Limit Multiplicity Problem

von der Fakultät für Mathematik und Informatik
der Universität Leipzig
angenommene

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

DOCTOR RERUM NATURALIUM
(Dr. rer. nat.)

im Fachgebiet
Mathematik
vorgelegt

von M.Sc. Vishal Gupta
geboren am 14.12.1990 in Yamuna Nagar, Indien

Die Annahme der Dissertation wurde empfohlen von:

1. Prof. Dr. Tobias Finis (Universität Leipzig)
2. Prof. Dr. Andreas Thom (TU Dresden)

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Dedication

To my parents—for everything

Epigraph

"God is in the detail."

Anonymous

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Abstract

Let G be a locally compact group (usually a reductive algebraic group over an algebraic number field F). The main aim of the theory of Automorphic Forms is to understand the right regular representation of the group G on the space $L^2(\Gamma\backslash G)$ for certain "nice" closed subgroups Γ. Usually, Γ is taken to be a lattice or even an arithmetic subgroup.

In the case of uniform lattices, the space $L^2(\Gamma\backslash G)$ decomposes into a direct sum of irreducible unitary representations of the group G with each such representation π occurring with a *finite* multiplicity $m(\Gamma, \pi)$. It seems quite difficult to obtain an explicit formula for this multiplicity; however, the limiting behaviour of these numbers in case of certain "nice" sequences of subgroups $(\Gamma_n)_n$ seems more tractable.

We study this problem in the global set-up where G is the group of adelic points of a reductive group defined over the field of rational numbers and the relevant subgroups are the maximal compact open subgroups of G. As is natural and traditional, we use the Arthur trace formula to analyse the multiplicities. In particular, we expand the geometric side to obtain the information about the spectral side—which is made up from the multiplicities $m(\Gamma, \pi)$.

The geometric side has a contributions from various conjugacy classes, most notably from the unipotent conjugacy class. It is this *unipotent* contribution that is the subject of Part III of this thesis. We estimate the contribution in terms of level of the maximal compact open subgroup and make conclusions about the limiting behaviour.

Part IV is then concerned with the spectral side of the trace formula where we show (under certain conditions) that the trace of the discrete part of the regular representation is the only term that survives in the limit.

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Part I

Introduction

Chapter 1

What is this Thesis about

> “Nearly all men can stand adversity, but if you want to test a man’s character, give him power.”
>
> Anonymous

This thesis is about *Limit Multiplicity Property*, so we take a closer look at what this concept is.

1.1 What is Limit Multiplicity Property

Definition 1.1. A Lie group G is called a ***semisimple Lie group*** if its Lie algebra $\mathfrak{g}$ is semisimple. ♣

Definition 1.2. Let G be a Lie group with a fixed invariant right Haar measure. A subgroup $\Gamma \subseteq G$ is called a ***lattice*** if it is discrete (in the induced topology) and such that $\operatorname{vol}(\Gamma\backslash G)$ is finite. ♣

Remark 1.3. Note that the definition of lattice requires that there *exist* a measure on $\Gamma\backslash G$ and it be finite. ❇

Definition 1.4. Let G be a Lie group and $\Gamma \subseteq G$ be a lattice. It is said to be a ***uniform lattice*** if $\Gamma\backslash G$ is a compact topological space. ♣

Definition 1.5. A Lie group G is called a ***linear Lie group*** if there exists a faithful continuous group homomorphism $\phi : G \to \boldsymbol{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$. ♣

Notation 1.6. Let G be a connected linear semisimple Lie group. We denote by $\widehat{G}$ the set of all equivalence classes of irreducible unitary representations of G. It is a topological space under the Fell topology (cf. [BdlHV08, §F.2]).

If $\Gamma \subseteq G$ is a lattice, then we can form the right regular representation of G on $L^2(\Gamma\backslash G)$, which is denoted by R_Γ. ♦

Definition 1.7. The ***discrete spectrum*** is defined as the closed span of irreducible closed subrepresentations of $L^2(\Gamma\backslash G)$. It is denoted by $L^2_{\text{disc}}(\Gamma\backslash G)$. ♣

Definition 1.8. The ***continuous spectrum*** is defined as the orthogonal complement in $L^2(\Gamma\backslash G)$of the discrete spectrum $L^2_{\text{disc}}(\Gamma\backslash G)$. It is denoted by $L^2_{\text{cont}}(\Gamma\backslash G)$. ♣

We will focus on the discrete part $L^2_{\text{disc}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$, namely the sum of all irreducible subrepresentations, and we denote by $R_{\Gamma,\text{disc}}$ the corresponding restriction of R_Γ. For any $\pi \in \widehat{G}$, let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma\backslash G)$. Thus,

$$m_\Gamma(\pi) \ := \ \dim \operatorname{Hom}_G(\pi, R_\Gamma) = \dim \operatorname{Hom}_G(\pi, R_{\Gamma,\text{disc}}).$$

These multiplicities will be finite in cases of interest to us.

Lemma 1.9. *Under the reduction theoretic assumptions on G and Γ mentioned in [OW81, Page 62], we have $\mu_\Gamma(A) < \infty$ for any bounded subset $A \subseteq \widehat{G}$.* ♣

Proof. We refer to [BG83]. ■

We define the discrete spectral measure on $\widehat{G}$ with respect to Γ by

$$\mu_\Gamma \ := \ \frac{1}{\operatorname{vol}(\Gamma\backslash G)} \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\, \delta_\pi$$

where δ_π is the Dirac measure at π. While one cannot hope to describe the multiplicity functions m_Γ on $\widehat{G}$ explicitly, it is feasible and interesting to study asymptotic questions.

The limit multiplicity problem concerns the asymptotic behaviour of μ_Γ as $\operatorname{vol}(\Gamma\backslash G) \to \infty$.

Definition 1.10. Let G be as above. The ***tempered dual*** of G is the support of Plancherel measure on $\widehat{G}$. It will be denoted by $\widehat{G}_{\text{temp}}$. A good reference for Plancherel measure is [Fö5, §3.4]. ♣

Theorem 1.11. *Up to a closed set of Plancherel measure zero, the topological space $\widehat{G}_{temp}$ is homeomorphic to a countable union of Euclidean spaces of bounded dimensions, and that under this homeomorphism the Plancherel density is given by a continuous function.* ♣

Proof. We refer to [Del86, §2.3]. ■

Definition 1.12. A *relatively quasi-compact* subset of $\widehat{G}$ is called ***bounded***. ♣

Remark 1.13. Note that sometimes a topological space X is said to be quasi-compact if it is compact (in the usual sense of open covers) but not necessarily Hausdorff. This terminology is used when one wants to insist that compact spaces be Hausdorff. Recall that a subspace $Y \subseteq X$ is called *relatively (quasi-)compact* if its closure (in X) is (quasi-)compact. ❇

Definition 1.14. A subset $A \subseteq \widehat{G}_{\text{temp}}$ is called ***Jordan measurable*** if it is bounded and its boundary has Plancherel measure zero. That is $\mu_{\text{pl}}(\partial A) = 0$, where $\partial A = \overline{A} \setminus A^{\text{int}}$. ♣

Definition 1.15. A ***Riemann integrable function*** on $\widehat{G}_{\text{temp}}$ is a bounded, compactly supported function which is continuous almost everywhere with respect to the Plancherel measure. ♣

Definition 1.16. Let $(\Gamma_n)_{n=1}^{\infty}$ be a sequence of lattices in G. We say that the sequence $(\Gamma_n)_{n=1}^{\infty}$ has ***the limit multiplicity property*** if the following conditions are satisfied

(LM1) For any Jordan measurable set $A \subseteq \widehat{G}_{\text{temp}}$, we have

$$\lim_{n\to\infty} \mu_{\Gamma_n}(A) = \mu_{\text{pl}}(A).$$

(LM2) For any bounded set $A \subseteq \widehat{G} \setminus \widehat{G}_{\text{temp}}$, we have

$$\lim_{n\to\infty} \mu_{\Gamma_n}(A) = 0.$$

♣

Remark 1.17. Note that we can rephrase the first condition by requiring that for any Riemann integrable function f on $\widehat{G}_{\mathrm{temp}}$,

$$\lim_{n\to\infty} \mu_{\Gamma_n}(f) = \mu_{\mathrm{pl}}(f),$$

where

$$\mu_{\mathrm{pl}}(f) \;:=\; \int_{\widehat{G}_{\mathrm{temp}}} f \,\mathrm{d}\mu_{\mathrm{pl}}$$

and similarly for $\mu_{\Gamma_n}(f)$. ❇

Now that we have seen what Limit Multiplicity Problem is about, we review what has already been done.

1.2 Previous Work

1.2.1 The Beginning

A great deal is known about the limit multiplicity problem for uniform lattices, where R_Γ decomposes discretely. Indeed we have,

Theorem 1.18. *Let G be as above, a connected linear semisimple Lie group and $\Gamma \subseteq G$ be a discrete cocompact subgroup. Then the right regular representation R_Γ of G on $L^2(\Gamma\backslash G)$ splits into a countable direct sum of unitary irreducible representations of G and the number of summands of any $\omega \in \widehat{G}$ in R_Γ is finite, denoted, as above by $m_\Gamma(\omega)$.* ♣

Proof. We refer to [OW81, Page 18–19] ■

The study of Limit multiplicity problem began with the classic article [DW78] by DeGeorge and Wallach in 1978 and continued in [DW79] and [Wal80]. We describe briefly what they did in this article. They considered a decreasing sequence of cocompact normal subgroups, $(\Gamma_n)_{n=1}^{\infty}$ such that $\cap_n \Gamma_n = \{1\}$ (such a sequence is called a 'tower') and studied for $S \subseteq \widehat{G}$, the quantity

$$\mu_n(S) \;:=\; \mathrm{vol}\,(\Gamma_n\backslash G)^{-1} \sum_{\omega\in S} m_\Gamma(\omega).$$

Their main results are pertaining to the limit formulae for μ_n as $n \to \infty$. In fact, one of their main results is

$$\lim_{n\to\infty} \mathrm{vol}\,(\Gamma_n\backslash G)^{-1}\, m_\Gamma\,(\omega) = \begin{cases} 0 & \text{if } \omega \text{ is not square integrable,} \\ d\,(\omega) & \text{if } \omega \text{ is square integrable,} \end{cases}$$

where $d(\omega)$ is the formal degree.

Subsequently, the limit multiplicity problem was settled in affirmative by Delorme in [Del86] in this case. Recently, there has been a lot of progress in proving the limit multiplicity for much more general sequences of uniform lattices [ABB$^+$11, ABB$^+$17].

1.2.2 Recent Work

We make a (very) brief survey of some of the more recent work—taken from [FLM15].

In the case of non-compact quotients $\Gamma\backslash G$, where the spectrum also contains a continuous part, much less is known. Here, the limit multiplicity property has been solved for normal towers of arithmetic lattices and discrete series $L-$packets $A \subseteq \widehat{G}$ by Rohlfs and Speh [RS87]. In this direction, the case of singleton sets A and normal towers of congruence subgroups has been solved by Savin [Sav89] (cf. also [Wal90]). Earlier results on the discrete series had been obtained by DeGeorge [DeG82] and Barbasch and Moscovici [BM83] for groups of real rank one, and by Clozel [Clo86]. The limit multiplicity problem for the entire unitary dual has been solved for the principal congruence subgroups of $\boldsymbol{SL}_2(\mathbb{Z})$ by Sarnak. A partial result for certain normal towers of congruence arithmetic lattices defined by groups of $\mathbb{Q}-$rank one has been shown by Deitmar and Hoffman in [DH99]. Finally, generalisations to the distribution of Hecke eigenvalues have been obtained by Sauvageot [Sau97], Shin [Shi12] and Shin and Templier [ST16].

The general analysis of non compact quotients was initiated in [FLM15] and continued in [FL17c]. In fact they consider reductive algebraic groups over number fields and look at their adelic points. As the description of their work requires a significant amount of preparatory notation to be set up, we defer it to §3.1.

1.3 The Goal of this Thesis

The main goal of this thesis is to make progress towards a proof of the Conjecture 3.6 conjecturally stating the limit multiplicity property for an arbitrary non-degenerate collection of compact open subgroups of a reductive algebraic group G. This is a natural extension of the work of [FLM15] and [FL17c]. Specifically, our goal in this thesis is to analyse the unipotent contribution that comes up in the analysis of the geometric side of the trace formula in this context for *maximal* compact open subgroups. The full solution to the Conjecture 3.6 will require the analysis of the non-unipotent contributions and the extension to non-maximal subgroups, to which we hope to come back to in a subsequent work.

The results we do indeed prove (cf. Chapter 4) also imply a (conditional) theorem on limit multiplicity for certain collections of lattices in the connected real group $G(\mathbb{R})$ with G as in Chapter 4.

Chapter 2

Notations, Conventions and Objects of Focus

"Give me six hours to chop down a tree and I will spend the first four sharpening the axe."

Anonymous

Here we collect the notations, conventions and our basic objects of study throughout the thesis. Needless to say, more objects will be introduced, conventions made and notations set later as needed; the following however fixed for the rest of the thesis (except otherwise noted, as in Chapter 10 or in Appendices, where things are more basic).

Fixed Objects 2.1. We fix once and for all the following objects:

1. A simply connected simple split group G defined over the field of rational numbers $\mathbb{Q}$. In principle, everything to follow could be taken to be defined over any algebraic number field F; we will however, stick to the case of field of rational numbers for simplicity.

2. A minimal Levi subgroup M_0 of G, also defined over $\mathbb{Q}$.

3. A minimal parabolic subgroup P_0 of G with Levi M_0. Then the Levi decomposition of P_0 is given by $P_0 = M_0 N_0$, with N_0 being the unipotent radical of P_0.

4. The split component (cf. Item 4) $A_0 := A_{M_0}$ of the centre of M_0.

✢

Remark 2.2. Traditional references for the theory of algebraic groups are the three books, all titled “Linear Algebraic Groups”, namely— [Bor91], [Hum75] and [Spr09]. A more gentle introduction is given by [MT11] and finally, an introduction in the modern language of Algebraic Geometry can be found in [Mil17]. ❇

Convention 2.3. We make the following conventions:

1. All subgroups of G are implicitly assumed to be defined over $\mathbb{Q}$.

2. A parabolic subgroup P or a Levi subgroup M would be said to be ***standard*** if it contains the fixed minimal parabolic subgroup P_0 or the fixed minimal Levi subgroup M_0, respectively.

3. Unless stated otherwise, the Levi decomposition of a parabolic subgroup P would be written as $P = M_P N_P$. That is, M_P would be understood as the Levi component (containing M_0) of the parabolic subgroup P without being explicitly mentioned. The same goes for the unipotent radical N_P.

4. By ***split component*** of a reductive group we will always mean the maximal *split* torus of the connected component of its centre.

5. The split component of the centre of M_P would be denoted by A_P.

✡

Notation 2.4. The following notation concerning subgroups will be fixed once and for all:

1. Unless stated otherwise, alphabets M and L would denote Levi subgroups of G.

2. For a Levi subgroup M and another Levi subgroup L containing M, we will write:

    a) $\mathscr{L}^L(M)$ for the set of all Levi subgroups of L containing M,

    b) $\mathscr{F}^L(M)$ for the set of parabolic subgroups of L containing M and

    c) $\mathscr{P}^L(M)$ for the set of groups in $\mathscr{F}^L(M)$ for which M is a Levi component.

3. When $L = G$, we usually omit the superscript G and denote the sets by $\mathscr{L}(M)$, $\mathscr{F}(M)$, and $\mathscr{P}(M)$ respectively.

♦

Notation 2.5. The following notation concerning characters and roots will be fixed once and for all:

1. $X(M_P)$ will denote the (additive) group of characters of M_P defined over $\mathbb{Q}$.

2. Then we will put
$$\mathfrak{a}_P := \operatorname{Hom}(X(M_P), \mathbb{R}),$$
which will then be a vector space over $\mathbb{R}$ of dimension equal to that of A_P. Its dual space $\mathfrak{a}_P^*$ would be identified with $X(M_P) \otimes \mathbb{R}$.

3. The simple roots of (P, A_P) will be denoted by Δ_P. They are elements in $X(A_P)$ and as such can be canonically embedded in $\mathfrak{a}_P^*$.

4. We will abbreviate Δ_{P_0} to Δ_0. It turns out that it is a base of a root system. As such, for every $\alpha \in \Delta_0$, we define a coroot $\alpha^\vee \in \mathfrak{a}_0 := \mathfrak{a}_{P_0}$.

♦

Notation 2.6. The following notation concerning number theoretic objects associated to the field $\mathbb{Q}$ will be fixed:

1. The ring of integers of the field $\mathbb{Q}$ will be denoted by $\mathbb{Z}$. Of course, $\mathbb{Z}$ is the usual ring of integers.

2. A *place* (in the sense of number theory) of $\mathbb{Q}$ is either the Archimedean absolute value: $|\cdot|_\infty$ inherited from $\mathbb{R}$ or for every prime p of $\mathbb{Z}$, a $p-$adic absolute value denoted by $|\cdot|_p$. The set of places of $\mathbb{Q}$ is thus the set $\{\infty, p \mid p \text{ a prime in } \mathbb{Z}\}$ with ∞ denoting the Archimedean place.

3. S will denote a finite set of places of $\mathbb{Q}$ containing the Archimedean or the *infinite* place. We will write S_{fin} for $S \setminus \{\infty\}$.

4. The completion of $\mathbb{Q}$ at a prime p is the field of $p-$adic numbers, denoted by $\mathbb{Q}_p$. On the other hand, the completion of $\mathbb{Q}$ with respect to the Archimedean absolute value is, of course, $\mathbb{R}$.

5. For S as above, we set

$$\mathbb{Q}_S \;:=\; \mathbb{R} \cdot \prod_{p \in S_{\mathrm{fin}}} \mathbb{Q}_p.$$

6. Then the $\mathbb{Q}_S$−points of G are given by

$$G(\mathbb{Q}_S) = G\left(\mathbb{R}\right) \cdot \prod_{p \in S_{\mathrm{fin}}} G\left(\mathbb{Q}_p\right).$$

♦

Notation 2.7. The following notation concerning adelic subgroups will be fixed once and for all:

1. The group of adeles of $\mathbb{Q}$ will be denoted by $\mathbb{A}$.

2. For a set S, we will write $\mathbb{A}^S$ for

$$\mathbb{A}^S \;:=\; \prod_{p \notin S}{}' \mathbb{Q}_p.$$

3. $\mathbb{A}_{\mathrm{fin}}$ will denote the set of finite adeles. That is,

$$\mathbb{A}_{\mathrm{fin}} \;:=\; \mathbb{A}^{\infty} = \prod_{p}{}' \mathbb{Q}_p.$$

4. The group of adelic points of G is then given by the restricted direct product

$$\begin{aligned} G(\mathbb{A}) &= G\left(\mathbb{R}\right) \cdot \prod_{p}{}' G\left(\mathbb{Q}_p\right), \\ G\left(\mathbb{A}^S\right) &= \prod_{p \notin S}{}' G\left(\mathbb{Q}_p\right), \\ G\left(\mathbb{A}_{\mathrm{fin}}\right) &= \prod_{p}{}' G\left(\mathbb{Q}_p\right) \end{aligned}$$

of the groups $G\left(\mathbb{Q}_p\right)$ with respect to the compact open subgroups $G\left(\mathbb{Z}_p\right)$, for primes p of $\mathbb{Z}$.

5. A maximal compact open subgroup $\mathbf{K}$ of $G(\mathbb{A})$

$$\mathbf{K} := \prod_{\nu} \mathbf{K}_{\nu}$$

satisfying the conditions enumerated in [Art78, Page 917] and additional conditions enumerated in [Art81, Page 9]. In particular,

   a) For any embedding of G into $\boldsymbol{GL}_n$, $\mathbf{K}_{\nu} = \boldsymbol{GL}_n(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)$ for almost all primes p.
   b) For every p, $\mathbf{K}_p$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$.
   c) For every parabolic subgroup P of G, we have $G(\mathbb{A}) = P(\mathbb{A})\mathbf{K} = M_P(\mathbb{A})N_P(\mathbb{A})\mathbf{K}$.

Such a maximal compact open subgroup is called ***admissible with respect to*** M_0 in Arthur's works.

6. For a set S, we define

$$\begin{aligned} \mathbf{K}_S &:= \mathbf{K}_{\mathbb{R}} \cdot \prod_{p \in S} \mathbf{K}_p, \\ \mathbf{K}_{S_{\mathrm{fin}}} &:= \prod_{p \in S_{\mathrm{fin}}} \mathbf{K}_p, \\ \mathbf{K}^S &:= \prod_{p \notin S} \mathbf{K}_p. \end{aligned}$$

♦

Remark 2.8. The standard, and very well-written, reference for adeles and their topology is [RV99] whereas for groups over adeles, we refer to [PR94]. ❇

Notation 2.9. The following will also be fixed once and for all:

1. For a parabolic subgroup P, the H ***function*** $H_P : M_P(\mathbb{A}) \to \mathfrak{a}_P$, defined by

$$\exp\left(\chi\left(H_P\left(m\right)\right)\right) = \left|\chi\left(m\right)\right| := \prod_{p} \left|\chi\left(m_p\right)\right|_p,$$

for

$$\chi \in X\left(M_P\right), m = \prod_{p} m_p \in M_P(\mathbb{A}).$$

2. The extension of the H function H_P to $G(\mathbb{A})$ by
$$H_P(x) := H_P(m), \qquad x = mnk \in G(\mathbb{A}) = M_P(\mathbb{A})N_P(\mathbb{A})\mathbf{K},$$
with $m \in M_P(\mathbb{A}), n \in N_P(\mathbb{A})$, and $k \in \mathbf{K}$.

3. The kernel $M_P(\mathbb{A})^1$ of H_P acting on $M_P(\mathbb{A})$. Thus, $M_P(\mathbb{A}) = M_P(\mathbb{A})^1 A_P(\mathbb{R})^\circ$. In fact, then H_P becomes a group isomorphism from $A_P(\mathbb{R})^\circ$ to $\mathfrak{a}_P$.

♦

Convention 2.10. The following conventions concerning Haar measures will be in place:

1. For any connected subgroup V of N_0, we take the Haar measure on $V(\mathbb{A})$ which assigns $V(\mathbb{Q})\backslash V(\mathbb{A})$ the volume one.
2. We take the Haar measure of $\mathbf{K}$ to be one.
3. We fix a Euclidean Haar measure on each of the vector spaces $\mathfrak{a}_P$ and the dual Haar measure on their dual spaces $\mathfrak{a}_P^*$.
4. The Haar measure on $A_P(\mathbb{R})^\circ$ will be fixed by the isomorphism with $\mathfrak{a}_P$.
5. Finally the Haar measure on $G(\mathbb{A})$ is fixed so that any compatibility conditions are satisfied.

✡

Notation 2.11. We will have two fix some more special sets:

1. $G(F_S)^1$ will denote
$$G(F_S)^1 := \bigcap_{\chi \in X(G)} \ker\left(|\chi|_S : G(F_S) \to \mathbb{R}^{>0}\right),$$
where $|\cdot|_S$ denotes the normalized absolute value on $\mathbb{Q}_S^\times$.
2. In the same vein, $G(\mathbb{A})^1$ will denote
$$G(\mathbb{A})^1 := \bigcap_{\chi \in X(G)} \ker\left(|\chi|_{\mathbb{A}} : G(\mathbb{A}) \to \mathbb{R}^{>0}\right),$$
where $|\chi|_{\mathbb{A}}$ denotes the normalized absolute value on $\mathbb{A}^\times$.
3. The function space $\mathcal{H}\left(G(F_S)^1\right)$ will denote the smooth compactly supported bi$-\mathbf{K}_S-$finite functions on $G(F_S)^1$.

♦

Chapter 3

Context and Proof Strategy

> "Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world."
>
> Albert Einstein, What Life means to Einstein

In this chapter, we first describe some recent work and state Conjecture 3.6, which is the principal guiding post of this thesis. This is the context of the current investigations and helps everything we do in the rest of the thesis into perspective. Then we discuss our strategy towards a possible proof.

3.1 Recent Work

Now we leave the realm of real Lie groups and work with algebraic groups over number fields. We will also consider the adelic setting and our main focus will be the adelic points of reductive algebraic groups defined over a number fields. We use the notation summarised in Chapter 2 freely.

For any compact open subgroup $\mathbb{K}$ of $G\left(\mathbb{A}^S\right)$, let $\mu_{\mathbb{K}} = \mu_{\mathbb{K}}^{G,S}$ be the measure on $\widehat{G(F_S)^1}$ given by

$$\mu_{\mathbb{K}} := \frac{1}{\mathrm{vol}\left(G(F)\backslash G(\mathbb{A})^1/\mathbb{K}\right)} \sum_{\pi\in\widehat{G(F_S)^1}} \dim \mathrm{Hom}_{G(F_S)^1}\left(\pi, L^2\left(G(F)\backslash G(\mathbb{A})^1/\mathbb{K}\right)\right)\delta_\pi.$$

Thus, the number $\mu_{\mathbb{K}}(\pi)$ just the multiplicity of $\pi \in \widehat{G(F_S)}^1$ in the regular representation.

Definition 3.1. Let μ_{pl} be the Plancherel measure on $\widehat{G(F_S)}^1$. We say that a collection $\mathcal{K}$ of open compact subgroups of $G(\mathbb{A}^S)$ has ***the limit multiplicity property*** if $\mu_{\mathbb{K}} \to \mu_{\mathrm{pl}}$, in the sense that

(LM1) for any Jordan measurable subset $A \subseteq \widehat{G(F_S)}^1{}_{\mathrm{temp}}$, we have $\mu_{\mathbb{K}}(A) \to \mu_{\mathrm{pl}}(A)$, $\mathbb{K} \in \mathcal{K}$, and,

(LM2) for any bounded subset $A \subseteq \widehat{G(F_S)}^1 \setminus \widehat{G(F_S)}^1{}_{\mathrm{temp}}$, we have $\mu_{\mathbb{K}}(A) \to 0$, $\mathbb{K} \in \mathcal{K}$.

Here, we write $\mu_{\mathbb{K}}(A) \to \mu_{\mathrm{pl}}(A)$ to mean that for every $\epsilon > 0$, there are only finitely many subgroups $\mathbb{K} \in \mathcal{K}$ such that $|\mu_{\mathbb{K}}(A) - \mu_{\mathrm{pl}}(A)| \geq \epsilon$. ♣

Then the main result of [FLM15] is:

Theorem 3.2. *([FLM15, Theorem 1.3]). Suppose that G satisfies (TWN) and (BD). Then the limit multiplicity holds for the collection of all principal congruence subgroups $\mathbb{K}^S(\mathfrak{n})$ of $\mathbb{K}^S$.* ♣

The two properties (TWN) and (BD) are defined and explored in [FLM15, §5.2]. The notion of principal congruence subgroup is defined in Definition 8.2 (see also [FLM15, Page 9]).

The main result of [FL17c] is:

Theorem 3.3. *([FL17c, Theorem 1.4]). Suppose that G satisfies (TWN) and (BD) and let $\mathbf{K}_0^S$ be a compact open subgroup of $G(\mathbb{A}^S)$. Then the limit multiplicity holds for any non-degenerate family $\mathcal{K}$ of open subgroups of $\mathbf{K}_0^S$.* ♣

The notion of non-degenerate family is defined in [FL17c, Definition 1.3] which we recall below, but first a little notation:

Notation 3.4. For any reductive group H, let $H(\mathbb{A})^+$ be the image of the map $H^{\mathrm{sc}}(\mathbb{A}) \to H(\mathbb{A})$, where H^{sc} is the simply connected cover of the derived group of H. The group $H\left(\mathbb{A}^S\right)^+$ is defined analogously. ♦

Definition 3.5. A collection $\mathcal{K}$ of compact open subgroups of $G\left(\mathbb{A}^S\right)$ is said to be ***non-degenerate***, if for any $F-$simple normal subgroup H of G, we have $\mathrm{vol}_{H(\mathbb{A}^S)^+}\left(K \cap H\left(\mathbb{A}^S\right)^+\right) \to 0, K \in \mathcal{K}$. ♣

The next natural extension of these results would be to consider arbitrary open compact subgroups instead of subgroups of a fixed open compact subgroup $\mathbf{K}_0^S$ which leads us to:

Conjecture 3.6. *Suppose that G satisfies (TWN) and (BD). Then the limit multiplicity holds for any non-degenerate family $\mathfrak{K}$ of compact open subgroups of $G\left(\mathbb{A}^S\right)$.* ❋

We do not quite prove this conjecture in this thesis, which is why, it remains a conjecture. However, we imitate the strategy of [FLM15] and [FL17c] and prove a weaker statement, which would hopefully go a long way in the eventual proof of Conjecture 3.6.

3.2 Strategy

As in [FLM15] and [FL17c], we interpret the limit multiplicity problem in terms of the trace formula.

Notation 3.7. For any $h \in \mathcal{H}\left(G(F_S)^1\right)$, let $\hat{h}$ be the function on $\widehat{G(F_S)}^1$ given by $\hat{h}(\pi) = \operatorname{tr}\pi(h)$. ♦

The crucial link between the limit multiplicity property and the trace formula is provided by the

Proposition 3.8. *([FL17c, Theorem 2.1]). Suppose that a collection $\mathcal{K}$ of compact open subgroups of $G\left(\mathbb{A}^S\right)$ has the property that for any function $h \in \mathcal{H}\left(G(F_S)^1\right)$, we have*

$$\mu_{\mathbb{K}}\left(\hat{h}\right) - \sum_{z \in Z(F)\cap\mathbb{K}} h(z) \to 0, \qquad \mathbb{K} \in \mathcal{K}. \tag{3.9}$$

Then the limit multiplicity holds for $\mathcal{K}$. ♣

This proposition does not directly involve the trace formula. However, denoting by R_{disc} the regular representation of $G(\mathbb{A})^1$ on the discrete part of $L^2(G(F)\backslash G(\mathbb{A})^1)$, we see that

$$\mu_{\mathbb{K}}\left(\hat{h}\right) = \frac{1}{\operatorname{vol}\left(G(F)\backslash G(\mathbb{A})^1\right)} \operatorname{tr} R_{\text{disc}}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right).$$

This is our first hint that the trace formula might be a useful tool since when we expand the Arthur distribution $\mathfrak{J}_G(f)$ spectrally, the main term is indeed

$\operatorname{tr} R_{\mathrm{disc}}(f)$. Furthermore, when we expand the Arthur distribution $\mathfrak{J}_G(f)$ geometrically, we run into the central distribution $\mathfrak{J}_{Z(F)}$ (Definition F.42), which is given by

$$\mathfrak{J}_{Z(F)}(f) = \operatorname{vol}\left(G(F)\backslash G(\mathbb{A})^1\right) \sum_{z \in Z(F)} f(z).$$

Thus, in order to prove Equation (3.9) for a collection $\mathcal{K}$, it is enough to prove the following two statements, which together (in view of the remark above) clearly imply it:

$$\forall h \in \mathcal{H}\left(G(F_S)^1\right), \quad \mathfrak{J}_G(h \otimes \mathbb{1}_{\mathbb{K}}) - \mathfrak{J}_{Z(F)}(h \otimes \mathbb{1}_{\mathbb{K}}) \to 0, \qquad \mathbb{K} \in \mathcal{K}, \tag{3.10}$$

$$\forall h \in \mathcal{H}\left(G(F_S)^1\right), \quad \mathfrak{J}_G(h \otimes \mathbb{1}_{\mathbb{K}}) - \operatorname{tr} R_{\mathrm{disc}}(h \otimes \mathbb{1}_{\mathbb{K}}) \to 0, \qquad \mathbb{K} \in \mathcal{K}. \tag{3.11}$$

Following [FLM15], we will call these relations ***geometric limit property*** and ***spectral limit property***, respectively.

The Parts III and IV study these two properties respectively, for collections mentioned in Chapter 4. We analyse the contribution from the unipotent orbits on the geometric side and deal with the full spectral property albeit for maximal compact open subgroups.

One could use the conjugation action of $G\left(\mathbb{A}^S\right)$ and the natural action of $G^{\mathrm{ad}}(\mathbb{Q})$ to prove the limit multiplicity for more general collections of compact open subgroups (hopefully eventually even the collection of *all* compact open subgroups of $G\left(\mathbb{A}^S\right)$). We will however, not try to do this here.

Part II

Overview

Chapter 4

Summary

> "I am unable to think of any critical, complex human activity that could be safely reduced to a simple summary equation."
>
> Jerome Powell

We summarise the main results of the thesis and the context surrounding it.

4.1 Geometric Limit Property

On the geometric side, we let G to be a connected simple simply connected algebraic group over the field $\mathbb{Q}$ of *rational numbers* and we consider the collection of *maximal* compact open subgroups $\mathbb{K} = \prod_\nu \mathbb{K}_\nu$ of $G(\mathbb{A})$ such that $\mathbb{K}_\nu = \mathbf{K}_\nu$ for primes ν of $\mathbb{Q}$ except one, say p. Moreover, if x_p denotes the point in the fundamental alcove in the apartment in the Bruhat Tits building such that $\mathbf{K}_p$ is stabiliser of x_p, then $\mathbb{K}_p$ is stabiliser of a *non-hyperspecial* point y_p in the *same* fundamental alcove.

We also assume the Hypothesis 6.10 on the local coefficients appearing in the expression $\mathfrak{J}_{\text{unip}}$. This hypothesis is a growth condition (with respect to the set S) on the coefficients $a^M(S, u)$ appearing in the Arthur's (local) expansion of the unipotent distribution (cf. [Art85, Corollary 8.3]).

For the above collection of subgroups, under Hypothesis 6.10, our main result on the geometric side is

Theorem. *(Corollary 6.15). Let G be a simply connected simple split group over the field of rational numbers. Let $\mathbb{K}$ be a maximal compact open subgroup of $G(\mathbb{A})$ such that $\mathbb{K}_\nu = \mathbf{K}_\nu$ for all primes ν except one, say p. Then for any $f_\infty \in \mathcal{H}\left(G(\mathbb{R})^1\right)$, we have that*

$$\left|\left(\mathfrak{J}_{\mathrm{unip}} - \mathfrak{J}_{\{1\}}\right)\left(f_\infty \otimes \mathbb{1}_{\mathbb{K}_p}\right)\right| \to 0$$

as $\mathrm{lev}(\mathbb{K}) = p \to \infty$. ♣

Here, $\mathfrak{J}_{\mathrm{unip}}$ refers to the unipotent contribution to the trace formula (cf. Equation (F.40)).

We hope that this will go a long way in the eventual proof of the conjectural Limit Multiplicity Property Equation (3.10).

4.2 Spectral Limit Property

On the spectral side, we let G be again a connected simple simply connected algebraic group over the field $\mathbb{Q}$ of *rational numbers* and we consider the collection $\mathfrak{K}^{\mathrm{spc,max}}$ of *maximal* compact open subgroups $\mathbb{K} = \prod_p \mathbb{K}_p$ of $G(\mathbb{A}_{\mathrm{fin}})$ such that $\mathbb{K}_p = \mathbf{K}_p$ for all but finitely many primes p, the set of which is denoted by $\tilde{S}(\mathbb{K})$. Moreover, if x_p denotes the point in the fundamental alcove in the apartment in the Bruhat Tits building such that $\mathbf{K}_p$ is stabiliser of x_p, then $\mathbb{K}_p$ is stabiliser of a *non-hyperspecial* point y_p in the *same* fundamental alcove for all $p \in \tilde{S}(\mathbb{K})$.

We also assume Hypothesis 10.27 on the polynomial boundedness, which is indeed satisfied for $G = \boldsymbol{Sp}_4$. This hypothesis asserts the polynomial boundedness of the collection of measures $\left\{\mu^{M,S_\infty}_{\mathbb{K}_{M,\gamma}}\right\}$ (cf. Chapter 10 for notation) for every proper Levi subgroup M.

For this collection of subgroups, our main result on the spectral side, conditional on Hypothesis 10.27, is

Theorem. *(Corollary 10.35). Let G be a simply connected simple split group over the field of rational numbers satisfying the property (TWN) (cf. §8.2). Let $\mathfrak{K}^{\mathrm{spc,max}}$ be the above collection of maximal compact open subgroups of $G\left(\mathbb{A}^S\right)$. Then for any $f_\infty \in \mathcal{H}\left(G(F_S)^1\right)$, we have*

$$\mathfrak{J}_{\mathrm{spec},G}\left(f_\infty \otimes \mathbb{1}_{\mathbb{K}}\right) \to 0, \qquad \mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}.$$

♣

We hope to extend this to non-maximal compact subgroups of general reductive groups.

Chapter 5

Outlook

> "Live Life in Crescendo! Your most important work is always ahead of you."
>
> Stephen Covey, The 3rd Alternative

In closing we mention a few future directions that could be explored. Indeed, Conjecture 3.6 was mentioned in the beginning and seems like a viable goal.

On the geometric side, this would involve estimating the non-unipotent contribution to the trace formula and then extend the results to non-maximal compact subgroups. Furthermore, one would like to consider the general non-split reductive groups over general number fields.

On the spectral side, this would involve estimating the spectral contribution from compact open subgroups which are not necessarily maximal. This thwarts our approach of using buildings to analyse the structure of the compact subgroups. However, we hope that one can follow the techniques explicated in [FL18] in a way that is suitable to the case in hand. Finally, as on the geometric side, general non-split reductive groups over number fields would be natural objects of study.

Finally, the properties (TWN) and (BD) are subjects of active research. They are proved for the groups $\boldsymbol{SL}_n$ and $\boldsymbol{GL}_n$ in [FLM15] and for many more cases in [FL17a] and [FL17b].

Once all these steps are done, Conjecture 3.6 will be settled in its full generality.

Part III

Geometric Limit Property

Chapter 6

The Geometric Limit Property

"Begin at the beginning", the King said gravely, "and go on till you come to the end; then stop."

Lewis Carroll, Alice in Wonderland

Here we begin the subject matter proper of the thesis by studying the geometric limit property. We are guided by Conjecture 3.6, which leads us to study the unipotent contribution and the chapter culminates with Theorem 6.13 and its Corollary 6.15.

Following [Art85], we break down the unipotent contribution into the so called weighted orbital integrals—the objects of focus of the next chapter.

We remind the reader about the notational conventions set down in Chapter 2.

