On the spin cobordism invariance of the homotopy type of the space $\mathcal{R}^{\mathrm{inv}}(M)$

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Introduction

The central topic of this thesis is the study of the homotopy type of the space of metrics $\mathcal{R}^{\mathrm{inv}}(M)$ on a connected closed spin manifold M for which the Dirac operator is invertible.

During the past century many ideas and concepts from physics were adopted by mathematicians and revealed themselves to be fruitful once applied to investigations unrelated to their original scope. An example is the Dirac operator: it was introduced in the twenties by the physicist P. A. M. Dirac to describe the possible energetic states of a fermionic relativistic particle. Almost fourty years later, the Dirac operator became a prominent subject of study in mathematics, mainly due to the possibility to intertwine with it different branches such as Riemannian geometry (holonomy and scalar curvature), analysis and topology of the underlying manifold M on which the Dirac operator lives.
This latter relation was enlightened by the work of M. Atiyah and I. Singer: they noticed that if the dimension of M is divisible by 4, the analytical index of D^g is related to a topological invariant of the underlying manifold M in the following way:

Theorem (Index Theorem). *[AS68] Let M^n be a closed spin manifold of dimension $n \equiv 0 \mod 4$ then*

$$\mathrm{ind} D^{g,+} = \dim \ker D^{g,+} - \dim \mathrm{coker} D^{g,+} = \int_M \hat{A}\, dv^g, \tag{0.0.1}$$

where $\hat{A} \in H^n(M; \mathbb{Q})$ is the A-hat genus and

$$D^g = \begin{bmatrix} 0 & D^{g,-} \\ D^{g,+} & 0 \end{bmatrix}$$

is the splitting induced by the spinor representation in even dimension.

The interesting feature of the above theorem is that, even though the Riemannian metric g appears in the definition of the Dirac operator, the right hand side is a purely topological quantity, unaffected by a change of the metric on M (in the present thesis all manifolds will be assumed to be connected).
The equation 0.0.1 was later generalized to manifolds of all dimensions as

$$\mathrm{ind}\mathfrak{D}^g = \alpha([M^n]), \tag{0.0.2}$$

where $\mathfrak{D}$ is the Clifford linear Dirac operator (a generalisation of the classical Dirac operator D^g) and

$$\alpha : \Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\}) \longrightarrow KO^{-n}(\{\mathrm{pt}\}) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \mod 4, \\ \mathbb{Z}_2, & n \equiv 1,2 \mod 8, \\ 0, & \text{otherwise} \end{cases}$$

is the map defined in [Mil63] and equals the A-hat genus in dimension $n \equiv 0 \mod 8$.
Since the Clifford linear Dirac operator $\mathfrak{D}$ is defined on a bundle $\mathfrak{S}$ which splits as a direct sum of spinor bundles, we obtain from (0.0.2) a topological lower bound for the dimension of the kernel of the classical

Dirac operator D^g:

$$\dim\ker D^g \geq \begin{cases} |\langle \hat{A}, [M]\rangle|, & n \equiv 0 \mod 4, \\ 1, & n \equiv 1 \mod 8 \text{ and } \alpha \neq 0, \\ 2, & n \equiv 2 \mod 8 \text{ and } \alpha \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{0.0.3}$$

Riemannian metrics for which the lower bound in (0.0.3) is attained are called D-minimal and are the elements of the space $\mathcal{R}^{\min}(M)$. When $\alpha([M]) = 0$, a D-minimal metric g has an associated Dirac operator with trivial kernel, i.e. D^g is invertible. Such metrics will be called D-invertible and we will specify it in the notation for the relative space: $g \in \mathcal{R}^{\text{inv}}(M)$.
The work of the present thesis is devoted to unveil topological properties of such space, in particular how the homotopy type of $\mathcal{R}^{\text{inv}}(M)$ changes when surgery is performed on M.
This problem has already been investigated for the subspace of positive scalar curvature metrics

$$\mathcal{R}^{\text{psc}}(M) \subsetneq \mathcal{R}^{\text{inv}}(M).$$

E. Schrödinger and A. Lichnerowicz [Lic63] computed the square of the Dirac operator for a metric g and noticed that it differs from the usual connection Laplacian by a curvature term:

$$(D^g)^2 = \nabla^*\nabla + \frac{\text{scal}^g}{4},$$

where scal^g is the scalar curvature of the Riemannian metric g.
From the above formula it is immediate to see that if a Riemannian manifold has strictly positive scalar curvature, then the Dirac operator is invertible and hence its index is forced to be 0.
An interesting question is which manifolds admit a metric of positive scalar curvature or more generally a metric with invertible Dirac operator.
To solve problems regarding the existence of a metric with prescribed properties on a given manifold M, a convenient method is to broaden the class of manifolds studied at once: from isometry classes to cobordism classes. The cobordism equivalence relation is tantamount for two manifold M and $\widetilde{M}$ to be related by a surgery transformation [Mil62]: this is a controlled way of changing the topology of a manifold with a modification around an embedded sphere S^k with trivial normal bundle $\nu_{S^k} \cong D^{n-k} \times S^k$:

$$\widetilde{M} = M \setminus \left(D^{n-k} \times S^k\right) \underset{S^k \times S^{n-k-1}}{\cup} \left(S^{n-k-1} \times D^{k+1}\right),$$

which, from the metric point of view, can be performed in such a way that it takes place only in a small region of M. This suggests that there is a chance to modify the metric g on M only in a small tubular neighbourhood of S^k

$$\text{Tub}_k : D^{n-k} \times S^k \to M$$

so that the resulting metric $\widetilde{g}$ on the surgery-performed manifold $\widetilde{M}$ preserves the desired property (positive scalar curvature [GL80] or invertibility of the Dirac operator [ADH09]).
The next natural question, after the existence and abundance one, is the connectivity of the space of such metrics with a specific property. In the case of positive scalar curvature metrics, V. Chernysh [Che04] and M. Walsh [Wal13] proved that

Theorem. *[Che04, Wal13] Let M^n and $\widetilde{M}$ be two closed manifolds of dimension n obtained one another via a sequence of surgery transformations of dimension $2 \leq k \leq n-3$ then the relative spaces of Riemannian* *metrics with positive scalar curvature have the same homotopy type:*

$$\mathcal{R}^{\mathrm{psc}}(M) \simeq \mathcal{R}^{\mathrm{psc}}(\widetilde{M}).$$

Our task is to obtain an analogous result for the space $\mathcal{R}^{\mathrm{inv}}(M)$. In this case we expect the range of surgeries allowed to be $(1 \leq k \leq n-2)$, which covers all the possible surgeries needed to connect two spin cobordant manifolds:

Conjecture 1. *Let M^n and $\widetilde{M}^n$ be two closed spin manifolds with $n \geq 3$. If $\widetilde{M}$ is spin cobordant to M, then*

$$\mathcal{R}^{\mathrm{inv}}(M) \simeq \mathcal{R}^{\mathrm{inv}}(\widetilde{M}).$$

The strategy we will follow is the same used in [Wal13] and [Che04]: we use a parametrized version of the construction to extend a metric with the desired property along a surgery of a given dimension k and build with that nullhomotopies of pairs of spaces of metrics. Then we can deduce from the long exact sequence of a pair of spaces that the inclusion is a weak homotopy equivalence (hence a strong homotopy equivalence by Whitehead theorem). The space on which we continuously deform $\mathcal{R}^{\mathrm{inv}}(M)$ can be chosen to be homeomorphic for surgery-related manifolds; e.g. the subspace of metrics which are isometric to the product metric of the standard round metric σ_k and the hemisphere metric hem_{n-k} in a tubular neighbourhood (diffeomorphic to $D^{n-k} \times S^k$) of the surgery sphere S^k. Such a metric restricts on the boundary to

$$\left(S^{n-k-1} \times S^k, \sigma_{n-k-1} \oplus \sigma_k\right) = \partial\left(D^{n-k} \times S^k, \mathrm{hem}_{n-k} \oplus \sigma_k\right) = \partial\left(D^{k+1} \times S^{n-k-1}, \mathrm{hem}_{k+1} \oplus \sigma_{n-k-1}\right).$$

The property of a metric being D-invertible cannot be determined solely by local data, it is a global problem. This introduces new difficulties that were not completely overcome so that a complete proof of Conjecture 1 is unavailable.
Nevertheless we give a detailed strategy for the proof and we manage to prove along the way that:

Theorem 1. *Let M^n be a closed spin manifold of dimension $n \geq 3$ and let S^k be an embedded sphere with trivial normal bundle of codimension $n-k \geq 2$. Define the space of half-flat metrics to be*

$$\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}(M) := \left\{ g \in \mathcal{R}^{\min}(M) \,\middle|\, \mathrm{Tub}^*_k(g) = \mathrm{flat} \oplus g_{|_{S^k}} \right\}.$$

Then

$$\mathcal{R}^{\min}(M) \simeq \mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}(M),$$

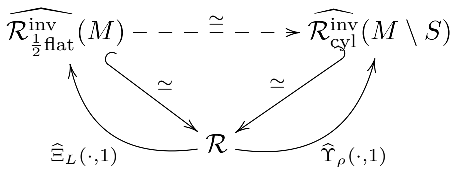
the homotopy equivalence being given by the inclusion $i : \mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}(M) \hookrightarrow \mathcal{R}^{\min}(M)$.

The next step to prove Conjecture 1 would be to show that $\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ is homotopy equivalent to

$$\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) := \Big\{ g \in \mathcal{R}^{\mathrm{inv}}(M \setminus S) \,\Big|\, \exists K \subset M \setminus S \text{ compact}, \left((M \setminus S) \setminus K,\, g_{|_{(M\setminus S)\setminus K}}\right) \overset{iso}{\cong} \left(\mathbb{R}_+ \times S^{n-k-1} \times S^k,\, du^2 \oplus \sigma_{n-k-1} \oplus \sigma_k\right) \Big\},$$

the space of metrics which are asymptotically (outside a compact subset K) cylindrical on $M \setminus S$. With this choice we have that $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \not\subseteq \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ and the method using the long exact sequence of pairs is then useless.
We take a detour and define an *ad hoc* space of metrics $\mathcal{R}$ so that it contains the spaces $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ and $\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$, respectively homotopy equivalent to $\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ and $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$ (compare Subsection 2.4.1 for the definition of $\mathcal{R}$), therefore we suggest to proceed by showing that they are both homotopy equivalent

to $\mathcal{R}$ by constructing nullhomotopies of pairs $\widehat{\Xi}_L$ and $\widehat{\Upsilon}_\rho$ so that the following diagram of weak homotopy equivalences commutes:



Regarding the map $\widehat{\Xi}_L$ we will show that it is indeed a nullhomotopy of pairs:

Theorem 2. *Let M^n be a closed spin manifold and S^k be an embedded submanifold with trivial normal bundle of codimension $n-k$ at least 2. If $k = 1$ suppose the embedded S^1 has the spin structure that bounds the disk. Then the spaces*

$$\mathcal{R} \overset{\text{weak}}{\simeq} \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$$

are weakly homotopy equivalent.

We will not manage to prove all the required properties required by the map $\widehat{\Upsilon}_\rho$ to be a nullhomotopy of pairs, we will prove that:

Theorem 3. *Let M^n be a closed spin manifold and let S^k be an embedded sphere with $n-k \leq 2$ and trivial normal bundle. Then for any compact family of metrics $B \subset \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$, there exists a value of $\rho > 0$ such that the map*

$$\Upsilon_\rho : B \times [0,1] \to \mathcal{R}$$

defined as in (2.4.7) *is well-defined and continuous.*

The missing part of the proof of Conjecture 1 is the property of the map $\widehat{\Upsilon}_\rho$ of preserving metrics of asymptotically cylindrical form. It provides, anyway, a good candidates for a parametrized version of the map in [ADH09, Subsection 3.2] that will serve as the desired nullhomotopy of pairs.
In case Conjecture 1 reveals to be true, one could use the spin cobordism invariance of the homotopy type of the space of D-invertible metrics to try to answer the question about existence of metrics with harmonic spinors:

Conjecture 2. *[Hit74, Bär96] On any closed spin manifold M^n of dimension at least 3 there exists a Riemannian metric g with*

$$\dim \ker D^g \neq 0.$$

This thesis is structured as follows:

In Chapter 1 are summarized the basic notions that will be needed throughout this thesis. We start with a quick review of Spin geometry, starting from the definition of the Spin group as a subgroup of a Clifford algebra. Then we pass to the differential-topological techniques exploited in the tentative proof of Conjecture 1: surgery and cobordism. Later we briefly recall the main analytical and topological properties of the Dirac operator and of the Cl_n-linear Dirac operator to obtain the lower bound in (0.0.3). To conclude we endow with a convenient topology the space of Riemannian metrics $\mathcal{R}iem(M)$ and describe its most relevant properties.

In Chapter 2, after recalling previous results on the space of D-minimal metrics $\mathcal{R}^{\min}(M)$, we explain the strategy to prove Conjecture 1. The remaining part of the chapter is devoted to such proof through the construction of maps between particular subspaces of metrics that will serve as nullhomotopies of pairs. Although along the way we will prove several technical lemmata and Theorems 1-3, we will not manage to prove that the map $\widehat{\Upsilon}_\rho$ fulfills all the properties of a nullhomotopy of pairs, making the proof of Conjecture 1 incomplete.

To conclude, in Chapter 3 are given further considerations about the results of this thesis and some possible future directions of research, e.g. how to employ Conjecture 1 to detect metrics with harmonic spinors on spin nullcobordant manifolds and answer positively to Conjecture 2.

Chapter 1

Preliminaries

In this chapter we plan to briefly recall the definition and the main properties of the Dirac operator which will be needed further in the thesis.
For this we first recall the definition of Spin group, spin manifold and spinor bundle. Then we defne surgery transformations and explain their relation to cobordism. In the third section we introduce the Dirac operator, describe how it is modified by a change of the Riemannian metric and state the Index Theorem, that intertwines analytical data of the Dirac operator to topological data coming from the underlying spin manifold in terms of its spin cobordism class. The last section will deal with the space of Riemannian metrics on a given manifold M (possibly open), describing in particular the conveninet topology it can be endowed with.
The standard reference for the material contained in Sections 1.1 and 1.3 is [LM90], while for Section 1.2 see [Kos93] or [Mil65] for a detailed proof of the equivalence of surgery and cobordism. The material in the last section can be found in [Hir97, Chapter 2] or [KM97].

1.1 Spin Geometry

The Dirac operator cannot be defined for any ordinary Riemannian manifold. It can be defined only for those Riemannian manifolds whose tangent bundle admits a larger structure group: $\mathrm{Spin}(n)$.
We provide two equivalent definitions for the group $\mathrm{Spin}(n)$; the first one needs only the concept of covering:

Definition 1.1.1. The spin group $\mathrm{Spin}(n)$ is the two-fold covering of $\mathrm{SO}(n)$. If $n \geq 3$,

$$\widetilde{Ad} : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \tag{1.1.1}$$

is the universal covering, in particular it is simply connected.

As a covering of the Lie group $\mathrm{SO}(n)$, also $\mathrm{Spin}(n)$ carries a structure of Lie group.
There is a second equivalent definition of $\mathrm{Spin}(n)$, with a more algebraic flavour. It involves the construction of a non commutative algebra called Clifford algebra. The choice for the name $\widetilde{Ad}$ for the covering map in (1.1.1) will become clear later.

Definition 1.1.2. Let V^n be an n-dimensional $\mathbb{K}$-vector space for some field $\mathbb{K}$ with characteristic different from 2, endowed with a positive definite $\mathbb{K}$-bilinear form g. Then the Clifford algebra $Cl(V, g)$ is defined as

$$Cl(V, g) := \frac{\bigoplus_{i=0}^{\infty} V^{\otimes i}}{\langle v \otimes w + w \otimes v + 2g(v, w) \rangle}.$$

The multiplication operation inside the algebra is usually denoted with $\cdot$.

The Clifford algebra $Cl(V, g)$ is isomorphic as a vector space to the Grassmann algebra $\wedge^* V$ (albeit as an algebra is not, unless $g \equiv 0$) and hence $\dim_{\mathbb{K}} Cl(V^n, g) = 2^n$.
There exist natural inclusions $V \hookrightarrow Cl(V, g)$ and $\mathbb{K} \hookrightarrow Cl(V, g)$.
A Clifford algebra always splits in two subspaces using an involution $\alpha : Cl(V, g) \to Cl(V, g)$ defined on an element $v \in V \hookrightarrow Cl(V, g)$ as

$$\alpha(v) = -v$$

and extended by linearity to the whole $Cl(V,g)$.
Using this involution, $Cl(V,g)$ can be given a $\mathbb{Z}_2$-grading according to the ± 1-eigenspaces of α, called even and odd elements:

$$Cl(V,g) = Cl^0(V,g) \oplus Cl^1(V,g), \tag{1.1.2}$$

and $Cl^i(V,g) \cdot Cl^j(V,g) \subseteq Cl^{i+j}(V,g)$ with $i,j \in \mathbb{Z}_2$.
A norm can be introduced on a Clifford algebra in the following way: first we define the transposition map that, for each monomial, inverts the order of its constituting factors:

$$\begin{aligned} (\,)^t : \quad Cl(V,g) &\longrightarrow Cl(V,g) \\ e_{i_1} \cdot \ldots \cdot e_{i_k} &\longmapsto e_{i_k} \cdot \ldots \cdot e_{i_1}, \end{aligned}$$

where $e_1, \ldots, e_n$ is a g-othonormal base of V. The map $(\,)^t$ linearly extends to all the elements of $Cl(V,g)$.
At this point we define, for any $x \in Cl(V,g)$,

$$\|x\|^2 := \alpha(x)^t \cdot x = x \cdot \alpha(x)^t.$$

Notice that we can use the norm and the transposition map to compute the inverse for any nonzero element $x \in Cl(V,g) \setminus \{0\}$:

$$x^{-1} := \frac{\alpha(x)^t}{\|x\|^2}.$$

From this point on we will deal only with $V^n = \mathbb{R}^n$ and $g = g_{\text{can}}$ the euclidean product and shorten the notation $Cl(\mathbb{R}^n, g_{\text{can}}) := Cl_n$.
We define the map

$$\widetilde{Ad} : Cl(V,g) \longrightarrow SO(n)$$

such that, for any $v \in \mathbb{R}^n$ and $x \in Cl_n$ we have

$$\widetilde{Ad}_x(v) := \alpha(x) \cdot v \cdot x^{-1}.$$

Definition 1.1.3. The group Spin(n) is the subgroup of the invertibles in the Clifford algebra of even degree and norm 1:

$$\text{Spin}(n) := \{x \in Cl_n \setminus \{0\} \,|\, x = x_1 \cdot \ldots \cdot x_k,\ \|x_i\| = 1,\ k \equiv 0 \mod 2\}.$$

Now the choice for the covering map name $\widetilde{Ad}$ (1.1.1) is clear, as Spin(n) is mapped to a composition of an even number of reflections in $\mathbb{R}^n$.
With the "algebraic" Definition 1.1.3 it is easy to prove that Spin(n) is a compact Lie group (elements of norm 1 in the finite dimensional subalgebra $Cl_n^0 \subset Cl_n$) whose Lie algebra $\mathfrak{spin}(n)$ is isomorphic to $\mathfrak{so}(n)$ through the Lie algebra isomorphism

$$\begin{aligned} \widetilde{Ad}^* : \mathfrak{spin}(n) &\longrightarrow \mathfrak{so}(n) \\ e_i \cdot e_j &\longmapsto 2e_i \wedge e_j. \end{aligned} \tag{1.1.3}$$

Consequently, as can be inferred also from Definition 1.1.1, Spin(n) have the same dimension of SO(n), namely $\dfrac{n(n-1)}{2}$.

Proposition 1.1.4. *The group* $\mathrm{Spin}(n)$ *for* n *odd admits a complex irreducible representation*

$$\rho : \mathrm{Spin}(n) \longrightarrow \Sigma_n$$

on a complex vector space Σ_n *(called complex spinor space) of complex dimension* $\dim_{\mathbb{C}} \Sigma_n = 2^{\left[\frac{n}{2}\right]}$*.*
Whenever n *is even, there are two irreducible inequivalent complex representations*

$$\rho_{\pm} : \mathrm{Spin}(n) \longrightarrow \Sigma_n^{\pm}$$

and $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$*.*

We can now use the Spin group to define new bundles on an orientable Riemannian manifold M^n. The lifting of the transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \subset M \longrightarrow \mathrm{SO}(n)$$

of the tangen bundle TM to $\mathrm{Spin}(n)$ will lead to topological obstructions detected by $\mathbb{Z}_2$-cohomology.

Definition 1.1.5. An orientable manifold M^n is said to be spin if its tangent bundle admits a reduction of the structure group as a vector bundle from the group $\mathrm{SO}(n)$ to the group $\mathrm{Spin}(n)$. Equivalently, this means that the principal bundle of orthonormal frames $P_{\mathrm{SO}}(M)$ admits an equivariant lifting to $P_{\mathrm{Spin}}(M)$ such that the following diagram commute:

$$\begin{array}{ccccc}
P_{\mathrm{Spin}}(M) \times \mathrm{Spin}(n) & \longrightarrow & P_{\mathrm{Spin}}(M) & & \\
\downarrow {\scriptstyle \Theta \times \widetilde{Ad}} & & \downarrow {\scriptstyle \Theta} & \searrow^{\pi} & \\
 & & & & M \\
 & & & \nearrow_{\pi} & \\
P_{\mathrm{SO}}(M) \times \mathrm{SO}(n) & \longrightarrow & P_{\mathrm{SO}}(M) & &
\end{array}$$

While any orientable manifold M^n can be endowed with a Riemannian metric (and henceforth the structure group of the tangent bundle TM can be reduced from $\mathrm{GL}(n)$ to $\mathrm{SO}(n)$), not every orientable manifold admits a spin structure, for example complex projective spaces $\mathbb{CP}^n$ for n even or $\mathbb{RP}^n$ with $n \neq 3$ mod 4. As one would expect, the obstruction is of topological nature. Such obstruction can be obtained from the short exact sequence of sheaves

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\widetilde{Ad}} \mathrm{SO}(n) \longrightarrow 1.$$

From this we get a long exact sequence in Čech cohomology:

$$\ldots \longrightarrow H^1(M; \mathrm{Spin}(n)) \xrightarrow{\widetilde{Ad}_*} H^1(M; \mathrm{SO}(n)) \xrightarrow{w_2} H^2(M; \mathbb{Z}_2) \longrightarrow \ldots$$

Hence it is necessary and sufficient, for a manifold M to be spin, that the map w_2, identified with the second Stiefel-Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$, vanishes.
Such vanishing of the second Stiefel-Whitney class ensures that in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma \subset M$ the lifting of the transition functions

$$\widetilde{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}(n), \quad \widetilde{Ad} \circ \widetilde{\tau}_{\alpha\beta} = \tau_{\alpha\beta}$$

is such that

$$\widetilde{\tau}_{\alpha\beta} \widetilde{\tau}_{\beta\gamma} \widetilde{\tau}_{\gamma\alpha} \equiv 1,$$

i.e. the cochain condition is satisfied. If $w_2 \neq 0$ then the maps $\widetilde{\tau}_{\alpha\beta}\widetilde{\tau}_{\beta\gamma}$ and $\widetilde{\tau}_{\alpha\gamma}$ may differ by a nontrivial deck transformation of the covering $\widetilde{Ad}$.

Moreover, the choice of lifting of the structure group is not unique, inequivalent spin structures are in 1:1 corrispondence with elements in $H^1(M;\mathbb{Z}_2)$. In conclusion we have:

Theorem 1.1.6. *A manifold is spin if and only if its first and second Stiefel-Whitney classes vanish, i.e.* $w_1(M) = w_2(M) = 0$. *The set of spin structures is in bijective corrispondence with* $H^1(M;\mathbb{Z}_2)$.

Remark 1.1.7. There is actually no need to introduce a metric g on the orientable manifold M^n and reduce the structure group of TM to $SO(n)$. We can consider the identity component $\mathrm{GL}^+(n)$ of the general linear group $\mathrm{GL}(n)$ and take its universal cover $\gamma : \widetilde{\mathrm{GL}^+}(n) \to \mathrm{GL}^+(n)$. A spin structure in this case is a choice of a lifting Θ such that the following diagram commutes:

$$\begin{array}{ccccc}
P_{\widetilde{\mathrm{GL}^+}}(M) \times \widetilde{\mathrm{GL}^+}(n) & \longrightarrow & P_{\widetilde{\mathrm{GL}^+}}(M) & & \\
\downarrow{\scriptstyle \Theta\times\gamma} & & \downarrow{\scriptstyle \Theta} & \searrow^{\pi} & \\
 & & & & M \\
 & & & \nearrow_{\pi} & \\
P_{\mathrm{GL}^+}(M) \times \mathrm{GL}^+(n) & \longrightarrow & P_{\mathrm{GL}^+}(M) & &
\end{array}$$

The drawback of this construction is encountered when building associated vector bundles to the principal bundle $P_{\widetilde{\mathrm{GL}^+}}(M)$, since the group $\widetilde{\mathrm{GL}^+}(n)$ has no finite dimensional faithful representations other than those which descend to $\mathrm{GL}^+(n)$.
The topological obstructions to the existence of such a spin structure is the same (the vanishing of the first two Stiefel-Whitney classes), as any Lie group is homotopy equivalent to its maximal compact subgroup: $\widetilde{\mathrm{GL}^+}(n) \simeq \mathrm{Spin}(n)$ and $\mathrm{GL}^+(n) \simeq \mathrm{SO}(n)$.

Remark 1.1.8. The spin condition is not very restrictive: all oriented manifolds with stably trivial tangent bundle have trivial characteristic classes (e.g. hypersurfaces in $\mathbb{R}^{n+1}$), 2-connected manifolds (i.e. manifolds M with $\pi_1(M) = \pi_2(M) = 0$), all orientable 3-manifolds.
Moreover the product of two spin manifolds is again spin, by the multiplication formula for the Stiefel-Whitney classes of a cartesian product $w_k(M \times N) = \sum_{i+j=k} w_i(M) \cap w_j(N)$.

Example 1.1.9. All the spheres S^k, $k \geq 1$ admit a spin structure as they are stably parallelizable and henceforth all their Stiefel-Whitney classes vanish. For $k \geq 2$ the sphere S^k is simply connected and hence $H^1(S^k;\mathbb{Z}_2) = 0$: it has only one spin structure. For the circle S^1, on the other hand, $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$ and hence there are two distinct spin structures. Θ_1 is the spin structure obtained by the restriction of the trivial one of the disk $S1 = \partial D^2 \overset{i}{\hookrightarrow} D^2$ with $\Theta_1 = i^*\Theta_{D^2}$, also referred to as the spin structure that bounds the disk. The second one, Θ_2, is the trivial spin structure

$$\Theta_2 : P_{\mathrm{Spin}}(S^1) \cong S^1 \times \{+1,-1\} \longrightarrow S^1 \times \{1\} \cong P_{\mathrm{SO}}(S^1),$$

which cannot be extended to the spin structure of the disk. The pair $\left(S^1, \Theta_2\right)$ is also denoted by $\bar{S}^1$, while we intend Θ_1 as the natural spin structure on the circle, so that we abbreviate $\left(S^1, \Theta_1\right) = S^1$.

We now have all the ingredients needed to define the domain of the Dirac operator, whose definition will be delayed to Section 1.3.
As a standard procedure in differential geometry, given the principal bundle $P_{\mathrm{Spin}}(M)$ with structure group

Spin(n), we can build an associated vector bundle using a representation $\rho : \mathrm{Spin}(n) \to \Sigma_n$ as in Definition 1.1.4. In the case of spin representations, the choice of a faithful representation forces us to use a complex vector space of dimension $2^{\left[\frac{n}{2}\right]}$.

Definition 1.1.10. The spinor bundle on the Riemannian spin manifold (M^n, g) is defined as the associated vector bundle to the principal bundle $P_{\mathrm{Spin}}(M)$

$$\Sigma^g M := P_{\mathrm{Spin}}(M) \times_\rho \Sigma_n$$

Proposition 1.1.11. *For* $\dim_{\mathbb{R}} M$ *even, the spin representation* (1.1.4) *splits the spinor bundle* $\Sigma^g M$ *into a direct sum*

$$\Sigma^g M = \Sigma^{g,+} M \oplus \Sigma^{g,-} M,$$

called half spinor bundles.

On $\Sigma^g M$ it can be defined a connection

$$\widetilde{\nabla} : \Gamma(\Sigma^g M) \longrightarrow \Gamma\left(T^*M \otimes \Sigma^g M\right),$$

lifted from the unique Levi-Civita connection defined on the bundle of orthonormal frames $P_{\mathrm{SO}}(n)$: as the Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{spin}(n)$ are isomorphic through $\widetilde{Ad}^*$ as in (1.1.3), the expression for the lifted Levi-Civita connection can be computed as follows: in local charts $(U_\alpha, \varphi_\alpha)$ on M and using a local g-orthonormal frame $e_1, \dots, e_n$ of TM the Levi-Civita connection is written as [Jos10, Section 3.3]

$$\nabla_{|U_\alpha} = d + \omega_\alpha,$$

with

$$\omega_\alpha = \sum_{i,j=1}^{n} \omega_{ij} e_i \wedge e_j \ \in \Gamma(T^*M_{|U_\alpha} \otimes \mathfrak{so}(n)).$$

Lifting it to the spinor bundle $\Sigma^g M$ the lifted connection $\widetilde{\nabla}$ is defined locally as

$$\widetilde{\nabla} = d + \widetilde{\omega}_\alpha$$

with

$$\widetilde{\omega}_\alpha = \sum_{i,j=1}^{n} \frac{1}{2} \omega_{ij} e_i \cdot e_j \ \in \Gamma(T^*M_{|U_\alpha} \otimes \mathfrak{spin}(n)).$$

From now on by ∇ we will mean the lifted Levi-Civita connection on the spinor bundle.
The spinor bundle admits an hermitian form $(\cdot, \cdot)$ (inducing the norm $|\cdot|$ on any fiber) which is invariant under the $Cl(n)$-action (in particular also Spin(n)-invariant) and, as a consequence, it is also ∇-parallel. With such product it is possible to complete the space of smooth sections of the spinor bundle $\Gamma(\Sigma^g M)$ to an Hilbert space:

Definition 1.1.12. On a complete Riemannian spin manifold (M, g) we define the Hilbert space of L^2-spinors

$$L^2(\Sigma^g M) := \overline{\Gamma(\Sigma^g M)}^{\|\ \|_{L^2}},$$

where the L^2-norm is given by the usual L^2-product

$$\|\phi\|^2_{L^2(\Sigma^g M)} = \int_M (\phi, \phi) dv^g,$$

dv^g being the volume form on M induced by g. We also define the Hilbert space $H^1(\Sigma^g M)$ as

$$H^1(\Sigma^g M) := \overline{\Gamma(\Sigma^g M)}^{\|\ \|_{H^1}},$$

with

$$\|\phi\|^2_{H^1(\Sigma^g M)} = \int_M (\phi,\phi) dv^g + \int_M (\nabla\phi,\nabla\phi) dv^g.$$

One can also use the whole Clifford algebra as a Cl_n-module itself, replacing Σ_n. This choice leads to the definition of Clifford bundle, where, in each fiber we have an additional binary operation inherited from the algebra structure of Cl_n.

Definition 1.1.13. We define the Clifford bundle on a spin manifold M^n to be the associated real vector bundle

$$Cl_n(M) := P_{\mathrm{SO}}(M^n) \times_\xi Cl_n,$$

where

$$\xi : \mathrm{SO}(n) \longrightarrow Cl_n$$

indicates the extension of the canonical representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ to the Clifford algebra $Cl_n \supset \mathbb{R}^n$.

The representation ξ exists since the action of $\mathrm{SO}(n)$ on $\bigoplus_i (\mathbb{R}^n)^{\otimes i}$ fixes the ideal $\langle v\otimes w + w\otimes v - 2g_{\mathrm{can}}(v,w)\rangle$.

