in A^{\sigma_i}$.

- The l-tuple $(\tau, \tau, \cdots, \tau)$, which corresponds to τ having l incoming arrows from all the σ_i's. For the above expression, this gives the value $\frac{1-\beta(l)}{l}$.
- The tuple $(\tau, \cdots, \tau, \tau^{\sigma_h}_r, \tau, \cdots, \tau)$ for all $h \neq i$ and for all $r = 2, \cdots, m_h$, which corresponds to τ having $l-1$ incoming arrows, from all the σ_i's except for σ_h. There are $l-1$ possibilities for h and $m_h - 1$ possibilities for each h. This gives the value $\frac{1-\beta(l-1)}{l-1} S_{i,1}$ for the above expression.
- The tuple $(\tau, \cdots, \tau, \tau^{\sigma_j}_p, \tau, \cdots, \tau, \tau^{\sigma_h}_q, \tau, \cdots, \tau)$ for all $h, j \neq i$, and for all $p = 2, \cdots, m_j,\ q = 2, \cdots, m_h$ corresponds to τ having $l-2$ incoming arrows from all σ_i's except from σ_j and σ_h. We get the value $\frac{1-\beta(l-2)}{l-2} S_{i,2}$ for the above expression.

⋮

- The tuple $(\tau, \tau^{\sigma_1}_{r_1}, \cdots, \tau^{\sigma_{i-1}}_{r_{i-1}}, \tau^{\sigma_{i+1}}_{r_{i+1}}, \cdots, \tau^{\sigma_{h-1}}_{r_{h-1}}, \tau, \tau^{\sigma_{h+1}}_{r_{h+1}}, \cdots, \tau^{\sigma_l}_{r_l})$ corresponds to τ having two incoming arrows, one from σ_i and the other from σ_h. Each $r_i = 2, \cdots, m_i$, for $i = 1, \cdots, l$. From this we get the value $\frac{1-\beta(2)}{2} S_{i,l-2}$ for the above expression.
- Finally, we get $(\tau, \tau^{\sigma_1}_{r_1}, \cdots, \tau^{\sigma_{i-1}}_{r_{i-1}}, \tau^{\sigma_{i+1}}_{r_{i+1}}, \cdots, \tau^{\sigma_l}_{r_l})$, which mean that τ has a single incoming arrow coming from σ_i, and this yields the value $S_{i,l-1}$ for the above expression.

Therefore, for $l \geq 1$,

$$\begin{aligned} P^{\sigma_i} &= \frac{1-\beta(m_i)}{|A|}\Big[\frac{1-\beta(l)}{l} + \frac{1-\beta(l-1)}{l-1}S_{i,1} + \frac{1-\beta(l-2)}{l-2}S_{i,2} \\ &\quad + \cdots + \frac{1-\beta(2)}{2}S_{i,l-2} + S_{i,l-1}\Big]. \end{aligned}$$

Observe that for $l=1$, $P^{\sigma_i} = P^{\sigma_1} = \frac{1-\beta(m_1)}{m_1}S_{1,0} = \frac{1-\beta(m_1)}{m_1}$, since we set $S_{i,0}=1$. One also refers to the forking case to see this.

To visualize for $l=2$, one refers to (6.27), (6.31), (6.29) and (6.30).

Now, observe that for all $E \in A^{\sigma_i}$, the elements $\tau^{\sigma_j} \in (E \setminus \{\tau\}) \cap AUn(\sigma_j)$ are exactly all the $\tau_p^{\sigma_j}$ for $j \in \{1,\cdots,l\} \setminus \{i\}$ and for $p = 2,\cdots,m_j$. Thus, for $h = 1,\cdots,l-2$,

$$S_{ij,h} = \sum_{e \in \Gamma_{ij}^h} \prod_{r \in e}(m_r - 1) \quad \text{for} \quad \Gamma_{ij}^h := C^h_{\{1,\cdots,l\}\setminus\{i,j\}},\ j \neq i.$$

Then

$$S_{i,h} = S_{ij,h} + (m_j - 1)S_{ij,h-1} \quad \text{for} \quad h = 1,\cdots,l-1,$$

where, $S_{ij,0}=1$, $S_{ij,l-1}=0$, and for $l=2$, $S_{ij,1}=0$.

To evaluate the expression

$$\sum_{E \in A^{\sigma_i}} \frac{1-\beta(\eta_E)}{\eta_E} \sum_{\substack{\tau^{\sigma_j} \neq \tau \\ \tau^{\sigma_j} \in E \cap AUn(\sigma_j)}} \bar{\partial}^t \tau^{\sigma_j},$$

we look at the different sets E such that we can reach $\tau_r^{\sigma_j}$ from σ_i. Thus these sets vary from the following:

- The tuple $(\tau,\cdots,\tau,\tau_r^{\sigma_j},\tau,\cdots,\tau)$ for all $j \neq i$ and for all $r = 2,\cdots,m_h$, which corresponds to τ having $l-1$ incoming arrows, from all the σ_i's except from σ_j. This gives the value $\frac{1-\beta(l-1)}{l-1}\bar{\partial}^t\tau_r^{\sigma_j}$ for the above expression.
- The tuple $(\tau,\cdots,\tau,\tau_r^{\sigma_j},\tau,\cdots,\tau,\tau_q^{\sigma_h},\tau,\cdots,\tau)$ for all $h,j \neq i$, $h \neq j$, and for all $q = 2,\cdots,m_h$ corresponds to τ having $l-2$ incoming arrows from all σ_i's except from σ_j and σ_h. For each such $\tau_q^{\sigma_h}$ there are $m_h - 1$ possibilities. We get for the above expression the value $\frac{1-\beta(l-2)}{l-2}S_{ij,1}\bar{\partial}^t\tau_r^{\sigma_j}$.

$\vdots$

- The tuple $(\tau, \tau^{\sigma_1}_{r_1}, \cdots, \tau^{\sigma_j}_{r}, \cdots, \tau^{\sigma_{i-1}}_{r_{i-1}}, \tau^{\sigma_{i+1}}_{r_{i+1}}, \cdots, \tau^{\sigma_{h-1}}_{r_{h-1}}, \tau, \tau^{\sigma_{h+1}}_{r_{h+1}}, \cdots, \tau^{\sigma_l}_{r_l})$ corresponds to τ having two incoming arrows, one from σ_i and the other from σ_h. For each $\tau^{\sigma_s}_q$ for $s \neq i, j, h$, we have $m_s - 1$ possibilities. From this we get the value $\frac{1-\beta(2)}{2} S_{ij,l-3} \bar{\partial}^t \tau^{\sigma_j}_r$ for the above expression.
- Finally, we get $(\tau, \tau^{\sigma_1}_{r_1}, \cdots, \tau^{\sigma_j}_{r}, \cdots, \tau^{\sigma_{i-1}}_{r_{i-1}}, \tau^{\sigma_{i+1}}_{r_{i+1}}, \cdots, \tau^{\sigma_l}_{r_l})$, which mean that τ has a single incoming arrow coming from σ_i. For each $\tau^{\sigma_h}_q$ for $h \neq i, j$, we have $m_h - 1$ possibilities, and this yields, for the above expression, the value $S_{ij,l-2} \bar{\partial}^t \tau^{\sigma_j}_r$.

Therefore,

$$Q = \sum_{j \neq i} Q^i_j \Big(\sum_{p=2}^{m_j} \bar{\partial}^t \tau^{\sigma_j}_p \Big),$$

where,

$$\begin{aligned} \text{for } l = 1, \quad Q^i_j &= 0, \\ \text{for } l \geq 2, \quad Q^i_j &= \frac{1-\beta(m_i)}{|A|} \Big(\frac{1-\beta(l-1)}{l-1} + \frac{1-\beta(l-2)}{l-2} S_{ij,1} \\ &\quad + \cdots + \frac{1-\beta(2)}{2} S_{ij,l-3} + S_{ij,l-2} \Big). \end{aligned}$$

Observe that for $l = 2$, $Q^i_j = \frac{1-\beta(m_i)}{|A|} S_{ij,0} = \frac{1-\beta(m_i)}{|A|}$ since we set $S_{ij,0} = 1$. To see this, one also refers to (6.27), (6.31), (6.29) and (6.30).
Hence,

$$\bar{\partial}^t \tau = (1 - \sum_i P^{\sigma_i}) \bar{\partial}^{t-1} \tau + \sum_{i=1}^{l} \Big(\frac{1-\beta(m_i)}{m_i} - \sum_{j \neq i, j=1}^{l} Q^j_i \Big) \sum_{p=2}^{m_i} \bar{\partial}^{t-1} \tau^{\sigma_i}_p, \tag{6.32}$$

$$\begin{aligned} \bar{\partial}^t \tau^{\sigma_i}_p &= (1 - \frac{1-\beta(m_i)}{m_i}) \bar{\partial}^{t-1} \tau^{\sigma_i}_p - \frac{1-\beta(m_i)}{m_i} \sum_{\substack{r \neq p \\ r=2}}^{m_i} \bar{\partial}^{t-1} \tau^{\sigma_i}_r \\ &\quad + P^{\sigma_i} \bar{\partial}^{t-1} \tau + \sum_{\substack{j \neq i \\ j=1}}^{l} Q^i_j \sum_{r=2}^{m_j} \bar{\partial}^{t-1} \tau^{\sigma_j}_r \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} \bar{\partial}^t \varsigma^{\sigma_i} &= \bar{\partial}^{t-1} \varsigma^{\sigma_i} + c_i \frac{1-\beta(m_i)}{m_i} \sum_{r=2}^{m_i} \bar{\partial}^{t-1} \tau^{\sigma_i}_r - c_i P^{\sigma_i} \bar{\partial}^{t-1} \tau \\ &\quad - c_i \sum_{\substack{j \neq i \\ j=1}}^{l} Q^i_j \sum_{r=2}^{m_j} \bar{\partial}^{t-1} \tau^{\sigma_j}_r . \end{aligned} \tag{6.34}$$

Using the fact that

$$\bar{\partial}_i^t = \bar{\partial}_i^{t-1} = 0 \ \text{ for } \ i \leq k \quad \text{and} \quad \bar{\partial}_j^t = \bar{\partial}_j^{t-1} = 0 \ \text{ for } \ j \geq k+2, \tag{6.35}$$

to prove that $\bar{\partial}^t \circ \bar{\partial}^t = 0$, it is enough to show

$$\bar{\partial}_k^t \circ \bar{\partial}_{k+1}^t = 0 \tag{6.36}$$

and

$$\bar{\partial}_{k+1}^t \circ \bar{\partial}_{k+2}^t = 0. \tag{6.37}$$

The equation in (6.36) follows by applying $\bar{\partial}_k^t$ to (6.34), (6.32) and (6.33), and using (6.35).