6.1 The Geometric Side of Trace Formula

We want to study the distribution $\mathfrak{J}_G - \mathfrak{J}_{Z(F)}$ for functions of the kind $(h \otimes \mathbb{1}_{\mathbb{K}})$. We recall that $\mathfrak{J}_G$ is the Arthur distribution defined in Equation (F.34) and $\mathfrak{J}_{Z(F)}$ is the central distribution defined in Equation (F.43). We also recall that the Arthur distribution can be expanded into a sum of distributions, one of which is the central distribution (cf. Theorem F.31). Thus, it is enough to analyse the non-central distribution $\mathfrak{J}_{\mathrm{nc}}$. As remarked before (Remark F.48), the unipotent distribution $\mathfrak{J}_{\mathrm{unip}}$ contributes to the non-central distribution. In fact, this is the distribution, which will be the major focus of this chapter. We do not study the non-unipotent contribution in this thesis; however, it seems to be the most important contribution as explained in [Art86].

6.2 The Unipotent Distribution

The unipotent distribution $\mathfrak{J}_{\mathrm{unip}}$, as defined by Equation (F.40), is the orbital distribution associated to the unipotent orbit. The orbital distribution is defined in terms of the truncated kernel (cf. Equations (F.29) and (F.35)). This formulation however, does not lend itself to an amenable analysis. Since we are working with functions of the form $(h \otimes \mathbb{1}_{\mathbb{K}})$, we are led to look for expressions which are defined locally in terms of places in S (as in Notation 2.6). This sort of analysis was first done by Arthur in 1985 in [Art85]. He was able to write down the globally defined distribution $\mathfrak{J}_{\mathrm{unip}}$ in terms of certain locally defined distributions, which he called the *weighted orbital integrals*. They are, as one would imagine, kind of orbital integrals with an additional 'weight function' thrown in the integrand. Additionally, they are defined locally (or at least for a finite set of places S) and are defined for every $M \in \mathscr{L}$. In this section, we give a general overview of the major results and philosophy of [Art85], reserving a more detailed explanation for §§ 7.1 and 7.3.

Convention 6.1. For the rest of Part III, we set the field F to be the field of rational numbers. All the notations and conventions set up in Chapter 2 still applies with the understanding that the underlying field is now $\mathbb{Q}$. ✡

6.2.1 Introducing the Weighted Orbital Integrals

As already mentioned the distributions defined in Appendix F.2 are all defined globally. That is, they are linear functionals on the space $C_c^\infty\left(G(\mathbb{A})^1\right)$. However, for any $f \in C_c^\infty\left(G(\mathbb{A})^1\right)$, there exists a finite subset S such that

$$f = f_S \otimes \mathbb{1}_{\mathbf{K}^S},$$

where $f_S \in C_c\left(G(\mathbb{Q}_S)^1\right)$.

Thus, it is sufficient to consider the distributions $\mathfrak{J}_G$ and $\mathfrak{J}_{\mathrm{unip}}$ on $G(\mathbb{Q}_S)^1$. This is what Arthur did in [Art85]. In doing so, he introduced the aforementioned weighted orbital integrals.

Definition 6.2. (Defining weighted orbital integrals, stage 1). Let M be a Levi subgroup of G and let S be a finite set of places containing the set of Archimedean places. Furthermore, let $\gamma \in M(\mathbb{Q}_S) \cap G(\mathbb{Q}_S)^1$. Then there is a distribution $\mathfrak{J}_M(\gamma)$ on $G(\mathbb{Q}_S)^1$, called a ***weighted orbital integral***. ♣

Remark 6.3. Of course, this is not a complete definition! This is just meant to give a taste of what kind of objects the weighted orbital integrals are. Thus, the import of the above "definition" is the following:

Once we have selected a Levi subgroup M (standard or not), a finite subset S and an element $\gamma \in M(\mathbb{Q}_S) \cap G(\mathbb{Q}_S)^1$, then there exists a distribution on $G(\mathbb{Q}_S)^1$. We denote the resulting distribution by $\mathfrak{J}_M(\gamma)$, with S being suppressed from the notation.

We will further refine the definition and give more properties of these objects in due course. In fact, Arthur himself does not define them precisely until 1988, when they make their appearance in [Art88b]. ❇

Coming back to our problem of estimating the unipotent distribution, how do the weighted orbital integrals relate to the unipotent distribution on $G(\mathbb{A})^1$? The answer is provided by

Theorem 6.4. *([Art85, Theorem 8.1]). For any S, there are uniquely determined numbers*

$$a^M(S,u)\,, M \in \mathscr{L}, u \in (\mathcal{U}_M(\mathbb{Q}))_{M,S}\,,$$

such that for any $L \in \mathscr{L}(M)$ and $f \in C_c^\infty\left(\left(L(\mathbb{Q}_S)^1\right)\right)$,

$$\mathfrak{J}^L_{\text{unip}}(f) = \sum_{M \in \mathcal{L}^L} \left|W_0^M\right| \left|W_0^L\right|^{-1} \sum_{u \in (\mathcal{U}_M(\mathbb{Q}))_{M,S}} a^M(S,u)\,\mathfrak{J}_M(u,f)\,. \tag{6.5}$$

♣

We do not need to worry about the undefined symbols here. They will be defined in §7.1 anyway. The import of the statement is that the unipotent contribution can be calculated in terms of the weighted orbital integrals.

In particular, specialising to $L = G$,

Theorem 6.6. *([Art85, Corollary 8.3]). For any $f \in C_c^\infty\left(G(\mathbb{Q}_S)^1\right)$,*

$$\mathfrak{J}_{\text{unip}}(f) = \sum_{M \in \mathscr{L}} \left|W_0^M\right| \left|W_0^G\right|^{-1} \sum_{u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S}} a^M(S,u)\,\mathfrak{J}_M(u,f)\,. \tag{6.7}$$

♣

The real advantage that the weighted orbital integrals offer, as already mentioned, is that they are defined locally instead of globally and hence are ‘easier’ to manipulate than the global objects. What Theorem 6.6 allows us to do is to analyse the locally defined weighted orbital integrals and conclude something about the globally defined unipotent distribution.

All the sums involved in Equation (6.7) are finite. Our strategy then, would be to estimate the weighted orbital integrals and impose a “reasonable” hypothesis on the coefficients $a^M(S,u)$ to estimate the unipotent integrals. However, to be able to estimate the weighted orbital integrals, we first need a precise definition of these objects. These are a little technical and separate from the current flow of ideas. Hence we relegate them to Chapter 7; the definition will be given in §7.3 and key estimates in §7.4. We first use Theorem 6.6 to finish the task at hand.

6.3 Estimating $\mathfrak{J}_{\text{unip}}(f)$

Armed with Equation (6.7), now we can estimate the unipotent distribution $\mathfrak{J}_{\text{unip}}(f)$ for the kind of functions f germane to our situation.

We recall, for $f \in C_c^\infty\left(G(\mathbb{Q}_S)^1\right)$, the formula (cf. Equation (6.7)),

$$\mathfrak{J}_{\text{unip}}(f) = \sum_{M \in \mathscr{L}} \left|W_0^M\right| \left|W_0^G\right|^{-1} \sum_{u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S}} a^M(S,u)\, \mathfrak{J}_M(u,f).$$

Since the sum over the Levi subgroups $M \in \mathscr{L}$ is finite, we can write

$$\mathfrak{J}_{\text{unip}}(f) \ll \sum_{M \in \mathscr{L}} \sum_{u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S}} \left|a^M(S,u)\, \mathfrak{J}_M(u,f)\right|. \tag{6.8}$$

Convention 6.9. We now specialise to the situation where $S = \{\infty, p\}$ for a fixed prime p and $f = f_\infty \otimes \mathbb{1}_{\mathbb{K}_p}$, with $\mathbb{K}$ being a maximal compact open subgroup of $G(\mathbb{A}_{\text{fin}})$ such that $\mathbb{K}_\nu = \mathbf{K}_\nu$ for all ν except p. Moreover, if x_p denotes the point in the fundamental alcove in the apartment in the Bruhat Tits building such that $\mathbf{K}_p$ is stabiliser of x_p, then $\mathbb{K}_p$ is stabiliser of a *non-hyperspecial* point y_p in the *same* fundamental alcove. ✡

We note that the number of classes u is bounded for $S = \{\infty, p\}$ independent of p. Hence, under Convention 6.9, we will be content to estimate the quantity on the right hand side of Equation (6.8) for a fixed Levi subgroup $M \in \mathscr{L}$.

We estimate the weighted orbital integral in §7.4 and as alluded to in §6.2.1, we impose the following hypothesis on the coefficients:

Hypothesis 6.10. *We hypothesise that there exists a $N > 0$ such that for every Levi subgroup M, every finite set of places (containing the infinite place) S, we have*

$$\left|a^M(S,u)\right| \ll \prod_{p\in S} (\ln(p))^N, \quad \forall u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S} \tag{6.11}$$

❄

Remark 6.12. The coefficients $a^M(S,u)$ have not been analysed in general. This hypothesis is "reasonable" in the sense that it is clearly satisfied for $\boldsymbol{GL}_2$ and $\boldsymbol{PGL}_2$ with $N = 1$.

This might be proven for certain low rank groups, like $\boldsymbol{SL}_2$, $\boldsymbol{SL}_3$ and $\boldsymbol{Sp}_4$ using the explicit formulæ given in [HW13]. ❇

This allows us to state the

Theorem 6.13. *Let $f = f_\infty \otimes \mathbb{1}_{\mathbb{K}_p} \in C_c^\infty\left(G(\mathbb{Q}_S)^1\right)$ with $\mathbb{K}$ being as above. Then there exists a constant $\mathfrak{N} > 0$ such that*

$$\left|\mathfrak{J}_{\text{unip}}(f) - \mathfrak{J}_{\{1\}}(f)\right| \ll \frac{\ln(p)^{\mathfrak{N}}}{\sqrt{p}}. \tag{6.14}$$

♣

Proof. This follows immediately from Equations (6.8), (6.11) and (7.120). ■

Now we are in a position to state our main result on the geometric side. This is a first step in attacking Equation (3.10).

Corollary 6.15. *For f and $\mathbb{K}$ as above,*

$$\left|\mathfrak{J}_{\text{unip}}(f) - \mathfrak{J}_{\{1\}}(f)\right| \to 0 \tag{6.16}$$

as $\operatorname{lev}(\mathbb{K}) \to \infty$. ♣

Proof. This follows from Equations (6.14) and (10.34). ■

Thus, to finish the proof, we are left with estimating the weighted orbital integral, $\mathfrak{J}_M(u,f)$. The next chapter endeavours to define, analyse and estimate this quantity.

Chapter 7

Weighted Orbital Integrals

> "If I have seen further than others, it is by standing upon the shoulders of giants."
>
> Isaac Newton

The purpose of the chapter is to define and analyse the weighted orbital integrals using the original [Art88b]. This is done in §7.3 whereas in §7.4 we develop some estimates for the weighted orbital integral which are useful for our purposes.

We start however, with analysing the unipotent distribution and the unipotent variety based on [Art85].

7.1 The Unipotent Variety

Definition 7.1. For a Levi subgroup M of G, let $\mathscr{U}_M$ be the Zariski closure in M of the unipotent set in $M(\mathbb{Q})$. It is a closed algebraic subvariety of M, which is defined over $\mathbb{Q}$. This is called the ***unipotent variety*** of M. The set

$$\mathfrak{o}_M := \mathscr{U}_M(\mathbb{Q})$$

of rational points of $\mathscr{U}_M$ consists, of course, of the unipotent elements in $M(\mathbb{Q})$. When $M = G$, we write $\mathfrak{o}$ for $\mathfrak{o}_G$. Thus, as a set $\mathfrak{o} = \tilde{\mathfrak{o}}_{\text{unip}}$. ♣

We briefly recall the expression for the unipotent distribution. Unpacking all the definitions in Appendix F, we see that for sufficiently regular $T \in$

$\mathfrak{a}_0^+$, $\mathfrak{J}^T_{\mathrm{unip}}(f)$ is the integral over x in $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ of the function

$$\sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\mathrm{unip}}(\delta x, \delta x)\, \hat{\tau}_P(H_P(\delta x) - T),$$

where

$$\begin{aligned} K_{P,\mathrm{unip}}(y,y) &:= K_{P,\mathfrak{o}}(y,y) \\ &= \sum_{\gamma \in \mathscr{U}_{M_P}(\mathbb{Q})} \int_{N_P(\mathbb{A})} f\left(x^{-1}\gamma n x\right) \mathrm{d}n. \end{aligned}$$

Thus, the leading term in the alternating sum is

$$\begin{aligned} K_{\mathrm{unip}}(x,x) &:= K_{G,\mathrm{unip}}(x,x) \\ &= \sum_{\gamma \in \mathscr{U}_G(\mathbb{Q})} f\left(x^{-1}\gamma x\right). \end{aligned}$$

Pitfall 7.2. We remark that

$$\mathfrak{J}^T_{\mathrm{unip}}(f) \neq \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} K_{\mathrm{unip}}(x,x)\, \mathrm{d}x.$$

In fact, the right hand side may not even be integrable, which was the reason for truncation in the first place. The kernel K_{unip} is important however, as one can indeed estimate $\mathfrak{J}^T_{\mathrm{unip}}(f)$ using it. The relationship nonetheless, is a little more complicated. It is enucleated in [Art85, Theorem 3.1]. We will not need the precise result. All we need to know is that we can estimate $\mathfrak{J}^T_{\mathrm{unip}}(f)$ in terms of K_{unip}.

We will now analyse the kernel K_{unip} by breaking down the orbit $\mathfrak{o}$. We know that although $\mathfrak{o}$ is a coarse geometric class, it is not a conjugacy class. Hence, we can further decompose it into conjugacy classes. In fact, we resort to geometric conjugacy classes.

The variety $\mathscr{U}_G$ is a finite union of (geometric) unipotent conjugacy classes of G. The Galois group, $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ operates on these conjugacy classes. We shall write $(\mathscr{U}_G)$ for the set of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ orbits. Then any $U \in (\mathscr{U}_G)$ is a locally closed subset of G, which is defined over $\mathbb{Q}$, and which consists of a finite union of unipotent conjugacy classes of G.

We can decompose the unipotent contribution in terms of these classes. In particular, we can write

$$\mathfrak{o} = \mathscr{U}_G(\mathbb{Q}) = \bigsqcup_{U \in (\mathscr{U}_G)} U(\mathbb{Q})$$

which leads us to write

$$K_{\mathrm{unip}}(x,x) = \sum_{U \in (\mathscr{U}_G)} K_U(x,x),$$

where

$$K_U(x,x) = \sum_{\gamma \in U(\mathbb{Q})} f\left(x^{-1}\gamma x\right).$$

The utility of this decomposition lies in the following

Theorem 7.3. *([Art85, Theorem 4.2]). There are distributions*

$$\left\{ \mathfrak{J}_U^T \;\middle|\; U \in (\mathscr{U}_G) \right\}$$

which are polynomials in T of total degree at most d_0 such that

$$\mathfrak{J}_{\mathrm{unip}}^T(f) = \sum_{U \in (\mathscr{U}_G)} \mathfrak{J}_U^T(f). \tag{7.4}$$

♣

Pitfall 7.5. Once again, we remark that

$$\mathfrak{J}_U^T(f) \neq \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} K_U(x,x)\,\mathrm{d}x.$$

In fact the relationship between $\mathfrak{J}_U^T(f)$ and K_U is the same as between $\mathfrak{J}_{\mathrm{unip}}^T(f)$ and K_{unip} (compare [Art85, Theorem 3.1] and [Art85, Equation (4.3), Theorem 4.2]). Thus, all we know at this point is that $\mathfrak{J}_U^T(f)$ is a polynomial in T.

We continue by setting

$$\mathfrak{J}_U := \mathfrak{J}_U^{T_0}, \qquad U \in (\mathscr{U}_G).$$

Just like the Arthur distribution, the orbital distribution and the unipotent distribution, the distributions $\mathfrak{J}_U$ are independent of the choice of fixed minimal parabolic subgroup P_0 and depend only on the fixed minimal Levi subgroup M_0 and the fixed maximal compact open subgroup $\mathbf{K}$. The distributions $\mathfrak{J}_{\mathrm{unip}}^T$ and $\mathfrak{J}_U$ are polynomials in T and [Art85, Theorems 3.1, 4.2] give formulae for them. However, Arthur goes on to express these distributions in terms of the locally defined objects. This is where the weighted orbital integrals come into the picture (cf. [Art85, §7]).

As we have already mentioned, a weighted orbital integral is a distribution

$$f \mapsto \mathfrak{J}_M(\gamma, f), \qquad f \in C_c^\infty\left(G(\mathbb{Q}_S)^1\right)$$

on $G(\mathbb{Q}_S)^1$ which is associated to a Levi subgroup $M \in \mathscr{L}$ and a conjugacy class γ in $M(\mathbb{Q}_S) \cap G(\mathbb{Q}_S)^1$. For our purposes, we will only need to study it for the case when γ is unipotent.

Notation 7.6. For any $u \in \mathscr{U}_G(\mathbb{Q})$, it can be embedded into $G(\mathbb{Q}_S)^1$. Let

$$u_S := \prod_{\nu \in S} u_\nu$$

be its image under this embedding. Denote the $G(\mathbb{Q}_S)^1$ conjugacy class of u_S by $[\,u_S\,]$. ♦

Definition 7.7. We define $(\mathscr{U}_G(\mathbb{Q}))_{G,S}$ to be the set of equivalence classes in $\mathscr{U}_G(\mathbb{Q})$ under the following relation:

> Two elements u and v in $\mathscr{U}_G(\mathbb{Q})$ are called $(G,S)-$***equivalent*** if the associated conjugacy classes $[\,u_S\,]$ and $[\,v_S\,]$ in $G(\mathbb{Q}_S)^1$ are the same.

♣

Any element $u \in \mathscr{U}_G(\mathbb{Q})$ is contained in a unique geometric conjugacy class $U_u = U_u^G$ in $(\mathscr{U}_G)$. It depends only on the (G,S) equivalence class of u. The set $U_u(\mathbb{Q}_S)$ breaks up into finitely many $G(\mathbb{Q}_S)^1$ conjugacy classes, one of which is $[\,u_S\,]$. In fact, a Lemma of Arthur ([Art85, Lemma 7.1]) makes sure that they are all of this form. Thus, for any $U \in (\mathscr{U}_G)$, the map $u \mapsto [\,u_S\,]$ is a bijection from the set

$$(\mathscr{U}_G(\mathbb{Q}))_{G,S,U} := \left\{ u \in (\mathscr{U}_G(\mathbb{Q}))_{G,S} \,\middle|\, U_u = U \right\}$$

onto the set of $G(\mathbb{Q}_S)^1-$orbits in $U(\mathbb{Q}_S)$.

Now we come to what we have been building up to. We can decompose the set $\mathscr{U}_G(\mathbb{Q}_S)$ into $G(\mathbb{Q}_S)^1$ conjugacy classes. We start by writing

$$\mathscr{U}_G(\mathbb{Q}_S) = \bigsqcup_{U \in (\mathscr{U}_G)} U(\mathbb{Q}_S).$$

However, as seen above, $U(\mathbb{Q}_S)$ can be broken down further into $G(\mathbb{Q}_S)^1$ orbits and these are then parametrised by the set $(\mathscr{U}_G(\mathbb{Q}))_{G,S,U}$. Thus, we can write

$$\mathscr{U}_G(\mathbb{Q}_S) = \bigsqcup_{U \in (\mathscr{U}_G)} \bigsqcup_{u \in (\mathscr{U}_G(\mathbb{Q}))_{G,S,U}} [\,u_S\,].$$

Now since every $u \in (\mathscr{U}_G(\mathbb{Q}))_{G,S}$ belongs to a unique $U \in (\mathscr{U}_G)$, we can finally write

$$\mathscr{U}_G(\mathbb{Q}_S) = \bigsqcup_{u \in (\mathscr{U}_G(\mathbb{Q}))_{G,S}} [\, u_S \,] .$$

Remark 7.8. At this point, Arthur abuses the notation and identifies u with its class $[\, u_S \,]$. To remain consistent with his notations and work, we will do the same and finally obtain

$$\mathscr{U}_G(\mathbb{Q}_S) = \bigsqcup_{u \in (\mathscr{U}_G(\mathbb{Q}))_{G,S}} u. \tag{7.9}$$

❇

Remark 7.10. Everything from Notation 7.6 onwards has an analogue if G is replaced by a Levi subgroup M. Thus, given a $u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S}$, we obtain a class $U_u^M \in (\mathscr{U}_M)$ and so on. ❇

Remark 7.11. Now it becomes clear that the symbol u appearing in $\mathfrak{J}_M(u, f)$ in Theorems 6.4 and 6.6 stands for the $M(\mathbb{Q}_S)^1$ conjugacy class of $u \in (\mathscr{U}_M)_{M,S}$. ❇

Notation 7.12. For any $u \in (\mathscr{U}_M(\mathbb{Q}))_{M,S}$, we shall write $U_u^G \in (\mathscr{U}_G)$ for the ***induced unipotent conjugacy class*** of G associated to $U_u^M \in (\mathscr{U}_M)$. It is the unique unipotent class in G which, for any $P = MN_P$, intersects $U_u^M \cdot N_P$ in a Zariski dense open subset (cf. [LS79]). The induced conjugacy class plays an important role in the eventual formula for $\mathfrak{J}_M(u, f)$. ♦

Now it would be ideal time to finally define the weighted orbital integrals. Before we can define them however, we need to introduce one more important concept.

7.2 $(G, M)-$family

The concept of $(G, M)-$family was first introduced in 1981 by Arthur in [Art81, §6]. We give here a brief summary.

Let G be a connected reductive algebraic group and M be a fixed Levi subgroup containing a fixed *minimal* Levi subgroup M_0.

Suppose that, for $P \in \mathscr{F}(M)$, $c_P(\lambda)$ is any smooth function on $\iota\mathfrak{a}_M^* = \iota\mathfrak{a}_P^*$. If $Q \supseteq P$, define

$$c_Q(\lambda) \;:=\; c_P(\lambda_Q),$$

where λ_Q denotes the projection of λ on $\iota\mathfrak{a}_Q^*$. Furthermore, we define the functions

$$c'_Q(\lambda), \qquad Q \supseteq P,$$

inductively by demanding that for all $Q \supseteq P$,

$$c_Q(\lambda)\,\theta_Q(\lambda)^{-1} = \sum_{\{R \mid R \supseteq Q\}} c'_R(\lambda)\,\theta_Q^R(\lambda)^{-1} \tag{7.13}$$

These functions can be calculated explicitly ([Art81, (6.3)]). We will not need the explicit formula for our purposes, however.

Remark 7.14. We regret the use of obsolete notation of writing $c_P(\lambda)$ as a function of λ instead of just c_P. However, this notation is entrenched in Arthur's work and we use it to be consistent with the notation in [Art81, Art88b] and hope it does not cause any confusions. ❇

Definition 7.15. Suppose that for every $P \in \mathscr{P}(M)$, $c_P(\lambda)$ is a smooth function on $\iota\mathfrak{a}_M^*$. We call the collection

$$\{\, c_P(\lambda) \mid P \in \mathscr{P}(M)\,\}$$

a $(G, M)-$***family*** if functions corresponding to adjacent parabolics coincide on the shared hyperplane of their corresponding chambers. ♣

Lemma 7.16. *If* $\{\, c_P(\lambda) \mid P \in \mathscr{P}(M)\,\}$ *is a* $(G, M)-$*family,*

$$c_M(\lambda) = \sum_{P \in \mathscr{P}(M)} c_P(\lambda)\,\theta_P(\lambda)^{-1}$$

can be extended to a smooth function on $\iota\mathfrak{a}_M^*$. ♣

Notation 7.17. For a $(G, M)-$ family $\{\, c_P(\lambda) \mid P \in \mathscr{P}(M)\,\}$, we shall denote the value of $c_M(\lambda)$ at $\lambda = 0$ by c_M. Likewise, if Q contains some group in $\mathscr{P}(M)$, we shall write c'_Q for $c'_Q(0)$.

These can be explicitly calculated [Art81, (6.5)]. We will not need the explicit formulas for our purposes, however, ♦

Now, fix a group $L \in \mathscr{L}(M)$. If $Q \in \mathscr{P}(L)$, $P \in \mathscr{P}(M)$, and $P \subseteq Q$, the function

$$\lambda \mapsto c_P(\lambda), \qquad \lambda \in \iota\mathfrak{a}_Q^*,$$

depends only on Q and not on P. We denote it by $c_Q(\lambda)$. Then

$$\{\, c_Q(\lambda) \mid Q \in \mathscr{P}(L)\,\}$$

is a $(G, L)-$family. Suppose that $Q \in \mathscr{P}(L)$ is fixed. If $R \in \mathscr{P}^L(M)$, let $Q(R)$ be the unique group in $\mathscr{P}(M)$ such that $Q(R) \subseteq Q$ and $Q(R) \cap L = R$. Define a function $c_R^Q(\lambda)$ on $\iota\mathfrak{a}_M^*$ by

$$c_R^Q(\lambda) \ := \ c_{Q(R)}(\lambda). \tag{7.18}$$

Then

$$\left\{ c_R^Q(\lambda) \;\middle|\; R \in \mathscr{P}^L(M) \right\}$$

is an $(L, M)-$family. In particular, we have the functions

$$c_M^Q(\lambda), \left\{ \left(c_R^Q\right)'(\lambda) \;\middle|\; R \in \mathscr{P}^L(M) \right\}$$

and their values

$$c_M^Q, \left(c_R^Q\right)'$$

at $\lambda = 0$.

In general, c_M^Q depends on Q, and not just on L. If it is indeed independent of Q, we shall sometimes denote it by c_M^L. If each of the functions $c_R^Q(\lambda)$ depends only on L and not on Q, we shall denote them by $c_R^L(\lambda)$.

Thus, to summarise, given a $(G, M)-$family

$$\{ c_P(\lambda) \mid P \in \mathscr{P}(M) \},$$

we have defined a $(G, L)-$family

$$\{ c_Q(\lambda) \mid Q \in \mathscr{P}(L) \}$$

and (for every $Q \in \mathscr{P}(L)$) an $(L, M)-$family

$$\left\{ c_R^Q(\lambda) \;\middle|\; R \in \mathscr{P}^L(M) \right\}.$$

Now suppose that $\{d_P(\lambda)\}$ is a second $(G, M)-$family. Then the point wise product

$$(cd)_P(\lambda) \ := \ c_P(\lambda)\, d_P(\lambda)$$

is also a $(G, M)-$family. We have a simple formula for the function $(cd)_M$:

Lemma 7.19. *([Art81, Lemma 6.3]). We have*

$$(cd)_M(\lambda) = \sum_{Q \in \mathscr{F}(M)} c_M^Q(\lambda)\, d_Q'(\lambda). \tag{7.20}$$

♣

In case the quantities c_M^Q are independent of Q for all $Q \in \mathscr{F}(M)$, we have a simplification:

Lemma 7.21. *([Art81, Corollary 6.5]). Suppose that for $L \in \mathscr{L}(M)$, the function*

$$c_M^L := c_M^Q, \qquad Q \in \mathscr{P}(L)$$

is independent of Q. Then

$$(cd)_M = \sum_{L \in \mathscr{L}(M)} c_M^L d_L. \tag{7.22}$$

♣

7.2.1 Special Kind of $(G, M)-$families

The concept of a $(G, M)-$family was first introduced in [Art81]. In [Art82, §7], Arthur defined and studied a particularly nice kind of $(G, M)-$family. We summarise here what we need.

Lemma 7.23. *Let M be a fixed Levi subgroup. Suppose that for each reduced root β of (G, A_M) that c_β is an analytical function on a neighbourhood of $\iota\mathbb{R}$ in $\mathbb{C}$ such that $c_\beta(0) = 1$. Define*

$$c_P(\lambda) := \prod_{\beta \in \Sigma_P^r} c_\beta\left(\lambda\left(\beta^\vee\right)\right), \qquad \lambda \in \iota\mathfrak{a}_M^*, \tag{7.24}$$

for each group $P \in \mathscr{P}(M)$.

Then the family of functions $\{ c_P(\lambda) \mid P \in \mathscr{P}(M) \}$ is a $(G, M)-$family. ♣

Proof. We refer to [Art82, Page. 1317]. ∎

For this special kind of family, the number c_M has a particularly nice expression:

Proposition 7.25. *([Art82, Lemma 7.1]). For a $(G, M)-$family defined by (7.24),*

$$c_M = \sum_F \operatorname{vol}\left(\mathfrak{a}_M^G / \mathbb{Z}\left(F^\vee\right)\right) \cdot \left(\prod_{\beta \in F} c'_\beta(0)\right), \tag{7.26}$$

where the sum is taken over all subsets F of $\Sigma^r(G, A_M)$ for which

$$F^\vee := \{ \beta^\vee \mid \beta \in F \}$$

is a basis of $\mathfrak{a}_M^G$, and $\mathbb{Z}(F^\vee)$ stands for the lattice in $\mathfrak{a}_M^G$ generated by $F^\vee$. ♣

One of the nice things about the $(G, M)-$family

$$\{\, c_P(\lambda) \mid P \in \mathscr{P}(M)\,\}$$

of the form (7.24) is that the associated $(G, L)-$family

$$\{\, c_Q(\lambda) \mid Q \in \mathscr{P}(L)\,\}$$

is also of the form (7.24) ([Art82, Page. 1321]). The number c_L has similarly an expression similar to (7.26):

Proposition 7.27. *([Art82, Corollary 7.3]).*

$$c_L = \sum_F \operatorname{vol}\left(\mathfrak{a}_L^G / \mathbb{Z}\left(F_L^\vee\right)\right) \cdot \left(\prod_{\beta \in F} c'_\beta(0)\right), \tag{7.28}$$

where the sum is taken over all subsets F of $\Sigma^r(G, A_M)$ for which $F_L^\vee$ is a basis of $\mathfrak{a}_L^G$. ♣

Now we turn to the associated $(L, M)-$family. In fact, we consider a little more general situation. Let $L_1 \in \mathscr{L}(L)$ and that $S \in \mathscr{P}(L_1)$. Then the $(L_1, L)-$family

$$c_T^S(\lambda), \quad \lambda \in \iota\mathfrak{a}_L^*, \qquad T \in \mathscr{P}^{L_1}(L), \tag{7.29}$$

is of the form (7.24). Furthermore,

Proposition 7.30.

$$c_L^S = \sum_F \operatorname{vol}\left(\mathfrak{a}_L^{L_1} / \mathbb{Z}\left(F_L^\vee\right)\right) \cdot \left(\prod_{\beta \in F} c'_\beta(0)\right), \tag{7.31}$$

where the sum is taken over all subsets F of $\Sigma^r(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. In particular, c_L^S depends only on L_1 and not on the group $S \in \mathscr{P}(L_1)$. ♣

7.3 Defining the Weighted Orbital Integral

Arthur first defined the weighted orbital integrals only in 1988 in [Art88b]. He defines it for a general element $\gamma \in M(\mathbb{Q}_S)$[1]. As noted, we will only need to consider the case when γ is unipotent. This simplifies the details considerably. The development of the definition and extracting out a suitable expression which we can work with still entails lengthy and laborious technical details. We will finally be interested in Equation (7.41). As such, we only present the details which are relevant for us to understand this expression.

Thus, we fix a Levi subgroup M of G, a set S and a unipotent element $u = \prod_{p \in S} u_p \in M(\mathbb{Q}_S)$ (we allow for the possibility of p being infinity).

The definition of a weighted orbital integral $\mathfrak{J}_M(u, f)$ for $f \in C_c^\infty\left(M(\mathbb{Q}_S)^1\right)$ is set up in two steps.

7.3.1 Step One

In the first step, we assume that the centraliser M_u of u in M equals the centraliser G_u of u in G. The definition of the weighted orbital integral in this case requires the use of a special $(G, M)-$family which we define below.

Notation 7.32. For any point $x \in G(\mathbb{Q}_S)$, and any $P \in \mathscr{F}(M)$, define

$$v_P(\lambda, x) \ := \ e^{-\lambda(H_P(x))}, \qquad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}. \tag{7.33}$$

Then for any $x \in G(\mathbb{Q}_S)$, the collection

$$\{\, v_P(\lambda, x) \mid P \in \mathscr{P}(M) \,\}$$

is a $(G, M)-$family of functions of $\lambda \in \iota\mathfrak{a}^*_M$.

As usual, this gives rise to the function (of λ) $v_M(\lambda, x)$ and we denote its value at $\lambda = 0$ by $v_M(x)$. ♦

Arthur now proceeds to define $\mathfrak{J}_M(u, f)$ for any element $u \in M(\mathbb{Q}_S)$ satisfying the above condition on centralisers.

Definition 7.34. Let $u \in M(\mathbb{Q}_S)$ be such that $M_u = G_u$. Then the ***weighted orbital integral*** $\mathfrak{J}_M(u, f)$ is defined by

$$\mathfrak{J}_M(u, f) \ := \ \int_{G_u(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f\left(x^{-1}ux\right) v_M(x) \, \mathrm{d}x. \tag{7.35}$$

♣

Remark 7.36. Arthur actually defines the weighted orbital integral differently [Art88b, (2.1)] and then derives the above formula. However, the above formulation is more suitable for us and we use it as the definition. ❇

Notation 7.37. Let $A_{M,\mathrm{reg}}$ be the set

$$\{\, a \in A_M \mid G_a \subseteq M \,\}.$$

It is an open subvariety of M, defined over $\mathbb{Q}$. ♦

Let γ be *any* element of $M(\mathbb{Q}_S)$. Then for any $a \in A_{M,\text{reg}}(\mathbb{Q}_S)$ which is close to $1, a\gamma$ will be a point in $M(\mathbb{Q}_S)$ with the property that $G_{a\gamma} = M_{a\gamma}$. The distribution $\mathfrak{J}_M(a\gamma)$ is thus defined. Arthur's strategy is then to take the limit as $a \to 1$. This is what ultimately gives the definition of a weighted orbital integral. The details are not as straightforward though. Before we state and explain the formula, we need some more preliminary notions.

7.3.2 Preparation for Step Two

In this subsection, we fix a prime p (possibly infinity) and work locally on field $\mathbb{Q}_p$.

We start with a $P_1 \in \mathscr{P}(M)$ and we write $N_1 := N_{P_1}$. We also write $\Sigma^r_{P_1}$ for the set of reduced roots of (P_1, A_M). If $a \in A_M$ and $u \in \mathcal{U}_M$, then

$$n \mapsto (au)^{-1} n^{-1} (au) n, \qquad n \in N_1,$$

is a polynomial mapping from N_1 to itself. It is invertible if $a \in A_{M,\text{reg}}$. Consequently, for any such a and any unipotent element

$$\pi = u\nu, \qquad u \in \mathcal{U}_M, \nu \in N_1,$$

in P_1, we can define $n \in N_1$ uniquely by

$$a\pi = n^{-1} aun. \tag{7.38}$$

Thus, we have a function $(a, \pi) \mapsto n$. We use it to study $v_P(\lambda, n)$ as a function of a and π.

We write $\mathrm{Wt}(\mathfrak{a}_M)$ for the set of elements of $\mathfrak{a}^*_M$ which are extremal weights of irreducible finite-dimensional representations of G. Then for any $\omega \in \mathrm{Wt}(\mathfrak{a}_M)$, we fix $(\Lambda_\omega, V_\omega, \phi_\omega, \|\cdot\|)$ with Λ_ω an irreducible representation of G on a vector space V_ω, ϕ_ω an extremal vector in V_ω with weight ω and $\|\cdot\|$ a norm function on $V_\omega(F)$ which is stabilised by $\mathbf{K}$ and for which ϕ_ω is a unit vector. Then Arthur shows that ([Art88b, Equation (3.3)] for any $x \in G(F)$,

$$v_P(\omega, x) = \left\| \Lambda_\omega \left(x^{-1}\right) \phi_\omega \right\|. \tag{7.39}$$

From this it follows that $v_P(\omega, n)$ as a function of (a, π) is a function of a times a polynomial in (a, π).

After these preliminaries, we can give the final formula for the weighted orbital integral.

7.3.3 Step Two

We state here, the awaited

Definition 7.40. ([Art88b, §5, 6]). The weighted orbital integral for a unipotent element $u \in M(\mathbb{Q}_S)$ equals

$$\mathfrak{J}_M(u, f) = \int_{\Pi_S} \left[\left(\int_{\mathbf{K}_S} f\left(k^{-1}\pi k\right) \mathrm{d}k \right) w_M(1, \pi) \right] \mathrm{d}\pi. \tag{7.41}$$

♣

We will define the undefined quantities shortly. However, we first pause to make a couple of remarks.

Remark 7.42. Again, this is not the definition Arthur gives. The definition is a little more technical and is given in [Art88b, Equation (6.5) on Page 254]. However, Equation (7.41) is what we need and can be extracted from the second formula (the one appearing on Page 256) in [Art88b, Corollary 6.2], if we specialise γ to be a unipotent element u. ❇

Remark 7.43. We also take the opportunity to point out a typographical error in the same formula in [Art88b, Corollary 6.2 (Page 256)]. The quantity $v_M^Q(1, \pi)$ should in fact be $w_M^Q(1, \pi)$. These quantities $w_M^Q(1, \pi)$ are analogues of $w_M(1, \pi)$ for a parabolic $Q \in \mathscr{F}(M)$. However, in our situation (when γ is a unipotent element), the sum over the parabolic subgroups $Q \in \mathscr{F}(M)$ appearing in Arthur's formula disappears and the only term that survives is $Q = G$, which is reflected in our simple formula (7.41). ❇

The Integration Domain Π_S

Recall that we have begun with a Levi subgroup M and a unipotent element $u \in M(\mathbb{Q}_S)$ with $u = \prod_{\nu \in S} u_\nu, u_\nu \in M(F_\nu)$.