Definition 1.1.14. The Clifford spinor bundle is the associated real vector bundle defined as

$$\mathfrak{S}(M^n) : P_{\mathrm{Spin}}(M^n) \times_\ell Cl_n$$

and ℓ is the representation given by left multiplication

$$\begin{aligned} \ell : \mathrm{Spin}(n) &\longrightarrow \mathrm{Aut}(Cl_n) \\ x &\longmapsto x\cdot \end{aligned}$$

Analogously to the spinor bundle $\Sigma^g M$, also the Clifford spinor bundle $\mathfrak{S}(M)$ can be made into an hermitian bundle once endowed with an hermitian product $(\cdot,\cdot)$. The hermitian product can be averaged over the generators of Cl_n to make it Cl_n-invariant.
All the symmetries of the Clifford spinor bundle are compatible with each other, as stated in the following proposition:

Proposition 1.1.15. *The Clifford spinor bundle $\mathfrak{S}(M^n)$ has the following properties:*

- *it carries a metric connection ∇ (the lifting of the Levi-Civita connection on $P_{\mathrm{SO}}(M^n)$),*
- *it is endowed with a right Cl_n fiberwise action which is parallel with respect to ∇,*
- *it carries a left action of $Cl_n(M^n)$ which commutes with the right action of Cl_n,*
- *as a vector bundle it is the direct sum of irreducible real spinor bundles over M.*
- *it inherits a $\mathbb{Z}_2$-grading from the Clifford algebra Cl_n as*

$$\mathfrak{S}(M^n) = \mathfrak{S}^0(M^n) \oplus \mathfrak{S}^1(M^n).$$

The $\mathbb{Z}_2$-grading is also intended with respect to the right Cl_n action, i.e.

$$\mathfrak{S}^i(M^n) \cdot Cl_n^j \subset \mathfrak{S}^{i+j}(M^n)$$

for $i, j \in \mathbb{Z}_2$.

We conclude this section underlining that the definition of the spinor bundle $\Sigma^g M$ depends on the metric g defined on the underlying manifold M. Nevertheless, J. P. Bourguignon and P. Gauduchon showed in [BG92] that there exist vector bundle isomorphisms between $P_{\mathrm{Spin}}(M, g)$ and $P_{\mathrm{Spin}}(M, h)$, principal bundles relative to different metrics, g and h on M. On the bundles of orthonormal frames $P_{\mathrm{SO}}(M, g)$ and $P_{\mathrm{SO}}(M, h)$ there exist bundle maps, indicated with b_h^g, which satisfy $g(b_h^g \cdot, b_h^g \cdot) = h(\cdot, \cdot)$. These maps admit liftings β_h^g to $\mathrm{Spin}(n)$ principal bundles that make the following diagram commute

$$
\begin{array}{ccccc}
P_{\mathrm{Spin}}(M, g) & \xrightarrow{\beta_h^g} & P_{\mathrm{Spin}}(M, h) & & \\
\downarrow \Theta & & \downarrow \Theta & \searrow^{\pi} & \\
 & & & & M \\
 & & & \nearrow_{\pi} & \\
P_{\mathrm{SO}}(M, g) & \xrightarrow{b_h^g} & P_{\mathrm{SO}}(M, h) & &
\end{array}
$$

The maps β_h^g with an abuse of notation operate also on the spinor bundle as isomorphisms

$$
\begin{aligned}
\beta_h^g : \Sigma^g M &\longrightarrow \Sigma^h M \\
[x, v] &\longmapsto [\beta_h^g x, v],
\end{aligned}
\tag{1.1.4}
$$

but in order to make them into isomorphism of Hilbert spaces $L^2(\Sigma^g M) \to L^2(\Sigma^h M)$ one has to multiply them by a factor $\frac{1}{f}$, where f is the unique smooth function for which $dv^g = f^2 dv^h$.

1.2 Surgery and Cobordism

A convenient setting to show that a certain class of objects satisfies a given property is to group these objects in finitely many equivalence classes and study each equivalence class separately. In the category of Riemannian manifolds is convenient to relax the equivalence relation from isometric to cobordant.

Definition 1.2.1. Two closed manifolds M^n and $\widetilde{M}^n$ are said to be cobordant whenever it exists an $n+1$ dimensional manifold W^{n+1} with boundary such that ∂W^{n+1} is diffeomorphic to $M^n \sqcup \widetilde{M}^n$.

The relation of cobordism is reflexive, symmetric and transitive, hence an equivalence relation. The set of representatives is indicated with $\Omega_n(\{\mathrm{pt}\})$ and it can be endowed with an abelian operation: the disjoint union of manifolds. This makes $\Omega_n(\{\mathrm{pt}\})$ an abelian group, while the operation of cartesian product of manifolds makes $\Omega_*(\{\mathrm{pt}\})$ a $\mathbb{Z}$-graded ring.
One can also consider various refinements of the previous definition, for example requiring that both manifolds' tangent bundles TM and $T\widetilde{M}$ have a bigger/smaller structure group and that the respective structure is inherited from the cobordism between M and $\widetilde{M}$:

Definition 1.2.2. Two closed spin manifolds (M^n, Θ) and $(\widetilde{M}^n, \widetilde{\Theta})$ are said to be spin cobordant if there exists an $n+1$ dimensional spin manifold (W^{n+1}, Θ_W) with boundary diffeomorphic to the disjoint union $M^n \sqcup \widetilde{M}^n$ and

$$i^*\Theta_W = \Theta, \qquad \widetilde{i}^*\Theta_W = \widetilde{\Theta},$$

where i and $\widetilde{i}$ are the embeddings $i : M^n \hookrightarrow W^{n+1}$ and $\widetilde{i} : \widetilde{M}^n \hookrightarrow W^{n+1}$, respectively.

In this case the literature talks about cobordism with decorations. Two of the most frequently used are oriented cobordism and spin cobordism, indicated respectively with $\Omega_n^{\mathrm{SO}}(\{\mathrm{pt}\})$ and $\Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$.

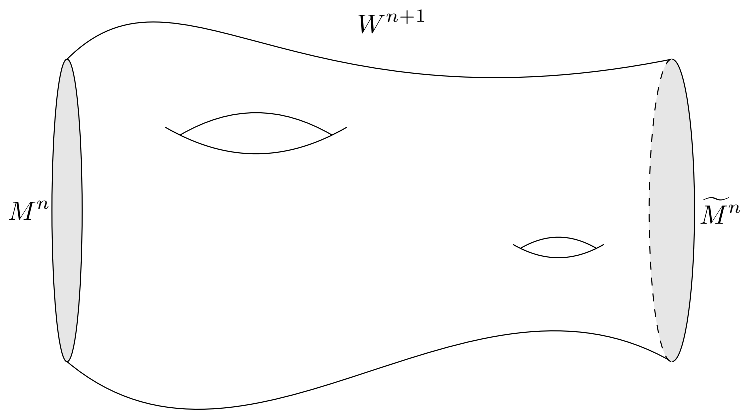


Figure 1.1: A cobordism between the manifolds M and $\widetilde{M}$.

A way of producing cobordant manifolds is to perform a surgery on a given manifold M, which is a controlled way to modify the homotopy groups of M, introduced in [Mil62]:

Definition 1.2.3. Consider two manifolds M^n and $\widetilde{M}^n$, a diffeomorhism $f \in \mathrm{Diff}(S^k \times S^{n-k-1})$, an embedding $\iota : S^k \to M$ with trivial normal bundle $\nu_{S^k} \cong D^{n-k} \times S^k$. We say that $\widetilde{M}$ is obtained from M by a surgery in dimension k (or a k-surgery) if

$$\widetilde{M}^n = (M^n \setminus S^k \times D^{n-k}) \underset{f}{\cup} D^{k+1} \times S^{n-k-1}.$$

The differentiable structure on $\widetilde{M}$ is well defined and independent from the diffeomorphism f, see [Kos93] for a detailed explanation.
The technique of surgery brought to major breakthroughs in the study of the homotopy type of manifolds during the '60. Recently surgery techniques appeared also in the solution of geometric problems, most notably the existence of positive scalar curvature metrics [GL80].
When dealing with Riemannian manifolds, we will have to ensure that also the metric is glued smoothly along the common submanifold $S^k \times S^{n-k-1}$.

Remark 1.2.4. From Remark 1.1.9 we know that there is a unique spin structure on $S^k \times S^{n-k-1}$ whenever $2 \leq k \leq n-3$. Nevertheless, surgery can be performed in a spin-compatible way also outside this range of k. It is in fact required that the spin structure of the surgery sphere S^k extends to the trivial spin structure of the disk. Whenever we will mention surgery on spin manifolds, we will consider the circle S^1 endowed with the spin structure bounding the disk.

Any surgery transformation in dimension k is reversible: it suffices to perform surgery in dimension $n-k-1$ as follows:

$$M^n \cong \overbrace{\left(M^n \setminus \iota(S^k \times D^{n-k}) \underset{S^k \times S^{n-k-1}}{\cup} D^{k+1} \times S^{n-k-1} \right)}^{\cong \widetilde{M}^n} \setminus \tilde{\iota}(S^{n-k-1} \times D^{k+1}) \underset{S^k \times S^{n-k-1}}{\cup} D^{n-k} \times S^k$$

Surgery and cobordism are closely related: in fact it holds the following:

Theorem 1.2.5. *Suppose M^n and $\widetilde{M}^n$ are two cobordant manifolds, then $\widetilde{M}$ can be obtained from M by a finite sequence of surgeries, and viceversa. In particular, in the case of spin manifolds, if the surgery is performed in a spin-compatible way (Remark 1.2.4), then the two manifolds are spin cobordant, i.e. $[M] = [\widetilde{M}]$ in $\Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$.*

To prove that surgery implies cobordism is straightforward: if $\widetilde{M}^n$ is obtained from M^n via a k-surgery, then we obtain the desired cobordism $Wn+1$ by attaching a $(k+1)$-hande $D^{k+1}\times D^{n-k}$ along its boundary $S^k \times D^{n-k}$ to the cylinder $M^n \times [0,1]$:

$$W = M \times [0,1] \underset{S^k\times D^{n-k}}{\cup} D^{k+1}\times D^{n-k}.$$

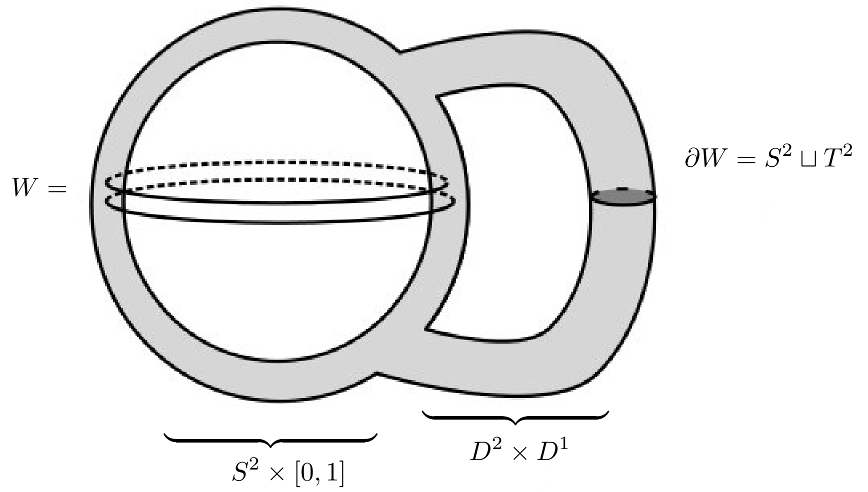


Figure 1.2: An example of a cobordism built between surgery-related manifolds.

The proof of the converse, that cobordant manifolds are related by a sequence of surgeries, involves a Morse function f defined on the cobordisms W^{n+1}: every critical point of f of index k will correspond to a surgery transformation of dimension k, where the attaching map of the k-handles $D^{n-k}\times D^k$ is given by the flow of $-\nabla f$. A complete proof is given in the book [Mil65].
The mathematical power of the cobordism relation lies in the possibility of computing the groups of representatives in terms of the stable homotopy groups of a particular topological space.
The spin cobordism groups $\Omega_*^{\mathrm{Spin}}(\{\mathrm{pt}\})$ have been calculated by D. Anderson, E. Brown and F. Petersen in [ABP67], while J. Milnor [Mil63] proved that in such groups it can only occurr 2-torsion. In the following tabular, taken from [Mil63], are listed the first nine spin cobordism groups with their relative generators:

n	$\Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$	generators
0	$\mathbb{Z}$	$\{\mathrm{pt}\}$
1	$\mathbb{Z}_2$	$\bar{S}^1$, see Remark 1.1.9
2	$\mathbb{Z}_2$	$\bar{S}^1 \times \bar{S}^1$
3	0	
4	$\mathbb{Z}$	Kummer surface $K3$
5	0	
6	0	
7	0	
8	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{HP}^2$ and ω_8 such that $\frac{1}{4}[\omega_8] = [K3]^2$

Table 1.1: The first nine spin cobordism groups and their generators.

The structure of $\Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$ for $n > 8$ is more complicated, we have in fact that, by [ABP67], the spin cobordism class of a spin manifold M^n is detected by its Stiefel-Whitney numbers [MS76] and its KO-Pontrjagin numbers. The latter are obtained in the following way: let $I = (i_1, \ldots, i_n)$ be a multi-index, then $p^I = p_1^{i_1} \cdots p_n^{i_n}$, where p_{i_k}, $k = 1, \ldots, n$ are the Pontrjagin classes of M^n, can be tought as an element of $KO^0(M)$. Any spin structure on a manifold M^n gives a KO-orientation $[M^n] \in KO_n(M^n)$. Then we obtain the KO-characteristic numbers of M^n by coupling with such orientation:

$$p^I(M^n) := \langle p^I(TM), [M^n] \rangle \in KO_n(\{\mathrm{pt}\}). \tag{1.2.1}$$

This way it is possible to establish the generators of the spin cobordism groups: let $I = (i_1, \ldots, i_n)$ be a multi-index and set $n(I) = \sum_{j=1}^{n} i_j$, then there exist spin manifolds M_I of dimension $4n(I)$ such that the generators of the spin cobordism groups modulo torsion for dimension greater than 8 are of the form

- $M_I \times \omega_8^k$ for $k \geq 0$
- $M_I \times K3 \times \omega_8^k$ for $k \geq 0$,

if all the indices i_j of the multi-index I in (1.2.1) are even, then M_I can be chosen as a product of quaternionic projective spaces.

A remarkable feature of the spin condition $w_1(M) = w_2(M) = 0$ is that, in low dimension, only some handles are allowed as building blocks of a spin cobordism W:

Theorem 1.2.6 ([Kir89], VII Theorem 3)**.** *If M^3 is closed and spin then it spin bounds a spin 4-manifold with only 0-handles and 2-handles.*

While for high dimensional manifolds (not necessarily spin) it holds that:

Theorem 1.2.7 ([Kos93], VIII Proposition 3.1)**.** *On any compact connected and simply connected cobordism W^{n+1}, $n \geq 4$ between M^n and $\widetilde{M}^n$ there exists a Morse function $f : W^{n+1} \to [0,1]$, with no critical points of index $k = n - 1$.*

Remark 1.2.8. The original Proposition 3.1 in [Kos93] actually states that there exists a Morse function $\tilde{f} : W^{n+1} \to [0,1]$ with no critical points of index 1 but "many" with index 3. In Theorem 1.2.7 we slightly change the original statement simply considering $f = 1 - \tilde{f} : W^{n+1} \to [0,1]$. It is immediate to see that the Morse functions $\tilde{f}, f$ have the same critical points $c_1, \ldots, c_k$ on W^{n+1} and the relation among the indeces at the mutual critical point c_i is

$$\mathrm{ind}_f(c_i) = n - \mathrm{ind}_{\tilde{f}}(c_i).$$

Remark 1.2.9. From the surgery perspective, Theorems 1.2.6 and 1.2.7 provide a bound on the dimension k of surgeries needed to obtain a spin manifold M^n from a spin cobordant one $\widetilde{M}^n$. We can in fact choose the cobordism between M^n and $\widetilde{M}^n$ to be connected and simply connected. Then we have that, for $n \geq 3$, any element $\widetilde{M}^n$ of a spin cobordism class $[M^n] \in \Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$ can be reached from M^n via a sequence of surgeries in dimension $1 \leq k \leq n-2$.

1.3 The Dirac operator

The physicist P.A.M. Dirac introduced the omonimous operator while searching a relativistic counterpart of the Klein-Gordon equation, a fist order differential operator D whose square is the dalambertian on $\mathbb{R}^4$:

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}.$$

Unfortunately, starting from dimension 3, the field of complex number is not sufficient to obtain such an operator, one has to enlarge the algebra of coefficients to the Clifford algebra Cl_n. Such algebra is noncommutative and encloses properties related to the bilinear form, as seen in Section 1.1.

Definition 1.3.1. The Dirac operator D^g on a Riemannian spin manifold (M, g) is defined as

$$\Gamma(\Sigma^g M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma^g M) \xrightarrow{g} \Gamma(TM \otimes \Sigma^g M) \xrightarrow{\cdot} \Gamma(\Sigma^g M)$$

where ∇ is the lift of the Levi-Civita connection to the spinor bundle $\Sigma^g M$ and $\cdot$ indicates the Clifford multiplication.

It follows from the definition that locally, on U_α, chosen an g-orthonormal frame $e_1, \ldots, e_n$ for the tangent bundle $TM^n{}_{|U_\alpha}$, we have that

$$D^g = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}.$$

It follows from the local expression of the Dirac operator that it satisfies a particular kind of Leibniz rule:

Lemma 1.3.2. *For any $f \in \mathcal{C}^\infty(M)$ and $\phi \in \Gamma(\Sigma^g M)$ it holds*

$$D^g(f\phi) = \nabla f \cdot \phi + f D^g \phi,$$

where ∇f is the Riemannian gradient of the function f.

From the analytical point of view, the Dirac operator has some remarkable properties, first of all:

Lemma 1.3.3. *The symbol of the Dirac operator D^g defined on a spin manifold (M, g) is*

$$\sigma_{p,\xi}(D^g) = -i\xi \cdot,$$

where $\cdot$ indicates the Clifford multiplication on the spinor bundle $\Sigma^g M$ and $\xi \in T_p^* M$.

We impose the domain of D^g to be $H^1(\Sigma^g M)$ (1.1.12).

Proposition 1.3.4. *On a closed spin manifold M, for $D^g : H^1(\Sigma^g M) \to L^2(\Sigma M)$ it holds*

- *the eigenspaces $E_\lambda(D^g)$ relative to an eigenvalue $\lambda \in Spec(D^g)$ are all finite dimensional,*
- *the operator D^g is Fredholm (kernel and cokernel are finite dimensional),*

- *it holds the decomposition*

$$L^2(\Sigma^g M) = \overline{\bigoplus_{\lambda \in Spec(D^g)} E_\lambda(D^g)}^{\| \|_{L^2}},$$

- D^g *is formally self-adjoint with respect to the* L^2 *product, i.e.*

$$\int_M (D^g\phi, \psi) dv^g = \int_M (\psi, D^g\psi) dv^g, \quad \forall \phi, \psi \in \Gamma(\Sigma^g M)$$

- *the spectrum* $SpecD^g$ *is real, unbounded on both sides and discrete.*

In even dimension the representation ρ splits the spinor space in a positive and a negative part (see Proposition 1.1.11). The Dirac operator $D^g : \Sigma^{g,\pm}M \to \Sigma^{g,\mp}M$ exchanges such subbundles and hence can be written as

$$D^g = \begin{bmatrix} 0 & D^{g,-} \\ D^{g,+} & 0 \end{bmatrix},$$

$D^{g,+}$ being the adjoint of $D^{g,-}$ and viceversa.
On complete noncompact manifolds an additional component of the spectrum appears, the essential spectrum:

Proposition 1.3.5. *On a complete Riemannian spin manifold* (M, g) *the spectrum of the Dirac operator* D^g *is real and splits as*

$$SpecD^g = essSpecD^g + disSpecD^g,$$

where the discrete spectrum $disSpecD^g$ *is the set of values* $\lambda \in \mathbb{R}$ *for which the equation* $D^g - \lambda = 0$ *has a strong solution in* $H^1(\Sigma^g M)$*, while the essential spectrum* $essSpecD^g$ *is the set of values* $\lambda \in \mathbb{R}$ *such that there exists a sequence of smooth* L^2*-orthonormal spinors* $\{\phi_i\}_i \subset L^2(\Sigma^g M)$ *such that* $D^g\phi_i - \lambda\phi_i \to 0$ *as* $i \to \infty$.

Whenever for the spectrum $SpecD^g$ holds the inclusion

$$SpecD^g \subset (-\infty, -a] \cup [a, \infty)$$

for a positive number $a \in \mathbb{R}_+$ we will say that D^g has a spectral gap of width a.
The ellipticity of the operator D^g implies that, by widening the section space to $H^1(\Sigma^g M)$, we can exploit standard elliptic regularity theory:

Lemma 1.3.6. *On a complete manifold* (M, g) *weak eigenspinors of the Dirac operator* $D : H^1(M) \to L^2(M)$*, i.e. solution of the equation*

$$\int_M (D^g\phi - \lambda\phi, \psi) dv^g = 0, \quad \forall \psi \in \Gamma(\Sigma^g M),\ \lambda \in \mathbb{R},$$

are smooth, hence strong solutions.

The focus of the second chapter will be on solutions of the equation $D^g = 0$ on a closed Riemannian spin manifold (M, g). In analogy with the Laplace-Beltrami operator, such solutions are called harmonic sections:

Definition 1.3.7. A spinor $\phi \in L^2(\Sigma^g M)$ is said to be harmonic if it satisfies the equation $D^g\phi = 0$ on M.

Let $\mathcal{C}^k(\Sigma^g M)$, $k = 0, 1, \ldots$, be the space of $\mathcal{C}^k$ sections of the spinor bundle $\Sigma^g M$ with the norm defined, for any $\phi \in \mathcal{C}^k(\Sigma^g M)$ as

$$\|\phi\|_{\mathcal{C}^k(\Sigma^g M)} \sum_{i=0}^{k} \sup_M |\nabla^i \phi|.$$

Lemma 1.3.8. *[ADH09, Lemma 2.2] Let (M, g) be a complete Riemannian spin manifold. Then for any compact subset $K \subset M$ there exists a constant $C = C(K, g)$ such that, for any solution of the equation $D^g\phi = 0$, it holds*

$$\|\phi\|_{L^2(\Sigma^g K)} \leq C\|\phi\|_{\mathcal{C}^2(\Sigma^g K)}.$$

Moreover, being elliptic, the eigenspinors of the Dirac operator satisfy nice extension properties: under certain codimension assumptions on the submanifold S, any weak solution of $D^g = 0$ on $M \setminus S$ can be extended to the whole manifold M.

Proposition 1.3.9. *[ADH09, Lemma 2.4] Let $\phi \in H^1(\Sigma^g M)$ be a weak solution of the equation $D^g\phi = 0$ on $M \setminus S$ with S a Riemannian submanifold of (M, g) of dimension k, with $n - k \geq 2$. Then ϕ can be extended to a weak solution on all of M.*

By the above lemmata regarding elliptic regularity, we have:

Corollary 1.3.10. *On a closed spin manifold (M, g), Ker D^g is finite dimensional and spanned by smooth sections of $\Sigma^g M$.*

Recall from Section 1.1 the change that undergoes the spinor bundle when changing the metric on M from g to h. Using the maps in (1.1.4) we can express the Dirac operator D^h in terms of D^g (see [BG92]): the image of the Dirac operator D^g on (M, g) through β^g_h is denoted by

$$^gD^h := \beta^g_h \circ D^g \circ \beta^h_g \tag{1.3.1}$$

and the resulting Dirac operator on (M, h) is related by the latter by

$$D^h = {^gD^h} + A^g_h \circ \nabla + B^g_h, \tag{1.3.2}$$

with $A^g_h \in \Gamma\left(TM \otimes \mathrm{End}(\Sigma^h M)\right)$ and $B^g_h \in \Gamma\left(\mathrm{End}(\Sigma^h M)\right)$, satisfying the following inequalities in terms of the norm of the section $g - h \in \Gamma(T^*M \odot T^*M)$ (see (1.4.1)):

$$|A^g_h| \leq C|g - h|_g, \qquad |B^g_h| \leq C(|g - h|_g + |\nabla^g(g - h)|_g). \tag{1.3.3}$$

In particular, for a conformal change of the metric like $h = F^2 g$, the Dirac operators D^g and D^h are related by

$$F^{\frac{n+1}{2}} D^h = D^g F^{-\frac{n-1}{2}} \tag{1.3.4}$$

From equation (1.3.4) it is evident that:

Theorem 1.3.11. *The dimension of the kernel of the Dirac operator D^g is invariant under conformal changes of the metric g.*

The most remarkable feature of the Dirac operator and the reason of its fame among mathematicians is the Index Theorem due to M. Atiyah and I. Singer [AS68]. The operator D^g, being Fredholm, has a finite index, defined as the difference between the dimension of the kernel and the cokernel. Recalling that the dimension of the cokernel for an operator equals the dimension of the kernel of the adjoint operator, we have that for a self adjoint operator such as D^g the index always vanishes. But interesting connections between analysis and topology can be obtained by looking in dimension divisible by 4 at the index of $D^{g,+}$:

Theorem 1.3.12 ([AS68])**.** *Let M^{4n} be a closed spin manifold. Then the index of the positive Dirac operator $D^{g,+}$ can be computed as*

$$\mathrm{ind} D^{g,+} = \langle \hat{A}, [M] \rangle = \int_M \hat{A}\, dv^g,$$

where the $\hat{A}$-genus is a polynomial in the Pontrjagin classes of M.

The value of the index is a purely topological quantity, depending only on the spin cobordism class of M, despite the definition of D^g, $\Sigma^g M$ and dv^g all involves the metric g.
Unfortunately Theorem 1.3.12 gives no precise information about the dimension of the space of harmonic spinors (compare with the dimension of the subspace of Laplace-Beltrami harmonic p-forms, which equals the p-th Betti number).
A lower bound on the dimension of the space of harmonic spinors can be obtained from the index of a modified version of the Dirac operator D^g.
Recall the Definition 1.1.14 of the bundle $\mathfrak{S}(M)$. In an analogous way as for the Dirac operator on $\Sigma^g M$ we can define a Dirac operator for the Clifford spinor bundle:

Definition 1.3.13. The Cl_n-linear Dirac operator $\mathfrak{D}$ on a spin manifold (M^n, g) is defined as the composition

$$\Gamma(\mathfrak{S}(M)) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathfrak{S}(M)) \xrightarrow{g} \Gamma(TM \otimes \mathfrak{S}(M)) \xrightarrow{\ell} \Gamma(\mathfrak{S}(M))$$

with ∇ the lifting of the Levi-Civita connection from $P_{\mathrm{SO}}(M)$ to $P_{\mathrm{Spin}}(M)$, g the musical isomorphism $T^*M \cong TM$ and ℓ the left action of $\mathrm{Spin}(n)$ on Cl_n.

The $\mathbb{Z}_2$-grading of $\mathfrak{S}(M^n)$ from Proposition 1.1.15 lets us write $\mathfrak{D}$ as

$$\mathfrak{D} = \begin{bmatrix} 0 & \mathfrak{D}^1 \\ \mathfrak{D}^0 & 0 \end{bmatrix}$$

for any $n = \dim M^n$ and

$$\mathfrak{D}^{0,1} : \Gamma(\mathfrak{S}^{0,1}(M^n)) \longrightarrow \Gamma(\mathfrak{S}^{1,0}(M^n))$$

with $\mathfrak{D}^1$ the adjoint operator of $\mathfrak{D}^0$ and viceversa. The operator $\mathfrak{D}^0$ is first order, real and elliptic; moreover, since the $Cl(M), Cl_n$ parallel actions commute with each other, also $\ker \mathfrak{D}$ is a $\mathbb{Z}_2$-graded Cl_n-module.
The set of isomorphism classes of $\mathbb{Z}_2$-graded Cl_n-modules $\mathcal{M}_n$ is closed under the semigroup commutative operation of direct sum. We define the subgroup $\mathcal{N}_n = \langle [V] + [W] - [V \oplus W] \rangle$ and consider the quotient

$$\mathfrak{M}_n := \frac{\mathcal{M}_n}{\mathcal{N}_n}.$$

The ring $[\mathfrak{M}_n, +]$ is addressed as Grothendieck group generated by the semigroup $[\mathcal{M}_n, \oplus]$, see [ABS64] for further details. The following theorem intertwines the groups $\mathfrak{M}_n$ with real topological K-theory:

Theorem 1.3.14. *[ABS64] For any n we have*

$$\frac{\mathfrak{M}_n}{i^*\mathfrak{M}_{n+1}} \cong KO^{-n}(\{\mathrm{pt}\}), \tag{1.3.5}$$

with $i : Cl_n \hookrightarrow Cl_{n+1}$ the canonical inclusion.

We define $\mathrm{ind}_n \mathfrak{D}$ to be

$$\mathrm{ind}_n \mathfrak{D} := [\ker \mathfrak{D}] \in \frac{\mathfrak{M}_n}{i^*\mathfrak{M}_{n+1}} \overset{(1.3.5)}{\cong} KO^{-n}(\{\mathrm{pt}\}).$$

In Table 1.2 below are expressed the groups $KO^{-n}(\{\mathrm{pt}\})$ appearing in (1.3.5):

$n \mod 8$	$KO^{-n}(\{\mathrm{pt}\})$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

Table 1.2: The KO-groups of a point.

The index $\mathrm{ind}_n\mathfrak{D}$ is a generalization of the classical index of a Fredholm operator and they coincide when $n = 0$. Indeed, recall that $Cl_0 \cong \mathbb{R}$, $Cl_1 \cong \mathbb{C}$ and that a $\mathbb{Z}_2$-graded Cl_0-module is just a real vector space $V^0 \oplus V^1$. Such space belongs to the image $i^*\mathfrak{M}_1$ if and only if it is a complex vector space, i.e. $V^0 \oplus V^1 = V \oplus iV$ for a real vector space V and hence $[V \oplus 0] = -[0 \oplus V] \in \mathfrak{M}_0/i^*\mathfrak{M}_1$. By the properties of the Cl_n-linear Dirac operator $\mathfrak{D}$ we have that $\ker\mathfrak{D} = \ker\mathfrak{D}^0 \oplus \ker\mathfrak{D}^1$ and each of the components are trivially Cl_0-modules. Then we can write

$$\mathrm{ind}_0\mathfrak{D} = [\ker\mathfrak{D}] \in \mathfrak{M}/i^*\mathfrak{M}_1 \overset{1.3.5}{\cong} KO^0(\{\mathrm{pt}\}) \cong \mathbb{Z}$$

and a chain of equalities

$$\begin{aligned}
\mathrm{ind}_0\mathfrak{D} &= [\ker\mathfrak{D}] \\
&= [\ker\mathfrak{D}^0 \oplus \ker\mathfrak{D}^1] \\
&= [\ker\mathfrak{D}^0 \oplus 0] + [0 \oplus \ker\mathfrak{D}^1] \\
&= [\ker\mathfrak{D}^0 \oplus 0] - [\ker\mathfrak{D}^1 \oplus 0] \\
&\cong \dim_{\mathbb{R}}\ker\mathfrak{D}^0 - \dim_{\mathbb{R}}\ker\mathfrak{D}^1 \\
&= \dim_{\mathbb{R}}\ker\mathfrak{D}^0 - \dim_{\mathbb{R}}\mathrm{coker}\mathfrak{D}^0.
\end{aligned}$$

The last two equalities comes from the fact that for $n = 0$, complex vector spaces in $\mathfrak{M}_0/i^*\mathfrak{M}_1$ are identified by their real dimension and the fact that $\mathfrak{D}^1$ is the adjoint operator of $\mathfrak{D}^0$. For further details, consult [LM90, II §7].
What written above is condensed in the index theorem for the Cl_n-linear Dirac operator $\mathfrak{D}$: it relates the analytical index $\mathrm{ind}_n\mathfrak{D}$ with the map

$$\alpha : \Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\}) \longrightarrow KO^{-n}(\{\mathrm{pt}\})$$

defined in [Mil63]:

Theorem 1.3.15. *[LM90, III Theorem 16.6] For a closed spin manifold M^n we have*

$$\mathrm{ind}_n\mathfrak{D} = \alpha([M^n]). \tag{1.3.6}$$

As $\mathfrak{S}(M^n)$ decomposes as a direct sum of complex spinor bundles, one obtains:

Theorem 1.3.16. *[LM90, II Theorem 7.10] Let (M^n, g) be a compact spin manifold, then*

$$\mathrm{ind}_n\mathfrak{D} = \begin{cases} \mathrm{ind}D^{g,+}, & n \equiv 0 \mod 8, \\ \frac{1}{2}\mathrm{ind}D^{g,+}, & n \equiv 4 \mod 8, \\ \dim\ker D^g, & n \equiv 1 \mod 8, \\ \dim\ker D^{g,+}, & n \equiv 2 \mod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, substituting according to (1.3.6) we get the desired lower bound for the dimension of $\ker D^g$:

$$\dim\ker D^g \geq \begin{cases} |\langle \hat{A}, [M]\rangle|, & n \equiv 0 \mod 4, \\ |\alpha(M)|, & n \equiv 1 \mod 8, \\ 2|\alpha(M)|, & n \equiv 2 \mod 8, \\ 0, & \text{otherwise.} \end{cases}$$

The equivalent for the Dirac operator of the Bochner formula relating the Laplace Beltrami operator to Ricci curvature is Scrödinger-Lichnerowicz theorem:

Theorem 1.3.17. *[Lic63] Suppose M is a complete spin manifold endowed with a metric g. Then it holds*

$$(D^g)^2 = \nabla^*\nabla + \frac{\mathrm{scal}^g}{4}. \tag{1.3.7}$$

It is an immediate consequence that if the scalar curvature scal^g is everywhere positive on a closed spin manifold (M, g) (or nonnegative but strictly positive in a neighbourhood of a point), then D^g has a spectral gap of width $\dfrac{\sqrt{\min_M(scal^g)}}{2}$.