We now show that (6.37) holds. This part of the proof will use the same techniques and ideas as in the proof Theorem 5.2.1.

Let $\omega^{(k+2)}$ be a critical cell.

a) If $\omega^{(k+2)}$ can reach $c\tau$ and/or $c_s^{\sigma_i}\tau_s^{\sigma_i}$, for c, $c_s^{\sigma_i} \in \mathbb{R}$, then there exist $\varsigma^{\sigma_1}, \cdots, \varsigma^{\sigma_l}$ (each is a linear combination of some critical cells) such that

$$\bar{\partial}^t \omega = b_\omega + \sum_{i=1}^{l} \varsigma^{\sigma_i} + c\tau + \sum_{i=1}^{l} \sum_{s \neq 1}^{m_i} c_s^{\sigma_i} \tau_s^{\sigma_i} = \bar{\partial}^{t-1}\omega,$$

where each ς^{σ_i} is such that

$$\bar{\partial}^{t-1}\varsigma^{\sigma_i} = b_{\varsigma^{\sigma_i}} - c\sigma_i - \sum_{s \neq 1}^{m_i} c_s^{\sigma_i}\sigma_i.$$

Since the fact that σ_i and ω are path-connected (and there are paths through τ and others through ς^{σ_i}) tells us that the induced orientation (induced from ω) from ς^{σ_i} onto σ_i has to be different from the one induced from τ. Similarly, the induced orientation (induced from ω) from ς^{σ_i} onto σ_i has to be different from the one induced from τ_s^{σ}. We then have:

$$0 = \bar{\partial}^{t-1} \circ \bar{\partial}^{t-1}\omega = \bar{\partial}^{t-1} b_\omega + \sum_{i=1}^{l} b_{\varsigma^{\sigma_i}} + c\, b_\tau + \sum_{i=1}^{l} \sum_{s \neq 1}^{m_i} c_s^{\sigma_i}\, b_{\tau_s^{\sigma_i}}. \tag{6.38}$$

Then using (just the first equalities) (6.32), (6.34) (with $c_i = c$), and the fact that $\bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega$, we get

$$\begin{aligned}\bar{\partial}^t \circ \bar{\partial}^t \omega &= \bar{\partial}^t b_\omega + \sum_{i=1}^{l} b_{\varsigma^{\sigma_i}} + c\, b_\tau + \sum_{i=1}^{l} \sum_{s \neq 1}^{m_i} c_s^{\sigma_i} b_{\tau_s^{\sigma_i}} \\ &= \bar{\partial}^{t-1} b_\omega + \sum_{i=1}^{l} b_{\varsigma^{\sigma_i}} + c\, b_\tau + \sum_{i=1}^{l} \sum_{s \neq 1}^{m_i} c_s^{\sigma_i} b_{\tau_s^{\sigma_i}}, \\ &= 0 \quad \text{from (6.38).}\end{aligned}$$

b) Observe that in the previous case, if $c_s^{\sigma_i} = 0$ for all i and for all s, the ς^{σ_i}'s can also be equal to τ^{σ_i}. In this case the idea is the same, but we would have: Each τ^{σ_i} is such that

$$\bar{\partial}^{t-1} \tau^{\sigma_i} = b_{\tau^{\sigma_i}} - \sigma_i$$

provided

$$\bar{\partial}^t \omega = b_\omega + c \sum_{i=1}^{l} \tau^{\sigma_i} + c_\tau = \bar{\partial}^{t-1} \omega.$$

Using the fact that $\bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega$, the result follows using (6.32), (6.33), and the fact that

$$\bar{\partial}^t \circ \bar{\partial}^t \omega = \bar{\partial}^t b_\omega + c \sum_i b_{\tau^{\sigma_i}} + c b_\tau = \bar{\partial}^{t-1} \circ \bar{\partial}^{t-1} \omega = 0.$$

c) If neither τ nor the τ^{σ_i}'s can be reached from ω, but instead only $c_i \sigma_i$ can, for $c_i \in \mathbb{R}$, there exist $\varsigma_1, \cdots, \varsigma_l$ and $\widetilde{\varsigma}_1, \cdots, \widetilde{\varsigma}_l$ (each is a linear combination of some critical cells) such that

$$\bar{\partial}^{t-1} \omega = b_\omega + \sum_i \varsigma_i + \sum_j \widetilde{\varsigma}_j = \bar{\partial}^t \omega,$$

where

$$\bar{\partial}^{t-1} \varsigma_i = b_{\varsigma_i} + c_i \sigma_i, \quad \text{and } \bar{\partial}^{t-1} \widetilde{\varsigma}_i = b_{\widetilde{\varsigma}_i} - c_i \sigma_i,$$

using the fact that the induced orientations (induced from ω) from ς_i onto σ_i has to be different from the one induced by $\widetilde{\varsigma}_i$. It follows immediately that

$$\bar{\partial}^t (\bar{\partial}^t \omega) = \bar{\partial}^{t-1} (\bar{\partial}^{t-1} \omega) = 0, \quad \text{since} \quad \bar{\partial}^t b_\omega = \bar{\partial}^{t-1} b_\omega.$$

□

Theorem 6.4.1, tells us that the boundary operator $\bar{\partial}$ satisfies $\bar{\partial}\circ\bar{\partial} = 0$, meaning that the homology groups are well defined using this boundary operator.

Let $\overline{m}_k := \dim \bar{C}_k$. We then have the following Morse-type inequalities.

Theorem 6.4.3 (Generalized Morse inequalities). *In the above settings, there exists $R(t)$, a polynomial in t with nonnegative integer coefficients such that*

$$\sum_{i=0}^{\dim \mathbb{K}} \overline{m}_i t^i = \sum_{i=0}^{\dim \mathbb{K}} b_i t^i + (1+t)R(t).$$

To prove this theorem, it is enough to show that $\bar{b}_i = b_i$, where $\bar{b}_i := \dim\left(\ker \bar{\partial}_i / \operatorname{im} \bar{\partial}_{i+1}\right)$, since the rest of the proof uses the same idea as in the proof of Proposition 5.2.3. Before proving it we give some examples.

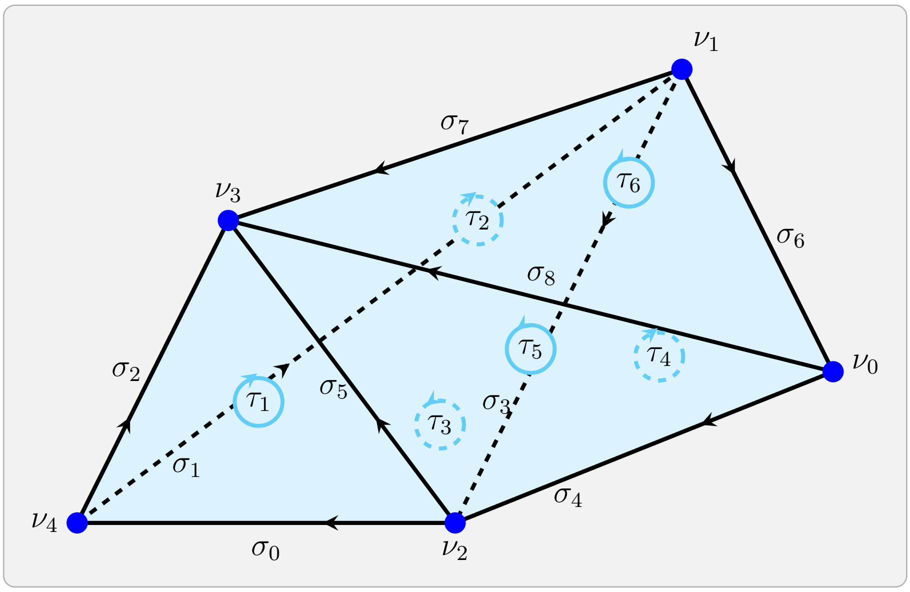


Figure 6.16: Initial orientation for Figure 6.17.