Now let $\mathfrak{m}$ be the Lie algebra of M. Then by the Jacobson-Morosov Theorem, there is, for every $\nu \in S$, a Lie algebra homomorphism

$$\Psi_\nu : \mathfrak{sl}_2 \to \mathfrak{m}$$

such that

$$u_\nu = \exp\left(\Psi_\nu\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right) \in M(F_\nu).$$

Let

$$H_\nu := \Psi_\nu\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \in \mathfrak{m}$$

and

$$\mathfrak{m}_{\nu,i} \;:=\; \{\, \xi \in \mathfrak{m} \mid \operatorname{ad}(H_\nu)\,\xi = i\xi \,\} \subseteq \mathfrak{m}, \quad i \in \mathbb{Z}.$$

Then

$$\mathfrak{m}_{\nu,\geq 0} \;:=\; \bigoplus_{i\geq 0} \mathfrak{m}_{\nu,i}$$

is a parabolic subalgebra of $\mathfrak{m}$ with Levi decomposition

$$\mathfrak{m}_{\nu,\geq 0} = \mathfrak{m}_{\nu,0} \oplus \mathfrak{u}_\nu^M, \quad \mathfrak{u}_\nu^M \;:=\; \bigoplus_{i>0} \mathfrak{m}_{\nu,i}.$$

This allows us to set the

Notation 7.44. Let

$$P_\nu^M \;:=\; \exp\left(\mathfrak{m}_{\nu,\geq 0}\right) \subseteq M$$

the Jacobson-Morosov parabolic subgroup of M associated to the orbit of u_ν with Levi decomposition

$$P_\nu^M = \tilde{M}_\nu U_\nu^M,$$

where $\tilde{M}_\nu \;:=\; \exp\left(\mathfrak{m}_{\nu,0}\right)$ and $U_\nu^M \;:=\; \exp\left(\mathfrak{u}_\nu^M\right)$. ♦

Notation 7.45. Furthermore, we set

$$Z_\nu \;:=\; \left\{\, p_\nu^{-1} u_\nu p_\nu \;\middle|\; p_\nu \in N_{M(F_\nu)}\left(\mathfrak{m}_{\nu,\geq 0}\left(F_\nu\right)\right) \,\right\} \subseteq M\left(F_\nu\right),$$

$$\mathfrak{u}_{\nu,\geq 2}^M \;:=\; \bigoplus_{i\geq 2} \mathfrak{m}_{\nu,i},$$

and

$$U_\nu^{M,\geq 2} \;:=\; \exp\left(\mathfrak{u}_{\nu,\geq 2}^M\right) \subseteq M.$$

Then Z_ν is an open subset of $U_\nu^{\geq 2}\left(F_\nu\right)$ ([Rao72, Lemmas 1, 4]).

We also write

$$Z_S \;:=\; \prod_{\nu\in S} Z_\nu \subseteq \prod_{\nu\in S} U_\nu^{M,\geq 2}\left(F_\nu\right) \subseteq M\left(F_S\right). \tag{7.46}$$

♦

Remark 7.47. We remark that everything above was strictly taking place inside the Levi subgroup M. Now we 'inflate' the domain Z_S to G, which would give us the final definition of the integration domain Π_S. ✻

Notation 7.48. Fix $R = \prod_{\nu \in S} R_\nu$, where for each $\nu \in S, R_\nu \in \mathscr{P}^G(M)$. Then let N_R be the unipotent radical of the parabolic R. Then we finally define

$$\Pi_S := Z_S N_R (F_S) \subseteq G(F_S). \tag{7.49}$$

♦

Remark 7.50. Thus, Π_S is a subset of $G(F_S)$ and not just $M(F_S)$. That means, in the weighted orbital integral associated to the Levi subgroup M, the integration occurs over a domain outside of M. ❇

The Function $w_M(1, \pi)$

Now we turn to defining the function $w_M(1, \pi)$. As expected there is a $(G, M)-$family

$$\begin{aligned} w_P(\lambda, a, \pi) &:= \prod_{\nu \in S} w_P(\lambda, a_\nu, \pi_\nu), \\ &\qquad \lambda \in \mathfrak{a}_M^*,\ a = \prod_{\nu \in S} a_\nu \in A_{M,\mathrm{reg}}(F_S),\ \pi = \prod_{\nu \in S} \pi_\nu \in \Pi_S. \end{aligned} \tag{7.51}$$

hiding behind its definition. Thus, we can define a function $w_M(a, \pi)$ as in Lemma 7.16. Then

Definition 7.52. The function $w_M(1, \pi)$ is defined as

$$w_M(1, \pi) := \lim_{a \to 1} w_M(a, \pi). \tag{7.53}$$

This limit exists as is shown in [Art88b, §5]. ♣

Of course we have not defined the $(G, M)-$families $w_P(\lambda, a_\nu, \pi_\nu)$. We will not a precise description of these for our purposes. They are defined in [Art88b, Equation (3.6)]. We will only need a quantitative estimate:

Proposition 7.54. *([Art88b, Lemma 5.4]). For any $a \in A_{M,\mathrm{reg}}(F_S)$ and $\pi \in \Pi_S$, we can write $w_M(a, \pi)$ as a finite sum*

$$\sum_{\Omega} c_\Omega \left(\prod_{(\omega, \nu) \in \Omega} \ln\left(\left\| W_{\omega_\nu}(a_\nu, \pi_\nu) \right\|\right) \right),$$

where each Ω is a finite disjoint union of pairs

$$(\omega, \nu) \in \mathrm{Wt}(\mathfrak{a}_M) \times S,$$

and W_ω is a $V_\omega-$valued polynomial (cf. [Art88b, Equation (3.8)]). ♣

The Relation between π and u

It might be, on first sight, a little surprising to note that in Equation (7.41), the unipotent element u does not appear on the right hand side. The surprise is only short lived however, as the assiduous reader realises that the quantity π is defined in terms of u by Equation (7.38), which of course involves a choice of element $a \in A_{M,\mathrm{reg}}(F_S)$ — which is again missing in Equation (7.41). In fact, Equation (7.41) is a little misleading. If one goes through [Art88b], one finds that one actually ought to write

$$\mathfrak{J}_M(u, f) = \lim_{a \to 1} \mathfrak{J}_M(au, f),$$

with

$$\mathfrak{J}_M(au, f) = \int_{\Pi_S} \left[\left(\int_{\mathbf{K}} f\left(k^{-1}\pi k\right) \mathrm{d}k \right) w_M(a, \pi) \right] \mathrm{d}\pi.$$

The similarity of this last formula to Equation (7.35) is no coincidence as the elements au satisfy the condition required for (7.35).

Thus, to summarise, $\mathfrak{J}_M(u, f)$ is defined in terms of $\mathfrak{J}_M(au, f)$, which is in turn given by a formula involving a and π but not u. However, π is defined in terms of a and u and hence there is no mystery.

The Measure $\mathrm{d}\pi$ on Π_S

We describe the measure on Z_S. We recall that $Z_S = \prod_{\nu \in S} Z_\nu$ with each $Z_\nu \subseteq \exp\left(\mathfrak{u}^M_{\nu,\geq 2}(\mathbb{Q}_\nu)\right)$. Then for

$$\zeta = \prod_{\nu \in S} \exp(X_\nu), \qquad X_\nu \in \mathfrak{u}^M_{\nu,\geq 2}(\mathbb{Q}_\nu),$$

set

$$\mathrm{d}\zeta := \prod_{\nu \in S} \left(|J_\nu(X_\nu)|_\nu^{\frac{1}{2}} \mathrm{d}X_\nu \right), \tag{7.55}$$

where J_ν is a polynomial on $\mathfrak{u}^M_{\nu,\geq 2}(\mathbb{Q}_\nu)$, defined by [Rao72, Lemma 2].

7.4 Estimating $\mathfrak{J}_M(u, f)$

We have seen in (7.41) what the weighted orbital integral associated to a Levi subgroup M and a unipotent element $u \in M(\mathbb{Q}_S)$ looks like. We repeat it here:

$$\mathfrak{J}_M(u, f) = \int_{\Pi_S} \left[\left(\int_{\mathbf{K}_S} f\left(k^{-1}\pi k\right) \mathrm{d}k \right) w_M(1, \pi) \right] \mathrm{d}\pi. \tag{7.56}$$

We will now estimate $\mathfrak{J}_M(u, f)$ for a fixed unipotent element $u \in M(\mathbb{Q}_S)$ with $(u, M) \neq (1, G)$.

Recalling Convention 6.9, we see that (cf. Equation (7.49)),

$$\Pi_S \ := \ Z_S N_R(\mathbb{Q}_S) = Z_\infty N_{R_\infty}(\mathbb{R})\, Z_p N_{R_p}(\mathbb{Q}_p) \ =: \ \Pi_\infty \Pi_p,$$

where (cf. Equation (7.46)),

$$Z_S \ := \ \prod_{\nu \in S} Z_\nu = Z_\infty Z_p,$$

with

$$Z_\nu \subseteq U_\nu^{M,\geq 2}(\mathbb{Q}_\nu) \ := \ \exp\left(\mathfrak{u}^M_{\nu,\geq 2}(\mathbb{Q}_\nu)\right).$$

Notation 7.57. We recall (cf. Notation 7.44) $P_p^M = \tilde{M}_p U_p^M$ being the Jacobson-Morosov parabolic subgroup associated to the unipotent orbit $u_p \in M(\mathbb{Q}_p)$. Then let P be the unique subgroup in $\mathscr{F}$ such that $P \in \mathscr{P}\left(\tilde{M}_p\right)$. Let $U \subseteq G$ be the unipotent radical of P. That is, $P = \tilde{M}_p U$.

Then we have that

$$U_p^{M,\geq 2} N_{R_p} \subseteq U \tag{7.58}$$

and

$$U_p^{M,\geq 2} \subseteq U^{\geq 2}. \tag{7.59}$$

Finally, we define

$$U^{\geq 2,\text{ext}} \ := \ U^{\geq 2} \cdot N_{R_p}, \tag{7.60}$$

which gives us

$$\Pi_p \subseteq U^{\geq 2,\text{ext}}(\mathbb{Q}_p). \tag{7.61}$$

♦

7.4.1 Estimating the Volume

Let T be the maximal torus of our reductive group G and let Φ be the root system of G with respect to T ([Mil17, Chapter 21]). We also let Φ^+ denote the set of positive roots with respect to a set of simple roots Δ.

Since G is a reductive group, we can use the theory of buildings to analyse the compact subgroups of $G(\mathbb{Q}_p)$. In particular, maximal compact subgroups are stabilisers of extremal points of chambers in the building of $G(\mathbb{Q}_p)$. Let $\mathbb{K}_p$ then be the stabiliser of the vertex x of the fundamental chamber C in the Bruhat-Tits building of the group $G(\mathbb{Q}_p)$. Using the description Proposition B.77 of chambers, we can assume that x is the point

$$\frac{\varpi_i}{p_i}$$

for i such that

$$\alpha_i(x) \neq 0$$

(in fact $\alpha_i(x) = \frac{1}{p_i}$). Also, since $\mathbb{K}_p$ is not hyperspecial, we have that $p_i > 1$.

Let $\Theta := \Delta \setminus \{\alpha_i\}$ and let $Q = LV$ be the standard parabolic corresponding to the set Θ ([CM93, Lemma 3.8.1]). That is, the root system generated by Θ is the root system of the Levi L. Then we define

$$\Phi_V := \left\{ \alpha \in \Phi^+ \,\middle|\, \varpi_i(\alpha) > 0 \right\},$$

and

$$\Psi := \left\{ \alpha \in \Phi^+ \,\middle|\, \varpi_i(\alpha) = p_i \right\}.$$

Clearly, $\Psi \subsetneq \Phi_V$ (since $p_i > 1$). Also, one may observe that the set Φ_V is exactly the set of positive roots that "occur" in the decomposition of the Lie algebra of V (the unipotent radical of Q) under the adjoint action of the maximal torus on the Lie algebra of V.

We remind the reader of Notation 7.57 and in particular, Equation (7.61). It is sufficient to estimate the volumes $U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p) \cap k\,\mathbb{K}_p\,k^{-1}$ for $k \in \mathbf{K}$. To this end, we let Φ_U be defined analogous to Φ_V. Then the group U is generated by the root subgroups $U_\alpha, \alpha \in \Phi_U$, where each U_α is isomorphic to the additive group $\mathbb{G}_\mathrm{a}$. This is a basic statement in the theory of Algebraic groups (cf. [Spr09, Proposition 8.1.1 and Corollary 8.1.2], cf. also [Mil17, Item 14.58]). Let $\chi_\alpha : \mathbb{G}_\mathrm{a} \to U_\alpha$ be the isomorphism of root subgroups U_α with $\mathbb{G}_\mathrm{a}$.

Then the group $U^{\geq 2,\mathrm{ext}}$ is generated by $U_\alpha, \alpha \in \Phi_{U^{\geq 2,\mathrm{ext}}}$ where

$$\Phi_{U^{\geq 2,\mathrm{ext}}} := \Phi_{M,\geq 2} \cup \Phi_{N_R}.$$

Furthermore, let $\overline{Q} = L\overline{V}$ be the opposite parabolic of Q. Then we have

Proposition 7.62. *The subgroup* $\mathbb{K}_p \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)$ *can be described as follows*

$$\mathbb{K}_p \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p) = \left\{ x = \prod_{\alpha \in \Phi_{U^{\geq 2,\mathrm{ext}}}} \chi_\alpha(x_\alpha) \right\}$$

such that

$$\begin{aligned} x_\alpha &\in p\mathbb{Z}_p, & \alpha &\in \Phi_{\overline{V}}, \\ x_\alpha &\in p^{-1}\mathbb{Z}_p, & \alpha &\in \Psi, \\ x_\alpha &\in \mathbb{Z}_p, & &\text{otherwise.} \end{aligned}$$

♣

Proof. This is clear from the description of the parahoric subgroup $\mathbb{K}_p$ in terms of the point x in the building and [Tit79, §3.3.1]. ∎

We will need the following corollary where $w \in W$ with W being the Weyl group of Φ.

Corollary 7.63. *The subgroup* $w\,\mathbb{K}_p\,w^{-1} \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)$ *can be described as follows*

$$w\,\mathbb{K}_p\,w^{-1} \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p) = \left\{ x = \prod_{\alpha \in \Phi_{U^{\geq 2,\mathrm{ext}}}} \chi_\alpha(x_\alpha) \right\}$$

such that

$$\begin{aligned} x_\alpha &\in p\mathbb{Z}_p, & w^{-1}(\alpha) &\in \Phi_{\overline{V}}, \\ x_\alpha &\in p^{-1}\mathbb{Z}_p, & w^{-1}(\alpha) &\in \Psi, \\ x_\alpha &\in \mathbb{Z}_p, & &\text{otherwise.} \end{aligned}$$

♣

Proof. This follows immediately from the previous proposition. ∎

Remark 7.64. The above two corollaries remain true if $U^{\geq 2,\mathrm{ext}}$ is replaced by U throughout (and $\Phi_{U^{\geq 2,\mathrm{ext}}}$ by Φ_U) with exactly the same justification. ❇

This allows us to state the

Corollary 7.65. *The volume of* $w\,\mathbb{K}_p\,w^{-1} \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)$ *is given by*

$$\operatorname{vol}\left(w\,\mathbb{K}_p\,w^{-1} \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\right) = p^{\left(\left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w(\Psi)\right| - \left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w\left(\Phi_{\overline{V}}\right)\right|\right)}. \tag{7.66}$$

♣

Proof. This is because we can normalise the measure on $\mathbb{Q}_p$ is given such that

$$\operatorname{vol}(\mathbb{Z}_p) = 1,$$
$$\operatorname{vol}(p\mathbb{Z}_p) = \frac{1}{p},$$

and

$$\operatorname{vol}\left(p^{-1}\mathbb{Z}_p\right) = p.$$

■

7.4.2 Technical Lemma

As before, let Q be the maximal parabolic subgroup corresponding to the root α_i. Recall that α_i was chosen before and the coefficient of α_i in $\tilde{\alpha}$ is positive. Let Φ_V be the set of roots "occurring" in the decomposition of the Lie algebra of subgroup V and let Ψ be the set of roots for which the α_i coefficient is maximal.

Lemma 7.67. *Let U be the unipotent radical for any standard parabolic subgroup P. If $\Phi_V \cap w^{-1}(\Phi_U) \neq \emptyset$, then*

$$(\Phi_V \setminus \Psi) \cap w^{-1}(\Phi_U) \neq \emptyset. \tag{7.68}$$

♣

Proof. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of the root system Φ of G with respect to a maximal torus T. Let $\Theta \subseteq \Delta$ be the subset defining the standard parabolic subgroup P. Using a suitable ordering, we can arrange that $\Theta = \{\alpha_1, \dots, \alpha_k\}$ for some $k \leq r$. Then $\Phi_U = \Phi^+ \setminus \Theta^+$.

Now let $\alpha_1, \alpha_2 \in \Phi^+$ be such that $\alpha_1 + \alpha_2 \in \Phi_U$. Then since Θ^+ is a closed subsystem, we have that either α_1 or α_2 must be in Φ_U. Thus, if the sum of two positive roots belongs to Φ_U, then so does at least one of them. The same remains true if we replace by Φ_U by its conjugate $w^{-1}(\Phi_U)$.

Now since any $\alpha \in \Psi$ can be written as $\alpha = \alpha_1 + \alpha_2$ with $\alpha_i \in \Phi_V \setminus \Psi$, we have the assertion. ■

Remark 7.69. The hypothesis on w in the above lemma is equivalent to saying that w is not in the $(W_P, W_Q)-$double coset of the longest Weyl element—that is, does not correspond to the biggest cell in the Bruhat decomposition.

Indeed, when w is the longest element, $w^{-1}\Phi_U$ consists only of negative roots and Φ_V only of the positive roots. Moreover, the longest element is the only element which maps all positive roots to negative roots. ❇

7.4.3 The Factor J

The polynomial J_p was introduced when we change the variables from the group to the Lie algebra level. It was introduced and explicated by R. Rao in [Rao72]. Indeed, the formulation is (cf. [Rao72, Lemma 2])

$$J_p(X) \ := \ \det\left([\,X, \cdot\,] : \mathfrak{u}_{-1} \to \mathfrak{u}_1\right) \tag{7.70}$$

for an element $X \in \ln\left(U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p) \cap w\,\mathbb{K}_p\,w^{-1}\right)$.

This being non-zero implies that there exists a bijection $\pi : \Phi_1 \to \Phi_1$ such that

$$\gamma + \pi(\gamma) \in \Phi_2 \qquad \forall \gamma \in \Phi_1.$$

The determinant above is then an integral linear combination of the monomials

$$\prod_{\gamma \in \Phi_1} \left(X_{\gamma+\pi(\gamma)}\right),$$

where $X_\alpha \in \mathbb{Q}_p$ denotes the root-coordinate of the element X with respect to the root α. Thus, we have

$$\left|J_p(X)\right|_p \leq \max_{\pi} \left\{ \prod_{\gamma \in \Phi_1} \left|\left(X_{\gamma+\pi(\gamma)}\right)\right|_p \right\}. \tag{7.71}$$

Hence we are reduced to calculating the $p-$adic norms of the root coordinates. We fix a map π and we give the root $\gamma + \pi(\gamma)$ weight 1 if $\gamma + \pi(\gamma) \in w(\Psi)$ and 0 otherwise. This is because of Corollary 7.63 and Remark 7.64.

We write

$$\delta_1 = \sum_{\gamma \in \Phi_1} \gamma$$

which implies

$$2\delta_1 = \sum_{\gamma \in \Phi_1} \left(\gamma + \pi(\gamma)\right)$$

and hence the total weight k of $2\delta_1$ is given by

$$\begin{aligned} k &= \sum_{\{\,\gamma \in \Phi_1 \,\mid\, \gamma+\pi(\gamma) \in w(\Psi)\,\}} 1 \\ &\leq \sum_{\{\,\gamma \in \Phi_1 \,\mid\, \gamma \in w(\Psi)\,\}} 1 + \sum_{\{\,\gamma \in \Phi_1 \,\mid\, \pi(\gamma) \in w(\Psi)\,\}} 1 \\ &\quad + \sum_{\{\,\gamma \in \Phi_1 \,\mid\, \gamma \in w(\Phi_V - \Psi)\,\}} \frac{1}{2} + \sum_{\{\,\gamma \in \Phi_1 \,\mid\, \pi(\gamma) \in w(\Phi_V - \Psi)\,\}} \frac{1}{2} \\ &= 2\left|\Phi_1 \cap w(\Psi)\right| + \left|\Phi_1 \cap w(\Phi_V - \Psi)\right|. \end{aligned}$$

The upshot of all this can be stated as

Proposition 7.72. *For $X \in \ln\left(U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p) \cap w\,\mathbb{K}_p\,w^{-1}\right)$, we have*

$$\sqrt{|J_p(X)|} \leq p^{\left(|\Phi_1 \cap w(\Psi)| + \frac{1}{2}|\Phi_1 \cap w(\Phi_V - \Psi)|\right)}. \tag{7.73}$$

♣

7.4.4 Estimating the Invariant Orbital Integral

Our goal in this subsection is to provide the relevant estimates for the function ϕ defined as

Notation 7.74. To simplify the estimation of Equation (7.56) we separate out the inner integral as:

$$\phi(x) := \int_{\mathbf{K}_S} f\left(k^{-1}xk\right)\mathrm{d}k, \qquad x \in G(\mathbb{Q}_S). \tag{7.75}$$

We can break it into infinite and finite parts and further write

$$\begin{aligned}\phi(x) &= \int_{\mathbf{K}_S} f\left(k^{-1}xk\right)\mathrm{d}k \\ &= \int_{\mathbf{K}_\infty} f_\infty\left(k^{-1}x_\infty k\right)\mathrm{d}k \cdot \int_{\mathbf{K}_p} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1}x_p k\right)\mathrm{d}k.\end{aligned}$$

We separate out the finite part in

$$\phi_p(x_p) := \int_{\mathbf{K}_p} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1}x_p k\right)\mathrm{d}k \qquad x_p \in G(\mathbb{Q}_p). \tag{7.76}$$

♦

We start by estimating the finite part ϕ_p. In this regard, we have the

Proposition 7.77. *For $x_p \in G(\mathbb{Q}_p)$, we have*

$$\phi_p(x_p) = \sum_{w \in W_P \backslash W / W_Q} C_w D_w(x_p), \tag{7.78}$$

where

$$C_w := \mathrm{vol}(\mathbf{K}_p \cap \mathbb{K}_p)\left(\left[(P(\mathbb{Q}_p) \cap \mathbf{K}_p) : \left(P(\mathbb{Q}_p) \cap w(\mathbf{K}_p \cap \mathbb{K}_p)w^{-1}\right)\right]\right). \tag{7.79}$$

and

$$D_w\left(x_p\right) \;:=\; \operatorname{vol}\left(\left\{\, l \in \left(P\left(\mathbb{Q}_p\right) \cap \mathbf{K}_p\right) \;\middle|\; x_p \in l\, w\, \mathbb{K}_p\, w^{-1}\, l^{-1} \,\right\}\right). \tag{7.80}$$

♣

Proof. We begin by decomposing the group $\mathbf{K}_p$ as

$$\mathbf{K}_p = \bigcup_{w \in W_P \backslash W / W_Q} \left(P\left(\mathbb{Q}_p\right) \cap \mathbf{K}_p\right) w \left(\mathbb{K}_p \cap \mathbf{K}_p\right).$$

This in turn allows us to write

$$\begin{aligned} \phi_p\left(x_p\right) &= \int_{\mathbf{K}_p} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1} x_p k\right) \mathrm{d}k \\ &= \sum_w \int_{\mathfrak{Z}_w} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1} x_p k\right) \mathrm{d}k, \end{aligned}$$

where we write $\mathfrak{Z}_w$ for $\left(P\left(\mathbb{Q}_p\right) \cap \mathbf{K}_p\right) w \left(\mathbb{K}_p \cap \mathbf{K}_p\right)$.

The result now follows from the estimate of the integral given by Equation (7.85). ■

Notation 7.81. We denote the quantity

$$\phi_{p,w}\left(x_p\right) \;:=\; \int_{\mathfrak{Z}_w} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1} x_p k\right) \mathrm{d}k. \tag{7.82}$$

so that

$$\phi_p\left(x_p\right) = \sum_w \phi_{p,w}\left(x_p\right). \tag{7.83}$$

♦

Proposition 7.84. *For $x_p \in G\left(\mathbb{Q}_p\right)$, we have*

$$\phi_{p,w}\left(x_p\right) = C_w \cdot D_w\left(x_p\right). \tag{7.85}$$

♣

Proof. We write $k \in \mathfrak{z}_w$ as $k = lw\tilde{k}$ with $l \in P(\mathbb{Q}_p) \cap \mathbf{K}_p$ and $\tilde{k} \in \mathbb{K}_p \cap \mathbf{K}_p$ and then we have

$$\begin{aligned}
\phi_{p,w}(x_p) &= \int_{\mathfrak{z}_w} \mathbb{1}_{\mathbb{K}_p}\left(k^{-1}x_pk\right) \mathrm{d}k \\
&= \mathrm{vol}\left(\left\{ k \in \mathfrak{z}_w \;\middle|\; k^{-1}x_pk \in \mathbb{K}_p \right\}\right) \\
&= \mathrm{vol}\left(\left\{ lw\tilde{k} \in \mathfrak{z}_w \;\middle|\; \left(lw\tilde{k}\right)^{-1} x_p \left(lw\tilde{k}\right) \in \mathbb{K}_p \right\}\right) \\
&= \mathrm{vol}\left(\left\{ lw\tilde{k} \in \mathfrak{z}_w \;\middle|\; w^{-1}l^{-1}x_plw \in \mathbb{K}_p \right\}\right) \\
&= \mathrm{vol}\left(\left\{ lw\tilde{k} \in \mathfrak{z}_w \;\middle|\; x_p \in l\,w\,\mathbb{K}_p\,w^{-1}\,l^{-1} \right\}\right) \\
&= \mathrm{vol}(\mathbf{K}_p \cap \mathbb{K}_p)\left(\left[(P(\mathbb{Q}_p) \cap \mathbf{K}_p) : \left(P(\mathbb{Q}_p) \cap w(\mathbf{K}_p \cap \mathbb{K}_p)w^{-1}\right)\right]\right) \\
&\quad \times \mathrm{vol}\left(\left\{ l \in (P(\mathbb{Q}_p) \cap \mathbf{K}_p) \;\middle|\; x_p \in l\,w\,\mathbb{K}_p\,w^{-1}\,l^{-1} \right\}\right) \\
&= C_w \cdot D_w(x_p).
\end{aligned}$$

■

7.4.5 Estimate on C_w

Lemma 7.86. *We have that*

$$\begin{aligned}
C_w \;:=\; &\mathrm{vol}(\mathbf{K}_p \cap \mathbb{K}_p) \cdot \left(\left[(P(\mathbb{Q}_p) \cap \mathbf{K}_p) : \left(P(\mathbb{Q}_p) \cap w(\mathbf{K}_p \cap \mathbb{K}_p)w^{-1}\right)\right]\right) \\
&\ll p^{-|\Phi_U \cap w(\Phi_V)|}.
\end{aligned} \tag{7.87}$$

♣

Proof. Let $\mathbf{F}_p$ denote the finite field of order p. Then we write

$$\mathrm{vol}(\mathbf{K}_p \cap \mathbb{K}_p) = [G(\mathbf{F}_p) : Q(\mathbf{F}_p)]^{-1} \tag{7.88}$$

and

$$\begin{aligned}
&\left[(P(\mathbb{Q}_p) \cap \mathbf{K}_p) : \left(P(\mathbb{Q}_p) \cap w(\mathbf{K}_p \cap \mathbb{K}_p)w^{-1}\right)\right] \\
&= \left[P(\mathbf{F}_p) : P(\mathbf{F}_p) \cap w\,Q(\mathbf{F}_p)\,w^{-1}\right].
\end{aligned} \tag{7.89}$$

Now using Nori's estimate for the number of points of algebraic groups over the finite field $\mathbf{F}_p$, (cf. [Nor87, Lemma 3.5]), we can estimate the quantity in Equation (7.88) by

$$\begin{aligned}
[G(\mathbf{F}_p) : Q(\mathbf{F}_p)]^{-1} &\ll p^{\dim(Q) - \dim(G)} \\
&= p^{-\dim(V)}.
\end{aligned} \tag{7.90}$$

Similarly, factor in Equation (7.89) can be estimated as

$$\left[P\left(\mathbf{F}_p\right) : P\left(\mathbf{F}_p\right) \cap w\, Q\left(\mathbf{F}_p\right) w^{-1}\right] \ll p^{\dim(P) - \dim\left(P \cap w\, Q\, w^{-1}\right)}. \tag{7.91}$$

The product C_w is then estimated as

$$\begin{aligned} C_w &\ll p^{-\dim(V) + \dim\left(P \cap w\, \overline{V}\, w^{-1}\right)} \\ &= p^{-\left|\Phi_U \cap w(\Phi_V)\right|}. \end{aligned} \tag{7.92}$$

This is what we wanted to show. ■

The Collective Estimate

Proposition 7.93. *We have, for $X \in \ln\left(U^{\geq 2,\mathrm{ext}}\left(\mathbb{Q}_p\right) \cap w\, \mathbb{K}_p\, w^{-1}\right)$,*

$$\operatorname{vol}\left(w\, \mathbb{K}_p\, w^{-1} \cap U^{\geq 2,\mathrm{ext}}\left(\mathbb{Q}_p\right)\right) \cdot C_w \cdot \sqrt{J_p\left(X\right)} \ll p^{-\frac{1}{2}} \tag{7.94}$$

for all $w \in W_Q \backslash W / W_P$. ♣

Proof. We have, by Equation (7.66),

$$\operatorname{vol}\left(w\, \mathbb{K}_p\, w^{-1} \cap U^{\geq 2,\mathrm{ext}}\left(\mathbb{Q}_p\right)\right) = p^{\left(\left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w(\Psi)\right| - \left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w\left(\Phi_{\overline{V}}\right)\right|\right)},$$

by Equation (7.73),

$$\sqrt{\left|J_p\left(X\right)\right|} \leq p^{\left(\left|\Phi_1 \cap w(\Psi)\right| + \frac{1}{2}\left|\Phi_1 \cap w(\Phi_V - \Psi)\right|\right)},$$

and by Equation (7.87),

$$C_w \ll p^{-\left|\Phi_U \cap w(\Phi_V)\right|}.$$

Now using the fact that

$$\Phi_U = \Phi_1 \dot{\bigcup} \Phi_{U^{\geq 2,\mathrm{ext}}}, \tag{7.95}$$

we can estimate the product by

$$p^{\left(\left|\Phi_U \cap w(\Psi)\right| - \left|\Phi_U \cap w\left(\Phi_{\overline{V}}\right)\right| + \frac{1}{2}\left|\Phi_1 \cap w(\Phi_V \backslash \Psi)\right| + \left|\Phi_1 \cap w\left(\Phi_{\overline{V}}\right)\right| - \left|\Phi_U \cap w(\Phi_V)\right|\right)}. \tag{7.96}$$

Now we consider the two cases:

Case 1: When w is not the longest element. In this case, we combine the terms in the exponent strategically in two groups as

$$|\Phi_U \cap w(\Psi)| + \frac{1}{2}|\Phi_1 \cap w(\Phi_V \setminus \Psi)| - |\Phi_U \cap w(\Phi_V)| \tag{7.97}$$

and

$$-\left|\Phi_U \cap w\left(\Phi_{\overline{V}}\right)\right| + \left|\Phi_1 \cap w\left(\Phi_{\overline{V}}\right)\right| \tag{7.98}$$

Now the terms in Equation (7.97) can be estimated as

$$\begin{aligned}(7.97) &\leq -|\Phi_U \cap w(\Phi_V \setminus \Psi)| + \frac{1}{2}|\Phi_U \cap w(\Phi_V \setminus \Psi)| \\ &\leq -\frac{1}{2},\end{aligned} \tag{7.99}$$

where we use the Lemma 7.67. Similarly, the terms in Equation (7.98) can be estimated as

$$\begin{aligned}(7.98) &= -\left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w\left(\Phi_{\overline{V}}\right)\right| \\ &\leq 0.\end{aligned} \tag{7.100}$$

Thus, we are left with an exponent of p which is at most $-\frac{1}{2}$, which is what we wanted to show.

Case 2: When w is the longest element. In this case, since w maps all positive roots onto negative roots, we have

$$\begin{aligned}\Phi_U \cap w(\Psi) &= \emptyset, \\ \Phi_1 \cap w(\Phi_V \setminus \Psi) &= \emptyset,\end{aligned}$$

and

$$\Phi_U \cap w(\Phi_V) = \emptyset$$

The two surviving terms in the exponent in Equation (7.96) combine to give

$$\begin{aligned}\left|\Phi_1 \cap w\left(\Phi_{\overline{V}}\right)\right| - \left|\Phi_U \cap w\left(\Phi_{\overline{V}}\right)\right| &= -\left|\Phi_{U^{\geq 2,\mathrm{ext}}} \cap w\left(\Phi_{\overline{V}}\right)\right| \\ &\leq -1.\end{aligned} \tag{7.101}$$

Thus, in this case too, we are left with a strictly negative exponent of p (in fact at most -1) and hence for the longest w, we have

$$\mathrm{vol}\left(w\,\mathbb{K}_p\,w^{-1} \cap U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\right) \cdot C_w \cdot \sqrt{J_p(X)} \leq p^{-1}. \tag{7.102}$$

Combining the two cases, we have our result. ■

7.4.6 Finishing the Estimation

We will need the following technical lemma.

Lemma 7.103. *Let $f_1, \dots f_r \in \mathbb{Q}_p\left[X_1, \dots X_s\right]^N$ be non-zero vector valued functions such that there exist a fixed constant B with*

$$|\nu_p(f_i)| \leq B, \quad 1 \leq i \leq r,$$

where $\nu_p(f)$ is the highest power of p dividing all of the coefficients of f. Let d be the maximum degree among the polynomials.

Let $L \subseteq \mathbb{Q}_p^s$ be a lattice and A be such that $p^A\mathbb{Z}_p^s \subseteq L \subseteq p^{-A}\mathbb{Z}_p^s$. Then

$$\left|\prod_{i=1}^{r} \ln \|f_i(x)\|_p\right| \mathrm{d}x \ll_{r,s,d,A,B} (\ln(p))^r \operatorname{vol}(L). \tag{7.104}$$

♣

Proof. We first make a change of variables to rewrite L in a more manageable form.

Let $\{\mathfrak{v}_1, \dots, \mathfrak{v}_s\}$ be a basis of $\mathbb{Z}_p^s$. Then we can write the lattice L as

$$L = \sum_{i=1}^{s} \mathbb{Z}_p\left(p^{\alpha_i}\mathfrak{v}_i\right),$$

with $\alpha_i \in \mathbb{Z}$ such that $|\alpha_i| \leq A$ for $1 \leq i \leq s$. Then an element $x \in L$ can be written as

$$x = \sum_{i=1}^{s} \xi_i\left(p^{\alpha_i}\mathfrak{v}_i\right), \qquad \xi_i \in \mathbb{Z}_p. \tag{7.105}$$

Define $\tilde{f}_i$ by $\tilde{f}_i(\xi) := f_i(x)$, where x and ξ satisfy Equation (7.105). Using this change of variables, we have the following modifications

$$\tilde{B} := \max_{1 \leq i \leq r} \left|\nu_p\left(\tilde{f}_i\right)\right| \leq B + A \cdot d,$$

and the measure $\mathrm{d}x$ on L is given as

$$\mathrm{d}x = \operatorname{vol}(L) \cdot \mathrm{d}\xi.$$

The upshot of this change of variables is that

$$\int_L \left|\prod_{i=1}^{r} \ln \|f_i(x)\|_p\right| \mathrm{d}x = \int_{\mathbb{Z}_p^s} \left|\prod_{i=1}^{r} \ln \left\|\tilde{f}_i(\xi)\right\|_p\right| \operatorname{vol}(L)\,\mathrm{d}\xi. \tag{7.106}$$

The right hand side is now estimated as an application of [FL18, Lemma A.9]. Indeed, we have for $1 \leq i \leq r$,

$$\left\|\tilde{f}_i(\xi)\right\|_p \leq p^{\tilde{B}}.$$

Now let $A_m \subseteq \mathbb{Z}_p^s$ be such that there exists an $m \in [-\infty, B) \cap \mathbb{Z}$ such that

$$\min_{1 \leq i \leq r} \left\|\tilde{f}_i(\xi)\right\|_p = p^m \quad \forall \xi \in A_m.$$

Then, for $\xi \in A_m$, we have

$$m \ln(p) \leq \ln \left\|\tilde{f}_i(\xi)\right\|_p \leq \tilde{B} \ln(p),$$

which implies

$$\prod_{i=1}^{r} \left|\ln \left\|\tilde{f}_i(\xi)\right\|_p\right| \leq \left(\max\left\{\tilde{B}, |m|\right\}\right)^r \cdot (\ln(p))^r, \qquad \xi \in A_m.$$

Now since $\mathbb{Z}_p^s = \cup_{m=-\infty}^{\tilde{B}} A_m$, we have that

$$\begin{aligned}\int_{\mathbb{Z}_p^s} \left|\prod_{i=1}^{r} \ln \left\|\tilde{f}_i(\xi)\right\|_p\right| \mathrm{vol}(L)\, \mathrm{d}\xi &= \sum_{m=-\infty}^{\tilde{B}} \int_{A_m} \left|\prod_{i=1}^{r} \ln \left\|\tilde{f}_i(\xi)\right\|_p\right| \mathrm{vol}(L)\, \mathrm{d}\xi \\ &\leq \mathrm{vol}(L) \ln(p)^r \cdot \sum_{m=-\infty}^{\tilde{B}} \left(\max\left\{\tilde{B}, |m|\right\}\right)^r \mathrm{vol}(A_m).\end{aligned} \tag{7.107}$$

Now using [FL18, Lemma A.9], we have for any $m \leq 0$,

$$\mathrm{vol}(A_m) \leq d^s \binom{-m+s-1}{s-1} p^{\frac{m}{d}}. \tag{7.108}$$

Using this in Equation (7.107), we get

$$\begin{aligned}\int_{\mathbb{Z}_p^s} \left|\prod_{i=1}^{r} \ln \left\|\tilde{f}_i(\xi)\right\|_p\right| \mathrm{vol}(L)\, \mathrm{d}\xi &\ll_{r,\tilde{B}} \sum_{m=-\infty}^{0} \left(\max\left\{\tilde{B}, |m|\right\}\right)^r d^s \binom{-m+s-1}{s-1} p^{\frac{m}{d}} \\ &\ll_{r,s,d,A,B} (\ln(p))^r \cdot \mathrm{vol}(L),\end{aligned} \tag{7.109}$$

since the series converges and $\tilde{B} = B + A \cdot d$.