Example 1.3.18. On (S^k, σ_k), $k \geq 2$ the scalar curvature is $\mathrm{scal}^{\sigma_k} = k(k-1)$. It follows that D^{σ_k} has a spectral gap of width $\dfrac{\sqrt{k(k-1)}}{2}$. The width of the spectral gap can be refined as in [ADH09, Lemma 2.5] to $\dfrac{k}{2}$.

Also the circle $S^1 = [0, 1]/\sim$ with the bounding spin structure and standard round metric $d\theta^2$ has no harmonic spinors: in this case the sections of the spinor bundle $\Sigma^{d\theta^2}S^1$ satisfy the antiperiodicity property $\phi(0) = -\phi(1)$, where we have considered $S^1 \subset \mathbb{C}$ and the covering map Θ to be

$$\Theta : \underbrace{P_{\mathrm{Spin}}(S^1)}_{\cong S^1} \longrightarrow \underbrace{P_{\mathrm{SO}}(S^1)}_{\cong S^1}$$
$$z \longmapsto z^2.$$

It follows that the only values λ for which there exist solutions for the equation

$$i\frac{d}{d\theta}\phi = \lambda\phi$$

are $\lambda = k + \frac{1}{2}$, $k \in \mathbb{Z}$; there are no harmonic spinors as constant spinors are not elements of $\Gamma(\Sigma^{d\theta^2}S^1)$. This fact, joint with Example 1.3.18 proves that:

Lemma 1.3.19. *The Dirac operator on a sphere S^n, $n \geq 1$ endowed with the bounding spin structure and standard round metric has a spectral gap of width $\dfrac{n}{2}$.*

We have seen in Remark 1.1.8 that the product of two spin manifolds is again spin. The Dirac operator, contrarily to the scalar curvature, does not split for product Riemannian manifolds, albeit its square does. We have in fact that the squared Dirac operator on $(M \times N,\, g \oplus h)$ splits as

$$\left(D^{g\oplus h}\right)^2 = \left(D^g\right)^2 + \left(D^h\right)^2. \tag{1.3.8}$$

It follows from Lemma 1.3.19 and the self-adjointness of the Dirac operator that on any complete manifold M crossed with a standard bounding sphere the Dirac operator will be invertible, with a spectral gap inherited from the spherical factor:

Corollary 1.3.20. *[ADH09, Proposition 2.6] Let $(M \times S^k,\, g \oplus \sigma_k)$ be the Riemannian product of a complete spin manifold (M, g) and the sphere S^k endowed with the bounding spin structure and standard round metric σ_k. Then $D^{g\oplus\sigma_k}$ has a spectral gap of width at least $\frac{k}{2}$.*

If the Dirac operator D^g defined on (M, g) has a spectral gap, then D^g is invertible and hence its index as a Fredholm operator is 0 and from Theorem 1.3.16 we get that also the index of the Cl_n-linear Dirac operator $\mathfrak{D}$ vanishes. The Index Theorem 1.3.6, together with Scrödinger-Lichnerowicz formula (1.3.7), provides then a topological obstruction to the existence of metrics of positive scalar curvature:

Corollary 1.3.21. *A manifold M with $\alpha([M]) \neq 0$ cannot admit a metric of positive scalar curvature.*

The converse statement, i.e. that manifolds with zero topological index admit metrics of positive scalar curvature is false in general (e.g. the torus T^n with the bounding spin structure, $n \leq 8$), but, due to the work of S. Stolz, the statement is true for simply connected manifolds of dimension at least 5:

Theorem 1.3.22. *[Sto92, Theorem A] A closed simply connected spin manifold M^n of dimension $n \geq 5$ admits a metric of uniformly positive scalar curvature if and only if $\alpha([M]) = 0$.*

1.4 The topology of the space of Riemannian metrics

In this section we want to define a topology on the space of complete Riemannian metrics on a manifold M. We start by giving a definition of a topology on the space of $\mathcal{C}^k$ mappings between two smooth manifolds:

Definition 1.4.1. [Hir97, Chapter 2] Let, M and N be smooth manifolds and (U, φ) and (V, ψ) be charts of M and N, respectively. Let $f \in \mathcal{C}^k(M, N)$ and let $K \subset U$ be a compact set such that $f(K) \subset V$ and let $\epsilon > 0$. The $\mathcal{C}^k$ compact-open topology on the space $\mathcal{C}^k(M, N)$ of $\mathcal{C}^k$ maps from M to N is the topology generated by the subbase of f-neighbourhoods

$$\mathcal{N}^k(f; (U, \varphi), (V, \psi), K, \epsilon)$$

whose elements are functions $g \in \mathcal{C}^k(M, N)$ with $g(K) \subset V$ and

$$\|D^i(\psi f \varphi^{-1})(p) - D^i(\psi g \varphi^{-1})(p)\| \leq \epsilon$$

for all $p \in \varphi(K)$ and all $i \in [0, k]$.
The $\mathcal{C}^\infty$ compact-open topology on $\mathcal{C}^\infty(M, N)$ is defined as the union of all the topologies induced by the inclusion $\mathcal{C}^\infty(M, N) \hookrightarrow \mathcal{C}^k(M, N)$ for finite k.

The space $\mathcal{C}^\infty(M, N)$ admits a Frechét manifold structure. We start describing it by recalling the definition of a Frechét space:

Definition 1.4.2. A Frechét space F is a complete metrizable locally convex vector space whose topology is induced by a family of seminorms.

The structure of Frechét space allows a definition of derivative Df of maps $f : F_1 \to F_2$ between Frechét spaces F_1 and F_2: we define the derivative of f at the point $x \in F_1$ in the direction of $v \in F_1$ to be the limit

$$Df = \lim_{t \to 0} \frac{f(x+tv)}{t}.$$

As for functions of a real variable, we say that f is $\mathcal{C}^1$ if $Df : F_1 \times F_1 \to F_2$ is continuous for any $x, v \in F_1$, f is $\mathcal{C}^2$ if $D^2 f : F_1 \times F_1 \times F_1 \to F_2$ is continuous for any $x, v \in F_1$ and so on; we can iterate this procedure to obtain $\mathcal{C}^\infty$ or smooth maps between Frechét spaces.
At this point we can use a Frechét space as a local model for a manifold M (possibly infinite dimensional) and mantain a notion of smoothness for the transition function of two overlapping charts:

Definition 1.4.3. A Hausdorff second countable topological space M is a Frechét manifold if there exist a Frechét space F and an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of M with $U_\alpha \subset M$ open and $\varphi_\alpha : U_\alpha \to V_\alpha \subset F$ such that, for any nonempty intersection $U_\alpha \cap U_\beta \subset M$ the maps

$$\varphi_\alpha \circ \varphi_\beta^{-1} : V_\beta \longrightarrow V_\alpha, \qquad \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \longrightarrow V_\beta$$

are smooth mappings of Frechét spaces.

The most relevant example of Frechét manifold is the space of smooth mappings between two smooth compact manifolds:

Theorem 1.4.4. *[KM97, §42] Let M and N be smooth finite dimensional manifolds. Suppose M is compact. Then the space $\mathcal{C}^\infty(M, N)$ of smooth mappings between M and N endowed with the $\mathcal{C}^\infty$ compact-open topology is a Frechét manifold.*

Definition 1.4.5. We define the topological space $\mathcal{R}iem(M)$ on a smooth manifold M (possibly noncompact) to be the space of Riemannian metrics on M endowed with the $\mathcal{C}^\infty$ compact-open topology.

Remark 1.4.6. Any metric g on a compact manifold M is a two-fold covariant symmetric tensor field on M. This allows us to consider the space of Riemannian metrics $\mathcal{R}iem(M)$ as a subset of the space of maps $\mathcal{C}^\infty(M, T^*M \odot T^*M)$ and by Theorem 1.4.4 inherits the structure of a Frechét manifold.

Remark 1.4.7. The space of sections $\Gamma(E)$ of any vector bundle $\pi : E \to M$ over a compact manifold M is a Frechét space when endowed with the countable family of seminorms

$$\|s\|_k = \sum_{i=0}^{k} \sup_M |\nabla^k s|, \quad s \in \Gamma(E).$$

Here ∇ denotes a fixed connection on E and $|\cdot|$ a fixed norm in the fiber.
The space $\mathcal{R}iem(M)$ is a subset of $\Gamma(T^*M \odot T^*M)$ and it inherits from the latter the topology induced by the countable family of seminorms

$$\|g\|_k = \sum_{i=0}^{k} \sup_M |\nabla^k g|, \quad g \in \mathcal{R}iem(M). \tag{1.4.1}$$

It follows that $\mathcal{R}iem(M)$ is a Frechét space.

By an abuse of notation ∇ in (1.4.1) indicates the Levi-Civita connection associated to the background metric g_0 on any tensor bundle $T^*M^{\odot k}$. The choice of the reference metric g_0 is unimportant for the definition of the topology, since different background metrics provide equivalent topologies on $\mathcal{R}iem(M)$ for compact manifolds M.

Lemma 1.4.8. *Let M be a compact manifold. Then the space $\mathcal{R}iem(M)$ is sequential, i.e. a map $f : \mathcal{R}iem(M) \to \mathcal{R}iem(M)$ is continuous if and only if for any sequence $\{g_i\}_i \subset \mathcal{R}iem(M)$ converging to $g \in \mathcal{R}iem(M)$ we have $f(g_i) \to f(g)$ as $i \to \infty$.*

Proof. Since the space $\mathcal{R}iem(M)$, by Remark 1.4.7 is a Frechét space, it is by definition metrizable and hence in particular first countable. This property implies that $\mathcal{R}iem(M)$ is a sequential topological space. □

Lemma 1.4.9. *For any diffeomorphism $\Phi \in \mathrm{Diff}(M)$ the pullback map*

$$\begin{aligned}\Phi^* : \mathcal{R}iem(M) &\longrightarrow \mathcal{R}iem(M)\\ g &\longmapsto \Phi^* g\end{aligned}$$

is continuous in the $\mathcal{C}^\infty$ compact-open topology.

Proof. A pullback can be written as a composition

$$\begin{aligned}\mathrm{Diff}(M) \times \mathcal{R}iem(M) &\longrightarrow \mathrm{Diff}(M) \times \mathcal{R}iem(M) \longrightarrow && \mathcal{R}iem(M)\\ (\Phi, g) &\longmapsto \qquad (\Phi, g(\Phi)) && \longmapsto g(\Phi)(D\Phi\,\cdot, D\Phi\,\cdot)\end{aligned}$$

It is evident from Definition 1.4.1 that the composition of constinuous functions is again continuous. □

Proposition 1.4.10. *Let $p \in (M^n, g)$, then the distance functions $d_g(\cdot, p)$ is continuous with respect to g in the $\mathcal{C}^\infty$ compact-open topology. In particular the exponential map $\exp_g : U \subset TM^n \to M^n$ is continuous with respect to g.*

Proof. On a Riemannian manifold (M, g) the distance from a point p is defined as

$$d_g(q, p) = \inf_{\Lambda_{p,q}} L(\gamma),$$

where $\Lambda_{p,q} = \{\gamma \in \mathcal{C}^\infty([0,1], M) \,|\, \gamma(0) = p,\ \gamma(1) = q\}$. The smallest length is achieved by geodesic curves $c(t) = (x_1(t), \ldots, x_n(t))$, which satisfy the system of second order nonlinear ODEs

$$\ddot{x}_k(t) + \sum_{i,j=1}^{n} \Gamma^k_{ij} \dot{x}_i(t) \dot{x}_j(t) = 0$$

with boundary condition $c(0) = p$, $c(1) = q$. The exponential map $\exp_g(p, v)$, on the other hand, is defined as $\gamma(1)$, where $\gamma : \mathbb{R} \to M$ is the unique geodesic satisfying $\gamma(0) = p$, $\dot{\gamma}(0) = v$. This shows that also the value of the exponential map is determined by a system of second order differential equations (See [Jos10, pag. 20]). Since any system of second order ODEs is equivalent to a system of coupled first order ODEs, to prove the statement of the proposition it suffices to show that, chosen a map $h : \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$, the map

$$\begin{aligned}\mathcal{S} : \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)\\ f &\longmapsto \mathcal{S}(f) := c(t), \text{ such that } \dot{c}(t) = f(t, c(t)) \text{ and } c(0) = h(f)\end{aligned}$$

is continuous.
If $c(t)$ is a solution of the above differential equation, we know that $c(t)$ can be written as $c(t) = \displaystyle\int_0^t f(s, c(s))ds+$

$c(0)$. Consider a sequence $f_i \to f$. Then

$$\begin{aligned}\|\mathcal{S}(f_i)(t) - \mathcal{S}(f)(t)\| &= \left\| \int_0^t f_i\left(s, c_i(s)\right) ds + h(f_i) - \int_0^t f\left(s, c(s)\right) ds - h(f) \right\| \\ &\le \left\| \int_0^t f_i\left(s, c_i(s)\right) ds - \int_0^t f_i\left(s, c(s)\right) ds \right\| \\ &\quad \underbrace{+ \int_0^t \underbrace{\|f_i\left(s, c(s)\right) - f\left(s, c(s)\right)\|}_{\to 0} ds + \underbrace{\|h(f_i) - h(f)\|}_{\to 0}}_{=: a_i \to 0}.\end{aligned}$$

Since by definition f_i is smooth for every i, it is also Lipschitz with a certain Lipschitz constant $L_i \to L$ on the compact subset $[0, t] \subset \mathbb{R}$. From $f_i \to f$ follows that for any $x, y \in \mathbb{R}^n$, for all $\epsilon > 0$ and i sufficiently big

$$\begin{aligned}\|f_i(x) - f_i(y)\| &\le \underbrace{\|f_i(x) - f(x)\|}_{\to 0} + \|f(x) - f(y)\| + \underbrace{\|f_i(y) - f(y)\|}_{\to 0} \\ &\le L\|x - y\| + \epsilon \end{aligned} \tag{1.4.2}$$

and we can choose a L_0 so that the inequality $\|f_i(x) - f_i(y)\| \le L_0 \|x - y\|$ holds for all i. Then

$$\begin{aligned}\varphi_i &:= \|c(t) - c_i(t)\| \le \left\| \int_0^t \left(f_i\left(s, c_i(s)\right) - f_i\left(s, c(s)\right)\right) ds \right\| + a_i \\ &\overset{(1.4.2)}{\le} \int_0^t L_0 \|c_i(t) - c(t)\| + a_i = a_i + L_0 \int_0^t \varphi_i(s) ds.\end{aligned}$$

By Gronwall's lemma we obtain

$$\|\mathcal{S}(f_i)(t) - \mathcal{S}(f)(t)\| = \|c_i(t) - c(t)\| = \varphi_i(t) \le a_i \exp(L_0 t) \to 0.$$

This concludes the first step of an induction argument: we have in fact, for the first derivative:

$$\begin{aligned}\left\| \frac{d}{dt}\mathcal{S}(f_i)(t) - \frac{d}{dt}\mathcal{S}(f)(t) \right\| &= \|f_i\left(t, c_i(t)\right) - f\left(t, c(t)\right)\| \\ &\le \underbrace{\|f_i\left(t, c_i(t)\right) - f_i\left(t, c(t)\right)\|}_{\le L_0 \varphi_i \to 0} + \underbrace{\|f_i\left(t, c(t)\right) - f\left(t, c(t)\right)\|}_{\to 0} \to 0\end{aligned}$$

Suppose then that

$$\left\| \frac{d^{k-1}}{dt^{k-1}}\mathcal{S}(f_i)(t) - \frac{d^{k-1}}{dt^{k-1}}\mathcal{S}(f)(t) \right\| \to 0. \tag{1.4.3}$$

For the k-th derivative we have

$$\begin{aligned}\left\| \frac{d^k}{dt^k}\mathcal{S}(f_i)(t) - \frac{d^k}{dt^k}\mathcal{S}(f)(t) \right\| &= \left\| \frac{d^{k-1}}{dt^{k-1}}\left(f_i\left(t, c_i(t)\right) - f\left(t, c(t)\right)\right) \right\| \\ &= \Big\| P\left(f_i, \dots, f_i^{(k-1)}, c_i(t), \dots, c_i^{(k-1)}(t), t\right) \\ &\qquad - P\left(f, \dots, f^{(k-1)}, c(t), \dots, c^{(k-1)}(t), t\right) \Big\|,\end{aligned}$$

where P is a polynomial obtained from consecutive derivation. By the induction hypothesis (1.4.3) we have that each derivative of $c_i(t)$ up to order $k - 1$ converges punctually to $c(t)$ and since polynomials are continuous functions, we obtain, in conclusion, that

$$\mathcal{S}(f_i) \to \mathcal{S}(f),$$

i.e. $\mathcal{S}$ is continuous.

□

Next comes a lemma which will be needed for the construction of homotopies in the space $\mathcal{R}iem(M)$: it states that segments are continuous in the compact-open topology.

Lemma 1.4.11. *Let M be a compact manifold, then the map*

$$L : \mathcal{R}iem(M) \times \mathcal{R}iem(M) \times [0,1] \longrightarrow \mathcal{R}iem(M)$$
$$(g_1, g_2, t) \longmapsto (1-t)g_1 + tg_2$$

is continuous.

Proof. We have from Remark 1.4.7 that $\mathcal{R}iem(M)$ is a Frechét space, which, by definition is a topological vector space. □

Remark 1.4.12. The space of sections $\Gamma(E)$ of any vector bundle $\pi : E \to M$ over a compact manifold M is contractible, since we can continuously deform any section s to the zero section through linear combination exploiting the fiberwise linear structure of E. The space of complete Riemannian metrics $\mathcal{R}iem(M)$ is an open cone in the space of sections $\Gamma(T^*M \otimes T^*M)$, i.e.

$$\forall a, b \in \mathbb{R}_+, \, \forall g_1, g_2 \in \mathcal{R}iem(M) \quad ag_1 + bg_2 \in \mathcal{R}iem(M).$$

It follows that the space of complete Riemannian metrics is contractible:

$$\mathcal{R}iem(M) \simeq \{\mathrm{pt}\}$$

We conclude this section with a remark on the space of complete metrics on a noncompact manifold M:

Remark 1.4.13. The space $\mathcal{R}iem(M)$ for noncompact smooth manifolds M has in general an infinite number of path components, as metrics with different behaviour at infinity cannot be continuously joined by a path. Nevertheless the situation changes if we focus our attention to complete metrics with a prescribed asymptotic behaviour. Consider for example the set of complete metrics for which there exists a compact set $K \subset M$ such that on $M \setminus K \cong N \times [0, \infty)$ each metric of the set has a cylindrical form $g_N \oplus dx^2$. We call this space $\mathcal{R}iem_{\mathrm{cyl}}(M)$. From the point of view of the $\mathcal{C}^\infty$ compact-open topology, the space $\mathcal{R}iem_{\mathrm{cyl}}(M)$ is homeomorphic to the space of Riemannian metrics $\mathcal{R}iem(K)$ on the compact manifold K with boundary $\partial K = N$, therefore the subspace $\mathcal{R}iem_{\mathrm{cyl}}(M) \subset \mathcal{R}iem(M)$ has only one path component (is actually contractible).

Chapter 2

The space of $D-$minimal Riemannian metrics

This chapter deals with the main focus of the thesis: the homotopy type of the space of Riemannian metrics on a spin manifold M with invertible associated Dirac operator. After introducing in detail the problem and listing the known results, we will state the strategy to prove Conjecture 1. The rest of the chapter is devoted to the proof of such result.

2.1 Statement of the problem and previous results

The celebrated Atiyah Singer Index Theorem [AS68], [LM90, III Theorem 13.2] states that, on a closed Riemannian manifold (M, g), the analytic index of any elliptic (hence Fredholm) operator P, defined as the difference

$$\text{ind } P = \text{ dim ker } P - \text{ dim coker } P = \text{ dim ker } P - \text{ dim ker } P^*$$

equals a topological quantity (independent of the Riemannian metric chosen on M). In the particular case of a Dirac operator defined on a compact spin manifold (M^{4n}, g) of dimension divisible by 4, the Index Theorem takes the form

$$\text{ind } D^{g,+} = \langle \hat{A}, [M] \rangle = \int_M \hat{A}\, dv^g,$$

where $\hat{A} \in H^{4n}(M; \mathbb{Q})$ is the so-called $\hat{A}$-genus, a top-degree cohomological class computed as a polynomial in the Pontrjagin classes.

Later on such result was extended to any dimension using the Cl_n-linear Dirac operator as

$$\text{ind}_n\, \mathcal{D} = \alpha([M^n]),$$

where $\alpha : \Omega^{\text{Spin}}_*(\{\text{pt}\}) \to KO^{-*}(\{\text{pt}\})$ is the α-genus of M, defined by Milnor in [Mil63] and Hitchin [Hit74]. By the properties listed in Proposition 1.1.15, the kernel of the Cl_n-linear Dirac operator $\mathcal{D}$ splits as a sum of copies of the kernel of the classical Dirac operator D^g. Hence the Atiyah Singer formula (1.3.12) only provides an equality between a topological invariant of M and the index of the Cl_n-linear Dirac operator, which implies only a lower bound for the dimension of the kernel of D^g:

$$\text{dim ker} D^g \geq \begin{cases} |\langle \hat{A}, [M] \rangle|, & n \equiv 0 \text{ mod } 4, \\ 1, & n \equiv 1 \text{ mod } 8, \quad \alpha([M]) \neq 0, \\ 2, & n \equiv 2 \text{ mod } 8, \quad \alpha([M]) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

In dimension 1 and 2 mod 8 the α-invariant only counts the parity of the dimension of ker D^g, since $KO^{-8k-1}(\{\text{pt}\}) \simeq KO^{-8k-2}(\{\text{pt}\}) \simeq \mathbb{Z}_2$ (see Tabular 1.2). Metrics for which the equality in (2.1.1) is attained will be called D-minimal metrics and they will form the set $\mathcal{R}^{\min}(M)$ (with the $\mathcal{C}^\infty$ compact-open topology induced by $\mathcal{R}iem(M)$, the set of all complete Riemannian metrics on M, see Section 1.3). When

$\alpha([M]) = 0$ the minimal dimension of the kernel of D^g is 0, hence in this case D-minimal metrics are called D-invertible metrics and the topological space they form will be indicated by $\mathcal{R}^{\mathrm{inv}}(M)$.
If we call $\mathbf{H}^g$ the dimension of the kernel of the Dirac operator D^g and $\mathbf{H}^{g,+}$, $\mathbf{H}^{g,-}$ the dimension of the space of harmonic spinors of $D^{g,+}$ and $D^{g,-}$ respectively, then $\dim\ker D^g, D^{g,\pm}$ is related to the α-invariant according to the following tabular [BD02, pag. 17]:

$n \mod 8$	$\alpha([M^n])$	
0,4	≥ 0	$\mathbf{H}^{g,+} = \langle \hat{A}, [M^n] \rangle, \quad \mathbf{H}^{g,-} = 0$
	< 0	$\mathbf{H}^{g,+} = 0, \quad \mathbf{H}^{g,-} = -\langle \hat{A}, [M^n] \rangle$
1	0	$\mathbf{H}^g = 0$
	1	$\mathbf{H}^g = 1$
2	0	$\mathbf{H}^{g,+} = \mathbf{H}^{g,-} = 0$
	1	$\mathbf{H}^{g,+} = \mathbf{H}^{g,-} = 1$
3,5,6,7	0	$\mathbf{H}^g = 0$

Table 2.1: The relation between the dimension of the space of harmonic spinors and the α-genus.

Aim of the chapter is to obtain informations on the homotopy type of the space $\mathcal{R}^{\min}(M)$ using surgery and spin cobordism, in the same fashion used for the space of positive scalar curvature metrics $\mathcal{R}^{\mathrm{psc}}(M)$ by V. Chernysh [Che04] and M. Walsh [Wal13].
The result concerning positive scalar curvature metrics suggests that a similar statement might be true also for metrics with invertible Dirac operator; in fact, using Schrödinger-Lichnerowicz formula (1.3.7)

$$(D^g)^2 = \nabla^*\nabla + \frac{\mathrm{scal}^g}{4},$$

we see immediatly that if a spin manifold M can be given a metric with $\mathrm{scal}^g > 0$, then the kernel of the Dirac operator is trivial and $\alpha([M]) = 0$.
Nevertheless positive scalar curvature is not a necessary condition to ensure triviality of the kernel of D: a counterexample is given by tori $\mathbb{T}^n$ endowed with a bounding spin structure. However tori do not admit any metric with positive scalar curvature [RS01].

2.1.1 The work of B. Ammann, M. Dahl and E. Humbert on D-minimal metrics

First of all, it is worth noticing that, from standard results in perturbation theory, if the space $\mathcal{R}^{\min}(M)$ is not empty, then it is open and dense in all the $\mathcal{C}^k$-topologies, $k \geq 2$, and open in the $\mathcal{C}^1$ [Kat76, Section VII 1.3].
The first question regarding D-minimal metrics is about their existence: can any closed connected spin manifold be endowed with a metric g for which the lower bound in (2.1.1) is attained?

The positive answer was given by B. Ammann, M. Dahl and E. Humbert in [ADH09]

Theorem 2.1.1. *[ADH09] Any closed connected spin manifold has a D-minimal metric.*

Remark 2.1.2. The hypothesis of connectedness of the manifold M cannot be removed. In fact the α-genus, being a spin cobordism invariant, is additive under disjoint union and the inverse with respect to such addition of any manifold M is given by the manifold itself with the opposite orientation M^-.
On the other hand, the kernel of a disjoint union of Riemannian manifolds is the direct sum of the kernels of

each component, independently of the orientation. Hence counterexamples of non-connected spin manifolds not admitting D-minimal metrics become easy to produce: for example consider $M = M_1 \sqcup M_1^-$ such that $\alpha([M_1]) = c \neq 0$. In this case $\alpha([M]) = 0$ but there exist at least $2c$ linearly independent harmonic spinors for any choice of metric g on M.

The question of existence of D-minimal metrics on any compact connected spin manifold belongs to the more general class of problem of the form:

Given a property P, show that any element X in the set Θ satisfies P.

If the cardinality of Θ does not allow to check that property P is satisfied case by case, it is better to focus on clusters inside of Θ, for example equivalence classes with respect to a convenient equivalence relation for which there exists a transformation T between elements of Θ such that property P happens to be preserved by T and such transformation, starting from any representative Y in an equivalence class $[Y]$, allows to reach any other element in $\pi^{-1}([Y])$. This expedient reduces the cases in Θ to the ones in its quotient $\Theta \xrightarrow{\pi} \Theta/\sim$, where $\sim$ is the aforementioned equivalence relation. Provided such $\sim$ and T exist, it will be sufficient to check that $\Theta/\sim$ consists of elements satisfying the property P.
Leaving abstraction apart and coming back to our particular problem, we will consider Θ to be the set of finite-dimensional closed connected spin manifolds, we will take the quotient with respect to the equivalence relation given by spin cobordism, property P will be the existence of a D-minimal metric and as T it will be used a surgery transformation of codimension ≥ 2 (see Section 1.2 for a definition of surgery transformation and cobordism relation).
The relation of bordism is much broader and identifies also manifolds with possibly different homotopy type. All the spin cobordism groups $\Omega_n^{\mathrm{Spin}}(\{\mathrm{pt}\})$ are finitely generated, hence the problem is reduced to study the existence problem on finitely many spin manifolds.
To prove Theorem 2.1.1, B. Ammann, M. Dahl and E. Humbert showed that a metric with D-minimal kernel can be transported along a surgery of the right codimension, i.e. starting from a metric g on M with minimal kernel, produce a metric $\widetilde{g}$ on the surgery performed manifold $\widetilde{M}$. A surgery transformation changes the topology of the underlying manifold whenever the embedded surgery sphere represents a generator in a certain homotopy group of the manifold. On the other hand, in the geometrical context we have a notion of length and volume given by the metric, hence a surgery transformation can be performed in a region of our manifold of arbitrarily small width δ around the surgery sphere S. Intuitively this should not change so much the geometry of the manifold and with it the Dirac operator and the spinor bundle on which it acts. It is reasonable to think that, paying the price of modifing the metric only is a small region, say a tube of radius δ around S, the manifold obtained via surgery will have the same amount of harmonic spinors: in fact, in the limit for $\delta \to 0$, the modified metric g_δ will differ from the original one only on a submanifold of codimension at least 2, a region which is irrelevant for L^2 sections.
In what follows we want to explain the authors's idea of proof; in Sections 2.3 and 2.4 we give more details in a parametrized version of this.
The original metric g on M is deformed in two steps: first one takes a convex combination with cut off function $\eta : M \to [0,1]$ with support in a tubular neighbourhood around S of radius δ between the original metric g and the product metric $dr^2 + r^2\sigma_{n-k-1} \oplus g_{|S}$ to create a half-flat neck close to the embedded surgery sphere S. Then we blow up the metric around S with a conformal change, so that we can transform the flat (euclidean) factor of the metric on D^{n-k} to one with product form $dr^2 + \sigma_{n-k-1}$ near the boundary ∂D^{n-k} and smoothly glue σ_{n-k-1} on the complementary S^{n-k-1} factor given by the syrgery transformation at a distance ρ from S. Now, the squared Dirac operator $\left(D^{g_1 \oplus g_2}\right)^2$ on a product Riemannian manifold

$(M \times N,\, g_1 \oplus g_2)$ is the sum of the squared Dirac operators, the horizontal and vertical parts:

$$\left(D^{g_1 \oplus g_2}\right)^2 = \left(D^{g_1}\right)^2 + \left(D^{g_2}\right)^2.$$

In a cartesian product with spheres S^m, $m \geq 1$, endowed with the standard round metric σ_m and the bounding spin structure, $\left(D^{\sigma_m \oplus g}\right)^2$, as stated in Corollary 1.3.20, has a spectral gap of $\frac{m^2}{4}$, $m = n - k - 1$ being the dimension of the surgery cosphere. This ensures that the L^2-norm of harmonic spinors relative to the "blown-up" metric g_ρ tends to avoid the region close to S and is uniformly bounded.
In both steps the deformation depends on a parameter, respectively δ and ρ, both positive, and taking sequences of metrics modified around S generates sequences of harmonic spinors bounded in $L^2(M)$. The compactness of M and the regularity of solutions of elliptic equations imply that such sequences converge in $\mathcal{C}^1_{\text{loc}}(M \setminus S)$ and S having codimension at least 2 in M allows the limit spinors, by Lemma 1.3.9 to be extended to weak harmonic spinors on the whole of M. Elliptic regularity in Lemma 1.3.6 ensures that the limit is actually smooth and therefore a strong harmonic spinor. Hence the two deformed metrics g_δ and g_ρ have a less or equal amount of harmonic spinors with respect to the starting metric g.
To conclude, notice that any two spin cobordant manifold can be obtained via codimension ≥ 2 surgeries (see Remark 1.2.9) and in any spin cobordism class there is a representative with a D-minimal metric: if $\alpha([M]) = 0$ then by a work of S. Stolz ([Sto92], Theorem B) we can even choose a representative (a bundle with fiber $\mathbb{HP}^1$) with a metric of positive scalar curvature. For $\alpha([M]) = 1$, C. Bär and M. Dahl built in [BD02] n-dimensional manifolds V_n, listed in the following tabular:

n	V_n
1	$\bar{S}^1$,
2	$\bar{S}^1 \times \bar{S}^1$,
4	$K3$ surface,
8	manifold with holonomy $Spin(7)$,
> 8	$V_m \times V_8$,

Table 2.2: Spin manifolds with α=1.

where V_m is one of the manifolds in $\{V_1, V_2, V_4, V_8\}$, with dimension satisfying $m + 8 = n$. We obtain representatives of spin cobordism groups with minimal metrics simply considering connected sum of the manifolds V_n in Table 2.2, according to the value of $\alpha([M])$. Infact, again by Theorem B of [Sto92] we have that the disjoint union of M^n with $|\alpha([M])|$ copies of V_n^- (with the reversed orientation) is spin cobordant to a manifold E with positive scalar curvature. Hence M is spin cobordant to $E\#\underbrace{V_n\#\ldots\#V_n}_{|\alpha([M])|-\text{times}}$, which also has a D-minimal metric.