Example 6.4.2. Using the computations in Example 6.4.1, we get the following:
$\ker \bar{\partial}_0 = \langle \nu_2\,,\, \nu_3\,,\, \nu_4 \rangle$, $\quad \operatorname{im} \bar{\partial}_1 = \langle \nu_2 - \nu_3\,,\, \nu_2 - \beta(2)\nu_4 + \frac{1-\beta(2)}{2}(\nu_3 + \nu_3)) \rangle$;
$\ker \bar{\partial}_1 = \langle \sigma_3\,,\, \sigma_2 - \sigma_6\,,\, \sigma_2 + \sigma_5\,,\, \sigma_4 - \sigma_8\,,\, \sigma_7 + \sigma_8 - \sigma_6\,,\, \sigma_7 + \sigma_4 + \sigma_5 \rangle$;
$\operatorname{im} \bar{\partial}_2 = \langle \sigma_3\,,\ \sigma_2 + \sigma_5\,,\, \sigma_7 + \sigma_4 + \sigma_5\,,\, \bar{\partial}\tau_5\,,\, \bar{\partial}\tau_3\,,\, \bar{\partial}\tau_2 \rangle$;
$\ker \bar{\partial}_2 = \langle \tau_0 + \tau_1 + \tau_4 + \tau_2\,,\, \tau_3 + \tau_4 + \tau_5 + \tau_6 \rangle$, $\quad \operatorname{im} \bar{\partial}_3 = \langle \beta(2)(\tau_1 + \tau_2 + \tau_0 + \tau_4)\,,\, \bar{\partial}\omega_1^{(3)} \rangle$;
$\ker \bar{\partial}_3 = 0 = \operatorname{im} \bar{\partial}_4$.
This yields

$$\bar{b}_0 = 1 = b_0,\ \bar{b}_1 = 0 = b_1,\ \bar{b}_2 = 0 = b_2,\ \bar{b}_3 = 0 = b_2.$$

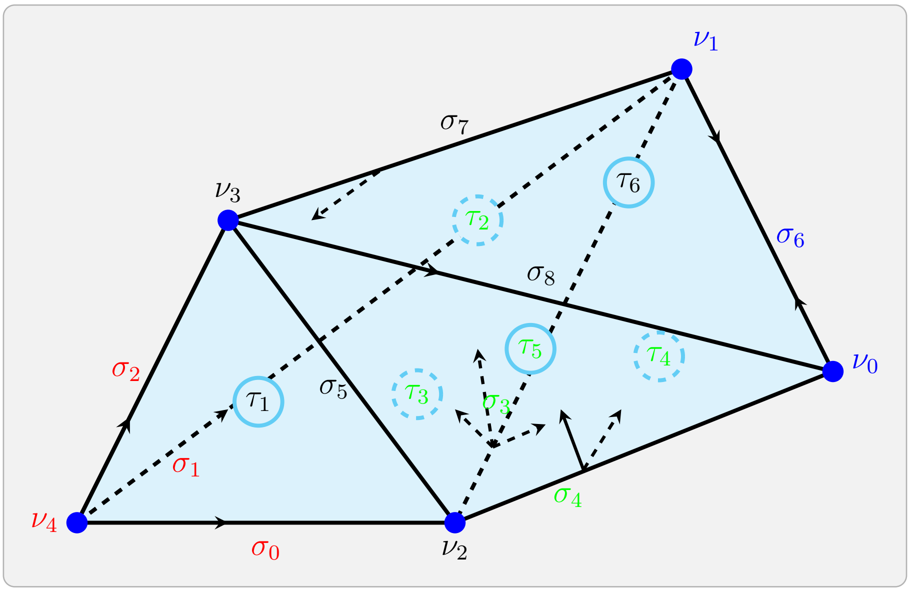


Figure 6.17: Another general example.

Also, we have

$$\overline{m}_0 = 3,\ \overline{m}_1 = 7,\ \overline{m}_2 = 7,\ \overline{m}_3 = 2,$$

and, we obtain

$$\sum_k \overline{m}_k t^k = 3+7t+7t^2+2t^3 = 1+(1+t)(2+5t+2t^2), \quad \text{that is} \quad R(t) = 2+5t+2t^2.$$

Example 6.4.3. Using Figure 6.17, with the initial orientation for the cells given by Figure 6.16, we get the following:
$\tau_1 = [\nu_2, \nu_4, \nu_3]$; $\tau_2 = [\nu_1, \nu_2, \nu_3]$; $\tau_3 = [\nu_1, \nu_4, \nu_2]$; $\tau_4 = [\nu_1, \nu_0, \nu_2]$;
$\tau_5 = [\nu_0, \nu_3, \nu_2]$; $\tau_6 = [\nu_1, \nu_3, \nu_0]$; $\sigma_0 = [\nu_2, \nu_4]$; $\sigma_1 = [\nu_4, \nu_1]$; $\sigma_2 = [\nu_4, \nu_3]$;
$\sigma_3 = [\nu_1, \nu_2]$; $\sigma_4 = [\nu_0, \nu_2]$; $\sigma_5 = [\nu_2, \nu_3]$; $\sigma_6 = [\nu_1, \nu_0]$; $\sigma_7 = [\nu_1, \nu_3]$;
$\sigma_8 = [\nu_0, \nu_3]$;
The edge σ_3 is abnormally upward noncritical with the cells τ_3, τ_2 and τ_4.
The edge σ_4 is abnormally upward noncritical with the cells τ_5 and τ_4.
The vertex ν_4 is abnormally upward noncritical with the edges σ_0, σ_1 and σ_2.
The cell τ_4 is abnormally downward noncritical with the edges σ_3 and σ_4.
The edge σ_6 is abnormally downward noncritical with the vertices ν_0 and ν_1.
We obtain the following set:

$$\bar{C}_o^{(0)} = \{\nu_2\},\ \bar{C}_o^{(1)} = \{\sigma_5\},\ \bar{C}_o^{(2)} = \{\tau_1\,,\ \tau_6\};$$

$$\bar{C}_{in}^{(1)} = \{\sigma_6\},\ \bar{C}_{in}^{(2)} = \{\tau_4\},\ \bar{C}_{ou}^{(0)} = \{\nu_4\},\ \bar{C}_{ou}^{(1)} = \{\sigma_3\,,\ \sigma_4\};$$

$$\bar{C}^{(0)}_{sin} = \{\nu_0, \nu_1\},\ \bar{C}^{(1)}_{sou} = \{\sigma_0, \sigma_1, \sigma_2\},\ \bar{C}^{(2)}_{sou} = \{\tau_2, \tau_3, \tau_4, \tau_5\}.$$

It then follows that

$$\begin{aligned} v^G(\sigma_4) &= \beta(2)\sigma_4 + \frac{1-\beta(2)}{2}\Big(-\sigma_5 + \frac{1}{3}\big(\beta(2)\sigma_4 + \frac{1-\beta(2)}{2}(\sigma_4 - \sigma_6 + \sigma_3) \\ &\quad + (-\sigma_6 - \sigma_5 - \sigma_1 + \sigma_2) + (-\sigma_6 - \sigma_1 - \sigma_0)\big)\Big), \end{aligned}$$

$$\begin{aligned} v^G(\sigma_3) &= \beta(3)\sigma_3 + \frac{1-\beta(3)}{3}\Big((-\sigma_5 - \sigma_1 + \sigma_2) + (-\sigma_1 - \sigma_0) \\ &\quad + \frac{1}{2}\big(\beta(2)\sigma_3 + \frac{1-\beta(2)}{2}(\sigma_3 + \sigma_6 + \sigma_4) + (\sigma_6 - \sigma_5)\big)\Big), \end{aligned}$$

$v^G(\nu_1) = \beta(2)\nu_1 + \frac{1-\beta(2)}{2}(\nu_1 + \nu_0),\ v^G(\nu_0) = \beta(2)\nu_0 + \frac{1-\beta(2)}{2}(\nu_1 + \nu_0),$

$v^G(\nu_4) = \beta(3)\nu_4 + \frac{1-\beta(3)}{3}\big(v^G(\nu_0) + v^G(\nu_1) + \nu_2\big);$

$\bar{\partial}(\tau_1) = \sigma_2 + \sigma_0 - \sigma_5, \quad \bar{\partial}(\tau_6) = -\sigma_1 - \sigma_6 + \sigma_2.$
We then have

$$\begin{aligned} \bar{\partial}(\tau_3) &= -\sigma_1 - \sigma_0 - v^G(\sigma_3) \\ &= -\beta(3)(\sigma_1 + \sigma_0 + \sigma_3) + \frac{1-\beta(3)}{3}\Bigg((-\sigma_0 + \sigma_5 - \sigma_2) \\ &\quad + \frac{1}{2}\Big(-\beta(2)(\sigma_0 + \sigma_1 + \sigma_3) - \frac{1-\beta(2)}{2}\big((\sigma_0 + \sigma_1 + \sigma_3) \\ &\quad + (\sigma_0 + \sigma_1 + \sigma_6 + \sigma_4)\big) + (\sigma_5 - \sigma_0 - \sigma_1 - \sigma_6)\Big)\Bigg), \end{aligned}$$

$$\begin{aligned} \bar{\partial}(\tau_2) &= -\sigma_2 + \sigma_1 + \sigma_5 + v^G(\sigma_3) \\ &= -\beta(3)(\sigma_1 + \sigma_5 - \sigma_2 + \sigma_3) + \frac{1-\beta(3)}{3}\Bigg((-\sigma_0 + \sigma_5 - \sigma_2) \\ &\quad + \frac{1}{2}\Big(\beta(2)(\sigma_5 + \sigma_1 - \sigma_2 + \sigma_3) + \frac{1-\beta(2)}{2}\big((\sigma_5 + \sigma_1 - \sigma_2 + \sigma_3) \\ &\quad + (\sigma_5 + \sigma_1 - \sigma_2 + \sigma_6 + \sigma_4)\big) + (\sigma_1 - \sigma_2 - \sigma_6)\Big)\Bigg), \end{aligned}$$

$\bar{\partial}(\tau_5) = -\sigma_5 - v^G(\sigma_4), \quad \bar{\partial}(\tau_4) = \sigma_6 + v^G(\sigma_4) - v^G(\sigma_3),$

$$\bar{\partial}\sigma_6 = v^G(\nu_0) - v^G(\nu_1) = \beta(2)(\nu_0 - \nu_1), \;\; \bar{\partial}\sigma_2 = v^G(\nu_0) - v^G(\nu_4),$$

$$\bar{\partial}\sigma_1 = v^G(\nu_1) - v^G(\nu_4), \quad \bar{\partial}\sigma_0 = v^G(\nu_4) - \nu_2; \quad \bar{\partial}\sigma_5 = v^G(\nu_0) - \nu_2,$$

$$\bar{\partial}\sigma_4 = \nu_2 - v^G(\nu_0); \quad \bar{\partial}\sigma_3 = \nu_2 - v^G(\nu_1).$$