The result now follows from Equation (7.106) and Equation (7.109). ∎

This allows us to state the

Corollary 7.110. *There exists a constant r such that*

$$\int_{U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\cap w\,\mathbb{K}_p\,w^{-1}} C_w \left|\prod_{i=1}^{r} \ln \left\| f_i\left(\ln\left(\pi_p\right)\right)\right\|_p\right| \mathrm{d}\pi_p \ll \frac{\left(\ln\left(p\right)\right)^r}{\sqrt{p}} \tag{7.111}$$

for polynomials $f_1,\ldots,f_r$. ♣

Proof. We estimate the integral by changing the variables, remembering to keep the factor introduced from the change of variables (cf. Equation (7.55)) and arrive at

$$\begin{aligned}\int_{\Pi_p} C_w \left|\prod_{i=1}^{r} \ln \left\| f_i\left(\ln\left(\pi_p\right)\right)\right\|_p\right| \mathrm{d}\pi_p &= \int_{\ln(\Pi_p)} \left|\prod_{i=1}^{r} \ln \left\| f_i\left(X\right)\right\|_p\right| \cdot C_w \sqrt{\left|J_p\left(X\right)\right|}\,\mathrm{d}X \\ &\leq \int_{\ln(L)} \left|\prod_{i=1}^{r} \ln \left\| f_i\left(X\right)\right\|_p\right| \cdot C_w \sqrt{\left|J_p\left(X\right)\right|}\,\mathrm{d}X\end{aligned}$$

where we write L for $U^{\geq 2,\mathrm{ext}}\left(\mathbb{Q}_p\right)\cap w\,\mathbb{K}_p\,w^{-1}$. The result now follows from Equations (7.94) and (7.104). ■

Proposition 7.112. *There exists a constant r such that*

$$\int_{U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\cap w\,\mathbb{K}_p\,w^{-1}} C_w \left|w_M\left(1,\pi_p\right)\right| \mathrm{d}\pi_p \ll \frac{\left(\ln\left(p\right)\right)^r}{\sqrt{p}}. \tag{7.113}$$

♣

Proof. By Proposition 7.54 and [Art88b, Page 239], we know that $w_M\left(1,\pi_p\right)$ is (a finite linear combination over the weights, of) the product of logarithm of the $p-$adic norms of a (finite number) of polynomials (say $f_1,\ldots f_r$). Hence we can use Corollary 7.110 for its estimates. Hence, writing L for the integration domain $U^{\geq 2,\mathrm{ext}}\left(\mathbb{Q}_p\right)\cap w\,\mathbb{K}_p\,w^{-1}$, we obtain

$$\begin{aligned}\int\limits_L C_w \left|w_M\left(1,\pi_p\right)\right| \mathrm{d}\pi_p &= \sum_{\Omega} c_\Omega \int\limits_L \left|\prod_{i=1}^{r} \ln \left\| f_i\left(\pi_p\right)\right\|_p\right| \cdot C_w\,\mathrm{d}\pi_p \\ &\ll \int\limits_L \left|\prod_{i=1}^{r} \ln \left\| f_i\left(\pi_p\right)\right\|_p\right| \cdot C_w\,\mathrm{d}\pi_p.\end{aligned}$$

The result now follows from Equation (7.111). ■

Proposition 7.114. *There exists an $r>0$ such that*

$$\sum_w \int_{\Pi_p} \left|C_w D_w(\pi_p)\, w_M(1,\pi_p)\right| \mathrm{d}\pi_p \ll \frac{(\ln(p))^r}{\sqrt{p}}. \tag{7.115}$$

♣

Proof. Since $\Pi_p \subseteq U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)$, we can write

$$\int_{\Pi_p} C_w \left|D_w(\pi_p)\, w_M(1,\pi_p)\right| \mathrm{d}\pi_p \leq \int_{U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)} C_w \left|D_w(\pi_p)\, w_M(1,\pi_p)\right| \mathrm{d}\pi_p$$

Now plugging in the definition of D_w (Equation (7.80)), the quantity on the right can be estimated by

$$\begin{aligned}&\int_{P(\mathbb{Q}_p)\cap\mathbf{K}_p}\int_{U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\cap l\, w\, \mathbb{K}_p\, w^{-1}\, l^{-1}} C_w\left|w_M(1,\pi_p)\right| \mathrm{d}\pi_p\, \mathrm{d}l\\ &=\int_{P(\mathbb{Q}_p)\cap\mathbf{K}_p}\int_{l\left(U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\cap w\, \mathbb{K}_p\, w^{-1}\right) l^{-1}} C_w\left|w_M(1,\pi_p)\right| \mathrm{d}\pi_p\, \mathrm{d}l.\end{aligned}$$

Now the measure on $\mathbf{K}_p$ being normalised and the adjoint action being measure preserving allows us to estimate the quantity on the right by

$$\int_{U^{\geq 2,\mathrm{ext}}(\mathbb{Q}_p)\cap w\, \mathbb{K}_p\, w^{-1}} C_w\left|w_M(1,\pi_p)\right| \mathrm{d}\pi_p. \tag{7.116}$$

The result now follows from Equation (7.113) and the fact that the sum over w is finite. ■

Proposition 7.117. *There exists an $r>0$ such that*

$$\sum_w \int_{\Pi_p} \left|C_w D_w(\pi_p)\cdot\prod_{i=1}^{r} \ln\left\|f_i(\ln(\pi_p))\right\|_p\right| \mathrm{d}\pi_p \ll \frac{(\ln(p))^r}{\sqrt{p}}. \tag{7.118}$$

♣

Proof. The proof follows exactly as the proof of Proposition 7.114 above except that in the end we appeal to Equation (7.111) instead of Equation (7.113). ■

Now we state our main result of this chapter.

Theorem 7.119. *There exists a $r > 0$ such that for $f = f_\infty \otimes \mathbb{1}_{\mathbb{K}_p}$ and $(1, G) \neq (u, M)$, we have that*

$$|\mathfrak{J}_M (u, f)| \ll_{f_\infty} \frac{(\ln (p))^r}{\sqrt{p}}. \tag{7.120}$$

♣

Proof. We have

$$\begin{aligned}\mathfrak{J}_M (u, f) &= \int_{\Pi_S} \left[\left(\int_{\mathbf{K}_S} f \left(k^{-1} \pi k \right) \mathrm{d}k \right) w_M (1, \pi) \right] \mathrm{d}\pi \\ &= \int_{\Pi_S} \phi (\pi) w_M (1, \pi) \mathrm{d}\pi.\end{aligned}$$

Since by Equation (7.51) w_M comes from a $(G, M)-$family which is a product of two $(G, M)-$families, we can use Arthur's formula ([Art88a, Corollary 7.4]) to write w_M as a finite linear combination of the products of finite and infinite parts. Breaking down into finite and infinite parts, we have

$$|\mathfrak{J}_M (u, f)| \ll_{f_\infty} \sum_{L \in \mathscr{L}(M)} \int_{\Pi_p} \left| \phi_p (\pi_p) w_M^Q (1, \pi_p) \right| \mathrm{d}\pi_p,$$

with Q being a section for L (cf. [Art88a, §7]). Now using the estimate for ϕ_p given in Equation (7.78), we have

$$|\mathfrak{J}_M (u, f)| \ll_{f_\infty} \sum_{w} \int_{\Pi_p} \left| C_w D_w (\pi_p) w_M^Q (1, \pi_p) \right| \mathrm{d}\pi_p.$$

However, the right hand side is exactly the quantity estimated in Proposition 7.114 and hence the result follows from Equation (7.115). Indeed, the main ingredient in the proof of Proposition 7.114 is that $w_M (1, \pi_p)$ can be written as a linear combination of product of logarithm of polynomials (which allowed us to use Corollary 7.110), which is also true for $w_M^Q (1, \pi_p)$. ■

Part IV

Spectral Limit Property

Chapter 8

Properties (TWN) and (BD)

This is a technical chapter, where we introduce the two properties (TWN) and (BD).

8.1 Level of a compact open subgroup

Let G be as usual, a connected reductive group over an algebraic number field F.

Notation 8.1. We fix once and for all, a faithful $F-$rational representation $\rho : G \to \boldsymbol{GL}(V)$ and an $\mathfrak{o}_F$ lattice Λ in the representation space V such that the stabiliser of $\widehat{\Lambda} = \widehat{\mathfrak{o}_F} \otimes \Lambda \subseteq \mathbb{A}_{\text{fin}} \otimes V$ in $G(\mathbb{A}_{\text{fin}})$ is the group $\mathbf{K}_{\text{fin}}$ (which is a fixed open compact subgroup of $G(\mathbb{A}_{\text{fin}})$). ♦

Definition 8.2. For any ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, we define the ***principal congruence subgroup of level*** $\mathfrak{n}$ to be

$$\mathbf{K}(\mathfrak{n}) \ := \ \left\{ g \in G(\mathbb{A}_{\text{fin}}) \,\middle|\, \rho(g)v \equiv v \pmod{\mathfrak{n}\widehat{\Lambda}} \quad \forall v \in \widehat{\Lambda} \right\}.$$

♣

Definition 8.3. Let $\mathfrak{n}_K$ be the largest ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ such that $\mathbf{K}(\mathfrak{n}) \subseteq \mathbb{K}$. Then the ***level*** of $\mathbb{K}$ is defined to be the norm $N(\mathfrak{n}_K)$ of $\mathfrak{n}_K$. That is,

$$\text{lev}(K) \ := \ N(\mathfrak{n}_K).$$

♣

Definition 8.4. A subgroup $\mathbb{H} \subseteq G(\mathbb{A})$ is called ***factorisable*** if it can be written as a product of groups $H_\nu \subseteq G(F_\nu)$ at local places. That is,

$$\mathbb{H} = \prod_\nu H_\nu.$$

♣

Example 8.5. According to Proposition 10.13, maximal compact subgroups of $G(\mathbb{A})$ are factorisable. ♠

Definition 8.6. Let M be a Levi subgroup of G containing M_0 and let $\mathbb{H}$ be a factorisable subgroup of $G(\mathbb{A}_{\text{fin}})$. Then for any compact subgroup $\mathbb{K} \subseteq M(\mathbb{A}_{\text{fin}})$, let $\mathfrak{n}_{\mathbb{K};\mathbb{H}}$ be the largest ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ such that $\mathbf{K}(\mathfrak{n}) \cap M(\mathbb{A}_{\text{fin}}) \cap \mathbb{H} \subseteq \mathbb{K}$. Then we define the ***relative level of*** $\mathbb{K}$ ***with respect to*** $\mathbb{H}$ to be

$$\operatorname{lev}_M(\mathbb{K};\mathbb{H}) \ := \ N(\mathfrak{n}_{\mathbb{K};\mathbb{H}}).$$

We also write

$$\operatorname{lev}_M(\mathbb{K}) \ := \ \operatorname{lev}_M(\mathbb{K}; G(\mathbb{A}_{\text{fin}})).$$

♣

Lemma 8.7. *If* $\mathbb{H}_1 \subseteq \mathbb{H}_2$*, then*

$$\operatorname{lev}_M(\mathbb{K};\mathbb{H}_1) \leq \operatorname{lev}_M(\mathbb{K};\mathbb{H}_2).$$

Moreover, for every $\mathbb{H}$*, we have*

$$\operatorname{lev}_M(\mathbb{K};\mathbb{H}) \leq \operatorname{lev}_M(\mathbb{K}).$$

♣

Proof. This is immediate from the definitions. ■

8.2 Tempered Winding Numbers

We recall the notation from [FLM15].

For $M \in \mathscr{L}, \alpha \in \Sigma_M$ and $\pi \in \widehat{M(\mathbb{A})}_{\text{disc}}$, we let $n_\alpha(\pi, s)$ be the global normalising factor defined by [FLM15, Equation 9]. Let U_α be the unipotent subgroup of G corresponding to α and let M_α be the subgroup generated by M and $U_{\pm\alpha}$ and let $\hat{M}_\alpha$ be the subgroup generated by $U_{\pm\alpha}$.

For any $\mathcal{F} \subset \widehat{\mathbf{K}_{M,\infty}}$, we denote by $\widehat{M(\mathbb{A})}^{\mathcal{F}}_{\text{disc}}$ the set

$$\widehat{M(\mathbb{A})}^{\mathcal{F}}_{\text{disc}} := \left\{ \pi = \pi_\infty \otimes \pi_{\text{fin}} \in \widehat{M(\mathbb{A})}_{\text{disc}} \;\middle|\; \pi_\infty \text{ contains a } \mathbf{K}_{M,\infty} - \text{type in } \mathcal{F} \right\}.$$

Let $H \subseteq G$ be a reductive algebraic subgroup normalised by M. For an irreducible representation π of $M(F_\infty)$, we will write

$$\Lambda_M(\pi; H) := 1 + \left(\lambda_\pi^H\right)^2 + \|\tau\|^2,$$

where λ_π^H is the eigenvalue of the Casimir operator of $M(F_\infty) \cap H(F_\infty)$, τ is a lowest $\mathbf{K}_\infty \cap M(F_\infty) \cap H(F_\infty)$ −type of π and $\|\cdot\|$ is the Vogan's norm (cf. [Vog81, Definition 5.4.1].

Definition 8.8. We say that a group G satisfies the property ***(TWN)*** (tempered winding numbers) if, for any $M \in \mathscr{L}, M \neq G$, and any finite subset $\mathcal{F} \subseteq \widehat{\mathbf{K}_{M,\infty}}$, there exists $k > 1$ such that for any $\alpha \in \Sigma_M$ and any $\epsilon > 0$ we have

$$\int_{\iota\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \ll_{\mathcal{F},\epsilon} \Lambda_M\left(\pi_\infty; \hat{M}_\alpha\right)^k \operatorname{lev}_M\left(\pi; \hat{M}_\alpha^+\right)^\epsilon \tag{8.9}$$

for any $\pi \in \widehat{M(\mathbb{A})}^{\mathcal{F}}_{\text{disc}}$. ♣

Now which groups do actually satisfy this property? The first result in this direction is

Proposition 8.10. *([FLM15, Proposition 5.1]). The groups $\boldsymbol{GL}_n$ and $\boldsymbol{SL}_n$ satisfy the property (TWN).* ♣

The next result, obtained by the same authors is

Proposition 8.11. *([FL17a, Theorem 3.11]). The following groups satisfy the property (TWN):*

1. *$\boldsymbol{GL}_n$ and its inner forms.*
2. *Quasi-split classical groups.*
3. *The exceptional group G_2.*

♣

8.3 Bounded Degree

Definition 8.12. We say that a group G satisfies the property ***(BD)*** (bounded degree) if there exists a constant c such that for any

- Levi subgroup $M \in \mathscr{L}, M \neq G$,
- $\alpha \in \Sigma_M$,
- finite place ν of F,
- open subgroup $K_\nu \subseteq \mathbf{K}_{M_\alpha,\nu}$,
- smooth irreducible representation π_ν of $M(F_\nu)$,

the degrees of the numerators (as a rational function of s) of the linear operators $R^{M_\alpha}_{\overline{P_\alpha}|P_\alpha}(\pi_\nu, s)^{K_\nu}$ are bounded by

$$\begin{cases} c \log_{q_\nu} \operatorname{lev}_{M_\alpha}\left(K_\nu; \hat{M}_\alpha^+\right), & \text{if } \mathbf{K}_\nu \text{ is hyperspecial,} \\ c\left(1 + \log_{q_\nu} \operatorname{lev}_{M_\alpha}\left(K_\nu; \hat{M}_\alpha^+\right)\right), & \text{otherwise.} \end{cases}$$

♣

We again ask for which groups this property is satisfied. The first result in this direction is

Proposition 8.13. *([FLM12], [FLM15, Theorem 5.15]). The groups $\boldsymbol{GL}_n$ and $\boldsymbol{SL}_n$ satisfy the property (BD).* ♣

This result was further extended for split groups of rank 2 and for inner forms of $\boldsymbol{GL}_n$ and $\boldsymbol{SL}_n$ in [FL17b, Corollary 1].

Remark 8.14. By [FL17b, Theorem 4], we see that the property (BD) is always satisfied if one excludes a finite set of places depending on G. This is sufficient for our purposes since we can choose our prime p to be outside this excluded set. ❇

Chapter 9

Preliminaries on Reductive Groups over Local Fields

In this chapter, we discuss some preliminaries related to reductive groups defined over local fields. In particular, we study Tits systems, the concept of a parahoric subgroup and a formula for calculating their volumes.

9.1 Volumes of Parahorics

In this section, we present a method of computing the volumes of parahoric subgroups of a reductive group G defined over a local field k. We will deduce that the volumes can be computed in terms of the constants '$d(v)$' associated to the Dynkin diagram of the group G and in terms of the cardinality q of the residue field of k.

We begin with the general theory of Tits system in abstract groups.

9.1.1 Input from the Theory of Tits System

Let G be an abstract group with an affine Tits system (G, B, N, R). Let (W, R) be the corresponding affine Coxeter system. Then we have a bijection (cf. Theorem D.9(b)) between the power set of R and the set of standard parahoric subgroups. Thus, any parahoric subgroup P is conjugate, for some $S \subseteq R$, to a subgroup of the form

$$P_S = \bigsqcup_{w \in W_S}^{\cdot} BwB;$$

and hence has the same volume as P_S.

Thus, to compute the volume of an arbitrary parahoric subgroup, it is sufficient to compute the volume of standard parahoric subgroups (which are of the form P_S as above).

9.1.2 Application to Algebraic Groups

Now let k be a local field and let G be the set of $k-$points of a reductive algebraic group defined over k. Recall that such groups have a Bruhat-Tits building $\mathcal{B}$ attached to them which gives rise to a Tits system (G, I, N, R) where I is the Iwahori subgroup (stabiliser of the fundamental chamber) and R corresponds to the set of reflections along the walls of the fundamental chamber.

The Iwahori subgroup I is compact in the topology induced on the group G by the field k. Hence, we can normalise the measure on the group G so that $\mathrm{vol}\,(I) = 1$. This is called the ***Iwahori normalisation*** and the corresponding volume (or measure) is denote by $\mathrm{vol}_{\mathrm{Iw}}$. In other words, the Iwahori volume of a standard parahoric subgroup P is just the index of I in P.

By [Tit79, 3.3.1], for $w \in W$, we can calculate the index of I in IwI which is finite. Recall that associated to G is an affine Dynkin diagram with $l + 1 = \mathrm{card}\,(R)$ vertices. The groups are then classified according to classification of the Dynkin diagrams. Also, in the diagrams every vertex v comes equipped with an integer $d(v)$ in the tables of [Tit79, 4.2, 4.3]. Then we have the following:

Proposition 9.1. *Let $S \subseteq R$ and let $w \in W_S$, with reduced decomposition $w = r_1 \cdots r_k$, with $r_i \in S$. Let v_i be the vertex of the affine Dynkin diagram corresponding to r_i. Then*

$$\mathrm{vol}_{\mathrm{Iw}}(P_S) = \mathrm{card}(P_S/I) = \sum_{w \in W_S} q^{\sum\limits_{i=1}^{i=k} d(v_i)}.$$

♣

Proof. We have

$$\mathrm{vol}_{\mathrm{Iw}}\,(P_S) = \sum_{w \in W_S} \mathrm{card}\,(IwI/I)\,.$$

Put

$$q_w \;:=\; \mathrm{card}\,(IwI/I).$$

Now by [Tit79, 3.3.1], we have that

$$q_w = q^{\sum\limits_{i=1}^{i=k} d(v_i)},$$

for $w = r_1 \cdots r_k$.

Putting these two together implies the result. ■

9.1.3 The case of Split Groups

Proposition 9.2. *In the case when the group is split, the formula simplifies to*

$$\mathrm{vol}_{\mathrm{Iw}}(P_S) = \sum_{w \in W_S} q^{l(w)} ,$$

where $l(w)$ denotes the length of the Weyl group element w. ♣

Proof. This is because when the group is split, $d(v_i) = 1$ for every vertex v_i and hence $q_w = q^{l(w)}$. That $d(v_i)$ is indeed 1 may be inferred from the tables in [Tit79, §4.3]. ■

Now we state the main result for the volumes of maximal compact subgroups in split groups.

Theorem 9.3. *Let G be a split reductive group defined over local field k with residue field with q elements. Let K be a maximal compact subgroup and stabilizer of a vertex x of a chamber in the building. Let Ψ be the root system corresponding to the Dynkin diagram obtained by removing the vertex x from the affine Dynkin diagram of G. Then we can bound $\mathrm{vol}_{\mathrm{Iw}}(K)$ from above and below as follows:*

$$\mathrm{vol}_{\mathrm{Iw}}(K) \leq |W_\Psi| \cdot q^{|\Psi^+|}, \qquad \mathrm{vol}_{\mathrm{Iw}}(K) \geq q^{|\Psi^+|},$$

where W_Ψ is the (finite) Weyl group of the root system Ψ and Ψ^+ denotes the set of positive roots in the root system Ψ. ♣

Proof. A maximal compact subgroup is a maximal parahoric subgroup and hence we have that a maximal compact K is stabilizer of a vertex (say x) of the chamber. Now, removing the vertex x from the affine Dynkin diagram Φ of G gives us another Dynkin diagram, corresponding to the root system Ψ (say). Let W_Ψ be the Weyl group associated to the root system Ψ. Thus by Proposition 9.2, we have

$$\mathrm{vol}_{\mathrm{Iw}}(K) = \sum_{w \in \langle R \setminus \{x\} \rangle} q^{l(w)} = \sum_{w \in W_\Psi} q^{l(w)}.$$

This allows us to bound $\mathrm{vol}_{\mathrm{Iw}}(K)$ from above and below:

$$\mathrm{vol}_{\mathrm{Iw}}(K) \leq |W_\Psi| \cdot q^{l(w_0)}, \qquad \mathrm{vol}_{\mathrm{Iw}}(K) \geq q^{l(w_0)},$$

where w_0 is the longest element of the Weyl group W_Ψ.

Now since the length of longest Weyl element is equal to the number of positive roots in the root system (Proposition C.29), we have that

$$\mathrm{vol}_{\mathrm{Iw}}(K) \leq |W_\Psi| \cdot q^{\left|\Psi^+\right|}, \qquad \mathrm{vol}_{\mathrm{Iw}}(K) \geq q^{\left|\Psi^+\right|}.$$

∎

This is all we will need from the theory of reductive groups and their buildings. Now after these preliminaries, we can at long last go on to analyse the spectral limit property.

Chapter 10

The Spectral Limit Property

> "The key to success is to focus our conscious mind on things we desire not things we fear."
>
> Brian Tracy

In this chapter, we study the spectral limit property for the special non-degenerate collection $\mathfrak{K}^{\mathrm{spc,max}}$ of all *maximal* compact open subgroups of G introduced in Notation 10.22. Recall that this means

$$\forall h \in \mathcal{H}\left(G(F_S)^1\right), \qquad J\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) - \operatorname{tr} R_{\mathrm{disc}}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) \to 0, \qquad \mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}.$$

10.1 Polynomially bounded collection of Measures

The technical concept of polynomial boundedness was introduced in [FLM15] following the work of Delorme [Del86]. We set up some notation:

Notation 10.1. Let θ be the Cartan involution of $G(F_\infty)$ defining $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g}_\infty = \operatorname{Lie} G(F_\infty) = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{k} = \operatorname{Lie} \mathbf{K}_\infty$. We fix an invariant bilinear form B on $\mathfrak{g}_\infty$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p} \cap \operatorname{Lie}(G(F_\infty)^1)$ and let $\|\cdot\|$ be the norm on $\mathfrak{a}$ induced by B.

For any $r > 0$, let $\mathcal{H}\left(G(F_\infty)^1\right)_r$ be the subspace of $\mathcal{H}\left(G(F_\infty)^1\right)$ consisting of all functions supported in the compact subset $\mathbf{K}_\infty \exp\left(\{\, x \in \mathfrak{a} \mid \|x\| \le r \,\}\right) \mathbf{K}_\infty$ of $G(F_\infty)^1$. ♦

Definition 10.2. Let K be a compact group and let $\mathcal{F} \subseteq \widehat{K}$. Let K act on a space X. Then we say that the K ***type of*** X ***is contained in*** $\mathcal{F}$ if X decomposes under the action of K as sum of representations from $\mathcal{F}$. That is,

$$X = \bigoplus_i (\pi_i, V_i),$$

where for every i, (π_i, V_i) is an element of $\mathcal{F}$. ♣

Notation 10.3. For any finite set $\mathcal{F} \subseteq \widehat{\mathbf{K}_\infty}$, we let $\mathcal{H}\left(G(F_\infty)^1\right)_{\mathcal{F}}$ be the subspace of $\mathcal{H}\left(G(F_\infty)^1\right)$ consisting of functions whose $\mathbf{K}_\infty \times \mathbf{K}_\infty-$types are contained in $\mathcal{F} \times \mathcal{F}$. Let also

$$\mathcal{H}\left(G(F_\infty)^1\right)_{r,\mathcal{F}} = \mathcal{H}\left(G(F_\infty)^1\right)_r \cap \mathcal{H}\left(G(F_\infty)^1\right)_{\mathcal{F}}.$$

For any $f \in \mathcal{H}\left(G\left(F_S\right)^1\right)$, let $\hat{f}$ be the function on $\widehat{G\left(F_S\right)}^1$ given by

$$\hat{f}\left(\pi\right) = \operatorname{tr} \pi(f).$$

For any finite $S \supseteq S_\infty$ and any open subgroup $K_S \subseteq \mathbf{K}_{S_{\text{fin}}}$, put

$$\begin{aligned} \mathcal{H}\left(G(F_S)^1\right)_{\mathcal{F},K_S} := & \left\{ f \in \mathcal{H}\left(G(F_S)^1\right) \;\middle|\; f \text{ is bi} - K_S - \text{invariant} \right\} \\ & \cap \left\{ f \in \mathcal{H}\left(G(F_S)^1\right) \;\middle|\; f\left(\cdot g\right) \in \mathcal{H}\left(G(F_\infty)^1\right)_{\mathcal{F}} \quad \forall g \in G(F_S)^1 \right\}. \end{aligned}$$

♦

Definition 10.4. A collection $\mathfrak{M}$ of Borel measures on $\widehat{G(F_\infty)}^1$ is called ***polynomially bounded*** if for any finite set $\mathcal{F} \subseteq \widehat{\mathbf{K}_\infty}$, the supremum

$$\sup_{\nu \in \mathfrak{M}} \left|\nu\left(\hat{f}\right)\right|$$

is a continuous seminorm on $\mathcal{H}\left(G(F_\infty)^1\right)_{r,\mathcal{F}}$. ♣

Remark 10.5. It is shown in [FLM15, Proposition 6.1] that this property is independent of $r > 0$. ❇

10.2 Integrals over Richardson Orbits

Let P be a parabolic subgroup of G and let $P = MU$ be its Levi decomposition. Let dp be the left Haar measure on P and let δ_P be the modulus function of $P(\mathbb{A}_{\text{fin}})$. The following three lemmas follow from standard theory of integration on Homogeneous spaces ([Wil91, Chapter 1]).

Lemma 10.6. *For any continuous function f on $G(\mathbb{A}_{\mathrm{fin}})$ such that $f(pg) = \delta_P(p)f(g)$ for all $p \in P(\mathbb{A}_{\mathrm{fin}}), g \in G(\mathbb{A}_{\mathrm{fin}})$, the integral $\int_{P(\mathbb{A}_{\mathrm{fin}})\backslash G(\mathbb{A}_{\mathrm{fin}})} f(g)\,\mathrm{d}g$ is well defined and is invariant under right translations by elements of $G(\mathbb{A}_{\mathrm{fin}})$.* ♣

Lemma 10.7. *The group $\mathbf{K}$ in Chapter 2 can be chosen so that $\mathbf{K}_{\mathrm{fin}}$ satisfies*

$$G(\mathbb{A}_{\mathrm{fin}}) = P(\mathbb{A}_{\mathrm{fin}})\mathbf{K}_{\mathrm{fin}}.$$

Moreover, we can arrange measures such that

$$\int\limits_{P(\mathbb{A}_{\mathrm{fin}})\backslash G(\mathbb{A}_{\mathrm{fin}})} f(g)\,\mathrm{d}g = \int\limits_{\mathbf{K}_{\mathrm{fin}}} f(k)\,\mathrm{d}k,$$

and

$$\int\limits_{G(\mathbb{A}_{\mathrm{fin}})} f(g)\,\mathrm{d}g = \int\limits_{P(\mathbb{A}_{\mathrm{fin}})\backslash G(\mathbb{A}_{\mathrm{fin}})} \int\limits_{P(\mathbb{A}_{\mathrm{fin}})} f(pg)\,\mathrm{d}p\,\mathrm{d}g$$

for all $f \in C_c(G(\mathbb{A}_{\mathrm{fin}}))$. ♣

Proof. The first part follows from [BT72, Chapter 4] and the second from standard theory of integration on Homogeneous spaces ([Wil91, Chapter 1]). ■

For any $f \in C_c(G(\mathbb{A}_{\mathrm{fin}}))$, define

$$\mathcal{OI}_P(f) := \int\limits_{P(\mathbb{A}_{\mathrm{fin}})\backslash G(\mathbb{A}_{\mathrm{fin}})} \int\limits_{U(\mathbb{A}_{\mathrm{fin}})} f(g^{-1}ug)\,\mathrm{d}u\,\mathrm{d}g = \int\limits_{\mathbf{K}_{\mathrm{fin}}} \int_{U(\mathbb{A}_{\mathrm{fin}})} f(k^{-1}uk)\,\mathrm{d}u\,\mathrm{d}k.$$

Finally, for any compact open subgroup $\mathbb{K} \subseteq G(\mathbb{A}_{\mathrm{fin}})$, set

$$\mathcal{OI}_{P,\mathbb{K}} := \mathcal{OI}_P(\mathbb{1}_{\mathbb{K}}).$$

Denote by proj_M the canonical projection $P(\mathbb{A}_{\mathrm{fin}}) \to M(\mathbb{A}_{\mathrm{fin}})$.

10.2.1 Bounding the Quantity $\mathcal{OI}_{P,\mathbb{K}}$

We start with the following formula for $\mathcal{OI}_{P,\mathbb{K}}$:

Lemma 10.8. *([FL17c, Lemma 4.3]). For any compact $\mathbb{K} \subseteq G(\mathbb{A}_{\mathrm{fin}})$, we have*

$$\mathcal{OI}_{P,\mathbb{K}} = \mathrm{vol}(\mathbb{K}) \sum_{\gamma \in P(\mathbb{A}_{\mathrm{fin}})\backslash G(\mathbb{A}_{\mathrm{fin}})/\mathbb{K}} \left(\mathrm{vol}_M\left(\mathrm{proj}_M\left(P(\mathbb{A}_{\mathrm{fin}}) \cap \gamma K \gamma^{-1}\right)\right)\right)^{-1}.$$

♣

We begin by bounding the quantity $\mathcal{OI}_{P,\mathbb{K}}$ for maximal $\mathbb{K}$ by using our volume computations in §9.1.

Notation 10.9. For the rest of the subsection we will use the notation $\mathbb{K}_M$ and $\mathbb{K}_{M,\gamma}$ to denote

$$\begin{aligned}\mathbb{K}_M &:= \operatorname{proj}_M (P(\mathbb{A}_{\text{fin}}) \cap \mathbb{K}),\\ \mathbb{K}_{M,\gamma} &:= \operatorname{proj}_M \left(P(\mathbb{A}_{\text{fin}}) \cap \gamma\mathbb{K}\gamma^{-1}\right).\end{aligned}$$

♦

As a first step in estimating the quantity $\mathcal{OI}_{P,\mathbb{K}}$, we have:

Proposition 10.10. *The double coset space $P(\mathbb{A}_{\text{fin}})\backslash G(\mathbb{A}_{\text{fin}})/\mathbb{K}$ is finite.* ♣

Proof. This follows since $P(\mathbb{A}_{\text{fin}})\backslash G(\mathbb{A}_{\text{fin}})$ is compact and $\mathbb{K}$ is open. ∎

Thus, now we just need to estimate the following quantity:

$$\frac{\operatorname{vol}(\mathbb{K})}{\operatorname{vol}_M(\mathbb{K}_{M,\gamma})}$$

for every $\gamma \in P(\mathbb{A}_{\text{fin}})\backslash G(\mathbb{A}_{\text{fin}})/\mathbb{K}$.

We shall illustrate with the case $\gamma = 1$. Then $\mathbb{K}_{M,\gamma} = \mathbb{K}_M$. First we introduce some new concepts and reduce the problem to the local case.

Definition 10.11. Let G be an abstract group with an affine Tits system (G, B, N, S) and let $\mathcal{B}$ be the affine building associated to it. Then a subgroup $K \subseteq G$ is called ***hyperspecial*** if it is the stabiliser of a hyperspecial point in the building. ♣

Remark 10.12. In the tables in [Tit79, §§4.3], the hyperspecial vertices in the Dynkin diagram (the vertices corresponding to hyperspecial vertices of the fundamental chamber [Tit79, §§1.9]) are marked as 'hs'. ❄

Proposition 10.13. *Any maximal compact open subgroup $\mathbb{K} \subseteq G(\mathbb{A}_{\text{fin}})$ is a restricted direct product of the maximal compact subgroups at local places,*

$$\mathbb{K} = \prod_{\nu} K_\nu,$$

with $K_\nu \subseteq G(F_\nu)$ hyperspecial at almost all places. ♣

Proof. It is clear that $\mathbb{K}$ has to be a product of maximal compact subgroups at local places. Almost all of them have to be hyperspecial since $\mathbb{K}$ is assumed to be open. ∎

Definition 10.14. Let $\mathbb{K} \subseteq G(\mathbb{A})$ be a maximal compact subgroup. Then $\mathbb{K}$ is called ***hyperspecial*** if every one of its local components K_ν is hyperspecial. A hyperspecial compact subgroup will be denoted by $\mathbb{K}_{\mathrm{hs}}$. ♣

Lemma 10.15. *We have that*

$$\frac{\mathrm{vol}(\mathbb{K})}{\mathrm{vol}_M(\mathbb{K}_M)} = \frac{\mathrm{vol}_{\mathrm{Iw}}(\mathbb{K})/\,\mathrm{vol}_{\mathrm{Iw}}(\mathbb{K}_M)}{\mathrm{vol}_{\mathrm{Iw}}(\mathbb{K}_{\mathrm{hs}})/\,\mathrm{vol}_{\mathrm{Iw}}(\mathbb{K}_{\mathrm{hs},M})},$$

where $\mathrm{vol}_{\mathrm{Iw}}$ *denotes the Iwahori normalization (where the Iwahori subgroup is given the volume* 1*) and* $\mathbb{K}_{\mathrm{hs}}$ *denotes the hyperspecial maximal compact subgroup.* ♣

Proof. This follows since in the usual normalisation the hyperspecial compact open subgroups have volume 1. ∎

Now we have a formula to compute Iwahori volumes in the local case (cf. §9.1) and it is enough to compute volumes in the local case as guaranteed by Proposition 10.13.

Notation 10.16. In view of Proposition 10.13, we introduce, for a maximal open compact subgroup $\mathbb{K}$, the notation:

$$\tilde{S}\,(\mathbb{K}) \;:=\; \{\,\nu \mid K_\nu \text{ is not hyperspecial}\,\}\,.$$

♦

Local Considerations

Here we go on to compute local volumes. We prove

Theorem 10.17. *Let G be a simply connected split reductive group defined over a local field k, the cardinality of whose residue field is q. Let $P = MU$ is a parabolic defined over k, K a maximal compact subgroup of G which is not hyperspecial, K_{hs} a hyperspecial maximal compact subgroup. Then*

$$\frac{\mathrm{vol}_{\mathrm{Iw}}(K)/\,\mathrm{vol}_{\mathrm{Iw}}(K_M)}{\mathrm{vol}_{\mathrm{Iw}}(K_{\mathrm{hs}})/\,\mathrm{vol}_{\mathrm{Iw}}(K_{\mathrm{hs},M})} \le \frac{1}{q}.$$

♣

We will need the following result from the theory of root systems:

Lemma 10.18. *Let Φ be an irreducible root system and let Δ be its basis. Let Ψ be a proper symmetric closed subset of Φ. Let Φ_M be the root system generated by any proper subset of Δ. Let Φ_M^+ be the set of positive roots in Φ_M. Clearly, it is a proper subset of all the positive roots Φ^+. Let $\Phi_U^+ = \Phi^+ \backslash \Phi_M^+$. Then*

$$\left|\Psi \cap \Phi_U^+\right| < \left|\Phi_U^+\right|.$$

♣

Proof. The proof is deferred to §10.3. ■

Proof of Theorem 10.17. Let $\tilde{\Phi}$ be the affine root system of G. Recall that this is obtained from the spherical root system Φ by adjoining the longest root. A quick glance at the tables of [Tit79, 4.3] shows us that the hyperspecial vertices in affine Dynkin diagrams are always the longest roots. Hence, the root system for K_{hs} is Φ. Let Φ_M denote the root system for M and Ψ be the root system of K. Since K is not hyperspecial, we see that $\Psi \subsetneq \Phi$. Then, using Theorem 9.3 we estimate

$$\frac{\operatorname{vol}_{\text{Iw}}(K)/\operatorname{vol}_{\text{Iw}}(K_M)}{\operatorname{vol}_{\text{Iw}}(K_{\text{hs}})/\operatorname{vol}_{\text{Iw}}(K_{\text{hs},M})} \leq \frac{|W_\Psi|}{|W_{\text{sph}}|} \frac{q^{\left|\Psi^+\right| - \left|\Psi^+ \cap \Phi_M\right|}}{q^{\left|\Phi^+\right| - \left|\Phi_M^+\right|}}.$$

Now, using Lemma 10.18, we see that

$$\left(\left|\Psi^+\right| - \left|\Psi^+ \cap \Phi_M\right|\right) = \left|\Psi \cap \Phi_U^+\right| < \left|\Phi_U^+\right| = \left(\left|\Phi^+\right| - \left|\Phi^+ \cap \Phi_M\right|\right)$$

and hence that

$$\frac{\operatorname{vol}_{\text{Iw}}(K)/\operatorname{vol}_{\text{Iw}}(K_M)}{\operatorname{vol}_{\text{Iw}}(K_{\text{hs}})/\operatorname{vol}_{\text{Iw}}(K_{\text{hs},M})} \leq \frac{|W_\Psi|}{|W_{\text{sph}}|} \frac{1}{q^n},$$

where $n := \left(\left|\Phi^+\right| - \left|\Phi_M^+\right|\right) - \left(\left|\Psi^+\right| - \left|\Psi^+ \cap \Phi_M\right|\right) \geq 1$. Since $|W_\Psi| \leq |W_{\text{sph}}|$, we have our result. ■

Global Considerations

Now we go back to our usual setting where everything is done over the adeles. To be precise, let G be a reductive group defined over a number field F. Let $\mathcal{P}(F)$ be the set of places of F and for every $\nu \in \mathcal{P}(F)$, let F_ν be the completion of F at ν and let q_ν be the cardinality of the residue field of the local field F_ν.