2.1.2 The work of M. Dahl on $\mathcal{R}^{\text{inv}}(M)$

The first result concerning the topology of the space $\mathcal{R}^{\min}(M^n)$ is due to Dahl [Dah08]: for $\alpha([M^n]) = 0$ and $n = 0, 1, 3, 7 \mod 8$, $n \geq 7$ he showed that the space $\mathcal{R}^{\text{inv}}(M^n)$ is not path connected, obtaining in a different way the existence of harmonic spinors in dimension $0, 1, 7 \mod 8$ by Hitchin [Hit74] and in dimension $3 \mod 4$ by Bär [Bär96].
First of all M. Dahl introduced the notion of invertibility of D^g for manifolds with boundary:

Definition 2.1.3. A metric g on a spin manifold M with boundary ∂M is D-invertible if invertibility for D^g holds in the usual L^2-sense: there exists an $\epsilon > 0$ such that

$$\frac{\|D\varphi\|^2_{L^2(M_\infty)}}{\|\varphi\|^2_{L^2(M_\infty)}} \geq \epsilon, \quad \forall \varphi \in L^2(\Sigma M_\infty)$$

on the manifold M_∞ obtained by attaching an half-infinite cylinder $\big([0,\infty) \times \partial M, dx^2 + g_{|\partial M}\big)$ along the boundary of M.

With this generalized notion of invertibility it is possible to introduce new equivalence relations on the space $\mathcal{R}^{\text{inv}}(M)$:

Definition 2.1.4. [Dah08] Two metrics $g_0, g_1 \in \mathcal{R}^{\text{inv}}(M)$ are said to be:

- *isotopic* if there exists a continous path $g(t)$, $t \in [0,1]$ such that $g(0) = g_0$ and $g(1) = g_1$.
- *concordant* if there exists a metric $\bar{g}$ on $[0,1] \times M$ with product form near the boundary with invertible Dirac operator such that $\bar{g}_{|_{\{i\} \times M}} = g_i$, $i = 0, 1$.

Moreover, two metrics $g_M \in \mathcal{R}^{\text{inv}}(M)$ and $g_{\widetilde{M}} \in \mathcal{R}^{\text{inv}}(\widetilde{M})$ are said to be

- *bordant* if there exists a D-invertible metric g_W, on the bordism W between M and $\widetilde{M}$, with product form near the boundary, such that $g_{W|_{\partial W}} = g_M \sqcup g_{\widetilde{M}}$.

These definitions induce equivalence classes of metrics on M and isotopy implies concordance which in turns implies bordance. We will collect these results in a single proposition:

Proposition 2.1.5. *For D-invertible metrics in $\mathcal{R}^{\text{inv}}(M)$ on a closed spin manifold the following statements hold:*

- *[Dah08, Corollary 2.2] concordance and bordance are equivalence relations,*
- *[Dah08, Corollary 2.4] isotopic metrics are concordant,*
- *[Dah08, Proposition 2.5] given a D-invertible metric g on M, there exists a D-invertible metric g_W on the trace W of a surgery of codimension ≥ 3 such that $g_{W|_M} = g$.*

Then the author, using exotic spheres Σ with $\alpha([\Sigma]) = 1$, creates a family of manifolds $\{Y_i\}_{i \in \mathbb{N}}$ in dimension $n+1$, $n \equiv 0, 1, 3, 7 mod 8$, with boundary diffeomorphic to S^n. Any Y_i has a metric g_i with invertible Dirac operator such that $\alpha([Y_i \cup_{S^n} Y_j]) = c_n(i-j)$, for a suitable nonzero constant c_n.
Now, for any compact spin manifold M^n with D-invertible metric g we consider the trace W_i of the zero dimensional surgery (connected sum) between (M^n, g) and $(M^n \sqcup S^n, g \sqcup g_{i|_{S^n}})$. The manifold W_i is diffeomorphic to the boundary connected sum of $[0,1] \times M^n$ and Y_i. One can repeat the same procedure for $j \neq i$. If g_i is concordant with g_j, we can glue the manifolds W_i and W_j endowing it with a D-invertible metric. But

$$\alpha([W_i \cup W_j^-]) = \alpha([S^1 \times M]) + \alpha([Y_i \cup_{S^n} Y_j]) = c_n(i-j) \neq 0,$$

which is a contradiction. Hence the metrics g_i and g_j are not concordant (in particular not isotopic), which implies:

Theorem 2.1.6. *[Dah08, Theorem 3.3+Corollary 2.4] Let M^n be a closed spin manifold of dimension n. Then*

- *if $n \equiv 3 \mod 4$, $n \geq 7$, the space $\mathcal{R}^{\text{inv}}(M)$ has infinitely many path components.*
- *if $n \equiv 0, 1 \mod 8$, $n \geq 8$, the space $\mathcal{R}^{\text{inv}}(M)$ has at least two path components.*

This result cannot include the case $n = 3$, since in this dimension, using the restrictions imposed by the Gromov-Lawson surgery theorem [GL80, Theorem A], one cannot build the family Y_i.
To overcome such inconvenience, in a subsequent article joint with N. Große [DG14], they showed that, starting with any D-invertible metric g on the manifold with boundary M, there is a procedure to define a D-invertible metric also on the manifold obtained from M via attachment of a handle $D^{k+1} \times D^{n-k}$ along the boundary, provided $n - k \geq 2$:

Theorem 2.1.7. *[DG14] Let M be a spin manifold with boundary and $g \in \mathcal{R}^{\text{inv}}(M)$ (D-invertible in the sense of Definition 2.1.3). Let M'' be obtained via handle attachment of $D^{k+1} \times D^{n-k}$ on M with $n - k \geq 2$. Then for any given neighbourhood U of the surgery sphere $\partial D^{k+1} = S^k \hookrightarrow \partial M$ there is a metric $g'' \in \mathcal{R}^{\text{inv}}(M'')$ such that $g'' = g$ outside U.*

As a consequence the authors are able to define a metric g^W on the trace of a surgery of codimension at least 2, improving [Dah08, Proposition 2.5]; now they are allowed to exploit a similar argument as in [Dah08], defining Y_i as the i-fold connected sum of $K3 \setminus D^4$, which is obtained by gluing 20 2-handles to a disk D^4. Choose an hemisphere metric on the disk and then extend it to $K3 \setminus D^4$ using the new handle attachment procedure. This way each Y_i will be endowed with a metric g_i with invertible Dirac operator, leading to the following theorem:

Theorem 2.1.8. *[DG14, Proposition 4.3] Suppose M is a 3-dimensional closed Riemannian spin manifold, then $\mathcal{R}^{\text{inv}}(M)$ has infinitely many path components.*

2.2 Motivation and strategy to investigate $\mathcal{R}^{\min}(M)$

One of the first application of surgery transformations in geometry is due to M. Gromov and B. Lawson [GL80]:

Theorem 2.2.1. *[GL80, Theorem A] Let M be a compact manifold which carries a Riemannian metric of positive scalar curvature. Then any manifold which can be obtained from M by performing surgeries in codimension ≥ 3 also carries a metric with positive scalar curvature.*

Given a metric g of positive scalar curvature on the closed manifold M, the authors first determine an embedding γ of M in $M \times \mathbb{R}$ that pinches a long neck around the surgery sphere S^k so that for $\gamma^* g$, in the tubular neighbourhood $D^{n-k} \times S^k$, the metric has an $SO(n-k)$ symmetry. If the neck is long enough, there exists a curve $g(t)$ of metrics with positive scalar curvature connecting the induced metric on the boundary of the tubular neighbourhood $S^{n-k-1} \times S^k$ to the product of the standard round metric on S^{n-k-1} of small radius and the induced metric on S^k. We can attach then the cylinder $[0,1] \times S^{n-k-1} \times S^k$ with the metric $dx^2 + g(t/a)$ for a certain value of $a > 0$ to make M ready for the surgery: on a open neighbourhood U of S^k the metric looks like $\sigma(\epsilon) \oplus g_{|_{S^k}}$. Since the scalar curvature of a product metric $g_1 \oplus g_2$ splits as $\text{scal}^{g_1 \oplus g_2} = \text{scal}^{g_1} + \text{scal}^{g_2}$ there exists a value of ϵ such that the manifold obtained by gluing to $M \setminus U$ the complementary $\widetilde{U} \simeq S^{n-k-1} \times D^{k+1}$ still has positive scalar curvature.
Since $\sigma_1(\epsilon)$ carries no metric of positive scalar curvature for any value of the radius ϵ, one has to assume that $n - k - 1 \geq 2$ and hence force the surgery to happen only in codimension $n - k \geq 3$.
As a corollary of [GL80, Theorem A] the authors provide many examples of manifolds admitting a metric of positive scalar curvature by extending the positive scalar curvature property along surgeries of codimension bigger than 3:

Theorem 2.2.2. *Let M be a closed simply connected manifold of dimension ≥ 5. Then*

- *[GL80, Theorem B] if M is spin cobordant to a manifold admitting positive scalar curvature, then also M does,*

- *[GL80, Corollary C] if M is not spin, then it admits a metric of positive scalar curvature.*

Starting with the seminal work of N. Hitchin [Hit74], the topology of the space $\mathcal{R}^{\mathrm{psc}}(M)$ of positive scalar curvature metrics attracted the interest of mathematicians. In recent years, V. Chernysh [Che04] and later M. Walsh [Wal13] proved, by parametrizing the construction of M. Gromov and B. Lawson, that the homotopy type of such space is the same for manifolds obtained by a sequence of surgeries in dimension $2 \leq k \leq n-3$, which are enough to reach any simply connected $n \geq 5$-dimensional spin manifold starting from the sphere S^n:

Theorem 2.2.3. *[Che04, Theorem 1.2][Wal13, Main Theorem] Let M be a closed Riemannian manifold and $S^k \hookrightarrow M$ an embedding with trivial normal bundle. Let $\widetilde{M}$ be a manifold obtained via k-surgery on M. If $n-k \geq 3$ and $k \geq 2$ then $\mathcal{R}^{\mathrm{psc}}(M)$ and $\mathcal{R}^{\mathrm{psc}}(\widetilde{M})$ are homotopy equivalent.*

The fundamental observation both authors start from is analogous to the one lying at the heart of topological surgery: as $\partial(D^{n-k} \times S^k) = S^{n-k-1} \times S^k = \partial(S^{n-k-1} \times D^{k+1})$, it also holds that both metrics $\mathrm{torp}_{n-k} \oplus \sigma_k$ and $\sigma_{n-k-1} \oplus \mathrm{torp}_{k+1}$, where torp_n is the hemisphere metric joined along the equator with a cylinder $([0,1] \times S^{n-1}, dx^2 + \sigma_{n-1})$, restrict to $\sigma_{n-k-1} \oplus \sigma_k$ on the boundary.
Therefore, let M and $\widetilde{M}$ be two manifolds obtained via a surgery on the embedded sphere $\iota : S^k \hookrightarrow M$ of dimension $2 \leq k \leq n-3$. Since $\iota(S^k) = S$ has a trivial normal bundle by hypothesis, we consider the tubular neighbourhood

$$\mathrm{Tub}_k : S^k \times D^{n-k} \hookrightarrow M.$$

Then the two spaces

$$\mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(M) := \left\{ g \in \mathcal{R}^{\mathrm{psc}}(M) \,\middle|\, \mathrm{Tub}_k^*(g) = \mathrm{torp}_{n-k} \oplus \sigma_k \right\}$$

and

$$\mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(\widetilde{M}) := \left\{ g \in \mathcal{R}^{\mathrm{psc}}(\widetilde{M}) \,\middle|\, \mathrm{Tub}_{n-k-1}^*(g) = \mathrm{torp}_{k+1} \oplus \sigma_{n-k-1} \right\}$$

are easily seen to be homeomorphic: it is sufficient to exchange the torpedo part with the standard round one.
Adapting the construction of M. Gromov and B. Lawson to work also for families of metrics parametrized by a compact space, it is shown, using Whitehead theorem and an accurate exam of positive scalar curvature metrics on the disk D^{n-k}, that $\mathcal{R}^{\mathrm{psc}}(M)$ is homotopy equivalent to the subspace $\mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(M)$. A composition of this homotopy with the homeomorphism above yields

$$\mathcal{R}^{\mathrm{psc}}(M) \simeq \mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(M) \simeq \mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(\widetilde{M}) \simeq \mathcal{R}^{\mathrm{psc}}(\widetilde{M}).$$

The equivalent of the geometric surgery in [GL80] in the case of D-minimal metrics is presented in [ADH09], where the authors extend a D-minimal metric along a surgery of codimension at least 2. The codimension can be reduced by one because the surgery cosphere S^{n-k-1} with the bounding spin structure has an invertible Dirac operator, which allows to define sequence of metrics with L^2-bounded harmonic spinors on compacts that exhaust M.
The aim of this PhD thesis is to propose a strategy and prove several steps along the way to obtain a parametrized version of Ammann-Dahl-Humbert construction as it was done for positive scalar curvature by V. Chernysh [Che04] and M. Walsh [Wal13], parametrizing the construction of M. Gromov and B. Lawson [GL80]. This would have the advantage of allowing surgeries in dimension $1 \leq k \leq n-2$, which are sufficient to reach any element $\widetilde{M}$ in a spin cobordism class $[M]$ starting from a fixed representative of the class M (Remark 1.2.9). It will then be possible to use the spin cobordism invariance of the homotopy type of $\mathcal{R}^{\mathrm{inv}}(M)$ to investigate the existence of metrics with harmonic spinors.

The major difficulty in extending the result of V. Chernysh and M. Walsh relating the homotopy type of the space of metrics with positive scalar curvature for cobordant manifolds is that the number of harmonic

spinors cannot be computed in terms of local informations: by means of modifications of the metric in an arbitrarily small neighbourhood of a point it is possible to both increase the dimension of the kernel of the Dirac operator (see [Hit74], [Bär96] or decrease it [ADH11]). On the other side, scalar curvature is a smooth function $\mathrm{scal}^g : M \to \mathbb{R}$ and its value at a point depends only on the local behaviour (up to 2nd derivative) of the metric g. In many cases, in particular for metrics with many isometries, the formula to compute scalar curvature simplifies considerably. Such situation has no analogue for the D-invertibility case.
In conclusion, whenever a perturbation of the metric in a small neighbourhood of a point is applied, one has to check that the newly defined Dirac operator acting on the spinor bundle (both change together with the metric) has the same number of eigespinors relative to the eigenvalue 0.
An example of this local/global difference is the comparison of the space of metrics with "standard" form around the surgery sphere for topologically different manifolds: for metrics whose scalar curvature is positive everywhere, starting from a metric of standard form near the surgery sphere S, swapping the torpedo factor with the spherically round one will preserve the positivity of the scalar curvature. On the other hand, just exchanging the torpedo factor with the spherically round one won't ensure that the resulting Riemannian metric on $\widetilde{M}$ will have an invertible Dirac operator, even though both the torpedo metric on the disk and the sphere with the standard round metric have no harmonic spinors.
We then have to use an expedient coming from conformal geometry: we can blow up the metric around the submanifold S, but fixing the cylindrical behaviour at infinity.
Let K, $\widetilde{K}$ be compact subsets of $M \setminus S$ and $\widetilde{M} \setminus \widetilde{S}$ respectively. From the chain of isometries

$$(M \setminus S) \setminus K \cong [0, \infty) \times S^{n-k-1} \times S^k \cong \left(\widetilde{M} \setminus \widetilde{S}\right) \setminus \widetilde{K}$$

we see that the spaces

$$\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}\left(M \setminus S\right) := \left\{ g \in \mathcal{R}^{\mathrm{inv}}\left(M \setminus S\right) \,\middle|\, \exists K \subset M \setminus S,\ g_{|_{(M\setminus S)\setminus K}} = du^2 + \sigma_k + \sigma_{n-k} \right\}$$

and

$$\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}\left(\widetilde{M} \setminus \widetilde{S}\right) := \left\{ g \in \mathcal{R}^{\mathrm{inv}}\left(\widetilde{M} \setminus \widetilde{S}\right) \,\middle|\, \exists \widetilde{K} \subset \widetilde{M} \setminus \widetilde{S},\ g_{|_{(\widetilde{M}\setminus \widetilde{S})\setminus \widetilde{K}}} = du^2 + \sigma_{n-k} + \sigma_k \right\}$$

are clearly homeomorphic, as can be seen by removing the common cylinder contained in the complementary of K and $\widetilde{K}$. They play the role of the spaces $\mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}\left(M\right)$, $\mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}\left(\widetilde{M}\right)$ defined at in the previous page for positive scalar curvature metrics.
Now that we detected homeomorphic subspaces of D-invertible metrics on $M \setminus S$ and $\widetilde{M} \setminus \widetilde{S}$, we are left to prove that the entire $\mathcal{R}^{\mathrm{inv}}\left(M\right)$ has the same homotopy type of $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$.
For this, we would like to exploit the long exact sequence in homotopy of pairs, but unfortunately $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \nsubseteq \mathcal{R}^{\mathrm{inv}}(M)$. We then have to break the proof in two parts, passing through intermediate spaces: in the first parte of the proof we enlarge the local isometry group (radial symmetry) of each metric $g \in \mathcal{R}^{\mathrm{inv}}(M)$ in the normal direction to the embedded surgery sphere S, provided we perform such change only in a small tubular neighbourhood of such S, whose polar coordinates are defined by the diffeomorphism $\mathrm{Tub}_k : D^{n-k} \times S^k \to M$. This is the space of half-flat metrics

$$\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}} := \left\{ g \in \mathcal{R}^{\mathrm{inv}}(M) \,\middle|\, \mathrm{Tub}^*_k(g) = \left(f'(s)\right)^2 ds^2 + f(s)^2 \sigma_{n-k} \oplus g_{|_S} \right\}. \qquad (2.2.1)$$

Here $f(s)$ is a positive smooth odd function with $f(0) = 0$ and $f'(0) = 1$ which equals s sufficiently close to S and $s + c$, for a positive constant c, sufficiently close to $\partial \mathrm{Tub}_k(D^{n-k \times S^k})$. It is immediate to see that under the change of coordinates

$$r = f(s)$$

the metric takes the form $dr^2 + r^2 \sigma_{n-k-1} \oplus g_{|_S} = \mathrm{flat} \oplus g_{|_S}$.

Consider the long exact sequence of the pair $\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)$, coming from the short exact sequence

$$\pi_*\left(\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{i^*} \pi_*\left(\mathcal{R}^{\text{inv}}(M)\right) \xrightarrow{p^*} \pi_*\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right),$$

that is:

$$\cdots \xrightarrow{\partial_{l+2}} \pi_{l+1}\left(\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{i^*} \pi_{l+1}\left(\mathcal{R}^{\text{inv}}(M)\right) \xrightarrow{p^*}$$

$$\pi_{l+1}\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{\partial_{l+1}} \pi_l\left(\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{i^*} \pi_l\left(\mathcal{R}^{\text{inv}}(M)\right) \xrightarrow{p^*}$$

$$\pi_l\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{\partial_l} \pi_{l-1}\left(\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) \xrightarrow{i^*} \cdots$$

By the very definition $\pi_l\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) = \left[\left(D^l, S^{l-1}\right), \left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)\right]$ and if one can prove that any map $f_l : \left(D^l, S^{l-1}\right) \to \left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)$ can be continuously shrunk to a map $F_l : D^l \to \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ (i.e. we can build a nullhomotopy of pairs H) then it would hold

$$\pi_l\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right) = 0 \quad \forall l \in \mathbb{N} \Leftrightarrow i^* \text{ is an isomorphism.}$$

We hence define a parametrized version of Ammann-Dahl-Humbert construction on the pair $\left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)$, to show that the inclusion is the required weak homotopy equivalence.
By the definition of nullhomotopy of pairs, we have to fullfill the following requirements for any $l \in \mathbb{N}$:

- the map $H \circ f_l : (D^l, S^{l-1}) \times [0,1] \to \left(\mathcal{R}^{\text{inv}}(M), \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)$ has to be continuous,
- $H(\cdot, 0) = \text{id}$,
- elements of the family parametrized by the boundary $S^{l-1} = \partial D^l$ never leave the subspace $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ on which we want to "squeeze" our family of metrics, i.e. $H(f_l(S^{l-1}), t) \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$, $\forall t \in [0,1]$.

All this properties are satisfied by linear combinations of metrics, allowed by the affine structure of the space $\mathcal{R}iem(M)$ (Remark 1.4.12). We will see that most of the homotopies that will appear below will be of this kind.
The abovementioned construction of nullhomotopies of pairs would just prove the inclusion $i : \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \hookrightarrow \mathcal{R}^{\text{inv}}(M)$ to be a weak homotopy equivalence. The topology of the space $\mathcal{R}iem(M)$ and its open subsets is not so far from the one of a CW-complex, in particular $\mathcal{R}iem(M)$ is dominated by a CW-complex [Pal66] and as such satisfies the hypothesis of Whitehead theorem: any weak homotopy equivalence is also a strong one.
For any value value $R > 0$ smaller than the injectivity radius of a metric g, we define the tubular neighbourhood $U_S(R)$ of S using the distance induced on M by the metric g:

$$U_S(R) := \{p \in M \mid d_g(p, S) \leq R\}.$$

The map $\mathbf{ADH}_\delta$ that we will present as parametrization of the Ammann-Dahl-Humbert construction will be shown to preserve the space

$$\overline{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}} := \left\{ g \in \mathcal{R}^{\text{inv}}(M) \,\middle|\, \exists \delta > 0,\ g_{|_{U_S(\delta)}} = \text{flat} \oplus g_{|_S} \right\}$$

rather than $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ as desired; nevertheless fixing a background metric and rescaling the exponential map $\exp^{\perp}_g$ through a smooth radial function on the disk provides an homeomorphism

$$\overline{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M) \cong \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M).$$

Even though the path seems to be established to reach the result, it is left the hard task of modifying a metric in $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ to one in $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ in a continuous way; since it does not form a pair of spaces the long exact sequence method is useless. This ulterior inconvenient can be overcome passing through an *ad hoc* space $\mathcal{R}$ containing both $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ and $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ as subspaces and using it as a second waypoint.
We have then to create a nullhomotopy of pairs $\left(\mathcal{R}, \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right)$ that generates an infinite cylindrical neck on the disk factor (which, in the space $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ reads as a multiplication by a conformal function $F(r)$ which is constantly 1 outside a small neighbourhood of S) and leaves invariant $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$.
Another nullhomotopy of pairs will be created to continuously "cap off" the cylindrical part of a metric $g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ to show that $\mathcal{R}$ is weakly homotopy equivalent to $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$. Such continuous deformation will also preserve metrics which are already of half-flat form, but we will not fully manage to show that compact families of metrics in $\mathcal{R}$ can be continuously deformed to asymptotically cylindrical ones.
Supposing that $\mathcal{R} \simeq \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ holds weakly, we obtain that $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ is weakly homotopy equivalent to $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ and concatenating the weak homotopy equivalences obtained, we would write the chain of homotopy equivalences as

$$\mathcal{R}^{\text{inv}}(M) \simeq \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \simeq \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S). \tag{2.2.2}$$

The chain can be prolonged on the right observing that any surgery of dimension k transforming M in $\widetilde{M}$ can be reverted by performing on $\widetilde{M}$ a surgery on an embedded sphere $\widetilde{S}$ with trivial normal bundle of dimension $n-k-1$:

$$\begin{aligned} M^n \cong & \big(\overbrace{M^n \setminus \bar{\iota}_1(S^k \times D^{n-k}) \underset{S^k \times S^{n-k-1}}{\cup} D^{k+1} \times S^{n-k-1}}^{\cong \widetilde{M}^n}\big) \\ & \setminus \bar{\iota}_2(S^{n-k-1} \times D^{k+1}) \underset{S^k \times S^{n-k-1}}{\cup} D^{n-k} \times S^k \end{aligned}$$

Since both surgeries have to satisfy the condition on the codimension greater or equal than two to make the chain of homotopy equivalences (2.2.2) hold, we have that

$$\begin{cases} n-k \geq 2 \\ n-(n-k-1) \geq 2 \end{cases} \quad \Leftrightarrow \quad 1 \leq k \leq n-2,$$

as aforementioned in Remark 1.2.9, these are all the surgeries that we need to span the entire spin cobordism class of the manifold M.
We finally arrive to the statement of the conjecture:

Conjecture 2.2.4. *Let M be an $n \geq 3$ closed spin manifold. Then the homotopy type of $\mathcal{R}^{\text{inv}}(M)$ is a spin cobordism invariant.*

Albeit the proof of Conjecture 2.2.4 is incomplete, we provide along the way a partial result, which is the main theorem of this thesis:

Theorem 2.2.5. *Let M^n be a closed spin manifold and $N^k \hookrightarrow M^n$ a closed submanifold of codimension $n-k$ at least 2 and with trivial normal bundle. Define*

$$\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M, N) := \left\{ g \in \mathcal{R}^{\min}(M^n) \,\middle|\, \text{Tub}^*_N(g) = \text{flat} \oplus g_{|_N} \right\}$$

for a fixed tubular neighbourhood $\text{Tub}_N : D^{n-k} \times N \to M$ *of* N*. Then*

$$\mathcal{R}^{\min}(M) \simeq \mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M, N).$$

In dimension 4 and 5, where the conjecture on the existence of metrics with harmonic spinors remains open, there are clues that any closed spin manifold $M^{4,5}$ can be endowed with such metric. Four dimensional spheres admits metrics with harmonic spinors [See00] as well as five dimensional ones [Dah08, Corollary 4.2], which makes sensible the quest for nontrivial homotopy classes of the space $\mathcal{R}^{\text{inv}}(S^{4,5})$. Provided Conjecture 2.2.4 is true, the conjecture about existence of metrics with harmonic spinors in any dimension could be proved to be true.

2.3 The homotopy equivalence $\mathcal{R}^{\min}(M) \simeq \mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M)$

Throughout this section the pair (M, g) will always denote a closed connected spin manifold M endowed with the Riemannian metric g, $\Sigma^g M$ will denote the associated complex spinor bundle. We will indicate with ∇^g and D^g the lift of the Levi-Civita connection to the spinor bundle and the Dirac operator, respectively. Some results will apply more generally for manifolds with nontrivial α invariant and in this case we will consider the space of D-minimal metrics on M. For such results it will appear the space $\mathcal{R}^{\min}$ instead of $\mathcal{R}^{\text{inv}}$ in the relative statements. The aim is to show that the space of Riemannian metrics with minimal kernel $\mathcal{R}^{\min}(M)$ of a spin manifold M is homotopy equivalent to the subspace of metrics which are of product form with $SO(n-k)$-symmetry on a fixed compact set around the embedded surgery sphere S, i.e. on the space $\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M)$. To do this, we have to build nullhomotopies for any map of pairs $(D^l, S^{l-1}) \to \left(\mathcal{R}^{\min}(M), \mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M)\right)$ for all $l \in \mathbb{N}$. These nullhomotopies will first flatten the metric g in the normal direction to the embedded surgery sphere S. The map will be well-defined provided the flattening is performed only sufficiently close to S. Then we will proceed rescaling the radial distance function $r = d_g(\cdot, S)$, so that the deformation will take place on a fixed neighbourhood of S and will not depend on the metric g anymore.

Fix an isometric embedding $\iota : S^k \to M$ with trivial normal bundle $\nu_S : TS^\perp \subset TM_{|S} \to S$ and denote $S = \iota(S^k)$. With an abuse of notation call $\iota : D^{n-k} \times S^k \to \nu_S$ also the trivialization of the normal bundle of S.

Consider the set of points

$$U_{S,g}(\delta) := \{p \in M \mid d_g(p, S) \leq \delta\},$$

where $d_g(\cdot, S)$ is the distance (induced from the metric g) from the embedded surgery sphere S. The dependence from the metric g will not be specified whenever a unique metric is involved or the metric considered is clear from the context.

For values of δ smaller than the injectivity radius of g and with the help of the exponential map $\exp^\perp$ restricted to the normal bundle ν_S of the submanifold S we define the diffeomorphism $\exp^\perp \circ \iota : D^{n-k} \times S^k \to U_S(\delta)$. In [ADH09] the metric

$$g_\delta = \eta_\delta((\exp^\perp \circ \iota)^{-1})^*(\text{flat} \oplus g_{|S}) + (1 - \eta_\delta) g$$

is defined, where the term flat is used to address the euclidean metric on D^{n-k} and $\eta_\delta : M \to [0, 1]$ is a smooth function s.t.

$$\eta_\delta(p) = \begin{cases} 1, & p \in U_{S,g}(\delta), \\ 0, & p \in M \setminus U_{S.g}(2\delta). \end{cases} \qquad |\nabla \eta_\delta|_g \leq \frac{2}{\delta}.$$

We define the space of half-flat D-minimal metrics with varying size to be

$$\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M) := \left\{ g \in \mathcal{R}^{\min}(M) \,\middle|\, \exists \delta > 0,\ g_{|_{U_S(\delta)}} = \text{flat} \oplus g_{|_S} \right\}. \tag{2.3.1}$$

Given a compact space B and a continuous map $\theta : B \hookrightarrow \mathcal{R}^{\min}(M)$, $\theta(b) = g_b$, we introduce the map

$$\begin{aligned} \mathbf{ADH}_{\delta,B} \circ (\theta \times \mathrm{id}_{[0,1]}) : B \times [0,1] &\longrightarrow \mathcal{R}^{\min}(M), \\ (b,t) &\longmapsto (1 - t\eta_\delta) g_b + t\eta_\delta(((\exp^\perp \circ \iota)^{-1})^* \text{flat} \oplus g_{|_S}) \end{aligned}$$

which will be called Ammann-Dahl-Humbert nullhomotopy and it is immediate to see that for all $g \in \mathcal{R}^{\min}(M)$ and $\delta > 0$

$$\begin{aligned} &\mathbf{ADH}_{\delta,B}(g,0) = g, \\ &\mathbf{ADH}_{\delta,B}(g,1) \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M)}. \end{aligned}$$

The subscript B will be dropped whenever clear from the context.

The properties that the maps $\mathbf{ADH}_{\delta,B}$ have to satisfy in order to be nullhomotopies are, by the very definition:

(*i*) the map $\mathbf{ADH}_{\delta,B}$ is well defined, i.e. for any value of $t \in [0,1]$ and any $b \in B$, $\mathbf{ADH}_{\delta,B}(g_b,t) \in \mathcal{R}^{\min}(M)$, hence it never leaves the subspace $\mathcal{R}^{\min}(M) \subset \mathcal{R}iem(M)$,

(*ii*) the map $\mathbf{ADH}_{\delta,B}(g_b,t)$ is continuous in $t \in [0,1]$ and $b \in B$,

(*iii*) whenever $B = (D^l, S^{l-1})$, metrics in the family $\{g_b\}$ parametrized by points on the sphere remain in $\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}$, $\forall t \in [0,1]$.

In the next two subsections the properties $(i) - (iii)$ will be verified.