From the above computations we get:
$\ker \bar{\partial}_0 = \langle \nu_0 , \nu_1 , \nu_2, \nu_4 \rangle, \quad \operatorname{im} \bar{\partial}_1 = \langle \beta(2)(\nu_0 - \nu_1) , v^G(\nu_4) - \nu_2 , v^G(\nu_0) - \nu_2 \rangle;$
$\ker \bar{\partial}_1 = \langle \sigma_0 - \sigma_5 + \sigma_2 , -\sigma_1 - \sigma_6 + \sigma_2 , \sigma_6 + \sigma_4 - \sigma_3 , \sigma_1 + \sigma_0 + \sigma_3 , \sigma_5 + \sigma_4 \rangle;$
$\operatorname{im} \bar{\partial}_2 = \langle -\sigma_1 - \sigma_6 + \sigma_2 , \sigma_2 + \sigma_0 - \sigma_5 , \bar{\partial}\tau_4 , \bar{\partial}\tau_2 , \bar{\partial}\tau_5 \rangle;$
$\ker \bar{\partial}_2 = \langle \tau_1 + \tau_3 + \tau_2 , \tau_2 + \tau_4 + \tau_5 + \tau_6 \rangle,$
This yields

$$\bar{b}_0 = 1 = b_0, \; \bar{b}_1 = 0 = b_1, \; \bar{b}_2 = 2 = b_2.$$

We also get

$$\overline{m}_0 = 4, \; \overline{m}_1 = 7, \; \overline{m}_2 = 6, \; \overline{m}_3 = 0,$$

and this gives

$$\sum_k \overline{m}_k t^k = 4 + 7t + 6t^2 = 1 + 2t^2 + (1+t)(3+4t), \quad \text{that is} \quad R(t) = 3 + 4t.$$

To prove Theorem 6.4.3, we assume for simplicity that the CW complex $\mathbb{K}$ has no noncritical cells that belong to Forman's framework. Collapsing the Forman-type noncritical cells preserves the homotopy type and hence the Betti numbers. Thus creating some noncritical cells in Forman's framework by adding single arrows in our framework to those cells having none, will not change the homotopy type of the CW complex.

Proof of Theorem 6.4.3. Let $\mathbb{K}$ be a CW complex with an arrow configuration satisfying the conditions given by Definition 6.1.2. Recall that

$$\bar{C}^{(k)} = \bar{C}_o^{(k)} \cup \bar{C}_{ou}^{(k)} \cup \bar{C}_{in}^{(k)} \cup \bar{C}_{sin}^{(k)} \cup \bar{C}_{sou}^{(k)}.$$

Let (C_*, ∂^c) be the cellular chain complex where, ∂^c is the cellular boundary operator. Using $(\bar{C}_*, \bar{\partial})$ as the chain complex obtained from $\bar{C}^{(k)}$, and the fact that there are no Forman-type noncritical cells, we get the equality $\bar{C}_k = C_k$ for all k, which already means that $m_k = \overline{m}_k$.

We know from the definition of v^G that, v^G is linear, which implies that, if $\partial^c\tau = 0$, then $0 = v^G(\sum_{\sigma<\tau} \sigma) = \sum_{\sigma<\tau} v^G(\sigma) = \bar{\partial}\tau$. Thus, $\bar{\partial}$ satisfies:

$$\text{if for } \; \varpi \in C_k, \; \partial^c(\varpi) = 0 \text{ then } \bar{\partial}(\varpi) = 0.$$

This tells us that $\ker\partial^c \subseteq \ker\bar{\partial}$.

Now we prove the opposite inclusion. We suppose that $\bar{\partial}\varpi = 0$ then we have to show that $\partial^c\varpi = 0$. We use the same induction idea we have been using to prove Theorem 6.4.1. This means that assuming that $\bar{\partial}^t\varpi = 0$, we need to show that $\bar{\partial}^{t-1}\varpi = 0$. It is enough to show this at dimension $k+1$. Let $\varsigma_q^{\sigma_i}$ be connected to $c_q^{\sigma_i}\sigma_i$, $c_q^{\sigma_i} \in \mathbb{R}$ for $q = 2, \cdots, s_i$ and for $i = 1, \cdots, l$ be as in the proof of Theorem 6.4.1. Suppose for α, α_h, $\alpha_p^{\sigma_i}$, $\alpha_q^{s_i} \in \mathbb{R}$,

$$\varpi = \alpha\tau + \sum_{i=1}^{l}\sum_{p=2}^{m_i}\alpha_p^{\sigma_i}\tau_p^{\sigma_i} + \sum_{i=1}^{l}\sum_{q=2}^{s_i}\alpha_q^{s_i}\varsigma_q^{\sigma_i} + \sum_h \alpha_h\varpi_h,$$

where the ϖ_h's are not connected in any way to the σ_i's. Then

$$\begin{aligned}
0 &= \bar{\partial}^t\varpi \qquad (6.39)\\
&= \alpha\,\bar{\partial}^t\tau + \sum_{i=1}^{l}\sum_{p=2}^{m_i}\alpha_p^{\sigma_i}\bar{\partial}^t\tau_p^{\sigma_i} + \sum_{i=1}^{l}\sum_{q=2}^{s_i}\alpha_q^{s_i}\bar{\partial}^t\varsigma_q^{s_i} + \sum_h \alpha_h\bar{\partial}^t\varpi_h\\
&= \Big(\big(1-\sum_i P^{\sigma_i}\big)\alpha + \sum_{i=1}^{l}P^{\sigma_i}\sum_{p=2}^{m_i}\alpha_p^{\sigma_i} - \sum_{i=1}^{l}P^{\sigma_i}\sum_{p=2}^{s_i}c_p^{\sigma_i}\alpha_p^{s_i}\Big)\bar{\partial}^{t-1}\tau\\
&\quad + \sum_{i=1}^{l}\sum_{p=2}^{m_i}\Big(\big(1-\frac{1-\beta(m_i)}{m_i}\big)\alpha_p^{\sigma_i} - \frac{1-\beta(m_i)}{m_i}\sum_{r\neq p, r=2}^{m_i}\alpha_r^{\sigma_i}\\
&\quad + \sum_{j\neq i, j=1}^{l}Q_i^j\sum_{r=2}^{m_j}\alpha_r^{\sigma_j} + \frac{1-\beta(m_i)}{m_i}\sum_{p=2}^{s_i}c_p^{\sigma_i}\alpha_p^{s_i} - \sum_{j\neq i,j=1}^{l}Q_i^j\sum_{p=2}^{m_j}c_p^{\sigma_j}\alpha_p^{s_j}\\
&\quad + \Big(\frac{1-\beta(m_i)}{m_i} - \sum_{j\neq i,j=1}^{l}Q_i^j\Big)\alpha\Big)\bar{\partial}^{t-1}\tau_p^{\sigma_i} + \sum_{i=1}^{l}\sum_{q=2}^{s_i}\alpha_q^{s_i}\bar{\partial}^{t-1}\varsigma_q^{s_i}\\
&\quad + \sum_h \alpha_h\bar{\partial}^{t-1}\varpi_h.
\end{aligned}$$

To conclude, we need to prove the following equalities:

$$\begin{aligned}
\alpha &= \Big(\big(1-\sum_i P^{\sigma_i}\big)\alpha + \sum_{i=1}^{l}P^{\sigma_i}\sum_{p=2}^{m_i}\alpha_p^{\sigma_i} - \sum_{i=1}^{l}P^{\sigma_i}\sum_{q=2}^{s_i}c_q^{\sigma_i}\alpha_q^{s_i}\Big) \qquad (6.40)\\
&= \alpha - \sum_{i=1}^{l}P^{\sigma_i}\Big(\alpha - \sum_{p=2}^{m_i}\alpha_p^{\sigma_i} + \sum_{q=2}^{s_i}c_q^{\sigma_i}\alpha_q^{s_i}\Big),
\end{aligned}$$

$$\begin{aligned}
\alpha_p^{\sigma_i} &= \Big((1-\frac{1-\beta(m_i)}{m_i})\alpha_p^{\sigma_i} - \frac{1-\beta(m_i)}{m_i}\sum_{r\neq p, r=2}^{m_i}\alpha_r^{\sigma_i} \\
&\quad + \sum_{j\neq i, j=1}^{l} Q_i^j \sum_{r=2}^{m_j}\alpha_r^{\sigma_j} + \frac{1-\beta(m_i)}{m_i}\sum_{q=2}^{s_i} c_q^{\sigma_i}\alpha_q^{s_i} - \sum_{j\neq i, j=1}^{l} Q_i^j \sum_{q=2}^{m_j} c_q^{\sigma_j}\alpha_q^{s_j}\Big) \\
&\quad + \Big(\frac{1-\beta(m_i)}{m_i} - \sum_{j\neq i, j=1}^{l} Q_i^j\Big)\alpha \\
&= \alpha_p^{\sigma_i} + \frac{1-\beta(m_i)}{m_i}\Big(\alpha - \sum_{r=2}^{m_i}\alpha_r^{\sigma_i} + \sum_{q=2}^{s_i} c_q^{\sigma_i}\alpha_q^{s_i}\Big) \\
&\quad - \sum_{j\neq i, j=1}^{l} Q_i^j\Big(\alpha - \sum_{r=2}^{m_j}\alpha_r^{\sigma_j} + \sum_{q=2}^{s_j} c_q^{\sigma_j}\alpha_q^{s_j}\Big).
\end{aligned} \tag{6.41}$$

Now, $\bar{\partial}^t\varpi = 0$ means in particular that the coefficient of σ_i in $\bar{\partial}^t\varpi$ has to be zero. The expression of ϖ above has four summands. We apply the boundary operator $\bar{\partial}^t$ to each summand and evaluate the coefficient of σ_i in each resulting expression.