Theorem 10.19. *Let $\mathbb{K} \subseteq G(\mathbb{A}_{\text{fin}})$ be an open maximal compact subgroup. Then*

$$\mathcal{OI}_{P,\mathbb{K}} \leq \prod_{\nu \in \tilde{S}(\mathbb{K})} \frac{|W_{\nu,\text{sph}}|}{q_\nu}.$$

The product is a finite product since $\tilde{S}(\mathbb{K})$ is a finite set. ♣

Proof. By Proposition 10.13, for $\mathbb{K} \subseteq G(\mathbb{A}_{\text{fin}})$, we have

$$\mathbb{K} = \prod_{\nu \in \tilde{S}(\mathbb{K})} K_\nu$$

with $K_\nu \subseteq G(F_\nu)$ hyperspecial at almost all places.

We similarly, break down $\mathcal{OI}_{P,\mathbb{K}}$ into local components and get

$$\mathcal{OI}_{P,\mathbb{K}} = \prod_{\nu \in \tilde{S}(\mathbb{K})} \mathcal{OI}_{P,K_\nu},$$

with

$$\mathcal{OI}_{P,K_\nu} = \sum_{\gamma \in P(F_\nu) \backslash G(F_\nu)/K_\nu} \left(\text{vol}_{M_\nu}\left(\text{proj}_{M_\nu}\left(P(F_\nu) \cap \gamma\, K_\nu\, \gamma^{-1}\right)\right)\right)^{-1}.$$

Thus, using Theorem 10.17, every summand on the right hand side at most $\frac{1}{q_\nu}$. Hence,

$$\begin{aligned}\mathcal{OI}_{P,\mathbb{K}} &\leq \prod_{\nu \in \tilde{S}(\mathbb{K})} \sum_{\gamma \in P(F_\nu) \backslash G(F_\nu)/K_\nu} \left(\frac{1}{q_\nu}\right) \\ &\leq \prod_{\nu \in \tilde{S}(\mathbb{K})} \left(\frac{|W_{\text{sph},\nu}|}{q_\nu}\right).\end{aligned}$$

since $|P(F_\nu) \backslash G(F_\nu)/K_\nu| \leq |W_{\text{sph},\nu}|$. ■

Corollary 10.20. *Let $\mathbb{K} \subset G(\mathbb{A}_{\text{fin}})$ be an maximal compact open subgroup. Then*

$$\mathcal{OI}_{P,\mathbb{K}} \leq \prod_{\nu \in \tilde{S}(\mathbb{K})} \frac{C}{q_\nu},$$

where C is a constant depending only on G. ♣

Proof. This follows from Theorem 10.19 and the fact that $|W_{\mathrm{sph},\nu}|$ is bounded independently of ν in terms of G. Indeed, the maximal value of $|W_{\mathrm{sph},\nu}|$ depends upon the type of $\mathbb{K}$ (for which there are only finitely many possibilities once the rank of G is given). ■

Corollary 10.21. *For every $\delta > 0$, we have*

$$\mathcal{OI}_{P,\mathbb{K}} \ll_\delta \prod_{\nu \in \tilde{S}(\mathbb{K})} \left(\frac{1}{q_\nu}\right)^{1-\delta}.$$

♣

Proof. Let $\delta > 0$ be given and put $s := \left|\tilde{S}(\mathbb{K})\right|$. Then we have by Corollary 10.20,

$$\mathcal{OI}_{P,\mathbb{K}} \leq C^s \prod_{\nu \in \tilde{S}(\mathbb{K})} \left(\frac{1}{q_\nu}\right).$$

Thus, we just need to prove that there exists a constant D (depending only on δ) such that

$$C^s \leq D \cdot \left(\prod_{\nu \in \tilde{S}(\mathbb{K})} q_\nu\right)^\delta.$$

Taking logarithms, this is equivalent to

$$s \ln(C) \leq \delta \cdot \sum_{\nu \in \tilde{S}(\mathbb{K})} \ln(q_\nu) + \ln(D).$$

We want this to hold no matter what $\tilde{S}(\mathbb{K})$ is. Note however, that $\tilde{S}(\mathbb{K}) \subseteq \mathcal{P}(F)$, the latter denoting the set of all places of F.

Now let $A := \left\{ \nu \in \mathcal{P}(F) \,\middle|\, \ln(q_\nu) < \frac{\ln C}{\delta} \right\}$ and choose D such that $\ln(D) \geq |A| \ln(C)$ and we are done. ■

10.3 Proof of Lemma 10.18

In this section, we prove the Lemma 10.18. We state it here again and then prove it.

Lemma. *Let Φ be an irreducible root system and let Δ be its basis. Let Ψ be a proper symmetric closed subset of Φ. Let Φ_M be the root system generated* *by any proper subset of* Δ*. Let* Φ_M^+ *be the set of positive roots in* Φ_M*. Clearly, it is a proper subset of all the positive roots* Φ^+*. Let* $\Phi_U^+ = \Phi^+ \backslash \Phi_M^+$*. Then*

$$\left|\Psi \cap \Phi_U^+\right| < \left|\Phi_U^+\right|.$$

♣

Proof. We prove the Lemma by contradiction. We assume that the assertion is false and then show that $\Psi = \Phi$.

Thus, assume that there exists a M such that $\Phi_U^+ \subsetneq \Psi$. Using this, we first prove that Φ_M^+ is contained in Ψ, which immediately implies that $\Phi^+ = \Psi^+$ and since Ψ is symmetric, this in turn implies that $\Psi = \Phi$.

Now, how do we prove that $\Phi_M^+ \subseteq \Psi$? For this, it is sufficient to show that every root in Φ_M^+ is a difference of two roots in Φ_U^+ (since Ψ is closed).

Thus, let $\alpha \in \Phi_M^+$. Now, since the set Φ_U^+ generates the underlying vector space V of the root system, there exists a $\beta \in \Phi_U^+$ which is not orthogonal to α. Then either $(\alpha + \beta)$ or $(\beta - \alpha)$ belongs to Φ_U^+ and since $\alpha = (\alpha + \beta) - \beta$ and also $\alpha = \beta - (\beta - \alpha)$, we have the assertion that α is a difference of two roots in Φ_U^+.

Thus, the above chain of arguments, gives us the result. ■

10.4 Bounds on Spectral Terms

10.4.1 Spectral Limit Property

We will prove the spectral limit property for the collection $\mathfrak{K}^{\mathrm{spc,max}}$ defined as

Notation 10.22. Let $\mathfrak{K}^{\mathrm{spc,max}}$ denote the non-degenerate collection of *maximal* compact open subgroups $\mathbb{K} = \prod_p \mathbb{K}_p$ of $G(\mathbb{A}_{\mathrm{fin}})$ such that $\mathbb{K}_p = \mathbf{K}_p$ for all but finitely many primes p, the set of which is denoted by $\tilde{S}(\mathbb{K})$. Moreover, if x_p denotes the point in the fundamental alcove in the apartment in the Bruhat Tits building such that $\mathbf{K}_p$ is stabiliser of x_p, then $\mathbb{K}_p$ is stabiliser of a *non-hyperspecial* point y_p in the *same* fundamental alcove for all $p \in \tilde{S}(\mathbb{K})$. ♦

Recall that this means

$$\forall h \in \mathcal{H}\left(G(F_S)^1\right), \qquad \mathfrak{J}_G\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) - \operatorname{tr} R_{\mathrm{disc}}(h \otimes \mathbb{1}_{\mathbb{K}}) \to 0, \qquad \mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}. \tag{10.23}$$

We also recall that the distribution $\mathfrak{J}_G$ depends on the choice of a maximal compact subgroup of $G(\mathbb{A})$ containing the fixed minimal Levi subgroup M_0.

We already fixed such a maximal compact subgroup $\mathbf{K} = \prod_{\nu} \mathbf{K}_\nu = \mathbf{K}_\infty \mathbf{K}_{\text{fin}}$ in Chapter 2.

Now to analyse (10.23), we expand the distribution $\mathfrak{J}_G$ spectrally and estimate the resulting quantity.

10.4.2 Spectral Side of Trace Formula

Recall that the trace formula is the equality $\mathfrak{J}_G = \mathfrak{J}_{\text{spec},G} = \mathfrak{J}_{\text{geo},G}$. Furthermore, recall that the spectral side $\mathfrak{J}_{\text{spec},G}$ can be expanded as

$$\mathfrak{J}_{\text{spec},G}(h) = \sum_{[M]} \mathfrak{J}_{\text{spec},M}(h), \qquad h \in C_c^\infty\left(G(\mathbb{A})^1\right),$$

with summation ranging over the conjugacy classes of Levi subgroups of G, represented by $M \supseteq M_0$. The term corresponding to $M = G$ is simply $\mathfrak{J}_{\text{spec},G}(h) = \operatorname{tr} R_{\text{disc}}(h)$. The other terms are computed in Arthur's work. We will not need the explicit descriptions for our purpose.

Thus, to prove that $\mathfrak{J}_{\text{spec},G}(h \otimes \mathbb{1}_{\mathbb{K}}) \to \operatorname{tr} R_{\text{disc}}(h \otimes \mathbb{1}_{\mathbb{K}})$, it is sufficient to prove that for every proper Levi M of G and every $h \in \mathcal{H}\left(G(F_S)^1\right)$,

$$\mathfrak{J}_{\text{spec},M}(h \otimes \mathbb{1}_{\mathbb{K}}) \to 0, \qquad \mathbb{K} \in \mathfrak{K}^{\text{spc,max}}.$$

We begin with the following estimate:

Proposition 10.24. *([FL17c, Proposition 4.5]) Suppose that G satisfies (TWN) and (BD). Let $\mathfrak{K}$ be a collection of compact open subgroups of $G(\mathbb{A}^S)$. Let M be a proper standard Levi subgroup of G defined over F. Assume that the collection of measures $\left\{\mu_{\mathbb{K}_M}^{M,S_\infty}\right\}$ is polynomially bounded, where $\mathbb{K}_M = \operatorname{proj}_M\left(P(\mathbb{A}_{\text{fin}}) \cap \gamma K_S \mathbb{K} \gamma^{-1}\right)$, K_S is an open subgroup of $\mathbf{K}_{S_{\text{fin}}}$, $\mathbb{K} \in \mathfrak{K}$, and $\gamma \in G(\mathbb{A}_{\text{fin}})$.*

Then for any finite set $\mathcal{F} \subseteq \widehat{\mathbf{K}_\infty}$ there exists an integer $k \in \mathbb{N}$ such that for any open subgroup $K_S \subseteq \mathbf{K}_{S_{\text{fin}}}$ and $\epsilon > 0$, we have:

$$\mathfrak{J}_{\text{spec},M}((h \otimes \mathbb{1}_{\mathbb{K}})) \ll_{\mathcal{F},\epsilon} \operatorname{vol}(K_S)^{-1} \|h\|_k \cdot \mathcal{OI}_{P,K_S\mathbb{K}} \cdot (\operatorname{lev}(K_S\mathbb{K}))^\epsilon, \quad (10.25)$$

for all $h \in \mathcal{H}\left(G(F_S)^1\right)_{\mathcal{F},K_S}$ and all $\mathbb{K} \in \mathfrak{K}$. ♣

We will use our estimates of $\mathcal{OI}_{P,\mathbb{K}}$ to further refine this estimate in the next subsection.

Remark 10.26. In fact, the source cited above ([FL17c, Proposition 4.5]) estimates the quantity in terms of lev $\left(K_S\mathbb{K}; G_M^+\right)$. However, in our case when G is simple and simply connected, $G_M^+ = G$ and hence we obtain a simpler estimate of Equation (10.25). ❇

We need to make a hypothesis about the polynomial boundedness of the measures in question.

Hypothesis 10.27. *We hypothesise that for any proper Levi subgroup $M \in \mathscr{L}$ of G, the collection of measures $\left\{\mu_{\mathbb{K}_{M,\gamma}}^{M,S_\infty}\right\}$, with $\mathbb{K}$ ranging over $\mathfrak{K}^{\text{spc,max}}$ and $\gamma \in G\left(\mathbb{A}_{\text{fin}}\right)$, is polynomially bounded.* ❄

Remark 10.28. This hypothesis is conjectured to hold for all groups G in [FLM15, FL17c]. Indeed, they also prove it for a certain collection of compact subgroups (principal congruence subgroups and subgroups of a fixed compact open subgroup respectively). Their proofs are based on induction on the Levi subgroup and the establishment of the full geometric limit property. We cannot however, in our set up, resort to these methods for two reasons:

- Levi subgroup of a simple group may not be simple (in contrast Levi subgroup of a reductive group is reductive, which are the objects of study in the aforementioned sources).

- We have not proven the full geometric property, rather only estimated the contribution of the unipotent elements.

Thus, we have to settle at the moment in assuming this as a hypothesis, which would later fall in line when one considers general reductive groups and estimates all the terms on the geometric side. Indeed this hypothesis is satisfied if G is of rank two. ❇

10.4.3 Estimates for Maximal Compact Subgroups

In this section, we establish bounds for the quantity $\mathcal{OI}_{P,\mathbb{K}}$ for an arbitrary *maximal* compact open subgroup $\mathbb{K} \subseteq G(\mathbb{A})$.

First we recall that for the trace formula, we have fixed a maximal compact subgroup $\mathbf{K}$ such that

$$\mathbf{K} = \prod_{\nu} \mathbf{K}_\nu,$$

with $\mathbf{K}_\nu \subseteq G(F_\nu)$ begin hyperspecial for *every* ν.

Notation 10.29. Now, let us say that $\mathbf{K}_\nu$ is the stabiliser of a point x_ν in the affine building of $G(F_\nu)$. Now assume that our maximal compact subgroup $\mathbb{K}$ is given by

$$\mathbb{K} = \prod_\nu K_\nu,$$

with (by conjugating if necessary) every K_ν being a stabiliser of a point y_ν adjacent to x_ν in the building of $G(F_\nu)$. We know that for almost all ν, K_ν is hyperspecial. Thus, we have that $y_\nu = x_\nu$ for almost all ν. Thus, $\tilde{S}(\mathbb{K})$ is the set

$$\tilde{S}(\mathbb{K}) \;:=\; \{\, \nu \mid y_\nu \neq x_\nu \,\}\,.$$

♦

Lemma 10.30. *The Iwahori subgroup I is contained in $\mathbb{K} \cap \mathbf{K}$.* ♣

Proof. The Iwahori subgroup is given by

$$I = \prod_\nu I_\nu,$$

with each I_ν being the stabiliser of a chamber $\mathbf{C}_\nu$, where $\mathbf{C}_\nu$ is such that x_ν and y_ν are vertices of $\mathbf{C}_\nu$.

Thus,

$$I_\nu \subseteq \mathbf{K}_\nu \cap K_\nu \qquad \forall\, \nu \in \mathcal{P}(F),$$

whence we have that

$$I \subseteq \mathbf{K} \cap \mathbb{K}.$$

■

Notation 10.31. Let ϖ_ν denote the uniformiser of the local field F_ν. By abuse of notation, we will also denote by ϖ_ν the ideal generated by the uniformiser (the unique maximal ideal of F_ν). ♦

Lemma 10.32. *The principal congruence subgroup $\mathbf{K}(\varpi_\nu)$ is contained in I_ν for almost all ν. That is,*

$$\mathbf{K}(\varpi_\nu) \subseteq I_\nu \qquad \textit{for almost all } \nu.$$

♣

Proof. This follows because the principal congruence subgroups are pro-p subgroups ([PR94, Lemma 3.8, Page 138]) and hence contained in a maximal pro-p subgroup. However, Iwahoris are normalisers of maximal pro-p subgroups and hence contain them. ■

Proposition 10.33. *Let $\mathbb{K}$ be as above. Let $\tilde{S}(\mathbb{K})$ be the set of places ν such that K_ν is not hyperspecial. Then we have that*

$$\operatorname{lev}(\mathbb{K}) = \prod_{\nu \in \tilde{S}} q_\nu. \tag{10.34}$$

♣

Proof. This follows from Lemma 10.32, Lemma 10.30 and the facts that $N(\varpi_\nu) = q_\nu$ and that ϖ_ν is the maximal ideal of F_ν. ■

Completion of Proof

Corollary 10.35. *Let $\mathfrak{K}^{\mathrm{spc,max}}$ be the collection defined in Chapter 4. Then for any $h \in \mathcal{H}\left(G(F_S)^1\right)$, we have*

$$\mathfrak{J}_{\mathrm{spec},M}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) \to 0, \qquad \mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}. \tag{10.36}$$

♣

Proof. Let $h \in \mathcal{H}\left(G(F_S)^1\right)$ be given. Then choose $\mathcal{F}$ and $K_S \subseteq \mathbf{K}_{S_{\mathrm{fin}}}$ such that $h \in \mathcal{H}\left(G(F_S)^1\right)_{\mathcal{F},K_S}$. Then by Proposition 10.24, we have that for finite set $\mathcal{F} \subseteq \widehat{\mathbf{K}_\infty}$ there exists an integer $k \in \mathbb{N}$ such that for any $\epsilon > 0$,

$$\mathfrak{J}_{\mathrm{spec},M}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) \ll_{\mathcal{F},\epsilon} \left(\operatorname{vol} K_S\right)^{-1} \cdot \|h\|_k \cdot \mathcal{OI}_{P,K_S\mathbb{K}} \cdot \operatorname{lev}\left(K_S\mathbb{K}\right)^\epsilon,$$

for all $\mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}$.

Now using Corollary 10.21 and Proposition 10.33 in the above estimate, we get:

$$\mathfrak{J}_{\mathrm{spec},M}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) \ll_{\mathcal{F},\epsilon} \left(\operatorname{vol} K_S\right)^{-1} \cdot \|h\|_k \left(\frac{1}{\operatorname{lev}\left(K_S\mathbb{K}\right)}\right)^{1-\epsilon-\delta},$$

whence we have (choosing ϵ and δ small enough):

$$\mathfrak{J}_{\mathrm{spec},M}\left(h \otimes \mathbb{1}_{\mathbb{K}}\right) \to 0,$$

since

$$\operatorname{lev}(\mathbb{K}) \to \infty, \qquad \mathbb{K} \in \mathfrak{K}^{\mathrm{spc,max}}.$$

■

Part V

Appendices

Appendix A

Hyperplane Arrangements

In this appendix, we review some concepts of affine geometry. In a later appendix, the connections with affine buildings will be explicated.

A.1 Affine Spaces

Definition A.1. Let $\mathbb{F}$ be a field and $\vec{\mathbf{A}}$ be a vector space over $\mathbb{F}$. Then a space $\mathbf{A}$ is called an ***affine space*** over $\vec{\mathbf{A}}$ if we have a transitive and free action of $\vec{\mathbf{A}}$ on $\mathbf{A}$. The vector space $\vec{\mathbf{A}}$ is called the ***space of translations*** and the elements of $\vec{\mathbf{A}}$ are called the ***translations***. The dimension of the vector space $\vec{\mathbf{A}}$ is called the ***dimension of the affine space*** $\mathbf{A}$. ♣

Remark A.2. Recall free and transitive means that, if we write the action additively $(v, a) \mapsto v + a, v \in \vec{\mathbf{A}}, a \in \mathbf{A}$, that for every $a, b \in \mathbf{A}$, there exists a unique $v \in \vec{\mathbf{A}}$ such that $v + a = b$. Thus, for every $a, b \in \mathbf{A}$, we can define $\vec{ba} := b - a := v$, where $v \in \vec{\mathbf{A}}$ is the unique vector satisfying $a + v = b$. Note that $\vec{ab} = a - b = -(b - a) = -\vec{ba}$. ✻

Notation A.3. In the rest of this chapter, $\mathbf{A}$ will always denote an affine space over $\vec{\mathbf{A}}$. We will also write $(\mathbf{A}, \vec{\mathbf{A}})$ for the pair and call it an affine space. ♦

Definition A.4. An ***affine subspace*** of an affine space $\mathbf{A}$ over $\vec{\mathbf{A}}$ is a subset $\mathbf{B}$ of $\mathbf{A}$ such that the set

$$\vec{\mathbf{B}} := \{b - a \mid a, b \in \mathbf{B}\}$$

is a vector subspace of $\vec{\mathbf{A}}$. ♣

Remark A.5. The above makes $\mathbf{B}$ into an affine space over the vector space $\vec{\mathbf{B}}$. Thus, in other words, affine subspaces of $\mathbf{A}$ are of the form,

$$a + \vec{\mathbf{B}} = \{a + v \mid v \in \vec{\mathbf{B}}\},$$

where $a \in \mathbf{A}$ and $\vec{\mathbf{B}}$ is a linear subspace of $\vec{\mathbf{A}}$. ❇

Definition A.6. An affine subspace $\mathbf{B}$ of an affine space $\mathbf{A}$ is called a ***hyperplane*** of $\mathbf{A}$ if $\dim(\mathbf{B}) = \dim(\mathbf{A}) - 1$. ♣

Definition A.7. If $\mathbf{B}$ is an affine subspace of $\mathbf{A}$, then the linear subspace $\vec{\mathbf{B}}$ is called its ***direction***, and two affine subspaces with the same direction are said to be ***parallel***. ♣

Definition A.8. Let $(\mathbf{A}, \vec{\mathbf{A}})$ and $(\mathbf{B}, \vec{\mathbf{B}})$ be two affine spaces over the same field. Then a map $\phi : \mathbf{A} \to \mathbf{B}$ is called an ***affine linear transformation*** or just an ***affine map*** if there exists a linear map $\vec{\phi} : \vec{\mathbf{A}} \to \vec{\mathbf{B}}$ such that

$$\phi(a + v) = \phi(a) + \vec{\phi}(v)$$

for every $a \in \mathbf{A}$ and $v \in \vec{\mathbf{A}}$. The map ϕ is called the ***linear part*** of the map $\vec{\phi}$. ♣

Remark A.9. Since $\vec{\mathbf{A}}$ acts transitively on $\mathbf{A}$, an affine map is completely determined by its linear part and its value on a single point. Thus, two affine maps having the same linear map differ only by a translation. ❇

Definition A.10. An affine map $\phi : \mathbf{A} \to \mathbf{A}$ is called a ***displacement*** or a ***translation*** of $\mathbf{A}$ if ϕ is bijective and the linear part of ϕ is a unitary map from $\vec{\mathbf{A}}$ to $\vec{\mathbf{A}}$. ♣

Definition A.11. Let $v \in \vec{\mathbf{A}}$. We denote by T_v the mapping $a \mapsto a + v$ of $\mathbf{A}$ onto itself. This is called the ***translation induced by the vector*** v. Clearly, it is a translation of $\mathbf{A}$ in the sense defined above. ♣

Definition A.12. An affine linear transformation is said to be ***non-singular*** or ***invertible*** if its linear part is invertible. Similarly, it is said to be ***singular*** if its linear part is singular. The set of all non-singular affine maps of $\mathbf{A}$ will be denoted by $\mathrm{Aut}(\mathbf{A})$. ♣

Proposition A.13. *Let $(\mathbf{A}, \vec{\mathbf{A}})$ be an affine space. Then*

$$\mathrm{Aut}(\mathbf{A}) = \vec{\mathbf{A}} \rtimes \boldsymbol{GL}(V)\vec{\mathbf{A}},$$

where $\boldsymbol{GL}(V)\vec{\mathbf{A}}$ acts on $\vec{\mathbf{A}}$ naturally. ♣

Proof. This is clear in view of Remark A.9. ■

Definition A.14. An affine map is called an ***isomorphism*** if its linear part is an isomorphism. In this case, the affine spaces are said to be ***isomorphic***. ♣

Definition A.15. A ***Euclidean space*** is a vector space V over $\mathbb{R}$ with a choice of an inner product on it. ♣

Definition A.16. An affine space $(\mathbf{A}, \vec{\mathbf{A}})$ is said to be an ***affine Euclidean space*** if $\vec{\mathbf{A}}$ is a Euclidean space. ♣

Proposition A.17. *If $(\mathbf{A}, \vec{\mathbf{A}})$ is an affine Euclidean space with $\vec{\mathbf{A}}$ having an inner product $\langle\cdot,\cdot\rangle$, then $\mathbf{A}$ is a a metric space.* ♣

Proof. We define the function $d : \mathbf{A} \to \mathbb{R}$ by

$$d(a,b) = \|b-a\|, \qquad a,b \in \mathbf{A},$$

where $\|\cdot\|$ is the norm on $\vec{\mathbf{A}}$ given by the inner product by

$$\|v\| \;:=\; \sqrt{\langle v,v\rangle}, \qquad v \in \vec{\mathbf{A}}.$$

Then one can easily check that d is a metric on $\mathbf{A}$. ■

Definition A.18. A map $f : \mathbf{A} \to \mathbf{A}$ is said to be a ***motion*** if it is an isometry of $\mathbf{A}$ as a metric space. ♣

Remark A.19. Note that as defined, a motion does not have to be an affine linear transformation. However, we have the following: ❇

Theorem A.20. *Every motion is an affine linear transformation.* ♣

Proof. We refer to [SR13, Chapter 8, Theorem 8.37]. ■

Theorem A.21. *Every motion of f of an affine Euclidean space $\mathbf{A}$ can be represented as*

$$f = T_v g,$$

where g is an affine transformation having a fixed point $O \in \mathbf{A}$ and its linear part $\vec{g}$ is an orthogonal transformation of $\vec{\mathbf{A}}$, while T_v is the translation by a vector $v \in \vec{\mathbf{A}}$ such that $\vec{g}(v) = v$. ♣

Proof. We refer to [SR13, Chapter 8, Theorem 8.39]. ■

A.2 System of Hyperplanes and Weyl Groups

In this section we make abstract the notion of a hyperplane arrangement and the Weyl group. The primary reference for this section is [Bou02, Chapter 5].

A.2.1 Basic Notions

Definition A.22. Let $\mathbf{A}$ be a a real affine space of finite dimension d over the vector space $\vec{\mathbf{A}}$. Then $\vec{\mathbf{A}}$ and hence $\mathbf{A}$ has a natural topology. A set $\mathfrak{H}$ of *affine* hyperplanes of $\mathbf{A}$ is called ***locally finite*** if every compact set K of $\mathbf{A}$ intersects only finitely many hyperplanes in $\mathfrak{H}$. ♣

Example A.23. Consider the affine space $\mathbb{R}^2$ over $\mathbb{R}^2$. Let $\mathfrak{H}$ be the three hyperplanes H_1, H_2, and H_3 given in Figure A.1. Since this is a finite collection of hyperplanes, this is clearly locally finite. ♠

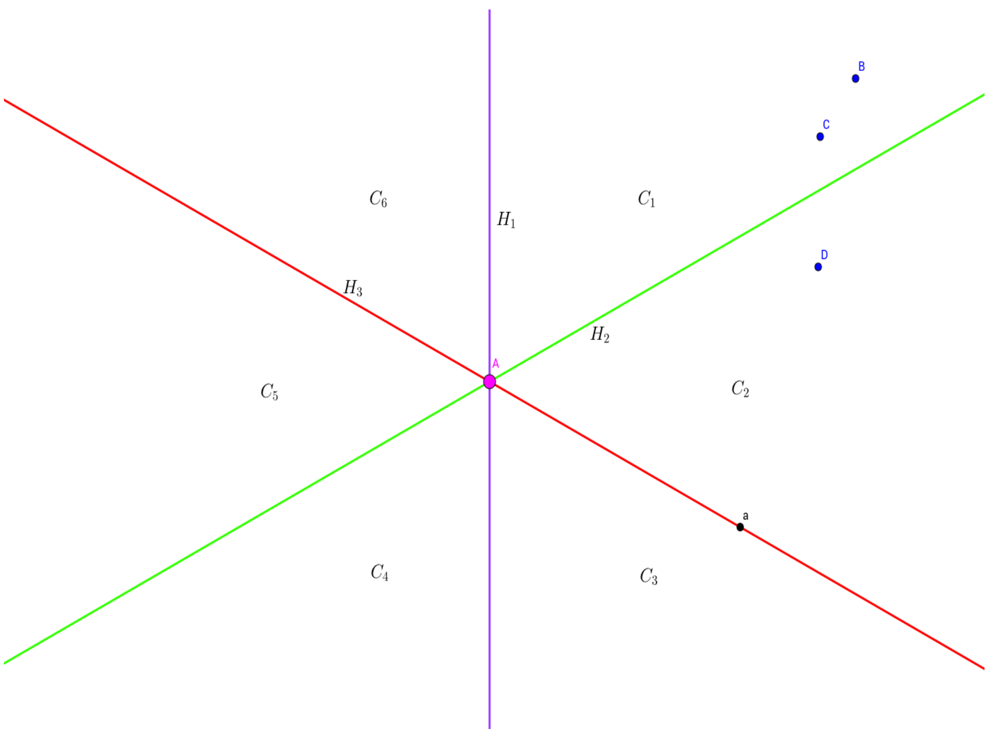

Figure A.1: The Hyperplane arrangement A_2

Example A.24. Now again considering the affine space $\mathbb{R}^2$ over $\mathbb{R}^2$, we let $\mathfrak{H}$ be the collection of hyperplanes H_1, H_2, and H_3 along with their integer

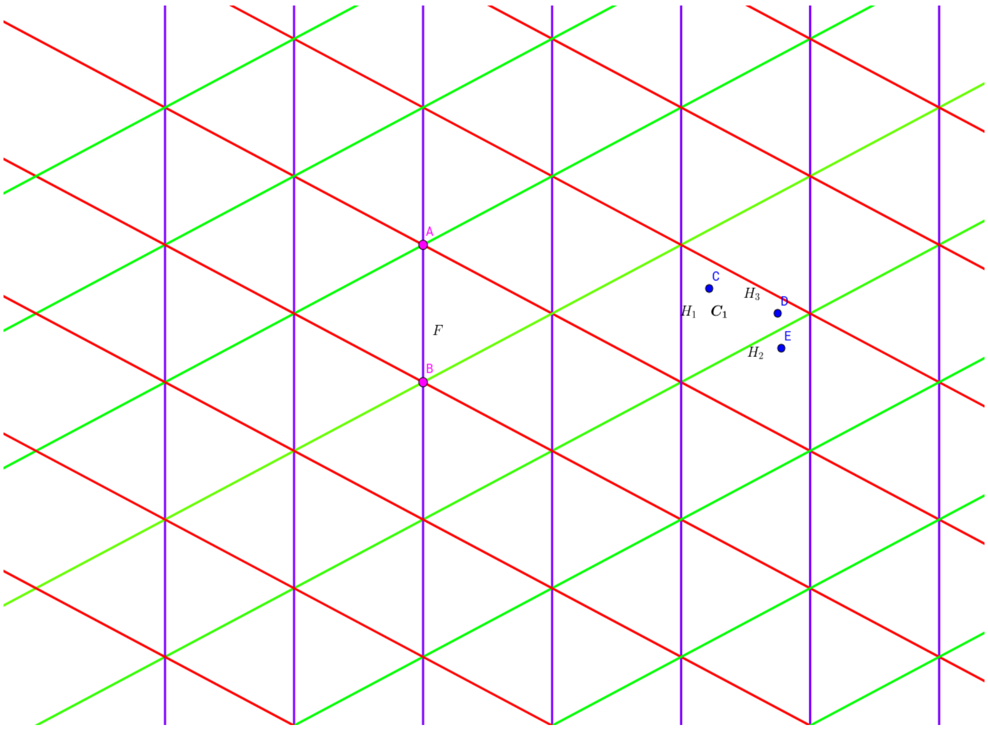

Figure A.2: The Hyperplane arrangement $\tilde{A}_2$

translates. That is, we also consider the hyperplanes $H_i + k$ for $k \in \mathbb{Z}$ and $i \in \{1, 2, 3\}$. This is shown in Figure A.2. It is clear once again that this collection is locally finite. ♠

Definition A.25. Let H be a hyperplane in $\mathbf{A}$. Recall that $\mathbf{A} \setminus H$ has two connected components. They are called the ***open half-spaces*** bounded by H. Their closures are called the ***closed half-spaces*** bounded by H. ♣

Example. In the first example, if we imagine the hyperplane H_1 as the $y-$axis in $\mathbb{R}^2$, then the open half-spaces bounded by H_1 are the two sets of points with $x-$coordinate strictly positive (or strictly negative). The closed half-spaces are the open half-spaces along with H_1. One computes the open (closed) half-spaces for H_2 and H_3 in a similar manner. This also applies to the second example. ♠

Definition A.26. Let H be a hyperplane of $\mathbf{A}$ and $x, y \in \mathbf{A}$. Then x and y are said to be ***strictly on the same side*** of H if they are contained in the same open half-space bounded by H. They are said to be on the ***opposite sides*** of H if x belongs to one of the half-spaces bounded by H and y to the other. ♣

Example. In the first example in Figure A.1, the points B and C are on the same side of H_2 (and H_1 and H_3) as well. However, the points B and D are on the opposite sides of H_2 (while still being on the same side of both H_1 and H_3).

In the second example, in Figure A.2, the points C and D are on the same side of H_2 (and H_1 and H_3 as well). However, the points D and E are on the opposite sides of H_2 (while still being on the same side of both H_1 and H_3). ♠

A.2.2 Facets

Given two points x and y of $\mathbf{A}$, denote by $R(x, y)$ the relation:

"For any hyperplane $H \in \mathfrak{H}$, either $x \in H$ and $y \in H$ or x and y are strictly on the same side of H."

Clearly, R is an equivalence relation on $\mathbf{A}$.

Definition A.27. A ***facet*** of $\mathbf{A}$ relative to $\mathfrak{H}$ is an equivalence class of the equivalence relation R defined above. Thus a facet is a subset F of $\mathbf{A}$ such that for every hyperplane $H \in \mathfrak{H}$, either $F \subseteq H$ or it is contained in one of the open half spaces bounded by H. ♣

Example. In the first example, the "chambers" marked C_1 to C_6 are the facets, so are the rays emanating from the point A and so is the singleton set consisting of point A. These are the only facets as one can check.

Situation is more complicated and interesting in the second example. Interior of all the 'triangles' are facets (such as C_1 shown in the figure); so is the open segment F between the points A and B (and any other such open segment); so are the sets $\{A\}$ and $\{B\}$ (and any other singleton whose element is a point which is intersection of the hyperplanes). The sets $\{C\}, \{D\}$ and $\{E\}$ however, are not facets. In fact, listed above are all the facets as is easily verified. ♠

Proposition A.28. *The set of facets is locally finite.* ♣

Proof. This is clear since $\mathfrak{H}$ is locally finite. ■

Definition A.29. Let F be a facet and a be a point of F. A hyperplane $H \in \mathfrak{H}$ contains F if and only if $a \in H$; the set $\mathfrak{F}$ of these hyperplanes is thus finite; their intersection is an affine subspace $\mathbf{L}$ of $\mathbf{A}$, which we shall call the ***affine support*** of F. The dimension of the support $\mathbf{L}$ will be called the ***dimension*** of F. If there are no hyperplanes containing a facet F, we will set its support to be the whole affine space $\mathbf{A}$. ♣

Example. In the first example, the affine support of the "chambers" $C_1, \ldots, C_6$ is the whole affine space $\mathbb{R}^2$. The affine support of the ray emanating from A is the corresponding hyperplane and the affine support of the facet $\{A\}$ is the set $\{A\}$. Thus, "chambers" are $2-$dimensional, rays are $1-$dimensional and the facet $\{A\}$ is $0-$dimensional.

In the second example, the affine support of the "chambers" (interior of triangles) is the whole affine space, the affine support of the facet F is the corresponding hyperplane and the affine support of facet $\{A\}$ is again the set $\{A\}$. Again "chambers" are $2-$dimensional, segments are $1-$dimensional and points are $0-$dimensional. ♠

Proposition A.30. *Let F be a facet and $\mathbf{L}$ be its affine support.*

(i) The set F is a convex open subset of the affine subspace $\mathbf{L}$ of $\mathbf{A}$.

(ii) The closure of F is the union of F and facets of dimension strictly smaller than that of F.

(iii) In the topological space $\mathbf{L}$, the set F is the interior of its closure.

♣

Proof. We refer to [Bou02, Chapter 5, §1, no. 2, Prop. 3]. ■

Example. All this is patently clear in our examples. Let us take the example facet F between the points A and B in our second example. It is a convex subset of the hyperplane containing it (which is its affine support); the closure of F is $F \cup \{A\} \cup \{B\}$ and $\{A\}$ and $\{B\}$ are facets of strictly smaller dimension; while F is the interior of its closure. ♠

Corollary A.31. *Let F_1 and F_2 be two facets. If $\overline{F_1} = \overline{F_2}$, then the facets F_1 and F_2 are equal.* ♣

Proof. This follows from Proposition A.30(iii). ■

A.2.3 Chambers

Definition A.32. A ***chamber*** of $\mathbf{A}$ relative to $\mathfrak{H}$ is a facet of $\mathbf{A}$ relative to $\mathfrak{H}$ that is not contained in any hyperplane belonging to $\mathfrak{H}$. ♣

Example. The facets marked C_i in the first example are chambers; so are the interior of triangles in the second example.

This explains our use of the word chambers in quotes in the examples above. ♠

Remark A.33. Let U be the open subset of $\mathbf{A}$ consisting of points that do not belong to any hyperplane of $\mathfrak{H}$. Since a hyperplane of $\mathfrak{H}$ must contain any facet that it meets, the chambers are the facets contained in U; every chamber is a convex, and hence connected open subset of $\mathbf{A}$ by Proposition A.30(i). Since chambers form a partition of U, they are exactly the *connected components* of U. Every convex subset A of U is connected, and thus contained in a chamber, which is unique if A is non-empty. It is clear that chambers are the facets with support $\mathbf{A}$ and Proposition A.30(iii) shows that every chamber is the interior of its closure. ❄

The next result shows that $\mathbf{A}$ is almost a disjoint union of chambers. Indeed their closures cover the whole space while the chambers themselves are disjoint.

Proposition A.34. *Every point of* $\mathbf{A}$ *is in the closure of at least one chamber.* ♣

Proof. We refer to [Bou02, Chapter 5, §1, no. 3 Proposition 6]. ■

Example. Again, all this is patently clear in our examples. In the first example, the points B and C are in chamber C_1, while D is in chamber C_2. The point a on the hyperplane belongs to the closure of both the chambers C_2 and C_3 while not being in either of these chambers. The point A is the closure of every chamber while not being in any of them.