2.3.1 Property (i) of the map ADH

We will start proving that, for a suitable value of $\delta > 0$ (in general depending on the compact space B) the nullhomotopy $\mathbf{ADH}_{\delta,B}$ never leaves the space of D-minimal metrics. Useful lemmas for the proof of property (ii) and (iii) will appear throughout this subsection and hence the proofs of the latter two properties will be postponed.
In [ADH09] the authors define

$$g_\delta := \eta_\delta \left((\exp^\perp \circ \iota)^* \right)^{-1} \left(\text{flat} \oplus g_{|_S} \right) + (1 - \eta_\delta) g,$$

which coincides with $\mathbf{ADH}_\delta(g,1)$, and prove the following proposition:

Proposition 2.3.1. *[ADH09, Proposition 3.2] The sequence g_{δ_i} for $\delta_i \to 0$ when $i \to \infty$ contains a metric g_δ with $\dim\ker D^{g_\delta} \leq \dim\ker D^g$.*

The proof relies on the fact that the modification, even though not $\mathcal{C}^1$-small, happens only in a small tubular neighbourhood of the surgery sphere S, in fact it holds:

Lemma 2.3.2. *[ADH09, Lemma 3.1] Given a Riemannian metric g on M, for sufficiently small $\delta > 0$ there exists a constant $C = C(g)$ so that on $U_S(2\delta)$ it holds*

$$\|g - g_\delta\|_{\mathcal{C}^0} \leq C\delta \qquad \|\nabla(g - g_\delta)\|_{\mathcal{C}^0} \leq C.$$

Our first task is to analogously find a δ such that the nullhomotopy $\mathbf{ADH}_\delta$ is well defined.
In our case, when working with compact families of metrics instead of a single one, one would expect that δ depends also on the metric in the family g_b and the variable t; nevertheless we will prove in Proposition 2.3.6 that we can choose a single value of δ for any compact family of metrics $\{g_b\}_{b\in B}$, i.e. δ is independent of $b \in B$ and $t \in [0,1]$.
Let us first prove that the result in Proposition 2.3.1 extends to the path of metrics $\mathbf{ADH}_{\delta,B}(g_b,t)$:

Proposition 2.3.3. *For any $t \in [0,1]$, $b \in B$, there exists a positive number $\delta = \delta(b,t)$ such that, for all $\delta' < \delta$ the metric $\mathbf{ADH}_\delta(g_b,t)$ belongs to $\mathcal{R}^{\min}(M)$.*

Proof. The proof will mimic the one of [ADH09, Lemmata 3.3 and 3.4]. From now on, for notation convenience, we will drop the notation of the parameter $b \in B$ and simply write g. Consider a sequence $\delta_i \to 0$ as $i \to \infty$ and write $\mathbf{ADH}_{\delta_i}(g,t)$ for the metric

$$(1 - t\eta_{\delta_i})g + t\eta_{\delta_i}((\exp^\perp \circ\iota)^{-1})^*\mathrm{flat} \oplus g_{|S}).$$

First of all, we choose a sequences of harmonic spinors φ_i where for each value of the index i, the spinor $\varphi_i \in \Gamma(\Sigma^{\mathbf{ADH}_{\delta_i}(g,t)}M)$ satisfies $D^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i = 0$, with $\displaystyle\int_M |\varphi_i|^2 dv^{\mathbf{ADH}_{\delta_i}(g,t)} = 1$.
Since for any value of i the harmonic spinors φ_i belongs to a different spinor bundle, we will have to use a reference spinor bundle $\Sigma^g M$ where comparing them and looking at their limit spinors. Recall from (1.3.3) that for any i there exist maps $\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)}$ which are spinor bundle isomorphism between $\Sigma^g M$ and $\Sigma^{\mathbf{ADH}_{\delta_i}(g,t)}M$.
Notice that, from Lemma 2.3.2 we have, for a certain value of the constant C and any value of δ, t:

$$\begin{aligned} \|g - \mathbf{ADH}_{\delta_i}(g,t)\|_{\mathcal{C}^0(M)} &= t\|g - g_{\delta_i}\|_{\mathcal{C}^0(M)} \le tC\delta_i, \\ \|\nabla^g(g - \mathbf{ADH}_{\delta_i}(g,t))\|_{\mathcal{C}^0(M)} &= t\|\nabla^g(\eta_{\delta_i}(g - g_{\delta_i}))\|_{\mathcal{C}^0(M)} \\ &\le \frac{2t}{\delta_i}\|g - g_{\delta_i}\|_{\mathcal{C}^0(M)} + t\|\nabla^g(g - g_{\delta_i})\|_{\mathcal{C}^0(M)} \le 2Ct. \end{aligned} \tag{2.3.2}$$

The value of $C = C(g_b)$, being defined as $\max_{p\in U_S(\delta)} |\nabla(g_b(p) - g_{b,\delta}(p))|$ is a continuous function of g_b and hence in particular varies continuously with respect to $b \in B$, but since the latter is a compact topological space, we can choose the maximum: $C = \max_{b\in B} C(g_b)$.
Now we can proceed showing that the sequence $\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i$ is bounded in $H^1(\Sigma^g M)$.
By contradiction, suppose that the sequence $\{\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i\}$ diverges in $H^1(\Sigma^g M)$, which means that in particular the sequence of the H^1-norms

$$\alpha_i := \sqrt{\int_M |\nabla^g(\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i)|_g^2 dv^g}$$

diverges for $i \to \infty$. Define $\psi_i := \alpha_i^{-1}\beta_g^{g_{\delta_i}}\varphi_i$; we have, using (1.3.1) that:

$$\begin{aligned} {}^g D^{\mathbf{ADH}_{\delta_i}(g,t)}\psi_i &= \beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} D^{\mathbf{ADH}_{\delta_i}(g,t)} \left(\beta^g_{\mathbf{ADH}_{\delta_i}(g,t)}\alpha_i^{-1}\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i\right) \\ &= \alpha_i^{-1}\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} D^{\mathbf{ADH}_{\delta_i}(g,t)}\varphi_i = 0, \end{aligned}$$

where we have used the fact that $\beta^g_{\mathbf{ADH}_{\delta_i}(g,t)} \circ \beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} = \mathrm{Id}_{\Sigma^{\mathbf{ADH}_{\delta_i}(g,t)}M}$.

We then get, using Schrödinger-Lichnerowicz formula (1.3.7) and equation (1.3.2),

$$\begin{aligned}
1 &= \int_M |\nabla^g \psi_i|^2 dv^g \\
&= \int_M \left(|D^g \psi_i|^2 - \frac{1}{4}\mathrm{scal}^g |\psi_i|^2 \right) dv^g \\
&\leq \int_M \left(2|{}^g D^{\mathbf{ADH}_{\delta_i}(g,t)} \psi_i|^2 + 2|A^g_{\mathbf{ADH}_{\delta_i}(g,t)} \nabla^g \psi_i|^2 + 2|B^g_{\mathbf{ADH}_{\delta_i}(g,t)} \psi_i|^2 \right) dv^g + C \int_M |\psi_i|^2 dv^g
\end{aligned}$$

The first integral can be estimated using the equations (1.3.3) and subsequently (2.3.2). Moreover, the $\mathcal{C}^0$ convergence of $\mathbf{ADH}_{\delta_i}(g,t)$ to g for $\delta_i \to 0$ and all $t \in [0,1]$ implies that also the volume $dv^{\mathbf{ADH}_{\delta_i}(g,t)}$ and the spinor bundle isomorphism $\beta^g_{\mathbf{ADH}_{\delta_i}(g,t)}$ will tend to dv^g and the identity respectively.
Hence we can write $\int_M |\psi_i|^2 dv^g = \alpha_i^{-2}(1 + \mathbf{o}_i(1))$ and substitute it in the estimate

$$\begin{aligned}
1 &\leq 2Ct^2\delta_i^2 \int_{U_S(2\delta_i)} |\nabla^g \psi_i|^2 dv^g + 2Ct^2 \int_{U_S(2\delta_i)} |\psi_i|^2 dv^g + C\alpha_i^{-2}(1 + \mathbf{o}_i(1)) \\
&\leq 2C\delta_i^2 + \alpha_i^{-2}(C + \mathbf{o}(1)) \to 0,
\end{aligned}$$

which is absurd.
Since for every sequence of $\delta_i \to 0$, the sequence $\{\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} \varphi_i\}_i$ is bounded in $H^1(\Sigma^g M)$, a subsequence $\{\beta_g^{g(t)_{\delta_{i_k}}} \varphi_{i_k}\}_k$ converges weakly in $H^1(\Sigma^g M)$ and, by the compactness of M and Sobolev embedding theorem [Jos10, Appendix A], a subsequence converges strongly in $L^2(\Sigma^g M)$. Fix an $\epsilon > 0$, then, for i big enough (so that $\epsilon > \delta_i$) the metrics g and $\mathbf{ADH}_{\delta_i}(g,t)$ will coincide on $M \setminus U_S(\epsilon)$, since $\eta_{\delta_i} \equiv 0$ on $M \setminus U_S(2\delta_i)$. By Lemma 1.3.8 we have that the spinor $\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} \varphi_i$ will belong also to $\mathcal{C}^2(M \setminus U_S(\epsilon))$ for any $\epsilon > 0$ and in $\mathcal{C}^1_{\mathrm{loc}}(M \setminus U_S(2\epsilon))$ by Ascoli-Arzéla theorem [Jos10, Appendix A].
The limit spinor Φ will be in $\mathcal{C}^1_{\mathrm{loc}}(M \setminus S)$ and hence satisfies the equation $D^g\Phi = 0$ on $M \setminus S$. Moreover Φ is an L^2 spinor of M and by Lemma 1.3.9 is everywhere (on all of M) weakly harmonic and smooth by Lemma 1.3.6.
To conclude, let $m := \liminf_i \dim \ker D^{\mathbf{ADH}_{\delta_i}(g,t)}$. Choose δ such liminf is attained at. Then there are $\varphi_\delta^1, \dots, \varphi_\delta^m$ in $\ker D^{\mathbf{ADH}_\delta(g,t)}$ such that

$$\int_{M\setminus S} (\varphi_\delta^j, \varphi_\delta^k) dv^{\mathbf{ADH}_\delta(g,t)} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Hence there are m sequences $\{\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} \varphi_i^j\}_{j=1,\dots,m}$ in $H^1(\Sigma^g M)$, with $\delta_i \to 0$, and harmonic spinors $\Phi^1, \dots, \Phi^m$ for the operator D^g such that for any $j = 1, \dots, m$, $\beta_g^{\mathbf{ADH}_{\delta_i}(g,t)} \varphi_i^j \to \Phi^j$ strongly in $L^2(\Sigma^g M)$. Such convergence preserves orthonormality of the limit spinors and linear independence: the relations are expressed by an integral over M and the limit spinors Φ^j satisfy such expressions on $M \setminus S$, S having n-dimensional measure zero.
This proves that for the chosen value of δ, $\dim \ker D^{\mathbf{ADH}(g,t)_\delta} \leq \dim \ker D^g$, but since $g \in \mathcal{R}^{\min}(M)$, we must have

$$\dim \ker D^{\mathbf{ADH}(g,t)_\delta} = \dim \ker D^g.$$

It is left to prove that any smaller $\delta' < \delta$ makes $\mathbf{ADH}_{\delta'}(g,t)$ D-minimal, i.e.

$$\exists \delta > 0 \text{ such that } \forall \delta' < \delta, \quad \dim \ker D^{\mathbf{ADH}_{\delta'}(g,t)} = m,$$

where m is, as before, the minimal dimension of the kernel (2.1.1) given by the α-invariant.
We will prove the statement by contradiction: suppose that

$$\forall \delta > 0,\ \exists \delta' < \delta \text{ such that } \dim\ker D^{\mathbf{ADH}_{\delta'}(g,t)} > m, \tag{2.3.3}$$

this means that for any sequence δ_i, the set of δ_i's between two consecutive values which give a lim inf in the sequence $D \dim\ker D^{\mathbf{ADH}_{\delta_i}(g,t)}$ contains a value δ' for which the kernel is not minimal. Also the sequence of such δ''s, indicated with δ'_{j_j} will converge to zero as $j \to \infty$ by (2.3.3) and, for all j, $\dim\ker D^{\mathbf{ADH}_{\delta'_j}} > m$. The lower bound is manteined in the limit, implying that $\liminf_j \dim\ker D^{\mathbf{ADH}_{\delta'_j}} > m$. If we apply the same diagonal sequence argument as in the first part of the proof, we get that

$$m < \dim\ker D^{\mathbf{ADH}_{\delta'_i}} \leq \dim\ker D^g = m,$$

which is a contradiction.

□

Remark 2.3.4. For families of metrics $\{\mathbf{ADH}_\delta(g_b,t)\}_{b\in B, t\in[0,1]}$, the choice of δ from Proposition 2.3.3 will depend in general on the metric considered and the value of t, hence $\delta = \delta(g_b,t)$.

Nevertheless, we will show that $\delta(g_b,t)$ can be chosen to be constant on $B \times [0,1]$. For this we first need the following estimate, showing that the same δ "fits" for nearby metrics:

Lemma 2.3.5. *For any two metrics g_1, $g_2 \in B \subset \mathcal{R}^{\min}(M)$ compact and values t_1, $t_2 \in [0,1]$ it holds*

$$\|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(M)} \leq C\left(|t_1 - t_2| + \|g_1 - g_2\|_{\mathcal{C}^\infty(M)}\right), \tag{2.3.4}$$

for a constant $C = C(B,M,S)$ and $\delta = \delta(g_1,t_1)$ being determined as in Proposition 2.3.3 for (g_1,t_1).

Proof. The proof of this lemma is just a careful computation of the distance of the image under the map $\mathbf{ADH}_\delta$ of two metrics g_1 and g_2. The estimate will rely on the fact that composition of maps and pullbacks are continuous in the compact open $\mathcal{C}^\infty$-topology, as stated in Lemma 1.4.9. We just have to pay attention to the fact that the sets $U_{S,g_1}(\delta(g_1,t_1))$ and $U_{S,g_2}(\delta(g_1,t_1))$ are in general different, since the two metrics g_1 and g_2 induce different distance functions $d_{g_1}(\cdot,S)$, $d_{g_2}(\cdot,S)$. Nevertheless, the difference $|d_{g_1}(\cdot,S) - d_{g_2}(\cdot,S)|$ is small whenever g_1 and g_2 are $\mathcal{C}^1$-close (in particular it happens for $\mathcal{C}^\infty$-close metrics), by Proposition 1.4.10. The distance of $\mathbf{ADH}_{\delta(g_1,t_1)}(g_1,t_1)$ from $\mathbf{ADH}_{\delta(g_1,t_1)}(g_2,t_2)$ can be then computed dividing M into regions according to the value of the cut-off function $\eta_{\delta(g_1,t_1)}$. To make the notation a little lighter, we fix from now on throughout this proof $\delta = \delta(g_1,t_1)$, see Remark 2.3.4.
We will then refer to $\eta_{i,\delta}$ for the cut-off function which is identically 1 on $U_{S,g_i}(\delta)$ and 0 on $M \setminus U_{S,g_i}(2\delta)$, with $i = 1,2$. Notice once more that the distance from the embedded surgery sphere S is not the same in general for different metrics and as a consequence also the support of the cut-off functions $\eta_{i,\delta}$, $i = 1,2$ will change accordingly. This difference separates M in seven regions, schematically depicted in Figure 2.1 below:

The estimate (2.3.4) will be obtained considering the contribution in any single region:

$$\|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(M)} = \max\left\{\|\ldots\|_{\mathcal{C}^\infty(i)}\right\},\ i = \mathrm{I},\ldots,\mathrm{VII}.$$

Let's then study the distance $\|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(M)}$ case by case (notice that the choice

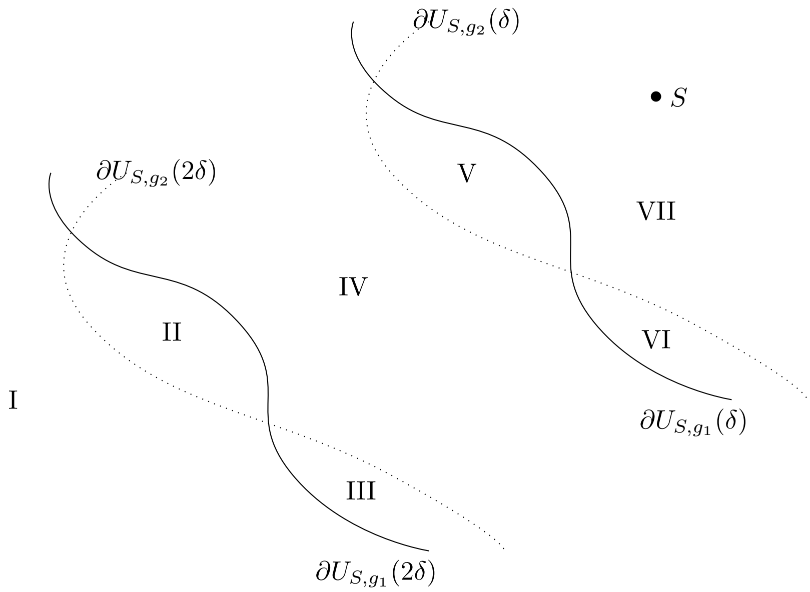


Figure 2.1: Different δ-neighbourhoods for different metrics g_1, g_2.

of t does not affect the width of the modification of the metric). We will define for shorthand notation

$$E(i) := \|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(i)}\,, \quad i = \{\mathrm{I},\ldots,\mathrm{VII}\}$$

and we will also introduce the positive constant C, which might vary its value from line to line, but always independent of the metric. This is made possible by the fact that both t and $g_{1,2}$ are parametrized by compact spaces. Therefore, whenever a quantity proportional to a continuous function $C(t,g_1,g_2)$ of t or $g_{1,2}$ appears (e.g. $t\,\|g_1\|_{\mathcal{C}^\infty(M)}$) in an estimate, we can bound $C(t,g_1,g_2)$ from above by its maximum attained on $B\times[0,1]$, making $C = \max\limits_{B\times[0,1]} C(t,g_1,g_2)$ independent of t, $g_{1,2}$.
Cases I and VII are easy to deal with, since the cutoff functions $\eta_{i,\delta}$ are both either 0 or 1. Hence, concerning region I

$$\|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(\mathrm{I})} = \|g_1-g_2\|_{\mathcal{C}^\infty(\mathrm{I})} \leq \|g_1-g_2\|_{\mathcal{C}^\infty(M)}$$

while in region VII

$$\left\|t_1\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right) - t_2\left(\left(\exp^\perp_{g_2}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_2{}_{|S}\right)\right\|_{\mathcal{C}^\infty(\mathrm{VII})}$$

adding and subtracting

$$t_2\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right) + t_1\left(\left(\exp^\perp_{g_2}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right) + t_1\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_2{}_{|S}\right),$$

then using the triangular inequality we have:

$$\begin{aligned}
E(\mathrm{VII}) \leq & \left\|(t_1-t_2)\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right)\right\|_{\mathcal{C}^\infty(\mathrm{VII})} \\
& + t_2\left\|\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right) - \left(\left(\exp^\perp_{g_2}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right)\right\|_{\mathcal{C}^\infty(\mathrm{VII})} \\
& + t_1\left\|\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_1{}_{|S}\right) - \left(\left(\exp^\perp_{g_2}\circ\iota\right)^{-1}\right)^*\left(\mathrm{flat}\oplus g_2{}_{|S}\right)\right\|_{\mathcal{C}^\infty(\mathrm{VII})} \\
\leq & C|t_1-t_2| + C\,\|g_1-g_2\|_{\mathcal{C}^\infty(M)}\,,
\end{aligned}$$

where we exploited the fact that the exponential maps for $\mathcal{C}^\infty$-close metrics are close (see Proposition 1.4.10) and taking the pullback of a smooth map is continuous in the smooth compact-open topology (Lemma 1.4.9). Moreover the metrics in region VII are of product form, the distance can be computed on each factor separately and it is straightforward that $\left\| {g_1}_{|S} - {g_2}_{|S} \right\|_{\mathcal{C}^\infty(\text{VII})} \leq \|g_1 - g_2\|_{\mathcal{C}^\infty(M)}$.
On region IV we have

$$\begin{aligned} E(\text{IV}) =& \Big\| (1 - t_1\eta_{1,\delta})g_1 + t_1\eta_{1,\delta} \left(\left(\exp^\perp_{g_1} \circ \iota\right)^{-1} \right)^* (\text{flat} \oplus {g_1}_{|S}) \\ &+ (1 - t_2\eta_{2,\delta})g_2 + t_2\eta_{2,\delta} \left(\left(\exp^\perp_{g_2} \circ \iota\right)^{-1} \right)^* (\text{flat} \oplus {g_2}_{|S}) \Big\|_{\mathcal{C}^\infty(\text{IV})} \\ \leq& \|(1 - t_1\eta_{1,\delta})g_1 - (1 - t_2\eta_{2,\delta})g_2\|_{\mathcal{C}^\infty(\text{IV})} \\ &+ \left\| t_1\eta_{1,\delta} \left(\left(\exp^\perp_{g_1} \circ \iota\right)^{-1} \right)^* (\text{flat} \oplus {g_1}_{|S}) - t_2\eta_{2,\delta} \left(\left(\exp^\perp_{g_2} \circ \iota\right)^{-1} \right)^* (\text{flat} \oplus {g_2}_{|S}) \right\|_{\mathcal{C}^\infty(\text{IV})} \end{aligned}$$

Similarly to region VII, we add and subtract the term $(1 - t_2\eta_{1,\delta})g_2 + t_2\eta_{1,\delta} \left(\left(\exp^\perp_{g_2} \circ \iota\right)^* \right)^{-1} \left(\text{flat} \oplus {g_2}_{|S}\right)$ and using again the triangular inequality one obtains

$$E(\text{IV}) \leq (1 + C)|t_1 - t_2| + C \|g_1 - g_2\|_{\mathcal{C}^\infty(M)} + C \|\eta_{1,\delta} - \eta_{2,\delta}\|_{\mathcal{C}^\infty(M)}.$$

The last term can be estimated exploiting the smoothness of the cutoff function η. Being smooth implies that all the derivatives are Lipschitz functions so that, for any $p \in M$ and $k \in \mathbb{N}$,

$$|\nabla^k(\eta_{1,\delta}(p) - \eta_{2,\delta}(p))| \leq L|d_{g_1}(p, S) - d_{g_2}(p, S)| \leq LC \|g_1 - g_2\|_{\mathcal{C}^\infty(M)}.$$

This follows again Proposition 1.4.10.
In "intermediate" regions II and III, since the expression is symmetric when exchanging (g_1, t_1) with (g_2, t_2), we will deal only with region II. The estimate for region III is performed in an analogous way.
Notice that here the cut-off function $\eta_{1,\delta}$ relative to g_1 is identically zero, while the other $\eta_{2,\delta}$ starts growing. Nevertheless we can add the term $t_1\eta_{1,\delta} \left(g_1 - \left(\left(\exp^\perp_{g_1} \circ \iota\right)^{-1} \right)^* (\text{flat} \oplus {g_1}_{|S}) \right)$ (which is zero) inside of

$$E(\text{II}) = \left\| g_1 - (1 - t_2\eta_\delta)g_2 + t_2\eta_\delta(\exp^\perp_{g_2} \circ \iota)^{-1})^*(\text{flat} \oplus {g_2}_{|S}) \right\|_{\mathcal{C}^\infty(\text{II})}.$$

The height of $\eta_{2,\delta}$ is bounded again by the Lipschitz property of smooth functions, in fact, as for region IV we have $|\eta_{2,\delta}| = |\eta_{2,\delta} - \eta_{1,\delta}| \leq LC \|g_1 - g_2\|_{\mathcal{C}^\infty(M)}$. We just displayed a constant L for the Lipschitz constant of $\nabla^k\eta_{1,\delta}$ for any k.

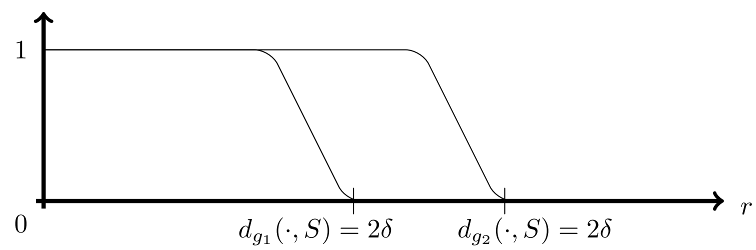


Figure 2.2: Different cutoff functions η relative to different metrics g_1, g_2.

The estimate relative to region II, adding and subtracting the mixed terms analogously to regions IV, takes again the form:

$$\begin{aligned}E(\mathrm{II}) \leq &\|(1-t_1\eta_\delta)g_1 - (1-t_2\eta_\delta)g_2\|_{\mathcal{C}^\infty(\mathrm{II})} \\ &+ \left\| t_1\eta_\delta g_1 - t_2\eta_\delta(\exp^\perp_{g_2}\circ\iota)^{-1})^*(\mathrm{flat}\oplus g_{2|_S})\right\|_{\mathcal{C}^\infty(\mathrm{II})} \\ \leq &(1+C+LC)\,\|g_1-g_2\|_{\mathcal{C}^\infty(M)} + C|t_1-t_2|.\end{aligned}$$

With the help of Figure 2.2 also the estimate in regions V and VI (as before one case is symmetric to the other) is easy. Let's do it in region V: here $\eta_{2,\delta}\equiv 1$

$$\begin{aligned}E(\mathrm{V}) = \Big\|&(1-t_2)g_2 + t_2\left(\left(\exp^\perp_{g_2}\circ\iota\right)^{-1}\right)^*(\mathrm{flat}\oplus g_{2|_S}) - \\ &(1-t_1\eta_{1,\delta})g_1 - t_1\eta_{1,\delta}\left(\left(\exp^\perp_{g_1}\circ\iota\right)^{-1}\right)^*(\mathrm{flat}\oplus g_{1|_S})\Big\|_{\mathcal{C}^\infty(\mathrm{V})}\end{aligned}$$

adding and subtracting the mixed terms and using once more the Lipschitzianity of all the derivatives of $\eta_{1,\delta}$ as for the other previously studied regions gives:

$$\|\mathbf{ADH}_\delta(g_1,t_1) - \mathbf{ADH}_\delta(g_2,t_2)\|_{\mathcal{C}^\infty(\mathrm{V})} \leq C\left(|t_1-t_2| + \|g_1-g_2\|_{\mathcal{C}^\infty(M)}\right),$$

concluding the proof. □

Corollary 2.3.6. *For any continuous compact family of metrics $B\hookrightarrow\mathcal{R}^{\min}(M)$ there exists a $\delta>0$ such that, for any $b\in B$ and $t\in[0,1]$, $\mathbf{ADH}_\delta(g_b,t)\in\mathcal{R}^{\min}(M)$.*

Proof. By the genericity of D-minimal metrics ([ADH09], [Kat76, Section VII 1.3]) we know that for any metric $g\in\mathcal{R}^{\min}(M)$ there exists a value $\epsilon>0$ such that all other metrics contained in $B_\epsilon(g)$ (a $\mathcal{C}^1$-ball of radius ϵ centered in g) are D-minimal. Moreover, from Lemma 2.3.5 we have that for a fixed metric g and value of t,

$$\mathbf{ADH}_{\delta(g,t)}(B_\epsilon(g,t)) \subset B_{C\epsilon}(\mathbf{ADH}_{\delta(g,t)}(g,t)),$$

where $B_\epsilon(g,t)$ is the ball of radius ϵ centered at $(g,t)\in\mathcal{R}^{\min}(M)\times[0,1]$ in the product topology.
Possibly choosing a smaller value of ϵ, we have that locally the same value $\delta(g,t)$ makes the map $\mathbf{ADH}_{\delta(g,t)}$ well defined on $B_\epsilon(g,t)$, i.e. $\mathbf{ADH}_{\delta(g,t)}$ preserves the D-minimality of all the metrics in a small ball around (g,t).
We can cover the compact set $B\times[0,1]\subset\mathcal{R}^{\min}(M)\times[0,1]$ with open balls $B_{\epsilon_i}(g_i,t_i)$ of radius ϵ_i centered at the points (g_i,t_i) on each of which the map $\mathbf{ADH}_{\delta(g_i,t_i)}$ is well defined.

By definition of compactness, from the open cover $\bigcup_i B_{\epsilon_i}(g_i,t_i)\supset B\times[0,1]$ we can extract a finite subcover $\{B_{\epsilon_{i_k}}(g_{i_k},t_{i_k})\}_k$ such that

$$\mathbf{ADH}_{\delta(g_{i_k},t_{i_k})}(g,t)\in\mathcal{R}^{\min}(M),\quad \forall(g,t)\in B_{\epsilon_{i_k}}(g_{i_k},t_{i_k}). \tag{2.3.5}$$

Compactness implies paracompactness for a topological space and hence we can find a partition of unity

$$\{\chi_k\}\subset\mathcal{C}^\infty(B\times[0,1],\mathbb{R}_+),\quad \mathrm{supp}\,(\chi_k)\Subset B_{\epsilon_{i_k}}(g_{i_k},t_{i_k}),\quad \sum_k\chi_k=1,$$

and patch together the finitely many different values of $\{\delta_{i_k}\}_k$, computed as in Proposition 2.3.3 for the centres (g_{i_k},t_{i_k}) of the balls of the cover $\{B_{\epsilon_{i_k}}(g_{i_k},t_{i_k})\}_k$, satisfying the property (2.3.5). The patching procedure is allowed since in the intersections of balls $B_{\epsilon_i}(g_i,t_i)\cap B_{\epsilon_j}(g_j,t_j)$ the last part of the proof of

Proposition 2.3.3 ensures that $\mathbf{ADH}_{(\chi_i\delta(g_i,t_i)+\chi_j\delta(g_j,t_j))}(g,t)$ is D-minimal for any point (g,t) belonging to the intersection of the balls, as $\chi_i\delta(g_i,t_i)+\chi_j\delta(g_j,t_j) \le \max\{\delta(g_i,t_i),\delta(g_j,t_j)\}$.
This way $\delta(g,t) := \sum_k \chi_k\delta(g_{i_k},t_{i_k})$ happens to be a continuous function defined on the compact set $B\times[0,1]$ and as such attains here its maximal and minimal value. Again by Proposition 2.3.3 the minimum

$$\delta = \min_{(g,t)\in B\times[0,1]} \delta(g,t)$$

fulfills the requirements, i.e.

$$\mathbf{ADH}_\delta(g,t) \in \mathcal{R}^{\min}(M) \quad \forall g \in B,\, t\in[0,1].$$

□

Remark 2.3.7. From the proof of the above Corollary 2.3.6 we see that the value of δ can be chosen to vary continuously on the whole space of D-minimal metrics; it defines therefore a continuous function

$$\delta : \mathcal{R}^{\min}(M) \longrightarrow \mathbb{R}_+.$$

2.3.2 Properties (ii) and (iii) of the map $\mathbf{ADH}_\delta$

We are now ready to prove that $\mathbf{ADH}_\delta$ is a continous map and preserves $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$ (see (2.3.1) for the definition):

Proposition 2.3.8. *Let $\theta : B \hookrightarrow \mathcal{R}^{\min}(M)$ be a compact family of Riemannian metrics with minimal Dirac kernel on M. Then the map*

$$\begin{aligned}\mathbf{ADH}_\delta \circ i : B\times[0,1] &\longrightarrow \mathcal{R}^{\min}(M),\\ (g_b,t) &\longmapsto (1-t\eta_\delta g_b) + t\eta_\delta((\exp^\perp\circ\iota)^{-1})^*\text{flat}\oplus g_{b|_S})\end{aligned}$$

for δ as in Corollary 2.3.6, is continuous in the $\mathcal{C}^\infty$ compact-open topology.