- Using (6.32), the coefficient of σ_i in $\bar{\partial}^t\tau$ is given by:

$$\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau = (1-\sum_i P^{\sigma_i}) - (m_i-1)\Big(\frac{1-\beta(m_i)}{m_i} - \sum_{j\neq i} Q_i^j\Big).$$

- Using (6.33), we get that the coefficient of σ_i

in $\bar{\partial}^t\tau_p^{\sigma_i}$ is $\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau_p^{\sigma_i} = -\big(1-\frac{1-\beta(m_i)}{m_i}\big) + (m_i-2)\frac{1-\beta(m_i)}{m_i} + P^{\sigma_i}$;

in $\bar{\partial}^t\tau_q^{\sigma_j}$ is given by $\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\tau_q^{\sigma_j} = P^{\sigma_j} - (m_i-1)Q_i^j$.

- Using (6.34), the coefficient of σ_i

in $\bar{\partial}^t\varsigma_q^{\sigma_i}$ is given by $\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\varsigma_q^{\sigma_i} = c_q^{\sigma_i}\Big(1-(m_i-1)\frac{1-\beta(m_i)}{m_i} - P^{\sigma_i}\Big)$;

in $\bar{\partial}^t\varsigma_p^{\sigma_j}$ is given by $\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\varsigma_p^{\sigma_j} = -\,c_p^{\sigma_j}\Big(P^{\sigma_j} - (m_i-1)Q_i^j\Big)$.

- The cell σ_i does not appear in the last summand of $\bar{\partial}^t\varpi$, that is

$$\mathrm{Proj}_{\sigma_i}\bar{\partial}^t\varpi_h = 0.$$

Thus, the coefficient of σ_i in $\bar{\partial}^t\varpi$ equated to zero yields: for $i = 1, \cdots, l$,

$$\begin{aligned}
0 &= \operatorname{Proj}_{\sigma_i} \bar{\partial}^t \varpi \qquad (6.42)\\
&= \Big((1 - \sum_i P^{\sigma_i}) - (m_i - 1)\Big(\frac{1-\beta(m_i)}{m_i} - \sum_{j\neq i} Q_i^j\Big)\Big)\alpha \\
&\quad + \Big(- (1 - \frac{1-\beta(m_i)}{m_i}) + (m_i - 2)\frac{1-\beta(m_i)}{m_i} + P^{\sigma_i}\Big) \sum_{p=2}^{m_i} \alpha_p^{\sigma_i} \\
&\quad + \sum_{j\neq i, j=1}^{l} \big(P^{\sigma_j} - (m_i - 1)Q_i^j\big) \sum_{q=2}^{m_j} \alpha_q^{\sigma_j} \\
&\quad + \sum_{p=2}^{s_i} \alpha_p^{s_i} c_p^{\sigma_i} \Big(1 - (m_i - 1)\frac{1-\beta(m_i)}{m_i} - P^{\sigma_i}\Big) \\
&\quad + \sum_{j\neq i, j=1}^{l} \sum_{q=2}^{s_j} \alpha_q^{s_j} c_q^{\sigma_j} \big(- P^{\sigma_j} + (m_i - 1) Q_i^j\big) \\
&= \Big(1 - P^{\sigma_i} - (m_i - 1)\frac{1-\beta(m_i)}{m_i}\Big)\Big(\alpha - \sum_{p=2}^{m_i} \alpha_p^{\sigma_i} + \sum_{p=2}^{s_i} c_p^{\sigma_i}\alpha_p^{s_i}\Big) \\
&\quad - \sum_{j\neq i, j=1}^{l} \Big(P^{\sigma_j} - (m_i - 1)Q_i^j\Big)\Big(\alpha - \sum_{p=2}^{m_j} \alpha_p^{\sigma_j} + \sum_{p=2}^{s_j} c_p^{\sigma_j}\alpha_p^{s_j}\Big).
\end{aligned}$$

Now we proceed with the analysis of the expression in (6.42) by proving the following lemmas.

Let $\mathbb{N}^* := \{1, 2, \cdots\}$,

$$K_i^j := \Big(P^{\sigma_j} - (m_i - 1)Q_i^j\Big) > 0 \quad \text{and} \quad K^i := \Big(1 - P^{\sigma_i} - (m_i - 1)\frac{1-\beta(m_i)}{m_i}\Big).$$

Lemma 6.4.4. *For $l \in \mathbb{N}^* \setminus \{1\}$ and $m_i \in \mathbb{N}^*$ for $i = 1, \cdots, l$,*

$$K_i^j = P^{\sigma_j} - (m_i - 1)Q_i^j > 0, \quad \textit{for} \quad j \neq i, \ i, j \in \{1, \cdots, l\}.$$

The condition on the arrow configuration excluding the Forman-type noncritical cells ensures that for the collection of natural numbers $l, m_1, \cdots, m_l$, at least one of them is greater than 1.

Proof. Since $l \geq 2$ it means that each m_i can also be equal to 1 which translates to the merging case in Section 6.3. In this case $K_i^j = P^{\sigma_j} = \frac{1-\beta(l)}{l} > 0$.

In general, the inequality is shown as follows.

Using the fact that $(m_i-1)S_{ij,h-1} = S_{j,h} - S_{ij,h}$, for $h = 1, \cdots, l-1$ with $S_{ij,0} = 1$, $S_{ij,l-1} = 0$ and for $l = 2$, $S_{ij,1} = 0$, we get:

$$\begin{aligned}
(m_i-1)Q_i^j &= \frac{1-\beta(m_j)}{|A|}\Big(\frac{1-\beta(l-1)}{l-1}\big(S_{j,1}-S_{ij,1}\big) + \frac{1-\beta(l-2)}{l-2}\big(S_{j,2}-S_{ij,2}\big) \\
&\quad + \cdots + \frac{1-\beta(2)}{2}\big(S_{j,l-2}-S_{ij,l-2}\big) + S_{j,l-1}\Big) \\
&= \frac{1-\beta(m_j)}{|A|}\Big(\frac{1-\beta(l-1)}{l-1}S_{j,1} + \frac{1-\beta(l-2)}{l-2}S_{j,2} \\
&\quad + \cdots + \frac{1-\beta(2)}{2}S_{j,l-2} + S_{j,l-1}\Big) \\
&\quad - \frac{1-\beta(m_j)}{|A|}\Big(\frac{1-\beta(l-1)}{l-1}S_{ij,1} + \frac{1-\beta(l-2)}{l-2}S_{ij,2} \\
&\quad + \cdots + \frac{1-\beta(2)}{2}S_{ij,l-2}\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
K_i^j &= P^{\sigma_j} - (m_i-1)Q_i^j \\
&= \frac{1-\beta(m_j)}{|A|}\Big(\frac{1-\beta(l)}{l} + \frac{1-\beta(l-1)}{l-1}S_{ij,1} + \frac{1-\beta(l-2)}{l-2}S_{ij,2} \\
&\quad + \cdots + \frac{1-\beta(2)}{2}S_{ij,l-2}\Big) > 0,
\end{aligned}$$

since, for $s \in \mathbb{N}^*$, $\beta(s)$ is such that $\beta(s) \in (0,1)$ if $s > 1$ and $\beta(1) = 0$. Also, the $S_{ij,h}$ are all nonnegative for $m_i \in \mathbb{N}^*$. □

Lemma 6.4.5. *For $l \in \mathbb{N}^*$ and any collection of natural numbers $l, m_1, \cdots, m_l$ such that at least one of them is greater than 1,*

$$K^i - \sum_{j\neq i} K_i^j > 0, \quad \textit{for } i,j \in \{1,\cdots,l\}.$$

Proof. Recall that

$$K_i^j = \Big(P^{\sigma_j} - (m_i-1)Q_i^j\Big) > 0 \quad \text{and} \quad K^i = \Big(1 - P^{\sigma_i} - (m_i-1)\frac{1-\beta(m_i)}{m_i}\Big).$$

For $l = 1$, $K_i^j = 0$ and $K^i = K^1 = \beta(m_1) > 0$, since if $l = 1$ then $m_1 \geq 2$. This translates to the forking case in Section 6.2.
In general, for $l \geq 2$,

$$K_i^j = P^{\sigma_j} - (m_i-1)Q_i^j$$

$$\begin{aligned}
&= \frac{1-\beta(m_j)}{|A|}\Big(\frac{1-\beta(l)}{l} + \frac{1-\beta(l-1)}{l-1}S_{ij,1} + \frac{1-\beta(l-2)}{l-2}S_{ij,2} \\
&\quad + \cdots + \frac{1-\beta(2)}{2}S_{ij,l-2}\Big) \\
&< \frac{1}{|A|}\Big(\frac{1}{l} + \frac{1}{l-1}S_{ij,1} + \frac{1}{l-2}S_{ij,2} + \cdots + \frac{1}{2}S_{ij,l-2}\Big),
\end{aligned}$$

since for $s \in \mathbb{N}^*$, $\beta(s) \in [0,1)$ is such that $\beta(s) = 0$ if $s = 1$. This then implies that

$$\begin{aligned}
\sum_{j\neq i} K_i^j &< \frac{1}{|A|}\Big(\sum_{j\neq i}\frac{1}{l} + \frac{1}{l-1}\sum_{j\neq i} S_{ij,1} + \frac{1}{l-2}\sum_{j\neq i} S_{ij,2} \\
&\quad + \cdots + \frac{1}{2}\sum_{j\neq i} S_{ij,l-2}\Big) \\
&= \frac{1}{|A|}\Big(\frac{l-1}{l} + \frac{l-2}{l-1}S_{i,1} + \frac{l-3}{l-2}S_{i,2} + \cdots + \frac{1}{2}S_{i,l-2}\Big),
\end{aligned} \tag{6.43}$$

since $\sum_{j\neq i} S_{ij,k} = (l-k-1)S_{i,k}$, for $k = 1, \cdots, l-2$.