In the second example, C and D are in chamber C_1 while E is in the "neighbouring" chamber. The points A and B are in the closure of six chambers each; whereas any point on the facet F is in the closure of two chambers (on either side of the facet). ♠

Definition A.35. Let $\mathbf{C}$ be a chamber of $\mathbf{A}$. A ***face*** of $\mathbf{C}$ is a facet contained in the closure of $\mathbf{C}$ whose support is a hyperplane. A ***wall*** of $\mathbf{C}$ is a hyperplane that is the affine support of a face of $\mathbf{C}$. ♣

Example. In the first example, let us consider the chamber C_1. It has two faces the rays emanating from A corresponding to hyperplanes H_1 and H_2. The hyperplanes H_1 and H_2, being the affine supports of these rays, are thus the walls of C_1.

In the second example, again consider the chamber C_1. The faces are the open segments of hyperplanes which form the 'edges' of the triangle. The walls are then the hyperplanes H_1, H_2, and H_3. ♠

Proposition A.36. *Every wall of a chamber* $\mathbf{C}$ *belongs to* $\mathfrak{H}$ *and every hyperplane in* $\mathfrak{H}$ *is a wall of at least one chamber* $\mathbf{C}$. ♣

Proof. We refer to [Bou02, Chapter 5, §1, no. 4 Proposition 8]. ∎

A.3 Groups Generated by Reflections

A.3.1 The Set-up

Let $(\mathbf{A}, \vec{\mathbf{A}})$ be an affine Euclidean space with $\vec{\mathbf{A}}$ endowed an inner product $\langle \cdot, \cdot \rangle$. Thus, $\mathbf{A}$ becomes a metric space as in Proposition A.17.

Let $\mathfrak{H}$ be a set of hyperplanes of $\mathbf{A}$ and $\mathbf{W}$ be the subgroup of $\mathrm{Aut}(\mathbf{A})$ generated by orthogonal reflections s_H with respect to the hyperplanes $H \in \mathfrak{H}$. We assume that the following conditions are satisfied :

(D1) For any $w \in \mathbf{W}$ and any $H \in \mathfrak{H}$, the hyperplane $w(H)$ belongs to $\mathfrak{H}$.

(D2) For any two compact subsets K and L of $\mathbf{A}$, the set of $w \in \mathbf{W}$ such that $w(K)$ meets L is finite.

Definition A.37. The subgroup $\mathbf{W}$ of $\mathrm{Aut}(\mathbf{A})$ generated by orthogonal reflections along the hyperplanes $H \in \mathfrak{H}$ is called the ***Weyl group*** of $(\mathbf{A}, \mathfrak{H})$ or just $\mathbf{A}$ when there is no ambiguity about $\mathfrak{H}$. ♣

Example. Our Examples A.23 and A.24 from §A.2 both satisfy these conditions. ♠

A.3.2 The Consequences

Lemma A.38. *Under the above assumptions, the set of hyperplanes* $\mathfrak{H}$ *is locally finite.* ♣

Proof. Let K be a compact subset of $\mathbf{A}$. If a hyperplane $H \in \mathfrak{H}$ meets K, then so does the set $s_H(K)$ since s_H fixes $H \cap K$. The set of $H \in \mathfrak{H}$ meeting K is thus finite by (D2). ■

Remark A.39. This allows us to speak of facets, chambers, walls etc. just as in §A.2, relative to $\mathfrak{H}$ or relative to $\mathbf{W}$. ❇

Proposition A.40. *Let $\mathbf{C}$ be a chamber.*

(i) For any $x \in \mathbf{A}$, there exists an element $w \in \mathbf{W}$ such that $w(x) \in \overline{\mathbf{C}}$.

(ii) For any chamber $\mathbf{C}'$, there is an element $w \in \mathbf{W}$ such that $w(\mathbf{C}') = \mathbf{C}$.

(iii) The group $\mathbf{W}$ is generated by the set of orthogonal reflections with respect to the walls of $\mathbf{C}$.

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 1, Lemma 2]. ■

Remark A.41. We saw in Proposition A.34 that every point is in closure of *some* chamber, and now in the special case where (D1) and (D2) are satisfied, Item (ii) ensures that the Weyl group acts transitively on the chambers and hence we have Item (i). Indeed, these two hypothesis are so strong that we get much more, such as the following ❇

Theorem A.42. *Let $\mathbf{C}$ be a chamber and let $\mathbf{S}$ be the set of reflections with respect to the walls of $\mathbf{C}$.*

(i) The pair $(\mathbf{W}, \mathbf{S})$ is a Coxeter system.

(ii) For any chamber $\mathbf{C}'$, there exists a unique $w \in \mathbf{W}$ such that $w(\mathbf{C}) = \mathbf{C}'$.

(iii) The set of hyperplanes H such that $s_H \in \mathbf{W}$ is equal to $\mathfrak{H}$.

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 2, Th. 1]. ■

A.3.3 Fundamental Domain and Stabilisers

Definition A.43. Let G be a group acting on a set X. Then a subset Y of X is called a ***fundamental domain*** for the action of G on X if :

(A) For every $x \in X$, there exists a $g \in G$ such that $g(x) \in Y$.

(B) If $x, y \in Y$ and $g \in G$ such that $y = g(x)$, then $x = y$.

♣

Remark A.44. This is equivalent to saying that Y is a set of representatives from each orbit of action of G on X. If X is a topological space, Y is usually chosen to be some topologically nice subset. ❇

The next two statements are the heart of this section.

Theorem A.45. *For any chamber* $\mathbf{C}$*, the closure* $\overline{\mathbf{C}}$ *of* $\mathbf{C}$ *is a fundamental domain for the action of* $\mathbf{W}$ *on* $\mathbf{A}$*.* ♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 3 Th. 2]. ■

Proposition A.46. *Let F be a facet and $\mathbf{C}$ a chamber such that $F \subseteq \overline{\mathbf{C}}$. Let $w \in \mathbf{W}$. The following conditions are equivalent :*

(i) $w(F)$ *meets* F*.*

(ii) $w(F) = F$*.*

(iii) $w(\overline{F}) = \overline{F}$*.*

(iv) w *fixes at least one point of* F*.*

(v) w *fixes every point of* $\overline{F}$*.*

(vi) w *belongs to the subgroup of* $\mathbf{W}$ *generated by the reflections with respect to the walls of* $\mathbf{C}$ *containing* F*.*

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 3 Prop. 1]. ■

Remark A.47. This fact that the Weyl group acts in rather restricted way on the facets and chambers was used crucially to deduce several results in Chapter 6. ❇

Notation A.48. For $w \in \mathbf{W}$, we denote by $\vec{w}$, the linear part (cf. Definition A.8) of the affine transformation w. We put

$$\vec{\mathbf{W}} := \{ \vec{w} \mid w \in \mathbf{W} \} .$$

We also denote by $U : \mathbf{W} \to \boldsymbol{O}(\vec{\mathbf{A}})$ the map given by $w \mapsto \vec{w}$. Then U is a group homomorphism from $\mathbf{W}$ to $\boldsymbol{O}(\vec{\mathbf{A}})$ and $\vec{\mathbf{W}} = U(\mathbf{W})$. ♦

We continue with this following finiteness result:

Theorem A.49. *We retain the notations above. Then*

(i) The set of walls of a chamber is finite.

(ii) The set of directions of hyperplanes belonging to $\mathfrak{H}$ is finite.

(iii) The set $\vec{\mathbf{W}} := \{ \vec{w} \mid w \in \mathbf{W} \}$ is finite and is a subgroup of group of orthogonal transformations of $\vec{\mathbf{A}}$.

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 6, Th. 3]. ■

Proposition A.50. *Let $\mathbf{C}$ be a chamber and let $\mathfrak{N}$ a set of walls of $\mathbf{C}$. Let $\mathbf{W}_{\mathfrak{N}}$ be the subgroup of $\mathbf{W}$ generated by the orthogonal reflections with respect to the elements of $\mathfrak{N}$. Then the following are equivalent:*

(a) The group $\mathbf{W}_{\mathfrak{N}}$ is finite.

(b) There exists a point of $\mathbf{A}$ invariant under every element of $\mathbf{W}_{\mathfrak{N}}$.

(c) The hyperplanes belonging to $\mathfrak{N}$ have a non-empty intersection.

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 6, Proposition 4]. ■

Example. The two examples in the beginning provide very contrasting situations when we take $\mathfrak{N}$ to be the set of *all* walls of a given chamber. In Example A.23, consider the chamber C_1. The set of walls $\mathfrak{N}$ is $\{H_1, \dots, H_3\}$ which is the set of all hyperplanes. The group $\mathbf{W}_{\mathfrak{N}}$ is thus equal to $\mathbf{W}$, which is finite and in fact $\mathfrak{S}_3$ in this case. The point A is invariant under every element of $\mathbf{W}$ while the set of hyperplanes have a non-empty intersection.

In contrast, consider the chamber C_1 in the second example. The set of walls is again the set of all hyperplanes $\{H_1, \dots, H_3\}$ and the Weyl group is infinite. No point of $\mathbf{A}$ is invariant under $\mathbf{W}$ and the hyperplanes intersect trivially. ♠

A.3.4 Structure of Chambers

Let $\mathbf{C}$ be a chamber, let $\mathfrak{M}$ be the set of walls of $\mathbf{C}$, and for $H \in \mathfrak{H}$, let e_H be the unit vector orthogonal to H on the same side of H as $\mathbf{C}$.

The following two results distinguish between finite and infinite Weyl groups.

Proposition A.51. *Assume that the group $\mathbf{W}$ is essential and finite. Then*

(a) *There exists a unique point $a \in \mathbf{A}$ invariant under $\mathbf{W}$.*

(b) *The family $(e_H)_{H\in\mathfrak{M}}$ is a basis of $\vec{\mathbf{A}}$.*

(c) *The chamber $\mathbf{C}$ is the open simplicial cone with vertex a defined by the basis $(f_H)_{H\in\mathfrak{M}}$ of $\vec{\mathbf{A}}$ such that $\langle e_H, f_K\rangle = \delta_{HK}$.*

♣

Example. This is illustrated by the spherical example Example A.23. ♠

Proposition A.52. *Assume that $\mathbf{W}$ is essential, irreducible and infinite. Then*

(a) *No point of $\mathbf{A}$ is invariant under $\mathbf{W}$.*

(b) *We have $\operatorname{card}\mathfrak{M} = \dim\vec{\mathbf{A}} + 1$, and there exist real numbers $c_H > 0$ such that $\sum_{H\in\mathfrak{M}} c_H.e_H = 0$.*

(c) *The chamber $\mathbf{C}$ is an open simplex.*

♣

Example. This is illustrated by the affine example Example A.24. ♠

A.3.5 Special Points

Definition A.53. A point $a \in \mathbf{A}$ is called ***special point*** if for every hyperplane $H \in \mathfrak{H}$, there exists a hyperplane $H' \in \mathfrak{H}$ parallel to H and such that $a \in H'$. ♣

Example. In Example A.23, there is only one special point, namely the point 'A'. In Example A.24, a little deliberation shows that every point is a special point. ♠

Example A.54. The above examples are rather extreme cases when either only one or all the points are special. In general, there are infinite number of both special and non-special points. We consider another hyperplane arrangement here, show in Figure A.3 and called $\tilde{C}_2$. Here all the pink points are special while the points representing the intersection of red and blue lines are not special. ♠

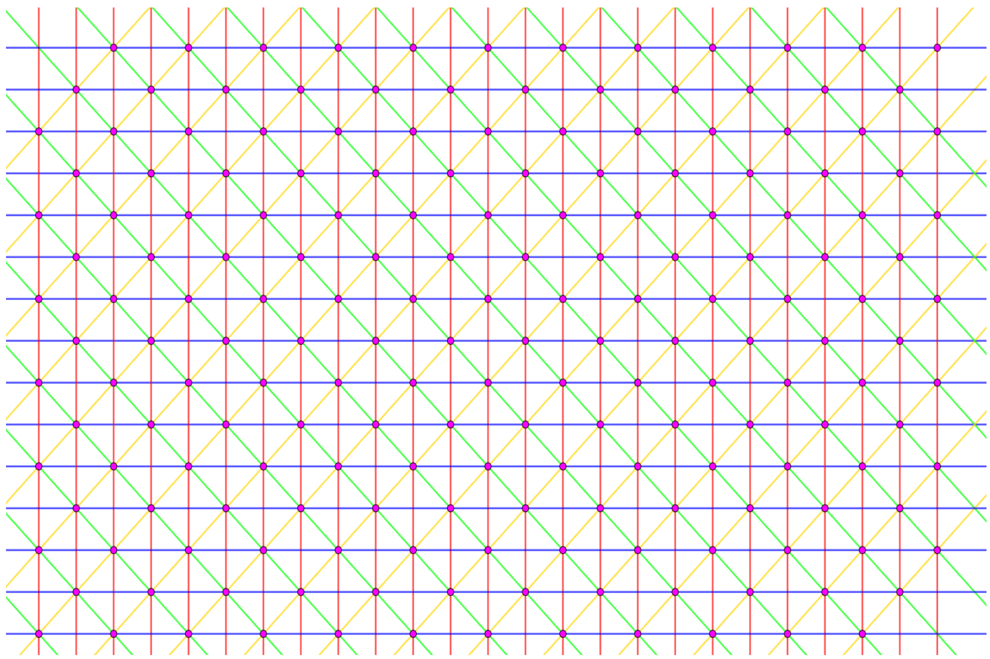

Figure A.3: The Hyperplane arrangement $\tilde{C}_2$

Notation A.55. Let L be the set of translations belonging to $\mathbf{W}$ and let Λ be the set of $v \in \vec{\mathbf{A}}$ such that the translation $x \mapsto v + x$ belongs to L. It is immediate that Λ is stable under $\vec{\mathbf{W}}$ and that L is a normal subgroup of $\mathbf{W}$. Since $\mathbf{W}$ acts properly on $\mathbf{A}$, the same holds for L, and it follows easily that Λ is a discrete subgroup of $\vec{\mathbf{A}}$. For any point $x \in \mathbf{A}$, denote by $\mathbf{W}_x$ the stabiliser of x in $\mathbf{W}$. That is

$$\mathbf{W}_x \;:=\; \{\, w \in \mathbf{W} \mid w(x) = x \,\}\,.$$

♦

Proposition A.56. *Let $a \in \mathbf{A}$. The following conditions are equivalent:*

(a) *We have that* $\mathbf{W} = \mathbf{W}_a.L$*;*

(b) *The restriction of the homomorphism* U *to* $\mathbf{W}_a$ *is an isomorphism from* $\mathbf{W}_a$ *to* $\vec{\mathbf{W}}$*.*

(c) *The point* a *is special.*

Recall that the map U *was defined in Notation A.48.* ♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 10, Proposition 9]. ■

Now that we are convinced that special points are important (and hence the adjective 'special' is justified), it is time to worry about their existence. We have

Proposition A.57. *There exists a special point for* $\mathbf{W}$*.* ♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 10, Proposition 10]. ■

Encouraged by the examples above, we have

Proposition A.58. *Assume that* $\mathbf{W}$ *is essential.*

(a) *If* $a \in \mathbf{A}$ *is special, there exists a chamber* $\mathbf{C}$ *such that* a *is an extremal point of* $\overline{\mathbf{C}}$*.*

(b) *If* $\mathbf{C}$ *is a chamber, there exists an extremal point of* $\overline{\mathbf{C}}$ *that is special.*

♣

Proof. We refer to [Bou02, Chapter 5, §3, no. 10, Corollary to Proposition 11]. ■

Remark A.59. An extremal point of a chamber is not necessarily special. For example, the chambers in the case of Figure A.3 are the right angled triangles bounded by the red, blue, and yellow(green) lines. Each of these chambers has two pink extremal points which are special and one extremal point represented by intersection of blue and red line which is not special. ❇

Appendix B

Root Systems

B.1 Definitions

Definition B.1. Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$. A ***root system*** in V is a *finite* set R such that

(R1) $0 \notin R$.

(R2) If $\alpha, \beta \in R$, then so is $s_\alpha(\beta)$ where s_α is the linear transformation defined by

$$s_\alpha v \; := \; v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \qquad v \in V.$$

(R3) The vectors in R span V.

The dimension of V is called the ***rank*** of the root system and the elements of R are called the ***roots***. ♣

Remark B.2. Since $s_\alpha \alpha = -\alpha$, we have that $-\alpha \in R$ whenever $\alpha \in R$. ❇

Remark B.3. We have given a very general definition of root system. Usually, some more conditions are imposed depending on one's interests. We describe the other conditions here.

(R4) If $\alpha \in R$ and $c\alpha \in R$ for any $c \in \mathbb{R}$, then $c = \pm 1$.

(R5) For all $\alpha, \beta \in R$, the quantity

$$2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer.

The root systems that additionally satisfy (R4) are called ***reduced*** root systems and those which additionally satisfy (R5) are called ***crystallographic*** root systems. Thus, for us, unadorned root systems will not necessarily be reduced or crystallographic. ❇

Remark B.4. The condition (R3) is one of convenience and does not really play a big role. If R does not span V, we can always take the subspace spanned by R and then the rank of R will be the dimension of that space. Since we do not want to write this subspace explicitly, we assume this condition in our definition. ❇

Lemma B.5. *Let $\alpha \in V$. Then s_α is the unique transformation of V such that $s_\alpha = -\alpha$ and $s_\alpha(\beta) = \beta$ whenever $\langle \alpha, \beta \rangle = 0$. Moreover, s_α is an orthogonal linear transformation, that is, it preserves the inner product.* ♣

Proof. This is a straightforward calculation. ■

This prompts the following:

Definition B.6. Let $\alpha \in V$ and let H_α be the hyperplane in V orthogonal to α. Then, the linear map s_α defined above is the ***orthogonal reflection*** with respect to the hyperplane H_α.

Note that if $H_\beta = H_\alpha$, then $s_\beta = cs_\alpha$ for some $c \in \mathbb{R}$ and hence the orthogonal reflection with respect to a hyperplane is unique up-to constant multiple. ♣

Definition B.7. If (V, R) is a root system, the ***Weyl group*** W of R is the subgroup of $\boldsymbol{GL}(V)$ generated by the reflections $s_\alpha, \alpha \in R$. ♣

Remark B.8. Since the reflections are all orthogonal transformations, the Weyl group is a subgroup of the orthogonal group $\boldsymbol{O}(V)$. ❇

Lemma B.9. *The Weyl group W of a root system (V, R) is a finite group.* ♣

Proof. By assumption, every s_α maps R into itself, indeed *onto* itself, since every $\beta \in R$ satisfies $\beta = s_\alpha(s_\alpha(\beta))$ for any $\alpha \in R$. Thus, every element of W maps R onto itself. Since V is spanned by R, a linear transformation of V is determined by its action on R. Thus, W can be considered as a subgroup of permutation group of R, which is finite, as R is finite. ■

Definition B.10. A root system (V, R) is called ***reducible*** if there exists an orthogonal decomposition $V = V_1 \oplus V_2$ with $\dim V_i > 0$ such that every element of R is either in V_1 or in V_2. If no such decomposition exists, R is said to be ***irreducible***. ♣

Definition B.11. Two root systems (V, R) and (U, S) are said to be ***isomorphic*** if there exists a vector space isomorphism $A : V \to U$ such that A maps R onto S and such that for all $\alpha \in R$ and $\beta \in V$, we have

$$A(s_\alpha \cdot \beta) = s_{A(\alpha)} \cdot (A(\beta)).$$

A map A with this property is called an ***isomorphism***. ♣

Remark B.12. The map A is not required to preserve inner products, just reflections along the roots. ❄

Example B.13. The simplest example of a non-trivial root system is the system (V, R) where $V = \mathbb{R}$ and $R = \{\alpha, -\alpha\}$ for some $\alpha \in \mathbb{R}$. This system is called A_1. The Weyl group is the symmetric group on 2 letters $\mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$. It is clearly irreducible, reduced and crystallographic. ♠

Example B.14. For a slightly more complicated example, we look at the root system A_2. It is the root system consisting of 6 vectors in a 2−dimensional vector space. Let V be the vector space $\mathbb{R}^2$ and let the roots be

$$R = \pm \{\alpha, \beta, \alpha + \beta\},$$

where $\alpha = (2, 0)$ and $\beta = (-1, \sqrt{3})$.

We compute the reflections s_α and s_β. We have

$$\begin{aligned} s_\alpha(\beta) &= \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \\ &= \beta - 2\frac{-2}{4}\alpha \\ &= \alpha + \beta \end{aligned}$$

Now since $s_\alpha^2 = 1$, we have that $s_\alpha(\alpha + \beta) = \beta$.

Similarly, we have that $s_\beta(\alpha) = \alpha + \beta$ and $s_\beta(\alpha + \beta) = \alpha$.

Using these computations, we will see that the Weyl group generated by the reflections is $\mathfrak{S}_3$.

Furthermore, one checks easily that this is a reduced irreducible crystallographic root system. ♠

Example B.15. For an example from a different 'family', we consider the root system B_2. It is the root system (R, V) where V is the vector space $\mathbb{R}^2$ and R consists of the following 8 vectors

$$R = \{\, \pm 2e_i, \pm e_i \pm e_j \mid 1 \leq i \leq 2, 1 \leq i < j \leq 2 \,\}.$$

It can be seen that this is a reduced irreducible crystallographic root system. ♠

Remark B.16. The terminology will become clear later when we discuss the classification of root systems. ❇

B.2 Duality

We introduce an important duality operation on root systems. First we introduce a duality relationship between an inner product space E and its dual E^*.

Definition B.17. Let E be a finite dimensional real inner product space and let E^* be its vector space dual. Then there is a canonical pairing between E and E^*, defined by $v \mapsto v^*$, where $v^* \in E^*$ is the linear transformation given by

$$v^*(w) \ := \ \langle w, v\rangle \qquad w \in E.$$

♣

Remark B.18. By Riesz representation theorem, the map $v \mapsto v^*$ is an isomorphism. ❇

Definition B.19. If (V, R) is a root system, then for each root $\alpha \in R$, the ***precoroot*** K_α is the vector given by

$$K_\alpha \ := \ 2\frac{\alpha}{\langle\alpha, \alpha\rangle}.$$

Thus, K_α is twice the unit vector in the direction of α. ♣

Remark B.20. The term *precorrot* is not standard in literature and is sometimes identified with the coroot defined below. ❇

Example B.21. For the root system A_1, let $\alpha = (1, 0)$. Thus, $R = \{(1, 0), (-1, 0)\}$. Then $K_\alpha = (2, 0)$ and $K_{-\alpha} = (-2, 0)$. ♠

Example B.22. For the root system A_2, both α and β have length 2, and hence $K_\alpha = \alpha$ and $K_\beta = \beta$. ♠

Definition B.23. If (V, R) is a root system, then for each root $\alpha \in R$, the ***coroot*** $\alpha^\vee$ is defined to be the vector $K_\alpha^* \in V^*$. ♣

Lemma B.24. *For $\alpha \in R$, the coroot $\alpha^\vee$ satisfies*

(a) $\alpha^\vee(\alpha) = 2$.

(b) $s_\alpha(v) = v - \alpha^\vee(v)\alpha = v - \langle\alpha, v\rangle K_\alpha$ *for all* $v \in V$.

♣

Proof. It follows from a simple calculation. ■

Definition B.25. The set of all coroots is denoted by $R^\vee$ and is called the ***dual root system*** to R. ♣

Proposition B.26. *If (V, R) is a root system, then so is $(V^*, R^\vee)$ and the Weyl group of both the root systems is same. Furthermore, $(R^\vee)^\vee = R$.* ♣

Proof. This again follows from an elementary calculation. We refer to [Hal15, Chapter 8, Prop. 8.11]. ■

Example B.27. We determine the dual root system $A_2^\vee$. First since $V = \mathbb{R}^2$ we have $V^* \cong \mathbb{R}^2$. Let $\{e_1, e_2\}$ be the standard basis of $V = \mathbb{R}^2$ and let $\{f_1, f_2\}$ be the basis (of V^*) dual to it. Then we compute

$$\begin{aligned}\alpha^\vee(e_1) &= K_\alpha^*(e_1)\\ &= \langle K_\alpha, e_1\rangle\\ &= \langle \alpha, e_1\rangle\\ &= 2.\end{aligned}$$

Also,

$$\begin{aligned}\alpha^\vee(e_2) &= K_\alpha^*(e_2)\\ &= \langle K_\alpha, e_2\rangle\\ &= \langle \alpha, e_2\rangle\\ &= 0.\end{aligned}$$

Hence, $\alpha^\vee = 2f_1$.

In a similar fashion, we compute that

$$\beta^\vee = -f_1 + \sqrt{3}f_2.$$

Since $\alpha = 2e_1$ and $\beta = -e_1 + \sqrt{3}e_2$, and they form the basis of the root respective root systems (cf. §B.3) we see that the root systems A_2 and $A_2^\vee$ are isomorphic. ♠

B.3 Bases and Chambers

Definition B.28. Let (V, R) be a root system. A subset Δ of R is called a ***base*** if the following conditions are satisfied:

(B1) Δ is a basis of V as a vector space.

(B2) Each root $\alpha \in R$ can be expressed as a linear combination of elements of Δ with integer coefficients and in such a way that the coefficients are either all non-negative or all non-positive.

♣

Definition B.29. Let Δ be a base of R. The roots whose coefficients with respect to Δ are all non-negative are called ***positive roots*** and the others (whose coefficients are all non-positive) are called ***negative roots***. The set of positive roots is denoted by R^+ and the set of negative roots by R^-. The elements of Δ are called the ***positive simple roots***. ♣

Remark B.30. Since Δ is a basis of V, every root α can be *uniquely* expressed as a linear combination of elements of Δ. We require furthermore that the coefficients in the expansion of each $\alpha \in R$ be integers and that all non-zero coefficients have the same sign. ❇

Example B.31. For the system A_1, we can take $\Delta = \{\alpha\}$. Then $R^+ = \{\alpha\}$ or equivalently, we can take $\Delta = \{-\alpha\}$ and then $R^+ = \{-\alpha\}$. ♠

Example B.32. For the system A_2, a choice of bases would be $\Delta = \{\alpha, \beta\}$. With this choice, $R^+ = \{\alpha, \beta, \alpha + \beta\}$. There are many other choices of bases. In fact, there are six of them, a fact that will become clear shortly. ♠

The notion of a base of a root system is an extremely important one. First we settle the question of existence.

Theorem B.33. *Let (V, R) be a root system. Then there exists a base Δ of R. In fact, there are many such basis.* ♣

Proof. We refer to [Hal15, Theorem 8.16]. ■

Proposition B.34. *If Δ is a base for R, then the set of coroots $\alpha^\vee, \alpha \in \Delta$ is a base for the dual root system $R^\vee$.* ♣

Definition B.35. A ***chamber*** or a ***Weyl chamber*** of a root system (V, R) is a connected component of

$$V \setminus \bigcup_{\alpha \in R} H_\alpha,$$

where H_α is the hyperplane through the origin orthogonal to α. ♣

Remark B.36. All elements in a chamber are on the 'same side' of every hyperplane. That is to say, for a chamber C, and a hyperplane H, $\langle x, h\rangle$ is either positive or negative for every $x \in C$ and every $h \in H$. ❇

Definition B.37. If $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ is a base for R, then the ***fundamental (Weyl) chamber*** in V (with respect to Δ) is the set of all $v \in V$ such that $\langle v, \alpha_j\rangle > 0$ for all $j = 1, \ldots, l$. ♣

Proposition B.38. *A Weyl chamber of R is an open convex non-empty subset of V.* ♣

Proof. This can be shown with elementary linear algebra. ■

Theorem B.39. *Each $w \in W$ is an orthogonal linear transformation that maps R to itself and hence maps the set of hyperplanes orthogonal to the roots to itself. It then readily follows that for each Weyl chamber C, the set $w \cdot C$ is another Weyl chamber. Thus, W acts on the set of chambers. In fact, it can be shown that W acts simply transitively on the set of all chambers.* ♣

Proof. This is the content of [Hal15, §8.5]. ■

For any base Δ we have defined the fundamental Weyl chamber to be the set of those elements which have positive inner product with each element of Δ. Now we see that one can reverse this process.

Proposition B.40. *For each chamber C, there exists a unique base Δ_C for R such that C is the fundamental chamber corresponding to Δ_C. The positive roots with respect to Δ_C are precisely those elements $\alpha \in R$ such that $\langle \alpha, c\rangle > 0$ for every $c \in C$. Thus, there is a one to one correspondence between bases and Weyl chambers.* ♣

Proof. We refer to [Hal15, Proposition 8.21]. ■

The next result shows that we do not need all the reflections to generate the Weyl group. Only the reflections corresponding to simple roots are sufficient to generate the Weyl group.

Proposition B.41. *If Δ is a base, then W is generated by the reflections s_α with $\alpha \in \Delta$.* ♣

Proof. We refer to [Hal15, Proposition 8.24] ■

Example B.42. We can now easily compute the Weyl group of the root system A_2. We have computed the reflections s_α and s_β and we can see that the group generated by them is the symmetric group $\mathfrak{S}_3$ of order 3.

In fact, this holds in general. For a root system of type A_n, the Weyl group is the symmetric group $\mathfrak{S}_n$ on n elements. ♠

B.4 Highest Root

Proposition B.43. *Assume that R is irreducible. Let C be a chamber of R and let $B(C) = \{\alpha_1, \ldots, \alpha_n\}$ be the corresponding basis. Then*

(i) There exists a root $\tilde{\alpha} = \sum_{i=1}^{l} n_i \alpha_i$ such that, for every root $\sum_{i=1}^{l} p_i \alpha_i$, we have $n_1 \geq p_1, n_2 \geq p_2, \ldots, n_l \geq p_l$. In other words, R has a largest element for the ordering defined by C.

(ii) $\tilde{\alpha} \in \overline{C}$.

(iii) We have $\langle \tilde{\alpha}, \tilde{\alpha} \rangle \geq \langle \alpha, \alpha \rangle$ for every $\alpha \in R$.

♣

Definition B.44. The root

$$\tilde{\alpha} = \sum_i n_i \alpha_i$$

is called the ***highest root*** of R (with respect to the given chamber C). ♣

B.5 Weights, Radical Weights

In this section we let (V, R) be a crystallographic root system. We define a notion of integrality of elements on the vector space V.

Definition B.45. Let (V, R) be a crystallographic root system with $l = \dim V$. Denote by $Q(R)$ the subgroup of V generated by R. The elements of $Q(R)$ are called the ***radical weights*** of R. ♣

Proposition B.46. *The group $Q(R)$ is a discrete subgroup of V (as an additive group) of rank l and every basis of R is a basis of $Q(R)$. Similarly, the group $Q(R^\vee)$ is a discrete subgroup of V^* of rank l.* ♣

Proof. This follows from the fact that every root is an integer linear combination of basis elements of a base of R and they span the whole vector space V. ■

Definition B.47. The elements of the set $P(R)$ defined by

$$\begin{aligned} P(R) &:= \{ x \in V \mid \alpha^\vee(x) \in \mathbb{Z} \quad \forall \alpha \in R \} \\ &= \{ x \in V \mid \langle K_\alpha, x \rangle \in \mathbb{Z} \quad \forall \alpha \in R \} . \end{aligned}$$

are called the ***integral elements*** or ***weights*** of the root system R. Similarly, we define the set of weights $P(R^\vee)$ of the root system $R^\vee$. ♣

Definition B.48. If (V, R) is a root system with a base Δ, then an element $x \in V$ is said to be ***dominant*** (relative to Δ) if

$$\langle \alpha, x \rangle \geq 0,$$

for all $\alpha \in \Delta$; it is said to be ***strictly dominant*** if

$$\langle \alpha, x \rangle > 0,$$

for all $\alpha \in \Delta$. ♣

Proposition B.49. *Let (V, R) be a root system with base Δ. Then the set $P(R)$ is a discrete subgroup of V containing $Q(R)$. As we know, the set $\Delta^\vee$ is a basis of $R^\vee$; then the basis of V dual to $\Delta^\vee$ is a basis of $P(R)$.* ♣

Proof. This is essentially clear from the definitions. ■

Example B.50. Let us go back to our favourite example of the root system A_2. Then we have $\Delta = \{\alpha, \beta\}$ is a basis of the root system. We computed $\Delta^\vee$ in Example B.27. Now we compute the basis of V dual to $\Delta^\vee$. We have to compute the elements $x, y \in V$ such that the following are satisfied

$$\alpha^\vee(x) = \beta^\vee(y) = 1, \quad \text{and} \quad \alpha^\vee(y) = \beta^\vee(x) = 0.$$

A little computation yields

$$x = \frac{1}{2}\left(1, \sqrt{3}\right), \quad \text{and} \quad y = \left(0, \frac{1}{\sqrt{3}}\right).$$

By Proposition B.49, $\{x, y\}$ generates $P(R)$ but clearly, $x, y \notin Q(R)$. Hence, $P(R)$ can be strictly bigger than $Q(R)$. ♠

Remark B.51. As the root system is crystallographic, the basis elements are always integral, and so is their integer linear combination. However, not all integral elements arise as the integer linear combination of basis elements, as shown in Example B.50. Hence, the sets $P(R)$ and $Q(R)$ may not be equal. However, $P(R)$ is not *too* big compared to $Q(R)$ as is shown below. ❇

Proposition B.52. *The groups*

$$P(R)/Q(R), \quad P(R^\vee)/Q(R^\vee)$$

are isomorphic and finite. ♣

Proof. We refer to [Bou90, Chapter 7, §2, no. 8]. ■

Definition B.53. The common order of these groups is called the ***connection index*** of R (or of $R^\vee$). ♣

Definition B.54. Let (V, R) be a root system and let C be a chamber of R and let B be the the corresponding basis of R. Then $B^\vee = \{\alpha^\vee\}_{\alpha\in B}$ is a basis of $R^\vee$. The *dual basis* $(\overline{\omega}_\alpha)_{\alpha\in B}$ of $B^\vee$ is thus a basis of the group of weights. These are called the ***fundamental weights*** of R (relative to B or C). If the elements of B are denoted by $(\alpha_1, \ldots, \alpha_l)$, the corresponding fundamental weights are denoted by $(\overline{\omega}_1, \ldots, \overline{\omega}_l)$. A weight $\overline{\omega}$ is said to be ***dominant*** if its coordinates with respect to the basis $(\overline{\omega_\alpha})_{\alpha\in B}$ are non-negative integers. ♣

Proposition B.55. *Let (V, R) be a root system and let Δ be its basis and let $\Lambda \subseteq \Delta$ be a subset. Let W be the vector space generated by Λ and let $S = R \cap W$. Then S is a root system and Λ is its basis.* ♣

Proof. We refer to [Bou02, Chapter 6, §1, no. 7, Corollary 4]. ■

This allows us to make the following definition.

Definition B.56. Let (V, R) be a root system. Put $\Lambda \subseteq \Delta$. Let W be the subspace of V spanned by Λ and $S = R \cap W$. Then the root system (W, S) is called the ***root system generated by*** Λ. ♣

B.6 Dynkin Diagrams

A Dynkin diagram is a convenient graphical way of encoding the structure of a base of a root system (V, R), and thus also of (V, R) itself. Before we define them, we study what relationship any two roots in a reduced crystallographic root system can have.

B.6.1 Two Roots

In a crystallographic root system, (R5) imposes severe restrictions between angles of any two roots. In fact, we have the following

Proposition B.57. *Let (V, R) be a root system and let $\alpha, \beta \in R$. Let θ be the angle between them and let $\|\alpha\|$, $\|\beta\|$ be their respective lengths. Then the following are the only possibilities for θ and the ratio of lengths squared $\|\beta\|^2 / \|\alpha\|^2$:*

$\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$	$\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$	θ	$\frac{\|\beta\|^2}{\|\alpha\|^2}$
0	0	$\pi/2$	*undetermined*
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

♣

Proof. The proof is elementary. We refer to [Hum72, §9.4]. ■

Proposition B.58. *Let α and β be non-proportional roots. If angle between them is strictly acute, then $\alpha - \beta$ is a root; if angle between them is strictly obtuse, then $\alpha + \beta$ is a root.* ♣

Proof. This is a matter of calculation using the table above. We refer to [Hum72, Lemma 9.4]. ■

The following corollary will be useful for the definition of Dynkin diagrams.

Corollary B.59. *If α, β are a part of a basis for the root system R, then the angle between them cannot be strictly acute.* ♣

Proof. If the angle were strictly acute, then $\alpha - \beta$ would be a root and by definition of root system, any root must have either all positive or all negative coefficients with respect to basis elements. ■

B.6.2 Dynkin Diagrams of Crystallographic Root systems

Definition B.60. If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a base of for a root system (V, R), the ***Dynkin diagram*** for R is a graph with vertices $v_1, \ldots, v_n$. Between two distinct vertices v_i and v_j, we place zero, one, two, or three edges according to whether the angle between α_i and α_j is $\pi/2, 2\pi/3, 3\pi/4$, or $5\pi/6$. In addition, if α_i and α_j have different lengths, then we decorate the edge(s) between v_j and v_k by an arrow pointing from the vertex associated to the longer root to the vertex associated to the shorter root.

In other words, a Dynkin diagram of a root system of rank n is a directed graph of n vertices with the number and direction of edges being dictated by the root system data. ♣

Definition B.61. Two Dynkin diagrams are said to be isomorphic if there is a bijective map of the vertices that preserves the number of edges and the direction of arrow.

In other words, two Dynkin diagrams are isomorphic if they are isomorphic as directed graphs. ♣

Lemma B.62. *Dynkin diagram corresponding to two different bases of a root system are isomorphic.* ♣

Proof. This follows since any two bases are transformed into each other by a Weyl group element which preserves the angles and lengths. ∎

The Dynkin diagram allows us to recover the root system unambiguously. We have

Proposition B.63. *(a) A root system is irreducible if and only if its Dynkin diagram is connected.*

(b) Two root systems R_1 and R_2 are isomorphic if and only if their Dynkin diagrams are isomorphic.