Proof. We will prove continuity separately in each variable.
Since we have proved above in Corollary 2.3.6 that δ can be chosen to be a constant function on $B\times[0,1]$, the proof that $\mathbf{ADH}_\delta(g_b,t)$ is a continuous map with respect to $b\in B$ is reduced to prove the continuity with respect to metrics g_b separately in the regions I,...,VII (see Figure 2.1). Here, a similar estimate to the one in Lemma 2.3.5 allows to conclude: eventhough the tubular neighbourhood $U_{S,g}(2\delta)$ of S might vary for different metrics, the value of δ is constant and the difference in the radial distance $r = d_g(\cdot,S)$ from the submanifold S depends in a $\mathcal{C}^\infty$-continuous fashion from g, as seen in Proposition 1.4.10. The continuity of the map $\mathbf{ADH}_\delta(\cdot,t) : B \to \mathcal{R}^{\min}(M)$ follows from the continuity of composition of continuous maps, as $\theta : B\to\mathcal{R}^{\min}(M)$ is continuous by hypothesis.
Regarding the t variable, as δ by Corollary 2.3.6 is independent from t, countinuity of $\mathbf{ADH}_\delta$ with respect to the variable t follows from the continuity of a convex combination proved in Lemma 1.4.11: infact the map $\mathbf{ADH}_\delta$ can also be written as the linear segment joining g to g_δ:

$$(1-t)g + t(1-\eta_\delta)g + t\eta_\delta((\exp^\perp\circ\iota)^{-1})^*\text{flat}\oplus g_{|_S}) = (1-t\eta_\delta g) + t\eta_\delta((\exp^\perp\circ\iota)^{-1})^*\text{flat}\oplus g_{|_S}),$$

so that also continuity with respect to the t variable is ensured. □

Remark 2.3.9. Even though the value of δ (and hence the definition of the $\mathbf{ADH}_\delta$ map) might vary for different maps $B\hookrightarrow\mathcal{R}^{\min}(M)$, its value, once fixed the family $\{g_b\}_{b\in B}$, will always be bounded away from zero since the parameter space B is compact and $\mathcal{R}^{\min}(M) \subset \mathcal{R}iem(M)$ is open.

The last property to check of the map $\mathbf{ADH}_\delta$ is the invariance of the space $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$. This is shown in the following lemma:

Lemma 2.3.10. *The map* $\mathbf{ADH}_\delta$ *preserves metrics which are already of half-flat form in a tubular neighbourhood* $U_S(R)$ *of* S*:*

$$\mathbf{ADH}_\delta\left(\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M), t\right) \subset \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M), \quad \forall t \in [0,1].$$

Proof. Since the homotopy $\mathbf{ADH}_\delta$ can be written as a linear segment joining g to g_δ, it turns out that any linear property of the metric is preserved. In our case, suppose that the metric $g \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$ is of the form flat $\oplus\, g_{|_S}$ on the compact set $U_S(R)$, while the width of the modification in the homotopy is δ. Then, on $U_S(\min\{\delta, R\})$ where g is of half-flat form and $\eta_\delta \equiv 1$ it holds

$$\mathbf{ADH}_\delta(g,t)_{|_{U_S(\min\{\delta,R\})}} = (1-t)\left(\text{flat} \oplus g_{|_S}\right) + t\left(\text{flat} \oplus g_{|_S}\right) \equiv \text{flat} \oplus g_{|_S}.$$

Up to shrinking the compact set appearing in the definition of $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$, the metric $\mathbf{ADH}_{\delta'}(g,t)$, with $g \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$, belongs to $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M)$. □

2.3.3 The final step

The construction of the nullhomotopy $\mathbf{ADH}_\delta$ reveals that there is no need to suppose that the embedded submanifold is a k-dimensional sphere:

Remark 2.3.11. The fact that the embedded submanifold is a sphere S^k is irrelevant to the definition and the properties satisfied by $\mathbf{ADH}_\delta$. Hence we can substitute in the previous part S with a submanifold of codimension at least 2 and with trivial normal bundle, by defining $U_{N,g}(\delta) := \{p \in M \mid d_g(p,N) \leq \delta\}$ and

$$\overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N) := \left\{ g \in \mathcal{R}^{\min}(M) \,\middle|\, g_{|_{U_N(\delta)}} = \text{flat} \oplus g_{|_N} \right\}.$$

We have proved so far that the map $\mathbf{ADH}_\delta$ possesses all the properties of an homotopy of pairs $(D^l, S^{l-1}) \to (\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N))$. We are then ready to prove the main theorem of this section:

Theorem 2.3.12. *Let* M *be a closed spin manifold and* $N \hookrightarrow M$ *a closed submanifold of codimension at least 2 and with trivial normal bundle. Then*

$$\mathcal{R}^{\min}(M) \simeq \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N).$$

Proof. Due to the freedom on the choice of the compact space B parametrizing the family of metrics g_b, we will parametrize the family of metrics in $\mathcal{R}^{\min}(M,N)$ with a particular pair of compact spaces: f : $(D^l, S^{l-1}) \to (\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M))$.
With this pair of spaces chosen, we have that any homotopy class

$$[f] \in \left[(D^l, S^{l-1}), (\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N))\right] = \pi_l(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N))$$

is trivial for any value of $l > 1$: the continuous map $\mathbf{ADH}$ will provide a nullhomotopy of pairs

$$\mathbf{ADH} : \left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N)\right) \times [0,1] \to \left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N)\right),$$

since $\mathbf{ADH}(g_b, 1) \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N) \subset \mathcal{R}^{\min}(M)$ and by Lemma 2.3.10 $\mathbf{ADH}(g_b,t) \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N)$ for all $t \in [0,1]$ whenever $b \in S^{l-1}$.

Since π_0 cannot be given in general a group structure, we have to prove that $i : \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N) \to \mathcal{R}^{\min}(M)$ is a bijection on connected components. Injectivity is immediate since $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N) \subset \mathcal{R}^{\min}(M)$; we are left to prove surjectivity. Consider two metrics $g_1, g_2 \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)$ connected by a path $f(s)$, $s \in [0,1]$ in $\mathcal{R}^{\min}(M)$. Then the curve

$$\bar{f}(t) = \begin{cases} \mathbf{ADH}(f(0), 3t) & t \in [0, 1/3], \\ \mathbf{ADH}(f(3t-1), 1) & t \in [1/3, 2/3], \\ \mathbf{ADH}(f(1), 3-3t) & t \in [2/3, 1], \end{cases}$$

lies entirely in $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)$ and connects g_1 with g_2, implying that they lie in the same path component also in $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)$. Hence, by the very definition of homotopy group of pairs

$$\pi_l\left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right) = 0, \quad \forall l \in \mathbb{N}. \tag{2.3.6}$$

In the long exact sequence of homotopy groups for the pair $\left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right)$, coming from the sequence

$$\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N) \xrightarrow{\ i\ } \mathcal{R}^{\min}(M) \xrightarrow{\ p\ } \left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right)$$

we will have:

$$\cdots \xrightarrow{\partial_{l+2}} \pi_{l+1}\left(\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right) \xrightarrow{i^*} \pi_{l+1}\left(\mathcal{R}^{\min}(M)\right) \xrightarrow{p^*}$$

$$\underbrace{\pi_{l+1}\left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right)}_{=0} \xrightarrow{\partial_{l+1}} \pi_l\left(\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M)\right) \xrightarrow{i^*} \pi_l\left(\mathcal{R}^{\min}(M)\right) \xrightarrow{p^*}$$

$$\underbrace{\pi_l\left(\mathcal{R}^{\min}(M), \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right)}_{=0} \xrightarrow{\partial_l} \pi_{l-1}\left(\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)\right) \xrightarrow{i^*} \cdots$$

where ∂_* are the connecting homomorphisms.
By (2.3.6), it follows that the map i^* is an isomorphism on homotopy groups, i.e. a weak homotopy equivalence .
Now, since the subspaces $\mathcal{R}^{\min}(M)$ and $\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)$ inherit from $\mathcal{R}iem(M)$ the property of being dominated by a CW-complex [Pal66], we can apply Whitehead's theorem [Bre93, Theorem 11.2] and conclude that, since the map $i : \mathcal{R}^{\min}(M) \to \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N)$ is a weak homotopy equivalence, then the inclusion i is also a strong homotopy equivalence. The theorem is then proved. □

For a reason that will be explained in Remark 2.4.5 we will have to choose the sets $U_{S,g}(\delta)$ where the metric $g \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M)$ is of half-flat form uniformly with respect to g and δ. To achieve this, we will use a background reference metric $g_0 \in \mathcal{R}iem(M)$ which we can assume, without loss of generality, has injectivity radius bigger than 1, and with the help of the normal exponential map $\exp^{\perp}_{g_0} : \nu_N \longrightarrow M$, we will deform any tubular neighbourhood $U_{N,g}(\delta)$ to a fixed one of radius 1.

Let

$$\begin{aligned}\mathrm{Tub}_N : D^{n-k} \times N^k &\longrightarrow M^n \\ (v,p) &\longmapsto (\exp^{\perp}_{g_0})_{\iota(p)}\iota(p,v)\end{aligned}$$

where $\iota : D^{n-k} \times N^k \longrightarrow \nu_N$ is the chosen embedding for the submanifold N^k extended to the trivial normal bundle ν_N (also in this case, there is no reason to impose that the submanifold is a sphere).

Proposition 2.3.13. *Let M^n be a closed spin manifold and $\iota : N^k \hookrightarrow M^n$ an embedded submanifold. Then the spaces*

$$\overline{\mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}}(M,N) \cong \mathcal{R}^{\min}_{\frac{1}{2}\mathrm{flat}}(M,N)$$

are homeomorphic.

Proof. Since the submanifold N^k has trivial normal bundle, with an abuse of notation we extend the embedding to be the bundle trivialization $\iota : D^{n-k} \times N^k \to \nu_N$.
Let us indicate with $D^{n-k}(\delta)$ the disk of radius δ in $\mathbb{R}^{n-k}$. We can rewrite $U_{N,g}(\delta) = \exp^{\perp}_g \circ \iota \left(D^{n-k(\delta) \times N^k} \right)$.
We start building the desired homeomorphism by choosing a radial diffeomorphism

$$\begin{aligned}\Phi_\delta : D^{n-k}(\delta) &\longrightarrow D^{n-k} \\ (r,\phi) &\longmapsto \begin{cases} (r,\phi), & 0 \le r \le \dfrac{\delta}{3}, \\ (1+r-\delta,\phi), & \dfrac{2\delta}{3} \le r \le \delta \end{cases}\end{aligned}$$

and smooth monotone increasing for $\dfrac{\delta}{3} \le r \le \dfrac{2\delta}{3}$.

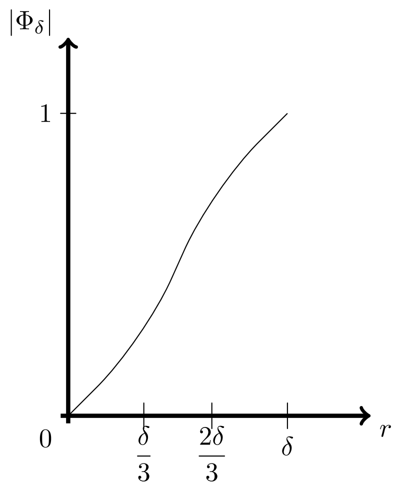


Figure 2.3: The function $|\Phi_\delta|(r)$.

The map

$$\begin{aligned}u_\delta : \exp^{\perp}_{g_0}\left(D^{n-k} \times N\right) &\longrightarrow U_{N,g}(\delta) \\ p &\longmapsto \exp^{\perp}_g \circ \iota \circ (\Phi^{-1}_\delta \times \mathrm{Id}_N) \circ \iota^{-1} \circ \left(\exp^{\perp}_{g_0}\right)^{-1}(p)\end{aligned} \tag{2.3.7}$$

is a diffeomorphism independent of the coordinates of the submanifold N. We extend it to be the identity outside $U_{N,g}(\delta)$ by [Hir97, Chapter 8 Theorem 1.7], obtaining the global diffeomorphism $U_\delta : M \longrightarrow M$ and then we take the pullback, arriving to the map uniforming the tubular neighbourhoods $U_{N,g}(\delta)$'s:

$$\mathcal{U} : \overline{\mathcal{R}^{\min}}(M,N) \longrightarrow \mathcal{R}^{\min}(M,N)$$
$$g \longmapsto U_\delta^* g.$$

This map depends on the metric $g \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N)$ through the value of δ, which, by Remark 2.3.7, depends continuously on the metric g. If we define $\text{Tub}_N : D^{n-k} \times N \longrightarrow M$ as

$$\text{Tub}_N = \exp^{\perp}_{g_0} \circ \iota : D^{n-k} \times N^k \longrightarrow M$$

we have immediately by (2.3.7) that, for any $g \in \overline{\mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}}(M,N)$,

$$\text{Tub}_N^* \mathcal{U}^*(g) = (f'(r))^2 dr^2 + f^2(r)\sigma_{n-k-1} \oplus g_{|_N},$$

for a smooth function $f(r)$ which equals r close to S and $r+c$, c constant, close to $\partial\left(\exp^{\perp}_{g_0} \circ \iota\right)(D^{n-k} \times N^k)$.
It follows that $\mathcal{U}^* g \in \mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M,N)$.
We build the inverse map of $\mathcal{U}$ for every metric $g \in \mathcal{R}^{\min}_{\frac{1}{2}\text{flat}}(M,N)$ as

$$\mathcal{U}^{-1} = (U_\delta^*)^{-1},$$

where the value of δ is obtained from the function $f(s) : [0,1] \longrightarrow \mathbb{R}$ from $\text{Tub}_N^*(g) = (f'(s))^2 ds^2 + f^2(s)\sigma_{n-k-1} \oplus g_{|_N}$ as $\delta = f(1)$.
The homeomorphism is well-defined, since for any diffeomorphism $\Psi : M \longrightarrow M$ and any metric $g \in \mathcal{R}iem(M)$, the metrics g and $\Psi^*(g)$ are isometric, hence $\dim\ker D^g = \dim\ker D^{\Psi^*(g)}$. □

As a corollary, composing the homotopy equivalence of Theorem 2.3.12 with the homeomorphism $\mathcal{U}$ of Proposition 2.3.13, we obtain:

Corollary 2.3.14. *Let M be a closed spin manifold and let N be a submanifold of codimension at least 2 with trivial normal bundle. Then*

$$\mathcal{R}^{\min}(M) \simeq \mathcal{R}^{\min}(M,N).$$

2.4 The weak homotopy equivalence $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \simeq \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$

Starting from this subsection all the statements will regard only the space of metrics with associated invertible Dirac operator, hence "min" will be replaced with "inv" in the notations for the various spaces of metrics.
This is due to the fact that minimal metrics are defined only for closed manifolds and, as we will see, some metrics appearing in the subsections will be defined on the open manifold $M \setminus S$.

As stated in Section 2.2, the major difference when dealing with metrics with invertible Dirac operator instead of positive scalar curvature ones is the global flavour of the property. This means in particular that the last consideration of [Che04, proof of Theorem 1.2], namely that for any pair of manifolds M and $\widetilde{M}$, related by surgeries in dimension $2 \le k \le n-3$

$$\mathcal{R}^{\text{psc}}_{\text{std}}(M) \cong \mathcal{R}^{\text{psc}}_{\text{std}}(\widetilde{M}),$$

does not hold in general for D-invertible metrics: the map

$$h : \mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(M) \to \mathcal{R}^{\mathrm{psc}}_{\mathrm{std}}(\widetilde{M})$$

obtained by exhanging the Riemannian factor $(D^{n-k} \times S^k,\ \mathrm{torp}_{n-k} \oplus \sigma_k)$ with $(S^{n-k-1} \times D^{k+1},\ \sigma_{n-k-1} \oplus \mathrm{torp}_{k+1})$ (see for more details [Che04, Theorem 1.2]) is not even well-defined in general; it is not clear whether it preserves D-invertibility or not.
To circumvent this problem we will define :

Definition 2.4.1. The space of asymptotically cylindrical metrics on the open manifold $M \setminus S$ is defined as

$$\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) := \left\{ g \in \mathcal{R}^{\mathrm{inv}}(M \setminus S) \,\middle|\, \exists K \subset M \setminus S \text{ minimal, } g_{|_{(M\setminus S)\setminus K}} \overset{\mathrm{iso}}{\cong} du^2 + \sigma_{n-k-1} + \sigma_k \right\},$$

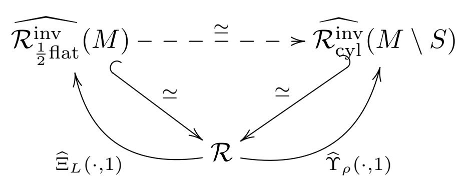
where the compact subset K being minimal means that if there exists another compact set $K' \subset M \setminus S$ with $g_{|_{(M\setminus S)\setminus K'}} \overset{\mathrm{iso}}{\cong} du^2 + \sigma_{n-k-1} + \sigma_k$, then it must hold $K \subseteq K'$. We also require that every factor of the hypersurface $S^{n-k-1} \times S^k$ is endowed with the spin structure that bounds the disk.

The strategy to show that there exists a weak homotopy equivalence

$$\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M) \dashrightarrow \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$$

is, as for Section 2.3, building nullhomotopies for any compact family of asymptotically cylindrical metrics $\theta : B \hookrightarrow \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$. Then the long exact sequence for homotopy groups of pairs would lead to the desired weak homotopy equivalence. Unfortunately for this machinery to work, we would need that $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \subset \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$, which is indeed not true.
We must then take a detour and define a new connected space $\mathcal{R}$ such that both $\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ and $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$ have homotopy equivalent subspaces $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ and $\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$ embedded in it.

Once defined such a space $\mathcal{R}$ we need to specify a method to pass from a generic metric in $\mathcal{R}$ to an asymptotically cylindrical one and another method to pass to an half-flat one. This will be done building a parametrized version $\widehat{\Upsilon}_\rho$ of the conformal blow up performed in [ADH09, Section 3.2]: we will blow up the metric in the normal direction around S using a conformal factor of the form $\dfrac{1}{d_g^2(S,p)}$ plus a correction term f_ρ^2 for the factor of g defined on S. The role of such auxiliary factor f_ρ^2 is to avoid the divergence of the volume of the S^k-factor, which would generate troubles when computing the L^2-norm of harmonic spinors. Such deformation exploits the nice behaviour of harmonic spinors under conformal changes of the metric (1.3.4) and for this, we need that the map $\widehat{\Upsilon}_\rho$ brings a metric $g \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ immediately after $t = 0$ to a metric which is asymptotically cylindrical.
On the other hand, we also have to continuously deform a metric in $\mathcal{R}$ to an half-flat one. For this we define the map $\widehat{\Xi}_L$ that closes the cylindrical end with a cap of the form $\widetilde{g} \oplus \sigma_k$, with $k \geq 1$. Provided the cap is glued after a sufficiently long piece of the cylindrical end, as in [Dah08, Proposition 2.1], we ensure that the metric remains D-invertible along the continuous deformation $\widehat{\Xi}_L$.
The maps briefly described above and the role of the space $\mathcal{R}$ are indicated in the following diagram:



We will not be able to define the map $\widehat{\Upsilon}_\rho$ for any compact family of metrics $B \subset \mathcal{R}$. In particular we will have troubles defining such map in a way that preserves, for all $t \in [0,1]$ the space $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$.
Supposing that the maps $\widehat{\Upsilon}_\rho$ and $\widehat{\Xi}_L$ satisfy all the properties of a nullhomotopy of pairs, we would use them to show that any homotopy class

$$[v] = \left[\left(D^l, S^{l-1}\right), \left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)}\right) \right]$$

and

$$[\xi] = \left[\left(D^l, S^{l-1}\right), \left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M \setminus S)\right) \right]$$

is actually trivial. This will automatically imply that the space $\mathcal{R}$ is weakly homotopic to both $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)}$ and $\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M \setminus S)$, which proves then, by the hypothesis on the homotopy type of $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ and $\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M \setminus S)$, that $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \overset{\text{weak}}{\simeq} \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$.
For technical reasons (about convergence of sequences of half-flat metrics) we will make the space $\mathcal{R}$ larger by pairing any metric g with a compact subset K of $M \setminus S$ as additional data.

2.4.1 A good guess for the space $\mathcal{R}$

This subsection is completely devoted to the definition of the space $\mathcal{R}$ aforementioned.
First of all, we make the following observation:

Remark 2.4.2. Any metric defined on $M \setminus S$ which restricts on some neighbourhood of S with the form $dr^2 + r^2\sigma_{n-k-1} \oplus g_{|_S}$, although incomplete, can be extended in a unique way to a smooth metric on the whole of M due to the $SO(n-k)$ symmetry around S and the constant factor $g_{|_S}$ defined on $S \times (0, \delta]$.
This implies that we can view

$$\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \subset \Gamma(T^{\odot 2}M \setminus S).$$

Such inclusion is continuous once we endow $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ and $\Gamma(T^{\odot 2}M \setminus S)$ with the $\mathcal{C}^\infty$ compact-open topology on M and $M \setminus S$ respectively.

To define the topology on $\mathcal{R}$ we will use the usual $\mathcal{C}^\infty$ compact-open topology on $M \setminus S$, but at the same time we want to avoid the existence of sequences $\{g_i\}_i \subset \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ with limit in $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$: $g_i \nrightarrow g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$: the existence of such sequences would make the quest for a candidate for the map $\widehat{\Upsilon}_\rho$ even harder.
For this, working with the space $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \sqcup \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S) \subset \Gamma(T^{\odot 2}M \setminus S)$ equipped with the $\mathcal{C}^\infty$ compact-open topology on $M \setminus S$ is not sufficient. We have to keep track also of the size of the neighbourhood of S where the metric g is of asymptotically cylindrical form: we will pair each metric in $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ with a compact set $K \subset M \setminus S$ isometric to

$$\left((M \setminus S) \setminus K, g_{|_{(M\setminus S)\setminus K}}\right) \overset{\text{iso}}{\cong} \left([0,\infty) \times S^{n-k-1} \times S^k, du^2 \oplus \sigma_{n-k-1} \oplus \sigma_k\right), \tag{2.4.1}$$

while metrics in $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ will be simply paired with $M \setminus S$.
We see, from (2.4.1) that the complement $(M \setminus S) \setminus K$ is spherically symmetric for any such compact K.

Definition 2.4.3. Let $\mathcal{K}$ be the set

$$\mathcal{K} := \{K \subset M \setminus S \,|\, K \text{ is compact }\}.$$

We define the space of neighbourhoods of S as

$$\overline{\mathcal{K}} := \mathcal{K} \cup \{M \setminus S\}.$$

We endow $\overline{\mathcal{K}}$ with the topology induced by the Hausdorff distance

$$d_H(K_1, K_2) := \inf\{\epsilon > 0 \,|\, K_1 \subseteq Tub_\epsilon(K_2) \text{ and } K_2 \subseteq Tub_\epsilon(K_1)\},$$

where

$$Tub_\epsilon(K) := \bigcup_{p \in K} \{q \in K \,|\, d_{g_0}(p,q) \leq \epsilon\}$$

and $d_{g_0}(\cdot,\cdot)$ is the distance induced on $M \setminus S$ by a reference metric $g_0 \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$.

The set K has to be tought of as a substitute for the set $U_S(\epsilon)$. While for half-flat metrics $U_S(\epsilon)$ is well defined for small ϵ, for asymptotically cylindrical metrics the latter makes no sense since the distance function from S, by completeness, in always infinite. For a sketched example of such compact set K, see the gray shaded part in Figure 2.5.

We require Hausdorff convergence of the compact sets around S because for different metrics g_1, g_2 the sets K_1, K_2 might have intersecting boundaries.

Definition 2.4.4. We define the space $\mathcal{R} \subset \Gamma(T^{\odot 2}M \setminus S) \times \overline{\mathcal{K}}$ to be

$$\mathcal{R} := \underbrace{\left\{(g, M\setminus S) \,\middle|\, g \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)\right\}}_{:=\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)} \cup \underbrace{\left\{(g, K) \,\middle|\, g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M\setminus S),\, K \in \mathcal{K},\; g_{|_{(M\setminus S)\setminus K}} \overset{\text{iso}}{\cong} du^2 \oplus \sigma_{n-k-1} \oplus \sigma_k\right\}}_{:=\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)}$$

and we endow such space with the product topology inherited as a subspace of $\Gamma(T^{\odot 2}M \setminus S) \times \overline{\mathcal{K}}$, i.e. a sequence $\{(g_i, K_i)\}_i$ converges to (g, K) in $\mathcal{R}$ if and only if

$$\begin{cases} g_i \to g \text{ in the usual } \mathcal{C}^\infty \text{ compact-open topology on } M \setminus S, \\ d_H(K_i, K) \to 0 \end{cases} \tag{2.4.2}$$

This way the space $\left(\mathcal{R}, \|\cdot - \cdot\|_{\mathcal{C}^\infty(M\setminus S)} + d_H(\cdot,\cdot)\right)$ is a metrizable space but since it is not a manifold, one cannot apply [Pal66, Theorem 14] and conclude that $\mathcal{R}$ is dominated by a CW-complex.
Nevertheless $\mathcal{R}$ endowed with its topology is first-countable and hence sequential, i.e. continuity of functions can be probed using sequences.

Remark 2.4.5. The following continuous embeddings hold:

$$\begin{aligned} \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) &\hookrightarrow \mathcal{R} \\ g &\mapsto (g, M \setminus S) \end{aligned} \tag{2.4.3}$$

by Remark 2.4.2 and

$$\begin{aligned} \mathcal{R}^{\text{inv}}_{\text{cyl}}(M\setminus S) &\hookrightarrow \mathcal{R} \\ g &\mapsto (g, K) \end{aligned} \tag{2.4.4}$$

with K the minimal compact set such that $g_{|_{(M\setminus S)\setminus K}}$ is cylindrical (see Definition 2.4.1. Moreover it holds the following homeomorphism:

$$\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \cong \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M), \tag{2.4.5}$$

as the map obtained by coupling a metric $g \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ with the set $\{M \setminus S\}$ is trivially continuous as

$$\|\cdot\|_{\mathcal{C}^\infty(M)} \geq \|\cdot\|_{\mathcal{C}^\infty(K)},$$

for any compact $K \subset M \setminus S$. Since the spaces $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ and $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ are sequential (Lemma 1.4.8) we prove that the obvious map $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M) \to \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ is continuous noticing that sequences g_i that converge on $M \setminus S$ but not on M in the compact open topology are ruled out by the very definition of $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$. In fact in Proposition 2.3.13 we fixed the tubular neighbourhood Tub_k of S on which a half-flat metric has the desired block form and as a consequence we have that for any sequence $\{g_i\} \subset \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ converging to $g \in \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ we have

$$\|g_i - g\|_{\mathcal{C}^\infty(M \setminus \mathring{\text{Tub}}_k(\mathring{D}^{n-k} \times S^k))} = \|g_i - g\|_{\mathcal{C}^\infty(M \setminus S)}.$$

If the compact set $\text{Tub}(D^{n-k} \times S^k)$ would not have been fixed, one could create sequences of metrics converging in $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ with the $\mathcal{C}^\infty$ compact open topology on $M \setminus S$ but not in $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ with the usual $\mathcal{C}^\infty$ compact open topology on all of M: fix a half-flat metric $g \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ with

$$g_{|_{\text{Tub}_k(D^{n-k} \times S^k)}} = dr^2 + r^2\sigma_{n-k-1} \oplus g_{|_S}.$$

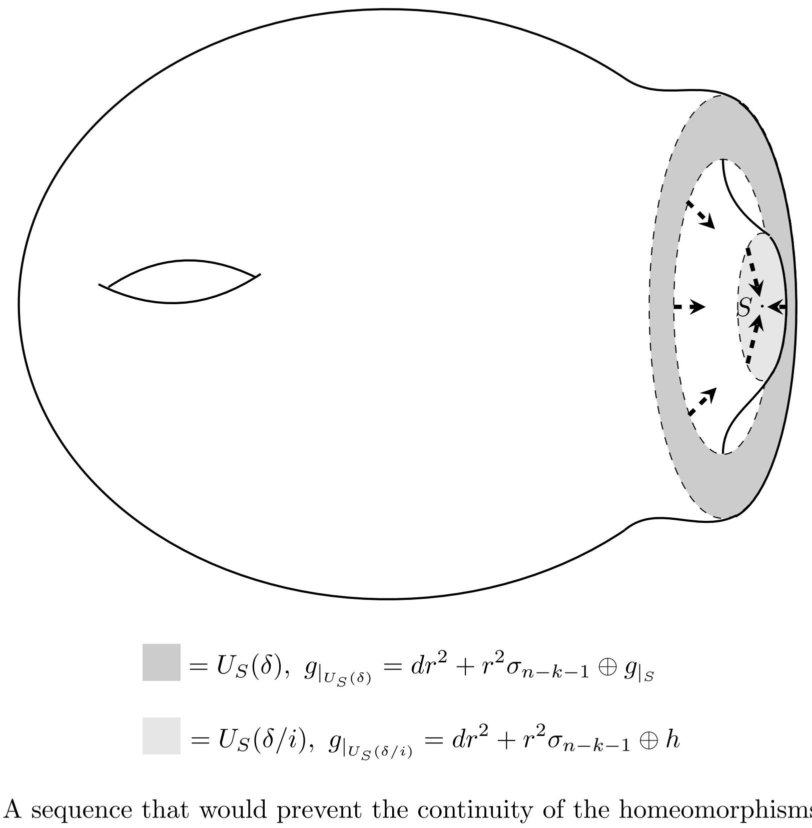


Figure 2.4: A sequence that would prevent the continuity of the homeomorphisms (2.4.5).

Sequences such that

$$g_{i|_{U_S(1/i)}} = dr^2 + r^2\sigma_{n-k-1} \oplus h$$

for a metric $h \in \mathcal{R}iem(S^k)$ and

$$g_{i|_{\text{Tub}(D^{n-k} \times S^k) \setminus U_S(2/i)}} = dr^2 + r^2\sigma_{n-k-1} \oplus g_{|_S}$$

pictured in Figure 2.4 are such that

$$\underset{\text{in the } \mathcal{C}^\infty_{\text{loc}}(M\backslash S) \text{ topology}}{g_i \to g} \qquad \text{but} \qquad \underset{\text{in the } \mathcal{C}^\infty(M)\text{-topology.}}{g_i \nrightarrow g}$$

Such sequences are not contained in $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ though, because the tubular neighbourhood $\text{Tub}(D^{n-k} \times S^k)$ on which the metrics g_i have to be of half-flat form have constant width (2.2.1).

It holds also the following homotopy equivalence

$$\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S) \simeq \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M \setminus S), \tag{2.4.6}$$

obtained by shrinking the compact set K of (g, K) to the minimal one $K_{\min}$ appearing in the Definition 2.4.1. The procedure of shrinking the compact set K does not modify the metric, hence the D-invertibility of g is preserved along the path $K \rightsquigarrow K_{\min}$ (see Figure 2.5 below).

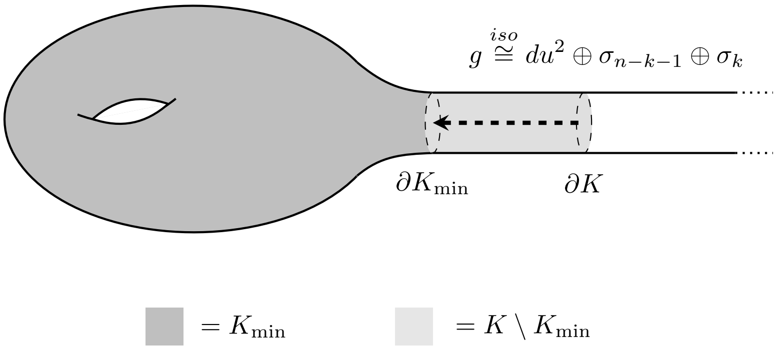


Figure 2.5: Any compact K can be shrunk on the minimal one $K_{\min}$.

With this topology defined on $\mathcal{R}$, also blow up procedures around S can be controlled and made continuous: the metrics in $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ are nothing more than some of the accumulation points of $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ with the topology of smooth convergence on compact sets on $M \setminus S$.

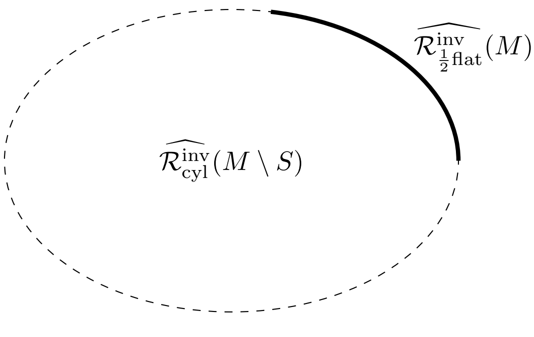


Figure 2.6: The space $\mathcal{R}$.