Also,

$$\begin{aligned}
P^{\sigma_i} &= \frac{1-\beta(m_i)}{|A|}\Big(\frac{1-\beta(l)}{l} + \frac{1-\beta(l-1)}{l-1}S_{i,1} \\
&\quad + \cdots + \frac{1-\beta(2)}{2}S_{i,l-2} + S_{i,l-1}\Big) \\
&< \frac{1}{|A|}\Big(\frac{1}{l} + \frac{1}{l-1}S_{i,1} + \cdots + \frac{1}{2}S_{i,l-2} + S_{i,l-1}\Big).
\end{aligned} \tag{6.44}$$

We also know that

$$\frac{m_i-1}{m_i}(1-\beta(m_i)) \leq \frac{m_i-1}{m_i},$$

since for $m_i \in \mathbb{N}^*$, $\beta(m_i) = 0$ if $m_i = 1$ and $\beta(m_i) \in (0,1)$ if $m_i > 1$.

Hence, using (6.43) and (6.44), we have

$$\begin{aligned}
K^i - \sum_{j\neq i} K_i^j &> 1 - \frac{m_i-1}{m_i} \\
&\quad - \frac{1}{|A|}\Big(\frac{1}{l} + \frac{1}{l-1}S_{i,1} + \cdots + \frac{1}{2}S_{i,l-2} + S_{i,l-1}\Big) \\
&\quad - \frac{1}{|A|}\Big(\frac{l-1}{l} + \frac{l-2}{l-1}S_{i,1} + \frac{l-3}{l-2}S_{i,2} + \cdots + \frac{1}{2}S_{i,l-2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m_i} - \frac{1}{|A|}\Big(1 + S_{i,1} + S_{i,2} + \cdots + S_{i,l-2} + S_{i,l-1}\Big) \\
&= 0
\end{aligned}
$$

since $1 + \sum_{k=1}^{l-1} S_{i,k} = \frac{|A|}{m_i}$ from Lemma 6.4.2.

Therefore, $K^i - \sum_{j\neq i} K_i^j > 0$. □

Lemma 6.4.6. *For $l \geq 1$ and any collection of natural numbers $(l, m_1, \cdots, m_l)$ such that at least one of them is greater than 1,*

$$
K^i = 1 - P^{\sigma_i} - (m_i - 1)\frac{1-\beta(m_i)}{m_i} > 0, \quad \text{for } i \in \{1, \cdots, l\}.
$$

Proof. From Lemma 6.4.5, we get $K^i > \sum_{j\neq i} K_i^j$. We also know from Lemma 6.4.4 that each $K_i^j > 0$ which implies that $\sum_{j\neq i} K_i^j > 0$. Hence $K^i > 0$. □

Now, back to the expression in (6.42), if we set $X_i := \alpha - \sum_{p=2}^{m_i} \alpha_p^{\sigma_i} + \sum_{p=2}^{s_i} c_p^{\sigma_i}\alpha_p^{s_i}$, then (6.42) becomes

$$
\begin{cases}
K^1 X_1 - K_1^2 X_2 - K_1^3 X_3 - \cdots - K_1^l X_l = 0 \\
-K_2^1 X_1 + K^2 X_2 - K_2^3 X_3 - \cdots - K_2^l X_l = 0 \\
-K_3^1 X_1 - K_3^2 X_2 + K^3 X_3 - \cdots - K_3^l X_l = 0 \\
\vdots \\
-K_l^1 X_1 - K_l^2 X_2 - K_l^3 X_3 - \cdots + K^l X_l = 0.
\end{cases}
$$

Using as variables the X_i's for $i = 1, \cdots, l$, the corresponding matrix of coefficients is given by

$$
M = \begin{pmatrix}
K^1 & -K_1^2 & \cdots & -K_1^l \\
-K_2^1 & K^2 & \cdots & -K_2^l \\
\vdots & \vdots & \ddots & \vdots \\
-K_l^1 & -K_l^2 & \cdots & K^l
\end{pmatrix}
$$

Showing that this system admits no non-zero solutions is equivalent to showing that the corresponding matrix M is nonsingular. To show this, we show that M is strictly diagonally dominant, that is,

$$|K^i| > \sum_{j\neq i} |K_i^j|,$$

since any strictly diagonally dominant matrix is nonsingular, see [30, P. 392]. This result is the well-known Levy-Desplanques Theorem stated below.

The fact that $|K^i| > \sum_{j\neq i} |K_i^j|$ follows from Lemma 6.4.4, Lemma 6.4.5 and Lemma 6.4.6. Hence the matrix M is strictly diagonally dominant. This means that

$$\alpha - \sum_{p=2}^{m_i} \alpha_p^{\sigma_i} + \sum_{p=2}^{s_i} c_p^{\sigma_i} \alpha_p^{s_i} =: X_i = 0, \quad \text{for} \quad i = 1, \cdots, l.$$

Substituting this in the RHS of (6.40) and (6.41) we get that the RHS of each equation is equal to the LHS, which in turn implies that the RHS of (6.39) is equal to $\bar{\partial}^{t-1}\varpi$, that is $0 = \bar{\partial}^t \varpi = \bar{\partial}^{t-1}\varpi$.

Observe that fact that the coefficients K^i and K_i^j are positive for all $i = 1, \cdots, l$ and for all $j \neq i$ is independent of the choice of orientation. Indeed, if the induced orientation of a cell σ_j is changed for the cell $\tau_q^{\sigma_j}$, then the coefficient $\alpha_q^{\sigma_j}$ will appear in (6.42), with the opposite sign. That is X_j will become $X_j := \alpha - \sum_{p\neq q, p=2}^{m_j} \alpha_p^{\sigma_j} + \alpha_q^{\sigma_j} + \sum_{p=2}^{s_j} c_p^{\sigma_i} \alpha_p^{s_j}$. In general changing orientations will only lead to a given coefficient M_{ij} being replaced by $-M_{ij}$.

For completeness, we now state and prove the Levy-Desplanques Theorem.

Lemma 6.4.7 (Levy-Desplanques Theorem). *Let A be an $n \times n$ strictly diagonally dominant matrix, that is, $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, for $i = 1, \cdots, n$, then A is nonsingular.*

Proof: If we suppose that A is singular, then there is a non-zero vector $x = (x_1, x_2, \cdots, x_n)$ s.t. $Ax = 0$. Among all the x_i's, pick x_j such that $|x_j| \geq |x_i|$ for all i. It implies that $x_j \neq 0$. Then looking at row j of the matrix A multiplied by x, we get:

$$a_{j1}\, x_1 + a_{j2}\, x_2 + \cdots + a_{jj}\, x_j + \cdots + a_{jn}\, x_n = 0,$$

which implies that

$$|a_{jj}\, x_j| = |\sum_{i\neq j} a_{ji}\, x_i| \leq \sum_{i\neq j} |a_{ji}||x_i| \leq |x_j| \sum_{i\neq j} |a_{ji}|,$$

which also implies that $|a_{jj}| \leq \sum_{i\neq j} |a_{ji}|$, but this contradicts the fact that A is strictly diagonally dominant.

$\square$

To summarize, we started with an arrow configuration more general than Forman's on a finite CW complex. We used as critical cells those that belong to either one of the following: the cells with no incoming and outgoing arrow; the abnormally downward noncritical cells; the cells having an outgoing arrow pointing to an abnormally downward noncritical cell; the abnormally upward noncritical cells; the cells having an incoming arrow from an abnormally upward noncritical cell. We then defined a boundary operator whose extracted Betti numbers coincide with the topological ones. Also, an analogue of the Morse inequalities hold.

We now move to the last section of this chapter which shows how we can retrieve the Poincaré polynomial of our CW complex using some systematically defined isolated invariant sets.

6.5 Conley theory

From the arrow configuration satisfying the same conditions as before, see Definition 6.1.2, we obtain the Poincaré polynomial of the CW complex from those of the well-defined isolated invariant sets, using Conley theory analysis.

Definition 6.5.1. *From the arrow configuration, we construct collections C_i consisting of:*

- *the singletons consisting of the cells without arrows,*
- *the collections $\{\tau\} \cup ADn(\tau)$, for $n_{in}(\tau) > 1$ and*
- *the collections $\{\sigma\} \cup AUn(\sigma)$, for $n_{ou}(\sigma) > 1$.*

Definition 6.5.2. *From the collections C_i given by Definition 6.5.1, we get disjoint collections $\widetilde{C}_i$ in the following way:*

- *if $C_i \cap C_j \neq \emptyset$, set $\widetilde{C}_i := C_i \cup C_j$ and $\widetilde{C}_j = \emptyset$,*
- *if $C_i \cap \big(\cup_{j\neq i} C_j\big) = \emptyset$, then $\widetilde{C}_i = C_i$.*

We repeat the procedure above until the collections $\widetilde{C}_i$ satisfy:

$$\widetilde{C}_i \cap \big(\cup_{j\neq i} \widetilde{C}_j\big) = \emptyset, \qquad \text{for all } i.$$

Remark 6.5.1. In Definition 6.5.2 above, since there is a symmetry between i and j, geometrically, we merge collections $\{\sigma\} \cup AUn(\sigma)$ for $n_{ou}(\sigma) > 1$ and $\{\tau\} \cup ADn(\tau)$ for $n_{in}(\tau) > 1$, whenever $\sigma \in ADn(\tau)$ (equivalently $\tau \in AUn(\sigma)$).

The resulting collections $\widetilde{C}_i$ are such that there is no path (following the arrows) that moves from a cell in $\widetilde{C}_i$ to another cell outside of $\widetilde{C}_i$, meaning that each of them is invariant. Since they constitute the building block for computing the Poincaré polynomial of the CW complex, we consider them to be isolated. Hence each collection $\widetilde{C}_i$ is an isolated invariant set. We then have the following definition.

Definition 6.5.3. *(i) the isolated invariant sets are given by the collections $I_i = \widetilde{C}_i$;*

(ii) the isolating neighborhood $N(I)$ of I is given by $N(I) = \cup_{\sigma \in I} \bar{\sigma}$;

(iii) the exit set $E(I)$ for each such I is given by $E(I) = N(I) \setminus I$.