♣

Proof. We refer to [Hal15, Proposition 8.2]. ∎

B.7 Classification

Theorem B.64. *Let R be a reduced irreducible crystallographic root system. Then it is one of the four classical root systems A_l, B_l, C_l, D_l for some l, or one of the five exceptional root systems E_6, E_7, E_8, F_4, G_2. Thus, up to* *isomorphism, these four families and the five exceptional root systems are the only reduced irreducible crystallographic root systems.* ♣

Proof. For the description of the root systems, we refer to [Hal15, §8.10] and for a proof to [Hal15, Theorem 8.49]. ■

B.8 Affine Weyl Groups

In this section, we construct the affine Weyl group of a root system, a group generated by reflections along *affine* hyperplanes. Hence, it would no longer consist of orthogonal transformations but of affine transformations.

Notation B.65. Let $(\mathbf{A}, \vec{\mathbf{A}})$ be an affine space and let $\left(\vec{\mathbf{A}}, R\right)$ be a reduced root system. For $\alpha \in R$ and $k \in \mathbb{Z}$, let $H_{\alpha,k}$ be the hyperplane of $\mathbf{A}$ defined by:

$$H_{\alpha,k} \ := \ \{\, x \in \mathbf{A} \mid \langle \alpha, x \rangle = k \,\}\,;$$

and let $s_{\alpha,k}$ be the orthogonal reflection with respect to $H_{\alpha,k}$. ♦

Lemma B.66. *We have that*

$$s_{\alpha,k}(x) = x - (\langle \alpha, x \rangle - k)\, K_\alpha = s_{\alpha,0}(x) + kK_\alpha,$$

for all $x \in \mathbf{A}$. In other words,

$$s_{\alpha,k} = T_{(kK_\alpha)} \circ s_\alpha,$$

where s_α is the orthogonal reflection with respect to the hyperplane H_α, that is, the reflection corresponding to the root α; and $T_{(kK_\alpha)}$ denotes the translation of $\mathbf{A}$ corresponding to the vector $kK_\alpha \in \vec{\mathbf{A}}$. ♣

Proof. This is an elementary calculation. ■

Definition B.67. The group of affine transformations of $\mathbf{A}$ generated by the reflections $s_{\alpha,k}$ for $\alpha \in R$ and $k \in \mathbb{Z}$ is called the ***affine Weyl group*** of the root system R and is denoted by $\mathbf{W}\,(R)$ (or simply by $\mathbf{W}$). ♣

Definition B.68. The usual Weyl group of R, generated by $s_{\alpha,0}$ for $\alpha \in R$ is also called the ***spherical Weyl group*** of the root system R and is denoted by $W\,(R)$ (or simply by W). ♣

Remark B.69. Since both W and $\mathbf{W}$ consist of linear transformations on $\vec{\mathbf{A}}$, we can consider these as subspaces of $\left(\vec{\mathbf{A}}\right)^*$. ❇

Proposition B.70. *The group* $\mathbf{W}$ *is the semi-direct product of* W *by* $Q(R)$ *(the group of radical weights).* ♣

Proof. We refer to [Bou02, Chapter 6, §2, no. 1, Proposition 1]. ■

Proposition B.71. *The group* $\mathbf{W}$ *with discrete topology, acts properly on* $\mathbf{A}$ *and permutes the hyperplanes* $H_{\alpha,k}$ *(for* $\alpha \in R$ *and* $k \in \mathbb{Z}$*).* ♣

Proof. We refer to [Bou02, Chapter 6, §2, no. 1, Proposition 2]. ■

We can thus apply the results Appendix A. Hence we can talk about special points with respect to $\mathbf{W}$ and hope to compute them in terms of the root system. Indeed, we have:

Proposition B.72. *The special points of* $\mathbf{W}$ *are the weights of* $R^\vee$*.* ♣

Proof. We refer to [Bou02, Chapter 6, §2, no. 2, Proposition 3]. ■

Thus, for a given root system, we have seen two Weyl groups — the spherical and the affine. Corresponding to both these are the hyperplane arrangements given by zero sets of roots (and their translates in affine case). Now we relate the chambers corresponding to these two arrangements. First we make a

Definition B.73. Let R be a root system and $\mathbf{W}$ be the affine Weyl group associated to it. Then the chambers (as defined in Definition A.32) corresponding to $\mathbf{W}$ are called ***alcoves***. ♣

Remark B.74. This definition, although standard in literature, seems superfluous as it is just renaming the chambers in a special case. However, it saves us the confusion when we need to talk about chambers from spherical Weyl group and affine Weyl group simultaneously. In the rest of this section, a chambers refers exclusively to the chamber of the spherical Weyl group. ❇

Proposition B.75. *Let* D *be a chamber of the root system* $R^\vee$*.*

(i) There exists a unique alcove C *contained in* D *such that* $0 \in \overline{C}$*.*

(ii) The union of the $w\left(\overline{C}\right)$ *for* $w \in W$ *is a neighbourhood of* 0 *in* $\mathbf{A}$*.*

(iii) Every wall of D *is a wall of* C*.*

♣

This is intuitively clear if one imagines the alcoves being the 'refinement' of chambers with being cut by extra hyperplanes present in the affine case.

Notation B.76. Now let Δ be a basis of R and let $\tilde{\alpha}$ be the highest root of R. ♦

Proposition B.77. *Let C be the alcove containing 0 in its closure and contained in a chamber D. Then C is the set*

$$C = \{\, x \in \mathbf{A} \mid \langle \alpha, x \rangle > 0 \quad \forall \alpha \in \Delta \quad \text{and } \langle \tilde{\alpha}, x \rangle < 1 \,\} \,.$$

♣

Proof. We refer to [Bou02, Chapter 6, §2, no. 2, Proposition 4]. ■

We have seen that root systems give rise to affine Weyl groups. Under certain circumstances we can reverse this process and obtain a root system from a given hyperplane arrangement. We will not need to go into this. For interested reader, we refer to We refer to [Bou02, Chapter 6, §2, no. 5, Proposition 8].

Appendix C

Coxeter Systems

C.1 Definitions and First Properties

Definition C.1. Let S be a *finite* set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ be a map such that $m(s, s) = 1$ for all $s \in S$ and $m(s, t) = m(t, s)$ for all $s, t \in S$. A ***Coxeter system*** is a pair (W, S) where S is a set of generators for a group W, and W has a presentation:

$$\begin{aligned} s^2 &= 1 \qquad \forall\, s \in S, \\ (st)^{m(s,t)} &= 1 \qquad \forall\, s, t \in S. \end{aligned}$$

We may refer to the function m as the ***Coxeter data***. By abuse of notation, we would also say that W is a ***Coxeter group***. The cardinality of the set S is called the ***rank*** of the Coxeter system (W, S). ♣

Definition C.2. Let $w \in W$. The ***length*** of w (with respect to S), denoted by $l_S(w)$ or simply by $l(w)$ is the smallest integer $q \geq 0$ such that w is the product of sequence of q elements of S. ♣

Definition C.3. Let $w \in W$. A ***reduced decomposition*** of w (with respect to S) is any sequence $\mathbf{s} = (s_1, \ldots, s_n)$ of elements of S such that $w = s_1 \cdots s_n$ and $n = l(w)$. ♣

Definition C.4. A ***Coxeter graph*** or a ***Coxeter diagram*** is a schematic diagram used to keep track of numbers $m(s, t)$. For each $s \in S$, we make a 'dot' and we connect the $s-$dot with a $t-$dot by a line if $m(s, t) > 2$ and label the line by $m(s, t)$. If $m(s, t) = 3$, we skip the label altogether (because this is the most common value for $m(s, t)$ if it is greater than 2). ♣

Definition C.5. A Coxeter graph is ***connected*** if for every $s, t \in S$, there is a sequence

$$s = s_1, s_2, \ldots, s_n = t$$

such that $m(s_i, s_{i+1}) > 2$. That is, the diagram is connected if every 'dot' can be reached from every other 'dot' by a finite number of lines. ♣

Remark C.6. Note that $m(s,t) \geq 2$ since for $s \neq t, st \neq 1$. Also, $m(s,t) = 2$ is equivalent to saying that s and t commute. ❄

Definition C.7. A Coxeter system (W, S) is called ***irreducible*** or ***indecomposable*** if its Coxeter diagram is connected and non-empty. Equivalently, S is non-empty and there exists no partition of S into two distinct subsets S_1 and S_2 such that every element of S_1 commutes with every element of S_2. ♣

C.2 Length Function

We investigate the length function of a Coxeter group more thoroughly. First of all, we have

Proposition C.8. *Let (W, S) be a Coxeter system. There is a unique epimorphism $\epsilon : W \to \{1, -1\}$ sending each generator $s \in S$ to -1. In particular, every generator $s \in S$ has order 2.* ♣

Proof. Let F be the free group generated by S. Define $\delta : F \to \{1, -1\}$ by sending each $s \in S$ to -1 and then extending by group operations. Each of the elements $(st)^{m(s,t)}$ lie in the kernel and hence the homomorphism δ descends to the quotient W, preserving the images of every $s \in S$. ■

We know that the length of an element w is defined as the minimum number of 'generating' elements needed to express w (also called its reduced decomposition). However, since w may have many reduced decompositions, the length functions has its subtleties. We collect the elementary properties of the length function below.

Proposition C.9. *Let (W, S) be a Coxeter system and let $l : W \to \mathbb{N}$ be the length function. Then*

(L1) $l(w) = l\left(w^{-1}\right)$.

(L2) $l(w) = 1$ *if and only if* $w \in S$.

(L3) $l(ww') \leq l(w) + l(w')$.

(L4) $l(ww') \geq l(w) - l(w')$.

(L5) $l(w) - 1 \leq l(ws) \leq l(w) + 1$, *for* $w \in W$ *and* $s \in S$.

♣

Proof. The proof is elementary. We refer to [Hum90, §5.2]. ■

Proposition C.10. *The homomorphism* ϵ *of Proposition C.8 is given by* $\epsilon(w) = (-1)^{l(w)}$. *As a result,* $l(ws) = l(w) \pm 1$ *for all* $w \in W$ *and* $s \in S$, *and similarly for* $l(sw)$. ♣

Proof. Let $w = s_1 \cdots s_r$ be a reduced expression. Then

$$\epsilon(w) = \epsilon(s_1) \cdots \epsilon(s_r) = (-1)^r = (-1)^{l(w)},$$

as required. Now $\epsilon(ws) = -\epsilon(w)$ implies that $l(ws) \neq l(w)$ and now by (L5) above, the lengths must differ by precisely 1. ■

C.3 Geometric Representation of Coxeter Systems

Coxeter groups are defined as abstract groups generated by involutions. In this section, we give a geometric representation of these groups viewing them as subgroups of $\boldsymbol{GL}(V)$ for a suitable vector space V.

C.3.1 The Coxeter Form

Definition C.11. Let (W, S) be a Coxeter system with rank n. Let V be a $n-$dimensional vector space over $\mathbb{R}$ with basis $(e_s)_{s\in S}$. Define a symmetric bilinear form on V by

$$B(e_s, e_t) \ := \ -\cos(\pi/m_{s,t}),$$

if $m_{s,t} < \infty$ and for $m_{s,t} = \infty$, it is taken to be -1; and then extend linearly to all of V. This is called the ***Coxeter form*** of the system. ♣

Remark C.12. One sees immediately that $B(e_s, e_s) = 1$ and $B(e_s, e_t) \leq 0$ for $s \neq t$. ❇

Remark C.13. We remark that this form may not be positive definite and hence *does not* in general make V into an inner product space. ❇

Remark C.14. We also remark that since e_s is non-isotropic, the subspace H_s orthogonal to e_s under the form B is complementary to the line $\mathbb{R}e_s$. ❇

Definition C.15. For each $s \in S$, we now define a ***reflection*** $\sigma_s : V \to V$ by

$$\sigma_s(v) \;:=\; v - 2B(e_s, v)e_s, \qquad v \in V.$$

Clearly, $\sigma_s e_s = -e_s$, while σ_s fixes H_s point-wise. ♣

Lemma C.16. *Each reflection preserves the Coxeter form B. That is*

$$B(\sigma_s v, \sigma_s w) = B(v, w),$$

for all $v, w \in V$ and for all $s \in S$.

Hence, the subgroup of $\boldsymbol{GL}(V)$ generated by these reflections also preserves the Coxeter form. ♣

Proof. This is an elementary calculation. ■

Proposition C.17. *There is a faithful homomorphism $\sigma : W \to \boldsymbol{GL}(V)$ sending s to σ_s and the group $\sigma(W)$ preserves the Coxeter form B on V.* ♣

Proof. We refer to [Hum90, Corollary §5.4]. ■

Definition C.18. This homomorphism will be called the ***geometric representation*** of the Coxeter group W. ♣

Definition C.19. A Coxeter system is called a ***spherical Coxeter system*** if its Coxeter form is *positive definite* and non-degenerate. It is said to be an ***affine Coxeter system*** if the Coxeter form is *positive semi-definite* but *not positive definite*. ♣

Remark C.20. Thus, these are two disjoint classes; they are however, not mutually exhaustive. That is, there are Coxeter systems that fall in neither of these classes (cf. [Hum90, §6.8]). ❇

Theorem C.21. *A Coxeter system is spherical if and only if the group W is finite.* ♣

Proof. We refer to [Kan01, Theorem 7-1]. ■

C.4 Roots

In this section, we obtain a precise criterion for $l(ws)$ to be greater or smaller than $l(w)$, in terms of action of W on V. This will be crucial in studying the combinatorial properties of W in terms of the set S.

Notation C.22. In this section, we write $w(e_s)$ instead of the more precise $\sigma(w)(e_s)$. ♦

Definition C.23. Let (W, S) be a Coxeter system and let $\sigma : W \to \boldsymbol{GL}(V)$ be its geometric representation. Then the set

$$\Phi \ := \ \{\, w(e_s) \mid w \in W, s \in S \,\}\,,$$

is called the ***set of roots*** of the Coxeter system (W, S). ♣

Remark C.24. The root system consists of unit vectors because W preserves the lengths. Also, one notes that $\Phi = -\Phi$. ❇

Definition C.25. Since e_s are the basis for V, any root $\alpha \in \Phi$ can be written uniquely as

$$\alpha = \sum_{s \in S} c_s e_s,$$

for a set of real numbers $(c_s)_{s \in S}$. A root is called a ***positive root*** if $c_s > 0$ for all $s \in S$. A ***negative root*** is similarly defined. ♣

The following Theorem is fundamental, in that it establishes the relationship between the length function and action on V.

Theorem C.26. *Let $w \in W$ and $s \in S$. If $l(ws) > l(w)$, then $w(e_s) > 0$. If $l(ws) < l(w)$, then $w(e_s) < 0$.* ♣

Proof. We refer to [Hum90, Theorem §5.3]. ■

Corollary C.27. *The set of roots Φ is a disjoint union of the positive and negative roots. That is,*

$$\Phi = \Phi^+ \bigsqcup \Phi^-.$$

♣

Proof. Every root is given as $w(e_s)$ and it is positive if $l(ws) > l(w)$ and negative otherwise. ■

Proposition C.28. *Let (W, S) be a Coxeter system and $\sigma : W \to \boldsymbol{GL}(V)$ be its geometric representation. Let Φ be the associated root system and Φ^+ be the set of positive roots. Then*

(a) If $s \in S$, then $s(e_s) = -e_s$ and it permutes all the other positive roots.

(b) For any $w \in W$, the length $l(w)$ of w is equal to the number of positive roots sent by w to negative roots.

♣

Proof. We refer to [Hum90, Proposition §5.6]. ■

This immediately leads to

Proposition C.29. *If the Coxeter group W is finite, then there exists a unique element $w_0 \in W$ of maximum length. This maximum length is equal to the number of positive roots; and w_0 sends every positive root to a negative root.* ♣

Proof. We refer to [Gar97, Corollary Page 10]. ■

Definition C.30. This element of maximal length is called the ***longest Weyl group element***. ♣

C.5 Special Subgroups

In this section, we study *special subgroups* of Coxeter groups. We begin with:

Theorem C.31. *Let (W, S) be a Coxeter system. Then*

(a) For every $T \subseteq S$, $(\langle T\rangle, T)$ is a Coxeter system.

(b) $\langle T\rangle \to T$ is a an inclusion preserving bijection

$$\{\langle T\rangle : T \subseteq S\} \to \{T \subseteq S\}$$

That is, no two subsets of S generate the same subgroup of W.

(c) For $T_1, T_2 \subseteq S$, we have

$$\langle T_1 \cap T_2\rangle = \langle T_1\rangle \cap \langle T_2\rangle$$

(d) The set S is a minimal generating set for the group W.

♣

Proof. We refer to [Gar97, Proposition §1.9]. ■

Definition C.32. These subgroups of W of the type $\langle T\rangle$ for $T \subseteq S$ are called ***special subgroups*** of W. Occasionally, we will write $W_T = \langle T\rangle \subseteq W$ for a subset $T \subseteq S$. ♣

Definition C.33. Let Γ is the Coxeter diagram of the Coxeter system (W, S). If $(\Gamma_i)_{i\in I}$ is the family of connected components of Γ, let S_i be the set of vertices of Γ_i and let $W_i := W_{S_i}$ be the subgroup of W generated by S_i. Then the Coxeter systems (W_i, S_i) are irreducible and are called the ***irreducible components*** of (W, S). Moreover, the group W is the *restricted direct product* of the subgroups W_i for $i \in I$. ♣

C.6 Seven Important Families of Coxeter Systems

Among all possible Coxeter systems (W, S), (there is a complete classification), there are seven infinite families that will be of special importance to us. They fall into two classes, *spherical Coxeter systems* and *affine Coxeter systems*, as define above. The Coxeter diagrams of these seven systems are given in Figure C.1 on Page 131.

C.6.1 Three Spherical Families

We will name and give the Coxeter data of the three spherical families.

The Family A_n: This is the single most important family. A Coxeter system (W, S) is said to be of ***type*** A_n when $S = \{s_1, \ldots, s_n\}$ and $m(s_i, s_{i+1}) = 3$ and otherwise the generators commute. It turns out that the Coxeter group W of type A_n is isomorphic to $\mathfrak{S}_{n+1}$. It will later appear in the study of spherical building attached to $\boldsymbol{GL}_{n+1}$.

The Family C_n: A Coxeter system (W, S) is said to be of ***type*** C_n when $S = \{s_1, \ldots, s_n\}$ with the data

$$3 = m(s_1, s_2) = m(s_2, s_3) = \cdots = m(s_{n-2}, s_{n-1})$$

while

$$4 = m(s_{n-1}, s_n)$$

and the generators commute otherwise. The Coxeter group W of type C_n appears in the study of spherical buildings attached to the symplectic group $\boldsymbol{Sp}_n$, as well as the spherical buildings of other isometry groups with the exception of certain orthogonal groups $\boldsymbol{O}_{n,n}$.

The Family D_n: A Coxeter system (W, S) is said to be of ***type*** D_n when $S = \{s_1, \ldots, s_n\}$ with the data

$$3 = m(s_1, s_2) = m(s_2, s_3) = \cdots = m(s_{n-2}, s_{n-1})$$

and

$$3 = m(s_{n-2}, s_n)$$

and all other pairs commute. Thus, unlike A_n and C_n, the element s_{n-2} has non-trivial relation with *three* other generators and hence the Coxeter diagram has a *branch*. This group occurs in the study of spherical buildings attached to some orthogonal groups.

C.6.2 Four affine Families

We will name and give the Coxeter data of the four affine families.

The family $\tilde{A_n}$: A Coxeter system (W, S) is said to be of ***type*** $\tilde{A_n}$ when $S = \{s_1, \ldots, s_{n+1}\}$ with the data

$$3 = m(s_1, s_2) = m(s_2, s_3) = \cdots = m(s_n, s_{n+1}) = m(s_n, s_1)$$

and all other pairs commute. The last relation forces the diagram to be a closed polygon and hence none of the generators can be distinguished in anyway. They are all on the same footing. This system appears in the study of *affine* building of $\boldsymbol{SL}_n$over a $p-$adic field.

The family $\tilde{C_n}$: A Coxeter system (W, S) is said to be of ***type*** $\tilde{C_n}$ when $S = \{s_1, \ldots, s_{n+1}\}$ with the data

$$3 = m(s_2, s_3) = m(s_3, s_4) = \cdots = m(s_{n-1}, s_n)$$

and

$$4 = m(s_1, s_2) = m(s_n, s_{n+1})$$

and all other pairs commute. This appears in the study of affine building of $\boldsymbol{Sp}_n$over a $p-$adic field.

The family $\tilde{B_n}$: We define $\tilde{B_2}$ to be $\tilde{C_2}$. Now let $n > 2$. A Coxeter system (W, S) is said to be of ***type*** $\tilde{B_n}$ when $S = \{s_1, \ldots, s_{n+1}\}$ with the data

$$3 = m(s_1, s_3) = m(s_2, s_3) = m(s_3, s_4) = \cdots = m(s_{n-1}, s_n)$$

and

$$4 = m(s_n, s_{n+1})$$

and all other pairs commute.

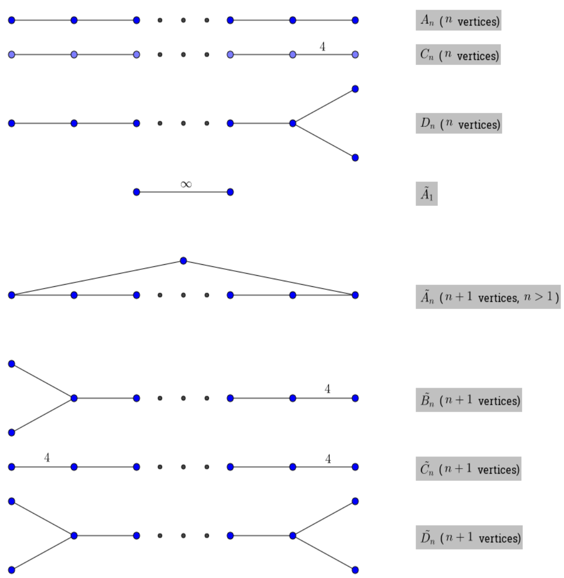

Figure C.1: The seven infinite families of Coxeter diagrams

The family $\tilde{D}_n$: Let $n \geq 4$. A Coxeter system (W, S) is said to be of ***type*** $\tilde{D}_n$ when $S = \{s_1, \ldots, s_{n+1}\}$ with the data

$$3 = m(s_1, s_3) = m(s_2, s_3) = m(s_3, s_4) = \cdots = m(s_{n-1}, s_n) = m(s_{n-1}, s_{n+1})$$

and all other pairs commute.

Appendix D

Tits System

D.1 Definitions and Notation

Definition D.1. Let G be an (abstract) group and B be a subgroup of G. The group $B \times B$ acts on G by $(b_1, b_2) \cdot g = b_1 g b_2^{-1}$ for $g \in G$ and $b_1, b_2 \in B$. The orbits of $B \times B$ on G are the sets BgB for $g \in G$, and are called the ***double cosets*** of G with respect to B. They form a *partition* of G and the corresponding quotient is denoted by $B\backslash G/B$. If C_1, C_2 are double cosets, $C_1 C_2$ is a union of double cosets. ♣

Definition D.2. A ***Tits system*** or a ***BN pair*** is a quadruple (G, B, N, S), where G is a group, B and N are subgroups of G and S is a subset of $N/(B \cap N)$, satisfying the following axioms:

(T1) The set $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.

(T2) The set S generates the group $W := N/(B \cap N)$ and consists of involutions (that is, elements of order 2).

(T3) $sBw \subseteq BwB \cup BswB$ for $s \in S$ and $w \in W$.

(T4) For all $s \in S, sBs \not\subseteq B$.

The group $W := N/(B \cap N)$ is called the ***Weyl group*** of the Tits system (G, B, N, S). ♣

Remark D.3. Every element of W is a coset modulo $B \cap N$, and thus is a subset of G; hence the products such as BwB make sense. More generally, for any subset $A \subseteq W$, we denote by BAB the subset $\cup_{w \in A} BwB$. ❇

Notation D.4. Throughout this Appendix, with (G, B, N, S) denoting a Tits system, we set $T := B \cap N$ and $W = N/T$. A double coset means a double coset of G with respect to B. For any $w \in W$, we set $C(w) := BwB$; this is a double coset. ♦

Example D.5. ([Bou02, Chapter 4, §2, n. 2]). Let k be a field, $n \in \mathbb{N}$ and (e_i) the canonical basis of k^n. Let $G = \boldsymbol{GL}_n(k)$, let B be the group of upper triangular matrices, and let N be the subgroup of G consisting of matrices having exactly one non-zero element in each row and column. An element of N permutes the lines ke_i; this gives rise to a surjective homomorphism $N \to \mathfrak{S}_n$ whose kernel is the subgroup $T = B \cap N$ of diagonal matrices, and allows us to identify the Weyl group $W = N/T$ with $\mathfrak{S}_n$. We denote by s_j $(1 \leq j \leq n-1)$ the element of W corresponding to the transposition of j and $j+1$; let S be the set of s_j. Then the quadruple (G, B, N, S) is a *spherical* Tits system. ♠

D.2 Bruhat Decomposition

Theorem D.6. *Let (G, B, N, S) be a Tits system. Then we have $G = BWB$. The map $w \mapsto C(w)$ is a bijection from W to the set $B\backslash G/B$ of double cosets of G with respect to B.* ♣

Proof. We refer to [Bou02, Chapter 4, §2, n. 3 Theorem 1]. ∎

What this really means is that G can be decomposed into 'cells' of the form BwB, which are *some of the* double cosets of G with respect to B. Clearly, *all* of the double cosets cover G so the main content of the theorem is that W is exactly the set of representatives of action of $B \times B$ on G.

D.3 Relations with Coxeter Systems

Theorem D.7. *The pair (W, S) is a Coxeter system. Moreover, for $s \in S$ and $w \in W$, the relations $C(sw) = C(s) \cdot C(w)$ and $l_S(sw) > l_S(w)$ are equivalent.* ♣

Definition D.8. A Tits system (G, B, N, S) is said to be an ***affine Tits system*** if its Weyl group W is an affine Weyl group (in the sense of Definition C.19). It is said to be a ***spherical Tits system*** if its Weyl group is a spherical Weyl group (again in the sense of Definition C.19). ♣

D.4 Subgroups Containing B

For any subset $X \subseteq S$, we denote by W_X the subgroup of W generated by X and by G_X the union BW_XB of the double cosets $C(w), w \in W_X$. We have $G_\emptyset = B$ and $G_S = G$ (the latter is the Bruhat decomposition, Theorem D.6).

Theorem D.9. *(a) For any subset X of S, the set G_X is a subgroup of G, generated by $\cup_{s\in X} C(s)$.*

(b) The map $X \mapsto G_X$ is a bijection from $\mathfrak{P}(S)$ to the set of subgroups of G containing B.

(c) Let $(X_i)_{i\in I}$ be a family of subsets of X. If $X = \cap_{i\in I} X_i$, then $G_X = \cap_{i\in I} G_{X_i}$.

(d) Let X and Y be two subsets of S. Then $G_X \subseteq G_Y$ (respectively. $G_X = G_Y$) if and only if $X \subseteq Y$ (respectively. $X = Y$).

♣

Proof. We refer to [Bou02, Chapter 4, §2, no. 5 Theorem 3] ■

Lemma D.10. *The set S consists of the elements $w \in W$ such that $w \neq 1$ and $B \cup C(w)$ is a subgroup of G.* ♣

Proof. We refer to [Bou02, Chapter 4, §2, no. 5 Corollary Theorem 3] ■

Remark D.11. The above lemma shows that the set S is completely determined by (G, B, N); thus we sometimes say that (G, B, N) is a Tits system or that (B, N) is a Tits system in G. ❇

D.5 Parabolic Subgroups

Definition D.12. A subgroup of G is called a ***parabolic*** if it contains a conjugate of B. ♣

Remark D.13. Clearly, any conjugate of B is a parabolic subgroup; they are in fact the minimal parabolic subgroups. Also, any subgroup containing a parabolic is itself parabolic. ❇

Proposition D.14. *Let P be a subgroup of G.*

(a). P is parabolic if and only if there exists a subset X of S such that P is conjugate to G_X.

(b). Let X_1, X_2 be subsets of S and $g_1, g_2 \in G$ be such that $P = g_1 G_{X_1} g_1^{-1} = g_2 G_{X_2} g_2^{-1}$. Then, $X_1 = X_2$ and $g_2 g_1^{-1} \in P$.

♣

Proof. We refer to [Bou02, Chapter 4, §2, no. 6 Proposition 4] ■

Definition D.15. If the parabolic subgroup P is conjugate to G_X, where $X \subseteq S$, then P is said to be of ***type*** X. Then, B (and its conjugates) are of type $\emptyset$ and G is of type S. ♣

Theorem D.16. *(a) Let P_1 and P_2 be two parabolic subgroups of G whose intersection is parabolic and let $g \in G$ be such that $gP_1g^{-1} \subseteq P_2$. Then $g \in P_2$ and $P_1 \subseteq P_2$.*

(b) Two parabolic subgroups whose intersection is parabolic are not conjugate.

(c) Let Q_1 and Q_2 be two parabolic subgroups of G contained in a subgroup Q of G. Then any $g \in G$ such that $gQ_1g^{-1} = Q_2$ belongs to Q.

(d) Every parabolic subgroup is its own normalizer.

♣

Proof. We refer to [Bou02, Chapter 4, §2, no. 6 Theorem 4] ■

Proposition D.17. *Let P_1 and P_2 be two parabolic subgroups of G. Then $P_1 \cap P_2$ contains a conjugate of T.* ♣

Proof. We refer to [Bou02, Chapter 4, §2, no. 6 Proposition 5] ■

D.6 Affine Tits System

In this section we study the affine Tits system more closely. Let G be an abstract group with an affine Tits system (G, B, N, S).

Definition D.18. The subgroup B or any of its conjugates is called an ***Iwahori subgroup*** of G. The subgroup B itself is called a ***standard Iwahori subgroup***. ♣

Definition D.19. A ***parahoric subgroup*** is any *proper* subgroup of G containing an Iwahori subgroup. A parahoric subgroup containing the standard Iwahori is called a ***standard parahoric subgroup***. ♣

Remark D.20. Note that we previously defined any subgroup containing a conjugate of B to be parabolic subgroup. Thus, a parahoric subgroup is just another name for a *proper* parabolic subgroup in case the Tits system is affine. ❇

The following Theorem is what makes the affine Tits system so useful in practice. It allows us to view the building attached to them (see Appendix E.5) as nothing more than some affine spaces glued together, which in turn are tessellated by some hyperplanes. This puts us back in the situation of Appendix A.

Theorem D.21. *Let (G, B, N, S) be an affine Tits system with Weyl group W. Then there exists an affine space $\mathbf{A}$ and a reduced irreducible root system $(\vec{\mathbf{A}}, R)$ such that W is isomorphic to the affine Weyl group $\mathbf{W}$ associated to the root system R. In fact there is a homomorphism $\nu : N \to \mathbf{W}$ such that $\ker \nu = T$. Moreover, the set S corresponds to the set $\mathbf{S}$ of reflections along the walls of a chamber $\mathbf{C}$.* ♣

Proof. This is just a restatement of previous results. ■

Corollary D.22. *There is a bijection between $\mathfrak{P}(S)$ and the set of facets of the chamber $\mathbf{C}$.* ♣

Proof. We know that the set of facets of the chamber $\mathbf{C}$ are in bijection with the set $\mathfrak{P}(\mathbf{S})$ which is again in bijection with the set $\mathfrak{P}(S)$, since by Theorem D.21, the sets S and $\mathbf{S}$ are in bijection with each other. ■

Theorem D.23. *The set of standard parahoric subgroups and facets of the chamber $\mathbf{C}$ are in one to one correspondence. For a subset X of S, write $\tilde{X}$ for the subset of $\mathbf{S}$ corresponding to it. If P is the standard parahoric subgroup G_X, then it corresponds to a facet F_P of $\mathbf{C}$, where F_P is defined by the reflections corresponding to the subset $\tilde{X}$.* ♣

Proof. This follows directly from Corollary D.22 and Theorem D.9 (b). ■

Appendix E

Buildings

E.1 Simplicial Complexes

Definition E.1. Let V be a set and X a subset of set of all finite subsets of V such that if $x \in X$ and $y \subseteq x$, then $y \in X$. Also, assume that for every $v \in V$, the singleton set $\{v\} \in X$. Then we say that X is a ***simplicial complex*** with vertex set V, and the elements $x \in X$ are called the ***simplices*** in X. ♣

Example E.2. Let V be the set $\mathbb{Z}$ of integers. Let X be the subset of $\mathfrak{P}(\mathbb{Z})$ consisting of all sets of cardinality 3 or lower. That is, X contains the singletons, sets with two elements and sets with three elements. We shall set up some convenient notation (not to be confused with any other standard notation):

$$\begin{aligned} A_i &:= \{i\}, \qquad i \in \mathbb{Z} \\ B_{i,j} &:= \{i,j\}, \qquad i,j \in \mathbb{Z} \\ C_{i,j,k} &:= \{i,j,k\} \qquad i,j,k \in \mathbb{Z}. \end{aligned}$$

Thus, $X = \{ A_i, B_{i,j}, C_{i,j,k} \mid i,j,k \in \mathbb{Z} \}$. To avoid confusion, we will denote this simplicial complex by $X_{\mathbb{Z}}$. ♠

Definition E.3. Let X be a simplicial complex and let $x \in X$ be a simplex. Then the ***dimension*** of x is defined to be one less than the cardinality of x. ♣

Example. In the example above, simplices A_i have dimension 0, $B_{i,j}$ have dimension 1 and $C_{i,j,k}$ have dimension 2. ♠

Definition E.4. If $y \subseteq x \in X$, then y is called a ***facet*** of x. The cardinality of the set $x \backslash y$ is called the ***codimension*** of y in x. If y is of codimension 1 in x, it is called a ***face*** of x. ♣

Example. In our example above, A_i is a facet of $C_{i,j,k}$ of codimension 2 while $B_{i,j}$ is a face. ♠

Definition E.5. Two simplices $x, y \in X$ are called ***adjacent*** if they have a common face (and not just a common facet). ♣

Example. In our example, simplices $C_{i,j,k}$ and $C_{i,j,l}$ are adjacent while $C_{i,j,k}$ and $C_{i,l,m}$ are not, provided $\{l, m\} \cap \{j, k\} = \emptyset$. ♠

Definition E.6. A simplex x is called ***maximal*** if it is not a facet of any other simplex. Clearly, every simplex is contained in a maximal simplex. ♣

Example. As should be clear by now, the simplices $C_{i,j,k}$ are maximal in our example. ♠

Definition E.7. A simplicial complex is called a ***chamber complex*** if for all maximal simplices $x, y \in X$, there exist maximal simplices $x_0, \ldots, x_n$ such that $x_0 = x, x_n = y$ and x_i is adjacent to x_{i+1} for all indices i. In this case, maximal simplices are called ***chambers*** and the sequence connecting the chambers is called a ***gallery***. ♣

Example. A moment's reflection shows that our example is indeed a chamber complex. ♠

Definition E.8. A chamber complex is called ***thin*** if each face is a face of exactly two chambers. It is called ***thick*** if every face is a face of at least three chambers. ♣

Example. Our example chamber complex is thick as $B_{i,j}$ is a face of $C_{i,j,k}$ for every $k \in \mathbb{Z}$ and there are more than 3 elements in $\mathbb{Z}$. ♠

Remark E.9. Note that any two adjacent simplices have the same dimension and hence in a chamber complex any two chambers have the same dimension. Further, the notions of thinness and thickness are not exhaustive. That is to say, there are chamber complexes that are neither thin nor thick. ❇

Definition E.10. A ***simplicial subcomplex*** of a simplicial complex X is a subset Y of X which is a simplicial complex 'in its own right', that is, with face relations from X. A ***chamber subcomplex*** is a simplicial subcomplex which is a chamber complex, and so that the chambers in the subcomplex were maximal simplices in the original complex. ♣

Example E.11. Now let $X_{\mathbb{N}}$ be the simplicial complex obtained in the similar manner as $X_{\mathbb{Z}}$ above but with $V = \mathbb{N}$ instead of $\mathbb{Z}$. Then one sees readily that $X_{\mathbb{N}}$ is a chamber subcomplex of $X_{\mathbb{Z}}$. ♠

Definition E.12. A ***morphism of simplicial complexes*** or a ***simplicial complex map*** $f : X \to Y$ is a set map on the set of vertices so that if x is simplex in X then $f(x)$ is a simplex in Y. The notion of isomorphism is the usual one as in any category. ♣

Definition E.13. A ***morphism of chamber complexes*** or a ***chamber complex map*** is a simplicial complex map from one *chamber complex* to another which sends chambers to chambers and preserves dimensions of simplices. ♣

Example. The absolute value map from $\mathbb{Z}$ to $\mathbb{N}$ gives a simplicial complex map between the corresponding simplicial complexes. It is even a chamber complex map. ♠

Definition E.14. A simplicial complex gives rise to a poset in a natural manner. The elements of the poset are the simplices and $x \leq y$ means that x is a facet of y, that is, $x \subseteq y$. However, not every poset is identifiable as that arising from a simplicial complex. We call a poset ***simplex-like*** if it is isomorphic to the poset of all non-empty subsets of some *non-empty finite* set, with inclusion as the order relation. ♣

Example. Let $V = \{1, 2, 3\}$ and let L be the poset consisting of all non-empty subsets of V. Then clearly, L is a simplex-like poset. ♠

Definition E.15. A ***labelling*** or ***typing*** of a poset P is a poset map $\lambda : P \to L$ from P to a simplex-like poset L (the ***labels*** or ***types***), so that $x \leq y$ in P implies $\lambda x \leq \lambda y$ in L. We say that a simplicial complex is ***labellable*** or ***typeable*** if the associated poset has a labelling. ♣

Example. For the simplicial complex, $X_{\mathbb{Z}}$ we define a map $\lambda : X_{\mathbb{Z}} \to L$ by:

$$\lambda(A_i) = \{1\}\,,$$

$$\lambda(B_{i,j}) = \{1, 2\}\,,$$

$$\lambda(C_{i,j,k}) = \{1, 2, 3\}\,.$$

for all $i, j, k \in \mathbb{Z}$. Clearly, this is a typing of $X_{\mathbb{Z}}$. ♠

We look at another example before proceeding further.

Example E.16. Let V be a set with a symmetric and reflexive relation $\sim$, an ***incidence relation***. Then define the ***flag complex*** X by taking the vertex set to be V itself, and the simplices to be subsets $x \subseteq V$ so that $j \sim k$ for all $j, k \in x$. That is, simplices are sets of *mutually incident* elements of V.