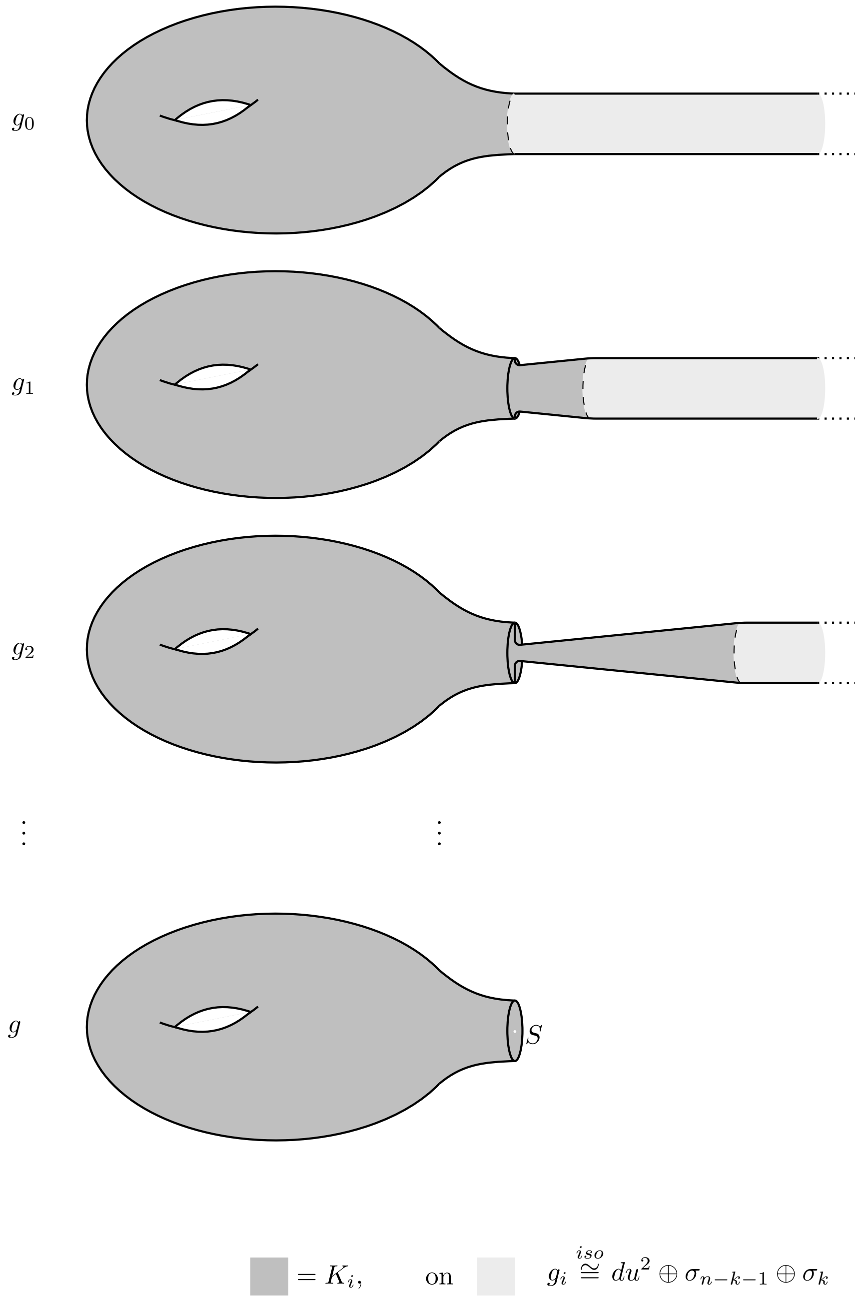


Figure 2.7: An example of a sequence $\{(g_i, K_i)\}_i \subset \widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$ converging to $(g, M \setminus S \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$.

In the next two subsections we will proceed by defining and proving some of the required properties of the maps $\widehat{\Upsilon}_\rho$ and $\widehat{\Xi}_L$.

2.4.2 Preparations for the weak homotopy equivalence $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \simeq \mathcal{R}$

The first nullhomotopy that we will try to construct, $\widehat{\Upsilon}_\rho$, will be defined separately for metrics which are asymptotically cylindrical (and have to remain asymptotically cylindrical as t varies from 0 to 1) and for those that are half-flat: to define it, first change coordinates inside the tubular neighbourhood Tub_k of S using the map $\mathcal{U}^{-1}$ of Proposition 2.3.13, this way the metric will have the form $dr^2 + r^2\sigma_{n-k-1} \oplus g_{|S}$ on

$U_S(\delta)$. We introduce new parameters $\rho < 2r_1 < r_0 < \delta$ (δ as in Corollary 2.3.6) and the function

$$F_t(r) = \begin{cases} \frac{1}{r}, & 0 < r < tr_1, \\ 1, & tr_0 \leq r \leq \delta \end{cases}$$

We define also the functions

$$\begin{aligned} f_{t,\rho} : [0,1] \times (0,\delta) \times \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) &\longrightarrow \mathbb{R} \\ (t,r,g) &\longmapsto \begin{cases} r, & \text{on } \{(t,r,g) \,|\, 0 < r \leq t\rho(g,t)\}, \\ 1, & \text{on } \{(t,r,g) \,|\, 2t\rho(g,t) \leq r \leq \delta\} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \eta_{t,\rho} : [0,1] \times (0,\delta) \times \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) &\longrightarrow \mathbb{R} \\ (t,r,g) &\longmapsto \begin{cases} 0, & \text{on } \{(t,r,g) \,|\, 0 < r \leq t\rho(g,t)\}, \\ 1, & \text{on } \{(t,r,g) \,|\, 2t\rho(g,t) \leq r \leq \delta\} \end{cases} \end{aligned}$$

We extend all the three of $F_t, f_{t,\rho}$ and $\eta_{t,\rho}$ to 1 on $M \setminus U_S(\delta)$.
These functions are then smooth in the M variables. Regarding the other variables (they depend also on g and t through $r_0, r_1 = r_0(g), r_1(g)$ and $\rho = \rho(g,t)$) continuity is ensured by the Urysohn lemma [Mun75, Theorem 33.1], while the continuity of $\rho = \rho(g,t)$ will be proved in the next Lemmata. We will simply write $\rho = \rho(g,t)$ for notation convenience.
Now we are ready to write down the map on $U_S(\delta)$ (on $M \setminus U_S(\delta)$ is the identity for all $t \in [0,1]$):

$$\begin{aligned} \Upsilon_\rho : \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M) \times (0,1] &\longrightarrow \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \\ (g,t) &\longmapsto F_t^2 \left(dr^2 + r^2\sigma_{n-k-1} \oplus f^2_{t,\rho} \left(\eta_{t,\rho} g_{|S} + (1 - \eta_{t,\rho})\sigma_k \right) \right), \end{aligned} \tag{2.4.7}$$

and $g_{|M \setminus U_S(\delta)}$ remains unchanged.
As we will see, the modifications happening on $U_S(2t\rho)$ are unimportant.
The conformal factor F_t and the correction term $f_{t,\rho}$, varying as t varies (in dotted lines), are drawn in Figure 2.8 below:

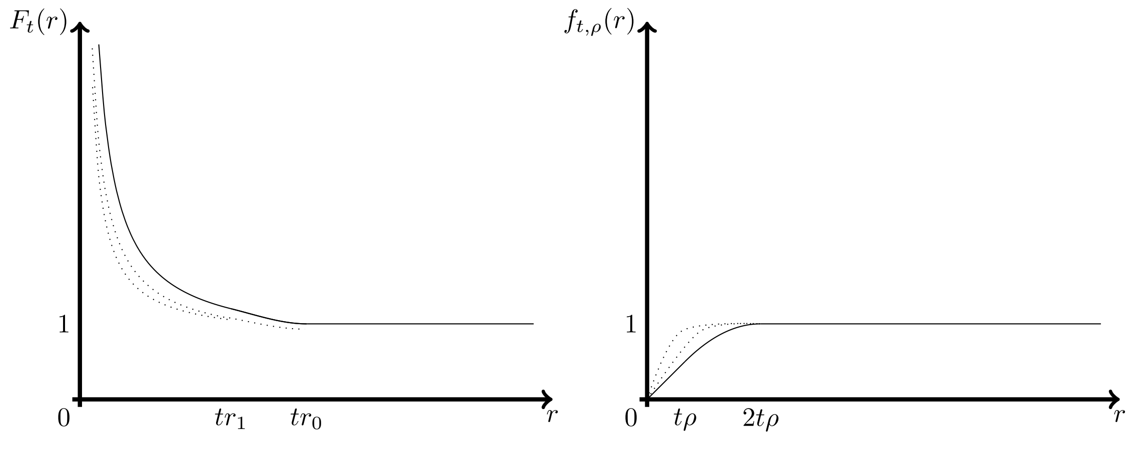


Figure 2.8: The funtions $F_t(r)$ and $f_{t,\rho}(r)$.

The choice of our particular conformal factor F_t (not defined for $p \in S$ for $t \neq 0$) as a drawback forces us to leave the space of metrics on M and pass to the ones on $M \setminus S$. As for the first part of the nullhomotopy $\mathbf{ADH}_\delta$ in Section 2.3, we will have to check that such map is continuous in both g and t variables and that the kernel of the Dirac operator remains trivial while t varies.

First of all we are due to prove that there exists a value of ρ so that the definition of the map in 2.4.7 is well-posed, meaning that the metric $\Upsilon_\rho(g,t) \in \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$ for all $t \in (0,1]$. For this we need the following proposition:

Proposition 2.4.6. *Let $g \in \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$, with associated value of δ as in Corollary 2.3.6. Then there exists a value $\rho = \rho(g,t)$ with $0 < \rho < \delta$ such that the metric*

$$\Upsilon_\rho(g,t) = F_t^2 \left(dr^2 + r^2 \sigma_{n-k-1} \oplus f_{t,\rho}^2 \left(\eta_{t,\rho} g_{|S} + (1 - \eta_{t,\rho}) \sigma_k \right) \right)$$

has an invertible associated Dirac operator, for all $\rho' \leq \rho$ and all $t \in (0,1]$.

The proof is almost the same of [ADH09, Proposition 3.2], except that here we perform no surgery but look at the blown up manifold $M \setminus S$ and that we introduce the parameter t. The manifold $M \setminus S$ happens to be noncompact although complete, due to the asymptotically cylindrical metric around S.

Proof. We will show by contradiction that there must exist such a ρ for any fixed value of $t \in (0,1]$. Suppose that $\forall \rho > 0$ the metric $\Upsilon_\rho(g,t)$ has at least one harmonic spinor ψ_ρ. Then we can build a sequence of $D^{\Upsilon_{\rho_i}(g,t)}$-harmonic spinors $\{F_t^{\frac{n-1}{2}} \psi_{\rho_i}\}_i$ defined on $\Sigma^g M \setminus S$ converging to an harmonic spinor on (M,g) for the Dirac operator D^g.
Suppose then that for any value of ρ, the metric $\Upsilon_\rho(g,t)$ has at least one harmonic spinor.
From Figure 3 above we see that we want to choose

$$0 \leq \rho \leq r_1 \leq r_0 \leq \delta,$$

with δ as in Corollary 2.3.6. We fix $t \in (0,1]$, choose $s \in (2t\rho, tr_1/2)$ and define a cut-off function

$$\chi(r) = \begin{cases} 1, & \text{on } U_S(s), \\ 0, & \text{on } M \setminus U_S(2s), \end{cases}$$

such that $|d\chi|_g \leq 2/s$ on $U_S(2s) \setminus U_S(s)$.
Consider an harmonic spinor for $\Upsilon_\rho(g,t)$, say ψ_ρ. Then $\chi\psi_\rho$ is not influenced by the behaviour of the metric outside $U_S(2s)$. The metric on $U_S(2s)$ is isometric through $r = -\ln u$ to

$$du^2 \oplus \sigma_{n-k-1} \oplus \widetilde{g},$$

$\widetilde{g}$ being a Riemannian metric on S whose form is unimportant at the moment.
As the squared of the Dirac operator for product manifolds $(M_1 \times M_2, g_1 \oplus g_2)$ splits as

$$\left(D^{g_1 \oplus g_2}\right)^2 = \left(D^{g_1}\right)^2 + \left(D^{g_2}\right)^2$$

we have that $D^{\Upsilon_\rho(g,t)}$, with domain restricted to smooth spinors with support in $U_S(2s)$ (like $\chi\psi_\rho$), has the same spectral gap of the standard round metric σ_{n-k-1} on the $(n-k-1)$-sphere (Corollary 1.3.20), i.e.

$$\frac{\int_{U_S(2s)} |D^{\Upsilon_\rho(g,t)}(\chi\psi_\rho)|^2 dv^{\Upsilon_\rho(g,t)}}{\int_{U_S(2s)} |\chi\psi_\rho|^2 dv^{\Upsilon_\rho(g,t)}} \geq \frac{1}{4}. \tag{2.4.8}$$

Since $D^{\Upsilon_\rho(g,t)}\psi_\rho = 0$ by hypothesis, it follows that

$$|D^{\Upsilon_\rho(g,t)}\chi\psi_\rho| = |d\chi|_{\Upsilon_\rho(g,t)}|\psi_\rho|,$$

as $\Upsilon_\rho(g,t)$ on $M \setminus U_S(2t\rho)$ is exactly $F_t^2 g$, which means $\frac{1}{r^2}g$. On the other hand, as supp $d\chi = U_S(2s) \setminus U_S(s) \subset M \setminus U_S(2t\rho)$, for the differential of χ holds:

$$|d\chi|^2_{\Upsilon_\rho(g,t)} = r^2 |d\chi|^2_g \leq \frac{4r^2}{s^2}$$

on $U_S(2s) \setminus U_S(s)$. Knowing this, we can use equation (1.3.4) to estimate the $L^2(U_S(2s))$-norm as

$$\begin{aligned}\int_{U_S(2s)} |D^{\Upsilon_\rho(g,t)}\chi\psi_\rho|^2 dv^{\Upsilon_\rho(g,t)} &\leq \frac{4}{s^2}\int_{U_S(2s)\setminus U_S(s)} r^{2-n} r^{n-1} |r^{-\frac{n-1}{2}}\psi_\rho|^2 dv^g \\ &\leq \frac{4}{s^2}\int_{U_S(2s)\setminus U_S(s)} r |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^{g,b} \\ &\leq \frac{8}{s}\int_{U_S(2s)\setminus U_S(s)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g,\end{aligned} \tag{2.4.9}$$

since by hypothesis $r \leq 2s$ on the domain of integration.
For the denominator of the Rayleigh quotient (2.4.8) we have $\Upsilon_\rho(g,t) = F_t^2 g$ and $0 \leq \chi \leq 1$, on $U_S(s) \setminus U_S(2t\rho)$:

$$\begin{aligned}\int_{U_S(2s)} |\chi\psi_\rho|^2 dv^{\Upsilon_\rho(g,t)} &\geq \int_{U_S(s)\setminus U_S(2t\rho)} |\psi_\rho|^2 dv^{\Upsilon_\rho(g,t)} \\ &= \int_{U_S(s)\setminus U_S(2t\rho)} r^{(n-1)-n} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g \\ &\geq \frac{1}{s}\int_{U_S(s)\setminus U_S(2t\rho)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g.\end{aligned} \tag{2.4.10}$$

Joining the inequalities (2.4.9) and (2.4.10) inside (2.4.8) we obtain

$$\frac{1}{4} \leq \frac{\frac{8}{s}\int_{U_S(2s)\setminus U_S(s)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g}{\frac{1}{s}\int_{U_S(s)\setminus U_S(2t\rho)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g},$$

from which it follows that

$$\int_{U_S(s)\setminus U_S(2t\rho)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g \leq 32 \int_{U_S(2s)\setminus U_S(s)} |F_t^{\frac{n-1}{2}}\psi_\rho|^2 dv^g. \tag{2.4.11}$$

The estimate (2.4.11) easily implies that the L^2-norm of any harmonic spinor ψ_ρ relative to the metric $\Upsilon_\rho(g,t)$ tends to avoid the asymptotically cylindrical neck.
We have infact that, for any $\psi_\rho \in \ker D^{\Upsilon_\rho(g,t)}$, $\varphi_\rho := F_t^{\frac{n-1}{2}}\psi_\rho \in \Gamma(\Sigma^g M \setminus S)$ with

$$\int_{M\setminus U_S(s)} F_t^{-1}|\psi_\rho|^2 dv^{\Upsilon_\rho(g,t)} = \int_{M\setminus U_S(s)} |\varphi_\rho|^2 dv^g = 1 \tag{2.4.12}$$

satisfies $D^g\varphi_\rho = 0$ on the compact set $M \setminus U_S(2t\rho)$ by (1.3.4) and satisfies the inequality (2.4.11)

$$\int_{U_S(2s)\setminus U_S(2t\rho)} |\varphi_\rho|^2 dv^g \leq 32 \int_{U_S(2s)\setminus U_S(s)} |\varphi_\rho|^2 dv^g.$$

Now, for any choice of $\mu \geq 2t\rho$ we notice that

$$\int_{U_S(s)\setminus U_S(2t\rho)} |\varphi_\rho|^2 dv^g \geq \int_{U_S(s)\setminus U_S(\mu)} |\varphi^i_\rho|^2 dv^g,$$

and hence

$$\int_{U_S(s)\setminus U_S(\mu)} |\varphi_\rho|^2 dv^g \leq 32 \int_{U_S(2s)\setminus U_S(s)} |\varphi_\rho|^2 dv^g.$$

By the properties of the Lebesgue integral, we can subdivide $M \setminus U_S(\mu)$ in disjoint sets to have:

$$\int_{M\setminus U_S(\mu)} |\varphi_\rho|^2 dv^g = \int_{M\setminus U_S(s)} |\varphi_\rho|^2 dv^g + \int_{U_S(s)\setminus U_S(\mu)} |\varphi_\rho|^2 dv^g,$$

then we finally get

$$\begin{aligned}\int_{M\setminus U_S(\mu)} |\varphi_\rho|^2 dv^g \leq & \underbrace{\int_{M\setminus U_S(s)} |\varphi_\rho|^2 dv^g}_{=1 \text{ by } (2.4.12)} + \int_{U_S(s)\setminus U_S(\mu)} |\varphi_\rho|^2 dv^g \\ \leq & (1+32) \int_{M\setminus U_S(s)} |\varphi_\rho|^2 dv^g \leq 33 \end{aligned} \tag{2.4.13}$$

Choose a sequence $\rho_i \to 0$ as $i \to \infty$. Inequality (2.4.13) tells us that as $\rho_i \to 0$ the sequence of D^g-harmonic spinors $\{\varphi_{\rho_i}\}_i$ on $M \setminus U_S(2t\rho)$ remains bounded in $L^2(\Sigma^g M \setminus S)$.
We will show now that the sequence of spinors $\{\varphi_{\rho_i}\}_i \subset \Gamma(\Sigma^g M \setminus S)$, converges to a smooth D^g-harmonic spinor on all of M.
As the sequence $\{\varphi_{\rho_i}\}_i$ is bounded in $L^2(\Sigma^g(M \setminus U_S(\mu)))$ it follows, by Lemma (1.3.8) that it is also $\mathcal{C}^2$-bounded on $M \setminus U_S(\mu+\epsilon)$ for any $\epsilon > 0$. Since the set $M \setminus U_S(\mu+\epsilon)$ is compact for any choice of ϵ, we can apply Ascoli-Arzéla theorem [Jos10, Appendix A] and obtain that, up to subsequence, $\varphi_{\rho_i} \to \Phi_0 \in \mathcal{C}^1(\Sigma^g(M \setminus U_S(\mu+\epsilon)))$. We build subsequences $i_{k_1}, \ldots, i_{k_n}, \ldots$ such that the harmonic spinor sequence $\varphi_{\rho_{i_j}} \to \Phi_j \in \mathcal{C}^1\left(\Sigma_g M \setminus U_S\left(\frac{\mu+\epsilon}{j}\right)\right)$. By passing to a diagonal sequence, we have that the limit spinor Φ lies in $\mathcal{C}^1_{\mathrm{loc}}(M \setminus S)$ and since φ_ρ is a D^{g_b}-harmonic spinor on $M \setminus U_S(2t\rho)$, it follows that Φ is D^g-harmonic on $M \setminus S$.
Equation (2.4.13) together with (2.4.12) implies that

$$1 \leq \|\Phi\|^2_{L^2(\Sigma^g M)} \leq 33$$

and hence by Proposition 1.3.9 we know that Φ is a weak harmonic spinor on all of M. By standard elliptic regularity (Lemma 1.3.6) Φ is also a harmonic spinor on M in the strong sense.
The normalization to 1 on $M \setminus U_S(s)$ prevents $\{\varphi_{\rho_i}\}$ to converge to 0, implying that the metric $g \in \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ has at least one harmonic spinor, which is absurd. Therefore there exists a value of ρ such that, for a fixed $t \in [0,1]$ the metric $\Upsilon_\rho(g,t) \in \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$.
For any value of $t \in (0,1]$ the metric $\Upsilon_\rho(g,t)$ on $M \setminus S$ is complete, but $M \setminus S$ is not compact. It might therefore happen that 0 is not a proper eigenvalue but still belongs to the essential spectrum (Proposition 1.3.5). This is not our case since the slice of the cylindrical end is isometric to $(S^{n-k-1} \times S^k, \sigma_{n-k-1} \oplus \widetilde{g})$ with $n-k-1 \geq 1$ and bounding spin structure on S^{n-k-1}. By [Bär00] the essential spectrum of $D^{\Upsilon_\rho(g,t)}$ is then empty, independently of the choice of ρ, $t \in (0,1]$ and $g \in \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$.
To conclude the proof, we show that for any other $\rho' \leq \rho$, $\Upsilon_{\rho'}(g,t) \in \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$. Suppose the countrary, then

$$\forall \rho > 0,\ \exists \rho' < \rho \text{ such that } \dim\ker \Upsilon_{\rho'}(g,t) \neq \{0\}.$$

By taking a sequence of such ρ' we can obtain a sequence of metrics $\Upsilon_{\rho'_i}(g,t)$ with non-invertible Dirac operator. Looking at the limit (possibly passing to subsequences as in the above proof of existence of ρ) we obtain the absurd statement $g \notin \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$. □

The following two lemmata, whose proofs are very similar to the one of Section 2.3 (namely Lemmata 2.3.5 and 2.3.6) prove that the map Υ_ρ restricted to compact families of metrics $B \hookrightarrow \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ can be extended to a continuous map on $B \times [0,1]$.

Lemma 2.4.7. *The value of $\rho = \rho(g,t)$ of Proposition 2.4.6 can be chosen to vary continuously with respect to $t \in (0,1]$ and $g \in B \subset \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$.*

Proof. The proof goes along the way of that of Lemma 2.3.5, showing that the same value of ρ fits for neighbouring metrics and t's. In this case it is again crucial that the distance function $d_g(S,\cdot)$ is $\mathcal{C}^\infty$-continuous with respect to g (Proposition 1.4.10).
We want to prove that, chosen $\rho = \rho(g_1,t_1)$,

$$\forall K \subset M \setminus S,\ \exists C = C(K) \text{ such that } \|\Upsilon_\rho(g_1,t_1) - \Upsilon_\rho(g_2,t_2)\|_{\mathcal{C}^\infty(K)} \le C\left(|t_1 - t_2| + \|g_1 - g_2\|_{\mathcal{C}^\infty(M)}\right). \tag{2.4.14}$$

Lipschitzianity of the $\mathcal{C}^\infty_{\mathrm{loc}}(M \setminus S)$ functions $F_t, f_{t,\rho}$ and η_ρ allows to reach an estimate of the desired form for some constant C' and L the biggest Lipschitz constant among $F_t, f_{t,\rho}$ and η_ρ (see the proof of Lemma 2.3.5). Even though the Lipschitz constant of any of the function $F_t, f_{t,\rho}$ and $\eta_{t,\rho}$ diverges for $t \to 0$, we have that the region $U_{S,g}(2t\rho)$ where L is unbounded converges to $M \setminus S$ as $t \to 0$, remaining bounded on any compact set K.
The constants C' and L might then depend from the compact set K appearing in (2.4.14), but are independent of g and t.
Then using a partition of unity of $B \times [0,1]$ we glue together the locally constant values of ρ like in the proof of Lemma 2.3.6 and we reach the end of the proof. □

Define now the map $\widehat{\Upsilon}_\rho$ as

$$\begin{aligned} \widehat{\Upsilon}_\rho : \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M) \times [0,1] &\longrightarrow \mathcal{R} \\ (g, M \setminus S, t) &\longmapsto \begin{cases} (g, M \setminus S), & t = 0, \\ (\Upsilon_\rho(g,t), M \setminus U_S(t\rho)), & t > 0 \end{cases} \end{aligned} \tag{2.4.15}$$

As a consequence of Definition 2.4.2, we have that the map $\widehat{\Upsilon}_\rho$ is continuous also in $t = 0$:

Lemma 2.4.8. *For any compact family of metrics $B \hookrightarrow \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$, the map $\widehat{\Upsilon}_\rho : B \times [0,1] \to \mathcal{R}$ is continuous.*

Proof. It is clear from the definition of $\widehat{\Upsilon}_\rho$ that for any sequence $t_i \to 0$ as $i \to \infty$ and for any $g \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ we have

$$\widehat{\Upsilon}_\rho\,(g, M \setminus S, t_i) = (\Upsilon_\rho(g,t_i), M \setminus U_S(t_i\rho)).$$

Since as $t_i \to 0$ $F_{t_i}, f_{t_i,\rho}, \eta_{t_i,\rho}$ tend to 1 in the $\mathcal{C}^\infty$ compact-open topology on $M \setminus S$ (which for functions from $M \setminus S$ to $\mathbb{R}$ is equivalent to the $\mathcal{C}^\infty_{\mathrm{loc}}(M \setminus S)$ topology) and $d_H\left(M \setminus U_S(t_i\rho), M \setminus S\right) \to 0$, we have that

$$\widehat{\Upsilon}_\rho(g, M \setminus S, t_i) \to (g, M \setminus S) = \widehat{\Upsilon}_\rho(g, M \setminus S, 0),$$

i.e. $\widehat{\Upsilon}_\rho$ is continuous in $t = 0$.
To prove the continuity with respect to g we notice that all the three functions $\eta, F_t, f_{t,rho}$ are smooth in the variables relative to M and continuous in the remaining ones. As for Proposition 2.3.8 on a compact

family of metrics $B \subset \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ one can achieve, using the continuity of the distance $d_g(\cdot, S)$ with respect to g stated in Proposition 1.4.10, the desired inequality

$$\|\Upsilon_\rho(g_1,t) - \Upsilon_\rho(g_2,t)\|_{\mathcal{C}^\infty(M\setminus S)} + d_H(M \setminus U_{S,g_1}(t\rho).M \setminus U_{S,g_2}(t\rho)) \leq C\,\|g_1 - g_2\|_{\mathcal{C}^\infty(M\setminus S)}$$

for some constant C.
To prove the continuity with respect to $t \in (0,1]$ we use once more the fact that the functions $F_t, f_{t,\rho}$ and $\eta_{t,\rho}$ are smooth and hence Lipschitz with respect to t. As the distance function $d_{\Upsilon_\rho(g_b,t)}$ converges to $d_{\Upsilon_\rho(g_b,\bar{t})}(S,\cdot)$ as $t \to \bar{t}$, for any $\bar{t} \in (0,1]$, by the continuity of the distance function with respect to the metric, the continuity of the map $\widehat{\Upsilon}_\rho$ over compact families of asymptotically cylindrical metrics is ensured. □

2.4.3 The missing part of $\widehat{\Upsilon}_\rho$

The map $\widehat{\Upsilon}_\rho$ is the prototype of a nullhomotopy of pairs $\left(\mathcal{R}, \widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)\right)$. What we miss and don't know how to define is the behaviour of $\widehat{\Upsilon}_\rho$ on metrics which are already of asymptotically cylindrical form.

Missing Lemma 2.4.9. *The deformation $\widehat{\Upsilon}_\rho$ is continuous as a map*

$$\widehat{\Upsilon}_\rho : B \times [0,1] \to \mathcal{R}$$

for any compact family of elements $B \subset \mathcal{R}$. Moreover it preserves metrics of asymptotically cylindrical form:

$$\widehat{\Upsilon}_\rho(\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S), t) \subset \widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S), \quad \forall t \in [0,1].$$

In case the Missing Lemma 2.4.9 was true, we would be ready at this point to prove that the space $\mathcal{R}$ is weakly homotopically equivalent to $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$:

Corollary 2.4.10. *Let M^n be a closed spin manifold and $\iota : S^k \hookrightarrow M^n$ a submanifold of codimension $n-k$ at least 2. Let $S = \iota(S^k)$, and suppose there exists an extension $\widehat{\Upsilon}_\rho$ to $\mathcal{R}$ for which Lemma 2.4.9 is true. Then it holds*

$$\mathcal{R} \simeq \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S).$$

Proof. The proof goes along the one of Corollary 2.3.14, *mutatis mutandis* the spaces of D-invertible metrics involved; we sketch it here for convenience of the reader.
The proof that $\pi_0(\mathcal{R})$ and $\pi_0\left(\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)\right)$ are in bijective correspondence is obtained, from any curve $f : [0,1] \to \mathcal{R}$ with $f(0), f(1) \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$ by noticing that the curve

$$\bar{f}(t) = \begin{cases} \widehat{\Upsilon}_\rho(f(0), 3t) & t \in [0,1/3], \\ \widehat{\Upsilon}_\rho(f(3t-1), 1) & t \in [1/3, 2/3], \\ \widehat{\Upsilon}_\rho(f(1), 3-3t) & t \in [2/3, 1], \end{cases}$$

is entirely contained in $\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$ and continuous.
To show that a bijection holds also on higher order homotopy groups, consider the long exact sequence of

the pair $\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right)$:

$$\cdots \xrightarrow{\partial_{l+2}} \pi_{l+1}\left(\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right) \xrightarrow{i^*} \pi_{l+1}\left(\mathcal{R}\right) \xrightarrow{p^*}$$

$$\underbrace{\pi_{l+1}\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right)}_{=0} \xrightarrow{\partial_{l+1}} \pi_{l}\left(\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right) \xrightarrow{i^*} \pi_{l}\left(\mathcal{R}\right) \xrightarrow{p^*}$$

$$\underbrace{\pi_{l}\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right)}_{=0} \xrightarrow{\partial_{l}} \pi_{l-1}\left(\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right) \xrightarrow{i^*} \cdots$$

The map $\widehat{\Upsilon}_\rho$, assuming by hypothesis that the Missing Lemma 2.4.9 holds and it has been proved in Proposition 2.4.6 and Lemma 2.4.8 to satisfy all the properties of a nullhomotopy of pairs. As such, the nullhomotopy $\widehat{\Upsilon}_\rho$ annihilates any homotopy class

$$[f] = \left[(D^l, S^{l-1}), \left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)\right)\right],$$

implying that the map i^* is a isomorphism of homotopy groups for any $l \in \mathbb{N}$.
We deduce that i^* is a weak homotopy equivalence.
To conclude, we compose the weak homotopy equivalence $\mathcal{R} \simeq \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)$ with the homotopy equivalence $\widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S) \overset{(2.4.6)}{\simeq} \mathcal{R}^{\text{inv}}_{\text{cyl}}(M\setminus S)$ to obtain

$$\mathcal{R} \simeq \mathcal{R}^{\text{inv}}_{\text{cyl}}(M\setminus S).$$

□

In the previous corollary we use a space of D-invertible metrics whose form is fixed on the subset $M \supset U_S(\rho) \cong (0,\rho)\times S^k \times S^{n-k-1}$ to be $r^{-2}dr^2 \oplus \sigma_k \oplus \sigma_{n-k-1}$. Hence we cannot provide a proof of the Corollary for a general submanifold N with trivial normal bundle as for Corollary 2.3.14 as N might not be endowed with the metric σ_k.