To proceed we show that we can find a discrete Morse-Bott function f such that the extracted vector field of f is exactly our arrow configuration.

Proposition 6.5.1. *Suppose we have a CW complex $\mathbb{K}$ together with the arrow configuration given by Definition 6.1.2. Then there exists a discrete Morse-Bott function whose extracted vector field coincides with this arrow configuration.*

Proof. We already know that it is always possible, from Lemma 6.1.1, to find a discrete function f such that the extracted vector filed of f yields the arrow configuration under consideration. We get a discrete Morse-Bott function f, see Definition 5.3.2, by requiring that for every isolated invariant set I:

- for every $\sigma,\ \tau \in I,\ \ f(\sigma) = f(\tau)$,
- for every cell $\sigma \in I$,

  (i) $f(\sigma) > \max\{f(\nu), \nu < \sigma, \nu \notin I\}$,
  (ii) $f(\sigma) < \min\{f(\tau), \tau > \sigma, \tau \notin I\}$.

- The remaining cells in the CW complex are those that belong to noncritical pairs, and we define f as:

  (iii) $f(\sigma) < f(\nu)$, for $\nu < \sigma$ s.t. $n_{in}(\sigma) = 1,\ n_{ou}(\nu) = 1$ and $\nu \to \sigma$,
  (iv) $f(\sigma) > f(\tau)$, for $\tau > \sigma$ s.t. $n_{ou}(\sigma) = 1,\ n_{in}(\tau) = 1$ and $\sigma \to \tau$.

One checks easily that this function f is discrete Morse-Bott and that each isolated invariant set is exactly a reduced collection (that is not a noncritical pair), see Definition 5.3.5. □

Before stating the main result we prove some facts below.

Lemma 6.5.2. *The set $N(I) \setminus I$ is a subcomplex.*

Proof. Although this follows from Theorem 5.4.2, we provide a proof that does not use the discrete Morse-Bott function, but only the arrows.

If I consists of a single cell the result follows immediately.

If $\sigma \in N(I) \setminus I$, then σ is the face of an element in I but it is not in I. Since $N(I)$ is a subcomplex, to conclude we are only left with showing that any $\nu < \sigma$ is not in I. Since there are no closed orbits in each such collection, in particular ν should be a regular face of σ which in turn is a regular face of some $\tau^* \in I$. If on the contrary $\nu \in I$ then, either

(i) there is a $\rho > \nu$, $\rho \in I$ s.t. there is an arrow $\nu \to \rho$, or

(ii) there is a $\upsilon^* < \nu$, $\upsilon^* \in I$ s.t. there is an arrow $\upsilon^* \to \nu$.

Note that from the definition of the I_i's we get $\min_{\sigma,\tau\in I} |\dim\sigma - \dim\tau| = 0$ and $\max_{\sigma,\tau\in I} |\dim\sigma - \dim\tau| = 1$. Indeed from the construction of I which is a union of intersecting collections of the form $\{\sigma, AUn(\sigma)\}$ and $\{\tau, ADn(\tau)\}$, where if $\dim\sigma = k$ (resp. $\dim\tau = k$), then $\dim\widetilde{\tau} = k+1$ for $\widetilde{\tau} \in AUn(\sigma)$ (resp. $\dim\widetilde{\sigma} = k-1$ for $\widetilde{\sigma} \in ADn(\tau)$). So they intersect in particular if $\tau \in AUn(\sigma)$ (resp. $\sigma \in ADn(\tau)$). So that the minimum in dimension difference will be zero while the maximum will be 1. From this fact we get that neither (i) nor (ii) can happen since $\nu < \sigma < \tau^* \in I$ yields $\dim\tau^* - \dim\nu = 2$ and $\upsilon^* < \nu < \sigma < \tau^*$ yields $\dim\tau^* - \dim\upsilon^* = 3$, and this is a contradiction. Hence $\nu \notin I$. □

Lemma 6.5.2 tells us that the boundary operator ∂^I, defined as in Definition 5.3.7, is a well-defined relative boundary operator of the pair $(N(I), E(I))$.

Recall that $\bar{m}_k := \dim\bar{C}_k$, and let

$$n_k^I := \sharp\{\sigma^{(k)} \in I\},$$

then we have the following.

Lemma 6.5.3. $\sum_k \overline{m}_k t^k = \sum_i \sum_k n_k^{I_i} t^k$.

Proof. It follows easily from the definitions of $\overline{m}_k$ and the I_i's, since the I_i's are disjoint and $\cup_i I_i^{(k)} = \bar{C}_k$, where $I^{(k)}$ denotes the set of k-cells of I. □

Let us recall that for an isolated invariant set I,

$$P_t(I) := \sum_{k=0}^{\dim\mathbb{K}} \dim\left(\ker\partial_k^I / \operatorname{im}\partial_{k+1}^I\right) t^k,$$

where, $\partial_k^{I_i}$ is given by Definition 5.3.7.

Lemma 6.5.4. *In the settings above, for each i there exist $r_i(t)$, a polynomial in t with nonnegative integer coefficients, such that*

$$\sum_k n_k^{I_i} t^k = P_t(I_i) + (1+t) r_i(t).$$

Proof. From Proposition 5.3.6, each $r_i(t) = \sum_{k=1}(n_k^{I_i} - \dim\ker \partial_k^{I_i})t^{k-1}$, where $\partial_k^{I_i}$ is also the relative boundary operator of $(N(I_i), E(I_i))$ since both $N(I_i)$ and $E(I_i)$ are subcomplexes. $\square$

We now give the main result in this section which states that we can retrieve the Poincaré polynomial of the CW complex from those of the isolated invariant sets, or from those of the index pairs of the isolated invariant sets.

Theorem 6.5.5. *Let I_i be the isolated invariant sets obtained as in Definition 6.5.2 using the arrow configuration given by Definition 6.1.2. Then there exists $\bar{R}(t)$, a polynomial in t with nonnegative integer coefficients, such that*

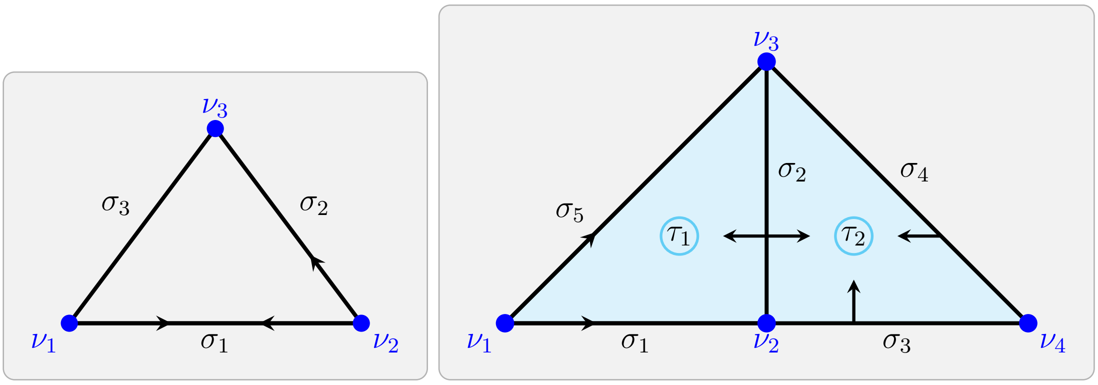
$$\sum_i P_t(N(I_i), E(I_i)) = \sum_i P_t(I_i) = P_t(\mathbb{K}) + (1+t)\bar{R}(t).$$

Proof. The first equality follows from the fact that ∂_k^I is the relative boundary operator of the pair $(N(I), E(I))$. Now, using Proposition 6.5.1, the fact that $\sum_i P_t(I_i) = P_t(\mathbb{K}) + (1+t)\bar{R}(t)$ follows from Theorem 5.3.7, and the fact that $\sum_i P_t(N(I_i), E(I_i)) = P_t(\mathbb{K}) + (1+t)\bar{R}(t)$ follows from Theorem 5.4.2. $\square$

We now give some examples to illustrate the result in Theorem 6.5.5. Since in this part there is no need for orientation, we denote by $(\nu_1, \cdots, \nu_k)$ the non-oriented simplex determined by the vertices ν_i for $i = 1, \cdots, k$.

Example 6.5.1. In Figure 6.18a,
$\sigma_1 = (\nu_1, \nu_2)$, $\sigma_2 = (\nu_2, \nu_3)$, $\sigma_3 = (\nu_1, \nu_3)$. We get collections
$C_1 = \{\sigma_1, \nu_1, \nu_2\}$, $C_2 = \{\nu_2, \sigma_1, \sigma_2\}$, $C_3 = \{\sigma_3\}$, $C_4 = \{\nu_3\}$.
The isolated invariant sets are:
$I_1 = C_1 \cup C_2 = \{\nu_1, \nu_2, \sigma_1, \sigma_2\}$, $I_2 = \{\sigma_3\}$, $I_3 = \{\nu_3\}$.
$P_t(I_1) = 0$, $P_t(I_2) = t$, and $P_t(I_3) = 1$. We know that $P_t(\mathbb{K}) = 1 + t$, hence $\bar{R}(t) \equiv 0$.

Example 6.5.2. In Figure 6.18b,
$\tau_1 = (\nu_1, \nu_2, \nu_3)$, $\tau_2 = (\nu_2, \nu_3, \nu_4)$, $\sigma_1 = (\nu_1, \nu_2)$, $\sigma_2 = (\nu_2, \nu_3)$, $\sigma_3 = (\nu_2, \nu_4)$,
$\sigma_4 = (\nu_3, \nu_4)$ and $\sigma_5 = (\nu_3, \nu_1)$.
We get the collections:
$C_1 = \{\nu_2\}$, $C_2 = \{\nu_3\}$, $C_3 = \{\nu_4\}$, $C_4 = \{\nu_1, \sigma_1, \sigma_5\}$, $C_5 = \{\sigma_2, \tau_1, \tau_2\}$,
$C_6 = \{\tau_2, \sigma_2, \sigma_3, \sigma_4\}$.