It is easy to check that this yields a simplicial complex. Some additional conditions would need to be imposed to ensure that the flag complex is a chamber complex. ♠

Example E.17. As a particular example of flag complexes, we consider flags of subspaces in a vector space. Let W be a finite dimensional vector space, of dimension n, say, and V be the set of subspaces of W. Define a relation $\sim$ on V as follows: for $x, y \in V$, $x \sim y$ iff either $x \subseteq y$ or $y \subseteq x$. Clearly, this is an incidence relation and then we form the flag complex as in the example above. In this situation, simplices are also called ***flags***. A flag would be a set of subspaces $\{W_1, \ldots, W_m\}$ with $W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_m$. The maximal simplices are the maximal flags, which are sets of the of subspaces $\{W_0, \ldots, W_n\}$ such that $0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n = W$ with $\dim(W_i) = i$.

This simplicial complex can be given a labelling as follows. First, let L be the poset of all non-empty subsets of the set $\{1, \ldots, n\}$. Now for a flag $\{W_1, \ldots, W_m\}$, define

$$\lambda(\{W_1, \ldots, W_m\}) \;:=\; \{\, \dim W_i \mid 1 \leq i \leq m \,\}\,.$$

That is, λ records the dimensions of subspaces appearing in the flag. ♠

E.2 Coxeter Complex

In this section, we define the Coxeter complex, a particularly important type of simplicial complex that is crucial to the theory of buildings.

Definition E.18. Let (W, S) be a Coxeter system with S *finite*. Let P be the poset of all subsets of W with *inclusion reversed*. The ***Coxeter poset*** associated to (W, S) is the sub-poset of P consisting of sets of the form $w\langle T\rangle$ for a *proper* (possibly empty) subset T of S and $w \in W$.

The associated ***Coxeter complex*** $\Sigma = \Sigma(W, S)$ is defined to be the simplicial complex associated to the Coxeter poset of (W, S). That is, $\Sigma(W, S)$ has simplices which are cosets in W of the form $w\langle T\rangle$ for a *proper* (possibly empty) subset T of S, with face relations *opposite* of subset inclusion in W. ♣

Thus, maximal simplices are of the form $w\langle\emptyset\rangle = \{w\}$ for $w \in W$, and the next-to-maximal simplices are of the form $w\langle s\rangle = \{w, ws\}$ for $s \in S$ and $w \in W$.

Since $\Sigma(W, S)$ is constructed as a collection of cosets $w\langle T\rangle$, there is a natural action of W on $\Sigma(W, S)$ by left multiplication.

Remark E.19. One has to verify that this indeed defines a simplicial complex. This, though not difficult is not entirely trivial and is done in [Gar97, §4.3]. ❇

We have the following theorem about Coxeter complexes ([Gar97, §4.3]):

Theorem E.20.

(a) A Coxeter complex $\Sigma(W, S)$ is a uniquely labellable thin chamber complex.

(b) The group W acts on $\Sigma(W, S)$ by type-preserving automorphisms.

(c) The group W acts transitively on the collection of simplices of a given type.

(d) The isotropy group in W of the simplex $w\langle T\rangle$ is $w\langle T\rangle w^{-1}$.

♣

Definition E.21. A Coxeter complex is said to be a ***spherical Coxeter complex*** or an ***affine Coxeter complex*** according to whether the associated Coxeter system is spherical or affine. ♣

E.3 Abstract Buildings

Definition E.22. A *thick* chamber complex X is called a ***building*** if there is a set $\mathcal{A}$ of chamber subcomplexes of X, called ***apartments***, so that each $A \in \mathcal{A}$ is a *thin* chamber complex, and

- given two simplices $x, y \in X$, there is an apartment $A \in \mathcal{A}$ containing both x and y.
- if two apartments $A, B \in \mathcal{A}$ both contain a simplex x and a chamber C, then there is a chamber complex isomorphism $\phi : A \to B$ which fixes both x and C *point-wise*, that is, not only fixes x and C but also fixes all simplices which are faces of x or of C.

The set $\mathcal{A}$ is a ***system of apartments*** in the chamber complex X. ♣

Theorem E.23. *Each apartment in a building is necessarily a* Coxeter complex $\Sigma(W, S)$ *for some Coxeter system* (W, S). *Furthermore, this Coxeter system does not depend on the choice of the apartment.* ♣

Proof. We refer to [Gar97, Chapter 4, Corollary 4.3]. ∎

Definition E.24. A building is called ***spherical*** if the Coxeter complex associated to its apartments is a spherical Coxeter complex. A building is called ***affine*** if the associated Coxeter complex is affine. ♣

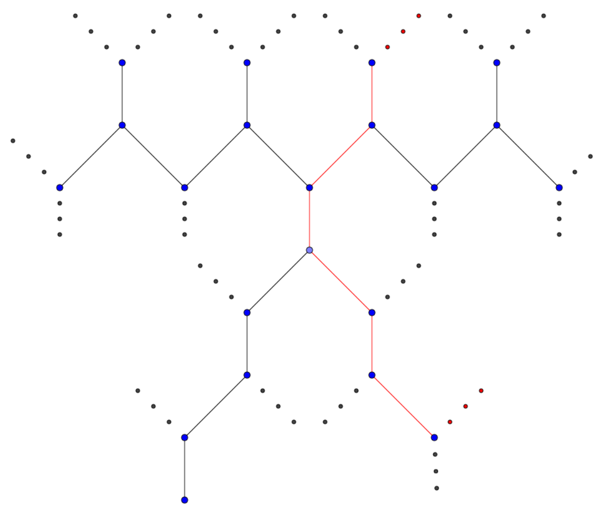

Figure E.1: A tree of degree 3

Example E.25. The simplest example of a building is a regular tree of degree 3. The chambers are are the edges (or any two vertex subsets which are adjacent) and an apartment is given by a sequence of vertices $\{x_1, x_2, \ldots\}$ where each x_i is adjacent to x_{i+1}. The faces are the vertices and by assumption every vertex is a part of exactly 3 edges and hence its a thick chamber complex. The Figure E.1 shows a pictorial representation. A chamber is any line segment between any two points. An apartment is any continuous system of line segments (an example in all red is shown). The dots mean that the tree continues in all directions. ♠

E.4 Spherical Building for $\boldsymbol{GL}_n$

In this section we will construct a spherical building where the group $\boldsymbol{GL}_n$ acts nicely.

E.4.1 Construction

Let V be a $n-$dimensional vector space over a field $\mathbb{F}$. Let $G = \boldsymbol{GL}_n(k) = \boldsymbol{GL}(V)V$. Let Ξ be the set of proper non-trivial subspaces of V (that is subspaces that are not V or $\{0\}$). We have an incidence relation $\sim$ on Ξ defined as follows: $x \sim y$ if either $x \subseteq y$ or $y \subseteq x$.

The Simplicial Complex: We define the associated *flag complex* X as the simplicial complex with vertices Ξ and the simplices which are mutually incident subspaces of Ξ. That is, the simplices in X are subsets σ of Ξ so that, for all $x, y \in \sigma, x \sim y$.
The maximal simplices in X are in bijection with sequences (maximal flags)

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1}$$

of subspaces V_i of V where V_i is of dimension i.

System of Apartments:

Definition E.26. A ***frame*** in V is an unordered $n-$tuple

$$\mathcal{F} = \{\lambda_1, \dots, \lambda_n\}$$

of one dimensional subspaces λ_i in V so that

$$\lambda_i \oplus \cdots \oplus \lambda_n = V.$$

♣

Given such a frame $\mathcal{F} = \{\lambda_1, \dots, \lambda_n\}$, we define a subcomplex $A = A_{\mathcal{F}}$ consisting of all simplices σ with vertices which are subspaces ξ expressible as

$$\xi = \lambda_{i_1} \oplus \cdots \oplus \lambda_{i_m}$$

for some $m-$tuple $i_1, \dots, i_m$.

We define a set $\mathcal{A}$ to be the set of all $A_{\mathcal{F}}$ for all possible frames $\mathcal{F}$.

Theorem E.27. *For every frame $\mathcal{F}$, the subcomplex $A_{\mathcal{F}}$ of X is a thin chamber complex. The simplicial complex X defined above is a thick building with the set $\mathcal{A}$ as its system of apartments. This building is of type A_{n-1}, that is the associated Coxeter complex is of type A_{n-1}.* ♣

Proof. We refer to [Gar97, §9.2]. ∎

E.5 Building from a Tits System

In this section, we construct a building for a group G with a Tits system (B, N). The building will be called an ***affine building*** if the Tits system is affine and it will be called a ***spherical building*** if the Tits system is spherical. We will mostly be concerned with affine buildings as they have richer properties and allow us to deduce many properties of groups over $p-$adic fields. Thus, here we only describe buildings associated to affine Tits system.

E.5.1 Construction of Building

Let G be a group with an affine Tits system (G, B, N, S). Recall that B is said to be the standard Iwahori subgroup and parabolic subgroups containing B are the standard parahoric subgroups. Recall (Theorem D.23) also the one to one correspondence between parahoric subgroups and facets of the fundamental chamber.

The Simplicial Complex

Definition E.28. For a parahoric subgroup P of G, let $F(P)$ be the facet of the chamber C corresponding to P. The ***building*** associated with the given Tits system is the set

$$\mathcal{B} \;:=\; \mathcal{B}_{(B,N)} \;:=\; \{\, (P, x) \mid P \text{ a parahoric of } G, \text{ and } x \in F(P)\,\}\,.$$

♣

Remark E.29. At this stage, this is just a set and we are not justified in calling it a building. However, we will continue to do so and prove later that it is indeed a building in the sense of Definition E.22. ❇

Definition E.30. With each parahoric subgroup P, we associate the subset $\mathfrak{F}(P)$ of $\mathcal{B}$, where

$$\mathfrak{F}(P) \;:=\; \{\, (P, x) \mid x \in F(P)\,\}\,.$$

Let $\mathcal{B}$ be the building associated to the given Tits system. A subset of the form $\mathfrak{F}(P)$ is called a ***facet of the building*** $\mathcal{B}$ of ***type*** $F(P)$. ♣

Remark E.31. Note that the set $\mathfrak{F}(P)$ is formally the same as the facet associated to P, but is now viewed as a subset of the building $\mathcal{B}$. ❇

Definition E.32. In particular, if P is an Iwahori subgroup, the set $\mathfrak{F}(P)$ is defined to be a ***chamber of the building*** $\mathcal{B}$. ♣

Definition E.33. Let $\mathfrak{F}(P)$ be a facet of the building $\mathcal{B}$. Then we define its 'closure' in the building to be

$$\overline{\mathfrak{F}(P)} \;:=\; \bigcup_{Q \supseteq P} \mathfrak{F}(Q)$$

♣

Remark E.34. Note that we put closure in quotes because we do not have a topology on the building yet. Furthermore, there are only finitely many parahoric subgroups containing P. ❇

Proposition E.35. *The building $\mathcal{B}$ defined above is a simplicial complex with open simplices $\mathfrak{F}(P)$ and closed simplices $\overline{\mathfrak{F}(P)}$.* ♣

The group G acts on the building $\mathcal{B}$ by conjugation action on the parahoric subgroups. Thus for $g \in G, (P,x) \in \mathcal{B}$, we define

$$g \cdot (P,x) \;:=\; (g\,P\,g^{-1}, x).$$

The System of Apartments

Definition E.36. A ***standard apartment*** of the building $\mathcal{B}$ is the subset $\mathcal{A}_0$ of $\mathcal{B}$ defined as

$$\mathcal{A}_0 \;:=\; \bigcup_{w \in W} \overline{\mathfrak{F}\,(w\,B\,w^{-1})}.$$

Equivalently, $\mathcal{A}_0$ is a union of the facets $n\mathfrak{F}(P)$ where $n \in N$ and $P \supseteq B$. ♣

Definition E.37. An ***apartment*** $\mathcal{A}$ of the building $\mathcal{B}$ is a set of the form $g \cdot \mathcal{A}_0$ for $g \in G$. ♣

Remark E.38. We note that $n \cdot \mathcal{A}_0 = \mathcal{A}_0$ and hence there are 'G/N-many' different apartments. ❇

Proposition E.39. *There exists a unique bijection $j : \mathbf{A} \to \mathcal{A}_0$ such that* *(i) for each facet F of the chamber* $\mathbf{C}$ *and each* $x \in F$,

$$j(x) = (P_F, x);$$

(ii) $j \circ w = w \circ j$ *for all* $w \in W$.

♣

What Proposition E.39 really says is this: the facet $\mathfrak{F}(B)$ corresponding to B is the chamber $\mathbf{C}$ as a set and an apartment is just these chambers glued together by N; which in turn makes the apartment in bijection with the affine space $\mathbf{A}$ because N surjects on $\mathbf{W}$ and $\mathbf{W}$ acts transitively on the chambers in $\mathbf{A}$. Thus ultimately, an apartment in the building is just an affine space tiled by the hyperplanes coming eventually from the root system associated to $\mathbf{A}$.

On the other hand, the building itself can be thought of as a collection of apartments glued together.

Theorem E.40. *The building* $\mathcal{B}$ *constructed above is a building in the sense of Definition E.22.* ♣

Proof. We refer to [Mac71, Chapter 2]. ■

Remark E.41. This is really not difficult and in fact almost clear from what we have done and a little reflection about the meanings of everything. We give a reference to not to have to write down every detail which is already written down in literature in a nice and readable fashion. ❇

Proposition E.42. *The stabiliser of a facet* $\mathfrak{F}(P)$ *is the parahoric* P. *Thus, maximal parahorics are the stabilisers of the vertices of chambers. Hence, there are* $l+1$ *many conjugacy classes of maximal parahorics, where* $l+1$ *is the number of vertices of the chamber* $\mathbf{C}$. ♣

Proof. We refer to [BT72, Proposition (2.1.5)] for the first statement. The second statement follows immediately. The third follows from the second and the fact that the Weyl group conjugates all chambers and all facets of a given type. ■

Theorem E.43. *For simply connected simple groups, the maximal compact subgroups are exactly the maximal parahoric subgroups and hence are stabilisers of points in the building.* ♣

Proof. We refer to [Mac71, Page 35]. ■

Appendix F

Trace Formula

The trace formula was developed by Selberg in his seminal paper [Sel56]. Then James Arthur further developed it during the period 1974–2005. Today it is one of the major tools used in Number Theory, Harmonic Analysis and various other areas of research.

We give here a (very) concise description of the trace formula, in so far it is useful for our purposes.

F.1 The Cocompact Case

Let G be a locally compact, unimodular topological group and let Γ be a discrete subgroup of G. The space $\Gamma\backslash G$ of right cosets has a right $G-$invariant Borel measure. Let R be the unitary representation of G by right translation on the corresponding Hilbert space $L^2(\Gamma\backslash G)$. Thus,

$$(R\,(y)\,\phi)\,(x) \;:=\; \phi\,(xy)\,, \qquad \phi \in L^2(\Gamma\backslash G), x, y \in G.$$

A fundamental problem in representation theory is to decompose this representation in terms of the irreducible representations of G. However, it is not a priori clear what this should mean except in the simplest case when the representation R decomposes discretely.

As is usual in representation theory, we study this representation closely by extending the domain of R from G to the space $C_c\,(G)$. That is, for $f \in C_c\,(G)$, we define

$$R\,(f) \;:=\; \int_G f(y)R(y)\,\mathrm{d}y.$$

This integral can be defined rigorously as the integral of a Banach valued function. However, we take the more naive approach and assume that that

$$(R(f)\phi)(x) = \int\limits_G (f(y)R(y)\phi)(x)\,\mathrm{d}y.$$

Expanding this expression further, we get

$$\begin{aligned}(R(f)\phi)(x) &= \int\limits_G (f(y)R(y)\phi)(x)\,\mathrm{d}y \\ &= \int\limits_G f(y)\phi(xy)\,\mathrm{d}y \\ &= \int\limits_G f\left(x^{-1}y\right)\phi(y)\,\mathrm{d}y \\ &= \int\limits_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f\left(x^{-1}\gamma y\right)\right)\phi(y)\,\mathrm{d}y,\end{aligned}$$

for any $\phi \in L^2(\Gamma\backslash G)$ and $x \in G$. It follows that $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f\left(x^{-1}\gamma y\right).$$

The sum over γ is finite for any x and y, since it may be taken over the intersection of the discrete group Γ with the compact subset $x\operatorname{supp}(f)\,y^{-1}$ of G.

For the rest of this section, we consider the special case when $\Gamma\backslash G$ is *compact*. The operators $R(f)$ then acquire two very amenable properties:

1. The representation R decomposes discretely into irreducible representations π with finite multiplicities $m(\pi,R)$. That is,

$$R = \bigoplus_{\pi\in\widehat{G}} m(\pi,R)\,\pi. \tag{F.1}$$

2. For many functions f, the operator $R(f)$ is of trace class with

$$\operatorname{tr} R(f) = \int\limits_{\Gamma\backslash G} K(x,x)\,\mathrm{d}x. \tag{F.2}$$

Spectral Decomposition. The decomposition in Equation (F.1) is called the spectral decomposition of R and it gives us a way to compute the trace of operator $R(f)$. Indeed, we have

$$\operatorname{tr} R(f) = \sum_{\pi} m(\pi, R) \operatorname{tr} \pi(f) \tag{F.3}$$

Geometric Decomposition. The formula (F.2) will ultimately give rise to what is called the geometric decomposition of $\operatorname{tr} \pi(f)$. Indeed, if we write $\{\Gamma\}$ for the set of representatives of conjugacy classes in Γ and A_γ for the centraliser of γ in $A \subseteq G$, we get

$$\begin{aligned}
\operatorname{tr}(R(f)) &= \int_{\Gamma\backslash G} K(x,x)\,\mathrm{d}x \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f\left(x^{-1}yx\right) \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f\left(x^{-1}\delta^{-1}\gamma\delta x\right)\mathrm{d}x \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f\left(x^{-1}\gamma x\right) \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f\left(x^{-1}u^{-1}\gamma u x\right)\mathrm{d}u\,\mathrm{d}x \\
&= \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}\left(\Gamma_\gamma\backslash G_\gamma\right) \int_{G_\gamma\backslash G} f\left(x^{-1}\gamma x\right).
\end{aligned}$$

These manipulations can be justified rigorously using measure theory which we skip here. Hence, we get

$$\operatorname{tr}(R(f)) = \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}\left(\Gamma_\gamma\backslash G_\gamma\right) \int_{G_\gamma\backslash G} f\left(x^{-1}\gamma x\right). \tag{F.4}$$

This is called the geometric decomposition of $\operatorname{tr}(R(f))$.

Trace Formula. Comparing the two formulas (F.3) and (F.4) gives us a *trace formula*:

$$\sum_{\pi} m(\pi, R) \operatorname{tr} \pi(f) = \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}\left(\Gamma_\gamma\backslash G_\gamma\right) \int_{G_\gamma\backslash G} f\left(x^{-1}\gamma x\right). \tag{F.5}$$

For more details about this cocompact case we refer to [DE14, Chapter 9] and also [Art05, §1].

This concludes our discussion when the quotient is compact. When the quotient is not compact, the analysis is significantly abstruse chiefly because the two properties mentioned above fail which necessitates a deeper investigation of both the spectral and the geometric sides. We proceed to do that below.

F.2 The Non Compact Case

As already mentioned when the quotient $\Gamma\backslash G$ is non-compact, the two amenable properties fail rather badly. In particular, the kernel K is not integrable over the diagonal, which leads to a complete breakdown of the analysis carried out above for the geometric side. We proceed to show what how Arthur modified the kernel K to tackle the situation. This was first carried out in [Art78, Art80] and then further modified and refined in his subsequent work. We provide the basic import of [Art78].

Arthur does everything in the more general context of adeles and adelic points of reductive groups, so we switch to that setting now. From now on, we again use the notation in Chapter 2 freely.

F.2.1 Modifying the Kernel

We want to modify the kernel function $K(x,x)$ on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ so that it is integrable.

Notation F.6. For a given standard parabolic subgroup P, we write τ_P for the characteristic function of the subset

$$\mathfrak{a}_P^+ = \{\, t \in \mathfrak{a}_P \mid \alpha(t) > 0,\ \alpha \in \Delta_P \,\}$$

of $\mathfrak{a}_P$. ♦

Example F.7. In the case of $\boldsymbol{SL}_3$, this subset is the open cone generated by $\varpi_1^\vee$ and $\varpi_2^\vee$ as shown in Figure F.1. ♠

Notation F.8. We also write $\hat{\tau_P}$ for the characteristic function of the subset

$$\left\{\, t \in \mathfrak{a}_P \;\middle|\; \varpi(t) > 0,\ \varpi \in \hat{\Delta_P} \,\right\}$$

of $\mathfrak{a}_P$. ♦

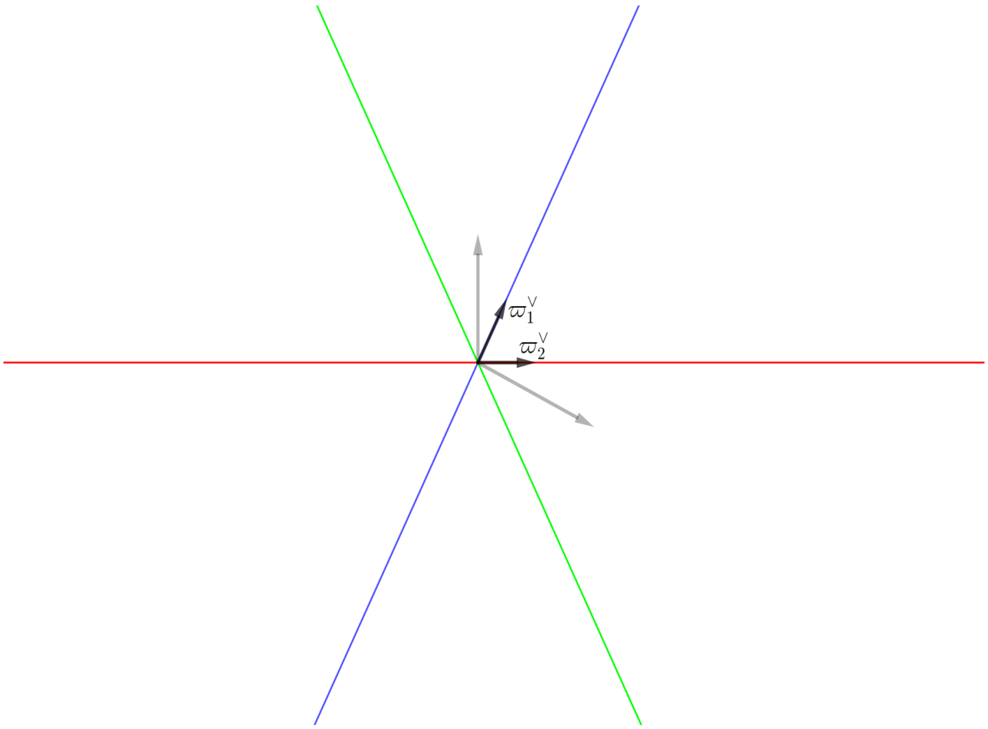

Figure F.1: The subset $\mathfrak{a}_P^+$ is spanned by $\varpi_1^\vee$ and $\varpi_2^\vee$ when $G = \boldsymbol{SL}_3$

Example F.9. In case $G = \boldsymbol{SL}_3$, this subset is the open cone generated by $\beta_1^\vee$ and $\beta_2^\vee$ in Figure F.2. ♠

Now we want to 'truncate' the kernel K so that it becomes integrable. We do this by selecting a suitable point in the cone associated to the minimal parabolic P_0.

Notation F.10. We will fix a parameter T in the cone $\mathfrak{a}_0^+ := \mathfrak{a}_{P_0}^+$ that is suitably regular, in the sense that $\beta(T)$ is large for each root $\beta \in \Delta_0$. ♦

Definition F.11. For any given T, we define

$$\begin{aligned} k^T(x) &:= k^T(x, f) \\ &:= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} K_P(\delta x, \delta x)\, \hat{\tau_P}(H_P(\delta x) - T)\,. \end{aligned} \tag{F.12}$$

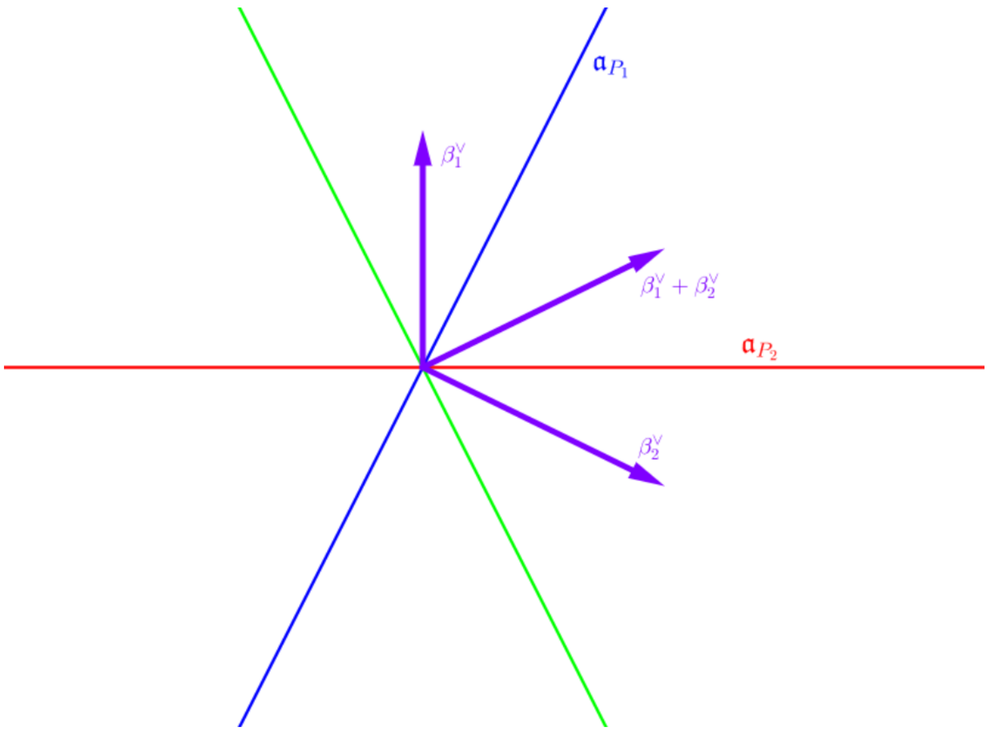

Figure F.2: $\hat{\tau_P}$ is the characteristic function of the open cone generated by $\beta_1^\vee$ and $\beta_2^\vee$ in case of $\boldsymbol{SL}_3$

Then the function k^T is called the ***modified kernel*** or the ***truncated kernel***. ♣

Remark F.13. One can show that for any x, the sum over δ in (F.12) may be taken over a finite set. Thus, $k^T(x)$ is given by a double sum over (P,δ) in a finite set. It is well defined function of $x \in G(\mathbb{Q}) \backslash G(\mathbb{A})^1$. ❇

Remark F.14. The term corresponding to $P = G$ in (F.12) is just $K(x,x)$. In case $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$ is compact, there are no proper parabolic subgroups P (over $\mathbb{Q}$) and hence $k^T = K$ in this case and the truncation is trivial. ❇

In a way, this truncation has solved our problem as the modified kernel is indeed integrable as shown by the following

Theorem F.15. *([Art05, Theorem 6.1]). The integral*

$$\mathfrak{J}_G^T(f) := \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x,f)\,\mathrm{d}x \tag{F.16}$$

converges absolutely for sufficiently regular T. ♣

Definition F.17. The distribution $\mathfrak{J}_G^T$ is called the ***truncated distribution***. ♣

F.2.2 The Coarse Geometric Expansion

Thus, truncation solved the problem of non-integrability. However, it has introduced another problem—the fact, that the integral now no longer represents the trace of an operator! Since the original trace formula was based on comparing the two expressions for the trace of an operator, that approach is now not available to us any more. The real genius of Arthur lies in the fact that instead of giving up at this stage, he instead modified both sides of Formula (F.5) in a way that he could still preserve the equality. However, none of the sides now represented the trace of an operator—but that was immaterial as the equality still contained valuable information. Nonetheless, the name trace formula has stuck in literature even long after trace has had something to do with it.

We briefly describe Arthur's further refinement of the modified kernel on the geometric side. We begin by recalling a result from the theory of algebraic groups.

Proposition F.18. *For any element $\gamma \in G(\mathbb{Q})$ we have a unique* **Jordan decomposition** *$\gamma = \mu\nu = \nu\mu$ with μ being semisimple and ν being unipotent.* ♣

Definition F.19. For $\gamma_1, \gamma_2 \in G(\mathbb{Q})$, write $\gamma_1 \sim_w \gamma_2$ if the semisimple parts of γ_1 and γ_2 are conjugate in $G(\mathbb{Q})$. The equivalence classes of this relation are called ***coarse geometric classes*** and are denoted by $\tilde{\mathfrak{o}}$. Let $\tilde{\mathfrak{O}}$ be the set of all equivalence classes under this relation. ♣

Remark F.20. This relation is weaker than that of the usual conjugacy relation (hence the adjective 'coarse'). The entire set of unipotent elements in $G(\mathbb{Q})$ is an equivalence class under the above relation. Any two unipotent elements may not, however, be conjugated. ❇

Although, not immediately relevant, we also make another definition which is important later on.

Definition F.21. For $\gamma_1, \gamma_2 \in G(\mathbb{Q})$, write $\gamma_1 \sim_g \gamma_2$ if there exists a $x \in G\left(\overline{\mathbb{Q}}\right)$ which conjugates them. That is, $x\,\gamma_1\,x^{-1} = \gamma_2$. The equivalence classes of this relation are called ***geometric conjugacy classes*** and are denoted by $\overline{\mathfrak{o}}$. The set of all geometric conjugacy classes will be denoted by $\overline{\mathfrak{O}}$. ♣

Finally,

Definition F.22. We shall say that a semisimple conjugacy class in $G(\mathbb{Q})$ is ***anisotropic*** or ***elliptic*** if it does not intersect $P(\mathbb{Q})$, for any $P \subsetneq G$. ♣

We will now breakdown the kernel K and the modified kernel k^T in terms of the coarse geometric classes.

Notation F.23. For $\tilde{\mathfrak{o}} \in \tilde{\mathfrak{O}}$, we put

$$K_{\tilde{\mathfrak{o}}}(x,x) := \sum_{\gamma\in\tilde{\mathfrak{o}}} f\left(x^{-1}\gamma x\right).$$

♦

Then we can write

$$K(x,x) = \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} K_{\tilde{\mathfrak{o}}}(x,x).$$

More generally, for any parabolic subgroup P,

$$\begin{aligned} K_P(x,x) &:= \sum_{\gamma\in M_P(\mathbb{Q})} \int_{N_P(\mathbb{A})} f\left(x^{-1}\gamma n x\right) \mathrm{d}n \\ &= \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} K_{P,\tilde{\mathfrak{o}}}(x,x), \end{aligned}$$

where

$$K_{P,\tilde{\mathfrak{o}}}(x,x) := \sum_{\gamma\in M_P(\mathbb{Q})\cap\tilde{\mathfrak{o}}} \int_{N_P(\mathbb{A})} f\left(x^{-1}\gamma n x\right) \mathrm{d}n.$$

Furthermore, put

$$\begin{aligned} k^T_{\tilde{\mathfrak{o}}}(x) &= k^T_{\tilde{\mathfrak{o}}}(x,f) \\ &= \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\tilde{\mathfrak{o}}}(\delta x,\delta x)\,\hat{\tau}_P\left(H_P(\delta x)-T\right). \end{aligned}$$

Definition F.24. We therefore have a decomposition

$$k^T(x) = \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} k^T_{\tilde{\mathfrak{o}}}(x) \tag{F.25}$$

This is called the ***coarse geometric expansion of the kernel***. ♣

So we have broken down the truncated kernel according to the coarse geometric classes. Now Theorem F.15 says that the truncated kernel is absolutely integrable. That is,

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \left| \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} k^T_{\tilde{\mathfrak{o}}}(x) \right| \mathrm{d}x$$

is finite for sufficiently regular T. The real utility of the decomposition (F.25) lies in the fact that we can take the sum out of the absolute value sign and the integral, thereby giving rise to the orbital distributions. We state

Theorem F.26. *([Art78, Theorem 7.1]). For all sufficiently regular T, the sum*

$$\sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \left| k^T_{\tilde{\mathfrak{o}}}(x) \right| \mathrm{d}x \tag{F.27}$$

is finite. ♣

This allows us to make the

Definition F.28. For $\tilde{\mathfrak{o}} \in \tilde{\mathfrak{O}}$, the ***truncated orbital distribution*** $\mathfrak{J}^T_{\tilde{\mathfrak{o}}}$ is defined by

$$\mathfrak{J}^T_{\tilde{\mathfrak{o}}}(f) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \left| k^T_{\tilde{\mathfrak{o}}}(x) \right| \mathrm{d}x. \tag{F.29}$$

♣

Thus, we can write

$$\mathfrak{J}^T_G(f) = \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} \mathfrak{J}^T_{\tilde{\mathfrak{o}}}(f), \tag{F.30}$$

for all sufficiently regular T. In particular, we can set $T = T_0$, where the point $T_0 \in \mathfrak{a}_0$ is characterised in [Art81, Lemma 1.1], and hence obtain the

Theorem F.31.

$$\mathfrak{J}_G = \sum_{\tilde{\mathfrak{o}}\in\tilde{\mathfrak{O}}} \mathfrak{J}_{\tilde{\mathfrak{o}}}. \tag{F.32}$$

This is called the **coarse geometric expansion of the Arthur distribution**. ♣

Here, of course, the Arthur distribution $\mathfrak{J}_G$ and the orbital distributions $\mathfrak{J}_{\tilde{\mathfrak{o}}}$ are defined by

Definition F.33.

$$\mathfrak{J}_G := \mathfrak{J}_G^{T_0}, \tag{F.34}$$

$$\mathfrak{J}_{\tilde{\mathfrak{o}}} := \mathfrak{J}_{\tilde{\mathfrak{o}}}^{T_0}. \tag{F.35}$$

These are called the ***Arthur distribution*** and the ***orbital distribution*** respectively. ♣

Remark F.36. The term "Arthur distribution" is not standard in literature. Usually the terms distributions and orbital distributions are unfortunately rather loosely thrown around. We take this opportunity to christen this term so as to avoid misunderstandings and honour James Arthur. ❇

Remark F.37. One might wonder at this stage that what, if any, the point of singling out the truncated distribution at a particular point T_0 is. The answer is that the truncated distributions depend on our initial choices of M_0, P_0 and $\mathbf{K}$. The distribution however, depends only on M_0 and $\mathbf{K}$ and is independent of P_0. This becomes important for later applications of the eventual trace formula. ❇

We next single two particularly important orbital distributions. We begin with the

Definition F.38. As remarked earlier, the set of unipotent elements in $G(\mathbb{Q})$ forms a coarse geometric class. We denote it by $\tilde{\mathfrak{o}}_{\text{unip}}$. This is called the ***unipotent orbit***. ♣

We single out the orbital distribution corresponding to the unipotent orbit.

Definition F.39. The ***unipotent distribution*** is defined as

$$\mathfrak{J}_{\text{unip}} := \mathfrak{J}_{\tilde{\mathfrak{o}}_{\text{unip}}}. \tag{F.40}$$

Similarly, we can define the ***truncated unipotent distribution*** $\mathfrak{J}_{\text{unip}}^T$ for a parameter T as

$$\mathfrak{J}_{\text{unip}}^T := \mathfrak{J}_{\tilde{\mathfrak{o}}_{\text{unip}}}^T. \tag{F.41}$$

♣

Every element $z \in Z(\mathbb{Q})$ in the centre of $G(\mathbb{Q})$ forms a coarse geometric class by itself. That is, for every $z \in Z(\mathbb{Q}), \{z\}$ is a coarse geometric class. We combine all of these orbital distributions in

Definition F.42. The ***central distribution*** $\mathfrak{J}_{Z(\mathbb{Q})}$ is defined as

$$\mathfrak{J}_{Z(\mathbb{Q})} \;:=\; \sum_{z \in Z(\mathbb{Q})} \mathfrak{J}_{\{z\}}. \tag{F.43}$$

Similarly, we can define the ***truncated central distribution*** $\mathfrak{J}^T_{Z(\mathbb{Q})}$ for a parameter T as

$$\mathfrak{J}^T_{Z(\mathbb{Q})} \;:=\; \sum_{z \in Z(\mathbb{Q})} \mathfrak{J}^T_{\{z\}}. \tag{F.44}$$

♣

Correspondingly, we can define the rest of the contribution from the 'non-central' elements.

Definition F.45. The ***non-central distribution*** $\mathfrak{J}_{\mathrm{nc}}$ is defined as

$$\mathfrak{J}_{\mathrm{nc}} \;:=\; \mathfrak{J}_G - \mathfrak{J}_{Z(\mathbb{Q})}. \tag{F.46}$$

Similarly, we can define the ***truncated non-central distribution*** $\mathfrak{J}^T_{\mathrm{nc}}$ for a parameter T as

$$\mathfrak{J}^T_{\mathrm{nc}} \;:=\; \mathfrak{J}^T_G - \mathfrak{J}^T_{Z(\mathbb{Q})}. \tag{F.47}$$

♣

Remark F.48. Since unipotent elements cannot be central, the unipotent distribution contributes solely to the non-central distribution. ❇

Thus, to summarise, we truncated the kernel so that it becomes integrable. Then we broke down the kernel according to the coarse geometric classes and then thanks to Theorem F.26, we were assured that each of the parts were themselves integrable. Thus, we got an (truncated) orbital distribution for each coarse geometric class which together sum to the (truncated) distribution.

This is just one half of the picture of course. We have just rewritten the geometric side in terms of the coarse geometric classes, which although useful is not in itself the trace formula. The real power comes from the fact that one can also decompose the spectral side in a similar manner in terms of what is called the "cuspidal automorphic data". Then equating the expansion of the distribution in terms of the coarse geometric classes on the geometric side and the cuspidal automorphic data on the spectral side gives us an identity, which is called the trace formula.

The great importance of the trace formula is underlined by the multitude of (expository) articles available on it. The case of $\boldsymbol{GL_2}$ was first discussed in [GJ79]. This article was written before Arthur's original 1978 article and hence the treatment is a little dated. A more modern treatment of the $\boldsymbol{GL}_2$ case may be found in [Gel96] and [Kna97]. Arthur's own exposition with proof ideas in case of $\boldsymbol{GL}_3$ may be found in [Art05].

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Notation Index

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

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