2.4.4 The weak homotopy equivalence $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \simeq \mathcal{R}$

In this subsection a particular choice of coordinates will help in the computations and in making the proofs clearer. Our attention will be restricted to the cylindrical neck of a metric, instead of the interior of a tubular neighbourhood around S.
We will use the coordinate $u \in [0,\infty)$ as in Definition 2.4.1, so that an asymptotically cylindrical metric $g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M\setminus S)$ outside a compact set $K \subset M\setminus S$ will be of the form $g_{|_{(M\setminus S)\setminus K}} = du^2 \oplus \sigma_{n-k-1}\oplus\sigma_k$.
Moreover we define the cylindrical segment $C_S(L_1, L_2)$ to be, for any $g \in \widehat{\mathcal{R}^{\text{inv}}_{\text{cyl}}}(M\setminus S)$,

$$C_{S,g}(L_1,L_2) := \left\{ p = (u,\phi,\theta) \in (M\setminus S)\setminus K_{\min} \overset{\text{iso}}{\cong} [0,\infty)\times S^{n-k-1}\times S^k \,\middle|\, u \in [L_1, L_2] \right\}$$

and we allow $L_2 = \infty$ meaning the half-infinite cylinder starting at $u = L_1$. The dependence on g (and hence on the additional data $K_{\min}$) will not be made explicit whenever no confusion can arise from the context.

Notice that, with this new notation, the manifold $M \setminus S$ endowed with any asymptotically cylindrical metric can be written as

$$M \setminus S \cong K_{\min} \cup C_S(0). \tag{2.4.16}$$

Recall that, as for surgery we use only spin-preserving embeddings of spheres S^k endowed with the bounding spin structure, also in this case the removed submanifold S^k will be intended endowed with the bounding spin structure. This assumption will be essential in the proof of the well-definedness of $\widehat{\Xi}_L$.
For the second nullhomotopy $\widehat{\Xi}_L$ we will have to pinch the cylindrical neck "at infinity" in a way that the *ad hoc* defined topology on $\mathcal{R}$ will ensure the continuity of $\widehat{\Xi}_L$.
We start by defining the map

$$\begin{aligned} \Xi_L : \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \times [0,1) &\longrightarrow \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \\ (g,t) &\longmapsto h_t \oplus \sigma_k \end{aligned} \tag{2.4.17}$$

where h_t is the smooth metric on $[0,\infty) \times S^{n-k-1}$

$$h_t = (1 - t\eta_{t,L})(du^2 \oplus \sigma_{n-k-1}) + t\eta_{t,L} \mathring{U}^*_{e^{-2L}} e^{-2u} \left(du^2 \oplus \sigma_{n-k-1}\right),$$

where $\eta_{t,L}(u)$ is a smooth cut-off function such that

$$\eta_{t,L}(u) = \begin{cases} 1, & 2L \leq u \leq \dfrac{4L}{1-t}; \\ 0, & 0 \leq u \leq L \,\cup\, u \geq \dfrac{5L}{1-t}. \end{cases}$$

and extended to 0 on the remaining part of $M \setminus S$, while $\mathring{U}_{e^{-2L}}$ is the diffeomorphism from Proposition 2.3.13 obtained by extension of the diffeomorphism

$$\begin{aligned} \mathring{u}_{e^{-2L}} : \exp^{\perp}_{g_0}\left((D^{n-k} \setminus \{0\}) \times S^k\right) &\longrightarrow U_S(e^{-2L}) \setminus S \\ p &\longmapsto \exp^{\perp}_{g} \circ \iota \circ (\Phi^{-1}_{\delta} \times \mathrm{Id}_N) \circ \iota^{-1} \circ \left(\exp^{\perp}_{g_0}\right)^{-1}(p) \end{aligned}$$

to all $M \setminus S$.

The value of the length $L = L(g)$ will be determined in the next proposition.

Proposition 2.4.11. *For any asymptotically cylindrical metric $g \in \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$ there exists a value of $L = L(g) > 0$ such that the Riemannian metric $\Xi_L(g,t)$ has an invertible Dirac operator for any value of $t \in [0,1)$, provided the submanifold $S = \iota(S^k)$ has dimension $k \geq 1$ and is endowed with the bounding spin structure.*

Proof. As noticed in (2.4.16), we can think of $M \setminus S$ endowed with an asymptotically cylindrical metric g as the union

$$(M \setminus S, g) \overset{\mathrm{iso}}{\cong} (K_{\min}, g_{|K_{\min}}) \cup (C_S(0), du^2 \oplus \sigma_{n-k-1} \oplus \sigma_k) \tag{2.4.18}$$

i.e. attaching an half-infinite cylinder $\left([0,\infty) \times \partial K_{\min}, du^2 \oplus g_{|\partial K_{\min}}\right)$ along the boundary of $K_{\min}$.
Therefore the spin manifold with boundary $(K_{\min}, g_{|K_{\min}})$ is D-invertible in the sense of Dahl [Dah08, Section 1]: there exists a $\epsilon^g > 0$ such that, for any L^2-spinor φ of $K_\infty := [0,\infty) \times \partial K_{\min}$ it holds the lower bound

$$\frac{\|D^g \varphi\|_{L^2(K_\infty)}}{\|\varphi\|_{L^2(K_\infty)}} \geq \epsilon^g$$

On the other hand, the deformation Ξ_L leaves the S^k-factor of the cylindrical end unmodified for all $t \in [0,1)$.

It follows that each one of the family of Dirac operators $\left(D^{\Xi_L(g,t)}\right)^2$ splits on $C_S(0)$ as a sum of operators $D_1^2 \oplus D_2^2$ since the metric is of block form

$$\underbrace{h_t}_{\text{block 1}} \oplus \underbrace{\sigma_k}_{\text{block 2}}$$

there. By hypothesis $k \geq 1$ and the sphere factor S^k is endowed with the bounding spin structure and the standard round metric, we have then by Corollary 1.3.20 that the family of operators $D^{\Xi_L(g,t)}$ has a uniform spectral gap of width at least $\frac{1}{4}$ when restricted to the neck $C_S(L)$ modified by Ξ_L. This implies that also the manifold with boundary

$$\left(C_S\left(L, \frac{5L}{1-t}\right), h_t \oplus \sigma_k\right)$$

is invertible in the sense of Dahl. We have then proved that the disconnected manifold

$$\left(K_{\min} \sqcup C_S\left(L, \frac{5L}{1-t}\right),\, g_{|K_{\min}} \sqcup h_t \oplus \sigma_k\right)$$

has a uniform lower bound on the width of the spactral gap $\min\left\{\epsilon^g, \frac{1}{4}\right\}$.
From [Dah08, Proposition 2.1] there exists a value of $L > 0$ such that the Dirac operator $D^{\Xi_L(g,t)}$ on

$$K_{\min} \cup \underbrace{C_S(0,L)}_{\cong [0,L]\times \partial K_{\min}} \cup C_S\left(L, \frac{5L}{1-t}\right) \cup C_S\left(\frac{5L}{1-t}\right) \cong M \setminus S$$

endowed with the asymptotically cylindrical metric

$$g_{|K_{\min}} \cup du^2 \oplus g_{|\partial K_{\min}} \cup h_t \oplus \sigma_k \cup du^2 \oplus \sigma_{n-k-1} \oplus \sigma_k$$

is D-invertible.
As Ξ_L does not modify the compact set $K_{\min}$, the value of ϵ^g, defined as the width of the spectral gap on the manifold K_∞, does not vary for $t \in [0,1)$. The same holds for the cylinder $h_t \oplus \sigma_k$ because the constant factor σ_k ensures a constant lower bound in the width of the spectral gap on $C_S\left(L, \frac{5L}{1-t}\right)$ of $\frac{1}{4}$, again by Corollary 1.3.20. It follows that also L is independent of t. □

In the same way we proved Lemma 2.4.7 we can also prove that the length of the cylindrical neck $L = L(g)$ we need to keep the operator $D^{\Xi_L(g,t)}$ invertible depends continuously with respect to g. In this case the proof is easier compared to the cases of δ and ρ.

Lemma 2.4.12. *Let $B \subset \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ be a compact family of metrics. Then the quantity $L = L(g_b)$ depends continuously with respect to g_b. In particular L can be chosen to be constant on B.*

Proof. As stated in the proof of Proposition 2.4.11 and in [Dah08, Proposition 2.1], the value of L such that

$$\frac{\left\|D^{\Xi_L(g_b,t)}\varphi\right\|^2_{L^2(M\setminus S)}}{\|\varphi\|^2_{L^2(M\setminus S)}} \geq \epsilon^{\Xi_L(g,t)} > 0$$

is related to the spectral gap ϵ^g of the manifold $(M \setminus S, g) \cong \left(K_{\min} \cup [0,\infty) \times \partial K_{\min},\, g_{|K_{\min}} \cup du^2 \oplus g_{|\partial K_{\min}}\right)$ through

$$\left(\frac{\epsilon^g}{4} - \frac{6}{L^2}\right) = \epsilon^{\Xi_L(g,t)}. \tag{2.4.19}$$

Since the width of the spectral gap ϵ^g on $K_{\min} \cong (M \setminus S) \setminus C_S(0)$ endowed with the metric $g_{|K_{\min}}$, depends in a $\mathcal{C}^\infty$ fashion from the metric g by [Bär96, proof of Proposition 7.1], we obtain the continuity of $L = L(g)$.

For compact families of asymptotically cylindrical metrics B we can fix a lower bound for the width of the spectral gap ϵ^g, say ϵ^B. Then formula (2.4.19) would give the desired constant value of L. □

In a completely analogous way to that of Lemma 2.4.8 we can prove that the map Ξ_L is continuous with respect to both g and t. We will only state the result, omitting the proof.

Lemma 2.4.13. *Let $B \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ be a compact family of asymptotically cylindrical metrics, then the map*

$$\Xi_L : B \times [0,1) \longrightarrow \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$$

is continuous.

For notation convenience, we refer to the set $K_{\min}$ relative to a metric $g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ as $M \setminus C_S(0)$, while if $g \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$, $U_S(0) = M \setminus S$.
We extend Ξ_L to a map $\widehat{\Xi}_L$, defined on the whole space $\mathcal{R}$ as

$$\widehat{\Xi}_L : \mathcal{R} \times [0,1] \longrightarrow \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$$

$$(g,K,t) \longmapsto \begin{cases} \begin{cases} (g, C_S(10tL)), & t \in \left[0, \frac{1}{2}\right], \\ \left(\Xi_L(g, 2t-1), C_S\left(\frac{5L}{2-2t}\right)\right), & t \in \left[\frac{1}{2}, 1\right), \\ (\cap(g), M \setminus S), & t = 1, \end{cases} & g \in \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S), \\ (g,K), & g \in \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M), \end{cases} \tag{2.4.20}$$

the map $\cap$ glues the submanifold S at the top of the cylindrical neck.
We will show in the next Lemma that such extension is continous and well-defined.

Lemma 2.4.14. *For any compact family of metrics $B \subset \mathcal{R}$ the map $\widehat{\Xi}_L$ as defined in* (2.4.20) *is continuous with respect to t and well-defined.*

Proof. We choose a constant L for the compact family B as in Lemma 2.4.12.
For $t \to 1$ the compact set $K_{\min} \to M \setminus S$. The fact that the capping at $t = 1$ is continuous is ensured by the topology on $\mathcal{R}$ which is uneffected by changes on S, which, in this case, it has to be considered like the "submanifold at infinity" added to compactify $M \setminus S$. Since, by hypothesis, S is endowed with the bounding spin structure, capping off the cylindrical neck preserves the original spin structure of M.
On the compact manifold M, for $t = 1$, we can change the coordinates

$$r = e^{-u}, \quad dr = -e^{-u}du, \tag{2.4.21}$$

so that

$$\widehat{\Xi}_L(g,1)_{|C_S(2L)\cup S} = U^*_{e^{-2L}}\left(e^{-2u}du^2 + e^{-2u}\sigma_{n-k-1} \oplus \sigma_k\right) = U^*_{e^{-2L}}\left(dr^2 + r^2\sigma_{n-k-1} \oplus \sigma_k\right).$$

.

Moreover we have that $\text{Tub}^*_k(\widehat{\Xi}_L(g,1)) = (f'(r))^2dr^2 + f^2(r)\sigma_{n-k-1} \oplus \sigma_k$. The manifold with boundary

$$\left(D^{n-k} \times S^k, \text{Tub}^*_k(\widehat{\Xi}_L(g,1))\right)$$

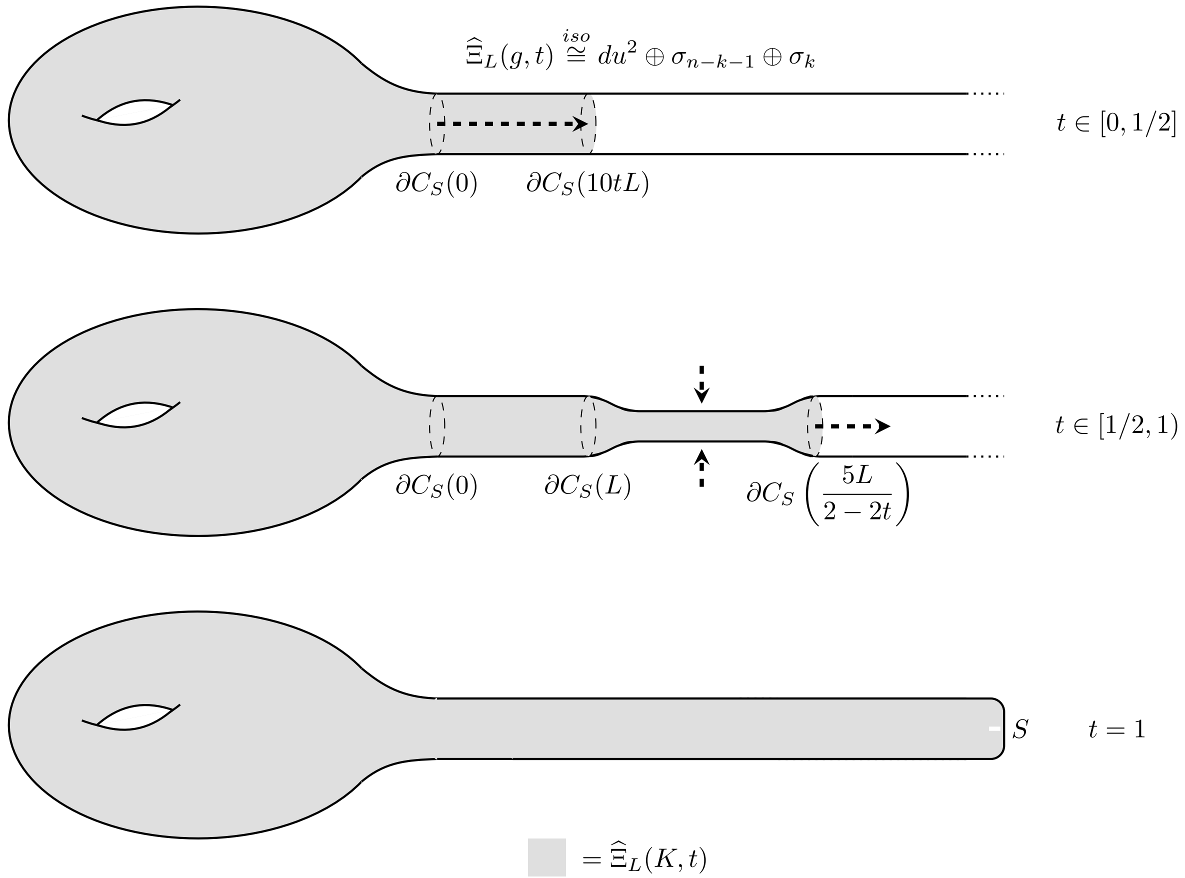


Figure 2.9: Schematic action of the map $\widehat{\Xi}_L$.

when attached an half-infinite cylinder $[0,\infty)\times S^{n-k-1}\times S^k$ to its boundary, has an invertible Dirac operator, as σ_k ensures that the width of the spectral gap in

$$\left(D^{n-k}\times S^k \cup [0,\infty)\times S^{n-k-1}\times S^k,\ f'(s)^2ds^2 + f^2(s)\sigma_{n-k-1}\oplus\sigma_k \cup du^2\oplus\sigma_{n-k-1}\oplus\sigma_k\right)$$

is at least $\frac{1}{4}$ by Corollary 1.3.20. Since by definition the metric $g_{|K_{\min}}$ on $K_{\min}$ has a spectral gap of width, say, ϵ^g, the disconnected manifold with boundary

$$\left(K_{\min}\sqcup \mathrm{Tub}_k(D^{n-k}\times S^k),\ g_{|K_{\min}}\sqcup \widehat{\Xi}_L(g,1)_{|_{\mathrm{Tub}_k(D^{n-k}\times S^k)}}\right)$$

has a spectral gap of width $\min\left\{\epsilon^g, \frac{1}{4}\right\}$.
An argument similar to that of the proof of Lemma 2.4.11 allows us to conclude that the manifold obtained attaching

$$K_{\min}\cup[0,L]\times S^{n-k-1}\times S^k\cup \mathrm{Tub}_k(D^{n-k}\times S^k)\cong M$$

endowed with the metric

$$g_{|K_{\min}}\cup du^2\oplus\sigma_{n-k-1}\oplus\sigma_k\cup\widehat{\Xi}_L(g,1)_{|_{\mathrm{Tub}_k(D^{n-k}\times S^k)}}$$

has an invertible Dirac operator.
This shows that the map $\widehat{\Xi}_L$ is well-defined and continuous for $t=1$. □

We have checked continuity with respect to t, we are going to prove it also holds with respect to g:

Lemma 2.4.15. *The map $\widehat{\Xi}_L$ is continuous with respect to $g \in \mathcal{R}$ for any compact family of metrics $B \subset \mathcal{R}$.*

Proof. We can prove that the map $\widehat{\Xi}_L$ is continuous on the space $\widehat{\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}}(M \setminus S)$ using once more smoothness of the metrics involved and the continuous dependence of the distance function with respect to the metric, as for Lemma 2.4.8. We will therefore focus on metrics in the space $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$.
Fix a value of $t \in (0,1)$ (for $t=0$ continuity is obvious, for $t=1$ it was proved in Lemma 2.4.14).
Since the space $\mathcal{R}$ is sequential, we will test the continuity of $\widehat{\Xi}_L$ along a sequence of elements $\{g_i, K_i\}_i \subset \mathcal{R}$ converging to $(g, M\setminus S) \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$. Since by Lemma 2.4.12 L can be chosen to be constant on the compact family of metrics B, convergence in $\mathcal{R}$ implies, by (2.4.2), that the compact sets K_i tend to exhaust the manifold $M \setminus S$ for $i \to \infty$. It follows that

$$\left\| \widehat{\Xi}_L(g_i, K_i, t) - g \right\|_{\mathcal{C}^\infty(M\setminus S)} + \underbrace{d_H(K_i, M \setminus S)}_{\to 0 \text{ by hypothesis}} \to 0$$

by the very definition of compact open topology on $M \setminus S$: for $i \to \infty$ the modification given by $\widehat{\Xi}_L$, taking place only inside $(M \setminus S) \setminus K_i$, will be constricted in the empty set:

$$\left\| \widehat{\Xi}_L(g_i, K_i, t) - g \right\|_{\mathcal{C}^\infty(K)} = 0, \quad \forall K \subset K_i \subset M \setminus S.$$

Letting $K_i \to M \setminus S$ proves the assertion. ☐

The last property of the map $\widehat{\Xi}_L$ that we have to show is the invariance of the space of half-flat metrics as t varies.

Lemma 2.4.16. *The deformation $\widehat{\Xi}_L$ preserves metrics in $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$, i.e.*

$$\widehat{\Xi}_L(g,t) \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M), \quad \forall g \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M),\, t \in [0,1].$$

Proof. Immediate from Definition 2.4.20: on the subspace $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M) \subset \mathcal{R}$ the map $\widehat{\Xi}_L(\cdot, t)$ is defined as the identity $\mathrm{Id}_{\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)}$. ☐

As a corollay of the above lemmata we get the desired weak homotopy equivalence:

Corollary 2.4.17. *Let M^n be a closed spin manifold of dimension at least 2 and $\iota : S^k \hookrightarrow M^n$ a submanifold of dimension k at least 1. Then*

$$\mathcal{R} \simeq \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M).$$

Proof. The proof goes along the one of Corollary 2.3.14, *mutatis mutandis* the spaces of D-invertible metrics involved; we sketch it here for convenience of the reader.
The proof that $\pi_0(\mathcal{R})$ and $\pi_0\left(\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)\right)$ are in bijective correspondence is obtained, from any curve $f : [0,1] \to \mathcal{R}$ with $f(0), f(1) \in \widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ by noticing that the curve

$$\bar{f}(t) = \begin{cases} \widehat{\Xi}_L(f(0), 3t) & t \in [0, 1/3], \\ \widehat{\Xi}_L(f(3t-1), 1) & t \in [1/3, 2/3], \\ \widehat{\Xi}_L(f(1), 3-3t) & t \in [2/3, 1], \end{cases}$$

is entirely contained in $\widehat{\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}}(M)$ and continuous.
To show that a bijection holds also on higher order homotopy groups, consider the long exact sequence of

the pair $\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right)$:

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\partial_{l+2}} & \pi_{l+1}\left(\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right) & \xrightarrow{i^*} & \pi_{l+1}(\mathcal{R}) \\
& & & \xrightarrow{p^*} & \\
\underbrace{\pi_{l+1}\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right)}_{=0} & \xrightarrow{\partial_{l+1}} & \pi_l\left(\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right) & \xrightarrow{i^*} & \pi_l(\mathcal{R}) \\
& & & \xrightarrow{p^*} & \\
\underbrace{\pi_l\left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right)}_{=0} & \xrightarrow{\partial_l} & \pi_{l-1}\left(\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right) & \xrightarrow{i^*} & \cdots
\end{array}$$

The map $\widehat{\Xi}_L$ has been proved in Proposition 2.4.11 and Lemmata 2.4.14, 2.4.15, 2.4.16 to satisfy all the properties of a nullhomotopy of pairs and, as such, annihilates any homotopy class

$$[f] = \left[(D^l, S^{l-1}), \left(\mathcal{R}, \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)\right)\right],$$

implying that the map i^* is a isomorphism of homotopy groups for any $l \in \mathbb{N}$.
We deduce that $i : \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M) \hookrightarrow \mathcal{R}$ is a weak homotopy equivalence. To conclude we compose the weak homotopy equivalence $\mathcal{R} \simeq \widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M)$ and the homeomorphism $\widehat{\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}}(M) \overset{2.4.5}{\cong} \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ to obtain

$$\mathcal{R} \simeq \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M).$$

□

Also in this case as for Corollary 2.4.10 a general choice of submanifold (N, g_N) instead of $(S^{n-k-1}, \sigma_{n-k-1})$ is not allowed since the operator D^{g_N} might not have a spectral gap.

2.5 Putting the pieces together

Suppose the map $\widehat{\Upsilon}_\rho$ is a nullhomotopy of pairs, i.e. suppose the Missing Lemma 2.4.9 is true. Then we have that the maps $\mathbf{ADH}_\delta$, $\widehat{\Xi}_L$ and $\widehat{\Upsilon}_\rho$ make the inclusion

$$\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \overset{i}{\hookrightarrow} \mathcal{R}^{\text{inv}}(M)$$

a homotopy equivalence and the two inclusions

$$\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M) \overset{i}{\hookrightarrow} \mathcal{R}, \qquad \mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S) \overset{i}{\hookrightarrow} \mathcal{R}$$

weak homotopy equivalences.
As both $\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S)$ and $\mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M)$ are weak homotopy equivalent to $\mathcal{R}$, by the transitive property of weak homotopy equivalences we have that

$$\mathcal{R}^{\text{inv}}_{\text{cyl}}(M \setminus S) \simeq \mathcal{R}^{\text{inv}}_{\frac{1}{2}\text{flat}}(M). \tag{2.5.1}$$

In this section we will put all the pieces together and prove the main theorem of the chapter:

Theorem 2.5.1. *Suppose the Missing Lemma 2.4.9 holds true. Then for a closed spin manifold M^n of dimension at least 3 the homotopy type of $\mathcal{R}^{\mathrm{inv}}(M^n)$ depends only on its spin cobordism class $[M^n] \in \Omega_n^{Spin}$.*

Proof. Consider a closed spin manifold M^n of dimension at least 3 and a submanifold $S = \iota(S^k)$, $\iota : S^k \hookrightarrow M^n$ of dimension $k \leq n-2$ with trivial normal bundle. Using the exponential map we define the tubular neighbourhood $\exp^\perp \circ \iota : D^{n-k} \times S \longrightarrow M$ of S.
As the codimension of the submanifold S is at least 2, we have by Corollaries 2.3.14 2.4.10 and 2.4.17 the following chain of weak homotopy equivalences:

$$\mathcal{R}^{\mathrm{inv}}(M) \overset{2.3.14}{\simeq} \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M) \overset{(2.5.1)}{\simeq} \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S).$$

Consider now the manifold $\widetilde{M}$ obtained from M via a surgery of dimension k on S. By definition $\widetilde{M} = M \setminus (S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1})$ and we indicate with $\widetilde{S} = \iota(S^{n-k-1})$ the submanifold with trivial normal bundle with tubular neighbourhood $\widetilde{\exp}^\perp \widetilde{\iota} : S^{n-k-1} \times D^{k+1} \longrightarrow M$.

To ensure that the chain of weak homotopy equivalences of Corollaries 2.3.14 2.4.10 and 2.4.17 for the subspace of D-invertible metrics on M subsists also for $\widetilde{M}$, we have to ensure that the dimension k of the surgery performed satisfies $n - (n-k-1) \geq 2$ which is equivalent to $k+1 \geq 2$:

$$\begin{cases} n-k \geq 2, & \text{to fulfill the hypothesis of Lemma 1.3.9 and have a local spectral gap in (2.4.8)} \\ k \geq 1, & \text{to have } D\text{-invertibility on the } S^k \text{ factor in Lemma 2.4.11} \\ k+1 \geq 2, & \text{to be able to apply Lemma 1.3.9 and have a local spectral gap also on } \widetilde{M}. \end{cases}$$

It is immediate to see that the solution of such system of inequalities is

$$1 \leq k \leq n-2. \tag{2.5.2}$$

When performing surgeries in the range (2.5.2), we have a chain of weak homotopy equivalences relating the space of D-invertible Riemannian metrics on M and on $\widetilde{M}$

$$\mathcal{R}^{\mathrm{inv}}(M) \simeq \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M) \simeq \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \overset{\text{by definition}}{\cong} \mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(\widetilde{M} \setminus \widetilde{S}) \simeq \mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(\widetilde{M}) \simeq \mathcal{R}^{\mathrm{inv}}(\widetilde{M}). \tag{2.5.3}$$

Reading the extrema of the chain of weak homotopy equivalences is (2.5.3) we obtain that whenever $\widetilde{M}$ is obtained from M via surgeries in dimension $1 \leq k \leq n-2$,

$$\mathcal{R}^{\mathrm{inv}}(M) \overset{\text{weak}}{\simeq} \mathcal{R}^{\mathrm{inv}}(\widetilde{M}).$$

To conclude, it is sufficient to recall (Remark 1.2.9) that for closed spin manifolds of dimension at least 3 it is possible, starting from any manifold M, to reach any other manifold $\widetilde{M}$ in the spin cobordism class $[M]$. This concludes the proof. ☐

Since Corollaries 2.4.10 and 2.4.17 do not work for minimal metrics, Theorem 2.5.1 holds only for spin manifolds with trivial α-genus, i.e. manifolds M with $\alpha([M]) = 0$. This class of spin manifolds includes of course all spin nullcobordant manifolds (in dimension $n \geq 9$ the map α is not injective [ABP67]). For such class of manifolds by (2.1.1) we have $\mathcal{R}^{\min}(M) = \mathcal{R}^{\mathrm{inv}}(M)$.

Chapter 3

Conclusions and future directions

Despite the small gap in the proof of Conjecture 2.2.4, the research started with the present thesis can be continued in two different directions (let alone the completion of the missing part of the proof of the aforementioned Conjecture 2.2.4): on one side, one can try to reformulate and prove Conjecture 2.2.4 also for manifolds with $\alpha([M]) \neq 0$:

Conjecture 3.0.1. *Let M^n be a closed spin manifold of dimension $n \geq 3$ and let $\widetilde{M}^n$ be obtained from M^n via a sequence of surgeries in dimension $1 \leq k \leq n-2$. Then*

$$\mathcal{R}^{\min}(M^n) \simeq \mathcal{R}^{\min}(\widetilde{M}^n).$$

To adapt the strategy shown in Section 2.4, one should first define minimal metrics with cylindrical ends on open manifolds like $M \setminus S$ and consequently adapt the proof [Dah08, Proposition 2.1].
Alternatively, Conjecture 2.2.4 can be used to explore the existence of metrics with "many" harmonic spinors. We know by [ADH09] that metrics with minimal kernel of the Dirac operator always exist in dimension at least 2 and form a dense subset of $\mathcal{R}iem(M)$. It would be interesting to investigate the existence problem for metrics such that the associated Dirac operator is strictly bigger than the minimal one prescribed by the lower bound in (2.1.1). This question was first posed by N. Hitchin in [Hit74] and then reformulated by C. Bär as

Conjecture 3.0.2. *[Bär96] The dimension of the space of harmonic spinors is not topologically obstructed, i.e. on any closed spin manifold of dimension at least 3, there is a metric which is not D-invertible.*

When $\alpha([M^n]) = 0$ a positive answer was given for dimension at least 6 in [CSS16] and for dimension 3 in [Bär96].
For the remaining dimensions 4 and 5, such existence problem can be solved by obtaining informations on the complementary space

$$\left(\mathcal{R}^{\text{inv}}(M)\right)^c = \mathcal{R}iem(M) \setminus \mathcal{R}^{\text{inv}}(M).$$

We could show for example that at least one of the homotopy groups

$$\pi_i\left(\mathcal{R}^{\text{inv}}(M)\right) \neq 0.$$

This, together with the contractibility of $\mathcal{R}iem(M)$ (see Remark 1.4.12) would imply

$$\left(\mathcal{R}^{\text{inv}}(M)\right)^c \neq \emptyset.$$

In higher dimension ($n \geq 9$) the map

$$\alpha : \Omega_n^{\text{Spin}}(\{\text{pt}\}) \longrightarrow KO^{-n}(\{\text{pt}\})$$

is not injective [ABP67], so manifolds M^n with $\alpha([M]) = 0$ are not necessarily spin cobordant to the sphere S^n. Restricting the attention to the case of spin nullcobordant manifolds, as a corollary of Conjecture 2.2.4 we have that the homotopy type of $\mathcal{R}^{\text{inv}}(M^n)$ is the same of $\mathcal{R}^{\text{inv}}(S^n)$. Then one could proceed as in [CSS16] and show that the pullback action map

$$\begin{aligned} P : \mathrm{Diff}^+(S^n) &\longrightarrow \mathcal{R}^{\mathrm{inv}}(S^n) \\ \phi &\longmapsto \phi^* g_0, \end{aligned}$$

for a fixed reference metric g_0, is nontrivial on homotopy groups. Then the required nontrivial homotopy group of $\mathcal{R}^{\mathrm{inv}}(S^n)$ would be pushed forward by the map P^*, as $\mathrm{Diff}^+(S^n)$, the space of orientation preserving diffeomorphisms of the sphere endowed with the $\mathcal{C}^\infty$ compact-open topology (see Definition 1.4.1), has the homotopy type

$$\mathrm{Diff}^+(S^n) \simeq \mathrm{SO}(n+1) \times \mathrm{Diff}(D^n \text{ rel } \partial),$$

where $\mathrm{Diff}(D^n \text{ rel } \partial)$ is the space of diffeomorphism of the disk which are the identity on the boundary.
The search for nontrivial homotopy classes of $\mathcal{R}^{\mathrm{inv}}(S^n)$ with $n = 4, 5$ is made sensible by the works of L. Seeger in [See00] and M. Dahl in [Dah08, Corollary 4.2], who proved respectively that in even dimension at least 4 and in dimension at least 5 spheres admit metrics with harmonic spinors.
The same method applied to the search for metrics with negative scalar curvature would produce results already known: analyzing the equation for the variation of the scalar curvature under a conformal change, J. Kazdan and F. Warner proved in [KW75] that any closed Riemannian manifold M admits a metric g with strictly negative scalar curvature. The space of such metrics $\mathcal{R}^{\mathrm{scal}<0}$ is even contractible by the work of J. Lokhamp [Lok92].

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