(a) (b)

Figure 6.18: Examples of vector fields satisfying Definition 6.1.2.

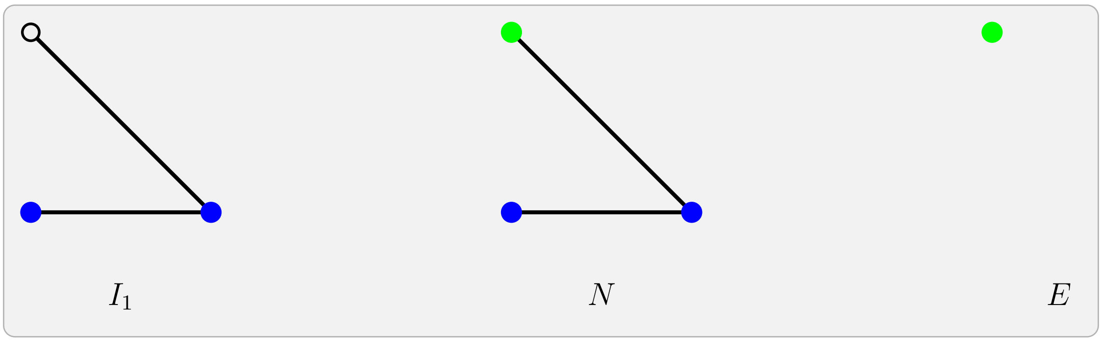


Figure 6.19: Corresponding isolating neighborhood and exit set for I_1 using Figure 6.18a and Example 6.5.1.

The isolated invariant sets are:
$I_1 = \{\nu_2\},\ I_2 = \{\nu_3\},\ I_3 = \{\nu_4\},\ I_4 = \{\nu_1, \sigma_1, \sigma_5\},\ I_5 = \{\tau_1, \tau_2, \sigma_2, \sigma_3, \sigma_4\}$.
For $i = 1, 2, 3,\ P_t(I_i) = 1,\ P_t(I_4) = t,\ P_t(I_5) = t$.
Thus,
$\sum_{i=1}^{4} P_t(I_i) = 3 + 2t = 1 + 2(1 + t) = P_t(\mathbb{K}) + (1 + t)R(t)$, that is $\bar{R}(t) \equiv 2$.

Example 6.5.3. In Figure 6.22,
$\tau_1 = (\nu_1, \nu_2, \nu_4)$, $\tau_2 = (\nu_2, \nu_3, \nu_4)$, $\tau_3 = (\nu_1, \nu_3, \nu_4)$, $\tau_4 = (\nu_1, \nu_2, \nu_3)$. $\sigma_1 = (\nu_1, \nu_4)$, $\sigma_2 = (\nu_1, \nu_2)$, $\sigma_3 = (\nu_2, \nu_4)$, $\sigma_4 = (\nu_2, \nu_3)$, $\sigma_5 = (\nu_3, \nu_4)$, $\sigma_6 = (\nu_1, \nu_3)$.
We get $I_i = \{\nu_i\}$, for $i = 1, \cdots, 4$, $I_5 = \{\tau_1, \sigma_1, \sigma_2\}$, $I_6 = \{\tau_4, \sigma_4, \sigma_6\}$. $I_7 = \{\tau_3\}$, $I_8 = \{\sigma_5\}$.
For $i = 1, \cdots, 4$, $P_t(I_i) = 1$, $P_t(I_5) = t$, $P_t(I_6) = t$, $P_t(I_7) = t^2$, $P_t(I_8) = t$.
$\sum_i P_t(I_i) = 4 + 3t + t^2 = 1 + t^2 + 3(1 + t)$, so $\bar{R}(t) = 3$, since $P_t(\mathbb{K}) = 1 + t^2$.

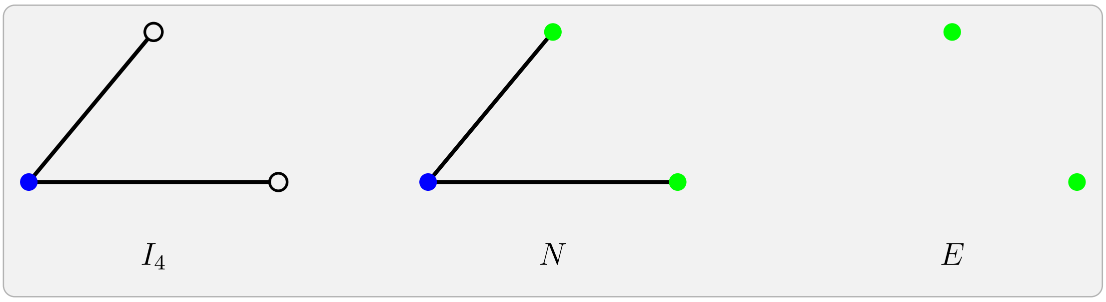


Figure 6.20: Isolating neighborhood and exit set for I_4 in Example 6.5.2 using Figure 6.18b.

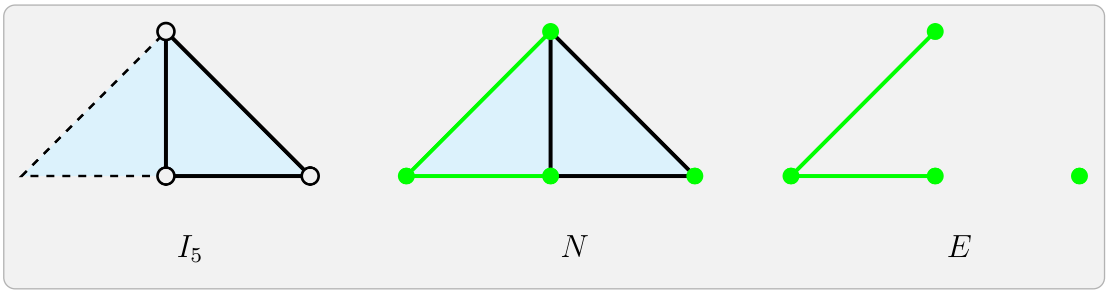


Figure 6.21: Isolating neighborhood and exit set for I_5 in Example 6.5.2 using Figure 6.18b.

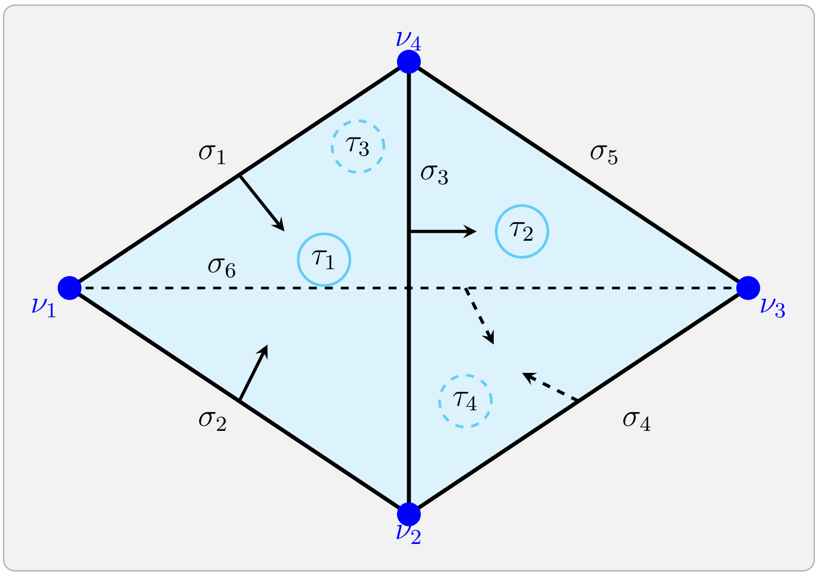


Figure 6.22: A vector field satisfying Definition 6.1.2.

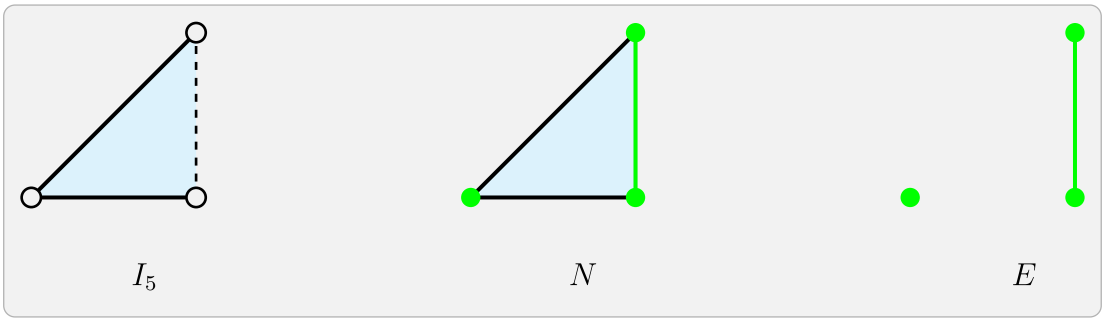


Figure 6.23: Isolating neighborhood and exit set for I_5 in Example 6.5.3 using Figure 6.22.

We have presented in this chapter three different ways to obtain the Euler number of a given finite CW complex.

1) The first one which is by means of the contribution function of each cell only uses the given vector field without any notion of boundary operator or Conley theory.

2) The second more dynamics related approach, which is the most important, shows how one can recover the Poincaré polynomial of the CW complex, using the notion of boundary operator. A new boundary operator is defined using a vector field more general than the one extracted from a discrete Morse function. The Betti numbers of this boundary operator coincide with the topological Betti numbers and we also get some Morse-type inequalities.

3) The third method uses Conley theory to analyze the topology of the CW complex. From the vector field at hand, a suitable definition for the isolated invariant sets is found. We recover the Poincaré polynomial of the CW complex by summing those of the respective index pairs of the isolated invariant sets up to some correction term.

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